Groups and Linear Algebra-SC220 Project

Q.8 Landau's Function

In Tutorial 3, we had a question about the maximum order of a permutation in any S_n as n is varied. Seen as a function of n, this is called the Landau's function. Study the properties of this function, and compute it for a few values of n.

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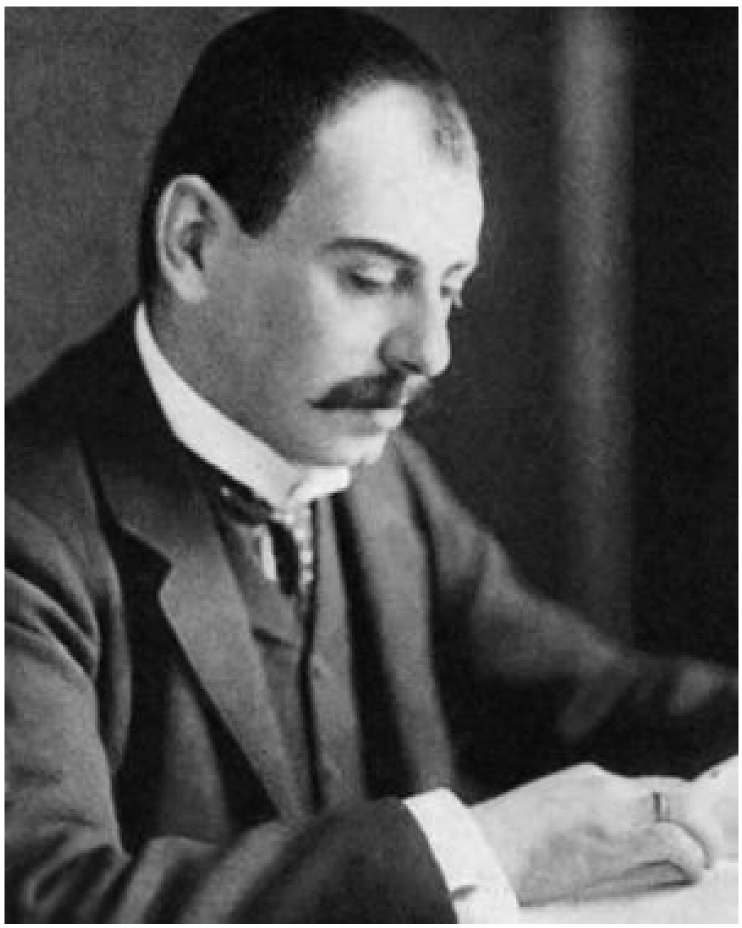
1 *Landau's Function*

1.1 Introduction,History, Definition & Notation:

$\Rightarrow$ Landau's function is named after Edmund Landau.And Landau's function is defined for every natural number n to be the largest order of an element of the symmetric group S_n. Landau's function is denoted as g(n). [3]



Edmund Landau

The landau's function is defined as the maximal order of an element of S_n.In number theory,Landau's function finds the largest LCM among all partitions of the given Number N.

Assume that
$S_n = m_1\ m_2\ m_3\ \ldots\ \ m_k$
$n = m_1 + m_2 + m_3 + \ldots\ + m_k$; Where m_i is the length of k^{th} cycle.

Then landau's function
$\mathbf{g(n) = max\ LCM(m_1, m_2, m_3, \ldots, m_k)}$

When n tends to ∞ (infinity)
$\log \mathbf{g(n)} \sim \sqrt{\mathbf{n} \log \mathbf{n}}$

The integer sequence
g(n=0) = 1, g(n=1) = 1, g(n=2) = 2, g(n=3) = 3, g(n=4) = 4, g(n=5) = 6, g(n=6) = 6, g(n=7) = 12,..., g(n=17) = 210, ...

$\lim_{\mathbf{n}\to\infty} \frac{\ln \mathbf{g(n)}}{\sqrt{\mathbf{n} \ln \mathbf{n}}} = \mathbf{1}$

For instance, 7 = 3 + 4 and LCM(4,3) = 12. No other partition of 7 yields a bigger LCM, so g(n=7) = 12.

An element of order 6 of group S_5 can be written in cycle notation as (1 2) (3 4 5). and same argument can be applied to the number 6,so g(6) = 6.

There are long sequences of consecutive numbers :
n, n + 1, . . . , n + m on which the function g is constant.

1.2 Properties :

1.2.1 Properties-1:

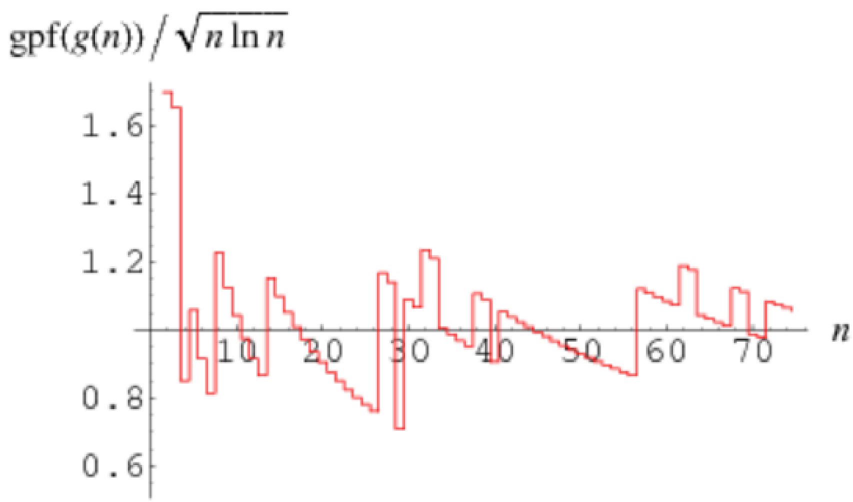
If $n_1 > n_2$ then $g(n_1) \geq g(n_2)$

Proof : If $n_1 > n_2$ then for all permutations of n_2 there were some images in n_1 for that n_2 numbers in n_1 are the same as n_2. Only $n_1 - n_2$ numbers are different. So that $g(n_1) \geq g(n_2)$.

Ex. assume that $n_1 = 12$ and $n_2 = 10$
Then $g(n_1) = g(12) = 60$
and $g(n_2) = g(10) = 30$
Here $12 > 10$ and $g(12) > g(10)$.

1.2.2 Properties-2:

$GPF(g(n)) \leq 2.86\sqrt{n \ln n}$; when $2 \leq n \leq 4$
$GPF(g(n)) \leq 1.328\sqrt{n \ln n}$; when $5 \leq n$
Where **GPF** is Greatest Prime Factor



Example:

i) For n=3
GPF(g(n))=3, g(n) =3
$\sqrt{n \ln n} = 1.82$
$GPF(g(n)) \leq 2.86\sqrt{n \ln n}$
$3 \leq 5.2025$

ii) For n=10
g(10)=30 ; from $LCM(5, 3, 2)$
GPF(30)=5
$\sqrt{n \ln n} = 4.8$
$GPF(g(n)) \leq 1.328\sqrt{n \ln n}$
$5 \leq 6.3744$

1.2.3 Properties-3:

For every prime p, there exists n such that the largest prime factor of g(n) is equal to p.

Proof:-
We have g(2) = 2, g(3) = 3.
If $p \geq 5$, let us choose $\rho = \frac{p}{\log p} > \frac{2}{\log 2}$.
if $\rho > \frac{2}{\log 2}$,
let us define $x > 4$ such that $\rho = \frac{x}{\log x}$ and

$$N_p = \prod_{p \leq x} p^{\alpha_p} = \prod_p p^{\alpha_p}$$

with

$$\alpha_p = \begin{cases} 0 \ \ if \ p > x \\ 1 \ \ if \ \frac{p}{\log p} \leq \rho < \frac{p^2 - p}{\log p} \\ k \geq 2 \ \ if \ \frac{p^k - p^{k-1}}{\log p} \leq \rho < \frac{p^{k-1} - p^k}{\log p} \end{cases}$$

Here, N_ρ belongs to $G \subset g(N)$, and its largest prime factor is p, which proves Proposition. [2]

1.3 Specific for finite permutation group :

Computing G(n) When n is Given.

For a small value of n, it is a routine exercise to calculate G(n). One merely recalls that every finite permutation can be decomposed (uniquely, up to the order in which the factors appear) as a product of disjoint cycles, and we will consider the order of a permutation as the least common multiple of the lengths of its disjoint cycles.

Then, using this, one enumerates the possible orders of a permutation on n elements. More explicitly, one considers all distinct representations of n as a sum of positive integers and, for each representation, computes the least common multiple of the integers in the representation.

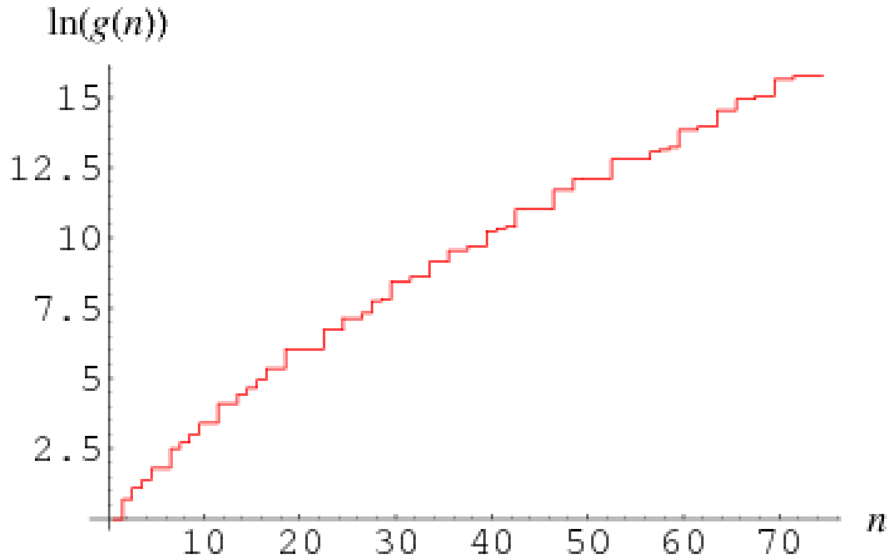
The largest number thus computed is G(n), and the integers in any representation corresponding to G(n) are the cycle lengths of a permutation on n elements having order G(n).

This tedious method can be streamlined; but calculating a particular value of G(n) involves substantial trial and error.

The table below displays the values of G(n) for $n < 20$ and gives the corresponding cycle structures of permutations (on n elements) with order G(n). [4]

Example:

n	$G(n)$	cycle lengths	n	$G(n)$	cycle lengths
2	2	2	11	30	1, 2, 3, 5 or 5, 6
3	3	3	12	60	3, 4, 5
4	4	4	13	60	1, 3, 4, 5
5	6	2, 3	14	84	3, 4, 7
6	6	1, 2, 3 or 6	15	105	3, 5, 7
7	12	3, 4	16	140	4, 5, 7
8	15	3, 5	17	210	2, 3, 5, 7
9	20	4, 5	18	210	1, 2, 3, 5, 7 or 5, 6, 7
10	30	2, 3, 5	19	420	3, 4, 5, 7


ln(g(n))
15
12.5
10
7.5
5
2.5
10
20
30
40
50
60
70
n

1.4 Algorithm to find Luanda's function

Here we generate all possible partitions for the given number N and we will find the max value of LCM from all the partitions:

1.get the integer N.

2. Declare Landau_function a global variable for storing Landau's function value

3. Recursive function with,
Base case: Find the LCM of the values in vector if we got 0 in recursive call and update Landau_function if LCM is greater than Landau_function
Recursive Call: Recursively Iterate i from value 1 to N, then Recursively iterate from [k, N – k]. Push the current element as k into the vector and recursively iterate for the next index and after this recursion ends then pop the element j inserted previously

4. At the end call print the Landau_function value. [1]

Link of CPP Code : https://drive.google.com/drive/folders/1tVXtIpxPeWfZuqVBLn-o9nfgesR4GqlU?usp=sharing

Link of Latex Code(Project) : https://www.overleaf.com/read/ztqnwrgmtbvg

2 *Bibliography :*

A very helpful references for the project are [3, 4, 1, 2].

References

[1] geeksforgeeks. Find Landau's function for a given number N. https://www.geeksforgeeks.org/find-landaus-function-for-a-given-number-n/.

[2] Jean-Louis Nicolas. On Landau's Function g(n). https://arxiv.org/pdf/1312.2569.pdf.

[3] Wikipedia. Landau's function From Wikipedia. https://en.wikipedia.org/wiki/Landau%27s_function.

[4] William Miller. The Maximum Order of an Element of a Finite Symmetric Group. http://www.ms.uky.edu/~sohum/ma561/notes/symorder.pdf.

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