Physics 2130: Assignment III

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Problem 1. Consider the driven, damped harmonic oscillator. Its equation of motion is:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 = A\cos(\omega t)$$

In the case of an underdamped oscillator, i.e., $\beta < \omega_0^2$ we found that the solution for the equation of motion is:

$$x(t) = x_c(t) + x_p(t)$$

Where:

$$x_c(t)e^{-\beta t}\left[c_1 e^{i\omega_1 t} + c_2 e^{-i\omega_1 t}\right] = Be^{-\beta t}\cos(\omega_1 t - \phi) = \Gamma(t)s(t)$$

And where:

$$\omega_1 = \sqrt{w_0^2 - \beta^2}$$

$$\Gamma(t) = Be^{-\beta t}$$

$$s(t) = (\omega_1 t - \phi)$$

The particular solution is instead:

$$x_p(t) = D\cos(\omega t - \delta)$$

Where:

$$D = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}}$$

$$\delta = \arctan\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right)$$

(a) Consider an underdamped driven oscillator that starts with the initial conditions $x(t=0) = x_0$ and $\dot{x}(t=0) = 0$. Find the analytical expressions for the unknown coefficients in $x(t)$ using these initial conditions.

(b) Write a python program that returns (and prints) the values of ω_1, β, ϕ, D and δ for a given x_0. This should be coded as a function called `harm_osc_params` that accepts as inputs ω_0, β, A, ω and x_0.

(c) In the same python program, now write a new function, `harm_osc_x_pos`, that calculates the array of positions x of the harmonic oscillator for a given array of times t. This function should receive the array t as an input as well as the values of ω_0, β, A, ω and x_0. It should call the previously written function `harm_osc_params` for the calculations of all the oscillation parameters. It should return the array of positions x of the harmonic oscillator for each value of t.

(d) In the same python program, now write a new function to plot the data, `harm_osc_x_plot_single`. The function receives as inputs the arrays t and x generated at the previous step.

(e) Pick three values of β (remember of the constraint $\beta < w_0$) and plot in the same graph $x(t)$ for the three chosen values. For reference use $w_0 = 1\text{s}^{-1}$ and $A = 1\text{s}^{-2}$ (but play with the values of A to see the relative importance of the driver and the damping). Comment on what effect β has on both the transient and steady-state solution.

(f) To help you distinguish the different effects, now write a function called `harm_osc_damp_drive` that plots in the same graph, $s(t)$, $\Gamma(t)$, $x_c(t)$ and $x_p(t)$. Comment on the results, specifically on the contribution of each component.

(g) Finally, write a new function called `harm_osc_euler_cromer` that receives has input parameters ω_0, β, A, ω, x_0, and the analytical solution $x(t)$. This function should calculate a new $x(t)$, called $x_{EC}(t)$, that uses the Euler-Cromer method to determine the position of the oscillator as a function of time for the same initial conditions. Additionally, the function should plot, on the same graph, the solutions $x(t)$ and $x_{EC}(t)$ obtained with the analytical and Euler-Cromer methods, respectively, and the residuals, i.e., the difference $x(t)-x_{EC}(t)$. Discuss how you choose a value of Δt that gives a sufficiently accurate answer (which means defining "sufficiently accurate").

Solution 1. (a)

$$
\begin{aligned}
&x(t=0) = B\cos(-\phi) + D\cos(-\delta) = x_0 \\
&\Rightarrow x_0 - D\cos(\delta) = B\cos(\phi) \\
&\Rightarrow \frac{x_0 - D\cos(\delta)}{\cos(\phi)} = B
\end{aligned}
$$

$$
\begin{aligned}
&\dot{x}(t=0) = -B\omega_1\sin(-\phi) - B\beta\cos(-\phi) - D\omega\sin(-\delta) = 0 \\
&= B\omega_1\sin(\phi) - B\beta\cos(\phi) + D\omega\sin(\delta) \\
&= \frac{x_0 - D\cos(\delta)}{\cos(\phi)}\omega_1\sin(\phi) - \frac{x_0 - D\cos(\delta)}{\cos(\phi)}\beta\cos(\phi) + D\omega\sin(\delta) \\
&= (x_0 - D\cos(\delta))\,\omega_1\tan(\phi) - (x_0 - D\cos(\delta))\,\beta + D\omega\sin(\delta) \\
&\Rightarrow -\frac{D\omega\sin(\delta)}{(x_0 - D\cos(\delta))} = \omega_1\tan(\phi) - \beta \\
&\Rightarrow \arctan\left(\frac{\beta - \frac{D\omega\sin(\delta)}{(x_0 - D\cos(\delta))}}{\omega_1}\right) = \phi
\end{aligned}
$$

(b)
```python
import numpy as np

def harm_osc_params(w_0: float,
                    beta: float,
                    A: float,
                    w: float,
                    x_0: float
                    ) -> tuple[float, float, float, float, float]:
    # "Given"
    w_1 = np.sqrt(w_0**2-beta**2)
    D = A/(np.sqrt((w_0**2-w**2)+4*(w**2)*beta**2))
    delta = np.atan((2*w*beta)/(w_0**2-w**2))

    # Unknowns
    phi = np.atan((beta-((D*w*np.sin(delta))/(x_0-D*np.cos(delta))))/w_1)
    B = (x_0-D*np.cos(delta))/(np.cos(phi))

    print(f"w_1: {w_1}", f"B: {B}", f"phi: {phi}", f"D: {D}", f"delta: {delta}", sep="\n")

    return w_1, B, phi, D, delta
```

(c)
```
import numpy as np

def harm_osc_x_pos(w_0: float,
                   beta: float,
                   A: float,
                   w: float,
                   x_0: float,
                   t: np.ndarray
                   ) -> list:
    w_1, B, phi, D, delta = harm_osc_params(w_0,
                                            beta,
                                            A,
                                            w,
                                            x_0)

    return B*np.exp(-beta*t)*np.cos(w_1*t-phi)+D*np.cos(w*t-delta)
```

(d) In the same python program, now write a new function to plot the data, `harm_osc_x_plot_single`. The function receives as inputs the arrays t and x generated at the previous step.

(e) Pick three values of β (remember of the constraint $\beta < w_0$) and plot in the same graph $x(t)$ for the three chosen values. For reference use $w_0 = 1\text{s}^{-1}$ and $A = 1\text{s}^{-2}$ (but play with the values of A to see the relative importance of the driver and the damping). Comment on what effect β has on both the transient and steady-state solution.

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