Calculus Cheat Sheet

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Revision 3

Limits

Existence

$\lim\limits_{x\to a} f(x) = L$ exists if $\forall \epsilon > 0 \exists \delta > 0$ s.t. $|x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$.

Properties

$\lim\limits_{x\to a}[cf(x)] = c\lim\limits_{x\to a}[f(x)]$
$\lim\limits_{x\to a}[f(x) \pm g(x)] = \lim\limits_{x\to a}[f(x)] \pm \lim\limits_{x\to a}[g(x)]$
$\lim\limits_{x\to a}[f(x)g(x)] = \lim\limits_{x\to a}[f(x)]\lim\limits_{x\to a}[g(x)]$
$\lim\limits_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a}[f(x)]}{\lim_{x\to a}[g(x)]}, \lim\limits_{x\to a}[g(x)] \neq 0$

Evaluation Techniques

Basics

$\lim\limits_{x\to a}[f(x)] = f(a)$ if f exists at a

L'Hôpital's Rule

If $\lim\limits_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{0}{0}$ **or** $\frac{\pm\infty}{\pm\infty}$ then $\lim\limits_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \lim\limits_{x\to a}\left[\frac{f'(x)}{g'(x)}\right]$

Factoring at Infinity

If $p(x)$ and $q(x)$ are polynomials, to evaluate $\lim\limits_{x\to\pm\infty}\left[\frac{p(x)}{q(x)}\right]$ factor the greatest power of x in $q(x)$ (the denominator) out of poth $p(x)$ and $q(x)$ then compute the limit, e.g. $\lim\limits_{x\to-\infty}\left[\frac{3x^2-4}{5x-2x^2}\right] = \lim\limits_{x\to-\infty}\left[\frac{\cancel{x^2}(3-\frac{4}{x^2})}{\cancel{x^2}(\frac{5}{x}-2)}\right]$ $\frac{3-0}{0-2} = -\frac{3}{2}$

Derivatives

Definition

$\frac{d}{dx}f(x) = f'(x) = \lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

Techniques

Sum Rule

$\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$

Product Rule

$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

Quotient Rule

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$

Power Rule

$\frac{d}{dx}(x^n) = nx^{n-1}$

Chain Rule

$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

Parametric

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ recurse y for nth derivative
$(x,y) + (x',y')t$ for parametric tangent

Common Derivatives

$\frac{d}{dx}(x) = 1$ $\frac{d}{dx}(\sin x) = \cos x$
$\frac{d}{dx}(\cos x) = -\sin x$ $\frac{d}{dx}(\tan x) = \sec^2 x$
$\frac{d}{dx}(\sec x) = \sec x \tan x$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$
$\frac{d}{dx}(\cot x) = -\csc^2 x$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\sin^{-1} x \neq \frac{1}{\sin x}$
$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\frac{d}{dx}(a^x) = a^x \ln a$
$\frac{d}{dx}(e^x) = e^x$ $\frac{d}{dx}(\ln x) = \frac{1}{x}, x > 0$
$\frac{d}{dx}(\ln|x|) = \frac{1}{x}, x \neq 0$ $\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$

Common Chain Rule Derivatives

$\frac{d}{dx}([f(x)]^n) = n[f(x)]^{n-1}f'(x)$
$\frac{d}{dx}(e^{f(x)}) = e^{f(x)}f'(x)$
$\frac{d}{dx}(\ln[f(x)]) = \frac{f'(x)}{f(x)}$
$\frac{d}{dx}(\sin[f(x)]) = f'(x)\cos[f(x)]$
$\frac{d}{dx}(\cos[f(x)]) = -f'(x)\sin[f(x)]$
$\frac{d}{dx}(\tan[f(x)]) = f'(x)\sec^2[f(x)]$
Trig derivatives, same old same old
$\frac{d}{dx}(f(x)^{g(x)}) = \frac{g(x)f'(x)}{f(x)} + \ln[f(x)]g'(x)$

Integrals

Definition

The integral of some function $f(x)$ is a function $f^\star(x)$ s.t. $f^{\star\prime}(x) = f(x)$. **Don't forget your constant!** The integral can be done using the Right Riemann Sum:

$\int_a^b f(x)\,dx = \lim\limits_{n\to\infty}\sum_{i=1}^n\left[\frac{b-a}{n} \cdot f(a + i \cdot \frac{b-a}{n})\right]$

U-Substitution

$\int_a^b f(g(x)) \cdot g'\,dx = \int_{g(a)}^{g(b)} f(u)\,du$. Using U-Substitution, $u = g(x)$ and $du = g'(x)dx$ ($dx = \frac{du}{g'}$).

Integration By Parts

$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$ and $\int_a^b u(x)v'(x)\,dx = uv|_a^b - \int_a^b u'(x)v(x)\,dx$.

Common Integrals

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$
$\int \frac{1}{x}\,dx = \ln|x| + C$
$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$
$\int \ln x\,dx = x\ln x - x + C$
$\int e^x\,dx = e^x + C$
$\int a^x\,dx = \frac{a^x}{\ln a} + C$
$\int \sin x\,dx = -\cos x + C$
$\int \cos x\,dx = \sin x + C$
$\int \tan x\,dx = \ln|\sec x| + C$
$\int \sec^2 x\,dx = \tan x + C$
$\int \sec x \tan x\,dx = \sec x + C$
$\int \csc x \cot x\,dx = -\csc x + C$
$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x\,dx = \ln|\sec x + \tan x| + C$
$\int \frac{1}{a^2+u^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{1}{\sqrt{a^2+u^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

Trig Reduction Formulae

(For $n \geq 2$)
$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$
$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$
$\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) + \frac{n-1}{n}\int \tan^{n-2}(x)\,dx$
$\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$

Trig Identities

$\sin^2(x) + \cos^2(x) = 1$
$1 + \tan^2(x) = \sec^2(x)$
$\sin(2x) = 2\sin(x)\cos(x)$
$\cos(2x) = \cos^2(x) - \sin^2(x)$
$= 2\cos^2(x) - 1$
$= 1 - 2\sin^2(x)$

Trig Substitutions

$\sqrt{a^2 - x^2} \rightsquigarrow x = a\sin(\theta)$
$\sqrt{a^2 + x^2} \rightsquigarrow x = a\tan(\theta)$
$\sqrt{x^2 - a^2} \rightsquigarrow x = a\sec(\theta)$

Partial Fractions & Polynomial Division

If the degree of the numerator is greater than that of a the denominator, use polynomial division to divide the denominator into the numerator ($denom\,\overline{)numer}$), then integrate the result. If the degree of the denominator is greater than that of the numerator, use partial fractions to decompose the integral as follows:
If the denominator contains different linear terms, break it down to $\frac{A}{ax+b}$
If it contains a repeated linear term ($(ax+b)^2$), break it down to $\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
If it contains an irreducible quadratic (x^2+bx+c), break it down to $\frac{A}{dx+o} + \frac{Bx+C}{x^2+bx+c}$
Then, set up equations for the coefficients of the powers of x in the numerator and solve them to determine A, B, C and so on.

Applications

Centroids

$C = (\hat{x}, \hat{y})$, $\hat{x} = \frac{\int xf(x)\,dx}{\int f(x)\,dx}$ & $\hat{y} = \frac{\int yf(y)\,dy}{\int f(y)\,dy}$

Arc Length

$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$
$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$
for parametric. Area is just this times $2\pi x$ or y for opposite of axis.

Volume

Use the variable parallel to the axis of revolution. $V = \int_a^b A(x)\,dx$ where $A(x)$ is a function which gives the area of one "slice" of the solid. Slices are generally circular and may have holes in them, area of a circle is πr^2, and the area of a washer is $\pi(r_{outer}^2 - r_{inner}^2)$

Tidbits

$\vec{v}$ here is the velocity vector (duh)
Unit tangent vector $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$
Curvature $\kappa = \left|\frac{d\vec{T}}{d\vec{s}}\right| = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$
Normal vector $\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$
For conservative fields $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}, \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}$.

Line Integrals and Work

For variable force $f(x, y, z)$ along curve $r(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$ work is $\int_C \vec{F} \cdot dr = \int_{t_i}^{t_f} f(g, h, k)\, |\vec{v}(t)|\ dt$.
For vector valued functions it simplifies to $\int_C \vec{F} \cdot dr \int_{t_i}^{t_f} \vec{F}(\vec{r}(t)) \cdot \vec{v}(t)\, dt$.
Alternatively for $\vec{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ we have $\int_C M dx + N dy + P dz$ where we change dx, y, z as we do in integration by Substitution.

Flux

Flux of field $\vec{F} = M\mathbf{i} + N\mathbf{j}$ through curve $C = g(t)\mathbf{i} + h(t)\mathbf{j}$ is given by $\int_C M dy - N dx$ which can be evaluated as seen above.

Green's Theorem

For $\vec{F} = M\mathbf{i} + N\mathbf{j}$:
Divergence is $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$.
$\oint_C \vec{F} \cdot d\vec{r} = \oint_C M\, dx + N\, dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dx\, dy$
and $\oint_C \vec{F} \cdot \vec{n} dS = \oint_C M\, dy - N\, dx = \iint_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\, dx\, d$
Integral of divergence over a region is the outward flux of the associated field over that region.

Surface Integrals

Area of $r(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$ is $\iint_R |r_u \times r_v|\ dA$ remember that cross product is det of matrix with first row of unit vectors, second of first vector and third of second vector. For a surface given as $z = f(x, y)$ we get $A = \iint_R \sqrt{1 + f_x^2 + f_y^2} dA$. For integrating a scalar function G over a surface if we have parametrized the surface we do $\iint G(f(u, v), g, h)\, |r_u \times r_v|\, dudv$, if the surface is given as $z = f(x, y)$ we do $\iint G(x, y, f(x, y)) \sqrt{f_x^2 + f_y^2 + 1} dxdy$.
For field $\vec{F}$ across surface $\vec{r}$ we do $\iint_R \vec{F} \cdot (\vec{r}_x \times \vec{r}_z)\ dxdz$