Physics 2250: Problem Set IX

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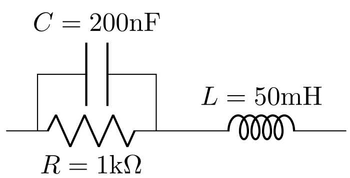
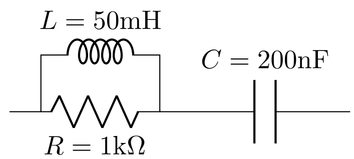
Problem 1. Determine both the DC impedance and the impedance at a frequency of $f = 1\text{kHz}$ of the circuits shown below.



(a)



(b)

Solution 1. Here $f = 1\text{kHz} \implies \omega = 2000\pi \text{rad}\,\text{s}^{-1}$

(a)

$$
\begin{aligned}
Z &= \left(\frac{1}{R} + j\omega C\right)^{-1} + j\omega L \\
&= \frac{R}{1 + j\omega CR} + j\omega L \\
&= \frac{R + (j\omega L)(1 + j\omega CR)}{1 + j\omega CR} \\
&= \frac{R(1 - j\omega CR) + (j\omega L)\left(1 + (\omega CR)^2\right)}{1 + (\omega CR)^2} \\
&= \frac{R - j\omega CR^2 + j\omega L + j\omega L(\omega CR)^2}{1 + (\omega CR)^2} \\
&= \frac{R}{1 + (\omega CR)^2} + \frac{-\omega CR^2 + \omega L + \omega L(\omega CR)^2}{1 + (\omega CR)^2} j
\end{aligned}
$$

Which means that the DC impedance ($\omega = 0$) is just R, and impedance with a frequency of 1kHz is $\approx 91.9997 - 25.1338j\Omega$

(b)

$$
\begin{aligned}
Z &= \left(\frac{1}{R}+\frac{1}{j\omega L}\right)^{-1}+\frac{1}{j\omega C}\\
&= \frac{j\omega LR}{R+j\omega L}+\frac{1}{j\omega C}\\
&= \frac{j\omega LR^2+(\omega L)^2 R}{R^2+(\omega L)^2}-\frac{j}{\omega C}\\
&= \frac{j\omega LR^2(\omega C)+(\omega L)^2 R(\omega C)}{(\omega C)\left(R^2+(\omega L)^2\right)}-\frac{j\left(R^2+(\omega L)^2\right)}{(\omega C)\left(R^2+(\omega L)^2\right)}\\
&= \frac{j\omega LR^2(\omega C)+(\omega L)^2 R(\omega C)-\left(R^2+(\omega L)^2\right)j}{(\omega C)\left(R^2+(\omega L)^2\right)}\\
&= \frac{(\omega L)^2 R}{R^2+(\omega L)^2}+\frac{L(\omega R)^2 C-\left(R^2+(\omega L)^2\right)}{(\omega C)\left(R^2+(\omega L)^2\right)}j
\end{aligned}
$$

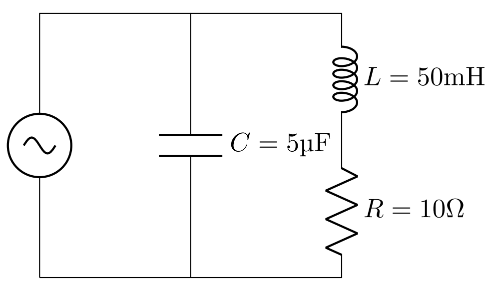
Which means that the DC impedance is infinite, and the impedance at a frequency of 1kHz is $89.8302 - 32.2716j\Omega$

Problem 2. A purely sinusoidal voltage of $V_s(t)$ of amplitude 10V and frequency $\omega = 300\text{rad}\,\text{s}^{-1}$ is applied in the circuit shown below.

(a) Find the equivalent circuit impedance, $\widetilde{Z}_{tot}$

(b) Find the total circuit current, $\widetilde{I}(t)$

(c) Find the average power expended in the circuit, and compare this to the DC power expended in the circuit (i.e. the power expended when powered by a simple 10V battery)



Solution 2.

(a)

$$
\begin{aligned}
Z &= \left(j\omega C + \frac{1}{R + j\omega L}\right)^{-1} \\
&= \left(\frac{1 + (R + j\omega L)\, j\omega C}{R + j\omega L}\right)^{-1} \\
&= \frac{R + j\omega L}{1 + (R + j\omega L)\, j\omega C} \\
&= \frac{R + j\omega L}{1 - \omega^2 LC + j\omega CR} \\
&= \frac{(R + j\omega L)(1 - \omega^2 LC - j\omega CR)}{(1 - \omega^2 LC)^2 + (\omega CR)^2} \\
&= \frac{j\omega L - j\omega^3 L^2 C + \omega^2 LRC + R - \omega^2 LRC - j\omega CR^2}{(1 - \omega^2 LC)^2 + (\omega CR)^2} \\
&= \frac{j\omega L - j\omega^3 L^2 C - j\omega CR^2 + R}{(1 - \omega^2 LC)^2 + (\omega CR)^2} \\
&= \frac{R}{(1 - \omega^2 LC)^2 + (\omega CR)^2} + \frac{\omega L - \omega^3 L^2 C - \omega CR^2}{(1 - \omega^2 LC)^2 + (\omega CR)^2} j \\
&= 10.4632 + 15.5367 i\Omega
\end{aligned}
$$

In phasor notation this is $\sqrt{10.4632^2 + 15.5367^2} e^{i \arctan\left(\frac{15.5367}{10.4632}\right)} = 18.7315 e^{0.9782j}$

(b) $V_s(t)$ in phasor notation is $10e^{300tj}$

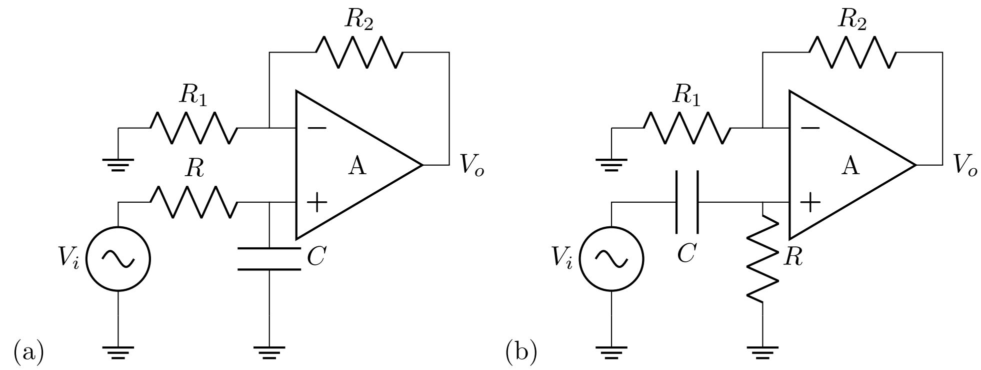
$$
\begin{aligned}
\widetilde{I}(t) &= \frac{\widetilde{V}_s(t)}{\widetilde{Z}_{tot}} \\
\widetilde{I}(t) &= \frac{10e^{300tj}}{18.7315 e^{0.9782j}} \\
\widetilde{I}(t) &= 0.5339 e^{(300t - 0.9782)j}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\langle P \rangle &= \frac{1}{2} \operatorname{Re}\left[\widetilde{V}_s(t) \widetilde{I}^*(t)\right] \\
&= \frac{1}{2} \operatorname{Re}\left[10e^{300tj} \cdot 0.5339 e^{-j(300t - 0.9782)}\right] \\
&= \frac{1}{2} \operatorname{Re}\left[5.339 e^{0.9782j}\right] \\
&= 2.6695 \operatorname{Re}\left[\cos(0.9782) + j \sin(0.9782)\right] \\
&= 1.4909\text{W}
\end{aligned}
$$

DC power is just $P = \frac{V^2}{R} = \frac{10\text{V}^2}{10\Omega} = 10\text{W}$

Problem 3. Consider the two Op-Amp filter circuits shown below. Each has a sinusoidal input, V_i, and component values of $C = 500\text{nF}$, $R = 1\text{k}\Omega$, $R_1 = 100\Omega$, $R_2 = 10\text{k}\Omega$.



For each filter:

(a) Use qualitative reasoning to predict the output (V_o) at low and at high input frequencies to determine the broad filter type.

(b) Find analytical expressions for

   i. The output voltage in terms of the input voltage, $V_o(V_i)$.

   ii. The magnitude of the output relative to the input, $\left|\frac{V_o}{V_i}\right|$.

   iii. The phase difference between the output and input, $\Delta\phi_{V_o - V_i}$

(c) Use any software of your choice to plot $\left|\frac{V_o}{V_i}\right|$ and $\Delta\phi_{V_o - V_i}$ for frequencies up to $\omega = 2\text{MHz}$. Make sure to annotate your plots with proper and legible labels. Plot $\left|\frac{V_o}{V_i}\right|$ on a log-log scale and $\Delta\phi_{V_o - V_i}$ on a linear-log scale.

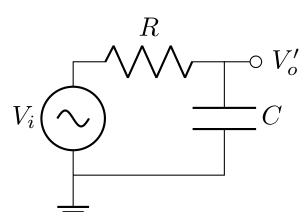
(d) Find the (exact or approximate) frequency and phase difference at which $\left|\frac{V_o}{V_i}\right| = 1$

Solution 3.

(a) At $\omega = 0$ in filter (a) the capacitor acts like an open meaning that the op-amp is in a non-inverting configuration where $V_o = \left(1 + \frac{R_2}{R_1}\right) V_i = 101 V_i$. As $\omega \to \infty$ in filter (a) the capacitor acts like a wire, meaning the op-amp does nothing so $V_o = 0$. So, filter (a) is a low pass filter with some gain. In filter (b) at $\omega = 0$ the capacitor acts like an open, dropping all of V_i, meaning that $V_o = 0$. As $\omega \to \infty$ the capacitor acts like a wire meaning that the op-amp is in a non-inverting configuration where $V_o = \left(1 + \frac{R_2}{R_1}\right) V_i = 101 V_i$. So, filter (b) is a high pass filter with some gain.

(b) I'm separating this into one part per circuit so it doesn't get super cluttered.

   i(a). Treating circuit (a) as a two stage circuit where the amplifier is a separate multiplier of 101 makes this conceptually much easier for me. To do this, I'll analyze the circuit below, finding $V_o'(V_i)$ and then just multiply V_o' by 101 to get V_o



We're trying to find the voltage dropped across the capacitor here which is given by $V_o' = \widetilde{I}(t)\widetilde{Z}_C$. Here $\widetilde{I}(t) = \dfrac{\widetilde{V}_i(t)}{\widetilde{Z}_R + \widetilde{Z}_C}$ so

$$\begin{aligned}
V_o' &= \frac{\widetilde{V}_i(t)\widetilde{Z}_C}{\widetilde{Z}_R + \widetilde{Z}_C} \\
&= \frac{\widetilde{V}_i(t)\left(-\frac{j}{\omega C}\right)}{R - \frac{j}{\omega C}} \\
&= \frac{\widetilde{V}_i(t)\left[(\omega C)^{-2} - R(\omega C)^{-1} j\right]}{R^2 + (\omega C)^{-2}} \\
&= \frac{\widetilde{V}_i(t)(\omega C)^{-2}}{R^2 + (\omega C)^{-2}} - \frac{\widetilde{V}_i(t)R(\omega C)^{-1}}{R^2 + (\omega C)^{-2}} j \\
\therefore V_o &= 101\widetilde{V}_i(t)\left[\frac{(\omega C)^{-2}}{R^2 + (\omega C)^{-2}} - \frac{R(\omega C)^{-1}}{R^2 + (\omega C)^{-2}} j\right]
\end{aligned}$$

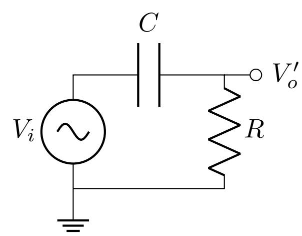
ii(a).

$$\begin{aligned}
&= 101\left|\frac{\cancel{\widetilde{V}_i(t)}\left[\frac{(\omega C)^{-2}}{R^2 + (\omega C)^{-2}} - \frac{R(\omega C)^{-1}}{R^2 + (\omega C)^{-2}} j\right]}{\cancel{\widetilde{V}_i(t)}}\right| \\
&= 101\sqrt{\left(\frac{(\omega C)^{-2}}{R^2 + (\omega C)^{-2}}\right)^2 + \left(\frac{R(\omega C)^{-1}}{R^2 + (\omega C)^{-2}}\right)^2} \\
&= \frac{101}{R^2 + (\omega C)^{-2}}\sqrt{(\omega C)^{-4} + R^2(\omega C)^{-2}} \\
&= \frac{101}{(\omega C)\left(R^2 + (\omega C)^{-2}\right)}\sqrt{\frac{1}{(\omega C)^2} + R^2} \\
&= \frac{101\sqrt{1 + R^2(\omega C)^2}}{(\omega C)^2\left(R^2 + (\omega C)^{-2}\right)}
\end{aligned}$$

iii(a).

$$\begin{aligned}
\Delta\phi_{V_o - V_i} &= \arctan\left(\frac{\mathrm{Im}\left[\frac{\widetilde{V}_o(t)}{\widetilde{V}_i(t)}\right]}{\mathrm{Re}\left[\frac{\widetilde{V}_o(t)}{\widetilde{V}_i(t)}\right]}\right) \\
&= \arctan\left(\frac{\mathrm{Im}\left[\frac{(\omega C)^{-2}}{R^2 + (\omega C)^{-2}} - \frac{R(\omega C)^{-1}}{R^2 + (\omega C)^{-2}} j\right]}{\mathrm{Re}\left[\frac{(\omega C)^{-2}}{R^2 + (\omega C)^{-2}} - \frac{R(\omega C)^{-1}}{R^2 + (\omega C)^{-2}} j\right]}\right) \\
&= \arctan\left(\frac{-\frac{R(\omega C)^{-1}}{R^2 + (\omega C)^{-2}}}{\frac{(\omega C)^{-2}}{R^2 + (\omega C)^{-2}}}\right) \\
&= \arctan\left(-R\omega C\right)
\end{aligned}$$

i(b). Treating circuit (b) as a two stage circuit where the amplifier is a separate multiplier of 101 makes this conceptually much easier for me. To do this, I'll analyze the circuit below, finding $V_o'(V_i)$ and then just multiply V_o' by 101 to get V_o



We're trying to find the voltage dropped across the capacitor here which is given by $V_o' = \widetilde{I}(t)\widetilde{Z}_R$. Here $\widetilde{I}(t) = \dfrac{\widetilde{V}_i(t)}{\widetilde{Z}_R + \widetilde{Z}_C}$ so

$$
\begin{aligned}
V_o' &= \frac{\widetilde{V}_i(t)\widetilde{Z}_R}{\widetilde{Z}_R + \widetilde{Z}_C} \\
V_o' &= \frac{\widetilde{V}_i(t)R}{R - j(\omega C)^{-1}} \\
V_o' &= \widetilde{V}_i(t)\frac{R^2 - jR(\omega C)^{-1}}{R^2 + (\omega C)^{-2}} \\
V_o' &= \widetilde{V}_i(t)\left[\frac{R^2}{R^2 + (\omega C)^{-2}} - \frac{R(\omega C)^{-1}}{R^2 + (\omega C)^{-2}}j\right] \\
\therefore V_o &= 101\widetilde{V}_i(t)\left[\frac{R^2}{R^2 + (\omega C)^{-2}} - \frac{R(\omega C)^{-1}}{R^2 + (\omega C)^{-2}}j\right]
\end{aligned}
$$

ii(b).

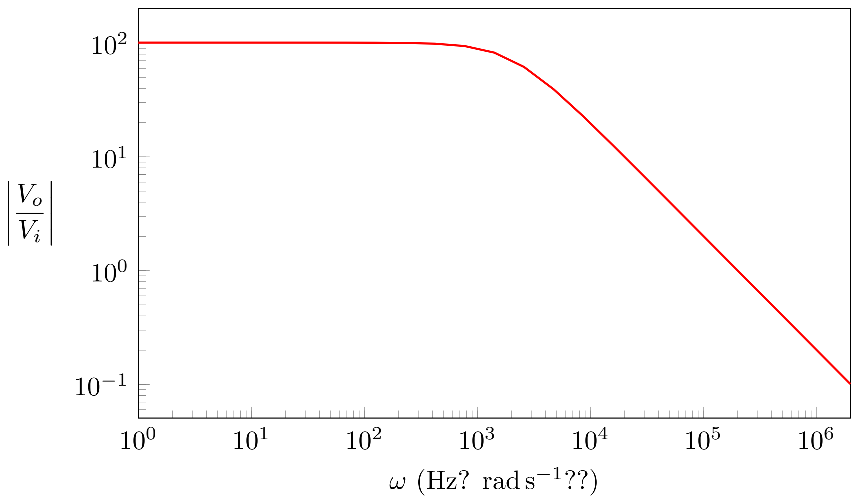
$$
\begin{aligned}
&= 101\left|\frac{R^2}{R^2 + (\omega C)^{-2}} - \frac{R(\omega C)^{-1}}{R^2 + (\omega C)^{-2}}j\right| \\
&= \frac{101}{R^2 + (\omega C)^{-2}}\sqrt{(R^2)^2 + (R(\omega C)^{-1})^2} \\
&= \frac{101}{R^2 + (\omega C)^{-2}}\sqrt{\frac{R^4(\omega C)^{-2} + R^2(\omega C)^{-2}}{(\omega C)^{-2}}} \\
&= \frac{101R\sqrt{1 + R^2}}{R^2 + (\omega C)^{-2}}
\end{aligned}
$$

iii(b).

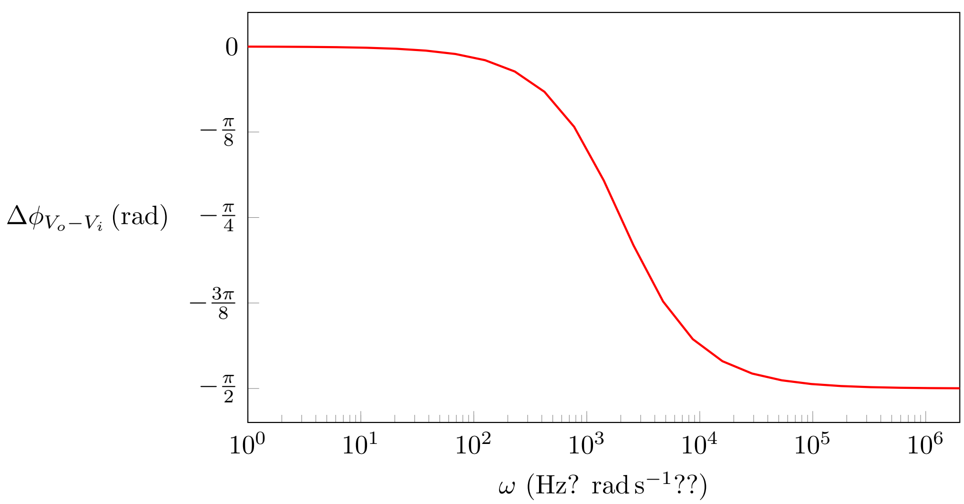
$$
\begin{aligned}
\Delta\phi_{V_o - V_i} &= \arctan\left(\frac{\mathrm{Im}\left[\frac{\widetilde{V}_o(t)}{\widetilde{V}_i(t)}\right]}{\mathrm{Re}\left[\frac{\widetilde{V}_o(t)}{\widetilde{V}_i(t)}\right]}\right) \\
&= \arctan\left(\frac{-\dfrac{R(\omega C)^{-1}}{R^2 + (\omega C)^{-2}}}{\dfrac{R^2}{R^2 + (\omega C)^{-2}}}\right) \\
&= \arctan\left(-\frac{1}{R\omega C}\right)
\end{aligned}
$$

(c) Plotted with TikZ

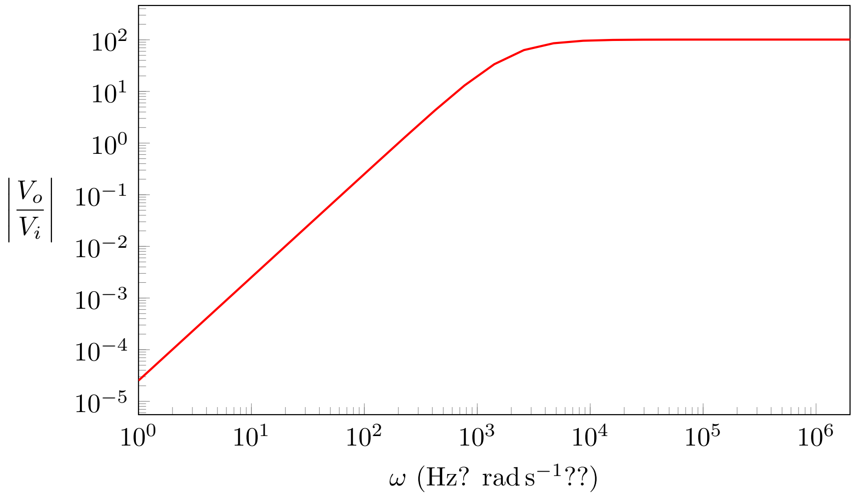
$\left|\frac{V_o}{V_i}\right|$ as a function of frequency for an active low-pass filter (circuit (a))



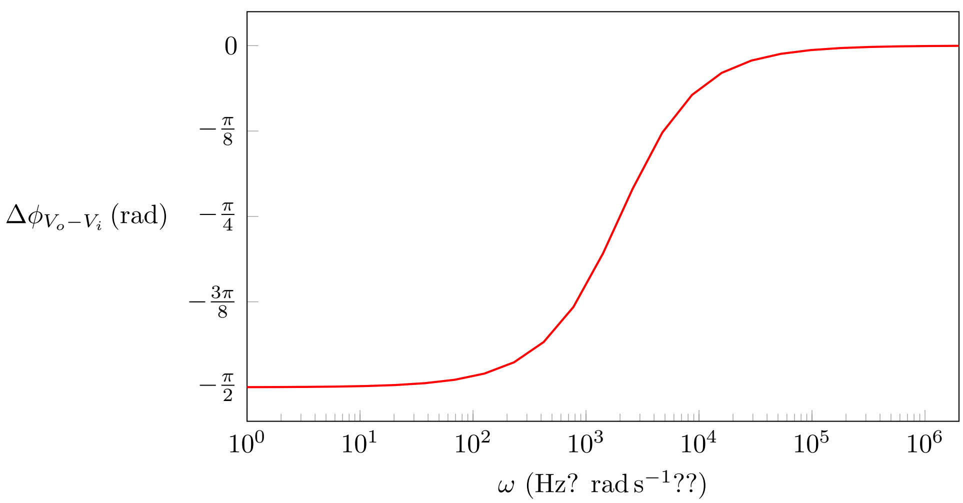
$\Delta\phi_{V_o - V_i}$ as a function of frequency for an active low-pass filter (circuit (a))



$\left|\dfrac{V_o}{V_i}\right|$ as a function of frequency for an active high-pass filter (circuit (b))



$\Delta\phi_{V_o - V_i}$ as a function of frequency for an active high-pass filter (circuit (b))



(d) For circuit (a)

$$
\begin{aligned}
&1 = \frac{101\sqrt{1 + R^2(\omega C)^2}}{(\omega C)^2\,(R^2 + (\omega C)^{-2})} \\
&(\omega C)^2 R^2 + 1 = 101\sqrt{1 + R^2(\omega C)^2} \\
&\frac{(\omega C)^2 R^2 + 1}{\sqrt{1 + R^2(\omega C)^2}} = 101 \\
&\omega = \sqrt{\frac{101^2 - 1}{(RC)^2}} = 201990 whatevertheunitofthegraphaboveis
\end{aligned}
$$

For circuit (b)

$$
\begin{aligned}
1 &= \frac{101R\sqrt{1+R^2}}{R^2+(\omega C)^{-2}} \\
R^2+(\omega C)^{-2} &= 101R\sqrt{1+R^2} \\
\omega &= \sqrt{\frac{1}{101RC^2\sqrt{1+R^2}-(RC)^2}} \approx 200
\end{aligned}
$$