Math 2120H: Assignment IV

Jeremy Favro (0805980)
Trent University, Peterborough, ON, Canada

March 14, 2025

Problem 1. Evaluate $\int_C (xy + y + z)\, ds$ along the curve $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + (2 - 2t)\mathbf{k}$, $0 \le t \le 1$.

Solution 1.

$$\begin{aligned}
\int_C (xy + y + z)\, ds &= \int_0^1 [(2t)(t) + t + 2 - 2t]\, |2\mathbf{i} + 1\mathbf{j} + (2 - 2t)\mathbf{k}|\, dt \\
&= \int_0^1 [(2t)(t) + t + 2 - 2t]\, |2\mathbf{i} + 1\mathbf{j} - 2\mathbf{k}|\, dt \\
&= \int_0^1 \left[2t^2 - t + 2\right] \sqrt{9}\, dt = \frac{13}{2}
\end{aligned}$$

Problem 2. Find the mass of a thin wire lying along the curve $\mathbf{r}(t) = (\sqrt{2})t\mathbf{i} + (\sqrt{2})t\mathbf{j} + (4 - t^2)\mathbf{k}$, $0 \le t \le 1$, if the density is $\delta = 3t$.

Solution 2. The mass of an object with a continuous density function is given by $\int_C \delta(x, y, z)\, ds$ so,

$$\begin{aligned}
\int_C \delta(x, y, z)\, ds &= \int_0^1 3t \left|(\sqrt{2})\mathbf{i} + (\sqrt{2})\mathbf{j} - 2t\mathbf{k}\right| dt \\
&= \int_0^1 3t\sqrt{2 + 2 + 4t^2}\, dt \\
&= \int_1^2 3t\sqrt{4 + 4t^2}\, dt \\
&= 3\int_0^1 \sqrt{u}\, du = 2^{5/2} - 2
\end{aligned}$$

Problem 3. Find the line integral of $F = 3y\mathbf{i} + 2x\mathbf{j} + 4z\mathbf{k}$ over the path $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}$, $0 \le t \le 1$.

Solution 3.

$$\begin{aligned}
&= \int_0^1 F(\mathbf{r}(t))\mathbf{r}'(t)\, dt \\
&= \int_0^1 \left[3t^2\mathbf{i} + 2t\mathbf{j} + 4t^4\mathbf{k}\right] \left[\mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k}\right] dt \\
&= \int_0^1 3t^2 + 4t^2 + 16t^7\mathbf{k}\, dt = \frac{13}{3}
\end{aligned}$$

Problem 4. Find the flux of the fields $F = 2x\mathbf{i} + (x - y)\mathbf{j}$ across the circle $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$, $0 \le t \le 2\pi$.

Solution 4.

$$\begin{aligned}
&= \int_C -(x - y)\, dx + 2x\, dy \\
&= a^2 \int_0^{2\pi} \cos(t)\sin(t) + \sin^2(t) + 2\cos^2(t)\, dt \\
&= a^2 \left[\frac{\sin^2(t)}{2} - \frac{\sin(2t) - 2t}{4} + \frac{\sin(2t)}{2} + t\right]_0^{2\pi} = 3a^2\pi
\end{aligned}$$