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Class Plan

Date	Plan	Completed
24.09	0.1, 0.2, 0.3, 1.1 (maybe 1.2)	
26.09	Finish 1.2, 1.3, preparation sheet 1	

0.1 About Me

- Name: Toby Lane
- Languages: Native English & French, Working Proficiency in German (enough to answer questions in German, albeit with mistakes)
- In final year of physics bachelor's
- Had Prof. Figalli in Analysis I as well
- Email: tolane@ethz.ch
- If you *really* want to know more, you can go look at my LinkedIn page

0.2 Important info

- Class Document: TODO
- Exercise classes on Wednesdays and Fridays
- You can change groups until the 1st October, so you can try out multiple TAs and see what style you like
  - The course organisers request that you go to the class you signed up for, but you don't have to
- My class in English, others in English or German, one in Italian
- From 3rd or 4th week: quizzes on Friday mornings (don't be late!)
  - you will get a bonus of up to 0.25 on your final AI grade
  - you won't need 100% correct to get the full bonus
  - MC questions or short calculations (like in 1st part of exam)
  - more info to come
- Plan (⚠ may (probably will) be subject to change):

Week	**Week n**	**Week $n+1$**		**Week $n+2$**	
Day	**Friday**	**Wednesday**	**Friday**	**Wednesday**	**Friday**
Sheets	n released	$n-1$ due	$n+1$ released	n due	$n+2$ released
Class plan	Quiz Correction $n-2$ Preperation $n-1$	Theory	Quiz Correction $n-1$ Preparation n	Theory	Quiz Correction n Preparation $n+1$

- You can find past papers on https://exams.vmp.ethz.ch
- You can find the script from 2 years ago on Metaphor (pretty much the same one as yours, but complete)

0.3 Tips for Solving Exercises

- Don't worry if you can't solve all of them! This is normal; they are meant to challenge you... The exams will be easier (go look at past papers to get an idea)!
- The important part is to try them all, see how far you can get, and make sure you understand the corrections. I'm always available for questions if need be!
- Use the internet to help! You are expected to, some exercises will be on topics not 100% covered in the lecture
- Generative AI (e.g. ChatGPT) can also be a great help if used correctly...
  - Ask for help solving a smaller part of the exercise you are stuck on
  - Ask for hints to solve an exercise if you're stuck (quite good at this)
  - Ask to look for mistakes (less good)
  - Don't ask the AI to solve it *for you*... Unless you're *really* stuck. But then make sure to understand what it says!

1 Mathematical Writing, Proofs and Rigour

1.1 Note-Taking

1.1.1 Personal recommendations

- Taking notes: I use an IPad or tablet (apps: GoodNotes, etc.) $\rightarrow$ lecturers often make mistakes or have to come back and modify things, it's much easier to manage!
- If you want to use a computer: LaTeX is OK, but Typst much quicker! (under no circumstances attempt to take maths notes e.g. in Word)… But ⚠️ much more time-consuming to do drawings
- I personally like to type up my notes after the course as part of my exam preparation -> I use Typst for this

1.1.2 Formatting Lecture Components

- Theorems:

(H) Hypotheses: what you assume to hold

(T) Thesis (or theses): what holds if (H) is true (what you want to proove)

(P) Proof

(Identical for lemmas, propositions, statements, etc.)

Example (we will go over the symbols again later):

Theorem 1.1.2.1: Property of a Group

(H) G a group
$g \in G$

(T_1) $\exists!\ a^{-1}$

(T_2) $\left(a^{-1}\right)^{-1} = a$

(P_1) …

(P_2) …

Also works for other types, e.g.:

Definition 1.1.2.2: Cartesian Product

(H) X, Y sets

(D) $X \times Y := \{(x, y) \mid x \in X, y \in Y\}$

Will make things much easier to follow and understand!

1.2 Mathematical Symbols

Don't worry about learning them all of by heart now, this will come naturally when you do exercises!

Symbol	**Meaning**	**Example**	**Remarks**
Logic			
$\Rightarrow, \Leftarrow$	implication	I am in ZH $\Rightarrow$ I am in CH	
$\Leftrightarrow$	equivalence	You can board a plane $\Leftrightarrow$ you have a ticket	
$\not\Rightarrow$, etc.	doesn't imply	I am wet $\not\Rightarrow$ it is raining	⚠ not same as implies not
$\neg$	logical negation		
$\wedge, \vee$	logical and / or		
$\exists \ldots (\ \$\ \ldots)$	exists … such that …	$\exists$ habitable planet	
$\exists! \ldots (\ \$\ \ldots)$	exists … such that … and it is unique	$\exists!$ habitable planet $\$$ we are on it	
$\nexists \ldots (\ \$\ \ldots)$	exists no … such that …	$\exists!\ \text{person}\ \$\ \text{person on mars}$	
$\forall \ldots, \ldots$	for all …, … holds	$\forall\ \text{person}, \text{person in Solar System}$	
Set Theory			
$x \in A$	the **element** x is in the **set** A	peter rabbit $\in$ mammals	
$X \subset A$ (also $\supset, \not\subset, \subsetneq, \subseteq$, etc.)	for all element x in the **set** X, x is in A	rabbits $\subset$ mammals	
$\{A \mid B\}$	the set of A such that B	mammals $=$ $\{$animals $\mid$ produce milk$\}$	
$\cap, \cup$	set of elements in A or / and B ("intersection" / "union")	animals $=$ vertebrates $\cup$ invertebrates	Also called set and / or – Notice similarity to logical and / or
$A \setminus B$	set of elements in A that are not in B	invertebrates $=$ animals $\setminus$ vertebrates	
$\emptyset$	set of nothing	birds that produce milk	
$[a, b]$	real numbers $a \leq x \leq b$		Intervals are sets
$[a, b[$ or $[a, b)$	real numbers $a \leq x < b$		
$\mathbb{N}$ ($\mathbb{N}_0$)	Natural numbers (0 inclusive)		
$\mathbb{Z}$	Integers		
$\mathbb{Q}$	Rationals		
$\mathbb{R}$	Irrationals		
$\mathbb{C}$	Complex numbers		
$\mathbb{I}$	Pure imaginary numbers		
$\mathbb{F}, \mathbb{K}$	General field		
$A^*, A^\times$	Remove 0		

A_+, A_-	Only positive/negative		Typically 0 included, but not in some conventions
Proof markings			
$\overset{?}{=}$, $\overset{?}{\Leftrightarrow}$	An assertion yet to be proven		
$\overset{!}{=}$, $\overset{!}{\Leftrightarrow}$	An assertion that must hold due to (H) or other results		
$:=$, $=:$, $:\Leftrightarrow$, $\Leftrightarrow:$	Defining objects / statements		⚠️ : always go next to object being defined: $X := A$ defines X as A $X =: A$ defines A as X
↯	contradiction, use at the end of proofs by contradiction	[your proof] $\Rightarrow A$ ↯ B, where A and B are logically incompatible	
$\#$, $\square$, QED, etc.	used to mark the end of a proof		
WTS ("Want To Show")	State what remains to be proven	WTS $x = y$	
WLOG ("Without Loss Of Generality")	used to reduce a symmetrical argument to a single case	WTS: for $a, b > 0$, $\frac{a+b}{2} \geq \sqrt{ab}$ Wlog, assume $a > b$ (if not, rename $a \leftrightarrow b$)	(ideally) requires justification But very powerful!

Any many more to come!

Use them, they'll make note-taking quicker and your notes easier to read! (See proof examples below)

1.3 Proofs

1.3.1 Formatting

Theorem 1.3.1.1: Triangle Inequality

(H) $x, y \in \mathbb{R}$

(T) $|x+y| \leq |x| + |y|$

$(\lvert x\rvert+\lvert y\rvert)^2 = \lvert x\rvert^2+\lvert y\rvert^2+2\ \lvert x\rvert\ \lvert y\rvert = x^2+y^2+2\ \lvert x\rvert\ \lvert y\rvert \quad (z^2 = \lvert z\rvert^2)$ $\lvert x+y\rvert^2 = (x+y)^2 = x^2+y^2+2xy$ $(\lvert x\rvert+\lvert y\rvert)^2 - \lvert x+y\rvert^2 = 2(\lvert x\rvert\ \lvert y\rvert - xy) = 2(\lvert xy\rvert - xy)$ (property of $\lvert\cdot\rvert$) $\lvert z\rvert \geq z \Rightarrow \lvert xy\rvert \geq xy$ $\Rightarrow (\lvert x\rvert+\lvert y\rvert)^2 - \lvert x+y\rvert^2 = 2(\lvert xy\rvert - xy) \geq 0$ $\Rightarrow (\lvert x\rvert+\lvert y\rvert)^2 \geq \lvert x+y\rvert^2$ $\Rightarrow \lvert x\rvert+\lvert y\rvert = \lvert\ \lvert x\rvert+\lvert y\rvert\ \rvert \geq \lvert x+y\rvert$ (property of $\cdot^2$) #
Let $x, y \in \mathbb{R}$ We observe: 1. $(\lvert x\rvert+\lvert y\rvert)^2 = \lvert x\rvert^2+\lvert y\rvert^2+2\ \lvert x\rvert\ \lvert y\rvert$ $= x^2+y^2+2\ \lvert x\rvert\ \lvert y\rvert \quad (z^2 = \lvert z\rvert^2)$ 2. $\lvert x+y\rvert^2 = (x+y)^2$ $= x^2+y^2+2xy$ $(1)-(2): \quad (\lvert x\rvert+\lvert y\rvert)^2 - \lvert x+y\rvert^2 = 2(\lvert x\rvert\ \lvert y\rvert - xy)$ $= 2(\lvert xy\rvert - xy)$ (property of $\lvert\cdot\rvert$) Now we know $\lvert z\rvert \geq z \Rightarrow \lvert xy\rvert \geq xy$ $\Rightarrow (\lvert x\rvert+\lvert y\rvert)^2 - \lvert x+y\rvert^2 = 2(\lvert xy\rvert - xy) \geq 0$ $\Rightarrow (\lvert x\rvert+\lvert y\rvert)^2 \geq \lvert x+y\rvert^2$ $\Rightarrow \lvert x\rvert+\lvert y\rvert = \lvert\ \lvert x\rvert+\lvert y\rvert\ \rvert \geq \lvert x+y\rvert$ (property of $\cdot^2$) #
Let x, y real numbers. Using $(z^2 = \lvert z\rvert^2)$, we observe that, first of all, $(\lvert x\rvert+\lvert y\rvert)^2 = \lvert x\rvert^2+\lvert y\rvert^2+2\ \lvert x\rvert\ \lvert y\rvert = x^2+y^2+2\ \lvert x\rvert\ \lvert y\rvert$. In addition, we also see that $\lvert x+y\rvert^2 = (x+y)^2 = x^2+y^2+2xy$. Subtracting the second equation from the first, using the fact that $\lvert xy\rvert = \lvert x\rvert\ \lvert y\rvert$ we obtain that $(\lvert x\rvert+\lvert y\rvert)^2 - \lvert x+y\rvert^2 = 2(\lvert x\rvert\ \lvert y\rvert - xy) = 2(\lvert xy\rvert - xy)$. Now we know that $\lvert z\rvert \geq z$; so it follows that $\lvert xy\rvert \geq xy$ and also $(\lvert x\rvert+\lvert y\rvert)^2 - \lvert x+y\rvert^2 = 2(\lvert xy\rvert - xy) \geq 0$. We can deduce from this that $(\lvert x\rvert+\lvert y\rvert)^2 \geq \lvert x+y\rvert^2$ and so also, using the properties of the square function, $\lvert x\rvert+\lvert y\rvert = \lvert\ \lvert x\rvert+\lvert y\rvert\ \rvert \geq \lvert x+y\rvert$ #

Notes:
- Align your equals and equivalence signs!
- Try to be as rigorous as possible

1.3.2 Proof Types

Type	**Description**	**Use Cases**
Direct	Directly proove the thesis	No better ideas
Contradiction	To show A holds, show that $\neg A$ leads to an impossible result	If it looks easier than direct proof

Contraposition	Instead of showing $A \Rightarrow B$, show $\neg B \Rightarrow \neg A$	If it looks easier than direct proof
Induction	1. Show holds for $n = 0$ or 1 2. Show if it holds for n, then also for $n + 1$	Something holds for every $n \in \mathbb{N}$ or $\mathbb{Z}$

And others to come!

⚠️ disclaimer: "uses cases" are hints as to when it could be a good idea to use them… But there are always exceptions!

Can also be used together to form a larger proof!

Examples:

- **Direct:**

Theorem 1.3.2.1: The sum of two even integers is even

(H) $n, m \in 2\mathbb{Z}$

(T) $n + m \in 2\mathbb{Z}$

(P) Let $n, m \in 2\mathbb{Z}$
Then $\exists k, l \in \mathbb{Z}$ \$ $n = 2k, m = 2l$
$\Rightarrow m + n = 2k + 2l = 2(k + l) = 2j, \quad j \in \mathbb{Z}$
$\Rightarrow m + n \in 2\mathbb{Z} \quad \#$

- **Contradiction:**

Theorem 1.3.2.2: There is no largest real

(T) There is no largest real number

(P) WTS: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ \$ $y > x$
By contradiction, assume $\exists X \in \mathbb{R}$ largest
But then by prop. of $\mathbb{R}$, $X + 1 \in \mathbb{R}$
Yet $X + 1 > X$ ↯ X largest real $\quad \#$

- **Contraposition**

Theorem 1.3.2.3: The product of odd numbers is odd

(H) $m, n \in 2\mathbb{Z} + 1$

(T) $mn \in 2\mathbb{Z} + 1$

(P) We will show the contraposition:
WTS: if $m \in 2\mathbb{Z}$ or $n \in 2\mathbb{Z}$, then $m, n \in 2\mathbb{Z}$
Wlog, assume $m \in 2\mathbb{Z}$ (if not, rename n to m)
Then $m = 2l, l \in \mathbb{Z}$ so $mn = 2ln \in 2\mathbb{Z} \quad \#$

- **Induction**

Theorem 1.3.2.4: Sum of odd numbers

H $n \in \mathbb{N}^*$

T $\sum_{k=1}^{n} (2k-1) = \underbrace{1 + ... + (2n-1)}_{n} = n^2$

P
- Holds for $n = 1$:
  $1 = 1^2$ ✓
- $(n) \Rightarrow (n+1)$:
  $$\begin{aligned} \underbrace{1 + ... + (2(n+1)-1)}_{n+1} &= 1 + ... + (2n-1) + (2n+1) \\ &= n^2 + (2n+1) \quad (\textbf{induction step}) \\ &= (n+1)^2 \quad \# \end{aligned}$$

Theorem: 6
Definition: 1
Lemma: 0
Corrolary: 0
Remark: 0
Property: 0
Example: 0
Statement: 0
Method: 0
Notation: 0
Result: 0
Formula: 0