Math 3310H: Assignment I

Jeremy Favro (0805980)
Trent University, Peterborough, ON, Canada

September 25, 2025

Problem 1. Define a relation $\mathbb{R} \times \mathbb{R}$ by $(a, b) \sim (c, d)$ if $2(a - c) - 3(b - d) = 0$

(a) Show that $\sim$ is an equivalence relation on $\mathbb{R}$.

(b) Give an example of two pairs $(a, b), (c, d) \in \mathbb{R} \times \mathbb{R}$, which lie in the same equivalence class, and two pairs that don't.

(c) This equivalence relation partitions the 2D plane $\mathbb{R} \times \mathbb{R}$ into subregions. What does the equivalence class (a, b) look like as a region of the plane?

Solution 1.

(a) For $\sim$ to be an equivalence relation it must satisfy the following properties for a set S (proofs included)

(i) Reflexivity: $x \sim x \, \forall x \in S$.

Proof. Let $(a, b) \in \mathbb{R} \times \mathbb{R}$, then

$$\begin{aligned} &(a, b) \overset{?}{\sim} (a, b) \\ \implies &2(a - a) - 3(b - b) = 0 \end{aligned}$$

Which satisfies our relation as defined. Therefore the relation is reflexive. □

(ii) Symmetry: $x \sim y \implies y \sim x \, \forall x, y \in S$

Proof. Let $(a, b), (c, d) \in \mathbb{R} \times \mathbb{R}$, then

$$\begin{aligned} &(a, b) \sim (c, d) \\ \implies &2(a - c) - 3(b - d) = 0 \\ \implies &2(a - c) = 3(b - d) \\ \implies &-2(a - c) = -3(b - d) \\ \implies &2(c - a) = 3(d - b) \\ \implies &2(c - a) - 3(d - b) = 0 \\ \implies &(c, d) \sim (a, b) \end{aligned}$$

□

(iii) Transitivity: $x \sim y \sim z \implies x \sim z \, \forall x, y, z \in S$

Proof. Let $(a, b), (c, d), (e, f) \in \mathbb{R} \times \mathbb{R}$, then

$$\begin{aligned}
&(a, b) \sim (c, d) \\
\implies &2(a - c) - 3(b - d) = 0
\end{aligned}$$

and

$$\begin{aligned}
&(c, d) \sim (e, f) \\
\implies &2(c - e) - 3(d - f) = 0
\end{aligned}$$

so

$$\begin{aligned}
&2(a - c) - 3(c - d) + 2(c - e) - 3(d - f) = 0 \\
\implies &2(a - c + c - e) - 3(b - d + d - f) = 0 \\
\implies &2(a - e) - 3(b - f) = 0 \\
\implies &(a, b) \sim (e, f)
\end{aligned}$$

□

Therefore $\sim$ is an equivalence relation on $\mathbb{R} \times \mathbb{R}$.

(b) For representative element $(1, 1)$ we get that for an element $(a, b) \in \mathbb{R} \times \mathbb{R}$ to belong to the associated equivalence class we must have

$$2(1 - a) - 3(1 - b) = 0$$

which can be rearranged to obtain

$$a = -\frac{1 - 3b}{2}$$

so for $b = \pm 1$ we get two members of the equivalence class represented by $(1, 1)$ under $\sim$, $(1, 1)$ and $(-2, 1)$. The elements (π, e) and (ϕ, i^i) where π, e take on their usual definitions, ϕ is the golden ratio and i^i is, interestingly, both transcendental *and* real!

(c) The equivalence class with representative (a, b) is the set $E = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x \sim (a, b)\}$. This gives the equation

$$2(a - x) - 3(b - y) = 0 \implies y = \frac{2(a - x) - 3b}{-3}$$

so the class looks like a line with slope $2/3$ and y-intercept $b - 2a/3$

Problem 2. For each of the following sets S, determine whether S is closed under addition modulo n, or multiplication modulo n, or both or neither. (Addition and multiplication modulo n are defined in Exercise Set 2).

(a) $S = \{0, 4, 8, 12\}, n = 16$.

(b) $S = \{0, 3, 6, 9, 12\}, n = 15$.

(c) $S = \{1, 2, 3, 4\}, n = 5$.

(d) $S = \{0, 2, 3, 4, 6, 8, 9, 10\}, n = 12$.

(e) $S = \{1, 5, 7, 11\}, n = 12$.

Solution 2.

(a)

$+_{16}$	0	4	8	12
0	0	4	8	12
4	4	8	12	0
8	8	12	0	4
12	12	0	4	8

$\cdot_{16}$	0	4	8	12
0	0	0	0	0
4	0	0	0	0
8	0	0	0	0
12	0	0	0	0

That these tables, being every possible combination of elements on each set with their respective operations contain no elements not members of S means that both are closed under $+_{16}$ and $\cdot_{16}$.

(b)

$+_{15}$	0	3	6	9	12
0	0	3	6	9	12
3	3	6	9	12	0
6	6	9	12	0	3
9	9	12	0	3	6
12	12	0	3	6	9

$\cdot_{15}$	0	3	6	9	12
0	0	0	0	0	0
3	0	9	3	12	6
6	0	3	6	9	12
9	0	12	9	6	3
12	0	6	12	3	9

Again because these tables contain only elements of S S is closed under both of their respective operations.

(c)

$+_5$	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

$\cdot_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Here because $0 \notin S \implies S$ is not closed under $+_5$ but is closed under $\cdot_5$ for the same reasons as previously.

(d)

$+_{12}$	0	2	3	4	6	8	9	10
0	0	2	3	4	6	8	9	10
2	2	4	5	6	8	10	11	0
3	3	5	6	7	9	11	0	1
4	4	6	7	8	10	0	1	2
6	6	8	9	10	0	2	3	4
8	8	10	11	0	2	4	5	6
9	9	11	0	1	3	5	6	7
10	10	0	1	2	4	6	7	8

$\cdot_{12}$	0	2	3	4	6	8	9	10
0	0	0	0	0	0	0	0	0
2	0	4	6	8	0	4	6	8
3	0	6	9	0	6	0	3	6
4	0	8	0	4	0	8	0	4
6	0	0	6	0	0	0	6	0
8	0	4	0	8	0	4	0	8
9	0	6	3	0	6	0	9	6
10	0	8	6	4	0	8	6	4

Here because $1, 5, 7, 11 \notin S$ S is not closed under $+_{12}$ but is closed under $\cdot_{12}$ for the same reasons as previously.

(e)

$+_{12}$	1	5	7	11
1	2	6	8	0
5	6	10	0	4
7	8	0	2	6
11	0	4	6	10

$\cdot_{12}$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

Here because $0, 2, 4, 6, 8, 10 \notin S$ S is not closed under $+_{12}$ but is closed under $\cdot_{12}$ for the same reasons as previously.

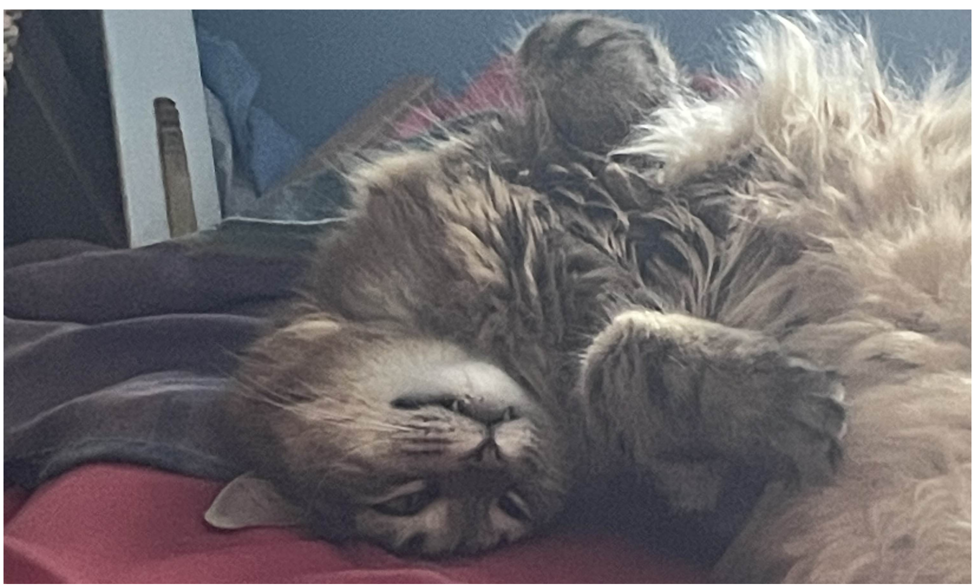

Figure 1: Space Filling Cat Picture

Problem 3. Determine whether the given binary operation $*$ is commutative, associative, both or neither. Justify your answers with proof.

(a) The operation $*$ on $\mathbb{Z}$ given by $a * b = a + b + ab$

(b) The operation $*$ on $\mathbb{R}$ given by $a * b = a + b - ab$

(c) The operation $*$ on $\mathbb{R}$ given by $a * b = a + 2ab$

(d) The operation $*$ on $\mathbb{Z} \times \mathbb{Z}$ given by $(a, b) * (c, d) = (ad + bc, bd)$

(e) The operation $*$ on $\mathbb{Z} \times \mathbb{Z}$ given by $(a, b) * (c, d) = (ad, bc)$

Solution 3.

(a) For commutativity,

Proof. Let $a, b \in \mathbb{Z}$, then

$$\begin{aligned} a * b &= a + b + ab \\ &= b + a + ba && \text{Commutativity of } + \text{ and } - \text{ on } \mathbb{Z} \\ &= b * a && \text{Definition of } * \end{aligned}$$

Therefore $*$ is commutative. $\square$

For associativity,

Proof. Let $a, b, c \in \mathbb{Z}$, then

$$\begin{aligned} a * (b * c) &= a * (b + c + bc) \\ &= a + b + c + bc + a(b + c + bc) \\ &= a + b + c + bc + ab + ac + abc && \cdot \text{ distributive on } \mathbb{Z} \end{aligned}$$

and

$$\begin{aligned} (a * b) * c &= (a + b + ab) * c \\ &= a + b + ab + c + (a + b + ab)c \\ &= a + b + ab + c + ac + bc + abc && \cdot \text{ distributive on } \mathbb{Z} \\ &= a + b + c + bc + ab + ac + abc && + \text{ commutative on } \mathbb{Z} \end{aligned}$$

Because the two are equal we have associativity. $\square$

(b) For commutativity,

Proof. Let $a, b \in \mathbb{Z}$, then

$$\begin{aligned} a * b &= a + b - ab \\ &= b + a - ba && \text{Commutativity of } + \text{ and } - \text{ on } \mathbb{Z} \\ &= b * a && \text{Definition of } * \end{aligned}$$

Therefore $*$ is commutative. ☐

For associativity,

Proof. Let $a, b, c \in \mathbb{Z}$, then

$$\begin{aligned} a * (b * c) &= a * (b + c - bc) \\ &= a + b + c - bc + a(b + c - bc) \\ &= a + b + c - bc + ab + ac - abc && \cdot \text{ distributive on } \mathbb{Z} \end{aligned}$$

and

$$\begin{aligned} (a * b) * c &= (a + b - ab) * c \\ &= a + b - ab + c + (a + b - ab)c \\ &= a + b - ab + c + ac + bc - abc && \cdot \text{ distributive on } \mathbb{Z} \end{aligned}$$

Because of the difference in sign on the ab terms these two cannot be made to be equal, therefore $*$ is not associative. ☐

(c) For commutativity,

Proof. Let $a, b \in \mathbb{Z}$, then

$$a * b = a + 2ab \neq b * a = b + 2ba$$

Therefore $*$ is not commutative here. ☐

For associativity,

Proof. Let $a, b, c \in \mathbb{Z}$, then

$$\begin{aligned} a * (b * c) &= a * (b + 2bc) \\ &= a + 2a(b + 2bc) \\ &= a + 2ab + 4abc && \cdot \text{ distributive on } \mathbb{Z} \end{aligned}$$

and

$$\begin{aligned} (a * b) * c &= (a + 2ab) * c \\ &= c + 2c(a + 2ab) \\ &= c + 2ca + 4cab && \cdot \text{ distributive on } \mathbb{Z} \end{aligned}$$

Which cannot be manipulated to be equal, therefore $*$ is not associative here. ☐

(d) For commutativity,

Proof. Let $(a,b),(c,d) \in \mathbb{Z} \times \mathbb{Z}$, then

$$\begin{aligned}
(a,b) * (c,d) &= (ad + bc, bd) \\
&= (cb + da, db) && \text{Associativity of } + \text{ and } \cdot \text{ on } \mathbb{Z} \\
&= (c,d) * (a,b) && \text{Definition of } * \text{ in reverse}
\end{aligned}$$

Therefore $*$ is commutative. $\square$

For associativity,

Proof. Let $(a,b),(c,d),(e,f) \in \mathbb{Z} \times \mathbb{Z}$, then

$$\begin{aligned}
(a,b) * ((c,d) * (e,f)) &= (a,b) * (cf + de, df) \\
&= (adf + b(cf + de), bdf) \\
&= (adf + bcf + bde, bdf) && \cdot\text{distributive on } \mathbb{Z}
\end{aligned}$$

and

$$\begin{aligned}
((a,b) * (c,d)) * (e,f) &= (ad + bc, bd) * (e,f) \\
&= ((ad + bc)f + bde, bdf) \\
&= (fad + fbc + bde, bdf) && \cdot\text{distributive on } \mathbb{Z} \\
&= (adf + bcf + bde, bdf) && \cdot\text{commutative on } \mathbb{Z}
\end{aligned}$$

As these two are equal, $*$ is associative. $\square$

(e) For commutativity,

Proof. Let $(a,b),(c,d) \in \mathbb{Z} \times \mathbb{Z}$, then

$$(a,b) * (c,d) = (ad, bc) \neq (c,d) * (a,b) = (cb, da).$$

$\square$

Therefore $*$ is not commutative. For associativity,

Proof. Let $(a,b),(c,d),(e,f) \in \mathbb{Z} \times \mathbb{Z}$, then

$$\begin{aligned}
(a,b) * ((c,d) * (e,f)) &= (a,b) * (cf, de) \\
&= (ade, bcf)
\end{aligned}$$

and

$$\begin{aligned}
((a,b) * (c,d)) * (e,f) &= (ad, bc) * (e,f) \\
&= (adf, bce)
\end{aligned}$$

As these two cannot be made equal $*$ is not associative. $\square$

Problem 4. Let S be a nonempty set. A binary algebraic structure $(S, *)$ is called a semigroup if $*$ is associative.

(a) Let S be the set of positive rational numbers. Show that $(S, *)$ is a commutative semigroup if

$$a * b = \frac{ab}{a+b}$$

(the usual operations on the right) for all $a, b \in S$

(b) Let S be a set containing more than one element. Define

$$a * b = b$$

for all $a, b \in S$. Show that $(S, *)$ is a noncommutative semigroup with no identity element.

Solution 4.

(a) For associativity (semigroupness)

Proof. Let $a, b, c \in S$, then

$$\begin{aligned} a * (b * c) &= a * \left(\frac{bc}{b+c}\right) \\ &= \frac{a(bc)}{a + (b + c)} \\ &= \frac{abc}{a + (b + c)} && \text{Associativity of } + \text{ and } \cdot \text{ on } \mathbb{Q}^{\geq 0} \end{aligned}$$

and

$$\begin{aligned} (a * b) * c &= \left(\frac{ab}{a+b}\right) * c \\ &= \frac{(ab)c}{(a+b)+c} \\ &= \frac{abc}{a+b+c} && \text{Associativity of } + \text{ and } \cdot \text{ on } \mathbb{Q}^{\geq 0} \end{aligned}$$

$\therefore (S, *)$ is a semigroup $\square$

For commutativity

Proof. Let $a, b \in S$, then

$$\begin{aligned} a * b &= \frac{ab}{a+b} \\ &= \frac{ba}{b+a} && \text{Commutativity of } + \text{ and } \cdot \text{ on } \mathbb{Q}^{\geq 0} \\ &= b * a && \text{Definition of } * \text{ in reverse} \end{aligned}$$

$\therefore (S, *)$ is commutative $\square$

(b) For associativity (semigroupness)

Proof. Let $a, b, c \in S$, then

$$\begin{aligned} a * (b * c) &= a * c \\ &= c \end{aligned}$$

and

$$\begin{aligned} (a * b) * c &= b * c \\ &= c \end{aligned}$$

$\therefore (S, *)$ is a semigroup □

For commutativity

Proof. Let $a, b \in S$, then

$$a * b = b \neq b * a = a$$

$\therefore (S, *)$ is not commutative □

For the identity element if we assume, by way of contradiction, that such an element, e, exists then it must satisfy

$$a * e = e * a = a \forall a \in S.$$

We can note however that by our definition of $*$ we have

$$a * e = e \neq e * a = a$$

which is only satisfied by $e = a$ which would not satisfy any other such equation where we swap out a for some other element of S unless we also swap out e which would violate the uniqueness requirement imposed by the definition of e as an identity element meaning that it must be the same for all expressions and therefore cannot exist in S.