Diff. Eq. Cheat Sheet

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Revision 3

Fundamentals

Classification

$\frac{d^n y}{dx^n} = f(x,y)$ denotes an ODE of order n. Note that $(\frac{dy}{dx})^n \neq \frac{d^n y}{dx^n}$. ODEs of order n will have n constants in their general form solutions.
A linear ODE is one which can be written in the form $a_n(x)\frac{d^n y}{d^n x} + a_{n-1}\frac{d^{n-1} y}{d^{n-1} x} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$.

Solutions

Given some IVP $\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0$ if f and $\frac{\partial f}{\partial y}$ are continuous in the rectangle $(x_0, y_0) \in \{(x,y) : a < x < b, c < y < d\}$ then the IVP has a unique solution $\phi(x)$ in some interval $(x_0 - h, x_0 + h),\ h \geq 0$

Solution Techniques $n = 1$

Direct Integration

Directly integrate ...

Seperable

For some ODE $\frac{dy}{dx} = f(x,y) = g(x)p(y)$ the differential can be split $s.t.$ $\frac{1}{p(y)}dy = g(x)dx$ which can be solved by direct integration. Note that when dividing by some function we assume that the function is nonzero. If there is a case (e.g. in an IVP) where the function is zero, the solution is lost.

Linear

For some linear ODE of the form $\frac{dy}{dx} + P(x)y = Q(x)$ we can multiply both sides of the ODE by $\mu(x) = \exp\left(\int P(x)\,dx\right)$ to obtain $\mu\frac{dy}{dx} + \mu P(x)y = \mu Q(x)$ which gives

$$y = \frac{\int \mu(x)Q(x)\,dx + C}{\mu}$$

Exact

Exact equations are ODEs of the form $M dx + N dy = 0$ where $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Then,
$f(x,y) = \int M\,dx + h(y) = C$ or
$f(x,y) = \int N\,dy + g(x) = C$ and
$\frac{d}{dy}\left(\int M\,dx + h(y)\right) = N$ or
$\frac{d}{dx}\left(\int N\,dy + g(x)\right) = M$

Non-Exact

In cases where something looks exact but $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ you can find an integrating factor

$$\mu(x) = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\,dx\right)$$

$$\mu(y) = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\,dy\right)$$

Homogeneous

If each term of the ODE is of equal order (e.g. the right hand side can be expressed as a function of only $\frac{y}{x}$) we can substitute $y = ux \implies dy = udx + xdu$. This should result in a seperable equation.

Bernoulli

If we have an equation of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$ we divide by y^n and substitute $u = y^{1-n} \implies \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ (you should know what $\frac{dy}{du}$ is here). This should result in a linear equation.

Linear Substituion

An ODE of the form $\frac{dy}{dx} = f(Ax + By + C),\ B \neq 0$ can be solved by

$$u = Ax + By + C$$
$$\Rightarrow \frac{du}{dx} = A + B\frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{dx} = (\frac{du}{dx} - A)\frac{1}{B}$$

Solution Techniques $n = 2$

Reduction of Order

If you solve a second order ODE and obtain a single solution y_1, $y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}dx$ where $P(x)$ is found in $y'' + P(x)y' + Q(x)y = g(x)$

Constant Coefficients

An equation of the form $ay'' + by' + cy = 0$ can be solved through the characteristic equation obtained by substituting $y = e^{mt}$ and solving for m. This gives a solution of the form $y_h = C_1 e^{m_1 t} + C_2 e^{m_2 t}$.

Undetermined Coefficients

To obtain the particular solution of $ay'' + by' + cy = g(x)$ we try

$g(x)$	$y_p(x)$
$Ce^{\alpha x}$	$x^s\left(Ae^{\alpha x}\right)$
$C_n x^n + \cdots + C_1 x + C_0$	$x^s\left(A_n x^n + \cdots + A_1 x + A_0\right)$
$C\cos(\beta x),\quad C\sin(\beta x)$	$x^s\left(A\cos(\beta x) + A_1\sin(\beta x)\right)$
$\left(C_n x^n + \cdots + C_1 x + C_0\right)e^{\alpha x}$	$x^s\left(A_n x^n + \cdots + A_1 x + A_0\right)e^{\alpha x}$

Variation of Parameters

For $y'' + P(x)y' + Q(x)y = g(x)$ if you have the homogeneous solutions $y_1(x)$ and $y_2(x)$, the particular solution $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ where

$$u_1(x) = -\int \frac{g(x)y_2(x)}{W\left[y_1, y_2\right]}dx$$

$$u_2(x) = \int \frac{g(x)y_1(x)}{W\left[y_1, y_2\right]}dx$$

where $W\left[y_1, y_2\right]$ is the Wronskian,

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Cauchy-Euler

An equation of the form $ax^2y'' + bxy' + cy = g(x)$ can be solved through the characteristic equation $am^2 + (b-a)m + c = 0$ obtained by substituting $y = x^{m66}$ and solving for m. In the case where m is complex here you end up with trig functions of logarithms.

Laplace Transform

If $f(t)$ has period T and is piecewise continuous on $[0, T]$ then $\mathcal{L}\left\{f(t)\right\} = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{sT}}$

Properties of the Laplace Transform

$\mathcal{L}\left\{f_1 + f_2\right\} = \mathcal{L}\left\{f_1\right\} + \mathcal{L}\left\{f_2\right\}$
$\mathcal{L}\left\{cf_1\right\} = c\mathcal{L}\left\{f_1\right\}$
$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$
$\mathcal{L}\left\{f'(t)\right\} = s\mathcal{L}\left\{f(t)\right\} - f(0)$
$\mathcal{L}\left\{f''(t)\right\} = s^2\mathcal{L}\left\{f(t)\right\} - sf(0) - f'(0)$
$\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n\frac{d^n F(s)}{ds^n} \implies f(t) = \frac{(-1)^n}{t^n}\mathcal{L}^{-1}\left\{\frac{d^n F(s)}{ds^n}\right\}$
$\mathcal{L}\left\{f(t-a)\mu(t-a)\right\} = e^{-as}F(s)$
$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)\mu(t-a)$

Solving Discontinuous IVPs with Laplace Transforms

For some ODE $ay'' + by' + cy = g(t)$ $\mathcal{L}\left\{g(t)\mu(t-a)\right\} = e^{-as}\mathcal{L}\left\{g(t+a)\right\}$

$$\mu(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$

Applications

Newton's Cooling

$$\frac{dT}{dt} = k(T - T_m) \implies T(t) = T_m + Ce^{kt}$$

where T is the temperature of an object, T_m the temperature of the medium in which the object sits, and k some cooling constant determined by initial/boundary conditions. C comes about as a result of solving the ODE and can also be determined using initial conditions.

Miscellaneous

Partial Fractions

$\frac{px+q}{(x-a)(x-b)} \to \frac{A}{x-a} + \frac{B}{x-b}$
$\frac{px+q}{(x-a)^2} \to \frac{A}{x-a} + \frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} \to \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)} \to \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)(x^2-bx+c)} \to \frac{A}{x-a} + \frac{B}{(x^2-bx+c)}$

Systems

$x' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = Ax$ Then guess that $x = e^{\lambda t} \implies A\vec{v} = \lambda\vec{v}$. To solve for eigenvalues find $det\,(A - \lambda I) = 0$. Then solve for eigenvectors $\vec{v_n}$ for each λ_n with $(A - \lambda_n I)\,\vec{v_n} = 0$. For real, distinct eigenvector-value pairs write $y(t) = C_1 e^{\lambda_1 t}\vec{v_1} + C_2 e^{\lambda_2 t}\vec{v_2}$. For a repeated eigenvalue write $y(t) = C_1 e^{\lambda_1 t}\vec{v_1} + C_2 e^{\lambda_1 t}(\vec{v_2} + t\vec{v_1})$ where $A\vec{v_2} - \lambda_1\vec{v_2} = \vec{v_1}$

Partial Differential Equations

Classification

If the coefficient of the highest derivative contains derivatives only up to the previous order then the PDE is called Quasilinear. However, if the coefficient of the highest derivative is not a function of the unknown or any of its derivatives then the PDE is called Semilinear. A PDE is Fully Nonlinear if it is nonlinear in the highest order.

Method of Characteristics

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Canonical Forms

For
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$
$\Delta = B^2 - 4AC$

$$\begin{cases} \Delta < 0 \implies & \text{Elliptic} \\ \Delta = 0 \implies & \text{Parabolic } A_1 = B_1 = 0 \\ \Delta > 0 \implies & \text{Hyperbolic } A_1 = C_1 = 0 \end{cases}$$

We transform to $A_1 u_{\xi\xi} + B_1 u_{\xi\eta} + C_1 u_{\eta\eta} + D_1 u_\xi + E_1 u_\eta + F_1 u = G_1$ with $A_1 = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$
$B_1 = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$
$C_1 = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$
$D_1 = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$
$E_1 = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$
$F_1 = F(\xi, \eta)$
$G_1 = G(\xi, \eta)$

Fourier Series

For some function $f(x)$ periodic with period $2L$:
$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$
$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$
$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$
and
$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$

Heat Equation

An equation of the form $u_{xx} + g = \frac{1}{k}u_t$ where g represents heat generation/loss inside the bar. We solve first for a steady state condition where $u_t = 0$ by writing $u = w(x,t) + v(x)$ and saying that $w(x, t \to \infty) = 0$. We then solve the homogenous problem for w where our initial conditions become zero and we subtract our steady-state solution v from the initial temperature distribution. We'll probably apply separation of variables to solve for the transient, w but that's easy.

Laplacians

Cartesian

$$\nabla^2 V = V_{xx} + V_{yy} + V_{zz} = 0.$$

Spherical

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}.$$

With Azimuthal symmetry this is solved by $V = \sum_n (A_n r^n + B_n/(r^{n+1}))P_n(\cos(\theta))$

Cylindrical

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial S}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}.$$

Bessel Functions

Normal

An ODE of the form $x^2\phi_{xx} + x\phi_x + (n^2 - \lambda^2 x^2)\phi = 0$ is solved by the Bessel functions $\phi_n = C_1 J_n(\lambda x) + C_2 Y_n(\lambda x)$. Generally $C_2 = 0$ as $Y_n(0) \to -\infty$.

Modified

An ODE of the form $x^2\phi_{xx} + x\phi_x - (n^2 + \lambda^2 x^2)\phi = 0$ is solved by the Bessel functions $\phi_n = C_1 I_n(\lambda x) + C_2 K_n(\lambda x)$. Generally $C_2 = 0$ as $K_n(0) \to \infty$. Note that $I_n(\infty) \to \infty$ as well.

Self-Adjoint Operators and Sturm-Liouville

A Sturm-Liouville problem is given by $L[\phi] + \lambda w(x)\phi(x) = 0$. L is a self-adjoint linear differential operator. For $[\phi] = P_0\phi_{xx} + P_1\phi_x + P_2\phi$ this means that $P_0' = P_1$. Distinct ϕ_n are orthogonal with respect to $w(x)$. The coefficients of the eigenfunction expansion for $f(x)$ are $a_n = \int_a^b w(x)f(x)\phi_n(x)\,dx / \int_a^b w(x)\phi_n^2(x)\,dx$. We can transform a non-self-adjoint L to a self adjoint one by multiplying by $P_0^{-1}\exp(\int P_1(x)/P_0(x)\,dx)$

Cartesian Laplacian Solution

In 2D the cartesian Laplacian for a rectangular membrane with $V(0,y) = g_1(y), V(L,y) = g_2(y)$ and $V(x,0) = f_1(x), V(x,H) = f_2(x)$ we get the solution $V = \sum_n [(a_n \sinh(n\pi(H-y)/L) + b_n \sinh(n\pi y/L))$ with $a_n = \frac{2}{L\sin(n\pi H/L)}\int_0^L f_1(x)\sin(n\pi x/L)\,dx$ and the other coefficients following the same theme. Those for y are with $g_{1,2}$ and are over y with H and L swapped.

TDNH

$A + x/L(B - A)$,
$\bar{Q} = Q - A' - x/L(B' - A')$