Physics 2130: Assignment III

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Problem 1. Consider the driven, damped harmonic oscillator. Its equation of motion is:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 = A\cos(\omega t)$$

In the case of an underdamped oscillator, i.e., $\beta < \omega_0^2$ we found that the solution for the equation of motion is:

$$x(t) = x_c(t) + x_p(t)$$

Where:

$$x_c(t)e^{-\beta t}\left[c_1 e^{i\omega_1 t} + c_2 e^{-i\omega_1 t}\right] = Be^{-\beta t}\cos(\omega_1 t - \phi) = \Gamma(t)s(t)$$

And where:

$$\omega_1 = \sqrt{w_0^2 - \beta^2}$$

$$\Gamma(t) = Be^{-\beta t}$$

$$s(t) = \cos(\omega_1 t - \phi)$$

The particular solution is instead:

$$x_p(t) = D\cos(\omega t - \delta)$$

Where:

$$D = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}}$$

$$\delta = \arctan\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right)$$

(a) Consider an underdamped driven oscillator that starts with the initial conditions $x(t=0) = x_0$ and $\dot{x}(t=0) = 0$. Find the analytical expressions for the unknown coefficients in $x(t)$ using these initial conditions.

(b) Write a python program that returns (and prints) the values of ω_1, β, ϕ, D and δ for a given x_0. This should be coded as a function called `harm_osc_params` that accepts as inputs ω_0, β, A, ω and x_0.

(c) In the same python program, now write a new function, `harm_osc_x_pos`, that calculates the array of positions x of the harmonic oscillator for a given array of times t. This function should receive the array t as an input as well as the values of ω_0, β, A, ω and x_0. It should call the previously written function `harm_osc_params` for the calculations of all the oscillation parameters. It should return the array of positions x of the harmonic oscillator for each value of t.

(d) In the same python program, now write a new function to plot the data, `harm_osc_x_plot_single`. The function receives as inputs the arrays t and x generated at the previous step.

(e) Pick three values of β (remember of the constraint $\beta < w_0$) and plot in the same graph $x(t)$ for the three chosen values. For reference use $w_0 = 1\text{s}^{-1}$ and $A = 1\text{s}^{-2}$ (but play with the values of A to see the relative importance of the driver and the damping). Comment on what effect β has on both the transient and steady-state solution.

(f) To help you distinguish the different effects, now write a function called `harm_osc_damp_drive` that plots in the same graph, $s(t)$, $\Gamma(t)$, $x_c(t)$ and $x_p(t)$. Comment on the results, specifically on the contribution of each component.

(g) Finally, write a new function called `harm_osc_euler_cromer` that receives has input parameters ω_0, β, A, ω, x_0, and the analytical solution $x(t)$. This function should calculate a new $x(t)$, called $x_{EC}(t)$, that uses the Euler-Cromer method to determine the position of the oscillator as a function of time for the same initial conditions. Additionally, the function should plot, on the same graph, the solutions $x(t)$ and $x_{EC}(t)$ obtained with the analytical and Euler-Cromer methods, respectively, and the residuals, i.e., the difference $x(t) - x_{EC}(t)$. Discuss how you choose a value of Δt that gives a sufficiently accurate answer (which means defining "sufficiently accurate").

Solution 1.

(a)

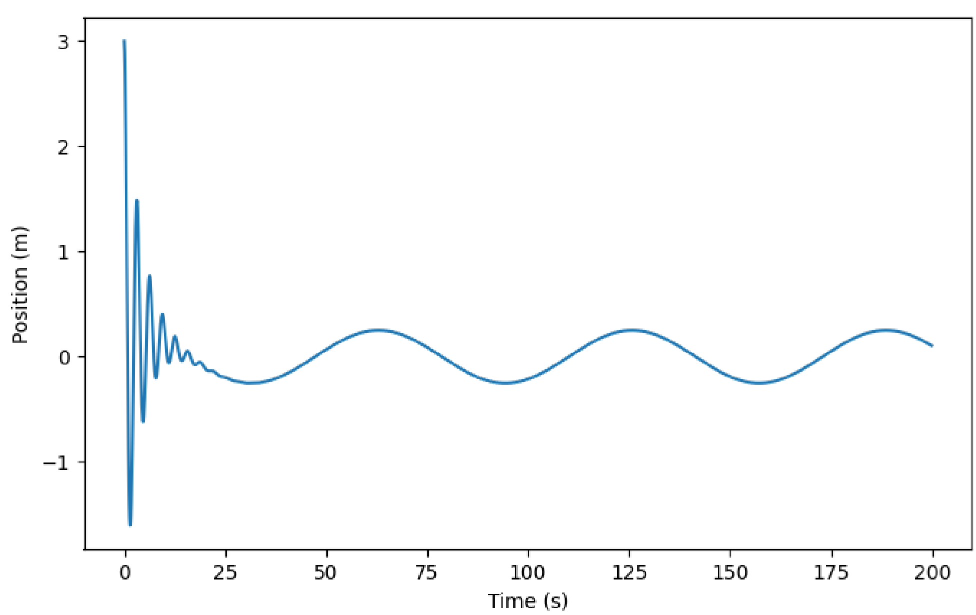
$$\begin{aligned}
&x(t=0) = B\cos(-\phi) + D\cos(-\delta) = x_0 \\
&\Rightarrow x_0 - D\cos(\delta) = B\cos(\phi) \\
&\Rightarrow \frac{x_0 - D\cos(\delta)}{\cos(\phi)} = B
\end{aligned}$$

$$\begin{aligned}
&\dot{x}(t=0) = -B\omega_1\sin(-\phi) - B\beta\cos(-\phi) - D\omega\sin(-\delta) = 0 \\
&= B\omega_1\sin(\phi) - B\beta\cos(\phi) + D\omega\sin(\delta) \\
&= \frac{x_0 - D\cos(\delta)}{\cos(\phi)}\omega_1\sin(\phi) - \frac{x_0 - D\cos(\delta)}{\cos(\phi)}\beta\cos(\phi) + D\omega\sin(\delta) \\
&= (x_0 - D\cos(\delta))\,\omega_1\tan(\phi) - (x_0 - D\cos(\delta))\,\beta + D\omega\sin(\delta) \\
&\Rightarrow -\frac{D\omega\sin(\delta)}{(x_0 - D\cos(\delta))} = \omega_1\tan(\phi) - \beta \\
&\Rightarrow \arctan\left(\frac{\beta - \frac{D\omega\sin(\delta)}{(x_0 - D\cos(\delta))}}{\omega_1}\right) = \phi
\end{aligned}$$

(b)
```python
import numpy as np


def harm_osc_params(w_0: float,
                    beta: float,
                    A: float,
                    w: float,
                    x_0: float
                    ) -> tuple[float, float, float, float, float]:
    # "Given"
    w_1 = np.sqrt(w_0**2-beta**2)
    D = A/(np.sqrt((w_0**2-w**2)**2+4*(w**2)*(beta**2)))
    delta = np.arctan2((2*w*beta), (w_0**2-w**2))

    # Unknowns
    print(((beta-(D*w*np.sin(delta)))/(x_0-D*np.cos(delta))))
    phi = np.arctan2(
        ((beta-(D*w*np.sin(delta)))/(x_0-D*np.cos(delta))),
        w_1)
    B = (x_0-D*np.cos(delta))/(np.cos(phi))

    print(f"w_1: {w_1}", f"B: {B}", f"phi: {phi}",
          f"D: {D}", f"delta: {delta}", sep="\n")

    return w_1, B, phi, D, delta
```

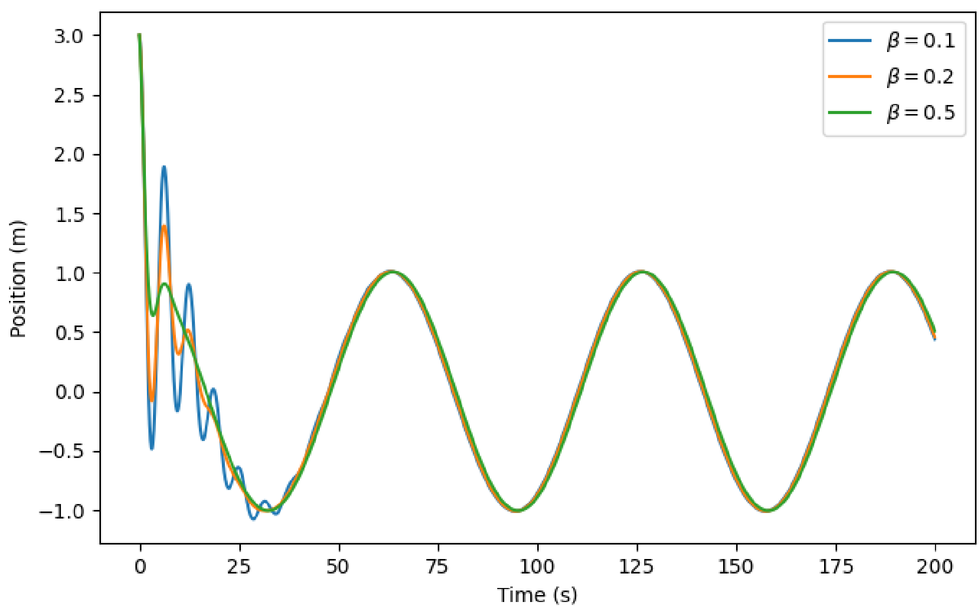
(c)
```python
import numpy as np
import q1b


def harm_osc_x_pos(w_0: float,
                   beta: float,
                   A: float,
                   w: float,
                   x_0: float,
                   t: np.ndarray
                   ) -> list:
    w_1, B, phi, D, delta = q1b.harm_osc_params(w_0,
                                                 beta,
                                                 A,
                                                 w,
                                                 x_0)

    return B*np.exp(-beta*t)*np.cos(w_1*t-phi)+D*np.cos(w*t-delta)
```

(d)
```python
import matplotlib.pyplot as pyplot
import numpy as np
import q1c


def harm_osc_x_plot_single(x: np.ndarray, t: np.ndarray) -> None:
    pyplot.plot(t, x)
    pyplot.xlabel("Time (s)")
    pyplot.ylabel("Position (m)")
    pyplot.show()


t_vals = np.arange(0, 200, 0.1)
x_vals = q1c.harm_osc_x_pos(2, 0.25, 1, 0.1, 3, t_vals)
harm_osc_x_plot_single(x_vals, t_vals)
```



The first regime, visible as the rapid oscillations from $t = 0$ to $t \approx 30$ is dominated by the $Be^{-\beta t}\cos(\omega_1 t - \phi)$ term. Because This term contains an exponential decay factor it will fade away as time progresses leaving the $D\cos(\omega t - \delta)$ term as the only oscillator. The first regime has a period of $\approx 3.2\mathrm{rad\,s^{-1}}$, which is consistent with $\omega_1 = \frac{2\pi}{\sqrt{w_0^2-\beta^2}} = \frac{2\pi}{\sqrt{2^2-0.25^2}} \approx 3.2\mathrm{rad\,s^{-1}}$. The second regime has a period of $\approx 63\mathrm{rad\,s^{-1}}$, which is consistent with $\omega = \frac{2\pi}{63} \approx 0.1$.

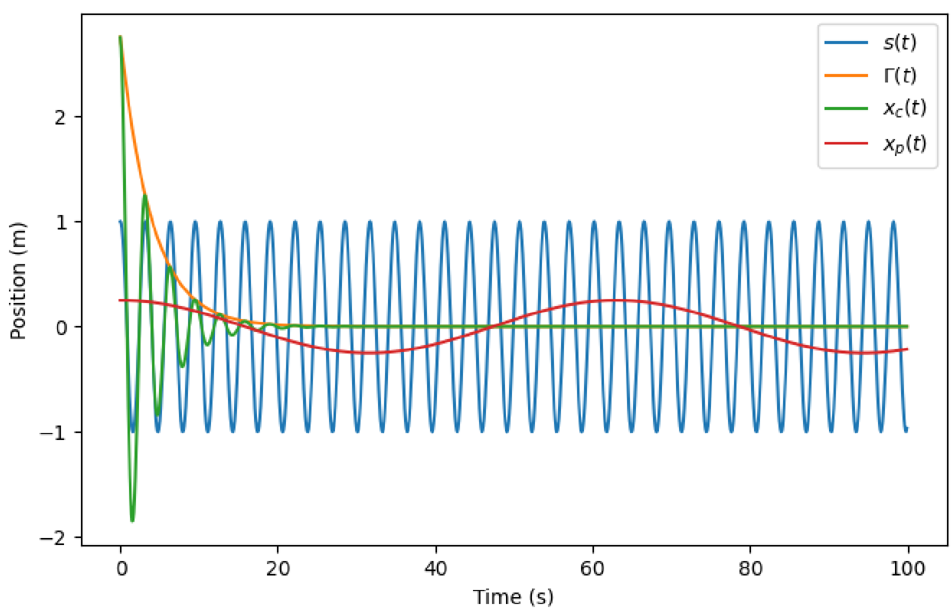
(e)
```python
import numpy as np
import matplotlib.pyplot as pyplot
import q1c


t_vals = np.arange(0, 200, 0.1)

for beta in [0.1, 0.2, 0.5]:
    x_vals = q1c.harm_osc_x_pos(1, beta, 1, 0.1, 3, t_vals)
    pyplot.plot(t_vals, x_vals, label=rf"$\beta={beta}$")

pyplot.xlabel("Time (s)")
pyplot.ylabel("Position (m)")
pyplot.legend()
pyplot.show()
```



As is expected here, greater values of β result in quicker die-off of the transient solution. β only slightly effects the steady-state solution, causing slight displacement as the transient solution never actually reaches zero.

(f)
```python
import numpy as np
import matplotlib.pyplot as pyplot
import q1b


t_vals = np.arange(0, 200, 0.1)


def harm_osc_damp_drive():
```

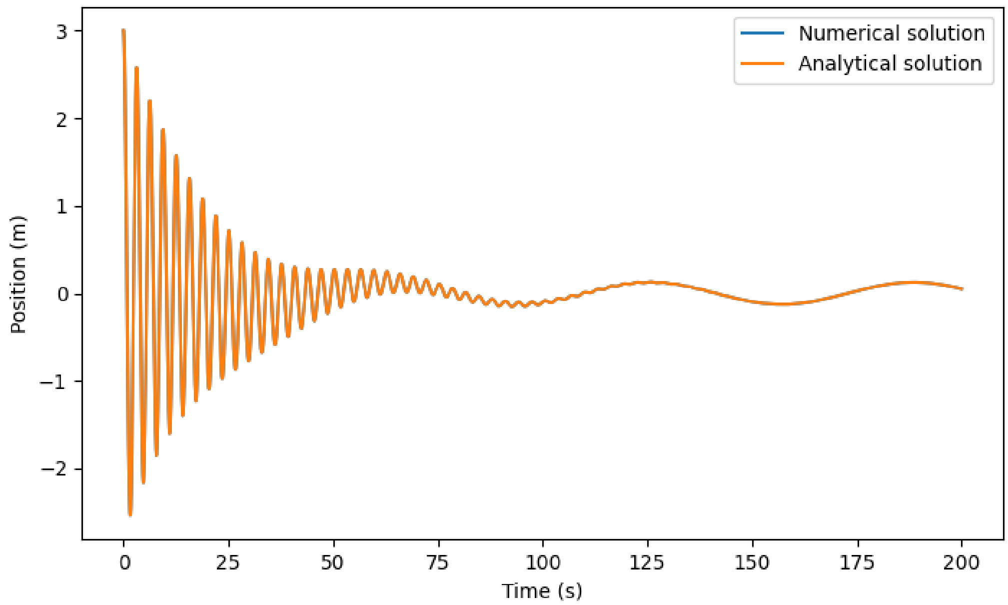
```
    t = np.arange(0, 100, 0.1)
    w_0 = 2
    beta = 0.25
    A = 1
    w = 0.1
    x_0 = 3
    w_1, B, phi, D, delta = q1b.harm_osc_params(w_0,
                                                beta,
                                                A,
                                                w,
                                                x_0)

    s = np.cos(w_1*t-phi)
    gamma = B*np.e**(-beta*t)
    x_c = s*gamma
    x_p = D*np.cos(w*t-delta)

    pyplot.plot(t, s, label=r"$s(t)$")
    pyplot.plot(t, gamma, label=r"$\Gamma(t)$")
    pyplot.plot(t, x_c, label=r"$x_c(t)$")
    pyplot.plot(t, x_p, label=r"$x_p(t)$")

    pyplot.xlabel("Time (s)")
    pyplot.ylabel("Position (m)")
    pyplot.legend()
    pyplot.show()

harm_osc_damp_drive()
```



$s(t)$ contributes the oscillation of the transient solution. $\Gamma(t)$ contributes the exponential decay envelope. $x_c(t)$ contributes, being the product of $s(t)$ and $\Gamma(t)$ produces the initial decaying transient solution. $x_p(t)$

contributes the steady-state solution.

(g)
```
import numpy as np
import matplotlib.pyplot as pyplot
import q1c


def harm_osc_euler_cromer(w_0: float,
                          beta: float,
                          A: float,
                          w: float,
                          x_0: float,
                          t: np.ndarray
                          ):
    dt = t[1]-t[0]
    x = np.full_like(t, x_0)
    x_r = q1c.harm_osc_x_pos(w_0, beta, A, w, x_0, t)
    v = np.zeros_like(t)
    for t_i in range(len(t) - 1):
        a = A*np.cos(w*t[t_i])-2*beta*v[t_i]-(w_0**2)*x[t_i] # Should depend on position,
        ↪   right?


        v[t_i+1] = v[t_i] + a*dt
        x[t_i+1] = x[t_i] + v[t_i+1]*dt

    pyplot.plot(t, x, label="Numerical solution")
    pyplot.plot(t, x_r, label="Analytical solution")
    pyplot.xlabel("Time (s)")
    pyplot.ylabel("Position (m)")
    pyplot.legend()
    pyplot.show()


t_vals = np.arange(0, 200, 0.01)
harm_osc_euler_cromer(2, 0.05, 1/2, 0.1, 3, t_vals)
```



Numerical solution
Analytical solution
Position (m)
Time (s)
3
2
1
0
−1
−2
0
25
50
75
100
125
150
175
200

Problem 2. We want to study now the behaviour of the **non-linear (physical) pendulum**. The analytical solutions we generally get for oscillating systems are derived under the small-angle approximation, which allows us to write $\sin(\theta) \approx \theta$. If one drops this hypothesis things change, and the motion now becomes dependent on the amplitude. The equation of motion is now:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin(\theta) - q\frac{d\theta}{dt} + F_d \sin(\Omega_D t)$$

Where the only difference from the linear case is that we now have $\sin\theta$ instead of θ. This equation of motion has no analytical solutions; therefore, we need to solve it numerically. We thus need to write the equations for angular acceleration and velocity and use them, e.g., with the Euler-Cromer method:

$$\frac{d\omega}{dt} = -\frac{g}{l}\sin(\theta) - q\frac{d\theta}{dt} + F_d \sin(\Omega_D t)$$

$$\frac{d\theta}{dt} = \omega$$

(a) Write a python program that calculates numerically $\theta(t)$ using the Euler-Cromer method.

(b) Write another python program (you can mostly base it on what you wrote in (2a)) that calculates two separate solutions $\theta_1(t)$ and $\theta_2(t)$, where $\theta_1(t)$ starts with an initial angle $\theta_0 = 10°$ while $\theta_2(t)$ starts with an ever so slightly different initial angle $\theta_0 = 10°$.

Solution 2.