Calculus II: Assignment 8

Jeremy Favro

March 14, 2024

Problem 1. Verify that the series $\sum_{n=0}^{\infty} \frac{2}{(4n+1)(4n+3)}$ converges using one or more of the convergence tests given in class.

Solution 1. Using the integral test which states that for some series $\sum_{n=k}^{\infty} f(n)$ converges or diverges as $\int_k^{\infty} f(x)\,dx$ converges or diverges. In our case we have $f(n) = \frac{2}{(4n+1)(4n+3)}$ which can be directly converted to a continuous function $f(x) = \frac{2}{(4x+1)(4x+3)}$ so

$$
\begin{aligned}
&= \int_0^{\infty} \frac{2}{(4x+1)(4x+3)}\,dx \\
&= \lim_{t\to\infty} \int_0^{t} \frac{2}{(4x+1)(4x+3)}\,dx \\
&= \lim_{t\to\infty} \int_0^{t} \frac{A}{(4x+1)} + \frac{B}{(4x+3)}\,dx \\
&= \lim_{t\to\infty} \int_0^{t} \frac{4Ax+3A+4Bx+B}{(4x+1)(4x+3)}\,dx \\
&= \lim_{t\to\infty} \int_0^{t} \frac{4(A+B)x+(3A+B)}{(4x+1)(4x+3)}\,dx \rightsquigarrow 3A+B=2,\ A+B=0 \implies A=-B,\ B=-1,\ A=1 \\
&= \lim_{t\to\infty} \left[\int_0^{t} \frac{1}{(4x+1)}\,dx - \int_0^{t} \frac{1}{(4x+3)}\,dx\right] \\
&= \frac{1}{4}\lim_{t\to\infty} \left[\ln(4x+1) - \ln(4x+3)\Big|_0^t\right] \\
&= \frac{1}{4}\lim_{t\to\infty} \left[\ln(4t+1) - \ln(4t+3) - (\ln(4(0)+1) - \ln(4(0)+3))\right] \\
&= \frac{1}{4}\lim_{t\to\infty} \left[\cancelto{0}{\ln\left(\frac{4t+1}{4t+3}\right)} - \ln\left(\frac{1}{3}\right)\right] \\
&= -\frac{1}{4}\ln\left(\frac{1}{3}\right)
\end{aligned}
$$

∴ Because the integral converges the series must also converge.

Problem 2. Use SageMath to to find the sum of the series in question 1

Solution 2.

[1]:
```
clear_vars()

n = var('n')

sum(2/((4*n+1)*(4*n+3)), n, 0, oo)
```

[1]: `1/4*pi`

Problem 3. What series involving powers of x has $\frac{1}{1+x^2}$ as its sum? For which values of x does this series converge?

Solution 3. Using the power series formula $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ we can tell that $a = 1$, so we just need to find a suitable r s.t. $\frac{1}{1-r} = \frac{1}{1+x^2}$ then plug that into the series.

$$= \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$$

$$\therefore \sum_{n=0}^{\infty} (-x^2)^n \text{ is the series of powers of } x \text{ which is equivalent to } \frac{1}{1+x^2}$$

The series converges for values of x which satisfy $\left|-x^2\right| < 1$, so any value on the interval $[-1, 1]$ (potentially excluding the endpoints). Testing the endpoints -1 and 1 to ensure the series doesn't also converge there: $\sum_{n=0}^{\infty}(-1^2)^n$ will diverge per the divergence test as $\lim_{n\to\infty}(-1^2)^n \neq 0$. The same is true for $\sum_{n=0}^{\infty}(1^2)^n$ as $\lim_{n\to\infty}(1^2)^n \neq 0$.

$$\therefore \sum_{n=0}^{\infty} (-x^2)^n \text{ is the series of powers of } x \text{ which is equivalent to } \frac{1}{1+x^2} \text{ which converges on } (-1, 1)$$

Problem 4. Since $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$ what series involving powers of x should be equal to $\arctan(x)$ when it converges? For which values of x does this series converge?

Solution 4. Knowing that $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$, we also know that $\arctan(x) = \int \frac{1}{1+x^2}\,dx$. Applying that to the series which resolves to $\frac{1}{1+x^2}$:

$$= \int \sum_{n=0}^{\infty} \left[(-x^2)^n\right] dx = \arctan(x)$$

$$= \sum_{n=0}^{\infty} \left[\int (-x^2)^n \, dx\right] = \arctan(x)$$

$$\therefore \sum_{n=0}^{\infty} \left[\int (-x^2)^n \, dx\right] = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \arctan(x)$$

Testing for convergence using the ratio test which we have yet to do in lecture but I think I've figured it out from the archive page:

$$\begin{aligned}
&= \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}x^{2n+2}}{2n+2}}{\frac{(-1)^n x^{2n+1}}{2n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}x^{2n+2}}{2n+2} \frac{2n+1}{(-1)^n x^{2n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \frac{2n+1}{2n+2} \frac{x^{2n+2}}{x^{2n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \frac{2n+1}{2n+2} \frac{x^{2n+2}}{x^{2n+1}} \right| \\
&= |(-1)(1)(x)| \\
&= |x|
\end{aligned}$$

So, by the ratio test the series converges for values of $|x| < 1$, so any value on the interval $[-1, 1]$ (potentially excluding the endpoints). Testing the endpoints for convergence:

Setting $x = 1$ yields $\frac{(-1)^n(1)^{2n+1}}{2n+1} = (-1)^n \frac{1}{2n+1}$ where $a_n = \frac{1}{2n+1}$ which is: always decreasing past $n = 0$, always greater than 0 past $n = 0$, and $\lim_{n\to\infty} \frac{1}{2n+1} = 0$. So by the alternating series test the series converges at $x = 1$.

Doing the same for $x = -1$ yields $\frac{(-1)^n(-1)^{2n+1}}{2n+1} = (-1)^{3n+1} \frac{1}{2n+1}$ where $a_n = \frac{1}{2n+1}$ which is always positive and nonzero past $n = 0$, decreasing past $n = 0$, and $\lim_{n\to\infty} \frac{1}{2n+1} = 0$ which means that by the alternating series test, the series converges on $[-1, 1]$.

$$\therefore \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \text{ is the series of powers of } x \text{ which is equivalent to } \arctan(x) \text{ which converges on } [-1, 1]$$

Problem 5. Given that $\arctan(1) = \frac{\pi}{4}$, what is the connection between the series in questions 1 and 4?

Solution 5. Let S_1 be the series in question 1 $\left(\sum_{n=0}^{\infty} \frac{2}{(4n+1)(4n+3)}\right)$ and S_2 be the series in question 4 $\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}\right)$ $\arctan(1) = \frac{\pi}{4}$ means that S_2 converges to $\frac{\pi}{4}$ when $x = 1$. S_1 always converges to $\frac{\pi}{4}$. There's a couple connections here but they all feel too obvious to warrant a question of their own. Just in case I'll list them anyway:

1. Both S_1 and S_2 are equal and converge when $x = 1$
2. Both S_1 and S_2 can be used to approximate π
3. $S_1 \pm S_2$ is convergent

Other than those three, obvious as they might be, I can't think of any other connection between the two