Math 2350H: Assignment I

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Problem 1. Demonstrate that there does not exist $\lambda \in \mathbb{C}$ such that

$$\lambda(2-3i, 5+4i, -6+i) = (12-5i, 7+22i, -32-9i)$$

Solution 1. Let $\lambda = a + bi$ then

$$\begin{aligned}
&\lambda(2-3i, 5+4i, -6+i) = (12-5i, 7+22i, -32-9i) \\
&\implies (2a+3b)+(-3a+2b)i = 12-5i \wedge (5a-4b)+(4a+5b)i = 7+22i \wedge (-6a-1b)+(1a-6b)i = -32-9i
\end{aligned}$$

Here the determinant of the matrix of the coefficients of i is

$$\begin{vmatrix} -3 & 2 & -5 \\ 4 & 5 & 2 \\ 1 & -6 & -9 \end{vmatrix} = -3(5\cdot(-9) - 22\cdot(-6)) + (-1)2(4\cdot(-9) - 1\cdot 22) - 5(4\cdot(-6) - 1\cdot 5) = 0$$

so there exist no a, b which satisfy all three coefficients of i and therefore there exists no λ that satisfies the original equation.

Problem 2. Let $V = \mathbb{R}^2$. If (x_1, x_2) and (y_1, y_2) are elements of V , and $\alpha \in \mathbb{R}$, define

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 y_2),$$

and

$$\alpha \cdot (x_1, x_2) = (\alpha x_1, x_2).$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Solution 2. For V to be a vector space over $\mathbb{R}$ with these operations the following must hold $\forall (x_1, y_1) \in V$, $\alpha \in \mathbb{R}$:

(i) $+$ must be commutative and associative

(ii) $\cdot$ must be associative

(iii) $0 \in V$

(iv) There must exist a multiplicative identity for scalar multiplication

(v) There must exist an additive inverse

(vi) The additive and multiplicative distributive law must hold

(i) Commutativity: Let $(x_1, x_2), (y_1, y_2) \in V$.
Given that $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 y_2)$,

$$\begin{aligned}
(y_1, y_2) + (x_1, x_2) &= (y_1 + x_1, y_2 x_2) \text{ (by definition)} \\
(y_1, y_2) + (x_1, x_2) &= (x_1 + y_1, x_2 y_2) \text{ (by commutativity in } \mathbb{R}) \therefore + \text{ under } V \text{is commutative}
\end{aligned}$$

Associativity: Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in V$.
Given that $(x_1, x_2) + ((y_1, y_2) + (z_1, z_2)) = (x_1, x_2) + (y_1 + z_1, y_2 z_2) = (x_1 + y_1 + z_1, x_2 y_2 z_2)$

$$\begin{aligned}
&= ((x_1, x_2) + (y_1, y_2)) + (z_1, z_2) \\
&= (x_1 + y_1, x_2 y_2) + (z_1, z_2) \text{ (by definition)} \\
&= (z_1, z_2) + (x_1 + y_1, x_2 y_2) \text{ (by previously demonstrated commutativity)} \\
&= (z_1 + x_1 + y_1, z_2 x_2 y_2) \\
&= (x_1 + y_1 + z_1, x_2 y_2 z_2) \text{ (by commutativity in } \mathbb{R}) \therefore + \text{ under } V \text{is associative}
\end{aligned}$$

(ii) Associativity: Let $(x_1, x_2) \in V$ and $\alpha, \beta \in \mathbb{R}$
Given that, by definition, $(\alpha \cdot \beta) \cdot (x_1, x_2) = (\alpha \cdot \beta \cdot x_1, x_2)$

$$
\begin{aligned}
&= \alpha \cdot (\beta \cdot (x_1, x_2)) \\
&= \alpha \cdot (\beta \cdot x_1, x_2) \\
&= (\alpha \cdot \beta \cdot x_1, x_2) \therefore \cdot \text{ under } V \text{is associative}
\end{aligned}
$$

(iii) Let $(x_1, x_2) \in V$

$$
\begin{aligned}
&= (x_1, x_2) + (0, 1) \,(\text{by handwaving}) \\
&= (x_1 + 0, x_2 \cdot 1) \,(\text{by definition}) \\
&= (x_1, x_2) \therefore (0, 1) = 0_V
\end{aligned}
$$

(iv) Let $(x_1, x_2) \in V$

$$
\begin{aligned}
&= 1 \cdot (x_1, x_2) \\
&= (1 \cdot x_1, x_2) \,(\text{by definition}) \\
&= (x_1, x_2) \therefore 1 \text{ is the multiplicative identity}
\end{aligned}
$$

(v) Let $v, v' \in V$. We are looking to show that $v + v' = 0_V$. By handwaving suppose that the additive inverse exists and is $v' = \left(-x_1, \frac{1}{y_1}\right)$ then,

$$
\begin{aligned}
v + v' &= (x_1, y_1) + \left(-x_1, \frac{1}{y_1}\right) \\
&= \left(x_1 - x_1, y_1 \frac{1}{y_1}\right) \text{ by definition} \\
&= (0, 1) \therefore \text{by rules for addition and multiplication under } \mathbb{R}, \left(-x_1, \frac{1}{y_1}\right) \text{ is the additive inverse}
\end{aligned}
$$

(vi) Additive: Let $v_1, v_2 \in V$, $\alpha \in \mathbb{R}$

$$
\begin{aligned}
&= \alpha(v_1 + v_2) \\
&= \alpha((x_1, y_1) + (x_2, y_2)) \\
&= \alpha((x_1 + x_2, y_1 y_2)) \,(\text{by definition}) \\
&= (\alpha(x_1 + x_2), y_1 y_2) \,(\text{by definition})
\end{aligned}
$$

Then,

$$
\begin{aligned}
&= \alpha v_1 + \alpha v_2 \\
&= \alpha(x_1, y_1) + \alpha(x_2, y_2) \\
&= (\alpha x_1, y_1) + (\alpha x_2, y_2) \,(\text{by definition}) \\
&= (\alpha x_1 + \alpha x_2, y_1 y_2) \,(\text{by definition}) \\
&= (\alpha(x_1 + x_2), y_1 y_2) \therefore \text{ the distributive law holds}
\end{aligned}
$$

Multiplicative: Let $v \in V$, $\alpha, \beta \in \mathbb{R}$

$$
\begin{aligned}
&= \alpha v + \beta v \\
&= (\alpha x_1, y_1) + (\beta x_1, y_1) \,(\text{by definition}) \\
&= (\alpha x_1 + \beta x_1, y_1 y_1) \,(\text{by definition}) \\
&= ((\alpha + \beta) x_1, y_1 y_1)
\end{aligned}
$$

Then,

$$
\begin{aligned}
&= (\alpha + \beta) v \\
&= ((\alpha + \beta) x_1, y_1) \,(\text{by definition}) \\
&= (\alpha x_1 + \beta x_1, y_1) \therefore \text{ the distributive law does not hold}
\end{aligned}
$$

$\therefore V$ is not a vector space over $\mathbb{R}$ with the given operations.

Problem 3. Let $V = \mathbb{R}^2$. If (x_1, y_1) and (x_2, y_2) are elements of V, and $\alpha \in \mathbb{R}$, define

$$(x_1, y_1) + (x_2, y_2) = (x_1 + 2x_2, y_1 + 3y_2),$$

and

$$\alpha \cdot (x_1, y_1) = (\alpha x_1, \alpha y_1).$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Solution 3. Nicely, this one fails on the first one I checked, additive commutativity: Let $(x_1, x_2), (y_1, y_2) \in V$. Given that $(x_1, x_2) + (y_1, y_2) = (x_1 + 2x_2, y_1 + 3y_2)$,

$$(y_1, y_2) + (x_1, x_2) = (x_2 + 2x_1, y_2 + 3y_1) \therefore + \text{ is not commutative}$$

$\therefore V$ is not a vector space over $\mathbb{R}$ with the given operations.

Problem 4. Which of the following sets are subspaces of $\mathbb{R}^3$ under the usual operations of addition and scalar multiplication in $\mathbb{R}^3$.

(a) $W_1 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 = x_2 \text{ and } x_3 = -x_2\right\}$.

(b) $W_2 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 - 4x_2 - x_3 = 0\right\}$.

(c) $W_3 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 | 5x_1^2 - 3x_2^2 + 6x_3^2 = 0\right\}$.

Solution 4. The subspace criterion state that a non-empty subset S is a subspace of V (with scalars F) iff $\forall v_1, v_2 \in S, \alpha \in F$

(i) $0_V \in S$

(ii) $v_1 + v_2 \in S$

(iii) $\alpha v_1 \in S$

Now,

(a) (i) $0_{\mathbb{R}^3} \in W_1$ because $x_1 = 0 \implies x_2 = 0 \implies x_3 = 0$

(ii) Let $v_1, v_2 \in W_1$,

$$\begin{aligned}
&= v_1 + v_2 \\
&= (x_1, x_2, x_3) + (y_1, y_2, y_3) \\
&= (x_2, x_2, -x_2) + (y_2, y_2, -y_2) \text{ (by definition)} \\
&= (x_2 + y_2, x_2 + y_2, -x_2 - y_2)
\end{aligned}$$

Then, by the definition of W_1 addition here is closed as $x_2 + y_2 = x_2 + y_2$ and $-(x_2 + y_2) = -x_2 - y_2$

(iii) Let $v_1 \in W_1$, $\alpha \in \mathbb{R}$,

$$\begin{aligned}
&= \alpha v_1 \\
&= \alpha(x_1, x_2, x_3) \\
&= (\alpha x_1, \alpha x_2, \alpha x_3) \\
&= (\alpha x_2, \alpha x_2, -\alpha x_2)
\end{aligned}$$

Which satisfies the definition of W_1 so W_1 is a subspace of $\mathbb{R}^3$ as it satisfies the subspace criterion.

(b) (i) $0_{\mathbb{R}^3} \in W_2$ because $x_1 = x_2 = x_3 = 0 \implies 0 - 4 \cdot 0 - 0 = 0$

(ii) Let $v_1, v_2 \in W_2$,

$$\begin{aligned}
&= v_1 + v_2 \\
&= (x_1, x_2, x_3) + (y_1, y_2, y_3) \\
&= (x_1 + y_1, x_2 + y_2, x_3 + y_3)
\end{aligned}$$

For this to be in W_2 it must satisfy the requirement $(x_1 + y_1) - 4(x_2 + y_2) - (x_3 + y_3) = 0$,

$$\begin{aligned}
&= (x_1 + y_1) - 4(x_2 + y_2) - (x_3 + y_3) \\
&= x_1 + y_1 - 4x_2 - 4y_2 - x_3 - y_3 \\
&= (x_1 - 4x_2 - x_3) + (y_1 - 4y_2 - y_3) \\
&= (0) + (0) = 0\,(v_1, v_2 \in W_2)
\end{aligned}$$

So this set is closed under addition.

(iii) Let $v_1 \in W_2$, $\alpha \in \mathbb{R}$,

$$\begin{aligned}
&= \alpha v_1 \\
&= (\alpha x_1, \alpha x_2, \alpha x_3)
\end{aligned}$$

Then, again, we must satisfy $\alpha x_1 - 4\alpha x_2 - \alpha x_3 = 0$,

$$\begin{aligned}
&= \alpha x_1 - 4\alpha x_2 - \alpha x_3 \\
&= \alpha(x_1 - 4x_2 - x_3) \\
&= \alpha(0) = 0
\end{aligned}$$

So W_2 is a subspace of $\mathbb{R}^3$ as it satisfies the subspace criterion.

(c) (i) $0_{\mathbb{R}^3} \in W_3$ because $x_1 = x_2 = x_3 = 0 \implies 5 \cdot 0^2 - 3 \cdot 0^2 + 6 \cdot 0^2 = 0$

(ii) Let $v_1, v_2 \in W_3$,

$$\begin{aligned}
&= v_1 + v_2 \\
&= (x_1, x_2, x_3) + (y_1, y_2, y_3) \\
&= (x_1 + y_1, x_2 + y_2, x_3 + y_3)
\end{aligned}$$

For this to be in W_3 it must satisfy the requirement $5(x_1 + y_1)^2 - 3(x_2 + y_2)^2 + 6(x_3 + y_3)^2 = 0$,

$$\begin{aligned}
&= 5(x_1 + y_1)^2 - 3(x_2 + y_2)^2 + 6(x_3 + y_3)^2 \\
&= 5(x_1^2 + 2x_1y_1 + y_1^2) - 3(x_2^2 + 2x_2y_2 + y_2^2) + 6(x_3^2 + 2x_3y_3 + y_3^2) \\
&= 5x_1^2 + 10x_1y_1 + 5y_1^2 - 3x_2^2 - 6x_2y_2 - 3y_2^2 + 6x_3^2 + 12x_3y_3 + 6y_3^2 \\
&= (5x_1^2 - 3x_2^2 + 6x_3^2) + (5y_1^2 - 3y_2^2 + 6y_3^2) + 10x_1y_1 - 6x_2y_2 + 12x_3y_3 \\
&= 10x_1y_1 - 6x_2y_2 + 12x_3y_3
\end{aligned}$$

Which is not necessarily equal to 0 (e.g. $v_1 = v_2 = (1, 1, 1)$) so this set is not closed under addition and therefore not a subspace of $\mathbb{R}^3$.