Physics 2130: Assignment III

Jeremy Favro

November 17, 2024

Problem 1. Consider the driven, damped harmonic oscillator. Its equation of motion is:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 = A\cos(\omega t)$$

In the case of an underdamped oscillator, i.e., $\beta < \omega_0^2$ we found that the solution for the equation of motion is:

$$x(t) = x_c(t) + x_p(t)$$

Where:

$$x_c(t)e^{-\beta t}\left[c_1 e^{i\omega_1 t} + c_2 e^{-i\omega_1 t}\right] = Be^{-\beta t}\cos(\omega_1 t - \phi) = \Gamma(t)s(t)$$

And where:

$$\omega_1 = \sqrt{w_0^2 - \beta^2}$$
$$\Gamma(t) = Be^{-\beta t}$$
$$s(t) = (\omega_1 t - \phi)$$

The particular solution is instead:

$$x_p(t) = D\cos(\omega t - \delta)$$

Where:

$$D = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}}$$
$$\delta = \arctan\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right)$$

Solution 1.

$$\begin{aligned}
&x(t=0) = B\cos(-\phi) + D\cos(-\delta) = x_0 \\
&\Rightarrow x_0 - D\cos(\delta) = B\cos(\phi) \\
&\Rightarrow \frac{x_0 - D\cos(\delta)}{\cos(\phi)} = B
\end{aligned}$$

$$\begin{aligned}
&\dot{x}(t=0) = -B\omega_1\sin(-\phi) - B\beta\cos(-\phi) - D\omega\sin(-\delta) = 0 \\
&= B\omega_1\sin(\phi) - B\beta\cos(\phi) + D\omega\sin(\delta) \\
&= \frac{x_0 - D\cos(\delta)}{\cos(\phi)}\omega_1\sin(\phi) - \frac{x_0 - D\cos(\delta)}{\cos(\phi)}\beta\cos(\phi) + D\omega\sin(\delta) \\
&= (x_0 - D\cos(\delta))\,\omega_1\tan(\phi) - (x_0 - D\cos(\delta))\,\beta + D\omega\sin(\delta) \\
&\Rightarrow -\frac{D\omega\sin(\delta)}{(x_0 - D\cos(\delta))} = \omega_1\tan(\phi) - \beta \\
&\Rightarrow \arctan\left(\frac{\beta - \frac{D\omega\sin(\delta)}{(x_0 - D\cos(\delta))}}{\omega_1}\right) = \phi
\end{aligned}$$