Math 3310H: Assignment III

Jeremy Favro (0805980)
Trent University, Peterborough, ON, Canada

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Problem 1. Show that a group G cannot be the union of two proper subgroups, in other words, if $G = H \cup K$ where H and K are subgroups of G, then $H = G$ or $K = G$.

Solution 1. Suppose, by way of contradiction, that $G = H \cup K$ and $H \neq G \neq K$. Then there are elements $a \in H$ and $b \in K$ but $a \notin K$ and $b \notin H$. Because $G = H \cup K$ and H, K, and G are closed by definition, $ab \in H$ or $ab \in K$. First then suppose that $ab \in H \implies a^{-1}ab \in H \implies eb \in H \implies b \in H$, but we began with the assumption that $b \notin H$, so unless $H = K = G$, K cannot be a subgroup. The same argument works in the other direction: Suppose $ab \in K \implies abb^{-1} \in K \implies ae \in K \implies a \in K$, but a was created to be something only in H, not K, meaning H is not closed unless $H = K = G$.

Problem 2. Let G be a group with identity e and $e \in G$. Show that if $a^n = e$ then the order of a divides n.

Solution 2. Let $|a| = k$ be the order of a. By the division algorithm we can write $n = qk + r$ for some $q, r \in \mathbb{Z}$ with $0 \leq r < k$. So

$$\begin{aligned}
e &= a^n \\
&= a^{qk+r} \\
&= a^{qk}a^r \\
&= (a^k)^q a^r \\
&= e^q a^r && a^k = e \text{ by definition.} \\
&= a^r.
\end{aligned}$$

For the expression $e = a^r$ to hold true r must be some multiple of the order of a, k. This means that our expression using the division algorithm becomes $n = qk + sk$ for $sk = r$ which means that $n/k = q + s$ which is an integer meaning that the order of a, k, divides n.

Problem 3. Let G be a cyclic group of order n with identity e. Suppose 15 divides n. How many solutions to $x^{15} = e$ are there in G?

Solution 3. Let $\langle g \rangle = G$. Then, as G is cyclic there is one subgroup of G of order 15 generated by $g^{n/15}$ where every element will satisfy $x^{15} = e$ as $x = g^{kn/15} \implies x^{15} = g^{kn} = e$. This means (I think, this was a really tough one) that there are 15 solutions in G.

Problem 4. Show that $H = \{\sigma \in S_n | \sigma(1) = 1\}$ is a subgroup of S_n.

Solution 4. For H to be a subgroup of S_n it must satisfy the following:

(i) Closure: This is fairly obvious, constructing any $\sigma'' = \sigma \circ \sigma'$ will always satisfy $\sigma''(1) = 1$ as both σ and σ' must map $1 \to 1$ to belong to H in the first place.

(ii) Contains the identity: The identity map looks like

$$\iota = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 2 & 3 & \dots & n \end{pmatrix}$$

which satisfies $\sigma(1) = 1$

(iii) Contains inverses: All inverses for a $\sigma \in H$ will map $1 \to 1$ by the definition of σ and so will belong to H.

Problem 5. Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 4 & 1 & 7 & 5 & 8 & 9 & 6 \end{pmatrix}$$

and

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 3 & 7 & 9 & 1 & 8 & 2 & 4 & 5 \end{pmatrix}.$$

(a) Compute σ^2, $\sigma\tau$, $\tau\sigma$, σ^{-1}, $\sigma\tau\sigma^{-1}$, and $\tau\sigma\tau^{-1}$.

(b) Find the order of τ

Solution 5.

(a)

$$\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 4 & 1 & 7 & 5 & 8 & 9 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 4 & 1 & 7 & 5 & 8 & 9 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 2 & 1 & 3 & 8 & 7 & 9 & 6 & 5 \end{pmatrix}$$

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 3 & 7 & 9 & 1 & 8 & 2 & 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 4 & 1 & 7 & 5 & 8 & 9 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 3 & 9 & 6 & 2 & 1 & 4 & 5 & 8 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 3 & 2 & 4 & 1 & 7 & 5 & 8 & 9 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 2 & 1 & 3 & 6 & 9 & 5 & 7 & 8 \end{pmatrix}$$

$$\begin{aligned}\sigma\tau\sigma^{-1} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 4 & 1 & 7 & 5 & 8 & 9 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 3 & 7 & 9 & 1 & 8 & 2 & 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 2 & 1 & 3 & 6 & 9 & 5 & 7 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 4 & 5 & 8 & 9 & 7 & 3 & 2 & 1 \end{pmatrix}\end{aligned}$$

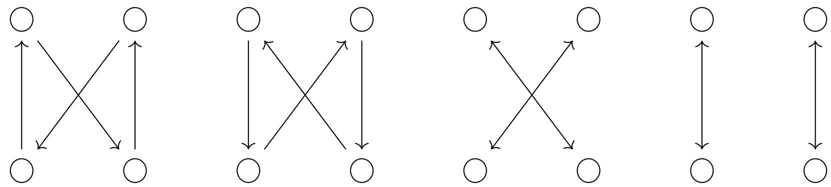
$$\begin{aligned}\tau\sigma\tau^{-1} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 3 & 7 & 9 & 1 & 8 & 2 & 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 4 & 1 & 7 & 5 & 8 & 9 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 7 & 2 & 8 & 9 & 1 & 3 & 6 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 3 & 5 & 8 & 7 & 9 & 1 & 6 \end{pmatrix}\end{aligned}$$

(b) I wrote some Python code to do this to check if I could get ι in a reasonable number of steps

```python
  tau = {
    1: 6,
    2: 3,
    3: 7,
    4: 9,
    5: 1,
    6: 8,
    7: 2,
    8: 4,
    9: 5
}
iota = [1, 2, 3, 4, 5, 6, 7, 8, 9]

t = [6, 3, 7, 9, 1, 8, 2, 4, 5]

cnt = 1
while t != iota:
    cnt = cnt + 1
    for i in range(0, 9):
        t[i] = tau[t[i]]
    print(t)

print(f"Got iota on {cnt}")
```

Which gives the order τ as 6.

Problem 6. Below are four recommended car tire rotation patterns.



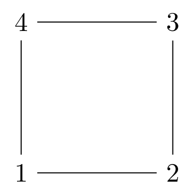
(a) Explain how these patterns can be represented as elements of S_4.

(b) Find the smallest subgroup H of S_4 that contains these four patterns.

(c) Is H abelian?

Solution 6.

(a) If we represent the "default" state of the tires as



then each rotation of the tires is a permutation of this default state. By definition S_4 is the group containing all permutations of 4 elements and so these will belong to S_4. Going from left to right in the previous figure:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}; \quad \sigma_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

(b) Let's assume that the smallest subgroup is just the σs, their inverses, and the identity ι. The inverses are

$$\sigma_1^{-1} = \begin{pmatrix} 3 & 4 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} = \sigma_2$$

$$\sigma_2^{-1} = \begin{pmatrix} 4 & 3 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} = \sigma_1$$

$$\sigma_3^{-1} = \begin{pmatrix} 3 & 4 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \sigma_3$$

$$\sigma_4^{-1} = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \sigma_4$$

So

$$H = \{\iota, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$$

But we need to check that this is closed. This involves checking all the combinations that we don't already know give an inverse. I'll set this up as a Cayley table,

	σ_1	σ_2	σ_3	σ_4
σ_1		ι		
σ_2	ι			
σ_3			ι	
σ_4				ι

$$\sigma_1\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

Which is not in H so we'll call it σ_5 and continue

	σ_1	σ_2	σ_3	σ_4
σ_1	σ_5	ι		
σ_2	ι			
σ_3			ι	
σ_4				ι

I again wrote some Python code to quickly fill these in,

```
class Permutation:
    def __init__(self, map: dict):
        self.map = map

    def __mul__(self, other):
        result = [1,2,3,4]
        for i in range(len(self.map)):
            result[i] = self.map[other.map[i+1]]
        return result

    def inverse(self):
        inv = {v: k for k, v in self.map.items()}
        return list(dict(sorted(inv.items())).values())


sig_1 = Permutation({
    1: 3,
    2: 4,
    3: 2,
    4: 1,
})

sig_2 = Permutation({
    1: 4,
    2: 3,
    3: 1,
    4: 2,
})

sig_3 = Permutation({
    1: 3,
    2: 4,
    3: 1,
    4: 2,
})

sig_4 = Permutation({
    1: 4,
    2: 3,
    3: 2,
    4: 1,
})
```

Which gives

	σ_1	σ_2	σ_3	σ_4
σ_1	σ_5	ι	σ_6	σ_7
σ_2	ι	σ_6	σ_7	σ_6
σ_3	σ_7	σ_6	ι	σ_5
σ_4	σ_6	σ_7	σ_5	ι

where

$$\sigma_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}; \quad \sigma_7 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}.$$

Now,

$$\sigma_5^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \sigma_5; \quad \sigma_6^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} = \sigma_6; \quad \sigma_7^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} = \sigma_7.$$

So our new smallest subgroup becomes

$$H = \{\iota, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7\}.$$

The Cayley table for this new group is

	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
σ_1	σ_5	ι	σ_6	σ_7	σ_2	σ_4	σ_3
σ_2	ι	σ_6	σ_7	σ_6	σ_1	σ_3	σ_4
σ_3	σ_7	σ_6	ι	σ_5	σ_4	σ_2	σ_1
σ_4	σ_6	σ_7	σ_5	ι	σ_3	σ_1	σ_2
σ_5	σ_2	σ_1	σ_4	σ_3	ι	σ_7	σ_6
σ_6	σ_3	σ_4	σ_1	σ_2	σ_7	ι	σ_5
σ_7	σ_4	σ_3	σ_2	σ_1	σ_6	σ_5	ι

So the group is closed and is therefore actually a subgroup. Yay!

(c) The Cayley table above is not diagonally symmetric and therefore H is not abelian.