Math 2120H: Assignment II

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Problem 1. Find the given limits.

$$\lim_{t\to\pi}\left[\left(\sin\frac{t}{2}\right)\mathbf{i}+\left(\cos\frac{2t}{3}\right)\mathbf{j}+\left(\tan\frac{5t}{4}\right)\mathbf{k}\right]$$

Solution 1. By the component-wise law for limits of vector-valued functions this limit is equivalent to the sum of each of its component limits along their respective unit vectors,

$$\begin{aligned}&=\lim_{t\to\pi}\left(\sin\frac{t}{2}\right)\mathbf{i}+\lim_{t\to\pi}\left(\cos\frac{2t}{3}\right)\mathbf{j}+\lim_{t\to\pi}\left(\tan\frac{5t}{4}\right)\mathbf{k}\\&=1\mathbf{i}-\frac{1}{2}\mathbf{j}+1\mathbf{k}\end{aligned}$$

Problem 2. Below $\mathbf{r}(t)$ is the position of a particle in space at time t. Find the particle's velocity and acceleration vectors. Then find the particle's speed at the given value of t.

$$\mathbf{r}(t)=(t+1)\mathbf{i}+(t^2-1)\mathbf{j}+2t\mathbf{k},\qquad t=1$$

Solution 2.

$$\begin{aligned}\mathbf{v}(t)\Big|_{t=1}&=\frac{d}{dt}\mathbf{r}(t)\Big|_{t=1}=\frac{d}{dt}(t+1)\Big|_{t=1}\mathbf{i}+\frac{d}{dt}(t^2-1)\Big|_{t=1}\mathbf{j}+\frac{d}{dt}(2t)\Big|_{t=1}\mathbf{k}\\&=1\Big|_{t=1}\mathbf{i}+2t\Big|_{t=1}\mathbf{j}+2\Big|_{t=1}\mathbf{k}\\&=\mathbf{i}+2\mathbf{j}+2\mathbf{k}\implies\text{speed}=\sqrt{1^2+2^2+2^2}=3\end{aligned}$$

then,

$$\begin{aligned}\mathbf{a}(t)\Big|_{t=1}&=\frac{d}{dt}\mathbf{v}(t)\Big|_{t=1}=\frac{d}{dt}\Big|_{t=1}\mathbf{i}+\frac{d}{dt}2t\Big|_{t=1}\mathbf{j}+\frac{d}{dt}2\Big|_{t=1}\mathbf{k}\\&=0\Big|_{t=1}\mathbf{i}+\frac{d}{dt}2\Big|_{t=1}\mathbf{j}+0\Big|_{t=1}\mathbf{k}\\&=0\mathbf{i}+2\mathbf{j}+0\mathbf{k}\end{aligned}$$

Problem 3. Below $\mathbf{r}(t)$ is the position of a particle in space at time t. Find the angle between the velocity and acceleration vectors at time $t = 0$.

$$\mathbf{r}(t) = (3t+1)\mathbf{i} + \sqrt{3}t\mathbf{j} + t^2\mathbf{k}$$

Solution 3.

$$\begin{aligned}
\mathbf{v}(t)\Big|_{t=0} &= \frac{d}{dt}\mathbf{r}(t)\Big|_{t=0} = \frac{d}{dt}(3t+1)\Big|_{t=0}\mathbf{i} + \frac{d}{dt}(\sqrt{3}t)\Big|_{t=0}\mathbf{j} + \frac{d}{dt}(t^2)\Big|_{t=0}\mathbf{k} \\
&= 3\Big|_{t=0}\mathbf{i} + \sqrt{3}\Big|_{t=0}\mathbf{j} + 2t\Big|_{t=0}\mathbf{k} \\
&= 3\mathbf{i} + \sqrt{3}\mathbf{j} + 0\mathbf{k}
\end{aligned}$$

then,

$$\begin{aligned}
\mathbf{a}(t)\Big|_{t=0} &= \frac{d}{dt}\mathbf{v}(t)\Big|_{t=0} = \frac{d}{dt}(3)\Big|_{t=0}\mathbf{i} + \frac{d}{dt}(\sqrt{3})\Big|_{t=0}\mathbf{j} + \frac{d}{dt}(2t)\Big|_{t=0}\mathbf{k} \\
&= 0\Big|_{t=0}\mathbf{i} + 0\Big|_{t=0}\mathbf{j} + 2\Big|_{t=0}\mathbf{k} \\
&= 0\mathbf{i} + 0\mathbf{j} + 2\mathbf{k}
\end{aligned}$$

Because these vectors are purely in the xy plane and purely along z they are orthogonal and therefore, by definition, have an angle of $\frac{\pi}{2}$rad between them.

Problem 4. Evaluate the integrals.

(a)

$$\int_0^1 t^3\mathbf{i} + 7\mathbf{j} + (t+1)\mathbf{k}\, dt$$

(b)

$$\int_1^4 \frac{1}{t}\mathbf{i} + \frac{1}{5-t}\mathbf{j} + \frac{1}{2t}\mathbf{k}\, dt$$

Solution 4.

(a)

$$\begin{aligned}
&= \int_0^1 t^3\mathbf{i} + 7\mathbf{j} + (t+1)\mathbf{k}\, dt \\
&= \int_0^1 t^3\, dt\mathbf{i} + \int_0^1 7\, dt\mathbf{j} + \int_0^1 (t+1)\, dt\mathbf{k} \\
&= \frac{t^4}{4}\Big|_0^1\mathbf{i} + 7t\Big|_0^1\mathbf{j} + \frac{t^2}{2} + t\Big|_0^1\mathbf{k} \\
&= \frac{1}{4}\mathbf{i} + 7\mathbf{j} + \frac{3}{2}\mathbf{k}
\end{aligned}$$

(b)

$$\begin{aligned}
&= \int_1^4 \frac{1}{t}\mathbf{i} + \frac{1}{5-t}\mathbf{j} + \frac{1}{2t}\mathbf{k}\, dt \\
&= \int_1^4 \frac{1}{t}\, dt\mathbf{i} + \int_1^4 \frac{1}{5-t}\, dt\mathbf{j} + \int_1^4 \frac{1}{2t}\, dt\mathbf{k} \\
&= \ln(t)\Big|_1^4\mathbf{i} - \ln(5-t)\Big|_1^4\mathbf{j} + 2\ln(2t)\Big|_1^4\mathbf{k} \\
&= \ln(4)\mathbf{i} + \ln(4)\mathbf{j} + \ln(16)\mathbf{k}
\end{aligned}$$

Problem 5. Solve the initial value problem.

$$\mathbf{r}'(t) = (t^3 + 4t)\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}, \qquad \mathbf{r}(0) = \mathbf{i} + \mathbf{j}$$

Solution 5.

$$\begin{aligned}
\mathbf{r}(t) &= \int (t^3 + 4t)\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}\,dt \\
&= \int (t^3 + 4t)\,dt\mathbf{i} + \int t\,dt\mathbf{j} + \int 2t^2\,dt\mathbf{k} \\
&= \left(\frac{t^4}{4} + 2t^2\right)\mathbf{i} + \frac{t^2}{2}\mathbf{j} + \frac{2t^3}{3}\mathbf{k} + \mathbf{C} \rightsquigarrow \mathbf{r}(t=0) = \mathbf{i} + \mathbf{j} \\
\implies \mathbf{i} + \mathbf{j} &= \left(\frac{0^4}{4} + 2\cdot 0^2\right)\mathbf{i} + \frac{0^2}{2}\mathbf{j} + \frac{2\cdot 0^3}{3}\mathbf{k} + \mathbf{C} \\
&= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} + \mathbf{C} \implies \mathbf{C} = \mathbf{i} + \mathbf{j} \\
\therefore \mathbf{r}(t) &= \left(\frac{t^4}{4} + 2t^2 + 1\right)\mathbf{i} + \left(\frac{t^2}{2} + 1\right)\mathbf{j} + \frac{2t^3}{3}\mathbf{k}
\end{aligned}$$