Math 3150H: Assignment I

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My student number is 0805980 so $p = 9$, $q = 5$, and $r = 22$.

Problem 1. Consider the second order linear PDE given by

$$pu_{xx} + 10pu_{xy} + 9pu_{yy} + qu_x + qu_y = 8pqx + e^{8ry}$$

(a) Find a canonical form of the PDE.

(b) Determine the general solution of the PDE.

(c) Show that the general solution you obtained satisfies the original equation.

Solution 1.

(a) Here we have

$$\Delta = B^2 - 4AC = 100p^2 - 4(p)(9p) = 64p^2 > 0$$

So the PDE is hyperbolic. Now we solve

$$\frac{dy}{dx} = \frac{B \pm \sqrt{64p^2}}{2A} = \frac{10p \pm 8p}{2p} = 5 \pm 4.$$

Which gives in the plus case

$$\frac{dy}{dx} = 9 \implies y = 9x + \xi \implies \xi = y - 9x$$

and in the minus case

$$\frac{dy}{dx} = 1 \implies y = x + \eta \implies \eta = y - x.$$

Now we do our partials

$$\begin{array}{lllll} \xi_x = -9 & \xi_{xx} = 0 & \xi_y = 1 & \xi_{yy} = 0 & \xi_{xy} = 0 \\ \eta_x = -1 & \eta_{xx} = 0 & \eta_y = 1 & \eta_{yy} = 0 & \eta_{xy} = 0. \end{array}$$

Now we find our new coefficients. We expect $A_1 = C_1 = 0$ but we'll check just to be sure,

$$\begin{aligned}
A_1 &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
&= p\cdot(-9)^2 + 10p\cdot(-9)\cdot(1) + 9p\cdot(1)^2 \\
&= 0 \\
B_1 &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y \\
&= 2p\cdot(-9)\cdot(-1) + 10p\cdot((-9)\cdot(1) + (1)\cdot(-1)) + 2\cdot(9p)\cdot(1)\cdot(1) \\
&= -64p \\
C_1 &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
&= p\cdot(-1)^2 + 10p\cdot(-1)\cdot(1) + 9p\cdot(1)^2 \\
&= 0 \\
D_1 &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
&= q\cdot(-9) + q\cdot(1) \\
&= -8q \\
E_1 &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
&= q\cdot(-1) + q\cdot(1) \\
&= 0 \\
F_1 &= 0 \\
G_1 &= pq\left(\eta - \xi\right) + e^{r(9\eta-\xi)}
\end{aligned}$$

Where for G_1 we've made the substitution

$$\begin{aligned}
x &= \frac{1}{8}\left(\eta - \xi\right) \\
y &= \frac{1}{8}\left(9\eta - \xi\right).
\end{aligned}$$

This gives our new canonical form PDE as (with some manipulation):

$$64pu_{\xi\eta} + 8qu_{\xi} = pq\left(\eta - \xi\right) + e^{r(9\eta-\xi)}$$

(b) First we integrate with respect to ξ,

$$\begin{aligned}
\int 64pu_{\xi\eta} + 8qu_{\xi}\, d\xi &= \int pq\left(\eta-\xi\right) + e^{r(9\eta-\xi)}\, d\xi \\
64pu_{\eta} + 8qu &= pq\left(\eta\xi - \frac{\xi^2}{2}\right) - \frac{e^{r(9\eta-\xi)}}{r} \\
u_{\eta} + \frac{8q}{64p}u &= \frac{pq}{64p}\left(\eta\xi - \frac{\xi^2}{2}\right) - \frac{e^{r(9\eta-\xi)}}{64pr}
\end{aligned}$$

Which is a linear first order ODE so we find an integrating factor μ,

$$\mu = \exp\left(\int \frac{8q}{64p}\, d\eta\right) = \exp\left(\frac{8q}{64p}\eta\right).$$

This gives us

$$\begin{aligned}
u\exp\left(\frac{8q}{64p}\eta\right) &= \int \exp\left(\frac{8q}{64p}\eta\right)\left[\frac{q}{64}\left(\eta\xi - \frac{\xi^2}{2}\right) - \frac{e^{r(9\eta-\xi)}}{64pr}\right]\, d\eta \\
&= -\frac{e^{9r\eta + \frac{q\eta}{8p} - r\xi}}{64pr\left(9r + \frac{q}{8p}\right)} + \frac{\xi\left(8pq\eta - 64p^2\right)e^{\frac{q\eta}{8p}}}{64q} - \frac{p\xi^2 e^{\frac{q\eta}{8p}}}{16}
\end{aligned}$$

Transforming this back to something in terms of x and y we get

$$u = \exp\left(-\frac{8q}{64p}(y-x)\right)\left(-\frac{e^{9r(y-x)+\frac{q(y-x)}{8p} - r(y-9x)}}{64pr\left(9r+\frac{q}{8p}\right)} + \frac{(y-9x)\left(8pq(y-x) - 64p^2\right)e^{\frac{q(y-x)}{8p}}}{64q} - \frac{p(y-9x)^2 e^{\frac{q(y-x)}{8p}}}{16}\right)$$

(c) For this we first calculate the partials,

$$u_x = -\frac{q\,(63x+y)}{64} \qquad u_{xx} = -\frac{63q}{64} \qquad u_{xy} = -\frac{q}{64}$$
$$u_y = \frac{q\,(2y-2x)}{128} - \frac{e^{8ry}}{8p} \qquad u_{yy} = \frac{q}{64} - \frac{re^{8ry}}{p}$$

Then evaluate the original equation with these values,

$$\begin{aligned}
&= pu_{xx} + 10pu_{xy} + 9pu_{yy} + qu_x + qu_y \\
&= p\left[-\frac{63q}{64}\right] + 10p\left[-\frac{q}{64}\right] + 9p\left[\frac{q}{64} - \frac{re^{8ry}}{p}\right] + q\left[-\frac{q\,(63x+y)}{64}\right] + q\left[\frac{q\,(2y-2x)}{128}\right] \\
&= -q\,(p+qx) - 9e^{8ry}r
\end{aligned}$$

$8pqx + e^{8ry}$