Physics 2605H: Assignment II

Jeremy Favro
Trent University, Peterborough, ON, Canada

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Problem 1. Two vectors are represented as

$$|\psi\rangle = \begin{pmatrix} 3+4i \\ 4+3i \end{pmatrix}, \qquad |\phi\rangle = \begin{pmatrix} 2+i \\ 1+2i \end{pmatrix}$$

(a) Find their scalar product, $\langle\phi|\psi\rangle$

(b) Find the products

  (i) $|\phi\rangle\langle\psi|$

  (ii) $|\psi\rangle\langle\phi|$

(c) Do (i) and (ii) commute?

(d) Find

  (i) The inner product of $|\psi\rangle$

  (ii) The inner product of $|\phi\rangle$

  (iii) The outer product of $|\phi\rangle$

  (iv) The outer product of $|\psi\rangle$

Solution 1.

(a)

$$\begin{aligned}\langle\phi|\psi\rangle &= \langle\phi|\psi\rangle = |\phi\rangle^\dagger |\psi\rangle \\ &= \begin{pmatrix} 2-i & 1-2i \end{pmatrix}\begin{pmatrix} 3+4i \\ 4+3i \end{pmatrix} \\ &= 20\end{aligned}$$

(b) Find the products

  (i) $\langle\psi| = |\psi\rangle^\dagger = \begin{pmatrix} 3-4i & 4-3i \end{pmatrix}$ So,

$$\begin{aligned}|\phi\rangle\langle\psi| &= \begin{pmatrix} 2+i \\ 1+2i \end{pmatrix}\begin{pmatrix} 3-4i & 4-3i \end{pmatrix} \\ &= \begin{pmatrix} (2+i)(3-4i) & (2+i)(4-3i) \\ (1+2i)(3-4i) & (1+2i)(4-3i) \end{pmatrix} \\ &= \begin{pmatrix} 10-5i & 11-2i \\ 11+2i & 10+5i \end{pmatrix}\end{aligned}$$

  (ii)

$$\begin{aligned}|\psi\rangle\langle\phi| &= \begin{pmatrix} 3+4i \\ 4+3i \end{pmatrix}\begin{pmatrix} 2-i & 1-2i \end{pmatrix} \\ &= \begin{pmatrix} (2-i)(3+4i) & (1-2i)(4+3i) \\ (2-i)(3+4i) & (1-2i)(4+3i) \end{pmatrix} \\ &= \begin{pmatrix} 10+5i & 11-2i \\ 11+2i & 10-5i \end{pmatrix}\end{aligned}$$

(c) I'm not quite sure what this is asking as this isn't an area where commutativity can be defined. They aren't equal but they also aren't the commuted forms of each other.

(d) Find

(i)

$$\begin{aligned}\langle\psi|\psi\rangle &= \begin{pmatrix}3-4i & 4-3i\end{pmatrix}\begin{pmatrix}3+4i\\4+3i\end{pmatrix}\\&= 50\end{aligned}$$

(ii)

$$\begin{aligned}\langle\phi|\phi\rangle &= \begin{pmatrix}2-i & 1-2i\end{pmatrix}\begin{pmatrix}2+i\\1+2i\end{pmatrix}\\&= 10\end{aligned}$$

(iii)

$$\begin{aligned}|\phi\rangle\langle\phi| &= \begin{pmatrix}2+i\\1+2i\end{pmatrix}\begin{pmatrix}2-i & 1-2i\end{pmatrix}\\&= \begin{pmatrix}|2+i|^2 & (2+i)(1-2i)\\(2-i)(1+2i) & |1+2i|^2\end{pmatrix}\\&= \begin{pmatrix}5 & 4-3i\\4+3i & 5\end{pmatrix}\end{aligned}$$

(iv)

$$\begin{aligned}|\psi\rangle\langle\psi| &= \begin{pmatrix}3+4i\\4+3i\end{pmatrix}\begin{pmatrix}3-4i & 4-3i\end{pmatrix}\\&= \begin{pmatrix}|3+4i|^2 & (3+4i)(4-3i)\\(3-4i)(4+3i) & |4+3i|^2\end{pmatrix}\\&= \begin{pmatrix}25 & 24+7i\\24-7i & 25\end{pmatrix}\end{aligned}$$

Problem 2.

(a) Using the results from 1(d), normalize $|\psi\rangle$ and $|\phi\rangle$ so that they belong to Hilbert space. Here write about what is Hilbert space.

(b) Identify which of the following belong to Hilbert space and state why?

(i) $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$

(ii) $|\psi\rangle = \frac{4}{5}|+\rangle + \frac{3}{5}|-\rangle$

Solution 2.

(a) Because a Hilbert space is a collection of orthonormal vectors (and these vectors are already orthogonal) we need to multiply them by the inverse of their inner product. So $|\psi\rangle_H = \frac{1}{5\sqrt{2}}\begin{pmatrix} 3+4i \\ 4+3i \end{pmatrix}$ and $|\phi\rangle_H = \frac{1}{\sqrt{10}}\begin{pmatrix} 2+i \\ 1+2i \end{pmatrix}$

(b) Identify which of the following belong to Hilbert space and state why?

(i) Because this is already in an orthonormal basis we only have to care about the magnitude of the vector which is (by inspection ;)) 1. So this vector belongs to a vector space.

(ii) Again this is an orthonormal basis so all we care about is if $\langle\phi|\phi\rangle = 1$,

$$\begin{aligned} \langle\phi|\phi\rangle &= \frac{1}{2}\begin{pmatrix} \frac{7}{5} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} \frac{7}{5} \\ \frac{1}{5} \end{pmatrix} \\ &= 1 \end{aligned}$$

So this vector is also part of a Hilbert space.

Problem 3.

(a) Would $|0\rangle$ and $|+\rangle$ together satisfy the criteria for a valid basis? Why?

(b) Review Stern and Gerlach Apparatus measurements from worksheet #2. To measure the difference between an electron in a spin state $|+\rangle$ and $|-\rangle$, which of the following one could use? Explain.

(I) A horizontal SGA.

(II) A vertical SGA.

(III) A 45° diagonal SGA.

Solution 3.

(a) Yes because they are both linearly independent (by inspection) and a spanning set as the matrix

$$\begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

has dimension 2 which implies that the vectors span $\mathbb{R}^2$.

(b) Because spin states are presented in basis $|\uparrow\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and $|\downarrow\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$ changing to basis $|+\rangle$ and $|-\rangle$ means we would need an angle such that $\sin(\theta) = \cos(\theta) = \frac{1}{\sqrt{2}} \implies \theta = \frac{\pi}{4}\text{rad} = 45°$.

Problem 4.

(a) Using the link given, obtain the mathematical form of a qubit with $\theta = \frac{2\pi}{3}$ and $\phi\frac{3\pi}{2}$

(b) Also find the corresponding orthonormal state for the state of qubit in (a).

(c) Show that the inner products of the two states lead to 1.

(d) Find their outer product.

Solution 4.

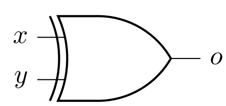
(a) $|\psi\rangle = \cos\left(\frac{\pi}{3}\right)|\uparrow\rangle + e^{i\frac{3\pi}{2}}\sin\left(\frac{\pi}{3}\right)|\downarrow\rangle = \frac{1}{2}|\uparrow\rangle - \frac{1}{2}|\downarrow\rangle$

(b) $|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ so the vector is already orthonormal (as expected as we used the Bloch sphere to create it).

(c) $\left[\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\right]\left[\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right] = \frac{1}{2} + \frac{1}{2} = 1$

(d) $|\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Problem 5.



(a) The figure shows two input XOR gate. Tabulate the outputs when, the inputs or (0,1), (1,0), (1,1), and (0,1) **assuming typo**

(b) Compare the functionality of XOR gate with AND gate and OR gate.

(c) Which gate is called as the Universal gate and why?

Solution 5.

(a)

x	y	o
0	1	1
1	0	1
1	1	0
0	0	0

(b) Exclusive or (XOR) is either but not both, AND is both, OR is either.

x	y	XOR	AND	OR
0	1	1	0	1
1	0	1	0	1
1	1	0	1	1
0	0	0	0	0

(c) NAND and NOR gates are sometimes called "universal" gates as any other gate can be constructed from some number of either.

Problem 6.

(a) Use matrix multiplication to show how applying an $\hat{X}$ gate flips:

(i) A qubit in the $|0\rangle$ state

(ii) A qubit in the general state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

(b) Use matrix multiplication to show how applying the $\hat{Z}$ gate to $|+\rangle$ changes it to $|-\rangle$

(c) Show that Hadamard gate is unitary and hence reversible.

Solution 6.

(a) Use matrix multiplication to show how applying an X gate flips:

(i)

$$\begin{aligned}\hat{X}|0\rangle &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}\\ &= \begin{pmatrix}(1)(0)+(1)(0)\\ (1)(1)+(0)(0)\end{pmatrix} & &= |1\rangle\end{aligned}$$

(ii)

$$\begin{aligned}\hat{X}|\psi\rangle &= \hat{X}\alpha|0\rangle + \hat{X}\beta|1\rangle\\ &= \alpha\hat{X}|0\rangle + \beta\hat{X}|1\rangle \rightsquigarrow \text{ skipping as I'm in a rush and this is trivial}\\ &= \alpha|1\rangle + \beta|0\rangle\end{aligned}$$

(b)

$$\begin{aligned}\hat{Z}|+\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}(1)(1)+(1)(0)\\ (1)(0)+(1)(-1)\end{pmatrix}\\ &= |-\rangle\end{aligned}$$

(c)

$$\begin{aligned}\hat{H}\hat{H} &= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}(1)(1)+(1)(1) & (1)(1)+(1)(-1)\\ (1)(1)+(-1)(1) & (1)(1)+(-1)(-1)\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\ &= I\end{aligned}$$

Problem 7. $|\psi\rangle = \frac{4}{5}|+\rangle + \frac{3}{5}|-\rangle$ is measured in basis $\{|0\rangle, |1\rangle\}$

(a) Perform the necessary basis change for the given state.

(b) What is the probability that the particle is in $|1\rangle$ state?

Solution 7. $|\psi\rangle = \frac{4}{5}|+\rangle + \frac{3}{5}|-\rangle$ is measured in basis $\{|0\rangle, |1\rangle\}$

(a)

$$\begin{aligned}&= \frac{4}{5}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{3}{5}\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\\ &= \frac{7\sqrt{2}}{10}|0\rangle + \frac{\sqrt{2}}{10}|1\rangle\end{aligned}$$

(b) $P(|1\rangle) = \left(\frac{\sqrt{2}}{10}\right)^2 = \frac{1}{50} = 0.02$