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Electricity price modeling with stochastic time change

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ABSTRACT

In this paper, we develop a novel approach to electricity price modeling, based on the powerful technique of stochastic time change. This technique allows us to incorporate the characteristic features of electricity prices (such as seasonal volatility, time varying mean reversion and seasonally occurring price spikes) into the model in an elegant and economically justifiable way. The stochastic time change introduces stochastic as well as deterministic (e.g., seasonal) features in the price process' volatility and in the jump component. We specify the base process as a mean reverting jump diffusion and the time change as an absolutely continuous stochastic process with seasonal component. The activity rate of the stochastic time change can be related to the factors that influence supply and demand. Here we use the temperature as a proxy for the demand and hence, as the driving factor of the stochastic time change, and show that this choice leads to realistic price paths. We derive properties of the resulting price process and develop the model calibration procedure. We calibrate the model to the historical EEX power prices and apply it to generating realistic price paths by Monte Carlo simulations. We show that the simulated price process matches the distributional characteristics of the observed electricity prices in periods of both high and low demand.

1. Introduction

1.1. Motivation

Electricity prices from liberalized markets exhibit features that are rarely observed in other commodity markets. Besides strong demand-related price seasonalities, electricity prices exhibit large spikes, which arise due to non-storability of electricity, non-elasticity of demand and, in case of renewable electricity generation, of supply. The occurrence of positive price spikes is related to supply interruptions and to the increased demand during periods of abnormally high or low temperatures. Negative spikes — which is a rather recent phenomenon — are related to power generation by e.g., wind farms, which cannot be "switched off". Such price spikes are usually short-lived as prices rapidly return to their "normal" levels once disruptions in supply are resolved. The electricity price volatility is also related to demand, increasing during periods of high demand. This leads to seasonal patterns not just in prices, but also in the volatility. All these features make electricity price modeling a challenging task.

A large variety of electricity price models have been described in the literature. The so-called *reduced-form models* range from jump-diffusions or Levy-driven diffusions (Geman and Roncoroni, 2006, Cartea and Figuera, 2005, Meyer-Brandis and Tankov, 2008, Klüppelberg et al., 2010, Barndorff-Nielsen et al., 2013) to regime switching (Huisman and Mahieu, 2001, Paraschiv et al., 2015) to time series models of the ARMA-GARCH type (Benth et al., 2014). Neural networks, agent-based models, fuzzy systems and other AI-based models also have been extensively applied to electricity prices. An important special class of models — the so-called *structural models* — incorporate external factors such as demand, capacity, load and fuel prices into the electricity price formation process. Weron (2014) provides an excellent recent survey of literature on electricity price modeling and forecasting.

Despite the voluminous literature on modeling electricity prices, there is no clear "winner" model. Such model should be versatile enough to generate important price characteristics (seasonalities, spikes and mean reversion), should incorporate at least some supply- and demand-related information, while it should be tractable enough to be useful in applications such as pricing of derivatives (electricity futures and options). In this paper, we suggest such a tractable continuous time model. This model can incorporate demand and supply proxies as the main driving price factors in a new

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and ingenious way. Our approach is based on the powerful technique of stochastic time change, previously successfully used in modeling other asset prices. The main goal of this modeling exercise is generating price paths with the correct stylized facts and distributional properties, particularly during different seasons (and hence, periods of different demand). This is useful for scenario simulations or any other applications where Monte Carlo generation of realistic price paths is required.

1.2. Stochastic time change

The concept of time-changing of a continuous time stochastic process is a powerful tool for building models in financial mathematics. It allows the model builder to introduce jumps and stochastic volatility into standard models based on the Brownian motion.

Clark (1973) was the first author who observed that prices in financial markets are more volatile on days when a lot of trading takes place. This observation lead to the idea that calender time might not always be the most appropriate way to measure time: in equal calender time intervals, the activity in the market can be very different. On the other hand, we can define a non-equidistant time grid, such that in each time interval the amount of trading activity is the same (the resulting new time scale is said to evolve in *operational time* or *business time*). Consequently, asset returns in each such interval should have the same distributional properties. This has been empirically confirmed by Ané and Geman (2000), who extended Clark's ideas and recovered normality of asset returns by stochastic time change based on order flow.

To define business time, the most important issue is to identify the triggers that determine the speed of the market. For example, Clark (1973) considers the accumulated traded volume as the relevant factor for cotton markets, while Ané and Geman (2000) conclude that in stock markets the appropriate time scale should be defined using the accumulated number of trades.

Stochastic time change can be defined in two distinctive ways. The first one is by means of the so-called *subordinators*, which are non-decreasing Levy processes (pure jump processes plus a linear deterministic drift). Time-changing a Brownian motion by such a subordinator results into another Levy process and hence, introduces jumps into an otherwise continuous stochastic process.

Another way to define it is by means of an absolutely continuous time change. Such time change is defined as the time integral over the so-called *activity rate*, which can be seen as a proxy for the (trading) activity in the market. On a day with a high trading activity, the activity rate is high as well and so, time evolves quicker.

In electricity markets, the spot prices are driven largely by demand, which in turn depends on the outside air temperature. In the summer months, electricity is used for air conditioning and in the winter for heating, which leads to an increased demand for electricity during periods of sufficiently high or low temperatures. A number of previous electricity price studies explore this relationship. For example, Bessec and Fouquau (2008) investigate the effect of temperature on electricity demand in Europe. As this effect depends on the regional climate as well as on heating and cooling habits in different countries, they differentiate between cold, medium and warm countries. In order to filter out the part of the demand that can be explained by the temperature, they remove the effect of other, non-climatic factors on electricity consumption: demographic and technological trends, monthly seasonality (in particular, the decrease in production during summer). They find that the functional relationship between the electricity demand and temperature has a parabolic form, with the minimum at around 16 °C: the "neutral" temperature with respect to energy demand, where neither heating nor airconditioning is needed in continental Europe.

There is also a well-documented positive relationship between energy price volatility and demand. Kanamura (2009) investigates this relationship for natural gas prices and explicitly models gas price volatility as a function of demand. Yang et al. (2002) study this relationship for crude oil prices in US, and a recent study by Jobling and Jamasb (2015) extend this to worldwide oil markets. Li and Flynn (2004) document the relationship between volatility and demand for various electricity markets.

In this paper, we specify an activity rate for the time change in electricity markets on the basis of a demand proxy, which is in our case the temperature. For that, we will explore the functional relationship found by Bessec and Fouquau (2008). However, this is not the only possible choice, and other choices for stochastic clock's activity are possible, for example, those related to the supply (such as wind speed or sunny hours for markets which are largely driven by the wind or solar energy).

When specifying a time-changed stochastic process, we need to decide on the general, basis features of the price evolution. Ané and Geman (2000) show that asset returns are normally distributed, when these are recorded not in calender time, but in business time. Other applications of time-changed processes in finance also try to recover normality in returns. These studies assume a (geometric) Brownian motion as the main driver of the price process in business time, so a Brownian-based diffusion often functions as the basis model for the stock price. This seems to be an important element of the success of those studies. However, electricity prices exhibit more complex features than stock prices, so a Brownian motion alone — neither in an arithmetic nor geometric form — would be appropriate as the basis model. Mean reversion and jumps should be inherent features of such a model. Therefore we choose a mean reverting jump diffusion as the price process in business time. Note that it is possible to introduce jumps into a continuous process (such as Brownian motion) by a Levy subordinator-based time change (but not by an absolutely continuous time change — see Barndorff-Nielsen and Shiryaev, 2010). However, our main goal is to mimic demand-driven seasonal and stochastic features in the price volatility and the jump components, which is possible to do by specifying the absolutely continuous time change with the activity rate related to the demand proxy. Combining mean reverting jump diffusion (the base process) with a stochastic and seasonal activity rate, our resulting spot price model will keep the properties of mean reversion and jumps, but will have stochastic and seasonal features in the jump intensity, rate of mean reversion and volatility.

In the existing finance literature, most studies that consider time changed models use a (geometric) Brownian motion or a general Lévy process as the base process (see e.g., Carr et al., 2003, Carr and Wu, 2004 or Kallsen and Muhle-Karbey, 2011). One exception is Li and Linetsky (2014), who combine a mean reverting model for commodity prices with a time change and derive futures and option prices in terms of Hermite expansions. They examine whether their spot price model leads to futures and option prices with empirically observed features as the Samuelson Effect for futures and implied volatility smiles for options. Lorig (2011) considers mean-reverting models with stochastic volatility, based on Fouque et al. (2000). However, the spot price model assumed in that paper is not itself mean reverting. As Li and Linetsky (2014), Lorig (2011) performs a time-change and, using spectral theory and singular perturbation techniques, derives an approximation for the price of a European option.

In this paper we apply a stochastic time change technique to electricity price modeling, as well as outline a calibration procedure and the empirical application to the German electricity market. Using a demand-based time change, we are able to have a solid economic interpretation of our model. Using the temperature (which can be accurately forecasted) as the proxy for the demand, the model can be applied to obtain distributional forecasts of the electricity price, by predicting its volatility and the probability of a large price spike, which is valuable especially during a cold season.

This paper is organized as follows. After giving the theoretical foundation of the model and its properties in Section 2, we move on to the empirical part in Section 3. Here we specify the time change and use it to estimate the parameters of the base process. In a simulation study in Section 4 we show that we indeed capture distributional properties of the electricity price data from the EEX. Section 5 concludes.

2. The model and its properties

Let $(X_s)_{s\geq 0}$ denote the base price process. Our goal is to specify a time change $T=(T_t)_{t\geq 0}$ so that the time changed process $(Y_t)_{t\geq 0}=(X(T(t))_{t\geq 0}$ has a more complex dynamics than $(X_t)_{t\geq 0}$, and captures characteristic features of electricity prices such as seasonal spikes, varying speed of mean reversion and stochastic volatility.

2.1. The base process and the time change

Consider a probability space $(\Omega,\mathcal{F},\mathbb{P})$ equipped with a filtration $(\widehat{\mathcal{F}}_s)_{s\geq 0}$. We postulate that the base process $X=(X_s)_{s\geq 0}$ is given by an Ornstein–Uhlenbeck process with a jump component:

$$dX(s)=\theta(\mu-X(s))ds+\sigma d\hat{B}_1(s)+d\hat{Z}(s)\ , \tag{2.1}$$

where $\hat{B}_1$ is a $(\widehat{\mathcal{F}}_s)$ − standard Brownian motion and $\hat{Z}=(\hat{Z}_s)_{s\geq 0}$ a compound Poisson process with Lévy measure $\nu(dx)=\lambda f(x)dx$, λ is the jump intensity and f the density of the jump size distribution.

In the time-changed setting, one often assumes that the base process is a Brownian motion or a Geometric Brownian motion. In our setting, we choose to add mean reversion and jumps as these are the distinguishing features of electricity spot prices. It is unlikely that these features are the result of market activity or business time. However, their magnitude depends on varying activity in the market. Also note that we specified the base process in terms of arithmetic and not Geometric Brownian motion, i.e., in terms of prices and not log-prices. However, a similar specification in terms of log-price is also possible.

In this paper we will use an absolutely continuous time change. Let τ be a positive stochastic process such that

$$\int_0^t \tau(u)\,du<\infty,\qquad \int_0^\infty \tau(u)\,du=\infty\ \forall t>0. \tag{2.2}$$

The process τ is called the *activity rate* and it describes the speed of the "economy", in the case of electricity, of its demand. We assume that this speed has a seasonal and a stochastic component. The time change $T(t)$ at time t is given by

$$T(t)=\int_0^t \tau(u)du\ . \tag{2.3}$$

The time changed process at time t is then

$$Y_t=X(T_t)\ . \tag{2.4}$$

Note that so far, we did not require that the base process X and the time change T are independent. With the time change T a new filtration is introduced by

$$\mathcal{F}_t:=\widehat{\mathcal{F}}_{T(t)}\ . \tag{2.5}$$

and the time changed process $(Y_t)_{t\geq 0}$ will be adapted to $(\mathcal{F}_t)_{t\geq 0}$, while the base process $(X_s)_{s>0}$ is adapted to the filtration $(\widehat{\mathcal{F}}_s)_{s\geq 0}$. We can construct the filtration $(\widehat{\mathcal{F}}_s)_{s\geq 0}$ reversely through

$$\widehat{\mathcal{F}}_s=\mathcal{F}_{\hat{T}(s)}\ ,$$

where $\hat{T}(s)=\inf\{t:T_t>s\}$ (see Chapter 1.2.1 in Barndorff-Nielsen and Shiryaev, 2010).

We give two examples of possible choices of the seasonal and stochastic components of the activity rate.

Example 2.1. Assume that the activity rate τ is given by

$$\tau(t)=s(t)+\epsilon(t)\ , \tag{2.6}$$

where $s(t)$ is a deterministic, time dependent and positive seasonal component (for example, a periodic trigonometric function) and $\epsilon(t)$ is a suitably chosen stochastic process, representing the stochastic part of the time change, such that Eq. (2.2) is satisfied.

Let $s(t)$ be a periodic trigonometric function, with the period equal to e.g., one year. Let $\epsilon(t)$ be given by the Cox–Ingersoll–Ross (CIR) process

$$d\epsilon(t)=\kappa(\eta-\epsilon(t))dt+\sqrt{\epsilon(t)}\sigma_2 dB_2(t)\ , \tag{2.7}$$

where B_2 is a standard Brownian motion (which we assume is independent on $\hat{B}_1$), η is the long term mean, and assume that $\frac{2\eta\kappa}{\sigma_2^2}\geq 1$ to ensure that the process is strictly positive.

Note that, in the above example, we specified seasonal component of the time change as additive (Eq. (2.6)); a multiplicative specification is possible as well, in which case we would have

$$\tau(t)=s(t)\epsilon(t)\ , \tag{2.8}$$

and in such a representation both $s(t)$ and $\epsilon(t)$ must be positive. The choice of additive or multiplicative representation of the time change is up to a model builder and it is similar in essence to the question of modeling periodic volatility in P-GARCH (periodic GARCH) model, where periodic component of the volatility also can be additive or multiplicative to the GARCH process.

The CIR process is an obvious choice for $\epsilon(t)$, as it is a positive process, and is therefore the standard example of an activity rate in the literature (see Cont and Tankov, 2004, Veraart and Winkel, 2010). If the positivity of the overall activity rate can be assured in a different way (e.g., by using some positive-valued transformation), then other specifications of $\epsilon(t)$ are possible.

In the empirical part, we will use a positive transformation of a mean reverting processes of Ornstein–Uhlenbeck type, as in the following example. Here, we assume that the activity rate is given by a suitable transformation of temperature.

Example 2.2. Let the temperature at time t be given by

$$C(t)=C_s(t)+C_r(t)\ ,$$

where C_r is an Ornstein–Uhlenbeck process

$$dC_r(t)=\theta_2(\mu_2-C_r(t))dt+\sigma_2 dB_2(t),$$

B_2 is as in the previous example, and C_s the seasonal component, given by a sinusoid function with a yearly period. Furthermore, let

$f:\mathbb{R}\to\mathbb{R}^+$ be a positive function. If condition (2.2) is satisfied, then we define the time change as

$$\tau(t) = h(C(t)) ,$$

where the function h describes the relationship between the temperature and market activity. Note that, in this example, the activity rate is not directly the sum of a seasonal and stochastic components (as in Eq. (2.6)), but it still possesses such a seasonal and stochastic components via its dependence on seasonal temperature $C(t)$.

The CIR specification satisfies Eq. (2.2): the expectation of the CIR (Eq. (2.7)) is finite for $t < \infty$, and goes to infinity for $t \to \infty$, hence,

$$\int_0^t \epsilon(u)\,du < \infty, \qquad \int_0^\infty \epsilon(u)\,du = \infty \;\forall t > 0.$$

The same holds for the Ornstein–Uhlenbeck process and so for specifications using (suitable) positive transformations of it.

Generally, we do not have to require that the base process and stochastic time change are independent — in our empirical application we will not assume that and will estimate the correlation between Brownian motions driving the base process and the stochastic time change. However, for obtaining the theoretical properties of the time changed process, which we will do below, we will impose this restriction.

2.2. The dynamics of the time changed process

The dynamics of the time changed process $(Y_t)_{t\geq 0}$ has the following form:

$$dY(t) = \theta(\mu - Y(t))dT(t) + \sigma d\hat{B}_1(T(t)) + d\hat{Z}(T(t)) \ . \tag{2.9}$$

This can be shown by considering the integral notation and formulas of integration by substitution (as shown in Appendix A).

As we have

$$dT(t) = \tau(t)dt \ ,$$

we immediately see that the mean reversion rate of our new process $(Y_t)_{t\geq 0}$ is given by $\theta\tau(t)$. It is now determined through the activity rate $(\tau_t)_{t\geq 0}$ (which is stochastic and time dependent), scaled by the mean reversion rate θ of the base process. More activity in the market leads to a higher speed of mean reversion: as more trading takes place, the price reverts to its equilibrium level faster.

Next, we investigate the behavior of the time-changed drivers $(\hat{B}_{T(t)})_{t\geq 0}$ and $(\hat{Z}_{T(t)})_{t\geq 0}$.

As the consequence of the famous Dambis, Dubins and Schwarz theorem (see e.g., Theorem 4.6 in Karatzas and Shreve, 1991, or Chapter 1.4, Corollary 1.3 in Barndorff-Nielsen and Shiryaev, 2010), the time change has a number of implications related to stochastic volatility. According to this theorem,

$$\hat{B}_1(T(t)) \tag{2.10}$$

is equivalent to

$$\int_0^t \sqrt{\tau(u)}\; dB_1(u) \tag{2.11}$$

for a Brownian motion B_1 with respect to $(\mathcal{F}_t)$. Starting with the expression for the stochastic volatility model (Eq. (2.11)), the Dambis–Dubins–Schwarz theorem tells how to construct a Brownian motion $\hat{B}_1$ such that Eqs. (2.10) and (2.11) are equal almost surely. For the converse statement starting with Eq. (2.10) the equality holds in distribution. Note that, for a continuous local martingale M of the form

$$M_t = \int_0^t \sqrt{\tau(u)}\; d\tilde{B}(u) \tag{2.12}$$

for a standard Brownian motion $\tilde{B}$, we can write the time change (Eq. (2.3)) as the quadratic variation of M:

$$T(t) = \langle M \rangle_t \ .$$

We now consider the time changed jump component. There is an analogue of the Dambis, Dubins, Schwarz theorem for counting processes (see for example Barndorff-Nielsen and Shiryaev, 2010, Chapter 1.4, Theorem 1.2). Assume that the Lévy density $k(x)$ exists and let its characteristic exponent be given by

$$\Phi_{\hat{Z}}(u) = \int_{-\infty}^{\infty} (e^{iux} - 1)k(x)dx.$$

Then $Z(t) = \hat{Z}(T(t))$ is a semimartingale with a martingale component that is a compensated jump process (see Carr et al., 2003):

$$Z(t) = Z(0) + \int_0^t \int_{-\infty}^{\infty} x\nu(dx, du) + \int_0^t \int_{-\infty}^{\infty} x(N(dx, du) - \nu(dx, du)),$$
$$\nu(dx, du) = \tau(u)k(x)dudx \ ,$$

where $N(dx, du)$ is a integer-valued random measure associated with the jumps of Z and $\nu(dx, du)$ is the compensator. The compensator is factorized into a time and a jump component. Then

$$\nu(dx, du) = \tau(u)\lambda f(x)dudx \ ,$$

where f is the density of the jump size distribution. So now we have a time dependent and stochastic jump intensity in a time interval $[t, t+1]$:

$$\lambda \int_t^{t+1} \tau(u)du = \lambda(T(t+1) - T(t)),$$

while the jump size distribution has not changed. This has a simple implication for the jump intensity: if the activity rate increases, so does the frequency of jumps.

Altogether, we arrive at the following dynamics for the time changed process $(Y_t)_{t\geq 0}$:

$$dY(t) = \theta\tau(t)(\mu - Y(t))dt + \sigma\sqrt{\tau(t)}\; dB_1(t) + dZ(t) \tag{2.13}$$

(note that $dZ(t)$ in the above formula is the time changed version of the compound Poisson process, i.e., $d\hat{Z}(T(t))$). The process $Y(t)$, specified above, is again a process of Ornstein–Uhlenbeck type, but now with time dependent (possibly seasonal) and stochastic

- speed of mean reversion, given by $\theta\tau(t)$,
- volatility, given by $\sigma\sqrt{\tau(t)}$, and
- jump intensity, given by $\lambda\tau(t)$.

All of these are altered according to the speed of economy $\tau(t)$ (which can be a combination of a seasonal and stochastic components, e.g. $\tau(t) = s(t) + \epsilon(t)$). Not altered are the level of mean reversion and the jump size distribution.

2.3. Properties of the time changed process

We now state some properties of the time changed process Y. As we are in the semimartingale framework, the general theory is well studied. Nevertheless, for further applications it is of interest to spell out some properties explicitly in our framework. For example, these properties can be useful for simulations of the time changed process, its calibration by moment-matching procedures and for obtaining derivatives prices under the time changed model.

Using Îto's formula, Eq. (2.13) can be solved analogously to the procedure for the classical Ornstein–Uhlenbeck process. In this way we derive the following proposition.

Proposition 2.3 (Explicit solution). *The explicit solution to Eq.* (2.13) *with* $T(0)=0$ *is given by*

$$
\begin{aligned}
Y(t) =& e^{-\theta T(t)}Y_0 + \mu(1-e^{-\theta T(t)}) + e^{-\theta T(t)}\int_0^t e^{\theta T(u)}\sigma\sqrt{\tau(u)}dB_1(u) \\
&+ e^{-\theta T(t)}\int_0^t e^{\theta T(u)}dZ(u)\,, \qquad (2.14)
\end{aligned}
$$

with $dZ(t)=\int_{\mathbb{R}} xN(dx,dt)$.

Proof. See Appendix A. □

The above proposition can be useful for e.g., simulating the future values of the time-changed process without having to simulate the entire paths.

Generally, one can distinguish between two different types of time changes: subordinators and absolutely continuous time changes (see Veraart and Winkel, 2010). Subordinators are positive increasing Lévy processes that can introduce jumps into a model. There are a number of popular pure jump Lévy processes, that can be written as a subordinated Brownian motion, and where the increments have a well known distribution as for example the NIG process and the NIG distribution. In contrast, here we use absolutely continuous time changes that consist of a time integral over a positive activity rate and lead to stochastic volatility models. In those models it is usually not possible to achieve closed form expressions for the distribution. Nevertheless, by conditioning on $T(t)$, it is possible to give some distributional properties if the time change and base process are independent. The next proposition is useful for Fourier-based approaches to option pricing or if one wants to employ the moment-matching procedure for calibration of the model parameters.

Proposition 2.4 (Characteristic function and moments). *Let* T *and* X *be independent. The characteristic function of* $Y(t)$ *is given by*

$$\mathbb{E}\left[e^{iuY_t}\right] = e^{\Psi_T(t,\Psi_X(u))}\,,$$

where $\Psi_X(u)=\log\mathbb{E}\left[e^{iuX(1)}\right]$ *and* $\Psi_T(t,u)=\log\mathbb{E}\left[e^{uT(t)}\right]$.

Expectation and covariance are given by

$$
\begin{aligned}
\mathbb{E}[Y_t] &= \mathbb{E}\left[e^{-\theta T(t)}\right]X_0 + \left(\mu + \frac{\mathbb{E}\left[\hat{Z}_1\right]}{\theta}\right)\left(1-\mathbb{E}\left[e^{-\theta T(t)}\right]\right)\,,\\
\mathrm{Cov}(Y_s,Y_t) &= \frac{\sigma^2+\mathbb{E}\left[\hat{Z}_1^2\right]}{2\theta}\left(\mathbb{E}\left[e^{-\theta(T(t)-T(s))}\right]-\mathbb{E}\left[e^{-\theta(T(s)+T(t))}\right]\right)\,, \qquad (2.15)
\end{aligned}
$$

for $s\le t$.

Proof. See Appendix A. □

Note that, for $t\to\infty$, the expectation of Y and X coincide:

$$\lim_{t\to\infty}\mathbb{E}[Y_t] = \mu + \frac{\mathbb{E}[\hat{Z}_1]}{\theta} = \lim_{t\to\infty}\mathbb{E}[X_t]\,,$$

as we can change limit and expectation operators by the bounded convergence theorem. Also, for the variance given by

$$\mathrm{Var}(Y_t) = \mathrm{Cov}(Y_t,Y_t) = \frac{\sigma^2+\mathbb{E}\left[\hat{Z}_1^2\right]}{2\theta}\left(1-\mathbb{E}\left[e^{-2\theta T(t)}\right]\right)\,,$$

we find that the long term variance of Y and X coincide:

$$\lim_{t\to\infty}\mathrm{Var}(Y_t) = \frac{\sigma^2+\mathbb{E}\left[\hat{Z}_1^2\right]}{2\theta} = \lim_{t\to\infty}\mathrm{Var}(X_t)\,.$$

These properties are logical, since, in the long term, the averages over the calendar and business time should be the same.

For the ordinary Ornstein–Uhlenbeck process X_s, one can calculate the autocorrelation function in the "stationary case". From

$$\mathrm{Cov}(X_s,X_t) = \frac{\sigma^2+\mathbb{E}\left[\hat{Z}_1^2\right]}{2\theta}\left(e^{-\theta(t-s)}-e^{-\theta(s+t)}\right)\,,$$

one deduces that for $s=t+h$, $h>0$, we have in the stationary case, that is for $t\to\infty$

$$\frac{\sigma^2+\mathbb{E}\left[\hat{Z}_1^2\right]}{2\theta}e^{-\theta h}\,, \qquad (2.16)$$

resulting in the well known exponential shape of the autocorrelation function: $\mathrm{acf}(h)=e^{-\theta h}$. Due to the time transformation, the time lag h autocovariance $\mathrm{Cov}(Y_{t+h},Y_t)$ (as in Eq. (2.15)) does not exist (and neither does the limit of Eq. (2.15)), so the standard notion of an autocorrelation function does not make sense for the time changed process Y.

Example 2.5. If one chooses for the random part of the time change the CIR process as in Example 2.1, the Laplace transform and the expectation of the integrated CIR process are know in closed form and given by

$$
\begin{aligned}
\mathbb{E}\left[\int_0^t \epsilon_u du\right] &= \eta t + \frac{(\epsilon_0-\eta)(1-e^{-\kappa t})}{\kappa}\\
\mathbb{E}\left[\exp\left\{-u\int_0^t \epsilon_u du\right\}\right] &= \frac{\exp\left(\frac{\kappa^2\eta t}{\tilde{\sigma}^2}\right)}{\left(\cosh\frac{\gamma t}{2}+\frac{\kappa}{\gamma}\sinh\frac{\gamma t}{2}\right)^{\frac{1\kappa\eta}{\tilde{\sigma}^2}}}\exp\left(-\frac{\epsilon_0 u}{\kappa+\gamma\coth\frac{\gamma t}{2}}\right)
\end{aligned}
$$

where $\gamma=\sqrt{\kappa^2+2\tilde{\sigma}^2u}$.

It is useful to know what class of processes Eq. (2.14) belongs to. The following proposition allows us to use the stochastic calculus machinery for semimartingales.

Proposition 2.6. *The process* Y *given by Eq.* (2.14) *is a semimartingale with*

$$Y_t = A_t + M_t^c + M_t^d\,, \qquad (2.17)$$

where

$$A_t = e^{-\theta T(t)}Y_0 + \mu\left(1 - e^{-\theta T(t)}\right) - \theta\int_0^t\int_0^u e^{\theta T(s)}\sigma\sqrt{\tau(s)}dB(s)e^{-\theta T(u)}\tau(u)du\ ,$$
$$+\int_0^t\int_{\mathbb{R}} x\nu(dx,du) - \theta\int_0^t\int_0^u\int_{\mathbb{R}} e^{\theta T(s)}xN(dx,ds)e^{-\theta T(u)}\tau(u)du$$
$$M^c = \sigma\int_0^t\sqrt{\tau(u)}dB(u)\ ,$$
$$M^d = \int_0^t\int_{\mathbb{R}} x(N-\nu)(dx,du)\ .$$

Proof. See Appendix A. □

The following proposition provides us with the expression for the quadratic variation of the time changed process (Eq. (2.14)). It is used as a measure for market activity, leading to the notion of realized variance.

Proposition 2.7 (Quadratic variation). *The quadratic variation of the time changed Ornstein–Uhlenbeck process (Eq. (2.14)) is given by*

$$\langle Y\rangle_t = \int_0^t\sigma^2\tau(u)du + \int_0^t\int_{\mathbb{R}} x^2N(dx,du)\ .$$

Proof. This follows directly from Eq. (2.17) and for example Cont and Tankov (2004), Example 8.10. □

3. Model estimation and empirical study

3.1. Data and deseasoning

We observe a discretely sampled price series $\{Y(1), Y(2), \ldots, Y(N)\}$ and assume that these are observations of the time changed process (Eq. (2.13)). Note that this process consists of two main components: the mean reverting base process (Eq. (2.1)) and the time change (Eq. (2.3)). The goal is to estimate the unknown parameters of both these processes. First, we are going to specify the time change (by the corresponding activity rate). Given the specified activity rate, we then go on to estimate the parameters of the base process.

As our historical dataset, we use the daily electricity spot prices from the German EEX, from 12.03.2009 until 31.12.2013. These are displayed in Fig. 1. We use weekdays prices as these were made available to us by EEX. We argue that excluding weekends is not a big issue here, as weekends are typically periods of lower demand, and characteristic price features such as spikes and high volatility are not affected by removing weekends.

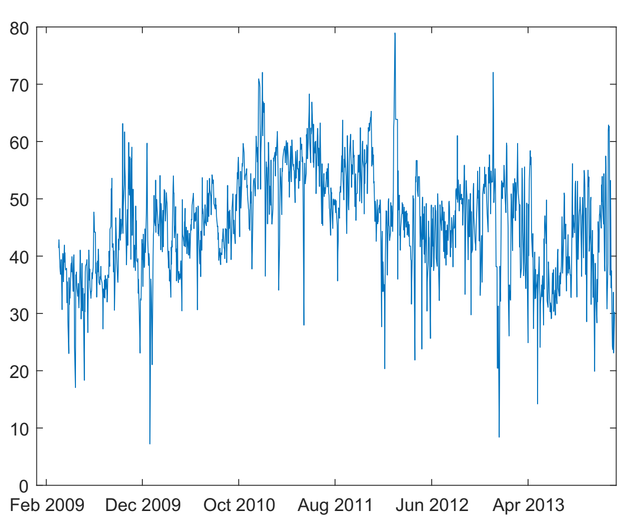

Fig. 1. EEX spot price from 12.03.2009 until 31.12.2012.

Due to the reform of the energy policy in Germany in 2008 (the so-called Erneuerbare Energien Gesetz), there is a structural break in the prices in 2008, following this reform. Therefore we use the prices from beginning of 2009. Nevertheless, in the first period of 2009 the prices seem to find their new price level. Therefore we use the data starting 50 weekdays after the first of January, that is from 12.03.2009. After liberalization of electricity markets in early 2000s, some dramatic spikes have been observed in electricity prices, with extreme returns reaching up to 1000%. However, such large spikes are not that prominent in European electricity prices anymore, but are still visible, as Fig. 1 shows. Nowadays, with wind power generation and other renewable electricity sources, negative spikes occur in electricity prices — something that has not been observed before. So in our model we should allow for both positive and negative price jumps.

It is well-known that electricity spot prices exhibit seasonalities, which we remove first. We fit a trigonometric function with a yearly and weekly period:

$$s_1(t) = a_1\sin\left(\frac{a_2 + 2\pi t}{5}\right) + a_3\sin\left(\frac{a_4 + 2\pi t}{251}\right) + a_5\ . \qquad (3.1)$$

We estimate and subtract the seasonal component; the estimated seasonal component parameters are in Table 1. Note that we remove seasonality only in price level — this does not interfere with seasonal patterns in volatility and spikes, which we will model by means of stochastic time change.

We assume that the deseasonalized price series is the series of observations of the process Y, given in Eq. (2.13), which consists of the base process X (as in Eq. (2.1)) and the time change specified through an activity rate.

Fig. 2 shows the plot of the empirical autocorrelation function of the deseasonalized price series. Mean reversion is clearly present. Recall that we mentioned in Section 2.3 that the notion of the autocorrelation function does not make sense for the time changed process Y. As a consequence, we do not have a one-to-one correspondence between the decay of the empirical autocorrelation function and the mean reversion parameter θ. Many authors argue that in energy prices, there are several speeds of mean reversion present, so a suitable model for a spot price should be a sum of mean reverting processes with different speeds of mean reversion. For example, Meyer-Brandis and Tankov (2008) provide an empirical study, while Schmeck (2016) investigates the impact of several mean reverting factors on pricing derivatives in energy markets. In this paper, our price dynamics (Eq. (2.13)) takes into account different mean reversion rates, as it is time dependent and stochastic. In times of high activity in the market, our model produces fast mean reversion and in times of low activity, we have slow mean reversion. In Section 3.3.2 we estimate the range of mean reversion speeds discuss the obtained estimates.

Now we move on to specifying the shape of the activity rate and estimating its parameters. Given the activity rate, we will be able to estimate the parameters of the base process.

Table 1
Estimated parameters of the seasonal component of the spot prices.

a_1	a_2	a_3	a_4	a_5
1.07 (0.41)	7.79 (1.90)	2.49 (0.40)	542.02 (43.11)	46.01 (0.29)

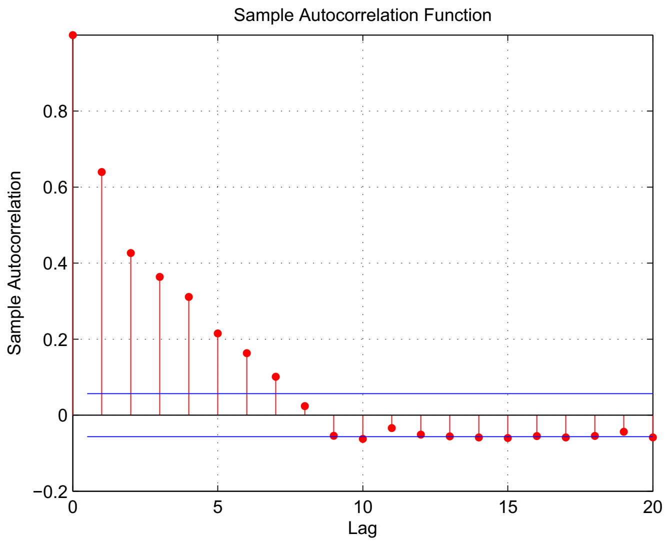

Fig. 2. Autocorrelation function of the deseasonalized spot price data.

3.2. Parameter estimation of the time change

We will specify an economical interpretable time change for our electricity price process. In calender time, the time series exhibits stochastic and seasonal patterns in volatility and in jump frequency. In business time, these features should be absent. The business time should be determined exogenously, i.e., by a factor other than the spot price, but the one that influences the spot price of an asset. As mentioned in the Introduction, in stock markets, empirical studies show that the business time can be determined by the traded volume (see e.g., Ané and Geman, 2000). However, here we consider energy markets, in particular electricity markets, and we want to specify the time change in a way that reflects the "activity" in these markets. For example, we can relate the activity rate τ to the demand for electricity, which is closely related to the outside air temperature. While the demand data is not always available, the historical temperatures are openly available (and predictable in the short term). So we will model the activity rate of the business time $\tau(t)$ as a deterministic function of the observed temperature (in °C), which we denote by $C(t)$. For the deterministic relationship between temperature and demand, we use a parabolic function, as in Fig. 4 of Bessec and Fouquau (2008). We fit this function using the data points from Bessec and Fouquau (2008), given in Table 2.

We assume that the relationship is given by a polynomial of order 2:

$$g(x) = p_1 x^2 + p_2 x + p_3 \ .$$

The resulting parameters are given in Table 3. Here, we shift the polynomial up so that it is always positive and enlarge the scale by a factor of 10.

For temperature data, we use historical temperatures in Cologne area of Germany. We use Cologne area as this region is quite central (in latitude) for Germany and hence is representative of an "average" temperature. Moreover, this area is the most industrialized area of Germany, with a large share of energy demand. In Fig. 3 the historical temperatures $C(t)$ from 01.01.2009 until 31.12.2012 in Cologne are depicted, as well as the estimated temperature-implied demand $g(C(t))$. In order to dampen the high peaks of the series $g(C(t))$, we apply the logarithmic transformation, in such a way that the activity rate remains positive.

Table 2
Data for estimating the quadratic relationship between temperature and demand.

C	−10	0	16	28
$g(C)$	3.7	1.7	0	0.7

Table 3
Estimated parameters of the demand function $g(C)$.

p_1	p_2	p_3
0.01	−0.17	1.57

The function $g(x)$ is such that the demand is minimal when the temperature is around 16°. As can be seen in Fig. 3, during warmer days the demand proxy is close to zero, and this would result in a volatility close to zero, which is unrealistic. Therefore, we increase the entire activity rate by a constant factor, which would correspond to the volatility during low-demand periods.

Taking all the above considerations into account, we set the activity rate to be

$$\tau(t) = \log\left(1 + g(C(t))\right) + k \ . \tag{3.2}$$

We set k such that the business time goes by as quickly as the calender time on average, meaning that the long-term average of τ is equal to 1. Therefore we take the average from 01.01.2009 until 31.12.2012, that is over four complete years, and find $k = 0.59$. The parameter k functions as the baseline and gives the lower bound for the activity rate. Fig. 4 shows the resulting activity rate, obtained by plugging in the observed temperature into Eq. (3.2)). We have that $\tau \in [0.59, 2.26]$, so the business time can go approximately half as fast as the calendar time in periods of low demand and more than twice as fast during high demand periods. For comparison, Fig. 5 shows the activity rate vs. calender time t.

Fig. 6 shows the fitted seasonal and stochastic parts of the historical activity rate $\tau(t)$ (obtained from the historical temperature using the relationship (3.2)). We see a pronounced cooling effect in the winter, while the heating effect in the summer is quite small. The random component of the activity rate $\epsilon(t)$ has higher volatility in the winter.

Here we specified the activity rate without specifying a model for the temperature. Nevertheless, for the purpose of simulation, we do so now.

Let $C(t)$ denote the temperature at time t in Celsius, and let it be the sum of a seasonal and a random components:

$$C(t) = C_s(t) + C_r(t) \ , \tag{3.3}$$

where

$$C_s(t) = b_1 \sin\left(\frac{b_2 + 2\pi t}{T}\right) + b_3 \tag{3.4}$$

is the seasonal component of the temperature. Here we have chosen a simple sinusoidal function for the seasonal component, as modeling the temperature is not our primary goal. The historical temperatures together with the fitted seasonal component are shown in Fig. 7. The fit of the seasonal component is worse in winter months, which is due to a higher volatility of the temperature in those periods.

The random part C_r, i.e., the series of temperature deviations from its seasonal shape (Fig. 8), is given by the mean reverting process (typically mean-zero):

$$dC_r(t) = \theta_2(\mu_2 - C_r(t))dt + \sigma_2 dB_2(t). \tag{3.5}$$

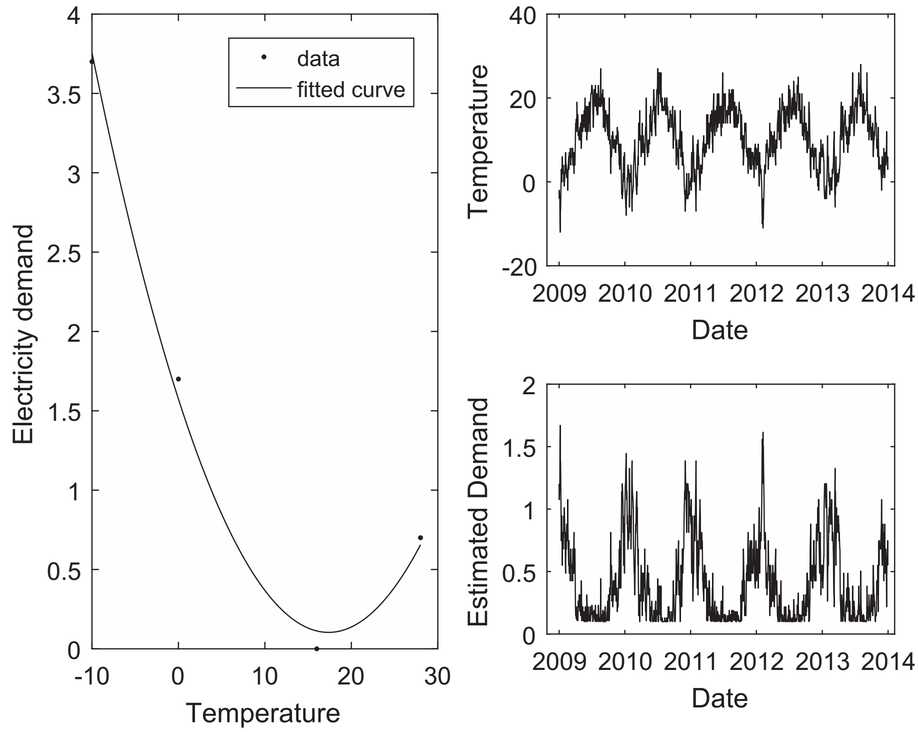

Fig. 3. Demand vs. temperature (left). Historical temperature (top right) and the estimated temperature-implied demand (bottom right); both 01.01.2009–31.12.2012.

Table 4 shows the estimated parameters for the temperature model (with standard errors in brackets).

3.3. Parameter estimation of the base process

Having specified the activity rate in the previous paragraph, we now move on to estimating the parameters of the base process. There are various approaches that estimate the parameters of jump-diffusions simultaneously, using e.g., maximum likelihood techniques or the generalized methods of moments (see, e.g., Huisman and Mahieu, 2001, Ball and Torous, 1983 or Aït-Sahalia, 2004). We apply several estimation methods here: Maximum Likelihood method, Generalized Method of Moments (GMM), Efficient Method of Moments (EMM) as well as suggest a two-step estimation approach, which combines jump filtering and a modified least squares estimation for the parameters of mean reversion, that is, a linear regression on unequally spaced observations.

First, we outline the two-step estimation approach and, at the end of the section, we compare the estimates to those obtained by MLE, GMM and EMM.

3.3.1. Jump component of the base process

In the suggested two-step estimation procedure, we first identify those price movements that can be considered as jumps. Disentangling diffusion component of a price process from the jump component has been studied by, e.g., Aït-Sahalia (2004), who uses both MLE and GMM for this purpose. It is well-known, however, that when these methods are applied to energy prices, they tend to severely overestimate the jump frequency and underestimate the mean jump size and diffusion volatility (this is pointed out in, e.g., Huisman and Mahieu, 2001, Weron, 2014 and other studies). For example, when we applied MLE method of Aït-Sahalia (2004) to EEX power prices, around 30% of daily price moves were identified as jumps, while electricity market participants estimate the jump frequency to be less than 10%.

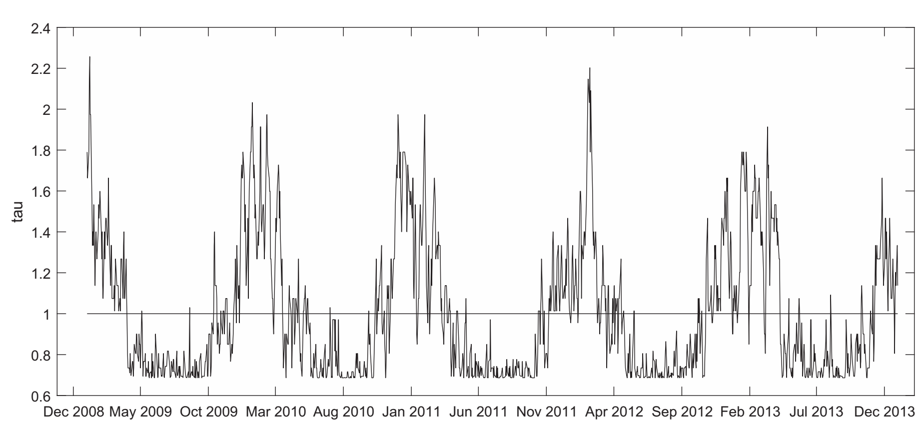

Fig. 4. Historical activity rate τ.

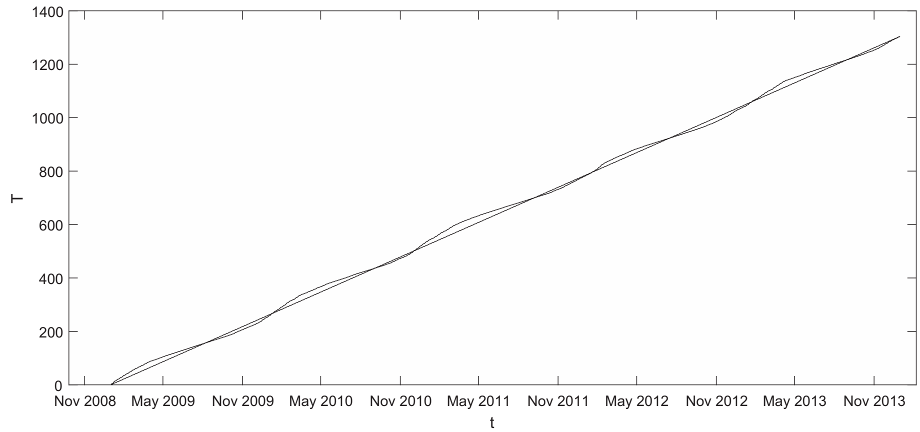

Fig. 5. Business time vs. calendar time.

Another way of disentangling jumps is suggested by Barndorff-Nielsen and Shephard (2008), who use a combination of the so-called bipower variation and realized variance for this purpose. Their method performs well for high-frequency data, as it relies heavily on good estimation of the realized variance. With daily data, such as in this study, the realized variance cannot be estimated reliably and, hence, their method does not perform that well.

Threshold-based methods for identifying jumps, such as the one studied by Mancini and Ren (2008), have been successfully applied to electricity and gas prices (see, e.g., Clewlow and Strickland, 2000, Borovkova and Permana, 2006 and others). Although such methods might seem ad-hoc, they provide us with the observed series of price jumps, which can then be analyzed for their distributional characteristics. Moreover, the threshold used for identifying jumps can be tailored to the "expected" (by market participants) jump frequency. However, it must be stressed that identifying the jumps without any restrictions is always an ad-hoc task by the researcher, possibly introducing sample selection error to the model.

We consider all daily price moves and calculate the volatility $\sigma_{20}(t)$ over the last 20 days and the moving average $m_{20}(t)$. We choose the window of 20 days as it corresponds to approximately one trading month. We then classify a price move as a potential jump if it exceeds the bounds $m_{20}(t) \pm TH * \sigma_{20}(t)$ and remove it from the data. In the next steps, we update the moving average mean and volatility and repeat the procedure until no such large price moves are detected. In the model we allow for both positive and negative jumps and, hence, we need to decide which (large) price moves, identified above, are due to the jump component and which are due to the mean reversion, i.e., price returning to the mean level after a jump. For this, we apply the following rule of thumb: after a jump, a subsequent price move in the opposite direction (occurring in the next two days) is considered as the price returning to its mean level due to mean reversion. This is motivated by a well-documented research on electricity price spikes, which have a half life of approximately two days; that is, after a price spike it takes at most two days for the electricity price to return towards its mean level.

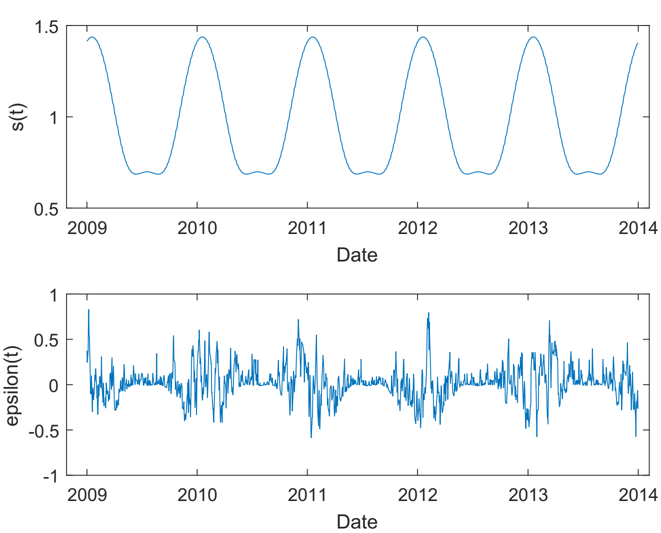

Fig. 6. Seasonal and random components of the activity rate.

The threshold TH is chosen so that the overall jump frequency is between 5 and 10%, as it is common in electricity markets. This results into the threshold value of 2 and the overall jump frequency of 8%. For comparison, if this threshold value is 2.5, the jump frequency would be 4%, and for the threshold value of 3 this frequency is only 1%. Fig. 9 shows the series of filtered jumps and Fig. 10 — the remaining diffusion component of the price process.

Now we can assess what distribution would be appropriate for the jump size. Lognormal or exponential distribution is often suggested for this purpose. We fit both the lognormal and exponential distributions separately to positive and negative jumps and consider the corresponding QQ-plots to decide which distribution gives a better fit (see Fig. 11). For negative jumps, we can choose either the lognormal distribution, to capture rare big jumps, or the exponential distribution, which would fit better the smaller jumps distribution. For positive jumps, we choose the exponential distribution. The resulting parameters for negative and positive jumps are given in Table 5 together with their standard errors. If exponential distribution is used also for negative jumps, the parameter of this jump size distribution is estimated at 5.54.

In fact, we can also use bootstrap to simulate jumps: in this case, instead of estimating a parametric distribution of jump sizes, we would use their empirical distribution, by drawing (with replacement) from the observed jumps. However, if the jump frequency is low and/or only a short period of historical prices is available, it is better to use the fitted parametric distribution for jump simulation.

Now we move on to estimating the diffusion parameters of the base process.

3.3.2. *Diffusion component of the base process*

The parameters μ, θ and σ of the base process have the function of scaling constants, as, for example, the stochastic volatility of the continuous driver of the time-changed process $Y(t)$ is given by $\sigma\sqrt{\tau(t)}$. We will estimate the parameters of the base process via an adjusted linear regression, that is, a linear regression on unequally

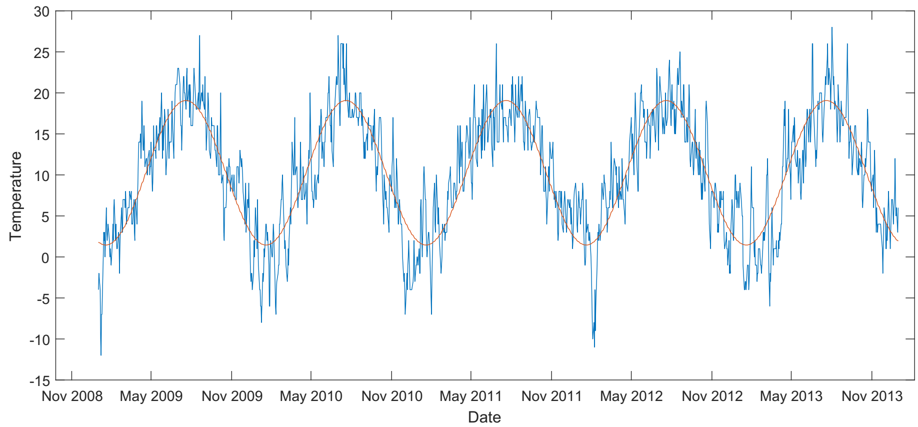

Fig. 7. Historical temperatures in Cologne on weekdays, 01.01.2009–31.12.1012, with the estimated seasonal function.

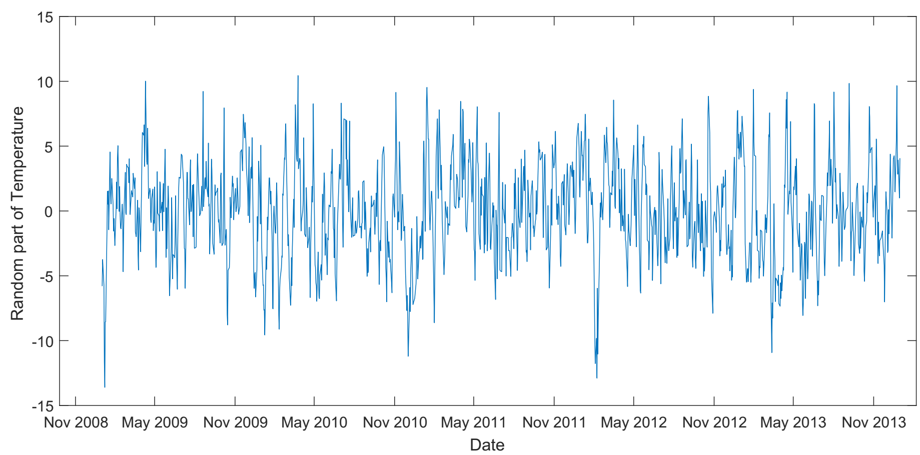

Fig. 8. Estimated random part of the temperature, 01.01.2009–31.12.1012.

spaced observations. The grid of the observed prices is determined by the time change.

Note that the sequence of observations of the time-changed process is equivalent to the series of non-equispaced observations on the base process. We will use this fact to adapt the standard least square procedure which is normally used for estimating the parameters of the mean reversion.

Denote $\tilde{Y}_1, \ldots, \tilde{Y}_N$ the observations of the diffusion component of the time changed process Y (i.e., after excluding jumps) — this component is displayed in Fig. 10. Note that, since $Y_t = X_{T(t)}$ and $X(t)$ is a mean reverting jump diffusion (Eq. (2.1)), these are the observations $\tilde{X}_{t_1}, \ldots, \tilde{X}_{t_n}$ with $t_i = T(i)$, where $\tilde{X}_t$ is the classical Ornstein–Uhlenbeck process

$$d\tilde{X}_t = \theta(\mu - \tilde{X}_t)dt + \sigma dB(t)\ . \tag{3.6}$$

So essentially we have a sequence of observations of a mean-reverting process, but on unequally spaced time intervals. Note that the business time observations moments $T(i) = \sum_{k=1}^{i} \tau(k)$ for $i \in \mathcal{N}$. Then the time step sizes between observations are $t_i - t_{i-1} = \tau(i)$.

Table 4
Estimated parameters of the temperature process.

b_1	b_2	b_3	μ_2	σ_2	θ_2
−8.81(0.15)	329.74 (4.31)	10.26 (0.10)	0.02 (0.07)	2.63 (0.04)	0.29 (0.02)

Discretizing Eq. (3.6), we get that, for any t_i and t_{i-1}, we have

$$\tilde{X}_{t_i} - \tilde{X}_{t_{i-1}} = \theta(\mu - \tilde{X}_{t_{i-1}})(t_i - t_{i-1}) + e_{t_i - t_{i-1}}\ ,$$

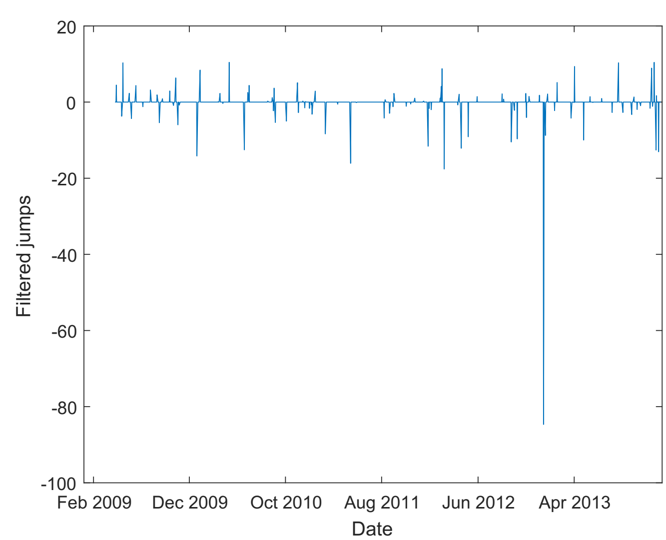

Fig. 9. Filtered jumps.

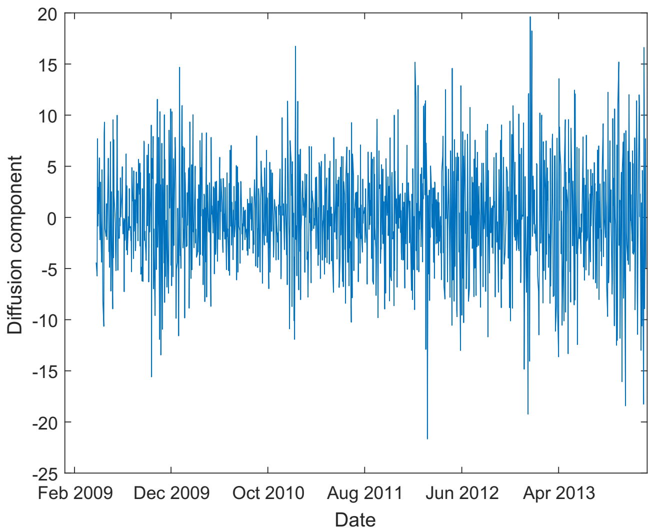

Fig. 10. Filtered diffusion component.

where $e_{t_i - t_{i-1}}$ is normally distributed with mean 0 and variance $\sigma_1^2(t_i - t_{i-1})$. This is equivalent to

$$\frac{\tilde{X}_{t_i} - \tilde{X}_{t_{i-1}}}{\sqrt{t_i - t_{i-1}}} = \theta\mu\sqrt{t_i - t_{i-1}} - \theta\tilde{X}_{t_{i-1}}\sqrt{t_i - t_{i-1}} + \tilde{e}_i \,,$$

where $\tilde{e}_i \sim \mathcal{N}(0, \sigma_1^2)$. So we have a linear regression setup of the form

$$\tilde{y}_i = \beta_1 x_i^{(1)} + \beta_2 x_i^{(2)} + \tilde{e}_i \,, \tag{3.7}$$

Table 5
Jump intensity and parameters of jump size distribution (positive jumps: exponential, negative jumps: lognormal distribution).

λ^+	m^+	λ^-	m^-	s^-
0.03 (0.01)	3.48 (0.47)	0.05 (0.01)	1.05 (0.19)	1.40 (0.14)

Table 6
Parameters of the base process' mean reversion.

θ	μ	σ
0.24 (0.02)	0.63 (0.01)	5.16 (0.10)

with

$$\tilde{y}_i = \frac{\tilde{Y}_i - \tilde{Y}_{i-1}}{\sqrt{t_i - t_{i-1}}} \,,$$
$$x_i^{(1)} = \sqrt{t_i - t_{i-1}} \,,$$
$$x_i^{(2)} = \tilde{Y}_{i-1}\sqrt{t_i - t_{i-1}} \,,$$

which we can estimate by the ordinary least squares procedure. We then find the estimates of the mean reversion parameters to be

$$\hat{\theta} = -\hat{\beta}_2 \quad \text{and} \quad \hat{\mu} = -\frac{\hat{\beta}_1}{\hat{\beta}_2} \,.$$

Noting that

$$\hat{e}_i = \tilde{y}_i - \hat{\beta}_1 x_i^{(1)} - \hat{\beta}_2 x_i^{(2)},$$

we can estimate the variance parameter by

$$\hat{\sigma}_1^2 = \mathrm{Var}(\hat{e}_i) \,.$$

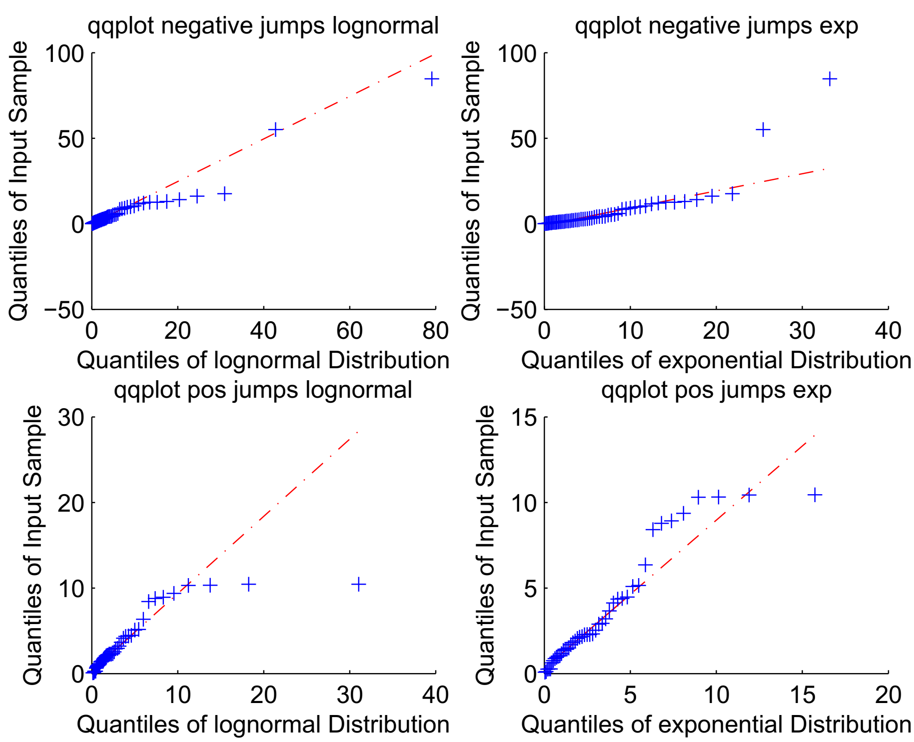

Fig. 11. QQ-plots of the jump size distributions.

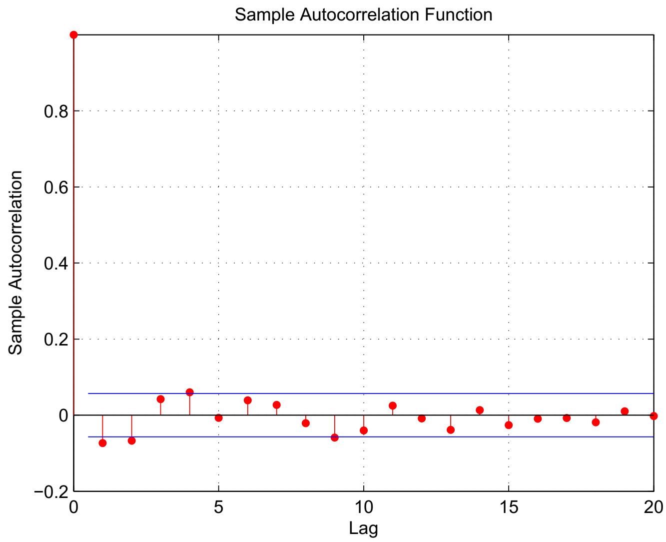

Fig. 12. Autocorrelation function of the residuals $\tilde{\varepsilon}$.

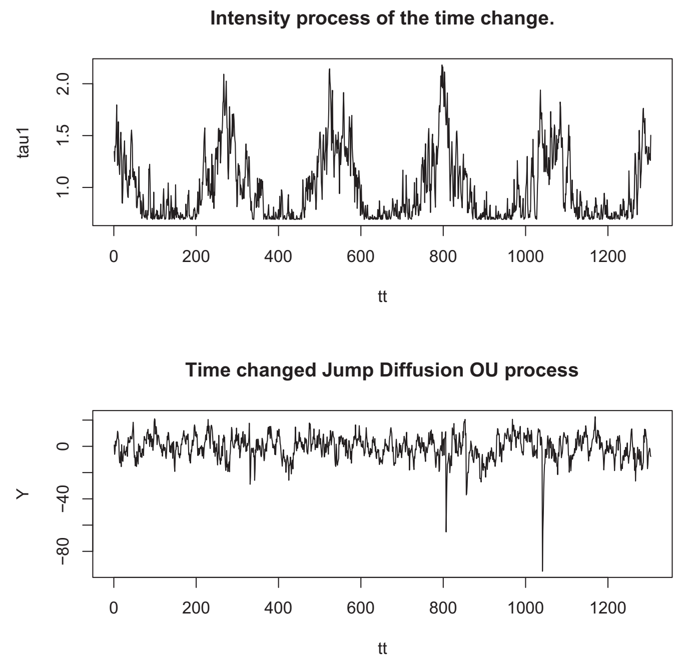

Fig. 13. Simulated trajectories of the time change and of Y.

The estimated parameters are $\hat{\theta} = 0.24$, $\hat{\mu} = 0.63$ and $\hat{\sigma}_1 = 5.16$ (these are also given in Table 6 together with their standard errors). The speed of mean reversion of the base process corresponds to the half life of 2.8 days. Note that this parameter is estimated from the Brownian component only. Including jumps might result in a faster mean reversion. For the time-changed process, the speed of mean reversion $\hat{\theta}\tau(t)$ varies between 0.14 and 0.54, that is a half life of 4.9 days in time of low activity, and 1.3 in times of high activity. As τ is scaled so that its mean is equal to 1, we have that the average mean reversion rate is $\theta = 0.24$. The "volatility" $\hat{\sigma}_1\sqrt{\tau}$ has a minimum level of 3.96, a maximum level of 7.75. Note that here we are modeling prices directly (and not log-prices), so σ is not the returns volatility in the usual sense.

In Fig. 12 we plot the autocorrelation function of the residuals of our model. We see that we have removed all the autocorrelation structure from the residuals.

Note that here we do not assume that the Brownian motions that drive the base process and the time change are independent; we estimated the correlation between these two and obtained a quite low (but significant) value of −0.11. The negative correlation makes sense: basically it reflects the intuitive relationship "temperature up–prices down".

Finally, we compare the estimated parameters to those obtained by other methods, such as MLE, GMM and EMM. When applying these methods, we did not separate positive and negative jumps, so the aggregated jump parameters are presented. All these parameters are given in Table 7.

Note that, particularly for GMM, the mean reversion parameters (mean reversion speed and diffusion volatility) are in good agreement with the two-step method. However, the jump component is not identified well by any other method: MLE suggests that over a quarter of all the price moves are (mostly small) jumps, while GMM classifies only 3% of price moves as jumps which are quite large in size (average jump size is −12 with the standard deviation of 11), and EMM does not identify jump component at all.

Table 7
Parameters of the base process estimated by alternative methods.

	θ	μ	σ	λ	m	s
MLE	0.16	−1.1	4.17	0.26	−0.34	5.38
GMM	0.22	0.04	5.98	0.03	−12.46	10.85
EMM	0.17	−2.01	6.56	0.00	–	–

Now we will proceed with the simulation study, where we compare the moments of the observed price series with series generated by our time-changed model (with parameters estimated by the two-step procedure) and by regular mean reverting jump-diffusion.

4. Simulation study

In this section, we describe a simulation study, where we simulate electricity price paths using the time changed model (which includes the temperature model (3.3)), a standard model of mean reverting jump diffusion (MRJD), often used for electricity prices, and compare the distributional characteristics (moments) of the simulated and observed prices in different seasons.

For simulation, we use the dynamics of $Y(t)$ given in Eq. (2.13) and discretize it using the Euler scheme. For a chosen step size Δt and the

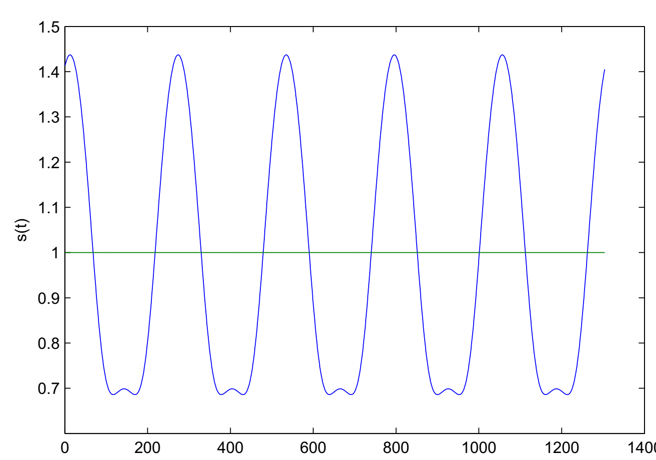

Fig. 14. Times of high and low activity.

Table 8
Distributional characteristics in times of high and low activity.

	Empirical			Simulation			
	$\tau_{seas} > 1$	$\tau_{seas} \leq 1$	overall	$\tau_{seas} > 1$	$\tau_{seas} \leq 1$	Overall	MRJD
Mean	0.13	−0.06	−0.02	0.02	−0.02	0.00	0.00
Standard deviation	8.50	5.93	7.23	7.43	5.44	6.37	5.71
Skewness	−1.93	−0.06	−1.23	−1.88	−1.42	−1.78	−1.42
Kurtosis	44.35	5.18	34.72	26.55	19.12	25.00	21.70
Jump intensity	0.1	0.06	0.08	0.1	0.06	0.07	0.07
Pos jump int	0.04	0.03	0.04	0.04	0.02	0.03	0.03
Neg jump int	0.05	0.04	0.05	0.06	0.04	0.04	0.04

number of steps n, the time horizon is Δt. Set $t_{i+1} = t_i + \Delta t$, $t_0 = 0$, $\Delta B_1(t_i) := B_1(t_i) - B_1(t_{i-1})$ and $\Delta Z(t_i) := Z(t_i) - Z(t_{i-1})$. Then

$$Y(t_{i+1}) = \theta\tau(t_i)(\mu - Y(t_i))\Delta t + \sigma\sqrt{\tau(t_i)}\Delta B_1(t_i) + \Delta Z(t_i)\ .$$

In order to simulate the Poisson process Z with stochastic and seasonal intensity, we need to simulate a trajectory of the activity rate first. Therefore, we need to simulate a trajectory of the temperature, $C(t) = C_s(t)+C_r(t)$, where the seasonal component $C_s(t)$ is given by Eq. (3.4). The random component $C_r(t)$ of the temperature, given in Eq. (3.5), is simulated again using the Euler scheme

$$C_r(t_{i+1}) = C_r(t_i) + \theta_2(\eta - C_r(t_i))\Delta t + \sigma_2\Delta B_2(t_i)\ .$$

Note that we do not assume that the Brownian motions driving the price process and the temperature B_1 and B_2 are independent — we can use the estimated correlation from Section 3 to simulate correlated Brownian motions.

The simulated trajectory of the activity rate is then given by

$$\tau(t_i) = \log(1 + g(C(t_i))) + k\ , \qquad i = 1, \ldots, n\ .$$

The time changed jump process $Z(t)$ is also called double stochastic process or Cox process. That is, its jump intensity $\lambda(t) := \theta\tau(t)$ is not only inhomogeneous, but also stochastic. Given a realization of the stochastic intensity, we can simulate a time-inhomogeneous Poisson process with that intensity realization by means of the rejection method. Set $\bar{\lambda} := \max_{i=1,\ldots,n}\lambda(t_i)$ and simulate successive arrival times $T_1^*, T_2^*, T_3^*, \ldots$ of a homogeneous Poisson process with intensity $\bar{\lambda}$. If we accept the ith arrival time with probability $\frac{\lambda(T_i^*)}{\bar{\lambda}}$ independently of all other arrivals, then the sequence $T_1, T_2, \ldots$ of the accepted arrival times forms a sequence of the arrival times of a inhomogeneous Poisson process with rate function $\lambda(t)$ (see Burnecki et al., 2004).

Combining these steps together, we arrive at the total simulated price path of the time changed process $Y(t)$. In Fig. 13 one such simulated trajectory is displayed in the bottom graph, where we used the estimated parameters from the previous section. Note that, to obtain simulated electricity prices from simulated $Y(t)$, we need to add the seasonal component of the price, given in Eq. (3.1) (and eventually a trend, if such is observed).

We compare the performance of the time changed process to the standard mean reverting jump diffusion process, often used to model electricity prices. The MRJD specification we use here is in terms of deseasonalized prices and it is given by

$$dX(s) = \theta(\mu - X(s))ds + \sigma dB(s) + dZ(s)\ , \tag{4.1}$$

where $X(s)$ is deseasonalized price, B is a $(\widehat{\mathcal{F}}_s)$ — standard Brownian motion and $Z = (Z_s)_{s\geq 0}$ a compound Poisson process with the jump intensity λ and some jump size distribution. Note that the regular MRJD is a special case of the time-changed price process with the activity rate that is constant and equal to 1.

If we estimate the MRJD parameters by MLE, the jump intensity is estimated at 0.12, with mean jump size of 1.02 and standard deviation of 1.35 (if we assume normal distribution for jump sizes). If, on the other hand, we use the threshold-based jump filtering procedure of the type we used for the time changed process, we estimate

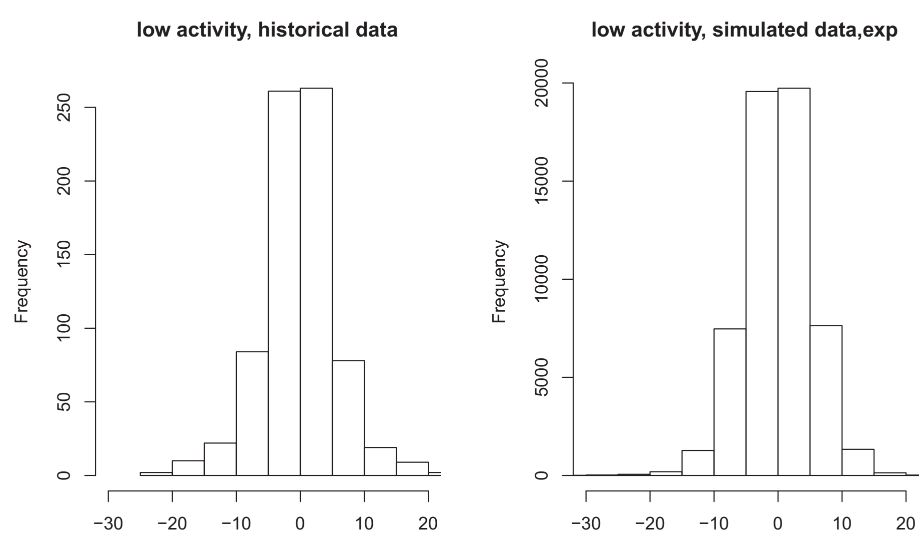

Fig. 15. Histogram low activity.

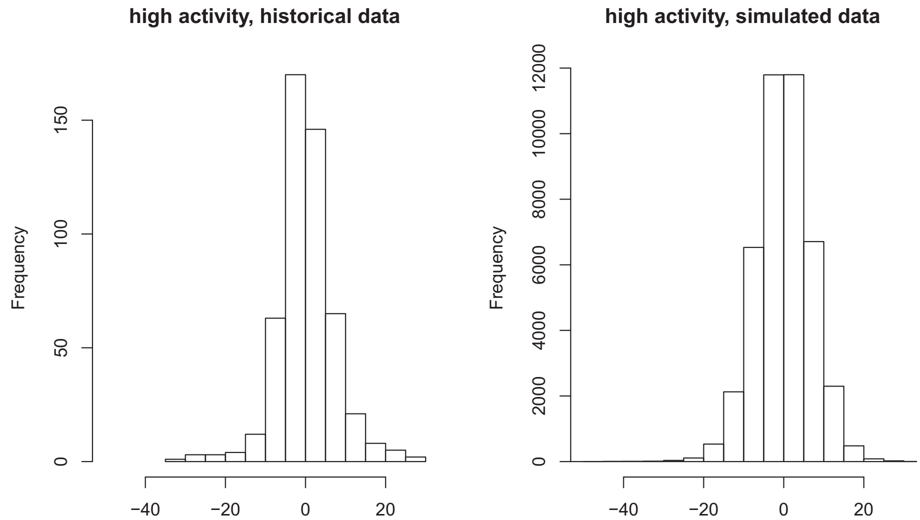

Fig. 16. Histogram high activity.

the intensity of negative jumps to be 0.05 and of positive jumps 0.03 (so overall jump intensity of 0.08). The estimated parameters of the MRJD diffusion component are $\theta = 0.27, \mu = 0.17$ and $\sigma = 4.96$.

We compare the distributional characteristics of the observed and simulated price returns, separately during periods of high and low activity (i.e., high and low demand). We say that the activity is high if the seasonal component of the activity rate is bigger than one, and low if it is smaller than one. This results in seven months with high activity (from November until March) and five months of low activity (from April until October), shown in Fig. 14. The mean activity rate in times of high activity is 1.29 and in times of low activity it is 0.81.

We estimate the mean, standard deviation, skewness and kurtosis of the electricity spot returns, using the deseasonalized and detrended spot price data. The empirical and the simulated quantities (we simulate 10 000 price paths) are shown in Table 4. Comparing the empirical and simulated moments, we find that our model captures the characteristic features of the prices in times of high and low activity well (Table 8). Although we underestimate the general level of the volatility and of kurtosis slightly, we do capture the higher/lower values of these characteristics in times of high/low activity.

We also look at the histograms of the returns in times of high and low activity (see Figs. 16 and 15). The two-sample Kolmogorov–Smirnov test was performed and in both cases (high and low activity) it does not reject the null hypothesis of the same distribution at the significance level of 5% (the p-values are 0.92 for low activity and 0.06 for high activity).

In our simulation study we see that, in comparison with the time changed OU process, the MRJD does perform quite well on average, that is, over the whole year without distinguishing between winter and summer month. Our time changed model though captures the differences between the seasons.

5. Conclusions and future work

Here we developed a model for electricity price using the technique of stochastic time change. We specified the activity rate of the business time via the demand proxy, given by a function of temperature. The resulting time-changed process has time-dependent, seasonal and stochastic parameters: rate of mean reversion, volatility and price spikes intensity — all the characteristic features observed in electricity prices. We estimated the model on the basis of EEX spot prices and German historical temperature data and interpreted the model parameters for the high and low activity periods. The simulation study shows that the characteristic features of observed prices and its moments during different seasons are well captured by the model.

Future work and applications of the presented model extend into several directions. Here we used the demand proxy (a function of temperature) for the activity rate. However, other choices are possible. For example, in electricity markets largely driven by wind, solar of hydro energy, we can also incorporate the supply proxies into the activity rate, in the form of e.g., wind speed, number of sunny hours or the rainfall. Also supply interruptions (such as planned power plant outages) can be incorporated into the activity rate.

The primary goal of the model in this paper was simulation, i.e., generation of realistic sample price paths with observed characteristics. However, if the activity rate is based on meteorological quantities such as temperature (which can be accurately forecasted nowadays), the model can be also used for electricity price forecasting. Although we cannot obtain point forecasts with it, we can obtain distributional forecasts, and accurately forecast quantities such as volatility or the probability of price spikes. Such forecasts can be useful for risk management purposes, planning of electricity production or of maintenance activities and for other applications.

We derived many useful properties of the model, such as the expressions for moments, characteristic function, quadratic variation and martingale representation. These properties can be used for developing derivative pricing tools (futures and options) on the basis of the time changed model. This is the subject of active research and the results of that will be available shortly.

Appendix A. Supplementary data

Supplementary data to this article can be found online at http://dx.doi.org/10.1016/j.eneco.2017.01.002.

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