RECENT DEVELOPMENT AND ANALYSIS OF TAYLOR SERIES

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TUNKU ABDUL RAHMAN UNIVERSITY COLLEGE KUALA LUMPUR

ACADEMIC YEAR
2021/2022

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A project report submitted to the
Faculty of Computing and Information Technology
in partial fulfillment of the requirement for the
Bachelor of Science (Hons.)
Management Mathematics with Computing
Tunku Abdul Rahman University College

Department of Mathematical and Data Science
Faculty of Computing and Information Technology
Tunku Abdul Rahman University College
Kuala Lumpur
2021/2022

DECLARATION

The project report submitted herewith is a result of my own efforts in totality and in every aspects of the works. All information that has been obtained from other sources had been fully acknowledged. I understand that any plagiarism, cheating or collusion of any sorts constitutes a breach of Tunku Abdul Rahman University College rules and regulations and would be subjected to disciplinary actions.

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APPROVAL FOR SUBMISSION

I certify that this project report entitled **"RECENT DEVELOPMENT AND ANALYSIS OF TAYLOR SERIES"** was prepared by **YEE JUN HOONG** and has met the required standard for submission in partial fulfillment of the requirements for the award of Bachelor of Science (Hons.) Management Mathematics with Computing at Tunku Abdul Rahman University College.

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Date: 15/12/2021 **Date:** 15/12/2021

ACKNOWLEDGEMENTS

It is always a great pleasure for me to remind the fine people in my surrounding for their help and sincere guidance to ensure my project success. My project would not have been completed this smoothly if without the dedicated assistance and help by those individuals during this year.

First and foremost, I would like to express my utmost appreciation to my research supervisor, Mr. Chong Kam Yoon for his invaluable suggestion, guidance and his enormous patience during the conduction of this project. With the constant help and support from Mr. Chong throughout this project ensured that I was able to continue my work all the time even though I was encountering some critical problem sometime. Moreover, I would also like to thank Mr. Chong for checking my project and his important feedback. In other words, my project and work were only possible successes because of his generous help and encouragement during the whole year. In short, I wish to convey my wholehearted gratitude and appreciation to Mr. Chong for contributing his advice, suggestions and in-depth knowledge in the field.

Besides that, I would like to thank my research moderator, Dr. Kee Boon Hui for pointing out my problems and mistakes made in this project, giving me a mountain of precious advice. Therefore, I was able to head the correct way of developing this project thanks to her rectification and advice.

In addition, I would like to give the highest acknowledgement to one of my classmates, Lim Hui Jing, for sharing her resources, opinions, knowledge, experience and skills in Mathematics so selflessly. My gratitude also goes out to the lecturers that show me the unwavering support which are Dr. Chin Wan Yoke, and Ms. Chong Voon Niang who have provided me with useful suggestions despite their busy schedule and heavy workload. They

have shown their most warming help possible to me when I faced a serious problem which is feeling stressful throughout the process of developing this project. They vigorously enlightened me and definitely brought me the motivation to continue to conduct this project.

Last but not least, I would also like to personally express my gratitude to my loving parents who vitally extended their assistance in completing this project by mentally supporting me and giving me encouragement all the time.

ABSTRACT

In this project, it is to understand the recent development of Taylor series by finding recent formal sources. The report of this project consists of five chapters.

In chapter one, it is an introduction of the Taylor series. It starts off with the background of the Taylor series which briefly explains what the Taylor series is, the history of the Taylor series, and the Maclaurin series. Besides that, the problem statements, project scope, and objectives of this project are also included in this chapter.

In chapter two, it is a literature review which reviews the well-crafted works of what the previous researchers have done, obtained, or achieved by using the Taylor series. Literature review is also a guideline in this project where it provides some references in order to carry out this project.

In chapter three, it is a methodology where mathematical tools will be involved and introduced. All the mathematical tools used in this project will be listed out in this chapter. The mathematical tools are used to generate some examples related to the Taylor series.

In chapter four, it contains the results that I produced from others' works. The results are obtained by completing the missing and reduced parts of few research materials in chapter two.

In chapter five, it is a conclusion that includes the conclusion of this project, problems and difficulties encountered, and future work.

Keywords: Taylor series, Recent development and analysis, Matlab.

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CHAPTER 1

INTRODUCTION

1.1 Background

In mathematics, the Taylor series of a function can be known as an infinite sum of terms that are expressed in terms of the function's derivatives at a single point. Briefly speaking, if the value of a function, and of all of its derivatives are known at a single point, the value of an entire function at every point can be easily calculated by using the Taylor series.

The concept of Taylor series is formulated by a Scottish mathematician named James Gregory. The Taylor series is named after an English mathematician, Brook Taylor, who formally introduced the Taylor series in 1715.

The Taylor series represents a function as an infinite sum of terms calculated from the values of its derivatives at a single point. The Taylor series can also be regarded as the limit of the Taylor polynomials.

If 0 is the point where the derivatives are considered, the Taylor series is known as a Maclaurin series, which can be recognized as a special case of the Taylor series. The Maclaurin series was named after a Scottish mathematician Colin Maclaurin. In the 18th century, Colin Maclaurin made extensive use of this Maclaurin series (UKEssays, 2018).

The partial sum formed by the first $n + 1$ terms of a Taylor series is a polynomial of degree n, which is called as the nth Taylor polynomial of the function. Taylor polynomials are approximations of a function, which become gradually more precise as n increases.

1.2 Problem Statements

Taylor series can be a bit mysterious the first time when we learn about them. The formula for the Taylor series of a function, $f(x)$ around a point $x = a$ is given by

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

Figure 1.2.1: Formula for Taylor series of a function

where $f(n)(a)$ denotes the nth derivative of the function $f(x)$ at $x=a$. To compute a Taylor series, we find the nth derivatives and substitute them into the formula $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

However, what Taylor series really is, what can Taylor series do, and is Taylor series useful and beneficial? In this project, recent formal sources that are related to Taylor series will be reviewed to get to know the recent development and analysis of Taylor series, thus we are able to understand the Taylor series better.

In this study, formal sources that are published under the Taylor series will be chosen as research materials that are going to be used in this project. After reviewing them, results will be obtained by extending the missing parts of few of the chosen research materials. The reason for doing so will be explained in the aims and objectives in the next page.

1.3 Aims and Objectives

Taylor series is fundamental to the understanding of mathematical analysis in both theoretical and practical terms in the fields of mathematics. Therefore, this project aims to **enhance current knowledge** of **Taylor series**, as well as to **further expand understanding** of **mathematical analysis**. Next, the project scope for this project will be discussed.

In order to learn what can the Taylor series do, and is it useful, an amount of ten research materials were collected that can bring the questions raised to light, some of these research materials included: **(1) A new type of Taylor series expansion** (Masjed-Jamei et al., 2018), **(2) Taylor series solution for a third order boundary value problem arising in Architectural Engineering** (He, 2020), and **(3) An improved C4.5 model classification algorithm based on Taylor series** (Idriss & Lawan, 2019).

After reviewing these papers, two lite summaries can be obtained: **(1) The Taylor series is able to solve real life problems that are involved in different areas such as in Architectural Engineering** and **(2) The Taylor series is very useful as it is very straightforward and simple to use**.

Besides that, a mathematical tool, Matlab will be used in this project to generate examples of Taylor series including showing the truncation order of Taylor series, calculating the expansion point of Taylor series equations at a certain point, and plotting graphs of Taylor series equations. Those examples will be shown in the methodology section.

Furthermore, results will be obtained by completing the missing and reduced parts of few research materials in chapter 2 that were published under the Taylor series. The reason to extend the missing parts of few research materials as results is because it is undeniable that unfortunately, the complete steps while solving a mathematics question is only written out when someone is learning the basics of a new mathematics topic.

Imagine that someone is learning Calculus, a topic of Pure Mathematics. While learning Calculus more advancedly, some of the steps will be naturally skipped. Although the reason to do so is because it can save up more time to do the more complex of the Calculus questions, it might gradually become solving the questions formulaically, which may lead to not understanding the questions, or even lead to a more serious problem, which is having poor understanding to the Calculus itself. It will result in someone who will only solve the Calculus questions rigidly, without any understanding. This is very likely to happen when learning not only a specific topic of mathematics, but more and more different topics of mathematics such as Algebra, Combinatorics, Number theory, Geometry and so on. In the view of above, extending those skippable, solving-in-mind steps of few research materials are the results in this project, as it can make it easier for everyone to understand how a step of the Taylor series equation is changing into the next step.

The objective of this project is to make sure that it is able to build a strong foundation and strengthen the knowledge of Taylor series during the conduction of this project. Besides that, learning about the trend and real-world application of Taylor series, such as how to use Taylor series in real life situations, and how to solve real life problems by using Taylor series is also the objective of this project. Lastly, this project aims to achieve a better understanding and a clearer vision about the development of Taylor series in recent years.

CHAPTER 2

LITERATURE REVIEW

The purpose of this literature review is to study the formal sources, ie., research papers, journals, articles and so on that are related to Taylor series in recent years, and use them as the research materials for this project.

2.1 A new type of Taylor series expansion

A variant of the classical integration by parts is presented to introduce a new type of Taylor series expansion.

Before discussing the new type of Taylor series expansion, this paper starts off with some Taylor series equations and explanations as an introduction. After that, this paper will present the general form of this expansion and consider some interesting cases of it leading to new closed forms for integrals involving Jacobi and Laguerre polynomials. An error analysis is given for the introduced expansion (Masjed-Jamei et al., 2018).

This paper is chosen as the one of the main research materials to produce the result in chapter 4. This is due to the fact that there are a lot of formulas shown in this paper. Besides that, the formulas shown in this paper have parts being skipped or reduced without showing the steps.

2.2 Taylor series solution for Lane-Emden equation

The Lane-Emden equation plays a significant role in physics, chemistry and astronomy. Ji-Huan & Fei-Yu (2019) conducted a study on a reaction diffusion process that can be modeled by the Lane-Emden equation, and the process can be optimized by its analytical solution.

Although there are many analytical methods for singular boundary value problems, it is relatively difficult to be solved analytically due to its singular property. In this research paper, it will show the suggestion of the simplest method to the problem, which is the Taylor series technology to solve the Lane-Emden equation (Ji-Huan & Fei-Yu, 2019).

This paper provides a suggestion of a simple approach to the Lane-Emden equation, with only a total of 18 steps. The idea in this paper can be extended to all differential equations with initial conditions and fractional calculus. This paper has concluded that the Taylor series method is straightforward compared with other analytical methods because it has a simple solutions process and accurate results. The best advantage of the Taylor series method is that the redundant terms will not be produced, and the series converges to the exact solution (Ji-Huan & Fei-Yu, 2019).

This paper is also chosen as one of the research materials to produce the result in chapter 4. This is due to the fact that there are some formulas that are easy to understand, which can be the great examples to be introduced to chapter 4.

2.3 Taylor series solution for a third order boundary value problem arising in Architectural Engineering

The boundary value problems can be used to model many architectural systems. Although there are many numerical and analytical methods to solve such problems, this research is able to suggest a simple yet effective way to the third-order ordinary differential equations by the Taylor series technology (He, 2020).

By clarifying the problem, giving an example and a summary discussion, this paper has successfully applied the technology to the third-order boundary value problems. A comparison is made between the approximate solution obtained in this paper with the variational iteration method, which shows that the result of the present method is simpler and more straightforward (He, 2020).

This paper shows the simple solution process and accurate results, which will prove that the Taylor series method is much more attractive for practical applications (He, 2020).

There are three obvious advantages stated in this paper which are that the solution procedure is simple and straightforward, making it valid for various boundary value problems and initial value problems. Besides that, the series solution converges to the exact solution, and any accuracy can be obtained for a practical problem. Furthermore, the method can be extended to complex boundary conditions (He, 2020).

As a conclusion, this paper has shown a great development of the Taylor series through the method in this paper. It also showed that the Taylor series is reliable and effective with the three advantages. Moreover, it indicated that the Taylor series is very useful in different areas.

2.4 An improved C4.5 model classification algorithm based on Taylor series

C4.5 is one of the most well known algorithms for rule base classification. The C4.5 algorithm is an improvement of the ID3 algorithm. This research paper proposes a technique that will handle the setback reported in C4.5. The performance of the proposed technique is measured based on better accuracy (Idriss & Lawan, 2019).

To overcome the limitations of C4.5, the researchers used Taylor series to modify the splitting information of C4.5, which brought the result of a modified model that can be called EC4.5. The researchers apply exponential splitting information, EC4.5, in utilizing the central attribute of the same dataset, which brought the result obtained on introducing Taylor series suggesting a far better result than when the C4.5 was introduced (Idriss & Lawan, 2019).

The proposed modification offers solutions to the limitations associated with C4.5 in terms of presenting an equivalent result with ID3 when the same number of attributes is used. Idriss & Lawan (2019) states that the result of the experiment shows that EC4.5 outperformed, with an accuracy of 99.40%, whereas C4.5 has an accuracy of 51.27%.

Based on the result obtained in this research paper, a summary can be made that the Taylor series has been proved to be advantageous and beneficial in a vast range of different areas such as bringing improvement and efficiency to the problems, and delivering a better and effective result. This is because the result obtained in this paper by using the Taylor series method (EC4.5) suggested a far better result than when the C4.5 was introduced.

Based on the result of this research, EC4.5 is selected as the optimal algorithm. Thanks to this research, future work can and is suggested to consider a hybrid approach to handle multi-dimensional data with large intervals using EC4.5 algorithm (Idriss & Lawan, 2019).

2.5 A numerical method based on Taylor series for bifurcation analyses within Föppl-von Karman plate theory

A combination of the asymptotic numerical method (ANW) and the Taylor meshless method (TMM) presents a new numerical technique for post-buckling analysis. These two methods are based on Taylor series, with respect to a scalar load parameter for ANM, and with respect to the space variables for TMM (Tian et al., 2018).

Tian et al. (2018) states that the advantage of ANM is an adaptive step length, which is very efficient near bifurcation points. The specificity of TMM is a quasi-exact solution of the partial differential equations inside the domain, which leads to a strong reduction of the number of degrees of freedom.

In short, the new method shown in this paper is very efficient to solve a quasi-perfect bifurcation response and this does not require a strong numerical expertise. It is possible to combine Taylor series in space and in loading parameters, which can be easily extended to other hyperelastic models or to Newtonian fluids. This double Taylor series leads to an efficient path following technique (Tian et al., 2018).

This paper provided very thorough explanations and examples while explaining their topic. The method described in this paper is assessed with only a single example. Last but not least, this paper definitely showed a good sign of the non-stoppable development of the Taylor series.

2.6 FPGA implementation of adaptive digital pre-distorter with improving accuracy of lookup table by Taylor series method

In this research paper, it shows the FPGA implementation with improving accuracy of the lookup table by using the Taylor series method.

Power amplifier (PA) is a critical component and inherently non-linear device of modern wireless base stations. Due to the non-linear behavior of the PA, the amplification of signals with fluctuating envelopes will inevitably lead to loss of adjacent spectrum, adjacent interference and the crossing intermodulation, in-band distortion as well as out-of-band spectral growth in the transmitted signal. The digital pre-distortion (DPD) with the adaptation and updating of the lookup table (LUT) can counteract these non-linear effects (Ren, 2018).

In this Letter, a low-complexity LUT implemented by FPGA of pre-distortion PA lineariser is proposed in order to obtain more accurate linearisation. The algorithm utilizes interpolation of the LUT with the method of Taylor series. This experiment showed that the method can be used to obtain the more accurate indexed value of LUT to estimate the PA behavior for effective DPD (Ren, 2018).

This Letter proposed a method for reducing the LUT indexed quantisation of a DPD system caused by the input value not always being located on the LUT index. Three sections are assigned in this Letter to discuss the detailed issues about the approximation of the LUT with the method of Taylor series implemented by FPGA to increase the performance of the adaptive digital predistorter (Ren, 2018).

This Letter has shown that the Taylor series is an efficient method that can be widely used in many other areas such as the FPGA implementation in this Letter.

2.7 Taylor’s series method for solving the nonlinear reaction-diffusion equation in the electroactive polymer film

This research paper analytically solves the nonlinear reaction-diffusion equation in the electroactive polymer film (Usha Rani & Rajendran, 2020).

This mathematical model describes a substrate to form a complex with the immobilized catalyst. The Taylor series method will be applied for an analytical approximation of the substrate in the electroactive polymer film (Usha Rani & Rajendran, 2020).

The obtained analytical solution is used to compare with the numerical results and it is found to be in satisfactory agreement. Present analytical expression is compared with the previous result. Usha Rani & Rajendran (2020) states that the Taylor series method yields a rapidly convergent, easily computable, and rapidly verifiable sequence of analytic approximations that are convenient for parametric simulations.

This paper has shown fascinating yet plentiful information when solving its title by using the Taylor series method. Taylor series is undoubtedly a simple, effective, and straightforward method.

2.8 Full-Wave Computation of the Electric Field in the Partial Element Equivalent Circuit Method Using Taylor Series Expansion of the Retarded Green's Function

This article presents new analytical formulas for the efficient computation of the full-wave electric field generated by conductive, dielectric, and magnetic media in the framework of the partial element equivalent circuit (PEEC) method (Kovacevic-Badstuebner et al., 2020).

To this aim, the full-wave Green's function is handled by the Taylor series expansion leading to three types of integrals for which new analytical formulas are provided in order to avoid slower numerical integration (Kovacevic-Badstuebner et al., 2020).

This article is indeed a very detailed article that explains its own title enormously well. It covers almost every aspect of the title by applying the Taylor series expansion wisely.

This article presents a fast computation of the electric field radiated by electrical currents, magnetization, and charges, in the framework of the PEEC full-wave modeling, assuming an orthogonal tessellation of volumes and surfaces (Kovacevic-Badstuebner et al., 2020).

2.9 Analytical Evaluation of Partial Elements Using a Retarded Taylor Series Expansion of the Green's Function

The computation of the fundamental magnetic and electric field coupling terms for the full-wave partial element equivalent circuit method is time consuming for large problems (Lombardi et al., 2018).

Therefore, this paper presents new full-wave analytical formulas for their computation, which are based on several different Taylor series expansion approaches for the case of rectangular elementary volumes and surfaces (Lombardi et al., 2018).

This paper provides a lot of examples, steps, graphs, plots, and so on, in order to give us the idea of how to apply the Taylor series expansion to its title. The algorithms in this paper represent a new approach for the computation of partial elements for PEEC with retardation (Lombardi et al., 2018).

The development of Taylor series has been successfully shown in this paper as the new formulation using the Taylor series expansion removes the frequency dependence from the integral part. Moreover, the errors in the evaluation of the formulas are sufficiently small for the analytical formulas (Lombardi et al., 2018).

2.10 Taylor series and twisting-index invariants of coupled spin-oscillators

In this paper, the list of invariants for the coupled spin-oscillator is completed by calculating higher order terms of the Taylor series invariant and by computing the twisting index (Alonso et al., 2019).

This paper successfully proves that the Taylor series invariant has certain symmetry properties that make the even powers in one of the variables vanish and allow to show superintegrability of the coupled spin-oscillator on the zero energy level (Alonso et al., 2019).

A lot of inspiring ideas are shown to us in this paper in order to achieve the results they want. A lot of theorems made in this paper come with very thorough and detailed explanations and examples.

This paper carries out a detailed analysis on the Taylor series, and successfully applied it on the title of this paper, and shows us a qualified and proficient development of Taylor series.

CHAPTER 3

METHODOLOGY

This chapter introduce a mathematical tool, Matlab. It will be used to generate examples related to Taylor series, ie., Taylor series expansion, plotting graphs for Taylor series expansion and so on.

3.1 Taylor Series Expansion

```
syms x
f = exp(-x)
fprintf('==============================================')

%------------------------------
taylor(f)
fprintf('==============================================')
% The default truncation order is 6.

pretty(taylor(f))
fprintf('==============================================')
% Prints the answer in a plain-text format that resembles typeset mathematics.

taylor(f, 'ExpansionPoint', 1)
taylor(f, x, 1)
% Find the Taylor series expansions at x=1 for f.
```

Figure 3.1.1: Example of the Taylor series expansion

The figure 3.1.1 indicates the code used in the Matlab. First, I declare the variable *x* by using *syms x*, then I let $f = e^{-x}$ which is written as $f = \exp(-x)$ in the code as shown in the figure 3.1.1. After that, I show the truncation order for $f = e^{-x}$ by using *taylor(f)*. It shows a default truncation order for f which is 6, generating the result $1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$ as shown in the figure 3.1.2. Next, the "pretty" symbolic expression is used to print the answer in a plain-text format. The difference in between is shown in the figure 3.1.2.

After that, the next example shows how to find the Taylor series expansion at $x = 1$ for f . The "ExpansionPoint" is used to specify a specific expansion point. For example, *taylor(f, 'ExpansionPoint', 1)* is to find the Taylor series expansion at $x = 1$ for f. In the view of the example, changing the number of the third argument can find the Taylor series expansion at x for a number that want to find. Alternatively, using the variable such as x in this example, can also replace the "ExpansionPoint" but work exactly the same and give the same result as shown in the figure 3.1.2.

$f = e^{-x}$

```
=============================================
ans =
```

$$1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$$

```
=============================================
         2    3    4     5
        x    x    x     x
1 - x + -- - -- + -- - ---
         2    6   24   120
=============================================
ans =
```

$$e^{-1} - e^{-1}(-1+x) + \frac{e^{-1}(-1+x)^2}{2} - \frac{e^{-1}(-1+x)^3}{6} + \frac{e^{-1}(-1+x)^4}{24} - \frac{e^{-1}(-1+x)^5}{120}$$

```
ans =
```

$$e^{-1} - e^{-1}(-1+x) + \frac{e^{-1}(-1+x)^2}{2} - \frac{e^{-1}(-1+x)^3}{6} + \frac{e^{-1}(-1+x)^4}{24} - \frac{e^{-1}(-1+x)^5}{120}$$

Figure 3.1.2: Result of the Taylor series expansion

3.2 Maclaurin Series of Univariate Expressions

```
syms x

T1 = taylor(exp(x))
T2 = taylor(sin(x))
T3 = taylor(cos(x))
% Find the Maclaurin series expansions

sympref('PolynomialDisplayStyle', 'ascend');
fprintf('==============================================')
T1
T2
T3
% Use the sympref function to modify the output order of symbolic
% polynomials, and redisplay it in ascending order.

sympref('default');
```

Figure 3.2.1: Example of the Maclaurin series of univariate expressions

First, It has the exact same start as the figure 3.1.1. A variable that we want to use is declared by using *syms*. With that, it can show the Maclaurin series expansions of e^x, sin x , and cos x using the declared variable, x.

The "sympref" function is used to alter the output order of symbolic polynomials, which redisplay the polynomials in ascending order, such as $T1 = \frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1$ will be redisplayed as $T1 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$, which is shown in the figure 3.2.2.

```
T1 =
```

$$\frac{x^5}{120}+\frac{x^4}{24}+\frac{x^3}{6}+\frac{x^2}{2}+x+1$$

```
T2 =
```

$$\frac{x^5}{120}-\frac{x^3}{6}+x$$

```
T3 =
```

$$\frac{x^4}{24}-\frac{x^2}{2}+1$$

```
==============================================
T1 =
```

$$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}$$

```
T2 =
```

$$x-\frac{x^3}{6}+\frac{x^5}{120}$$

```
T3 =
```

$$1-\frac{x^2}{2}+\frac{x^4}{24}$$

Figure 3.2.2: Result of the Maclaurin series of univariate expressions

3.3 Specify Truncation Order for Maclaurin series expansion

```
syms x

g = sin(x)/x
fprintf('=============================================')
T6 = taylor(g)
% The default truncation order is 6.

T10 = taylor(g, x, 'Order', 10)
T14 = taylor(g, x, 'Order', 14)
% Use Order to control the truncation order.

%----------------------------
fprintf('=============================================')
fplot([T6 T10 T14 g])
xlim([-4 4])
grid on

legend('approximation of sin(x)/x up to O(x^6)', ...
    'approximation of sin(x)/x up to O(x^{10})', ...
    'approximation of sin(x)/x up to O(x^{14})', ...
    'sin(x)/x', 'Location', 'Best')
title('Taylor Series Expansion')
% Plot the original expression g and its approximations T6, T10, and T14.
```

Figure 3.3.1: Example of the specify truncation order for the Maclaurin series expansion

The figure 3.3.1 shows a way to find the Maclaurin series expansion for $g = \sin(x)/x$. I declare $T6 = \text{taylor}(g)$ because the default truncation order for this function g is 6.As the Taylor series approximation of this expression does not have a fifth-degree term, therefore, the taylor approximates this expression with the fourth-degree polynomial, which is $\frac{x^4}{120} - \frac{x^2}{6} + 1$ as shown in the figure 3.3.2.

The original expression $g = \sin(x)/x$ and its approximations $T6 = \frac{x^4}{120} - \frac{x^2}{6} + 1$, $T10 = \frac{x^8}{362880} - \frac{x^6}{5040} + \frac{x^4}{120} - \frac{x^2}{6} + 1$, and $T14 = \frac{x^{12}}{6227020800} - \frac{x^{10}}{39916800} + \frac{x^8}{362880} - \frac{x^6}{5040} + \frac{x^4}{120} - \frac{x^2}{6} + 1$ are plotted in the figure 3.3.2 to show the accuracy of the approximation depends on the truncation order.

g =

$$\frac{\sin(x)}{x}$$

==

T6 =

$$\frac{x^4}{120}-\frac{x^2}{6}+1$$

T10 =

$$\frac{x^8}{362880}-\frac{x^6}{5040}+\frac{x^4}{120}-\frac{x^2}{6}+1$$

T14 =

$$\frac{x^{12}}{6227020800}-\frac{x^{10}}{39916800}+\frac{x^8}{362880}-\frac{x^6}{5040}+\frac{x^4}{120}-\frac{x^2}{6}+1$$

==

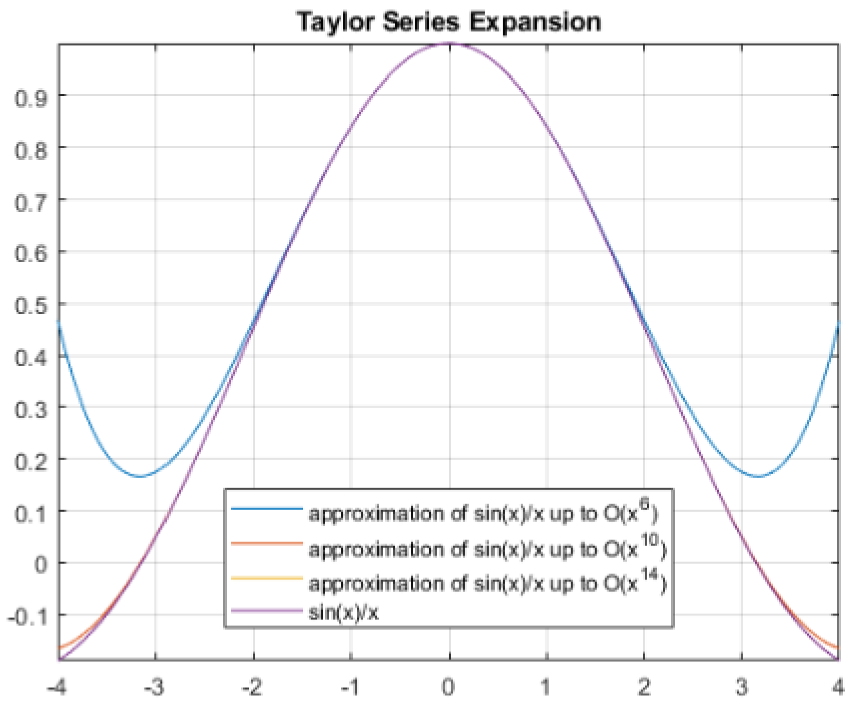

Figure 3.3.2: Result of the specify order for the Maclaurin series expansion

In this figure, we can see that there is a difference between the original expression $g = \sin(x)/x$ and its approximations $T6 = \frac{x^4}{120} - \frac{x^2}{6} + 1$ based on the shape of the graph. However, the $T10 = \frac{x^8}{362880} - \frac{x^6}{5040} + \frac{x^4}{120} - \frac{x^2}{6} + 1$, and $T14 = \frac{x^{12}}{6227020800} - \frac{x^{10}}{39916800} + \frac{x^8}{362880} - \frac{x^6}{5040} + \frac{x^4}{120} - \frac{x^2}{6} + 1$ become more close to its original expression $g = \sin(x)/x$. This is due to the fact that when the order, or more specifically, the n increases, the accuracy of the approximation will also increase, which brings an approximation result that is very close to the original answer as shown in the figure 3.3.2.

3.4 Plotting the Taylor series expansion

```
% Create an x vector

x = -2:0.1:2;

y = exp(x);

fig = figure();
set(fig, 'color', 'white')
plot(x, y, 'LineWidth', 2)
grid on
xlabel('x')
ylabel('y')

% Use x vector and create a y estimate using the Taylor series expansion
N = 1;
yest = 0 * y;

for n = 0:N
    yest = yest + (x.^n)./factorial(n);
end

hold on
plot(x, yest, 'r-', 'LineWidth', 2)
legend('Actual Function', 'Taylor Series Expansion')
```

Figure 3.4.1: Example of the plotting of the Taylor series expansion - 1

The figure 3.4.1 shows the code of plotting the Taylor series expansion. First, I let $x = -2: 0.1: 2$ and $y = \exp(x)$. Next, I let $N = 1$, and declare that $yest = 0 * y$, where yest means the estimated value of y.

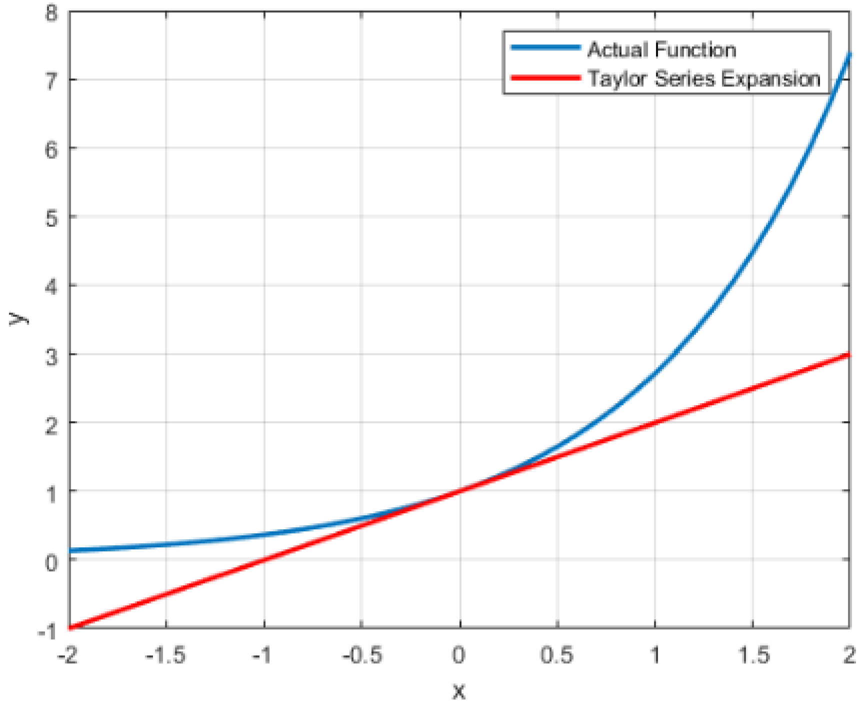

Figure 3.4.2: Result of the plotting of the Taylor series expansion - 1

The graph shown in the figure 3.4.2 is the result of the Taylor series expansion when N = 1. As we know that the Maclaurin series for e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$, the N = 1 indicates $\frac{x^0}{0!} + \frac{x^1}{1!} = 1 + x$, which form the approximation of the Taylor series expansion as shown in the figure 3.4.2. From the graph, we can see that there are some differences in values between the actual function, f = e^x and its approximation where N = 1.

By changing the N, the "Taylor Series Expansion" line will be getting closer and closer to the "Actual Function" line, as shows in the diagrams below:

```
N = 3;
yest = 0 * y;

for n = 0:N
    yest = yest + (x.^n)./factorial(n);
end

hold on
plot(x, yest, 'r-', 'LineWidth', 2)
legend('Actual Function', 'Taylor Series Expansion')
```

Figure 3.4.3: Example of the plotting of the Taylor series expansion - 2

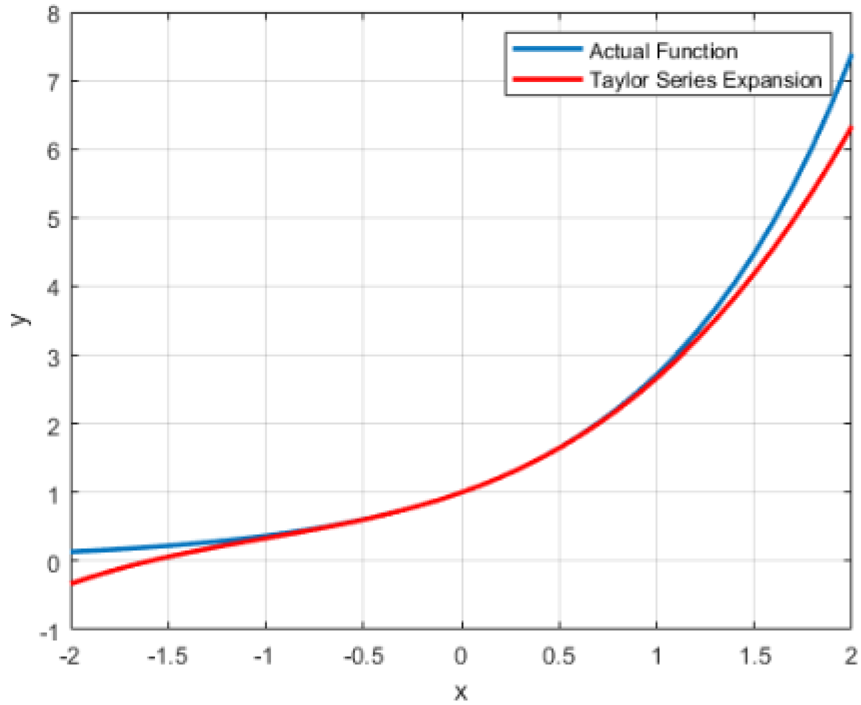

Figure 3.4.4: Result of the plotting of Taylor series expansion - 2

In the figure 3.4.3, the value of N is changed to N = 3. The graph shown in the figure 3.4.4 is the result of the Taylor series expansion when N = 3. This time, the N = 3 indicates $\frac{x^0}{0!}+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}= 1 + x + \frac{x^2}{2}+\frac{x^3}{6},$ which form the approximation of the Taylor series expansion as shown in the figure 3.4.4. From the graph, we can see the difference in values between the actual function, f = e^x and its approximation where N = 3 becomes smaller than when N = 1.

```
N = 5;
yest = 0 * y;

for n = 0:N
    yest = yest + (x.^n)./factorial(n);
end

hold on
plot(x, yest, 'r-', 'LineWidth', 2)
legend('Actual Function', 'Taylor Series Expansion')
```

Figure 3.4.5: Example of the plotting of the Taylor series expansion - 3

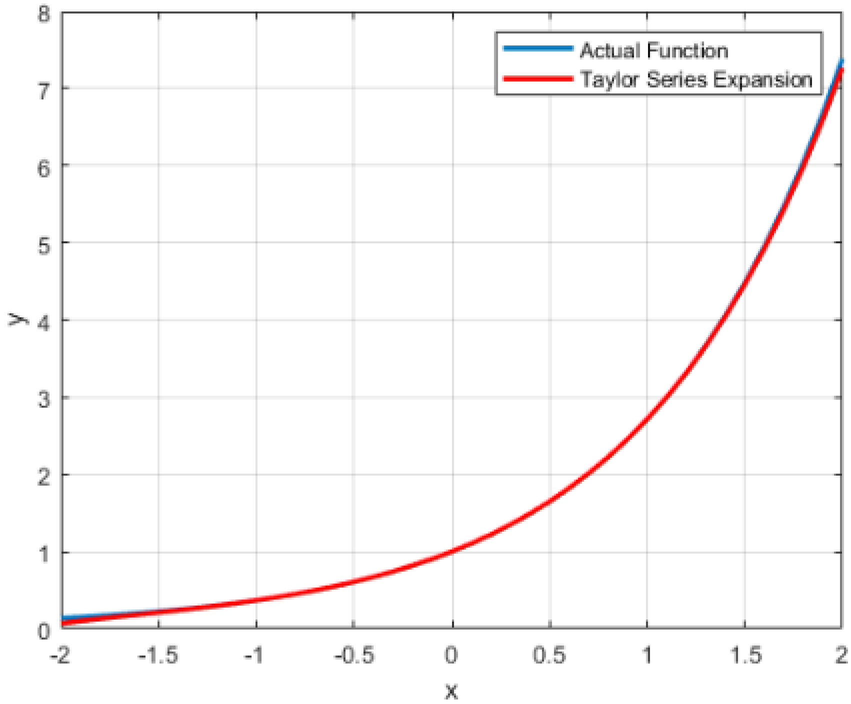

Figure 3.4.6: Result of the plotting of the Taylor series expansion - 3

In the figure 3.4.5, the value of N is changed to N = 5. The graph shown in the figure 3.4.6 is the result of the Taylor series expansion when N = 5. The N = 5 indicates $\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$, which form the approximation of the Taylor series expansion as shown in the figure 3.4.6. From the graph, we can see that there is almost no difference in values between the actual function, $f = e^x$ and its approximation where N = 5. In the view of above, we can conclude that, if the N keeps on increasing, where we will get $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots$, the line of the approximation will get infinitely close to the actual function. This will lead to the increment of the accuracy in value of the approximation, which means the approximated value will be very close to the actual value.

CHAPTER 4

RESULTS AND DISCUSSION

In this chapter, I will use a few of the research materials to produce the result throughout extending the missing and reduced steps.

4.1 Taylor series solution for Lane-Emden equation

In this 4.1, I will use the research paper, Taylor series solution for Lane-Emden equation (Ji-Huan & Fei-Yu, 2019) to produce the results by completing the missing parts of Taylor series equations shown in this paper.

$$2u'' + xu''' - v^3(u^2+1) - 2xv^3uu' - x3v^2v'(u^2+1) = 0$$

Figure 4.1.1: Example 1.1

$$2u''(0) - v^3(0)(u^2(0)+1) = 0$$

Figure 4.1.2: Example 1.2

The formula shown in the figure above is labeled as formula **(5)**, which changed into the figure 4.1.2 by setting x = 0 in the formula **(5)**, and labeled as formula **(7)** in the paper.

$$2u'' + xu''' - v^3(u^2+1) - 2xv^3uu' - x3v^2v'(u^2+1) = 0$$

$$2u'' + (0)u''' - v^3(u^2+1) - 2(0)v^3uu' - (0)3v^2v'(u^2+1) = 0$$

$$2u'' + 0 - v^3(u^2+1) - 0 - 0 = 0$$

$$2u''(0) - v^3(0)(u^2(0)+1) = 0$$

Figure 4.1.3: Result 1

The figure shown above is the result of example 1.1 and example 1.2. It shows the steps of how the example 1.1 is changed into the example 1.2. By letting $x = 0$, $2u'' + xu''' - v^3(u^2+1) - 2xv^3uu' - x3v^2v'(u^2+1) = 0$ will become $2u'' + (0)u''' - v^3(u^2+1) - 2(0)v^3uu' - (0)3v^2v'(u^2+1) = 0$. After multiplying the 0 , it will become $2u'' + 0 - v^3(u^2+1) - 0 - 0 = 0$, which can be reduced into the answer shown in the figure 4.1.2.

$$4v'' + xv''' + v^5(u^2+3) + 2xv^5uu' + 5xv^4v'(u^2+3) = 0$$

Figure 4.1.4: Example 2.1

$$4v''(0) + v^5(0)(u^2(0)+3) = 0$$

Figure 4.1.5: Example 2.2

The formula shown in the figure above is labeled as formula **(6)**, which changed into the figure 4.1.2 by setting $x = 0$ in the formula **(6)**, and labeled as formula **(8)** in the paper. These examples are the same with examples 1.1 and 1.2.

$$4v'' + xv''' + v^5(u^2+3) + 2xv^5uu' + 5xv^4v'(u^2+3) = 0$$
$$4v'' + (0)v''' + v^5(u^2+3) + 2(0)v^5uu' + 5(0)v^4v'(u^2+3) = 0$$
$$4v'' + 0 + v^5(u^2+3) + 0 + 0 = 0$$
$$4v''(0) + v^5(0)(u^2(0)+3) = 0$$

Figure 4.1.6: Result 2

The figure shown above is the result of example 2.1 and example 2.2. It can be done in the same way as the result 1 shown in the figure 4.1.3. By letting $x = 0$, $4v'' + xv''' + v^5(u^2+3) + 2xv^5uu' + 5xv^4v'(u^2+3) = 0$ will become $4v'' + (0)v''' + v^5(u^2+3) + 2(0)v^5uu' + 5(0)v^4v'(u^2+3) = 0$. After multiplying the 0, it will become $4v'' + 0 + v^5(u^2+3) + 0 + 0 = 0$, which can be reduced into the answer shown in the figure 4.1.5.

4.2 A new type of Taylor series expansion

In this part, I will use the research paper, A new type of Taylor series expansion (Masjed-Jamei et al., 2018) to produce the results by completing the missing parts of Taylor series equations shown in this paper.

$$\int u\,dv = uv - \int v\,du.$$

Figure 4.2.1: General rule

This general rule is shown in the paper, at the start of explaining a new type of Taylor series expansion. This general rule can also be called the product rule. The concept of this product rule plays an important role in this paper when the authors introduced the new type of Taylor series expansion. Although this product rule usually comes up as a formula, it actually has steps before it becomes this form, which will be shown in the figure 4.2.2.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\int \frac{d}{dx}(uv)dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$\mathrm{uv} = \int udv + \int vdu$$

$$\int udv = \mathrm{uv} - \int vdu$$

Figure 4.2.2: Result of expanding the general rule

In the figure 4.2.2, it shows the steps before the general rule becomes $\int udv = \mathrm{uv} - \int vdu$. While a simple equation such as f(x) = $2x^3 + 5x$ can be differentiate or integrated normally, an equation like g(x) = x^2e^x is not able to be differentiate or integrated through the normal way. The product rule is needed when calculating the equation like g(x).

In the figure 4.2.2, there is the product rule of the differentiation, $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$. If the formula is integrated, it will become $\int \frac{d}{dx}(uv)dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$. After the integration, the formula will become uv = $\int udv + \int vdu$. By changing the $\int vdu$, it will become the general rule, $\int udv = \mathrm{uv} - \int vdu$ that is shown in the paper.

By showing the steps, it indicates that this product rule did not come out of nothing, but it actually makes sense as it shows that the differentiation and the integration are related to each other.

$$\int_a^b F(t)G(t)\,dt = \left(F(t)G_1(t) - F'(t)G_2(t) + \cdots + (-1)^{n-1}F^{(n-1)}(t)G_n(t)\right)\Big|_a^b + (-1)^n \int_a^b F^{(n)}(t)G_n(t)\,dt$$
$$= \sum_{k=0}^{n-1}(-1)^k \left(F^{(k)}(t)G_{k+1}(t)\right)\Big|_a^b + (-1)^n \int_a^b F^{(n)}(t)G_n(t)\,dt,$$

Figure 4.2.3: Example 1

The equation shown in the figure 4.2.3 is obtained by using the general rule as mentioned in the paper. This equation is labeled as formula **(1)** in the paper.

$$\int_a^b F(t)G(t)dt$$
$$= [F(t)G_1(t) - F^{'}(t)G_2(t) + \ldots + (-1)^{n-1}F^{(n-1)}(t)G_n(t)]|_a^b + (-1)^n \int_a^b F^{(n)}(t)G_n(t)\,dt$$
$$= \big(F(t)G_1(t) - F^{'}(t)G_2(t) + F^{''}(t)G_3(t) - F^{'''}(t)G_4(t) + F^{''''}(t)G_5(t) - \ldots$$
$$+ (-1)^{n-1}F^{(n-1)}(t)G_n(t)\big)|_a^b + (-1)^n \int_a^b F^{(n)}(t)G_n(t)\,dt$$
$$= \textstyle\sum_{k=0}^{n-1}(-1)^k[F^{(k)}(t)G_{k+1}(t)]|_a^b + (-1)^n \int_a^b F^{(n)}(t)G_n(t)\,dt$$

Figure 4.2.4: Result 1

In the figure 4.2.4, it shows that $\int_a^b F(t)G(t)dt = [F(t)G_1(t) - F^{'}(t)G_2(t) + \ldots + (-1)^{n-1}F^{(n-1)}(t)G_n(t)]|_a^b + (-1)^n \int_a^b F^{(n)}(t)G_n(t)\,dt$, where the +...+ is expandable. If continued from the $F(t)G_1(t) - F^{'}(t)G_2(t) + \ldots$, it will be $F(t)G_1(t) - F^{'}(t)G_2(t) + F^{''}(t)G_3(t) - F^{'''}(t)G_4(t) + F^{''''}(t)G_5(t) -$ After expanding the function, it shows a clearer idea of how it changes into $(-1)^{n-1}F^{(n-1)}(t)G_n(t)$. As the flow of the function is $+, -, +, -, +, -, \ldots$, if it goes infinitely, the function will become $(-1)^{n-1}$, where $n \geq 1$.

When $n = 1$, it is $(-1)^{1-1} = (-1)^0 = 1$, which indicates that the first term, $F(t)G_1(t)$ is positive. Likewise, When $n = 2$, it is $(-1)^{2-1} = (-1)^1 = -1$, which indicates that the second term, $-F'(t)G_2(t)$ is negative as shown in the figure 4.2.4. By expanding the function in this equation, it brings a more clearer idea of how a step of this function changed into the next step.

Let $f \in C^{n+1}[a,b]$ *and* $x_0 \in [a,b]$. *Then, for all* $a \leq x \leq b$, *we have*

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!}(x - x_0)^k f^{(k)}(x_0) + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \, dt.$$

Figure 4.2.5: Example 2

The figure 4.2.5 is a Theorem 2.1, as the formula in the figure 4.2.3 provides a straightforward proof for Taylor's theorem with an integral remainder term, which is mentioned in the paper. The formula in the figure 4.2.5 is labeled as formula **(2)** in the paper.

$$\begin{aligned} f(x) \\ &= \frac{1}{0!}(x - x_0)^0 f^{(0)}(x_0) + \frac{1}{1!}(x - x_0)^1 f^{(1)}(x_0) + \frac{1}{2!}(x - x_0)^2 f^{(2)}(x_0) + \frac{1}{3!}(x - x_0)^3 f^{(3)}(x_0) \\ &\qquad + \ldots + \frac{1}{n!}(x - x_0)^n f^{(n)}(x_0) + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \, dt \\ &= f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \frac{1}{6}(x - x_0)^3 f'''(x_0) + \ldots + \frac{1}{n!}(x - x_0)^n f^{(n)}(x_0) \\ &\qquad + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \, dt \\ &= \sum_{k=0}^{n} \frac{1}{k!}(x - x_0)^k f^{(k)}(x_0) + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \, dt \end{aligned}$$

Figure 4.2.6: Result 2

The summation form, $\sum_{k=0}^{n} \frac{1}{k!}(x - x_0)^k f^{(k)}(x_0)$ shown in the figure above can be expanded to the form before it becomes the summation form. When comparing figure 4.2.3 to figure 4.2.5, a commonality can be found, which is that they both have the summation form in the formula. From the summation form, it gives the information that the first term of this equation starts with k=0. Therefore, the first term will be $\frac{1}{0!}(x - x_0)^0 f^{(0)}(x_0)$, where k will not only stop at 0, but keep on increasing until ∞, which forms the equation $\frac{1}{0!}(x - x_0)^0 f^{(0)}(x_0) + \frac{1}{1!}(x - x_0)^1 f^{(1)}(x_0) + \frac{1}{2!}(x - x_0)^2 f^{(2)}(x_0) + \frac{1}{3!}(x - x_0)^3 f^{(3)}(x_0) + \ldots + \frac{1}{n!}(x - x_0)^n f^{(n)}(x_0)$ as shown in the first step of the figure 4.2.6.

Afterwards, the second step is the simplification of the equation in the first step, where it is further simplified in the third step, and formed the equation into the summation form.

$$f(x) = \frac{1}{n!c_n}\left(\sum_{k=0}^{n} p_n^{(k)}(x-a)f^{(n-k)}(a) - \sum_{k=0}^{n-1} k!c_k f^{(n-k)}(x) + \int_a^x p_n(x-t)f^{(n+1)}(t)\,dt\right).$$

Figure 4.2.7: Example 3.1

$$f(x) = \sum_{k=0}^{n} \frac{(x-a)^{n-k}}{(n-k)!} f^{(n-k)}(a) + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt,$$

Figure 4.2.8: Example 3.2

In this "A new type of Taylor series expansion" (Masjed-Jamei et al., 2018) paper, a formula labeled as formula **(3)** is written as $f(x) = \frac{1}{n!c_n}(\sum_{k=0}^{n} P_n^{(k)}(x-a)f^{(n-k)}(a) - \sum_{k=0}^{n-1} k!\, c_k f^{(n-k)}(x) + \int_a^x P_n(x-t)f^{(n+1)}(t)\,dt)$, which is shown in the figure above. After that, the formula is reduced to $f(x) = \sum_{k=0}^{n} \frac{(x-a)^{n-k}}{(n-k)!} f^{(n-k)}(a) + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$ as shown in the figure 4.2.8. The steps of reducing will be shown in this example 3.

$$f(x) = \frac{1}{n!\,c_n}\left(\sum_{k=0}^{n} P_n^{(k)}(x-a)f^{(n-k)}(a) - \sum_{k=0}^{n-1} k!\,c_k f^{(n-k)}(x) + \int_a^x P_n(x-t)f^{(n+1)}(t)\,dt\right)$$

$$f(x) = \frac{1}{n!\left(\frac{1}{n!}\right)}\left(\sum_{k=0}^{n} \frac{1}{(n-k)!}(x-a)^{(n-k)}f^{(n-k)}(a) - \sum_{k=0}^{n-1} k!\,(0)f^{(n-k)}(x) + \int_a^x \frac{1}{n!}(x-t)^n f^{(n+1)}(t)\,dt\right)$$

$$f(x) = 1\left(\sum_{k=0}^{n} \frac{(x-a)^{(n-k)}}{(n-k)!}f^{(n-k)}(a) - 0 + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt\right)$$

$$f(x) = \sum_{k=0}^{n} \frac{(x-a)^{n-k}}{(n-k)!}f^{(n-k)}(a) + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$$

Figure 4.2.9: Result 3

The reduced steps are shown in this figure 4.2.9. In the paper, it let the $P_n(x-t) = \frac{1}{n!}(x-t)^n$, $c_j = 0$ for every $j = 0,1,2,\dots,n-1$, and $c_n = \frac{1}{n!}$. In this case, the formula, $f(x) = \frac{1}{n!c_n}\left(\sum_{k=0}^{n} P_n^{(k)}(x-a)f^{(n-k)}(a) - \sum_{k=0}^{n-1} k!\,c_k f^{(n-k)}(x) + \int_a^x P_n(x-t)f^{(n+1)}(t)\,dt\right)$ can be reduced into this formula, $f(x) = \sum_{k=0}^{n} \frac{(x-a)^{n-k}}{(n-k)!}f^{(n-k)}(a) + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$ by doing the substitution and simplification.

Dividing this formula into four parts can be easier to understand. For example, (a) $\frac{1}{n!c_n}$, (b) $\sum_{k=0}^{n} P_n^{(k)}(x-a)f^{(n-k)}(a)$, (c) $\sum_{k=0}^{n-1} k!\,c_k f^{(n-k)}(x)$, and (d) $\int_a^x P_n(x-t)f^{(n+1)}(t)\,dt$. First, the (a) can become $\frac{1}{n!\frac{1}{n!}}$ by substituting the $c_n = \frac{1}{n!}$, which will become 1 by eliminating the n!. The next part is substituting the $P_n(x-t) = \frac{1}{n!}(x-t)^n$ into the (b). By doing so, the $P_n^{(k)}(x-a)$ will become $\frac{1}{(n-k)!}(x-a)^{(n-k)}$. The third part is the easiest one, because it contains the c_j which is equal to 0. Lastly, the (d) will change into $\frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$ by substituting the $\frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$. After that, a simplification of the second step is performed in the third step. This is the result of the reduced steps and how the formula in the figure 4.2.7 changed into the formula in the figure 4.2.8.

$$\sum_{k=0}^{n} k! C_k^{(\alpha,\beta,n)} f^{(n-k)}(x)$$
$$= \frac{1}{\Gamma(\alpha+\beta+n+1)} \sum_{k=0}^{n} (x-a)^k \sum_{j=0}^{n-k} \frac{1}{2^j} f^{(n-j)}(a) \Gamma(\alpha+\beta+n+1+j) C_k^{(\alpha+j,\beta+j,n-j)}$$
$$+ \sum_{k=0}^{n} C_k^{(\alpha,\beta,n)} \int_a^x (x-t)^k f^{(n+1)}(t)\, dt.$$

Figure 4.2.10: Example 4

The formula shown in the figure above is obtained according to the formula **(3)** in the paper, and is labeled as the formula **(4)**. For this example, one more step will be further expanded.

$$\sum_{k=0}^{n} k!\, C_k^{(\alpha,\beta,n)} f^{(n-k)}(x)$$
$$= \frac{1}{\gamma(\alpha+\beta+n+1)} \sum_{k=0}^{n} (x-a)^k \sum_{j=0}^{n-k} \frac{1}{2^j} \gamma(\alpha+\beta+n+1+j) C_k^{(\alpha+j,\beta+j,n-j)} f^{(n-j)}(a)$$
$$+ \sum_{k=0}^{n} C_k^{(\alpha,\beta,n)} \int_a^x (x-t)^k f^{(n+1)}(t)\, dt$$
$$= \frac{1}{(\alpha+\beta+n)!} \sum_{k=0}^{n} (x-a)^k \sum_{j=0}^{n-k} 2^{-j} (\alpha+\beta+n+j)!\, C_k^{(\alpha+j,\beta+j,n-j)} f^{(n-j)}(a)$$
$$+ \sum_{k=0}^{n} C_k^{(\alpha,\beta,n)} \int_a^x (x-t)^k f^{(n+1)}(t)\, dt$$

Figure 4.2.11: Result 4

The differences of the extra step shown in the figure above are the $\frac{1}{\gamma(\alpha+\beta+n+1)}$, $\gamma(\alpha+\beta+n+1+j)$ and $\frac{1}{2^j}$. For the $\frac{1}{2^j}$, it is a minor change, which is written as 2^{-j}. According to the gamma function, $\gamma(n) = (n-1)!$, therefore, the $\frac{1}{\gamma(\alpha+\beta+n+1)}$ is changed into the $\frac{1}{(\alpha+\beta+n)!}$ and the $\gamma(\alpha+\beta+n+1+j)$ is changed into the $(\alpha+\beta+n+j)!$.

$$
\begin{aligned}
&\sum_{k=0}^{n} k! C_k^{(n)} f^{(n-k)}(x) \\
&= \sum_{k=0}^{n} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \cos\big(n \arccos(x-a)\big) f^{(n-k)}(a) + \int_a^x \cos\big(n \arccos(x-t)\big) f^{(n+1)}(t)\, dt \\
&= \frac{2^{2n}}{(n-1)!} \sum_{k=0}^{n} (x-a)^k \sum_{j=0}^{n-k} 2^{-j} (n+j-1)! C_k^{(j-\frac{1}{2}, j-\frac{1}{2}, n-j)} f^{(n-j)}(a) \\
&\quad + \int_a^x \cos\big(n \arccos(x-t)\big) f^{(n+1)}(t)\, dt, \quad n \ge 1.
\end{aligned}
$$

Figure 4.2.12: Example 5

The formula in the figure above is shown in the Remark 2 and before the formula **(5)** in the paper. This formula is obtained by reducing $p_n(x) = T_n(x)$ in the formula **(3)** in the paper.

$$
\begin{aligned}
&\sum_{k=0}^{n} k!\, C_k^{(n)} f^{(n-k)}(x) \\
&= \sum_{k=0}^{n} \frac{d^k}{dx^k} \cos\left(n\, arccos(x-a)\right) f^{(n-k)}(a) + \int_a^x \cos\left(n\, arccos(x-t)\right) f^{(n+1)}(t)\, dt \\
&= 2^{2n} \left(\frac{1}{\gamma(n)}\right) \sum_{k=0}^{n} (x-a)^k \sum_{j=0}^{n-k} \frac{1}{2^j} \gamma(n+j) C_k^{(\alpha+j, \beta+j, n-j)} f^{(n-j)}(a) \\
&\qquad + \int_a^x \cos\left(n\, arccos(x-t)\right) f^{(n+1)}(t)\, dt \\
&= \frac{2^{2n}}{(n-1)!} \sum_{k=0}^{n} (x-a)^k \sum_{j=0}^{n-k} 2^{-j} (n+j-1)!\, C_k^{(j-\frac{1}{2}, j-\frac{1}{2}, n-j)} f^{(n-j)}(a) \\
&\qquad + \int_a^x \cos\left(n\, arccos(x-t)\right) f^{(n+1)}(t)\, dt
\end{aligned}
$$

Figure 4.2.13: Result 5

An extra step is shown in the figure above. Before becoming a $\frac{2^{2n}}{(n-1)!}\sum_{k=0}^{n}(x-a)^k \sum_{j=0}^{n-k} 2^{-j}(n+j-1)!\, C_k^{(j-\frac{1}{2},j-\frac{1}{2},n-j)} f^{(n-j)}(a) + \int_a^x \cos(n\, arccos(x-t))\, f^{(n+1)}(t)\, dt$, there is a step in between which is similar to the example 4 shown in the figure 4.2.10. The formula will be divided into four parts for easier explaining and better understanding. There are (I) $\frac{2^{2n}}{(n-1)!}$, (II) $\sum_{k=0}^{n}(x-a)^k$, (III) $\sum_{j=0}^{n-k} 2^{-j}(n+j-1)!\, C_k^{(j-\frac{1}{2},j-\frac{1}{2},n-j)} f^{(n-j)}(a)$, and (IV) $\int_a^x \cos(n\, arccos(x-t))\, f^{(n+1)}(t)\, dt$, where the (2) and the (4) remain the same. By gamma function, the (I) will be $2^{2n}\,(\frac{1}{\gamma(n)})$. The same apply to the $(n+j-1)!$ of the (III), where it will be $\gamma(n+j)$. As the paper lets the α and the β equal to $-\frac{1}{2}$, therefore, the $C_k^{(\alpha+j,\beta+j,n-j)}$ will change into $C_k^{(j-\frac{1}{2},j-\frac{1}{2},n-j)}$.

CHAPTER 5

CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

Throughout the development of this project, it definitely brings a better understanding of mathematical analysis in both theoretical and practical terms in the field of mathematics.

As the title for this project is "Recent Development and Analysis of Taylor series", a total of ten formal sources including research papers, electronics letters, journals and articles are used in this project as the research materials. The research materials are reviewed, and two of them are chosen to produce results shown in chapter 4 by completing and expanding the missing or reduced parts of Taylor series equations shown in those research materials.

A mathematical tool, Matlab is chosen for this project. It is used to generate examples related to Taylor series, ie., Taylor series expansion, plotting graphs for Taylor series expansion and so on. It best the online mathematics calculators such as Wolfram Alpha, Symbolab and so on when it comes to calculate, expand, and simplify Taylor series equations. It helps to save a lot of time when calculating the Taylor series equations. It also allows to produce clear and concise results, such as plotting graphs for Taylor series expansion.

Throughout the duration of completing this project, it brings a better conceptual understanding about the Taylor series as a strong foundation in the understanding of Taylor series is built. For example, the Taylor series allows one to easily calculate the complicated functions and even the most complex functions. Whenever a complicated function is approximated numerically, all it takes is to take the first few terms of the Taylor series expansion of the function to get a nice polynomial approximation.

As a total of ten research materials that are related to the Taylor series are reviewed throughout the development of this project, a clearer vision about the development of Taylor series in recent years is achieved. For example, there are researches such as a new type of Taylor series expansion, solving the Lane-Emden equation by using Taylor series, solving the nonlinear reaction-diffusion equation in the electroactive polymer film by using Taylor series method, and so on.

5.2 Future Work and Recommendation

Nothing in this world is perfect, as the perfection itself will also be clarified as imperfect. The same goes for this project. Although the duration for this project is quite long, starting from March until December, this project actually conducts concurrently with other subjects. Therefore, there are still improvement that can be made for this project.

As there are only two research materials out of ten chosen to produce the results, the reason is because the other research materials are too difficult to complete the missing steps of the Taylor series equations. Therefore, the recommendation of this project is adding more research materials into this project to complete the missing parts of them. By doing so, it can help to increase the depth of this project.

As a conclusion, this project hopes to become one of the references for those who want to learn Taylor series, and those who want to relearn back the Taylor series by looking at the “skippable”, “solving-in-mind” steps of Taylor series formula shown in the chapter 4, in the future to provide a better understanding towards the Taylor series.

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APPENDICES

APPENDIX A: Taylor Series Example by Using Matlab

```
%syms x
%f = 1/(5 + 4*cos(x));
%F = taylor(f, 'Order', 8)
%fprintf('==============================================')

h = exp(x*sin(x));
H = taylor(h, 'ExpansionPoint', 2, 'Order', 12);

size(char(H))
fprintf('==============================================')

H = simplify(H);
size(char(H))
fprintf('==============================================')

xd = 1:0.05:3;
yd = subs(h, x, xd);

fplot(H, [1, 3])

hold on
plot(xd, yd, 'r-.')

title('Taylor Approximation vs Actual Function')
legend('Taylor', 'Function')
```

Figure 6.1: Extra example

The first few commands generate the first 12 nonzero terms of the Taylor series for g at the expansion point of $x = 12$. size(char(H)) is entered to find that H has about 100,000 characters in its printed form as shown in the figure 6.2. In order to proceed with using H, H = simplify(H) is used to simplify its presentation.

Next, a plot of these function together shows how well the Taylor approximation compares to the actual function h, as shown in the diagram below:

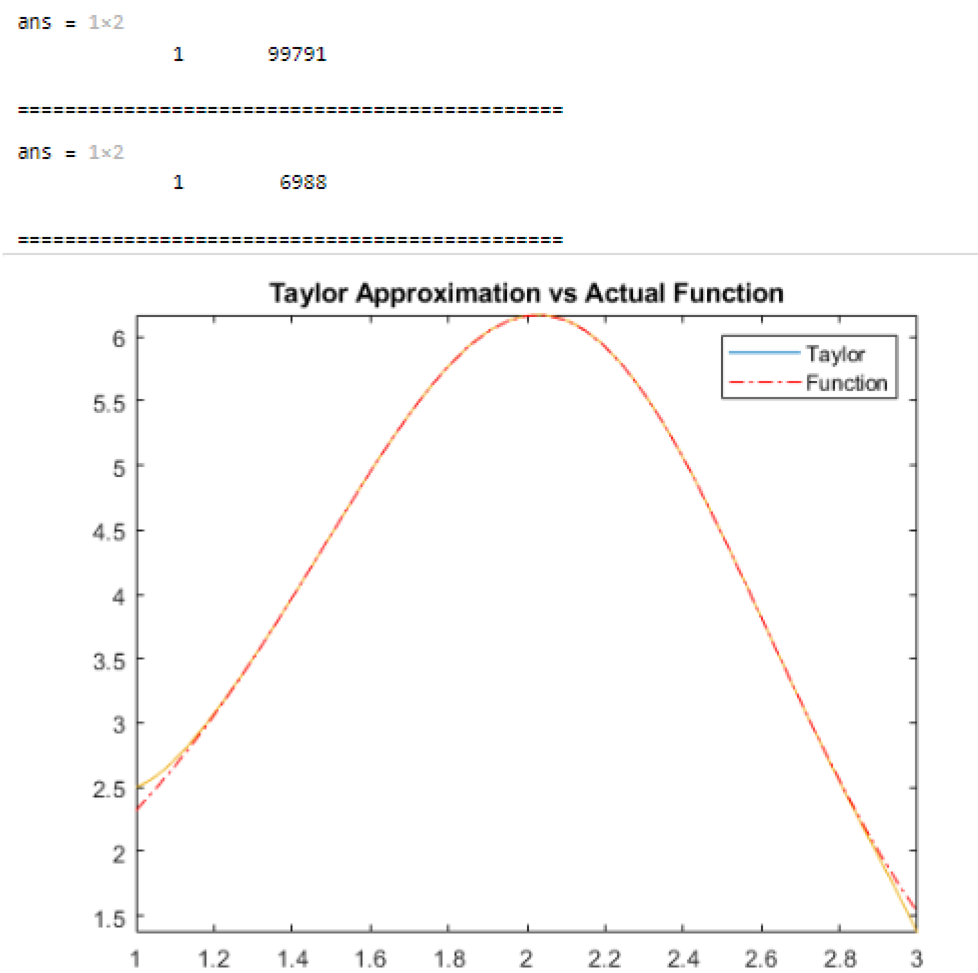

Figure 6.2: Result of the extra example