Math 2150: Assignment II

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Note: I've kept p, q, and r throughout my solutions and only substituted the actual numbers in at the end. This is because I find it easier, especially when dealing with things that might cancel nicely, to deal with variables rather than the numbers they represent. In my case my student number is 0805980 so $p = 9$, $q = 5$, and $r = 22$.

Problem 1.

(a) Determine $\mathcal{L}^{-1}\left\{\frac{(s-2)}{(s^2+1)(s^2-qs)}\right\}$

(b) Determine $\mathcal{L}^{-1}\left\{\frac{2s+1}{(s^2+4)(s^2+q^2)}\right\}$

(c) Determine $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s-q)}\right\}$

(d) Determine $\mathcal{L}\left\{t\sin^2(qt)\right\}$

(e) Determine $\mathcal{L}\left\{e^{-qt}\sin(pt)\sin(qt)\right\}$

(f) Determine $\mathcal{L}\left\{e^{-qt}t\sin\left(t-\frac{\pi}{6}\right)\right\}$

(g) Determine $\mathcal{L}^{-1}\left\{\ln\left(\frac{7+ps}{9+qs}\right)\right\}$

Solution 1.

(a)

$$= \mathcal{L}^{-1}\left\{\frac{(s-2)}{s(s^2+1)(s-q)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s-q)}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)(s-q)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{A_1 s + B_1}{(s^2+1)} + \frac{C_1}{(s-q)}\right\} - 2\mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{Bs+C}{(s^2+1)} + \frac{D}{(s-q)}\right\}$$

Which I'll solve seperately and then recombine

$$= \mathcal{L}^{-1}\left\{\frac{A_1 s + B_1}{(s^2+1)} + \frac{C_1}{(s-q)}\right\}$$

$$\Rightarrow A_1 s(s-q) + B_1(s-q) + C_1(s^2+1) = 1$$

$$\Rightarrow s = q \implies C_1(q^2+1) = 1 \implies C_1 = \frac{1}{q^2+1}$$

$$\Rightarrow s = 0 \implies B_1(-q) + \frac{1}{q^2+1} = 1 \implies B_1 = \frac{1 - \frac{1}{q^2+1}}{-q}$$

$$\Rightarrow s = 1 \implies A_1(1-q) + \frac{(1-q)\left(1-\frac{1}{q^2+1}\right)}{-q} + \frac{2}{q^2+1} = 1 \implies A_1 = \frac{1 - \frac{(1-q)\left(1-\frac{1}{q^2+1}\right)}{-q} - \frac{2}{q^2+1}}{(1-q)}$$

$$\Rightarrow f_1(t) = A_1\cos(t) + B_1\sin(t) + C_1 e^{qt}$$

And

$$= \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{Bs+C}{(s^2+1)} + \frac{D}{(s-q)}\right\}$$

$$\Rightarrow A\left(s^2+1\right)(s-q) + Bs(s)(s-q) + C(s)(s-q) + D(s)\left(s^2+1\right) = 1$$

$$\Rightarrow s = q \implies D(q)\left(q^2+1\right) = 1 \implies D = \frac{1}{q\left(q^2+1\right)}$$

$$\Rightarrow s = 0 \implies A\left(-q\right) = 1 \implies A = \frac{1}{-q}$$

$$\Rightarrow A\left(s^2+1\right)(s-q) + Bs(s)(s-q) + C(s)(s-q) + D(s)\left(s^2+1\right) = 1$$

$$\implies As^3 - Aqs^2 + As - Aq + Bs^3 - Bqs^2 + Cs^2 - Cqs + Ds^3 + Ds = 1$$

$$\implies A + B + D = 0, \ -Aq - Bq + C = 0, \ A - Cq + D = 0, \ -Aq = 1$$

$$\implies B = \frac{1}{q} - \frac{1}{q\left(q^2+1\right)}, \ C = -1 + q\left(\frac{1}{q} - \frac{1}{q\left(q^2+1\right)}\right)$$

$$\Rightarrow f_2(t) = A + De^{qt} + C\sin(t) + B\cos(t)$$

So,

$$f(t) = f_1(s) - 2(f_2(s))$$

$$= A_1\cos(t) + B_1\sin(t) + C_1 e^{qt} - 2A - 2De^{qt} - 2C\sin(t) - 2B\cos(t)$$

$$= (A_1 - 2B)\cos(t) + (B_1 - 2C)\sin(t) + (C_1 - 2D)e^{qt} - 2A$$

$$= \left(\frac{1 - \frac{(1-q)\left(1-\frac{1}{q^2+1}\right)}{-q} - \frac{2}{q^2+1}}{(1-q)} - 2\frac{1}{q} + 2\frac{1}{q\left(q^2+1\right)}\right)\cos(t)$$

$$+ \left(\frac{1 - \frac{1}{q^2+1}}{-q} + 2 - 2q\left(\frac{1}{q} - \frac{1}{q\left(q^2+1\right)}\right)\right)\sin(t)$$

$$+ \left(\frac{1}{q^2+1} - 2\frac{1}{q\left(q^2+1\right)}\right)e^{qt} + \frac{2}{q}$$

$$= -\frac{11}{26}\cos(t) - \frac{3}{26}\sin(t) + \frac{3}{130}e^{5t} + \frac{2}{5}$$

(b)

$$= \mathcal{L}^{-1}\left\{\frac{2s+1}{\left(s^2+4\right)\left(s^2+q^2\right)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2s}{\left(s^2+4\right)\left(s^2+q^2\right)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{\left(s^2+4\right)\left(s^2+q^2\right)}\right\}$$

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Which I will again solve seperately and then add

$$= \mathcal{L}^{-1} \left\{ \frac{2s}{(s^2+4)(s^2+q^2)} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{A_1 s + B_1}{(s^2+4)} + \frac{C_1 s + D_1}{(s^2+q^2)} \right\}$$

$$\Rightarrow (A_1 s + B_1)(s^2+q^2) + (C_1 s + D_1)(s^2+4) = 2s$$

$$\implies A_1 s^3 + A_1 q^2 s + B_1 s^2 + B_1 q^2 + C_1 s^3 + 4C_1 s + D_1 s^2 + 4D_1 = 2s$$

$$\implies A_1 + C_1 = 0 \implies A_1 = -C_1$$

$$\implies B_1 + D_1 = 0 \implies B_1 = -D_1$$

$$\implies A_1 q^2 + 4C_1 = 2 \implies A_1 = \frac{2}{q^2-4} \implies C_1 = -\frac{2}{q^2-4}$$

$$\implies B_1 q^2 + 4D_1 = 0 \implies B_1, D_1 = 0$$

$$\Rightarrow f_1(t) = A_1 \cos(2t) + C_1 \cos(qt)$$

And

$$= \mathcal{L}^{-1} \left\{ \frac{1}{(s^2+4)(s^2+q^2)} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{As+B}{(s^2+4)} + \frac{Cs+D}{(s^2+q^2)} \right\}$$

$$\Rightarrow (As+B)(s^2+q^2) + (Cs+D)(s^2+4) = 1$$

$$\implies As^3 + Aq^2 s + Bs^2 + Bq^2 + Cs^3 + 4Cs + Ds^2 + 4D = 1$$

$$\implies A + C = 0 \implies A = -C$$

$$\implies B + D = 0 \implies B = -D$$

$$\implies Aq^2 + 4C = 0 \implies A, C = 0$$

$$\implies Bq^2 + 4D = 1 \implies B = \frac{1}{q^2-4} \implies D = -\frac{1}{q^2-4}$$

$$\Rightarrow \mathcal{L}^{-1} \left\{ \frac{B}{(s^2+4)} + \frac{D}{(s^2+q^2)} \right\}$$

$$\Rightarrow \frac{B}{2} \mathcal{L}^{-1} \left\{ \frac{2}{(s^2+4)} \right\} + \frac{D}{q} \mathcal{L}^{-1} \left\{ \frac{q}{(s^2+q^2)} \right\}$$

$$\Rightarrow f_2(t) = \frac{B}{2} \sin(2t) + \frac{D}{q} \sin(qt)$$

So,

$$f(t) = f_1(t) + f_2(t)$$

$$= A_1 \cos(2t) + C_1 \cos(qt) + \frac{B}{2} \sin(2t) + \frac{D}{q} \sin(qt)$$

$$= \frac{2}{q^2-4} \cos(2t) - \frac{2}{q^2-4} \cos(qt) + \frac{\frac{1}{q^2-4}}{2} \sin(2t) + \frac{-\frac{1}{q^2-4}}{q} \sin(qt)$$

$$= \frac{2}{21} \cos(2t) - \frac{2}{21} \cos(5t) + \frac{1}{42} \sin(2t) - \frac{1}{105} \sin(5t)$$

(c) Here e^{-2s} can be "eliminated" using the theorem

$$\mathcal{L}^{-1} \left\{ e^{-\alpha s} F(s) \right\} = f(t-\alpha)\mu(t-\alpha)$$

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where $\mu(t)$ is the Heaviside function.

$$= \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s-q)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2(s-q)}\right\}\mu(t-2)$$

$$= \mathcal{L}^{-1}\left\{\frac{As+B}{s^2}+\frac{C}{s-q}\right\}\mu(t-2)$$

Then

$$Cs^2 + As(s-q) + B(s-q) = 1$$

$$\Rightarrow s = -q \implies Cq^2 = 1 \implies C = \frac{1}{q^2}$$

$$\Rightarrow s = 0 \implies B(-q) = 1 \implies B = \frac{1}{-q}$$

$$\Rightarrow s = 1 \implies \frac{1}{q^2} + A(1-q) - \frac{1}{q}(1-q) = 1$$

$$\implies A = \frac{1 - \frac{1}{q^2} + \frac{1}{q}(1-q)}{1-q}$$

Then

$$f(t) = A + Bt + Ce^{qt}$$

$$= \frac{1 - \frac{1}{q^2} + \frac{1}{q}(1-q)}{1-q} + Bt + Ce^{qt}$$

$$= -\frac{1}{25} - \frac{1}{5}t + \frac{1}{25}Ce^{5t}$$

So the full solution is

$$\left(-\frac{1}{25} - \frac{1}{5}(t-2) + \frac{1}{25}Ce^{5(t-2)}\right)\mu(t-2)$$

(d) Here the t can be "eliminated" through the theorem $\mathcal{L}\{t^n f(t)\} = (-1)^n \dfrac{d^n F(s)}{ds^n}$ so,

$$= \mathcal{L}\left\{\sin^2(qt)\right\}$$

$$F(s) = \frac{2q^2}{s(s^2+4q^2)} \rightsquigarrow \text{ by table}$$

Now because $n = 1$ here we must differentiate $F(s)$ once and switch its sign,

$$= -\frac{d}{ds}\left[\frac{2q^2}{s(s^2+4q^2)}\right]$$

$$= \frac{2q^2(3s^2+4q^2)}{s^2(s^2+4q^2)^2}$$

$$= \frac{50(3s^2+100)}{s^2(s^2+100)^2}$$

(e) Here the e^{-qt} can be "eliminated" through the theorem $\mathcal{L}\{f(t)e^{at}\} = F(s-a)$ so,

$$= \mathcal{L}\left\{\sin(pt)\sin(qt)\right\}$$

$$= \frac{1}{2}\mathcal{L}\left\{\cos((p-q)t) - \cos((p+q)t)\right\}$$

$$= \frac{1}{2}\mathcal{L}\left\{\cos((p-q)t) - \cos((p+q)t)\right\}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+(p-q)^2} - \frac{s}{s^2+(p+q)^2}\right]$$

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Then, applying the shift,

$$F(s) = \frac{1}{2}\left[\frac{(s+q)}{(s+q)^2 + (p-q)^2} - \frac{(s+q)}{(s+q)^2 + (p+q)^2}\right]$$

$$= \frac{1}{2}\left[\frac{(s+5)}{(s+5)^2 + 16} - \frac{(s+5)}{(s+5)^2 + 196}\right]$$

(f) The expression here can be simplified using the two theorems from the previous questions, $\mathcal{L}\{f(t)e^{at}\} = F(s-a)$ and $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$

$$= \mathcal{L}\left\{e^{-qt}t\sin\left(t - \frac{\pi}{6}\right)\right\}$$

$$= \mathcal{L}\left\{\sin\left(t - \frac{\pi}{6}\right)\right\}$$

$$= \mathcal{L}\left\{\sin(t)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)\cos(t)\right\}$$

$$= \mathcal{L}\left\{\frac{\sqrt{3}}{2}\sin(t) - \frac{1}{2}\cos(t)\right\}$$

$$= \frac{\sqrt{3}}{2}\mathcal{L}\{\sin(t)\} - \frac{1}{2}\mathcal{L}\{\cos(t)\}$$

$$= \frac{\sqrt{3}}{2(s^2+1)} - \frac{s}{2(s^2+1)}$$

Then we need to apply the derivative and shift (I don't think order should matter here),

$$= -\frac{d}{ds}\left[\frac{\sqrt{3}}{2(s^2+1)} - \frac{s}{2(s^2+1)}\right]$$

$$= \frac{s^2 - 2\sqrt{3}s - 1}{2(s^2+1)^2}$$

$$\Rightarrow F(s) = \frac{(s+q)^2 - 2\sqrt{3}(s+q) - 1}{2\left((s+q)^2 + 1\right)^2}$$

$$= \frac{(s+5)^2 - 2\sqrt{3}(s+5) - 1}{2\left((s+5)^2 + 1\right)^2}$$

(g)

$$= \mathcal{L}^{-1}\left\{\ln\left(\frac{7+ps}{9+qs}\right)\right\}$$

$$= \mathcal{L}^{-1}\{\ln(7+ps) - \ln(9+qs)\}$$

$$= \mathcal{L}^{-1}\{\ln(7+ps)\} - \mathcal{L}^{-1}\{\ln(9+qs)\}$$

Here we can use the theorem $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n} \implies f(t) = \frac{(-1)^n}{t^n}\frac{d^n F(s)}{ds^n}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{7+ps}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{9+qs}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{\frac{7}{p}+s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{\frac{9}{q}+s}\right\}$$

$$= e^{-\frac{7}{p}t} - e^{-\frac{9}{q}t}$$

Then, using the theorem

$$= \frac{1}{t}\left[-e^{-\frac{9}{q}t} - e^{-\frac{7}{p}t}\right]$$

$$f(t) = \frac{1}{t}\left[-e^{-\frac{9}{5}t} - e^{-\frac{7}{9}t}\right]$$

Problem 2. Use the **Laplace Transform** method to solve the following initial value problems:

(a) $y'' + 2qy' + q^2 y = t^p e^{-qt}$ $y(0) = y'(0) = 0$

(b) $y'' - 3qy' + 2q^2 y = e^{qt} \cos(pt)$ $y(0) = y'(0) = 0$

Solution 2.

(a)

$$\mathcal{L}\left\{y'' + 2qy' + q^2 y\right\} = \mathcal{L}\left\{t^p e^{-qt}\right\}$$
$$\mathcal{L}\left\{y''\right\} + \mathcal{L}\left\{2qy'\right\} + \mathcal{L}\left\{q^2 y\right\} = \mathcal{L}\left\{t^p e^{-qt}\right\}$$
$$s^2 Y(s) + 2qsY(s) + q^2 Y(s) = \frac{p!}{(s+q)^{p+1}}$$

$$Y(s) = \frac{p!}{\left(s^2 + 2qs + q^2\right)(s+q)^{p+1}}$$

Then

$$y(t) = \mathcal{L}^{-1}\left\{\frac{p!}{\left(s^2 + 2qs + q^2\right)(s+q)^{p+1}}\right\}$$

$$= p!\mathcal{L}^{-1}\left\{\frac{1}{(s+q)^2 (s+q)^{p+1}}\right\}$$

$$= p!\mathcal{L}^{-1}\left\{\frac{1}{(s+q)^{p+3}}\right\}$$

$$= p!e^{-qt}\mathcal{L}^{-1}\left\{\frac{1}{s^{p+3}}\right\}$$

$$= \frac{p!e^{-qt}}{(p+2)!}\mathcal{L}^{-1}\left\{\frac{(p+2)!}{s^{p+3}}\right\}$$

$$= \frac{p!e^{-qt}}{(p+2)!}t^{p+2}$$

$$= \frac{e^{-qt}}{(p+1)(p+2)}t^{p+2}$$

$$= \frac{e^{-5t}}{110}t^{11}$$

(b)

$$\mathcal{L}\left\{y'' - 3qy' + 2q^2 y\right\} = \mathcal{L}\left\{e^{qt}\cos(pt)\right\}$$
$$\mathcal{L}\left\{y''\right\} - \mathcal{L}\left\{3qy'\right\} + \mathcal{L}\left\{2q^2 y\right\} = \mathcal{L}\left\{e^{qt}\cos(pt)\right\}$$
$$Y(s)\left(s^2 - 3qs + 2q^2\right) = \frac{(s-q)}{(s-q)^2 + k^2}$$

$$Y(s) = \frac{(s-q)}{\left(s^2 - 3qs + 2q^2\right)\left((s-q)^2 + p^2\right)}$$

Then,

$$= \mathcal{L}^{-1} \left\{ \frac{1}{(s - 2q)\left((s - q)^2 + p^2\right)} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{A}{s - 2q} + \frac{Bs + C}{s^2 - 2qs + q^2 + p^2} \right\}$$

$$\Rightarrow A\left(s^2 - 2qs + q^2 + p^2\right) + Bs\left(s - 2q\right) + C\left(s - 2q\right) = 1$$

$$\Rightarrow s = 2q \implies A\left(q^2 + p^2\right) = 1 \implies A = \frac{1}{q^2 + p^2}$$

$$\Rightarrow s = 0 \implies 1 + C\left(-2q\right) = 1 \implies C = 0$$

$$\Rightarrow s = 1 \implies \frac{1}{q^2 + p^2}\left(1 - 2q + q^2 + p^2\right) + B\left(1 - 2q\right) = 1 \implies B = \frac{1 - \frac{1}{q^2 + p^2}\left(1 - 2q + q^2 + p^2\right)}{\left(1 - 2q\right)}$$

$$= \mathcal{L}^{-1} \left\{ \frac{A}{s - 2q} + \frac{Bs}{(s - q)^2 + p^2} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{A}{s - 2q} + \frac{B\left(s - q\right)}{(s - q)^2 + p^2} + \frac{Bq}{(s - q)^2 + p^2} \right\}$$

$$= Ae^{2qt} + Be^{qt}\cos\left(pt\right) + \frac{Bq}{p}e^{qt}\sin\left(pt\right)$$

$$= \frac{1}{106}e^{10t} - \frac{1}{106}e^{5t}\cos\left(9t\right) - \frac{5}{954}e^{5t}\sin\left(9t\right)$$