Calculus II: Assignment 6

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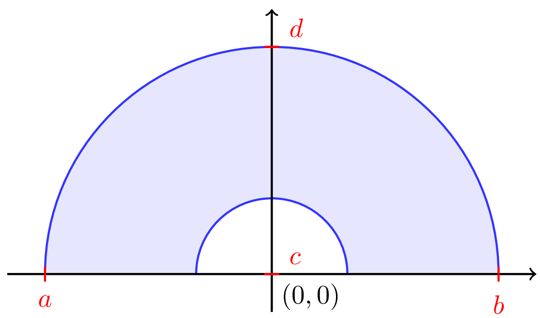
Problem 1. Find the centroid of the region $S = \{(x, y) | 1 \leq x^2 + y^2 \leq 9 \text{ and } y \geq 0\}$



Solution 1.
As can be seen in the above diagram the given shape is a washer meaning that we expect the centroid to have an x-coordinate of 0 because the area on either side will "cancel". The y-coordinate should be somewhere above $y = 0$. The centroid is given by

$$(\bar{x}, \bar{y}) = \left(\frac{\int_a^b x f(x)\, dx}{\int_a^b f(x)\, dx}, \frac{\int_c^d y f(y)\, dy}{\int_c^d f(y)\, dy} \right)$$

The area of the washer is actually just the area of the outer circle minus the area of the inner circle, meaning that the centroid can be found as follows

$$(\bar{x}, \bar{y}) = \left(\frac{\int_{-3}^3 x\sqrt{9 - x^2}\, dx - \int_{-1}^1 x\sqrt{1 - x^2}\, dx}{\int_{-3}^3 \sqrt{9 - x^2}\, dx - \int_{-1}^1 \sqrt{1 - x^2}\, dx}, \frac{\int_0^3 y\sqrt{9 - y^2}\, dy - \int_0^1 y\sqrt{1 - y^2}\, dy}{\int_0^3 \sqrt{9 - y^2}\, dy - \int_0^1 \sqrt{1 - y^2}\, dy} \right)$$

Which is pretty nasty looking, but quickly simplifies nicely. Starting with the numerator of $\bar{x}$

$$\begin{aligned}
&= \int_{-3}^3 x\sqrt{9 - x^2}\, dx - \int_{-1}^1 x\sqrt{1 - x^2}\, dx \\
&= \int_{-3}^3 x\sqrt{u}\, dx - \int_{-1}^1 x\sqrt{w}\, dx \rightsquigarrow \text{ let } u = 9 - x^2 \text{ and } w = 1 - x^2 \\
&= -\frac{1}{2}\int_{-3}^3 \sqrt{u}\, du + \frac{1}{2}\int_{-1}^1 \sqrt{w}\, dw \\
&= \left. -\frac{1}{3}\left(9 - x^2\right)^{\frac{3}{2}} \right|_{-3}^3 + \left. \frac{1}{3}\left(1 - x^2\right)^{\frac{3}{2}} \right|_{-1}^1 \\
&= 0
\end{aligned}$$

Which means we don't have to care about integrating the denominator of $\bar{x}$ because it'll end up being 0 as we expected. So, we currently sit knowing

$$(\bar{x}, \bar{y}) = \left(0, \frac{\int_0^3 y\sqrt{9 - y^2}\, dy - \int_0^1 y\sqrt{1 - y^2}\, dy}{\int_0^3 \sqrt{9 - y^2}\, dy - \int_0^1 \sqrt{1 - y^2}\, dy} \right)$$

Now to integrate the numerator of $\bar{y}$

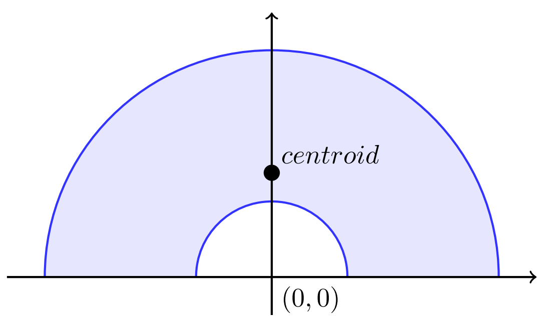
$$
\begin{aligned}
&= \int_0^3 y\sqrt{9-y^2}\,dy - \int_0^1 y\sqrt{1-y^2}\,dy \\
&= \int_0^3 y\sqrt{9-y^2}\,dy - \int_0^1 y\sqrt{1-y^2}\,dy \rightsquigarrow \text{ let } q = 9-y^2 \text{ and } r = 1-y^2 \\
&= -\frac{1}{2}\int_0^3 \sqrt{q}\,dq + \frac{1}{2}\int_0^1 \sqrt{r}\,dr \\
&= \ -\frac{1}{3}\left(9-y^2\right)^{\frac{3}{2}}\Big|_0^3 + \ \frac{1}{3}\left(1-y^2\right)^{\frac{3}{2}}\Big|_0^1 \\
&= \frac{26}{3}
\end{aligned}
$$

And the denominator of $\bar{y}$

$$
\begin{aligned}
&= \int_0^3 \sqrt{9-y^2}\,dy - \int_0^1 \sqrt{1-y^2}\,dy \\
&= \ \frac{1}{2}\sqrt{-y^2+9}y + \frac{9}{2}\ \arcsin\left(\frac{1}{3}\,y\right)\Big|_0^3 - \ \frac{1}{2}\sqrt{-y^2+1}y - \frac{1}{2}\ \arcsin\left(y\right)\Big|_0^1 \\
&= 2\pi
\end{aligned}
$$

Which I did partially with Sage because this assignment slipped my mind and I'm doing it kind of late on a Friday after a couple midterms :). Anyway, we now know that the centroid of the region S is

$$\left(0, \frac{13}{3\pi}\right)$$



at least I think, not much time to check as I've got to be up early tomorrow!