Physics 3610H: Assignment IX

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Problem 1. Prove that $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$

Solution 1. Recall the definition of the commutator,

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}.$$

Applying this then to $[\hat{A}, \hat{B}\hat{C}]$,

$$\begin{aligned}
[\hat{A}, \hat{B}\hat{C}] &= \hat{A}(\hat{B}\hat{C}) - (\hat{B}\hat{C})\hat{A} \\
&= \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} - \hat{B}\hat{A}\hat{C} + \hat{B}\hat{A}\hat{C} \\
&= \hat{A}\hat{B}\hat{C} - \hat{B}\hat{A}\hat{C} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{C}\hat{A} \\
&= (\hat{A}\hat{B} - \hat{B}\hat{A})\hat{C} + \hat{B}(\hat{A}\hat{C} - \hat{C}\hat{A}) \\
&= [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]
\end{aligned}$$

Problem 2. For the (normalized) wavefunction

$$\psi(x) = \left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2},$$

what is $\Delta x \Delta p_x$?

Solution 2. Generally for an operator $\hat{A}$

$$\Delta A = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}.$$

So in order to determine the product $\Delta x \Delta p_x$ we need

$$\langle \hat{x}^2 \rangle; \quad \langle \hat{x} \rangle^2; \quad \langle \hat{p}_x^2 \rangle; \quad \langle \hat{p}_x \rangle^2.$$

Working these out then,

$$\begin{aligned}
\langle \hat{x}^2 \rangle &= \left(\frac{2\alpha}{\pi}\right)^{1/2} \int_{\mathbb{R}} e^{-\alpha x^2} x^2 e^{-\alpha x^2} \, dx \\
&= \left(\frac{2\alpha}{\pi}\right)^{1/2} \int_{\mathbb{R}} x^2 e^{-2\alpha x^2} \, dx \\
&= \frac{1}{4\alpha}
\end{aligned}$$

where the big step is provided by equation 43 in https://mathworld.wolfram.com/GaussianIntegral.html.

$$\begin{aligned}
\langle \hat{x} \rangle^2 &= \frac{2\alpha}{\pi} \left(\int_{\mathbb{R}} e^{-\alpha x^2} x e^{-\alpha x^2} \, dx \right)^2 \\
&= \frac{2\alpha}{\pi} \left(\int_{\mathbb{R}} x e^{-2\alpha x^2} \, dx \right)^2 \\
&= 0
\end{aligned}$$

where again the big step comes from Wolfram.

$$\begin{aligned}
\langle \hat{p}_x^2 \rangle &= \left(\frac{2\alpha}{\pi}\right)^{1/2} \int_{\mathbb{R}} e^{-\alpha x^2} (-\hbar^2) \frac{\partial^2}{\partial x^2} e^{-\alpha x^2}\, dx \\
&= (-\hbar^2) \left(\frac{2\alpha}{\pi}\right)^{1/2} \int_{\mathbb{R}} e^{-2\alpha x^2} \left(4\alpha^2 x^2 - 2\alpha\right)\, dx \\
&= (-\hbar^2) \left(\frac{2\alpha}{\pi}\right)^{1/2} \sqrt{\frac{\pi\alpha}{2}} - \sqrt{2\pi\alpha} \\
&= (-\hbar^2) \left(\frac{2\alpha}{\pi}\right)^{1/2} \frac{-\sqrt{\pi\alpha}}{\sqrt{2}} \\
&= \hbar^2 \alpha
\end{aligned}$$

and

$$\begin{aligned}
\langle \hat{p}_x \rangle^2 &= \left(\frac{2\alpha}{\pi}\right)^{1/2} \left(\int_{\mathbb{R}} e^{-\alpha x^2} i\hbar \frac{\partial^2}{\partial x^2} e^{-\alpha x^2}\, dx\right)^2 \\
&= -2\alpha\hbar \left(\frac{2\alpha}{\pi}\right)^{1/2} i \left(\int_{\mathbb{R}} x e^{-2\alpha x^2}\, dx\right)^2 \\
&= 0.
\end{aligned}$$

So,

$$\Delta x = \sqrt{\left(\frac{1}{4\alpha}\right) - 0} = \frac{1}{2\sqrt{\alpha}}; \quad \Delta p_x = \sqrt{\left(\hbar^2 \alpha\right) - 0} = \hbar\sqrt{\alpha}$$

and so

$$\Delta x \Delta p_x = \frac{1}{2\sqrt{\alpha}} \cdot \hbar\sqrt{\alpha} = \hbar/2.$$

Problem 3. Using $[\hat{a}_-, \hat{a}_+] = 1$ and $\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + 1/2\right)$, show that $\left[\hat{H}, \hat{a}_+\right] = +\hbar\omega\hat{a}_+$ and $\left[\hat{H}, a_-\right] = -\hbar\omega\hat{a}_-$.

Solution 3.

$$\begin{aligned}
\left[\hat{H}, \hat{a}_+\right] &= \left[\hbar\omega\left(\hat{a}_+\hat{a}_- + 1/2\right), \hat{a}_+\right] && \text{Definition of } \hat{H} \\
&= \hbar\omega\left[\hat{a}_+\hat{a}_- + 1/2, \hat{a}_+\right] && [C\hat{A}, \hat{B}] = C[\hat{A}, \hat{B}] \\
&= -\hbar\omega\left[\hat{a}_+, \hat{a}_+\hat{a}_- + 1/2\right] && [\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}] \\
&= -\hbar\omega([\hat{a}_+, \hat{a}_+\hat{a}_-] + \cancelto{0}{[\hat{a}_+, 1/2]}) && \\
&= -\hbar\omega([\hat{a}_+, \hat{a}_+]\,\hat{a}_- + \hat{a}_+\,[\hat{a}_+, \hat{a}_-]) && \text{Proved in question 1} \\
&= \hbar\omega\hat{a}_+ &&
\end{aligned}$$

and

$$\begin{aligned}
\left[\hat{H}, \hat{a}_-\right] &= \left[\hbar\omega\left(\hat{a}_+\hat{a}_- + 1/2\right), \hat{a}_-\right] \\
&= \hbar\omega\left[\hat{a}_+\hat{a}_- + 1/2, \hat{a}_-\right] \\
&= -\hbar\omega\left[\hat{a}_-, \hat{a}_+\hat{a}_- + 1/2\right] \\
&= -\hbar\omega\left([\hat{a}_-, \hat{a}_+\hat{a}_-] + \cancelto{0}{[\hat{a}_-, 1/2]}\right) \\
&= -\hbar\omega\left([\hat{a}_-, \hat{a}_+\hat{a}_-]\right) \\
&= -\hbar\omega\left([\hat{a}_-, \hat{a}_+]\,\hat{a}_- + \hat{a}_+\,[\hat{a}_-, \hat{a}_-]\right) \\
&= -\hbar\omega\left([\hat{a}_-, \hat{a}_+]\,\hat{a}_-\right) \\
&= -\hbar\omega\hat{a}_-
\end{aligned}$$