Math 3770H: Assignment II

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Problem 1. Write the function

$$f(z) = z + \frac{1}{z} \qquad (z \neq 0)$$

in the form $f(z) = u(r, \theta) + iv(r, \theta)$

Solution 1. Writing f first in polar form to switch to a dependence on $r = |z|, \theta = \arctan\left(\operatorname{Im} z / \operatorname{Re} z\right)$,

$$z = re^{i\theta} = r\cos\theta + ir\sin\theta.$$

This is actually really nice here as it means we can easily write the inverse without using a fraction,

$$z^{-1} = r^{-1}e^{-i\theta} = r^{-1}\cos\left(-\theta\right) + ir^{-1}\sin\left(-\theta\right).$$

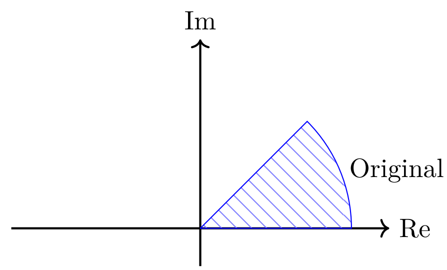
Which can be reworked further using the even/odd nature of cos and sin respectively giving:

$$\begin{aligned}
f(z) &= r\cos\theta + ir\sin\theta + r^{-1}\cos\theta - ir^{-1}\sin\theta \\
&= r\cos\theta + r^{-1}\cos\theta + ir\sin\theta - ir^{-1}\sin\theta \\
&= \left(r + r^{-1}\right)\cos\theta + \left(r + r^{-1}\right)\sin\theta
\end{aligned}$$

Problem 2. Sketch the region onto which the sector $r \leq 1$, $0 \leq \theta \leq \pi/4$ is mapped by the transformation

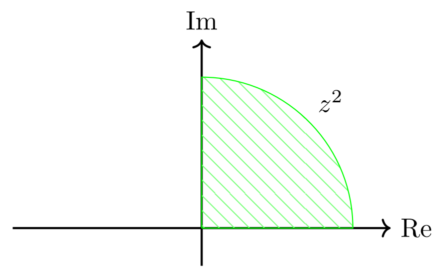
$$\text{(a) } w = z^2; \qquad \text{(b) } w = z^3; \qquad \text{(c) } w = z^4.$$

Solution 2. Generally the transform $w = z^n$ will give $w = r^n e^{in\theta}$, just by exponentiation laws. Our original region looks like:

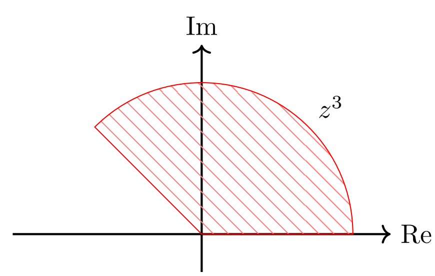


Because our region only extends out to $r = 1$ the radius won't change.

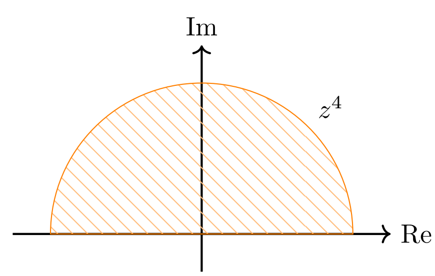
(a)



(b)



(c)



Problem 3. Use definition (2), Sec.15, of the limit to prove that

$$\text{(a) } \lim_{z\to z_0} \operatorname{Re} z = \operatorname{Re} z_0; \qquad \text{(b) } \lim_{z\to z_0} \overline{z} = \overline{z_0}; \qquad \text{(c) } \lim_{z\to 0} \frac{\overline{z}^2}{z} = 0.$$

Solution 3.

(a) Let $\epsilon > 0$. Suppose $|z - z_0| < \delta$. If we take $z = x + iy$ and $z_0 = x_0 + iy_0$ then we have

$$|\operatorname{Re} z - \operatorname{Re} z_0| = |x - x_0| \leq |z - z_0|.$$

We can make that last statement because

$$|x - x_0| \leq |x - x_0 + iy - iy_0| = |z - z_0|.$$

This means that

$$|\operatorname{Re} z - \operatorname{Re} z_0| = |z - z_0| \leq \delta$$

by the $\epsilon - \delta$ definition we are using. Having this means that provided $|z - z_0| \leq \delta = \epsilon$, then $|\operatorname{Re} z - \operatorname{Re} z_0| < \epsilon$ by transitivity.

(b) Let $\epsilon > 0$. Suppose $|z - z_0| < \delta$. We have

$$|\overline{z} - \overline{z_0}| = |\overline{z - z_0}| = |z - z_0| < \delta$$

all by properties of the complex conjugate. So provided $|z - z_0| < \delta = \epsilon$, then $|\overline{z} - \overline{z_0}| < \epsilon$.

(c) Let $\epsilon > 0$. Suppose $|z - z_0| < \delta$. We have

$$\left| \frac{\overline{z}^2}{z} - \frac{\overline{z_0}^2}{z_0} \right| = \left| \frac{\overline{z}\overline{z}}{z} - \frac{\overline{z_0}\overline{z_0}}{z_0} \right|$$

$\lim_{z\to 0} = 0$.

Problem 4. With the aid of the theorem in Sec. 17, show that when

$$T(z) = \frac{az + b}{cz + d} \qquad (ad - bc) \neq 0$$

(a) $\lim_{z\to\infty} T(z) = \infty \quad \text{if } c = 0;$

(b) $\lim_{z\to\infty} T(z) = \frac{a}{c} \quad \text{and} \quad \lim_{z\to -d/c} T(z) = \infty \quad \text{if } c \neq 0$

Solution 4.

Problem 5. Use the method in Example 2, Sec. 19, to show that $f(z)$ does not exist at any point z when (a) $f(z) = \operatorname{Re} z$; (b) $f(z) = \operatorname{Im} z$.

Solution 5.

Problem 6. Use the theorem in Sec. 24 to show that each of these functions is differentiable in the indicated domain of definition, and also to find $f'(z)$:

(a) $f(z) = 1/z^4$ $(z \neq 0)$;

(b) $f(z) = e^{-\theta} \cos(\ln r) + ie^{-\theta} \sin(\ln r)$ $(r > 0.0 < \theta < 2\pi)$.

Solution 6.

Problem 7. With the aid of the theorem in Sec. 21, show that each of these functions is nowhere analytic: (a) $f(z) = xy + iy$; (b) $f(z) = 2xy + i(x^2 - y^2)$; (c) $f(z) = e^y e^{ix}$.

Solution 7.

Problem 8.A. Let the function $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D, and consider the families of level curves $u(x, y) = c_1$ and $v(x, y) = c_2$, where c_1 and c_2 are arbitrary real constants. Prove that these families are orthogonal. More precisely, show that if $z_0 = (x_0, y_0)$ is a point in D which is common to two particular curves $u(x, y) = c_1$ and $v(x, y) = c_2$ and if $f'(z_0) \neq 0$, then the lines tangent to those curves at (x_0, y_0) are perpendicular.

Problem 8.B. Sketch the families of level curves of the component functions u and v when $f(z) = 1/z$, and note the orthogonality described in Exercise 2.

Solution 8.A.

Solution 8.B.