Physics 3610H: Assignment IV

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Problem 1. Consider the following one-dimensional potential:

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x > 0 \end{cases}$$

Determine the transmission coefficient (assuming this is actually supposed to be reflection) R for a wave of energy $E > V_0 > 0$ incident from the left, following these steps: The solutions to the time-independent Schrodinger equation have the following form in the two regions:

$x < 0$	$x > 0$
e^{+ikx} and/or e^{-ikx}	e^{+iqx} and/or e^{-iqx}

(a) Write expressions for k and q in terms of E and V_0.

(b) Consider a wave incident from the left. Write expressions for the wavefunction in the two regions. For $x < 0$, there are two terms, both an incident wave and a reflected wave, while for $x > 0$ there is only one, the transmitted wave. Use the coefficient A for the incident wave, B for the reflected wave, and C for the transmitted wave.

(c) List all the conditions which the full wavefunction must satisfy

(d) Is the energy quantized?

(e) Apply the conditions in (c) to determine the reflection coefficient R. Recall that R is the ratio of the reflected probability current density to the incident probability current density. In the notation established here that means $R = |B|^2/|A|^2$. Note that you do not need to determine each coefficient, only this ratio. Your answer should be expressed in terms of E and V_0.

Solution 1. (a) As usual

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

because the particle has relative energy E in the region where $V(x) = 0$. For the $V(x) = V_0$ region the particle has relative energy $E - V_0$ as we've been told $E > V_0$ so

$$q = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$$

(b) We have

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \\ Ce^{iqx} & x > 0 \end{cases}$$

because the positive exponents (when coupled with the time part) give a wave moving to the right and the negatives one moving to the left. Because the transmitted wave is moving exclusively to the left we use only the positive exponential.

(c) The full wavefunction must be continuous, smooth, and normalized (though this isn't a normalizable state?). In our case because our functions are obviously smooth and continuous by nature of being exponentials the only place we need to worry about smoothness and continuity is at the interface $x = 0$. Using the definitions of smoothness and continuity we get then that

$$\psi(0^-) = \psi(0^+) \implies A + B = C$$
$$\left.\frac{d\psi}{dx}\right|_{0^-} = \left.\frac{d\psi}{dx}\right|_{0^+} \implies Aik - Bik = Ciq$$

(d) No, because there is no restriction imposed by the above conditions that would imply (or rather, require) quantization.

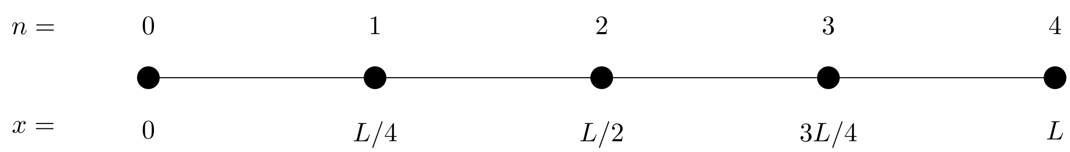
(e) Starting with our second condition,

$$Aik - Bik = Ciq \implies A - B = C\frac{q}{k}.$$

Then using our first condition,

$$\begin{aligned}
&A - B = (A + B)\frac{q}{k} \\
\implies &A - A\frac{q}{k} = B + B\frac{q}{k} \\
\implies &A\left(1 - \frac{q}{k}\right) = B\left(1 + \frac{q}{k}\right) \\
\implies &|B|^2/|A|^2 = |B/A|^2 = \left|\frac{1 - \frac{q}{k}}{1 + \frac{q}{k}}\right|^2 = \left|\frac{k - q}{k + q}\right|^2 \\
\implies &\left|\frac{\sqrt{\frac{2mE}{\hbar^2}} - \sqrt{\frac{2m(E - V_0)}{\hbar^2}}}{\sqrt{\frac{2mE}{\hbar^2}} + \sqrt{\frac{2m(E - V_0)}{\hbar^2}}}\right|^2 \\
\implies &\left|\frac{\sqrt{E} - \sqrt{E - V_0}}{\sqrt{E} + \sqrt{E - V_0}}\right|^2
\end{aligned}$$

Problem 2. Consider a system consisting of five beads equally spaced along a massless, elastic string.



The beads on each end are held fixed. The three beads in the middle can move up and down (but not along the string or in or out of the page). This system has only three degrees of freedom: the up and down motion of each of the three central beads. The following three functions form a complete set, in terms of which any allowed displacement of these beads can be expressed.

$$g_1(x) = \frac{1}{\sqrt{2}}\sin\frac{\pi x}{L}; \qquad g_2(x) = \frac{1}{\sqrt{2}}\sin\frac{2\pi x}{L}; \qquad g_3(x) = \frac{1}{\sqrt{2}}\sin\frac{3\pi x}{L}$$

Note that we consider in this problem only the positions of the beads and not the string. The x values in this problem are therefore discrete: $x_n = nL/4$ with $n = 1, 2, 3$.

(a) Show that $g_1(x)$ is normalized

(b) Show that $g_1(x)$ and $g_2(x)$ are orthogonal

(c) Consider a triangle wave, such as you might get from plucking the string in the middle.

$$f(x_1) = L/4; \qquad f(x_2) = L/2; \qquad f(x_3) = L/4.$$

Determine the values of a_k such that

$$f(x) = \sum_k a_k g_k(x).$$

(d) Check your result by plotting the positions of the bead generated by this sum.

Solution 2.

(a) For normalization we require that the self-inner product be 1,

$$\sum_{n=1}^{3} g_1(x_n)g_1(x_n) = \sum_{n=1}^{3} g_1^2(x_n) = 1.$$

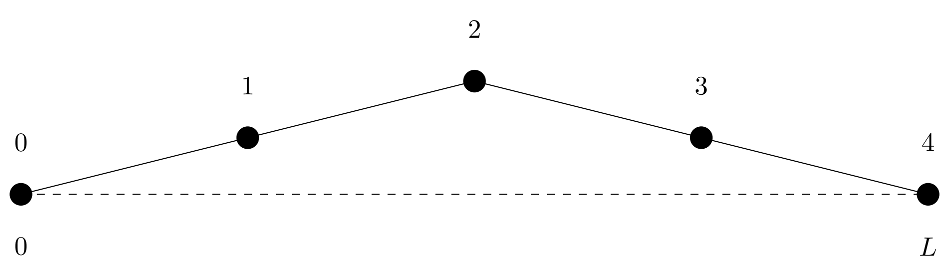
Evaluating this we get

$$\begin{aligned}
&= \sum_{n=1}^{3} g_1^2(x_n) \\
&= \frac{1}{2}\left[\sin^2\frac{\pi x_1}{L} + \sin^2\frac{\pi x_2}{L} + \sin^2\frac{\pi x_3}{L}\right] \\
&= \frac{1}{2}\left[\sin^2\frac{\pi L/4}{L} + \sin^2\frac{\pi L/2}{L} + \sin^2\frac{\pi 3L/4}{L}\right] \\
&= \frac{1}{2}\left[\sin^2\frac{\pi}{4} + \sin^2\frac{\pi}{2} + \sin^2\frac{3\pi}{4}\right] \\
&= \frac{1}{2}\left[\frac{1}{2} + 1 + \frac{1}{2}\right] = 1
\end{aligned}$$

(b) For orthogonality we require that the inner product of g_1 and g_2 be zero,

$$\begin{aligned}
&= \sum_{n=1}^{3} g_1(x_n)g_2(x_n) \\
&= \frac{1}{2}\left[\sin\frac{\pi x_1}{L}\sin\frac{2\pi x_1}{L} + \sin\frac{\pi x_2}{L}\sin\frac{2\pi x_2}{L} + \sin\frac{\pi x_3}{L}\sin\frac{2\pi x_3}{L}\right] \\
&= \frac{1}{2}\left[\sin\frac{\pi L/4}{L}\sin\frac{2\pi L/4}{L} + \sin\frac{\pi L/2}{L}\sin\frac{2\pi L/2}{L} + \sin\frac{\pi 3L/4}{L}\sin\frac{2\pi 3L/4}{L}\right] \\
&= \frac{1}{2}\left[\sin\frac{\pi}{4}\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\sin\pi + \sin\frac{3\pi}{4}\sin\frac{3\pi}{2}\right] \\
&= \frac{1}{2}\left[\frac{1}{\sqrt{2}} + 0 - \frac{1}{\sqrt{2}}\right] = 0
\end{aligned}$$

(c) This wave looks like



Generally we are looking to write

$$f(x) = \frac{1}{\sqrt{2}}\left[a_1\sin\frac{\pi x}{L} + a_2\sin\frac{2\pi x}{L} + a_3\sin\frac{3\pi x}{L}\right]$$

which gives three equations to use to determine $a_{1,2,3}$:

$$f(L/4) = \frac{1}{\sqrt{2}}\left[\frac{a_1}{\sqrt{2}} + a_2 + \frac{a_3}{\sqrt{2}}\right] = L/4 \tag{1}$$
$$f(L/2) = \frac{1}{\sqrt{2}}\left[a_1 - a_3\right] = L/2 \tag{2}$$
$$f(3L/4) = \frac{1}{\sqrt{2}}\left[\frac{a_1}{\sqrt{2}} - a_2 + \frac{a_3}{\sqrt{2}}\right] = L/4 \tag{3}$$
$$\tag{4}$$

Setting (1) and (3) equal,

$$\frac{1}{\sqrt{2}}\left[\frac{a_1}{\sqrt{2}} + a_2 + \frac{a_3}{\sqrt{2}}\right] = \frac{1}{\sqrt{2}}\left[\frac{a_1}{\sqrt{2}} - a_2 + \frac{a_3}{\sqrt{2}}\right]$$
$$\implies \frac{a_1}{\sqrt{2}} + a_2 + \frac{a_3}{\sqrt{2}} = \frac{a_1}{\sqrt{2}} - a_2 + \frac{a_3}{\sqrt{2}}$$
$$\implies a_2 = -a_2 \implies a_2 = 0$$

Then solving (2) for a_1,

$$\frac{1}{\sqrt{2}}\left[a_1 - a_3\right] = L/2$$
$$\implies a_1 = L\sqrt{2}/2 + a_3$$

Which, using (1), gets us

$$\frac{a_1}{\sqrt{2}} + \frac{a_3}{\sqrt{2}} = L\sqrt{2}/4$$
$$\implies a_1 + a_3 = L/2$$
$$\implies L\sqrt{2}/2 + a_3 + a_3 = L/2$$
$$\implies a_3 = L/4 - L\sqrt{2}/4 = \frac{1+\sqrt{2}}{4}L.$$

Which gives

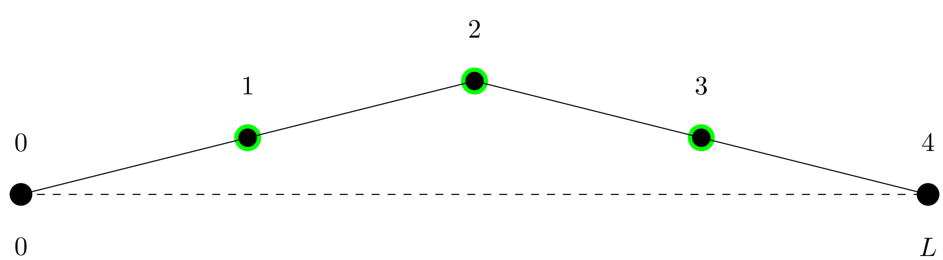
$$a_1 = L\sqrt{2}/2 + a_3 = L\sqrt{2}/2 + L/4 - L\sqrt{2}/4 = \frac{1-\sqrt{2}}{4}L.$$

We could probably have come up with these using some kind of variation on Fourier's trick but this method was what first popped in to my head.

(d) We've written

$$f(x) = \frac{1}{\sqrt{2}}\left[\frac{1+\sqrt{2}}{4}L\sin\frac{\pi x}{L} + \frac{1-\sqrt{2}}{4}L\sin\frac{3\pi x}{L}\right] = \frac{\left(\sqrt{2}-2\right)L\sin\left(\frac{3\pi x}{L}\right) + \left(\sqrt{2}+2\right)L\sin\left(\frac{\pi x}{L}\right)}{8}.$$

Which gives the following plot where green circles represent $f(x)$ and black dots represent the expected positions by the definition of $f(x)$,



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