Math 2350H: Assignment II

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Problem 1. Let $T \in \mathcal{L}(V)$. Prove that $T^2 = T_0$ iff $\operatorname{range} T \subseteq \operatorname{null} T$.

Solution 1. Assuming

$$\begin{aligned}
&\operatorname{range} T \subseteq \operatorname{null} T \\
\implies& T = T_0 \\
\implies& T(v \in V) = 0_V \\
\implies& T(T(v)) = T^2 = 0_V \; (T : V \mapsto V) \\
\implies& T^2 = T_0
\end{aligned}$$

In the other direction assuming

$$\begin{aligned}
&T^2 = T_0 \\
\implies& T(T(v \in V)) = 0_V \\
\implies& T(v' \in V) = 0_V \; (T : V \mapsto V) \\
\implies& \operatorname{range} T \subseteq \operatorname{null} T
\end{aligned}$$

$\therefore T^2 = T_0 \iff \operatorname{range} T \subseteq \operatorname{null} T$

Problem 2. Let U, V, W be vector spaces over field F, and let $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$.

(a) Show that if $T \circ S$ is injective, then S is injective.

(b) Give an example showing that if $T \circ S$ is injective then T need not be injective.

(c) Show that if $T \circ S$ is surjective, then T is surjective.

(d) Give an example showing that if $T \circ S$ is surjective then S need not be surjective.

Solution 2.

(a) Under the assumption that $T \circ S$ is injective we start by assuming, by way of contradiction, that S is *not* injective. This means

$$\exists a, b \in U,\ a \neq b \text{ such that } S(a) = S(b)$$

Because $S \circ T$ is valid it follows that

$$T(S(a)) = T(S(b))$$

However we assumed that $T \circ S$ is injective but because $a \neq b$ here the composition is not injective, a contradiction. Therefore, to avoid said contradiction, S must be injective.

(b) For $T : \mathbb{R}^2 \mapsto \mathbb{R}$ such that $T(x, y) = x$ and $S : \mathbb{R} \mapsto \mathbb{R}^2$ such that $S(x) = (x, 0)$. Here T is not injective as $(1, 1)$ and $(1, 2)$ both map to 1. S is injective by inspection as any x will generate a unique $(x, 0)$. The composition $T \circ S : \mathbb{R} \mapsto \mathbb{R}^2$ gives $T(S(x)) = x$ which is injective as any input x will map to itself, which is unique.

(c) Under the assumption that $T \circ S$ is surjective we are guaranteed that for all $b \in W \exists a \in U$ such that $T(S(a)) = b$ (every element of the co-domain maps to at least one element of the domain). Because we know that $S(a) \in V$ we can denote $S(a) = c \in V$ now $T(c)$ is surjective by our initial assumption which means that T itself is surjective.

(d) For $S : \mathbb{R} \mapsto \mathbb{R}^3$ such that $S(x) = (x, x, x)$ and $T : \mathbb{R}^3 \mapsto \mathbb{R}$ such that $T(x, y, z) = x$ S is not surjective as $\operatorname{range} S = \operatorname{span} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \neq \mathbb{R}^3$ whereas T is surjective as $\operatorname{range} T = \operatorname{span}(1) = \mathbb{R}$. The entire map is surjective as it is the identity map.

Problem 3. Let $x_0, x_1, \dots, x_n$ be $n+1$ distinct real numbers. For $i = 0, \dots, n$, the set of polynomials $p_0, \dots, p_n$, defined by

$$p_i(x) = \frac{(x - x_0)(x - x_1)\dots(x - x_{i-1})(x - x_{i+1})\dots(x - x_n)}{(x_i - x_0)(x_i - x_1)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)}$$

are called the *Lagrange polynomials* associated to $x_0, x_1, \dots, x_n$.

(a) Find the Lagrange polynomials associated to $-5, -1, 0, 3$.

(b) Notice that $p_i(x_j) = 0$ for $i \neq j$ and $p_i(x_i) = 1$ (verify for yourself). Use this to prove that the set of Lagrange polynomials $p_0, \dots, p_n$ associated to $x_0, x_1, \dots, x_n$, is a basis for $\mathcal{P}_n(\mathbb{R})$.

(c) Let $(x_0, y_0), \dots, (x_n, y_n)$ be $n+1$ points in the plane $\mathbb{R}^2$, where $x_0, x_1, \dots, x_n$ are distinct. Verify that the polynomial

$$f(x) = y_0p_0(x) + \dots + y_np_n(x)$$

is the unique polynomial in $\mathcal{P}_n(\mathbb{R})$ which passes through these $n+1$ points.

(d) Use the Lagrange polynomials found in part (a) to find a polynomial of degree at most 3 passing through

$$(-5, 4), (-1, 1), (0, 2), (3, -3).$$

(show how you obtain your answer).

(e) Suppose $g \in \mathcal{P}_n(\mathbb{R})$ and $g(r_i) = 0$ for $n+1$ distinct scalars $r_0, r_1, \dots, r_n$. Use the results above to show that g is the zero polynomial. This shows that a nonzero polynomial of degree n cannot have more than n distinct roots.

Solution 3.

(a)

$$\begin{aligned}
p_0(x) &= \frac{(x+1)(x)(x-3)}{(-5+1)(-5)(-5-3)} = \frac{x^3 - 2x^2 - 3x}{-160} \\
p_1(x) &= \frac{(x+5)(x)(x-3)}{(-1+1)(-1)(-1-3)} = \frac{x^3 + 2x^2 - 15x}{16} \\
p_2(x) &= \frac{(x+1)(x+5)(x-3)}{(5)(1)(-3)} = \frac{x^3 + 3x^2 - 13x}{-15} + 1 \\
p_3(x) &= \frac{(x+1)(x+5)(x)}{(3+5)(3+1)(3)} = \frac{x^3 + 6x^2 + 5x}{96}
\end{aligned}$$

(b) For a set to form a basis it must be both linearly independent and a spanning set. Proving linear independence here is sufficient to imply the set is a spanning set as it means we have a linearly independent set of dimension $n+1$ for $\mathcal{P}_n$. For a set to be linearly independent we have

$$c_0p_0(x) + \dots c_np_n(x) = 0 \implies c_0 = \dots = c_n = 0$$

If we evaluate the above expression for every x_i $i \leq n$ we obtain n expressions where all polynomials $p_j(x_i) = 0$ for $i \neq j$ and a single polynomial $p_i(x_i) = 1$. This single polynomial has coefficient c_i which, to satisfy the previous expression for linear independence must be 0. So, applying this to each x_i we obtain that all scalars $c_0, \dots c_n$ must be zero. This implies linear independence which as mentioned above implies span due the nature of polynomial spaces which then implies that the set of Lagrange polynomials generated by distinct real numbers $x_0, x_1, \dots, x_n$ is a basis for $\mathcal{P}_n(\mathbb{R})$.

(c) Again using the fact that $p_i(x_i) = 1$ it is evident that $y_ip_i(x_i) = y_i$ and so because we are also guaranteed that $p_i(x_j) = 0$ for $i \neq j$ $\sum_{i=0}^{n} y_ip_i(x_i)$ will be y_i at all corresponding x_i because only a single polynomial in the sum will be nonzero at that point. Uniqueness is guaranteed here due to the uniqueness of linear combinations.

(d) From part (c) we simply sum all of the polynomials found in part (a) and multiply them by the given y-coordinate

$$\begin{aligned}
&= 4 \cdot \frac{x^3 - 2x^2 - 3x}{-160} + 1 \cdot \frac{x^3 + 2x^2 - 15x}{16} + 2 \cdot \left(\frac{x^3 + 3x^2 - 13x}{-15} + 1 \right) - 3 \cdot \frac{x^3 + 6x^2 + 5x}{96} \\
&= \frac{-61x^3 - 198x^2 + 343x}{480} + 2
\end{aligned}$$

(e) Because the Lagrange polynomials associated with $r_0, r_1, \ldots, r_n$ form a basis for $\mathcal{P}_n(\mathbb{R})$ we can express $g(x)$ as a linear combination of these polynomials, $g(x) = \sum_{j=0}^{n} c_j p_j(x)$, which will have all zero terms except one when evaluating at $x = r_j$, but this point is zero by our definition of $g(x)$ which implies that $c_j = 0$ therefore $g(x)$ is a linear combination of n zeros, e.g. the zero polynomial $g(x) = 0$.