Math 3770H: Assignment I

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Problem 1. Verify that each of the two numbers $z = 1 \pm i$ satisfies the equation $z^2 - 2z + 2 = 0$.

Solution 1.

$$\begin{aligned} &= (1-i)^2 - 2(1-i) + 2 \\ &= -2i - 2 + 2i + 2 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} &= (1+i)^2 - 2(1+i) + 2 \\ &= 2i - 2 - 2i + 2 \\ &= 0 \end{aligned}$$

Problem 2. Prove that multiplication of complex numbers is commutative, as stated at the beginning of Sec. 2.

Solution 2. *Proof.* Let $a, b \in \mathbb{C}$ with $a = x_1 + iy_1$ and $b = x_2 + iy_2$ where $x_n, y_n \in \mathbb{R}$. Then,

$$\begin{aligned} a \cdot b &= (x_1 + iy_1) \cdot (x_2 + iy_2) \\ &= x_1x_2 + iy_1x_2 + ix_1y_2 + i^2y_1y_2 && \text{Definition, Commutativity of addition and multiplication under } \mathbb{R} \\ &= (x_1x_2 - y_1y_2) + i(x_2y_1 + x_1y_2) && \text{Commutativity of addition and multiplication under } \mathbb{R} \end{aligned}$$

and

$$\begin{aligned} b \cdot a &= (x_2 + iy_2) \cdot (x_1 + iy_1) \\ &= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 && \text{Definition, Commutativity of addition and multiplication under } \mathbb{R} \\ &= (x_1x_2 - y_1y_2) + i(x_2y_1 + x_1y_2) && \text{Commutativity of addition and multiplication under } \mathbb{R} \end{aligned}$$

□

Problem 3. Reduce each of these quantities to a real number:

$$\text{(a) } \frac{1+2i}{3-4i} + \frac{2-i}{5i}; \qquad \text{(b) } \frac{5i}{(1-i)(2-i)(3-i)}; \qquad \text{(c) } (1-i)^4.$$

Solution 3.

(a)

$$\begin{aligned}
&= \frac{1+2i}{3-4i} + \frac{2-i}{5i} \\
&= \frac{1+2i}{3-4i}\frac{3+4i}{3+4i} + \frac{2-i}{5i}\frac{-5i}{-5i} \\
&= \frac{-5+10i}{25} + \frac{-5-10i}{25} = -2/5
\end{aligned}$$

(b)

$$\begin{aligned}
&= \frac{5i}{(1-i)(2-i)(3-i)} \\
&= \frac{5i}{-10i} = -1/2
\end{aligned}$$

(c)

$$\begin{aligned}
&= (1-i)^4 \\
&= (\sqrt{2}e^{-i\pi/4})^4 \\
&= (4e^{-i\pi}) \\
&= -4
\end{aligned}$$

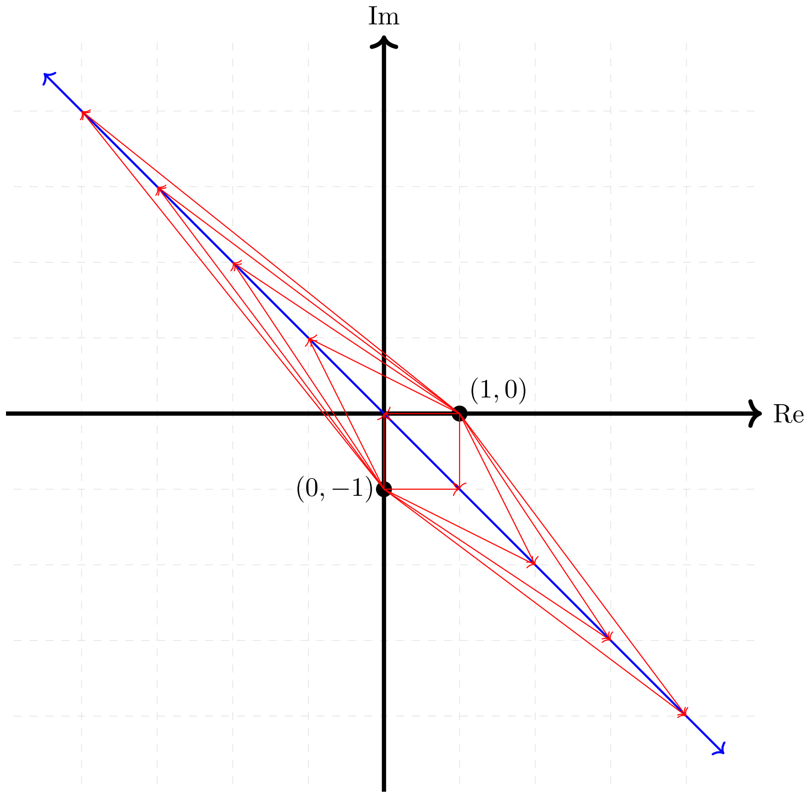
Problem 4. Verify that $\sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$

Solution 4. Let $z = x + iy$. Then,

$$\begin{aligned}
&\sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z| \\
&\implies \sqrt{2}\sqrt{x^2+y^2} \geq |x| + |y| \\
&\implies 2x^2 + 2y^2 \geq (|x| + |y|)^2 \\
&\implies 2x^2 + 2y^2 - (|x| + |y|)^2 \geq 0 \\
&\implies 2x^2 + 2y^2 - |x|^2 - |y|^2 - 2|xy| \geq 0 \\
&\implies 2x^2 + 2y^2 - x^2 - y^2 - 2|xy| \geq 0 \\
&\implies x^2 + y^2 - 2|xy| \geq 0 \\
&\implies (|x|^2 - |y|^2) \geq 0
\end{aligned}$$

Problem 5. Using the fact that $|z_1 - z_2|$ is the distance between two points z_1 and z_2, give a geometric argument that $|z-1| = |z+i|$ represents the line through the origin whose slope is -1.

Solution 5.



As we can see in the figure, the only points equidistant to 1 and $-i$ are the points on the line $-x$. We can immediately recognize that the values of z will form a line when plotted as that is the only way that the distance from two fixed points can be equal.

Problem 6. By factoring $z^4 - 4z^2 + 3$ into two quadratic factors and using inequality (2), Sec. 5, show that if z lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \leq \frac{1}{3}$$

Solution 6. We begin by factoring as $z^4 - 4z^2 + 3 = (z^2 - 1)(z^2 - 3)$. If we want our inequality to hold then $\left|z^4 - 4z^2 + 3\right| \geq 3$ so $\left|(z^2 - 1)(z^2 - 3)\right| \geq 3$ and because $|z|^2 = z^2$ we can substitute in our value for $|z|$ directly giving us

$$\begin{aligned} &\left|(2^2 - 1)(2^2 - 3)\right| \geq 3 \\ \implies &|3| \geq 3 \end{aligned}$$

and taking the reciprocal will flip the $\geq$ as we used at the beginning.

Problem 7. Find the principal argument $\operatorname{Arg} z$ when

$$\text{(a) } z = \frac{-2}{1+\sqrt{3}i}; \qquad \text{(b) } z = \left(\sqrt{3}-i\right)^6.$$

Solution 7. To do this we just need to determine the real and imaginary components and take the arctangent of $\operatorname{Im}/\operatorname{Re}$:

(a)

$$\begin{aligned} &= \frac{-2}{1+\sqrt{3}i}\frac{1-\sqrt{3}i}{1-\sqrt{3}i} \\ &= \frac{-2+2\sqrt{3}i}{4} = -1/2+\sqrt{3}i/2 \end{aligned}$$

which gives $\operatorname{Arg} z = \arctan\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right) = -\pi/3$ which isn't quite right when we look at the number on the Argand plane. This is because arctan will only give us the "correct" answer when our point lies in the first and fourth quadrants. Here this isn't the case so the actual value of $\operatorname{Arg} z$ is the value arctan "would" take on if it were defined in the quadrant we are currently in. This gives $\operatorname{Arg} z = 2\pi/3$ which is the correct value.

(b) Here we can take a bit of a shortcut and notice that $\operatorname{Arg} z = 6 \operatorname{Arg}\left(\sqrt{3}-i\right)$ through properties of exponentiation when writing z in polar form. Using this and being careful of the arctan ambiguity we get $\operatorname{Arg} z = \pi$

Problem 8. Find $\left(-8-8\sqrt{3}i\right)^{1/4}$, express the roots in rectangular coordinates, exhibit them as the vertices of a certain square, and point out which is the principal root.

Solution 8.

$$\begin{aligned} &= \left(-8-8\sqrt{3}i\right)^{1/4} \\ &= \left(16e^{i(4\pi/3+2k\pi)}\right)^{1/4} && k \in \mathbb{Z} \\ &= 2e^{i(\pi/3+k\pi/2)} = 1+\sqrt{3}i, -\sqrt{3}+i, -1-\sqrt{3}i, \sqrt{3}-i \end{aligned}$$

The principal root here is the case where $k=0$ which gives $1+\sqrt{3}i$.