Rational points on a circle

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1 Circles

Now we consider the case of a circle. This consists of several sub cases:

1. A circle of rational radius, and rational centre
2. A circle of irrational centre
3. A circle of irrational radius and rational centre

1.1 Rational radius and rational centre point

First we consider the case of a circle of rational radius with a rational centre. Here we shall use polar co-ordinates. Hence the co-ordinates are written as follows: $(x, y) = (p + rcos(\theta), q + rsin(\theta))$ where $p, q, r \in \mathbb{Q}$. Now we can see that the points on the circle are rational only when we also have the condition that $cos(\theta)$ and $sin(\theta)$ are rational. It can be proved that this occurs when the sides of the right angled triangle of hypotenuse r and angle θ are Pythagorean triples. This also holds for any constant multiple of Pythagorean triples as they can be scaled accordingly.

Take the unit circle as a simple example, the equation of which is $x^2+y^2 = 1$. Rational points occur when x and y are both rational numbers. Take $x = \frac{a}{c}$ and $y = \frac{b}{c}$ using Cartesian co-ordinates. Then the equation is now $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$, or alternatively $a^2 + b^2 = c^2$ which is Pythagoras' theorem. a,b,c are a Pythagorean triple as they are all integers by the assumption that the fractions substituted as x and y are in their simplest form.

1.2 Irrational centre points

Now we shall consider the case of a circle with an irrational centre point. Let the centre point, O, be an irrational point. Then assume there exist three distinct rational points P,Q,R which all lie on the circumference of the circle. Consider the line segments PQ and QR. Let the midpoint of PQ be a point U. This is a rational point as it is the midpoint of two other rational points. Consider the perpendicular bisect of this line segment PQ. It can be proved using geometry that the perpendicular bisect of a chord such as PQ will pass through the centre of the circle. Therefore we shall refer to this perpendicular bisect as the line segment OU. Given that the coefficients of this chord PQ are all rational, the same can be said for the bisect OU.

Let the midpoint of QR be a point V. This is also a rational point since it is the midpoint of two other rational points. Using the same principles as the previous chord, there also exists a perpendicular bisect OV which passes through the origin. OV is also a line with rational coefficients. However, the intersection of two lines of rational coefficients occurs only at a rational point. Since the origin is defined by our assumptions to be an irrational point, we have a contradiction, and hence it is proved that there can be no more than two rational points on a circle with an irrational centre.

Theorem 1. *A perpendicular bisector of a chord passes through the centre of a circle.*

Proof. Label any two points on a circle A,B as shown in the diagram, and the centre, O.

Join the chord AB, and the radii OA and OB, also illustrated in the diagram. Then by the definition of a radius OA and OB are equal in length.

Bisect the chord AB and label the point of bisection C. By definition of a bisect, the line segments AC and CB are equal in length. Construct a line segment joining the centre, O, with this new point C.

Then there are now two triangles formed. $\triangle AOC$, and $\triangle OBC$. Since the side OC is common, —OA—=—OB—, and —AC—=—CB—, we can use the side-side-side rule to state that these two triangles are similar.

As such, we can see that the two angles ∠ACO, and ∠OCB are equal. By definition, if two angles made from bisecting a straight line are equal, the angles are right angles.

Therefore it has been proved that a bisector of a chord passing through the centre is always a perpendicular bisector. conversely, if you bisect a chord perpendicularly, that bisector will pass through the centre. □