Math 3310H: Assignment I

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Problem 1. Define a relation $\mathbb{R} \times \mathbb{R}$ by $(a, b) \sim (c, d)$ if $2(a - c) - 3(b - d) = 0$

(a) Show that $\sim$ is an equivalence relation on $\mathbb{R}$.

(b) Give an example of two pairs $(a, b), (c, d) \in \mathbb{R} \times \mathbb{R}$, which lie in the same equivalence class, and two pairs that don't.

(c) This equivalence relation partitions the 2D plane $\mathbb{R} \times \mathbb{R}$ into subregions. What does the equivalence class (a, b) look like as a region of the plane?

Solution 1. (a) For $\sim$ to be an equivalence relation it must satisfy the following properties for a set S (proofs included)

(i) Reflexivity: $x \sim x \, \forall x \in S$.

Proof. Let $(a, b) \in \mathbb{R} \times \mathbb{R}$, then

$$\begin{aligned} & (a, b) \overset{?}{\sim} (a, b) \\ \implies & 2(a - a) - 3(b - b) = 0 \end{aligned}$$

Which satisfies our relation as defined. Therefore the relation is reflexive. □

(ii) Symmetry: $x \sim y \implies y \sim x \, \forall x, y \in S$

Proof. Let $(a, b), (c, d) \in \mathbb{R} \times \mathbb{R}$, then

$$\begin{aligned} & (a, b) \sim (c, d) \\ \implies & 2(a - c) - 3(b - d) = 0 \\ \implies & 2(a - c) = 3(b - d) \\ \implies & -2(a - c) = -3(b - d) \\ \implies & 2(c - a) = 3(d - b) \\ \implies & 2(c - a) - 3(d - b) = 0 \\ \implies & (c, d) \sim (a, b) \end{aligned}$$

□

(iii) Transitivity: $x \sim y \sim z \implies x \sim z \, \forall x, y, z \in S$

Proof. Let $(a, b), (c, d), (e, f) \in \mathbb{R} \times \mathbb{R}$, then

$$\begin{aligned} & (a, b) \sim (c, d) \\ \implies & 2(a - c) - 3(b - d) = 0 \end{aligned}$$

and

$$\begin{aligned} & (c, d) \sim (e, f) \\ \implies & 2(c - e) - 3(d - f) = 0 \end{aligned}$$

so

$$
\begin{aligned}
& 2(a-c)-3(c-d)+2(c-e)-3(d-f)=0 \\
\implies & 2(a-c+c-e)-3(b-d+d-f)=0 \\
\implies & 2(a-e)-3(b-f)=0 \\
\implies & (a,b)\sim(e,f)
\end{aligned}
\qquad \square
$$

Therefore $\sim$ is an equivalence relation on $\mathbb{R}\times\mathbb{R}$.

(b) For representative element $(1,1)$ we get that for an element $(a,b)\in\mathbb{R}\times\mathbb{R}$ to belong to the associated equivalence class we must have

$$2(1-a)-3(1-b)=0$$

which can be rearranged to obtain

$$a=-\frac{1-3b}{2}$$

so for $b=\pm1$ we get two members of the equivalence class represented by $(1,1)$ under $\sim$, $(1,1)$ and $(-2,1)$. The elements (π,e) and (ϕ,i^i) where π,e take on their usual definitions, ϕ is the golden ratio and i^i is, interestingly, both transcendental *and* real!

(c) The equivalence class with representative (a,b) is the set $E=\{(x,y)\in\mathbb{R}\times\mathbb{R}|x\sim(a,b)\}$. This gives the equation

$$2(a-x)-3(b-y)=0 \implies \frac{2(a-x)-3b}{-3}$$

Problem 2. For each of the following sets S, determine whether S is closed under addition modulo n, or multiplication modulo n, or both or neither. (Addition and multiplication modulo n are defined in Exercise Set 2).

(a) $S=\{0,4,8,12\}, n=16$.

(b) $S=\{0,3,6,9,12\}, n=15$.

(c) $S=\{1,2,3,4\}, n=5$.

(d) $S=\{0,2,3,4,6,8,9,10\}, n=12$.

(e) $S=\{1,5,7,11\}, n=12$.

Solution 2.

Problem 3. Determine whether the given binary operation $*$ is commutative, associative, both or neither. Justify your answers with proof.

(a) The operation $*$ on $\mathbb{Z}$ given by $a*b=a+b+ab$

(b) The operation $*$ on $\mathbb{R}$ given by $a*b=a+b-ab$

(c) The operation $*$ on $\mathbb{R}$ given by $a*b=a+2ab$

(d) The operation $*$ on $\mathbb{Z}\times\mathbb{Z}$ given by $(a,b)*(c,d)=(ad+bc,bd)$

(e) The operation $*$ on $\mathbb{Z}\times\mathbb{Z}$ given by $(a,b)*(c,d)=(ad,bc)$

Solution 3.

Problem 4. Let S be a nonempty set. A binary algebraic structure $(S,*)$ is called a semigroup if $*$ is associative.

(a) Let S be the set of positive rational numbers. Show that $(S,*)$ is a commutative semigroup if

$$a*b=\frac{ab}{a+b}$$

(the usual operations on the right) for all $a,b\in S$

(b) Let S be a set containing more than one element. Define

$$a*b=b$$

for all $a,b\in S$. Show that $(S,*)$ is a noncommutative semigroup with no identity element.

Solution 4.