ODE Cheat Sheet

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Revision 1

Fundamentals

Classification

$\frac{d^n y}{dx^n} = f(x, y)$ denotes an ODE of order n. Note that $(\frac{dy}{dx})^n \neq \frac{d^n y}{dx^n}$. ODEs of order n will have n constants in their general form solutions.
A linear ODE is one which can be written in the form $a_n(x)\frac{d^n y}{d^n x} + a_{n-1}\frac{d^{n-1}y}{d^{n-1}x} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$.

Solutions

Given some IVP $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ if f and $\frac{\partial f}{\partial y}$ are continuous in the rectangle $(x_0, y_0) \in \{(x, y) : a < x < b, c < y < d\}$ then the IVP has a unique solution $\phi(x)$ in some interval $(x_0 - h, x_0 + h)$, $h \geq 0$

Solution Techniques $n = 1$

Direct Integration

Directly integrate …

Seperable

For some ODE $\frac{dy}{dx} = f(x, y) = g(x)p(y)$ the differential can be split $s.t.$ $\frac{1}{p(y)}dy = g(x)dx$ which can be solved by direct integration. Note that when dividing by some function we assume that the function is nonzero. If there is a case (e.g. in an IVP) where the function is zero, the solution is lost.

Linear

For some linear ODE of the form $\frac{dy}{dx} + P(x)y = Q(x)$ we can multiply both sides of the ODE by $\mu(x) = \exp\left(\int P(x)\,dx\right)$ to obtain $\mu\frac{dy}{dx} + \mu P(x)y = \mu Q(x)$ which is equivalent to $\mu y = \mu Q(x)$ which can be solved by direct integration.

Exact

Exact equations are ODEs of the form $M dx + N dy = 0$ where $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Then,
$f(x, y) = \int M\,dx + h(y) = C$ or
$f(x, y) = \int N\,dy + g(x) = C$ and
$\frac{d}{dy}\left(\int M\,dx + h(y)\right) = N$ or
$\frac{d}{dx}\left(\int N\,dy + g(x)\right) = M$

Non-Exact

In cases where something looks exact but $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ you can find an integrating factor

$$\mu(x) = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\,dx\right)$$

$$\mu(y) = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\,dy\right)$$

Homogeneous

If each term of the ODE is of equal order (e.g. the right hand side can be expressed as a function of only $\frac{y}{x}$) we can substitute $y = ux \implies dy = udx + xdu$. This should result in a seperable equation.

Bernoulli

If we have an equation of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$ we divide by y^n and substitute $u = y^1 - n \implies \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$. This should result in a linear equation.

Linear Substituion

An ODE of the form $\frac{dy}{dx} = f(Ax + By + C)$, $B \neq 0$ can be solved by

$$\begin{aligned} u &= Ax + By + C \\ \Rightarrow \frac{du}{dx} &= A + B\frac{dy}{dx} \\ \Rightarrow \frac{dy}{dx} &= (\frac{du}{dx} - A)\frac{1}{B} \end{aligned}$$

Applications

Newton's Cooling

$$\frac{dT}{dt} = k(T - T_m) \implies T(t) = T_m + Ce^{kt}$$

where T is the temperature of an object, T_m the temperature of the medium in which the object sits, and k some cooling constant determined by initial/boundary conditions. C comes about as a result of solving the ODE and can also be determined using initial conditions.

Circuit Theory

$$V_{Resistor} = RI = R\frac{dQ}{dt}$$

$$V_{Inductor} = L\frac{dI}{dt} = L\frac{d^2Q}{dt^2}$$

$$V_{Capacitor} = \frac{Q}{C} = L\frac{d^2Q}{dt^2}$$