Physics 3610H: Assignment VII

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Problem 1. In class we showed that the eigenfunctions of the harmonic oscillator Hamiltonian have the form $\psi(\xi) = H(\xi)e^{i\xi^2/2}$ where $\xi = \alpha x$. Considering just the even solutions, we expanded $H(\xi)$ in a power series

$$H(\xi) = \sum_{k=0}^{\infty} c_k \xi^{2k},$$

and we derived a recursion relation for the coefficients of this series

$$c_{k+1} = \frac{4k + 1 - \lambda}{2\,(k+1)\,(2k+1)} c_k.$$

Use this recursion relation to calculate $H_6(\xi)$. Be sure to scale your result, following con- vention, such that the coefficient of ξ^6 is 2^6.

Solution 1. We want terms up to and including ξ^6 so we want k up to and including $k = 3$. This means we'll have four terms total so $\lambda = 4 \cdot 3 + 1 = 13$. Now applying the recursion relation,

$$\begin{aligned}
c_0 &= c_0 \\
c_1 &= \frac{4 \cdot 0 + 1 - 13}{2\,(0+1)\,(0+1)} c_0 = -6c_0 \\
c_2 &= \frac{4 + 1 - \lambda}{2\,(2)\,(3)} c_1 = 4c_0 \\
c_3 &= \frac{8 + 1 - \lambda}{2\,(3)\,(5)} c_2 = -\frac{8}{15} c_0.
\end{aligned}$$

Higher terms are zero as c_4 is zero, as we wanted. Now we use that

$$H_6(\xi) = \sum_{k=0}^{3} c_k \xi^{2k} = c_0 + c_1\xi^2 + c_2\xi^4 + c_3\xi^6.$$

by plugging in the obtained c_ks:

$$H_6(\xi) = c_0 - 6c_0\xi^2 + 4c_0\xi^4 - \frac{8}{15}c_0\xi^6.$$

We want $-\frac{8}{15}c_0 = 2^6 \implies c_0 = -120$ so we have

$$H_6(\xi) = -120 + 720\xi^2 - 480\xi^4 + 64\xi^6.$$

Problem 2. Now consider the odd series

$$H(\xi) = \sum_{k=0}^{\infty} d_k \xi^{2k+1}.$$

Using the equation for $H(\xi)$ which we derived in class, derive the recursion relation for the coefficients in this series. Explain your steps. You should obtain

$$d_{k+1} = \frac{4k+3-\lambda}{2\left(k+1\right)\left(2k+3\right)} d_k.$$

Solution 2. In class we found that

$$H(\xi) = \sum_{k=0}^{\infty} d_k \xi^{2k+1}.$$

We also found that

$$\frac{d^2 H}{d\xi} - 2\xi \frac{dH}{\xi} + H\left(\lambda - 1\right) = 0.$$

Setting up to plug in to our expression for H,

$$\frac{dH}{d\xi} = \sum_{k=0}^{\infty} \left(2k+1\right) d_k \xi^{2k}$$

$$\frac{d^2 H}{d\xi^2} = \sum_{k=0}^{\infty} 2k\left(2k+1\right) d_k \xi^{2k-1}.$$

So we have

$$\sum_{k=0}^{\infty} 2k\left(2k+1\right) d_k \xi^{2k-1} - 2\xi \sum_{k=0}^{\infty} \left(2k+1\right) d_k \xi^{2k} + \sum_{k=0}^{\infty} d_k \xi^{2k+1}\left(\lambda - 1\right) = 0.$$

We are trying to use Frobenius' method so we need to pull out the dependence on ξ and turn all the coefficients into one big coefficient so we can use the fact that

$$\sum_{k=0}^{\infty} C_k \xi^k = 0 \forall \xi \implies C_k = 0 \forall k.$$

We can start by simplifying a bit to obtain

$$\sum_{k=0}^{\infty} 2k\left(2k+1\right) d_k \xi^{2k-1} - 2\sum_{k=0}^{\infty} \left(2k+1\right) d_k \xi^{2k+1} + \sum_{k=0}^{\infty} d_k \xi^{2k+1}\left(\lambda - 1\right) = 0$$

which means that it's only the first term,

$$\sum_{k=0}^{\infty} 2k\left(2k+1\right) d_k \xi^{2k-1},$$

preventing us from pulling out a ξ^{2k+1}. To get around this we first note that the first term of this sum will be zero because of the $2k$ out front so we can change the starting index without issue

$$\sum_{k=1}^{\infty} 2k\left(2k+1\right) d_k \xi^{2k-1}.$$

Now this is the tricky bit: We want to find a mapping for $k \to k'$ so that $\xi^{2k-1} \to \xi^{2k'+1}$. Obviously we can do this by setting $k' = k - 1$. We are actually able to make this change of index in the summation because the new sum

$$\sum_{k'=0}^{\infty} 2\left(k'+1\right)\left(2k'+3\right) d_{k'+1} \xi^{2k'+1}$$

is the same as the old sum except we've just called the index something different. The indices still line up so it's no problem to recombine the sums as

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} 2(k+1)(2k+3)\, d_{k+1}\xi^{2k+1} - 2\sum_{k=0}^{\infty}(2k+1)\, d_k\xi^{2k+1} + \sum_{k=0}^{\infty} d_k\xi^{2k+1}(\lambda-1) \\ &= \sum_{k=0}^{\infty}\left[2(k+1)(2k+3)\, d_{k+1} - 2(2k+1)\, d_k + d_k(\lambda-1)\right]\xi^{2k+1}. \end{aligned}$$

As we know this means that, for all k,

$$2(k+1)(2k+3)\, d_{k+1} - 2(2k+1)\, d_k + d_k(\lambda-1) = 0$$

and so

$$d_{k+1} = \frac{4k+3-\lambda}{2(k+1)(2k+3)} d_k$$

Problem 3. Using Dirac notation:

(a) If $\hat{A}$ and $\hat{B}$ are two linear operators show that $\left(\hat{A}\hat{B}\right)^{\dagger} = \hat{B}^{\dagger}\hat{A}^{\dagger}$.

(b) If $\hat{C}$ and $\hat{D}$ are two Hermitian operators, is the product $\hat{C}\hat{D}$ always Hermitian? (You may use your result from (a)).

Solution 3.

(a) We know that for states ψ and χ $\langle\psi|\,\hat{A}\,|\chi\rangle = \left\langle \hat{A}^{\dagger}\psi \middle| \chi\right\rangle$. Applying this then to the product we obtain

$$\langle\psi|\,\hat{A}\hat{B}\,|\chi\rangle = \langle\psi|\,\hat{A}\,|\chi'\rangle$$

where $|\chi'\rangle = \hat{B}\,|\chi\rangle$. Then,

$$\langle\psi|\,\hat{A}\,|\chi'\rangle = \left\langle \hat{A}^{\dagger}\psi \middle| \chi'\right\rangle = \langle\psi'|\,\hat{B}\,|\chi\rangle = \hat{B}^{\dagger}\,\langle\psi'|\chi\rangle\,.$$

Here we've written $\left\langle \hat{A}^{\dagger}\psi \right| = \langle\psi'|$. undoing this we obtain

$$\langle\psi|\,\hat{A}\hat{B}\,|\chi\rangle = \hat{B}^{\dagger}\,\langle\psi'|\chi\rangle = \hat{B}^{\dagger}\hat{A}^{\dagger}\,\langle\psi|\chi\rangle\,.$$

Because $|\psi\rangle$ and $|\chi\rangle$ are generic we can say then that $\left(\hat{A}\hat{B}\right)^{\dagger} = \hat{B}^{\dagger}\hat{A}^{\dagger}$.

(b) As we know from part (a) linear operators in general satisfy $(\hat{C}\hat{D})^{\dagger} = \hat{D}^{\dagger}\hat{C}^{\dagger}$. If this product were Hermitian we could say that $(\hat{C}\hat{D})^{\dagger} = \hat{C}\hat{D}$ because we could remove the † directly by saying $\hat{C}\hat{D} = \hat{F}$ and $\hat{F}$ Hermitian would mean that $\hat{F}^{\dagger} = \hat{F}$. Provided these operators do not commute then we have no guarantee that the product will be Hermitian.