Math 2350H: Assignment III

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Problem 1. Let $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{M}_{2\times 2}(\mathbb{R})$ be the linear transformation given by

$$T(a + bx + cx^2 + dx^3) = \begin{pmatrix} 3a + 7b - 2c - 5d & 8a + 14b - 2c - 11d \\ -4a - 8b + 2c + 6d & 12a + 22b - 4c - 17d \end{pmatrix}$$

(a) Find the matrix representation $[T]_\beta^\gamma$ for bases

$$\beta = \left\{1, x, x^2, x^3\right\}, \qquad \gamma = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}.$$

(b) Find the matrix representation $[T]_\mathcal{B}^\mathcal{C}$ for bases

$$\mathcal{B} = \left\{1 + x - x^2 + 2x^3, -1 + 2x + 2x^3, 2 + x - 2x^2 + 3x^3, 1 + x + 2x^3\right\},$$

$$\mathcal{C} = \left\{\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\right\}.$$

(c) Let $p(x) = 3 - x + 2x^2 - 5x^3$. Find $[p(x)]_\beta$ and $[p(x)]_\mathcal{B}$.

(d) Find the image of $p(x)$ under T in the following three ways:

(i) By computing $T(p(x))$ directly

(ii) By computing $[T]_\beta^\gamma [p(x)]_\beta = [T(p(x))]_\gamma$

(iii) By computing $[T]_\mathcal{B}^\mathcal{C} [p(x)]_\mathcal{B} = [T(p(x))]_\mathcal{C}$

(e) Compute the matrix representations $[I]_\beta^\mathcal{B}$ and $[I]_\mathcal{B}^\beta$, where I is the identity map on $\mathcal{P}_3(\mathbb{R})$. Show that $\left([I]_\beta^\mathcal{B}\right)^{-1} = [I]_\mathcal{B}^\beta$

(f) Compute the matrix product $[I]_\beta^\mathcal{B} [p(x)]_\beta$. What do we notice about the result?

Solution 1.

(a) Here we start by applying T to each element of the input basis, β, and expressing the result as a linear combination of vectors in the output basis, γ,

$$T(1) = \begin{pmatrix} 3 & 8 \\ -4 & 12 \end{pmatrix} = 3\gamma_1 + 8\gamma_2 - 4\gamma_3 + 12\gamma_4 \rightsquigarrow \text{(treating } \gamma \text{ as ordered)}$$
$$T(x) = \begin{pmatrix} 7 & 14 \\ -8 & 22 \end{pmatrix} = 7\gamma_1 + 14\gamma_2 - 8\gamma_3 + 22\gamma_4$$
$$T(x^2) = \begin{pmatrix} -2 & -2 \\ 2 & -4 \end{pmatrix} = -2\gamma_1 - 2\gamma_2 + 2\gamma_3 - 4\gamma_4$$
$$T(x^3) = \begin{pmatrix} -5 & -11 \\ 6 & -17 \end{pmatrix} = -5\gamma_1 - 11\gamma_2 + 6\gamma_3 - 17\gamma_4$$

These linear combinations give us the columns of

$$[T]_\beta^\gamma = \begin{pmatrix} 3 & 7 & -2 & -5 \\ 8 & 14 & -2 & -11 \\ -4 & -8 & 2 & 6 \\ 12 & 22 & -4 & -17 \end{pmatrix}$$

(b) Applying the same process as in part (a) (again treating $\mathcal{C}$ as ordered),

$$T(1 + x - x^2 + 2x^3) = \begin{pmatrix} 2 & 2 \\ -2 & 4 \end{pmatrix} = 2\mathcal{C}_1 + 0\mathcal{C}_2 + 0\mathcal{C}_3 + 0\mathcal{C}_4$$
$$T(-1 + 2x + 2x^3) = \begin{pmatrix} 1 & -2 \\ 0 & -2 \end{pmatrix} = 0\mathcal{C}_1 - 1\mathcal{C}_2 + 0\mathcal{C}_3 + 0\mathcal{C}_4$$
$$T(2 + x - 2x^2 + 2x^3) = \begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix} = 0\mathcal{C}_1 + 0\mathcal{C}_2 + 1\mathcal{C}_3 + 0\mathcal{C}_4$$
$$T(1 + x + 2x^3) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0\mathcal{C}_1 + 0\mathcal{C}_2 + 0\mathcal{C}_3 + 0\mathcal{C}_4$$

$$\implies [T]_\mathcal{B}^\mathcal{C} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(c) Again we represent $p(x)$ as a linear combination where the coefficients give us the columns of the matrix representation. For $[p(x)]_\beta$ we get $p(x) = 3 \cdot 1 - 1 \cdot x + 2 \cdot x^2 - 5 \cdot x^3 \implies [p(x)]_\beta = \begin{pmatrix} 3 \\ -1 \\ 2 \\ -5 \end{pmatrix}$. Then for $[p(x)]_\mathcal{B}$ we have a less obvious solution (obtained by row-reduction of the associated matrix) of $p(x) = 32\mathcal{B}_1 - 7\mathcal{B}_2 - 17\mathcal{B}_3 - 2\mathcal{B}_4 \implies [p(x)]_\mathcal{B} = \begin{pmatrix} 32 \\ -7 \\ -17 \\ -2 \end{pmatrix}$

(d) Find the image of $p(x)$ under T in the following three ways:

(i) $T(p(x)) = T(3 - x + 2x^2 - 5x^3) = \begin{pmatrix} 23 & 61 \\ -30 & 91 \end{pmatrix}$

(ii)

$$
\begin{aligned}
[T]_\beta^\gamma \, [p(x)]_\beta &= \begin{pmatrix} 3 & 7 & -2 & -5 \\ 8 & 14 & -2 & -11 \\ -4 & -8 & 2 & 6 \\ 12 & 22 & -4 & -17 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 2 \\ -5 \end{pmatrix} \\
&= \begin{pmatrix} 3 \cdot 3 + 7 \cdot (-1) + (-2) \cdot 2 + (-5)^2 \\ 8 \cdot 3 + 14 \cdot (-1) + (-2) \cdot 2 + (-11) \cdot (-5) \\ -4 \cdot 3 + (-8) \cdot (-1) + 2 \cdot 2 + 6 \cdot (-5) \\ 12 \cdot 3 + 22 \cdot (-1) + (-4) \cdot 2 + (-17) \cdot (-5) \end{pmatrix} \\
&= \begin{pmatrix} 23 \\ -49 \\ -30 \\ 91 \end{pmatrix}
\end{aligned}
$$

(iii)

$$
\begin{aligned}
[T]_\mathcal{B}^\mathcal{C} \, [p(x)]_\mathcal{B} &= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 32 \\ -7 \\ -17 \\ -2 \end{pmatrix} \\
&= \begin{pmatrix} 64 \\ 7 \\ -17 \\ 0 \end{pmatrix}
\end{aligned}
$$

(e) For $[I]_\beta^\mathcal{B}$,

$$
\begin{aligned}
I(1) &= 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\
I(x) &= 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\
I(x^2) &= 0 \cdot 1 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 \\
I(x^3) &= 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 1 \cdot x^3
\end{aligned}
$$

solving the associated matrices for coefficients of the elements of $\mathcal{B}$ yields

$$
[I]_\beta^\mathcal{B} = \begin{pmatrix} 4 & 8 & -1 & -6 \\ 1 & 1 & 0 & -1 \\ -2 & -4 & 0 & 3 \\ 0 & -1 & 1 & 1 \end{pmatrix}.
$$

Then for $[I]_\mathcal{B}^\beta$,

$$
\begin{aligned}
I(1 + x - x^2 + 2x^3) &= 1 \cdot 1 + 1 \cdot x - 1 \cdot x^2 + 2 \cdot x^3 \\
I(-1 + 2x + 2x^3) &= -1 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 2 \cdot x^3 \\
I(2 + x - 2x^2 + 3x^3) &= 2 \cdot 1 + 1 \cdot x - 2 \cdot x^2 + 3 \cdot x^3 \\
I(1 + x + 2x^3) &= 1 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + 2 \cdot x^3
\end{aligned}
$$

which yields

$$
[I]_\mathcal{B}^\beta = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & -2 & 0 \\ 2 & 2 & 3 & 2 \end{pmatrix}.
$$

To find $\left([I]_\beta^\mathcal{B}\right)^{-1}$ we work the matrix $\left[[I]_\beta^\mathcal{B} \mid I\right]$ to $\left[I \mid \left([I]_\beta^\mathcal{B}\right)^{-1}\right]$ which **TYPESET THIS :(**

(f) $[I]_\beta^{\mathcal{B}}\,[p(x)]_\beta = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & -2 & 0 \\ 2 & 2 & 3 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 2 \\ -5 \end{pmatrix} = \begin{pmatrix} 32 \\ -7 \\ -17 \\ -2 \end{pmatrix} = [p(x)]_{\mathcal{B}}$

Problem 2. If β and γ are two bases for a finite dimensional vector space V, and I is the identity map on V, the matrix $[I]_\gamma^\beta$ is *called a change of basis matrix* (*or change of coordinates matrix*). It is always invertible because I is invertible.

1. Let $S \in \mathcal{L}(V)$. Show that
$$[S]_\gamma = \left([I]_\gamma^\beta\right)^{-1}[S]_\beta\,[I]_\gamma^\beta$$
by using the properties (given in class) which relate matrix multiplication and composition of linear maps
2. Let $T \in \mathcal{L}(\mathcal{M}_{2\times 2}(\mathbb{R}))$ be the map given by
$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -17a+11b+8c-11d & -57a+35b+24c-33d \\ -14a+10b+6c-10d & -41a+25b+16c-23d \end{pmatrix}.$$
Find the matrix representations $[T]_\beta$ and $[T]_\gamma$ for bases
$$\beta = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\},$$
$$\gamma = \left\{\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix}\right\}.$$
3. Find the change of basis matrix $[I]_\gamma^\beta$ and use it to verify what was shown in part (a) using the matrices computed in part (b).

Solution 2.

1. $$\left([I]_\gamma^\beta\right)^{-1}[S]_\beta\,[I]_\gamma^\beta = [I]_\beta^\gamma\,[S]_\beta^\beta\,[I]_\gamma^\beta$$
Because each of these are matrix representations of linear maps and composition of linear maps in matrix form is matrix multiplication the above expression can be re-written as
$$I_{\beta\gamma}(S(I_{\gamma\beta}))$$
which maps per $\gamma \to \beta \to \beta \to \gamma$, however the $\gamma \to \beta$ and $\beta \to \gamma$ steps are done using identity maps which do not change the mapped vector and can be removed from the expression yielding
$$[I]_\beta^\gamma\,[S]_\beta^\beta\,[I]_\gamma^\beta \implies I_{\beta\gamma}(S(I_{\gamma\beta})) = S \implies [S]_\gamma.$$
2. For $[T]_\beta$,
$$\begin{aligned} T(\beta_1) &= \begin{pmatrix} -17 & -57 \\ -14 & -41 \end{pmatrix} = -17\beta_1 - 57\beta_2 - 14\beta_3 - 41\beta_4 \\ T(\beta_2) &= \begin{pmatrix} 11 & 35 \\ 10 & 25 \end{pmatrix} = 11\beta_1 + 35\beta_2 + 10\beta_3 + 25\beta_4 \\ T(\beta_3) &= \begin{pmatrix} 8 & 24 \\ 6 & 16 \end{pmatrix} = 8\beta_1 + 24\beta_2 + 6\beta_3 + 16\beta_4 \\ T(\beta_4) &= \begin{pmatrix} -11 & -33 \\ -10 & -23 \end{pmatrix} = -11\beta_1 - 33\beta_2 - 10\beta_3 - 23\beta_4 \end{aligned}$$
so,
$$[T]_\beta = \begin{pmatrix} -17 & 11 & 8 & -11 \\ -57 & 35 & 24 & -33 \\ -14 & 10 & 6 & -10 \\ -41 & 25 & 16 & -23 \end{pmatrix}.$$

Then for $[T]_\gamma$ we get

$$\begin{aligned}
T(\gamma_1) &= \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = 2\gamma_1 + 0\gamma_2 + 0\gamma_3 + 0\gamma_4 \\
T(\gamma_2) &= \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} = 0\gamma_1 + 2\gamma_2 + 0\gamma_3 + 0\gamma_4 \\
T(\gamma_3) &= \begin{pmatrix} -1 & -3 \\ -2 & -3 \end{pmatrix} = 0\gamma_1 + 0\gamma_2 - 1\gamma_3 + 0\gamma_4 \\
T(\gamma_4) &= \begin{pmatrix} -4 & -12 \\ -2 & -8 \end{pmatrix} = 0\gamma_1 + 0\gamma_2 + 0\gamma_3 - 2\gamma_4
\end{aligned}$$

so,

$$[T]_\gamma = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

3. Applying I to the input basis,

$$\begin{aligned}
I(\gamma_1) = \gamma_1 &= 0\beta_1 + 1\beta_2 + 0\beta_3 + 1\beta_4 \\
I(\gamma_2) = \gamma_1 &= 1\beta_1 + 1\beta_2 + 1\beta_3 + 0\beta_4 \\
I(\gamma_3) = \gamma_1 &= 1\beta_1 + 3\beta_2 + 2\beta_3 + 3\beta_4 \\
I(\gamma_4) = \gamma_1 &= 2\beta_1 + 6\beta_2 + 1\beta_3 + 4\beta_4
\end{aligned}$$

so $[I]_\gamma^\beta = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 6 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 4 \end{pmatrix}$. Now,

$$\begin{aligned}
&= \left([I]_\gamma^\beta\right)^{-1} [T]_\beta \, [I]_\gamma^\beta \\
&= \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 6 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 4 \end{pmatrix}^{-1} \begin{pmatrix} -17 & 11 & 8 & -11 \\ -57 & 35 & 24 & -33 \\ -14 & 10 & 6 & -10 \\ -41 & 25 & 16 & -23 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 6 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 4 \end{pmatrix} \\
&= \begin{pmatrix} -11 & 7 & 4 & -6 \\ -4 & 3 & 2 & -3 \\ 1 & -1 & 0 & 1 \\ 2 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -17 & 11 & 8 & -11 \\ -57 & 35 & 24 & -33 \\ -14 & 10 & 6 & -10 \\ -41 & 25 & 16 & -23 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 6 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 4 \end{pmatrix} \\
&= \begin{pmatrix} -22 & 14 & 8 & -12 \\ -8 & 6 & 4 & -6 \\ -1 & 1 & 0 & -1 \\ -4 & 2 & 2 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 6 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = [T]_\gamma
\end{aligned}$$

Problem 3. Let $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{M}_{2\times 2}(\mathbb{R})$ be the linear transformation given by

$$T(a + bx + cx^2 + dx^3) = \begin{pmatrix} a+b & a-2c \\ d & b-d \end{pmatrix}$$

(a) Find the null space of T.

(b) Show that T is invertible without giving an explicit inverse.

(c) Find the matrix representation $[T]_\beta^\gamma$ for bases

$$\beta = \{1, x, x^2, x^3\}, \qquad \gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

(d) Compute $\left([T]_\beta^\gamma\right)^{-1}$ and use this to find an expression for T^{-1}; i.e. find $T^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(e) Consider the 2×2 identity matrix

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Find its pre-image in $\mathcal{P}_3(\mathbb{R})$ under T in two ways:

(i) By computing $T^{-1}(I_2)$ is the result from (d)

(ii) By first finding $[I_2]\,\gamma$, then computing $\left([T]_\beta^\gamma\right)^{-1}[I_2]_\gamma$.

Solution 3.

(a) $\operatorname{null} T$ is all $p(x) = a + bx + cx^2 + dx^3 \in \mathcal{P}_3(\mathbb{R})$ such that $a + b = a - 2c = d = b - d = 0$ which has only the trivial solution, $a = b = c = d = 0$ so $\operatorname{null} T = \{0\}$.

(b) Because $\operatorname{range} T = \mathcal{M}_{2\times 2}(\mathbb{R})$ and $\operatorname{null} T = \{0\}$ we are guaranteed that T is both invertible and bijective.

(c) Applying T to the elements of β,

$$\begin{aligned}
T(1) &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \gamma_1 + \gamma_2 + 0\gamma_3 + 0\gamma_4 \\
T(x) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1\gamma_1 + 0\gamma_2 + 0\gamma_3 + 1\gamma_4 \\
T(x^2) &= \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} = 0\gamma_1 - 2\gamma_2 + 0\gamma_3 + 0\gamma_4 \\
T(x^3) &= \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} = 0\gamma_1 + 0\gamma_2 + 1\gamma_3 - 1\gamma_4
\end{aligned}$$

so,

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

(d) Compute $\left([T]_\beta^\gamma\right)^{-1}$ and use this to find an expression for T^{-1}; i.e. find $T^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(e) Consider the 2×2 identity matrix

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Find its pre-image in $\mathcal{P}_3(\mathbb{R})$ under T in two ways:

(i) By computing $T^{-1}(I_2)$ is the result from (d)

(ii) By first finding $[I_2]\,\gamma$, then computing $\left([T]_\beta^\gamma\right)^{-1}[I_2]_\gamma$.