# Physics 2130: Assignment 1

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**Problem 1.** A particle moving in two dimensions has an acceleration:

$$a = \left[-\frac{3}{1+t}\hat{\boldsymbol{\imath}} + 20\cos(5t)\hat{\boldsymbol{\jmath}}\right]$$

where $t$ is measured in seconds. The particle starts from rest at the origin.

**Solution 1.** Initial conditions are that $\int\int a(t=0)\,dt\,dt = 0$ and $\int a(t=0)\,dt = 0$. a) Finding $x(t)$ (position in the $\hat{\boldsymbol{\imath}}$ direction)

$$\begin{aligned}
&= \int\int -\frac{3}{1+t}\,dt\,dt \\
&= -3\int\int \frac{1}{1+t}\,dt\,dt \rightsquigarrow \text{let}\, u = 1+t \implies \frac{du}{dt} = 1 \implies du = dt \\
&= -3\int\int \frac{1}{u}\,du\,dt \\
&= -3\int \ln(u)\,dt \\
&= -3\int \ln(1+t)\,dt \rightsquigarrow \text{let}\, u = 1+t \implies \frac{du}{dt} = 1 \implies du = dt \\
&= -3\int \ln(u)\,du \\
&= -3u\left[\ln(u) - 1\right] + C \\
&= -3(1+t)\left[\ln(1+t) - 1\right] + C
\end{aligned}$$

At $x(t=0) = 0$ thus

$$\begin{aligned}
&-3(1+0)\left[\cancelto{0}{\ln(1+0)} - 1\right] + C = 0 \\
&3 + C = 0 \\
&C = -3
\end{aligned}$$

$\therefore x(t) = -3(1+t)\left[\ln(1+t) - 1\right] - 3$

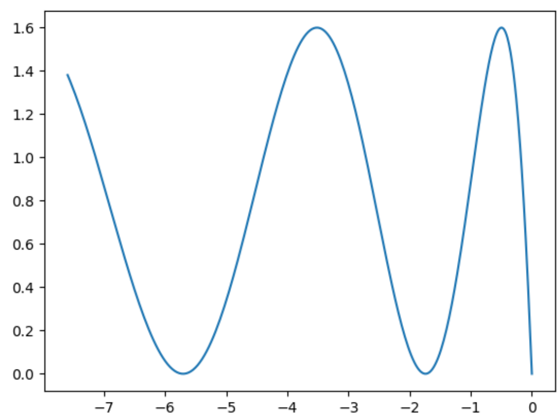
Next finding Finding $y(t)$ (position in the $\hat{\boldsymbol{\jmath}}$ direction)

$$
\begin{aligned}
&= \int\int 20\cos(5t)\,dt\,dt \\
&= 20\int\int \cos(5t)\,dt\,dt \rightsquigarrow \text{let } u = 5t \implies \frac{du}{dt} = 5 \implies \frac{1}{5}du = dt \\
&= 4\int\int \cos(u)\,du\,dt \\
&= 4\int \sin(5t)\,dt \rightsquigarrow \text{let } u = 5t \implies \frac{du}{dt} = 5 \implies \frac{1}{5}du = dt \\
&= 4\int \sin(u)\,dt \rightsquigarrow \text{let } u = 5t \implies \frac{du}{dt} = 5 \implies \frac{1}{5}du = dt \\
&= -\frac{4}{5}\cos(5t) + C
\end{aligned}
$$

At $y(t = 0) = 0$ thus

$$
\begin{aligned}
-\frac{4}{5}\cos\left(\cancelto{0}{5t}\right) + C &= 0 \\
-\frac{4}{5} + C &= 0 \\
C &= \frac{4}{5}
\end{aligned}
$$

$\therefore y(t) = -\frac{4}{5}\cos(5t) + \frac{4}{5}$
b)




```
import matplotlib.pyplot as pyplot
import numpy as np

t=np.arange(0,3,0.01)
def x(t):
    return -3*(1+t)*(np.log(1+t)-1) - 3
def y(t):
    return -(4/5)*np.cos(5*t) + 4/5 # Note: numpy trig
    ↪  takes radians!!!

pyplot.plot(x(t),y(t))
pyplot.show()
```

**Problem 2.** Assume that a projectile of mass $m$ is short with initial speed $v_0$ and angle $\theta$ from an initial height $h_0$

**Solution 2.**
a)

$$
\begin{aligned}
&F_x = \cancel{m}\ddot{x} = -k\cancel{m}v \\
&\Rightarrow \frac{dv_x}{dt} = -kv_x \\
&\Rightarrow \int \frac{1}{v_x}\,dv_x = -k\int\,dt \\
&\Rightarrow \ln|v_x| = -kt + C \rightsquigarrow v_x(t = 0) = v_0\cos\theta \implies \ln(v_0\cos\theta) = C \\
&\Rightarrow v_x = e^{-kt}e^{\ln(v_0\cos\theta)} \\
&\therefore v_x(t) = (v_0\cos\theta)e^{-kt}
\end{aligned}
$$

$$
\begin{aligned}
&F_y = \cancel{m}\ddot{y} = -\cancel{m}g - k\cancel{m}v \\
&\Rightarrow \frac{dv_y}{dt} = -g - kv_y \\
&\Rightarrow \int \frac{1}{g + kv_y}\, dv_y = -\int dt \\
&\Rightarrow \frac{1}{k}\ln|g + kv_y| = -t + C \rightsquigarrow v_y(t=0) = v_0\sin\theta \implies \frac{1}{k}\ln(g + kv_0\sin\theta) = C \\
&\Rightarrow g + kv_y = e^{-kt}e^{\ln(g+kv_0\sin\theta)} \\
&\therefore v_y(t) = \frac{1}{k}(g + kv_0\sin\theta)e^{-kt} - \frac{g}{k}
\end{aligned}
$$

b)

$$
\begin{aligned}
x(t) &= \int v_x(t)\, dt \\
&= \int (v_0\cos\theta)e^{-kt}\, dt \\
&= -\frac{1}{k}(v_0\cos\theta)e^{-kt} + C \rightsquigarrow x(t=0) = 0 \implies C = \frac{1}{k}(v_0\cos\theta) \\
\therefore x(t) &= \frac{(v_0\cos\theta)}{k}\left[1 - e^{-kt}\right]
\end{aligned}
$$

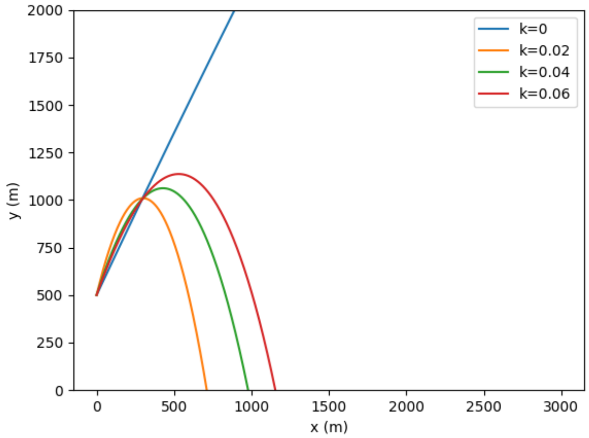
$$
\begin{aligned}
&y(t) = \int v_y(t)\, dt \\
&= \int \frac{1}{k}(g + kv_0\sin\theta)e^{-kt} - \frac{g}{k}\, dt \\
&= -\frac{1}{k^2}(g + kv_0\sin\theta)e^{-kt} - \frac{gt}{k} + C \rightsquigarrow y(t=0) = h_0 \implies C = h_0 + \frac{1}{k^2}(g + kv_0\sin\theta) \\
&\therefore y(t) = -\frac{1}{k^2}(g + kv_0\sin\theta)e^{-kt} - \frac{gt}{k} + h_0 + \frac{1}{k^2}(g + kv_0\sin\theta) = \frac{1}{k^2}(g + kv_0\sin\theta)(1 - e^{-kt}) + h_0 - \frac{gt}{k}
\end{aligned}
$$

c)

$$
\begin{aligned}
y(t = t_f) = 0 &= \frac{1}{k}(g + kv_0\sin\theta)(1 - e^{-kt_f}) + h_0 \\
-h_f &= \frac{1}{k}(g + kv_0\sin\theta)(1 - e^{-kt_f}) \\
-\frac{h_f k}{g + kv_0\sin\theta} - 1 &= e^{-kt_f} \\
-\frac{1}{k}\ln\left(-\frac{h_f k}{g + kv_0\sin\theta} - 1\right) &= t_f
\end{aligned}
$$

$$
R = x(t = t_f) = \frac{(v_0\cos\theta)}{k}\left[1 - e^{\ln\left(-\frac{h_f k}{g + kv_0\sin\theta} - 1\right)}\right]
$$

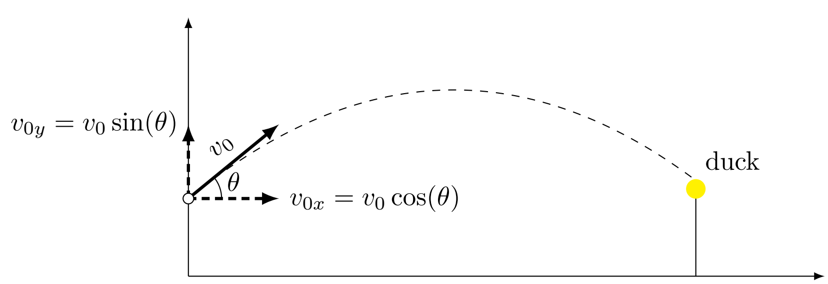
d)




```python
import matplotlib.pyplot as pyplot
import numpy as np

m = 1
v_0 = 600
h_0 = 500
theta = 60 * (np.pi/180)
g = 9.81
k_vals = [0, 0.02, 0.04, 0.06]

t = np.arange(0, 10, 0.01)

def x(t, k):
    return (v_0*np.cos(theta))*t if k == 0 else
    ↪ ((v_0*np.cos(theta))/k)*(1-np.e**(-k*t)) # k=0 is
    ↪ a special case where there is no drag so we use
    ↪ the plain kinematic equation

def y(t, k):
    return h_0 + (v_0*np.sin(theta))*t - (1/2)*g*t**2 if
    ↪ k == 0 else
    ↪ (1/k**2)*(g+k*v_0*np.sin(theta))*(1-np.e**(-k*t))
    ↪ + h_0 - (g*t**2)/k

for k in k_vals:
    pyplot.plot(x(t, k), y(t, k), label=f"k={k}")

pyplot.legend()
pyplot.ylim(bottom=0, top=2000)
pyplot.xlabel("x (m)")
pyplot.ylabel("y (m)")
pyplot.show()
```

**Problem 3.** To practice his skills, an archer is shooting at a rubber duck located on top of a wooden fence. The fence is located at a distance $x_d = 45$m away, and it stands $h_d = 1.5$m tall. The archer is shooting at an angle of $\theta = 35°$ holding the bow at a height of $h_0 = 1.2$m from the ground.
Determine the initial velocity needed for the arrow to hit the duck. Ignore air resistance. (Note: start from Newton's equations and show your work)

**Solution 3.** Note: I defined some variables in the problem statement




Using Newton's equations

$$
\begin{aligned}
&F_x = m\ddot{x} = 0 \\
&\Rightarrow \frac{dv_x}{dt} = 0 \\
&\Rightarrow dv_x = 0 \\
&\Rightarrow \int 1\, dv_x = \int 0\, dt \\
&\Rightarrow v_x(t) = 0 + C \rightsquigarrow v_x(t=0) = 0 + v_0 \cos\theta \implies C = v_0 \cos\theta \\
&\therefore v_x(t) = v_0 \cos\theta
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= \int v_x(t)\,dt \\
&= \int v_0 \cos\theta\,dt \\
&= (v_0\cos\theta)t + C \rightsquigarrow x(t=0) = 0 = \cancelto{0}{(v_0\cos\theta)t} + C \implies C = 0 \\
&\therefore x(t) = (v_0\cos\theta)t
\end{aligned}
$$

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$$
\begin{aligned}
&F_y = \cancel{m}\ddot{y} = -\cancel{m}g \\
&\Rightarrow \frac{dv_y}{dt} = -g \\
&\Rightarrow dv_y = -gdt \\
&\Rightarrow \int 1\,dv_y = \int -g\,dt \\
&\Rightarrow v_y(t) = -gt + C \rightsquigarrow v_y(t=0) = -g\cdot 0 + v_0\sin\theta \implies C = v_0\sin\theta \\
&\therefore v_y(t) = -gt + v_0\sin\theta
\end{aligned}
$$

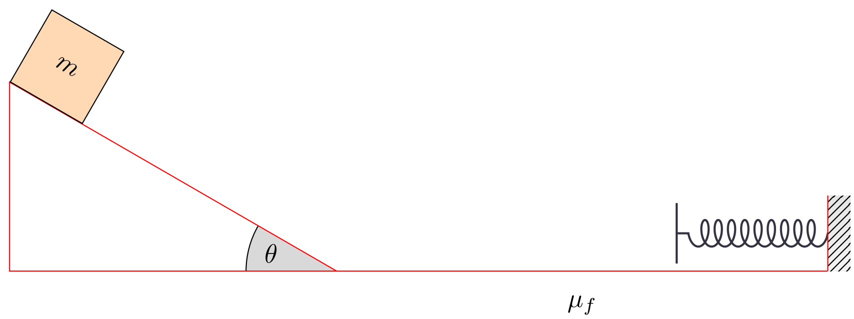
$$
\begin{aligned}
y(t) &= \int v_y(t)\,dt \\
&= \int -gt + v_0\sin\theta\,dt \\
&= -\frac{gt^2}{2} + (v_0\sin\theta)t + C \rightsquigarrow y(t=0) = h_0 = -\cancelto{0}{\frac{gt^2}{2}} + \cancelto{0}{(v_0\sin\theta)t} + C \implies C = h_0 \\
&\therefore y(t) = -\frac{gt^2}{2} + (v_0\sin\theta)t + h_0
\end{aligned}
$$

Now we have two equations $x(t)$ and $y(t)$ with two unknowns which can be solved to determine $v_0$. Solving $x(t)$ for $t$ yields $t_f = \dfrac{x_d}{v_0\cos\theta}$. Substituting into $y(t)$

$$
\begin{aligned}
&y(t=t_f) = h_d = -\frac{g\left(\frac{x_d}{v_0\cos\theta}\right)^2}{2} + (v_0\sin\theta)\left(\frac{x_d}{v_0\cos\theta}\right) + h_0 \\
&h_d - h_0 = -\frac{g\left(\frac{x_d}{v_0\cos\theta}\right)^2}{2} + \left(\frac{x_d\cancel{v_0}\sin\theta}{\cancel{v_0}\cos\theta}\right) \\
&h_d - h_0 = -\frac{g\left(\frac{x_d}{v_0\cos\theta}\right)^2}{2} + x_d\tan\theta \\
&2\left(h_d - h_0 - x_d\tan\theta\right) = -g\left(\frac{x_d}{v_0\cos\theta}\right)^2 \\
&-\frac{2}{g}\left(h_d - h_0 - x_d\tan\theta\right) = \frac{x_d^2}{{v_0}^2\cos^2\theta} \\
&-\frac{2\cos^2\theta}{gx_d^2}\left(h_d - h_0 - x_d\tan\theta\right) = \frac{1}{{v_0}^2} \\
&\sqrt{\left[-\frac{2\cos^2\theta}{gx_d^2}\left(h_d - h_0 - x_d\tan\theta\right)\right]^{-1}} = v_0 \\
&v_0 = 19.7\,\mathrm{m\,s^{-1}}\ 35^\circ \text{ above the horizontal}
\end{aligned}
$$

**Problem 4.** A block of mass $m$ is at rest on top of a slope that forms an angle $\theta$ with the horizontal. At time $t = 0$s, the block is let go down the slope. Note that the slope has a coefficient of friction $\mu_f$ and the block also experiences a drag proportional to the speed $|F_d| = \tilde{k}v$.

**Solution 4.**

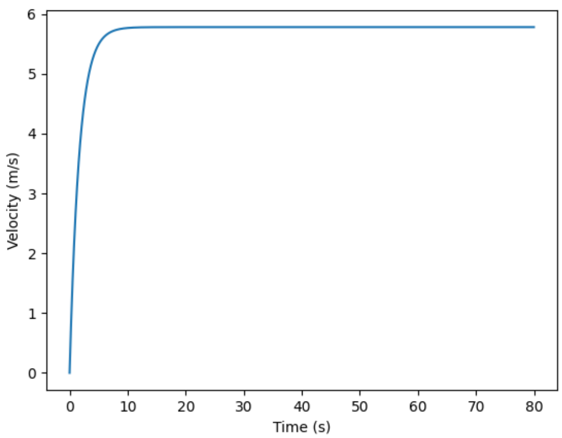



a)

$$\begin{aligned}
&F_x = \cancel{m}\ddot{x} = \cancel{m}g\sin\theta - \mu\cancel{m}g\cos\theta - k\cancel{m}v_x \\
&\Rightarrow \frac{dv_x}{dt} = g\sin\theta - \mu g\cos\theta - kv_x \\
&\Rightarrow \int \frac{1}{g\sin\theta - \mu g\cos\theta - kv_x}\, dv_x = \int dt \\
&\Rightarrow -\frac{1}{k}\ln(g\sin\theta - \mu g\cos\theta - kv_x) = t + C \rightsquigarrow v_x(t=0) = 0 \implies C = -\frac{1}{k}\ln(g\sin\theta - \mu g\cos\theta) \\
&\Rightarrow -\frac{1}{k}\ln(g\sin\theta - \mu g\cos\theta - kv_x) = t - \frac{1}{k}\ln(g\sin\theta - \mu g\cos\theta) \\
&\Rightarrow \ln(g\sin\theta - \mu g\cos\theta - kv_x) = -kt + \ln(g\sin\theta - \mu g\cos\theta) \\
&\Rightarrow g\sin\theta - \mu g\cos\theta - kv_x = (g\sin\theta - \mu g\cos\theta)e^{-kt} \\
&\Rightarrow -kv_x = (g\sin\theta - \mu g\cos\theta)e^{-kt} + \mu g\cos\theta - g\sin\theta \\
&\therefore v_x = -\frac{1}{k}((g\sin\theta - \mu g\cos\theta)e^{-kt} + \mu g\cos\theta - g\sin\theta)
\end{aligned}$$

$$\begin{aligned}
&F_y = \cancel{m}\ddot{y} = 0 \\
&\Rightarrow \frac{dv_y}{dt} = 0 \\
&\Rightarrow v_y = 0 + C \rightsquigarrow v_y(t=0) = 0 \implies C = 0 \\
&\therefore v_y(t) = 0
\end{aligned}$$

b) After a long time the block reaches its terminal velocity of approx. $10\,\mathrm{m\,s^{-1}}$. The block never reaches $40\,\mathrm{m\,s^{-1}}$ because that is more than the terminal velocity




```python
import matplotlib.pyplot as pyplot
import numpy as np

max_time = 80
theta = 45 * (np.pi/180)
g = 9.81
mu = 0.5
k = 0.6
m = 10

t = np.arange(0,max_time,0.01)

N = g*np.cos(theta)

def v_x(t):
     return (-1/k)*((g*np.sin(theta)-mu*N)*np.e**(-k*t)
     +mu*N-g*np.sin(theta))

pyplot.plot(t, v_x(t))
pyplot.xlabel("Time (s)")
pyplot.ylabel("Velocity (m/s)")
pyplot.show()
```

c)

$$
\begin{aligned}
&F_x = \cancel{m}\ddot{x} = -\mu\cancel{m}g \\
&\Rightarrow \frac{dv_x}{dx}\frac{dx}{dt} = -\mu g \\
&\Rightarrow v_x\frac{dv_x}{dx} = -\mu g \\
&\Rightarrow \frac{v_x^2}{2} = -\mu gx + C \rightsquigarrow v_x(x=0) = v_e \implies C = \frac{v_e^2}{2} \\
&\Rightarrow \frac{v_x^2}{2} = -\mu gx + \frac{v_e^2}{2} \\
&\Rightarrow v_x^2 = -2\mu gx + v_e^2 \\
&\therefore v_x = \sqrt{v_e^2 - 2\mu gx}
\end{aligned}
$$

Now, by conservation of energy

$$
\begin{aligned}
&E_{tot} = E'_{tot} \\
&\frac{1}{2}m(v_e^2 - 2\mu gd) = \frac{1}{2}\beta x^2 \\
&m(v_e^2 - 2\mu gd) = \beta x^2 \\
&\sqrt{\frac{m}{\beta}(v_e^2 - 2\mu gd)} = x
\end{aligned}
$$

Note: this checks out by the first tenet of "street fighting physics", dimensional analysis, which is good because I don't feel confident in it. I've been looking at all of these equations for too long.

**Problem 5.** An object is tossed vertically down a cliff with initial speed $v_0$. Besides gravity, it experiences a resisting, drag force that is proportional to the square of the velocity (i.e., $kmv^2$). Show that the distance travelled as the object accelerates from the initial speed $v_0$ to the final speed $v_1$ is given by the expression:

$$\frac{1}{2k}\ln\left|\frac{g - kv_0^2}{g - kv_1^2}\right|$$

**Solution 5.** We're looking for an expression of position in terms of velocity which we can get using the so-called "fuckery with the differentials" which still confuses me.

$$
\begin{aligned}
&\ddot{x} = -g + kv^2 \rightsquigarrow \text{ I'm using the "wrong" axis here because it's what I chose initially} \\
&\frac{dv}{dt} = -g + kv^2 \\
&\frac{dv}{dx}\frac{dx}{dt} = -g + kv^2 \rightsquigarrow \text{ Here's where the } x \propto v \text{ comes up} \\
&v\frac{dv}{dx} = -g + kv^2 \\
&\frac{v}{-g + kv^2} dv = dx \\
&\int \frac{v}{-g + kv^2}\, dv = x + C \rightsquigarrow C \text{ will be zero here as } v_0, x_0 = 0 \\
&\frac{1}{2k}\int \frac{\cancel{v}}{s\cancel{v}}\, ds = x \rightsquigarrow s = -g + kv^2 \text{ because I can't tell the difference between } u \text{ and } v \text{ in my handwriting} \\
&x(v) = \frac{1}{2k}\ln\left(kv^2 - g\right) \implies x(v)\Big|_{v_1}^{v_0} = \Delta x = \frac{1}{2k}\left[\ln\left(kv_0^2 - g\right) - \ln\left(kv_1^2 - g\right)\right] = \frac{1}{2k}\ln\left|\frac{kv_0^2 - g}{kv_1^2 - g}\right|
\end{aligned}
$$

Which, as noted at the beginning is "wrong" because I used the opposite axis (negative down).