Physics 3200Y: Assignment I

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Problem 1. Let $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$ be the separation between $\mathbf{r}$ and $\mathbf{r}'$, where $\mathbf{r}' = (x', y', z')$ is a fixed point and $\mathbf{r} = (x, y, z)$. Let $\mathscr{r} = |\boldsymbol{\mathscr{r}}|$ be the magnitude of the separation.

(a) Show that $\nabla\left(\mathscr{r}^2\right) = 2\boldsymbol{\mathscr{r}}$.

(b) Show that $\nabla \exp\left(\vec{k} \cdot \vec{\boldsymbol{\mathscr{r}}}\right) = \vec{k} \exp\left(\vec{k} \cdot \vec{\boldsymbol{\mathscr{r}}}\right)$, where $\vec{k}$ is a vector constant.

(c) Show that $\nabla \exp(k\mathscr{r}) = k\hat{\boldsymbol{\mathscr{r}}} \exp(k\mathscr{r})$.

(d) Show that $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$.

Solution 1.

(a) *Proof.*

$$
\begin{aligned}
&= \nabla\left(\mathscr{r}^2\right) \\
&= \left[\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right] \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}^{\,2} \\
&= \left[\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right] \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right] \\
&= \frac{\partial}{\partial x}(x-x')^2\,\hat{x} + \frac{\partial}{\partial y}(y-y')^2\,\hat{y} + \frac{\partial}{\partial z}(z-z')^2\,\hat{z} \quad \text{Note}^1 \\
&= 2(x-x')\,\hat{x} + 2(y-y')\,\hat{y} + 2(z-z')\,\hat{z} \quad \text{By chain rule} \\
&= 2((x-x'), (y-y'), (z-z')) = 2\boldsymbol{\mathscr{r}}
\end{aligned}
$$

□

(b) *Proof.*

$$
\begin{aligned}
&= \nabla \exp\left(\vec{k} \cdot \vec{\boldsymbol{\mathscr{r}}}\right) \\
&= \nabla \exp\left(k_x(x-x') + k_y(y-y') + k_z(z-z')\right) \\
&= k_x \exp\left(k_x(x-x') + k_y(y-y') + k_z(z-z')\right)\hat{x} \\
&\quad + k_y \exp\left(k_x(x-x') + k_y(y-y') + k_z(z-z')\right)\hat{y} \\
&\quad + k_z \exp\left(k_x(x-x') + k_y(y-y') + k_z(z-z')\right)\hat{z} \\
&= (k_x, k_y, k_z) \exp\left(k_x(x-x') + k_y(y-y') + k_z(z-z')\right) \\
&= \vec{k} \exp\left(\vec{k} \cdot \vec{\boldsymbol{\mathscr{r}}}\right)
\end{aligned}
$$

□

[1] The partials kill the terms that don't contain their variable of differentiation, omitted for brevity

(c) *Proof.*

$$
\begin{aligned}
&= \nabla \exp\left(k\text{\textit{r}}\right)\\
&= \nabla \exp\left(\sqrt{k_x^2+k_y^2+k_z^2}\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right)\\
&= \nabla \exp\left(\sqrt{k_x^2+k_y^2+k_z^2}\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right)\\
&= \exp\left(\sqrt{k_x^2+k_y^2+k_z^2}\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right)\\
&\quad \vec{k}\nabla\sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \quad \text{By chain rule}\\
&= \frac{\cancel{1}/\cancel{2}\,\vec{k}\exp\left(\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right)\left[\cancel{2}(x-x')\hat{x}+\cancel{2}(y-y')\hat{y}+\cancel{2}(z-z')\hat{z}\right]}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\\
&= \vec{k}\exp\left(k\text{\textit{r}}\right)\frac{\left[(x-x')\hat{x}+(y-y')\hat{y}+(z-z')\hat{z}\right]}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\\
&= \vec{k}\hat{\boldsymbol{r}}\exp\left(k\text{\textit{r}}\right)
\end{aligned}
$$

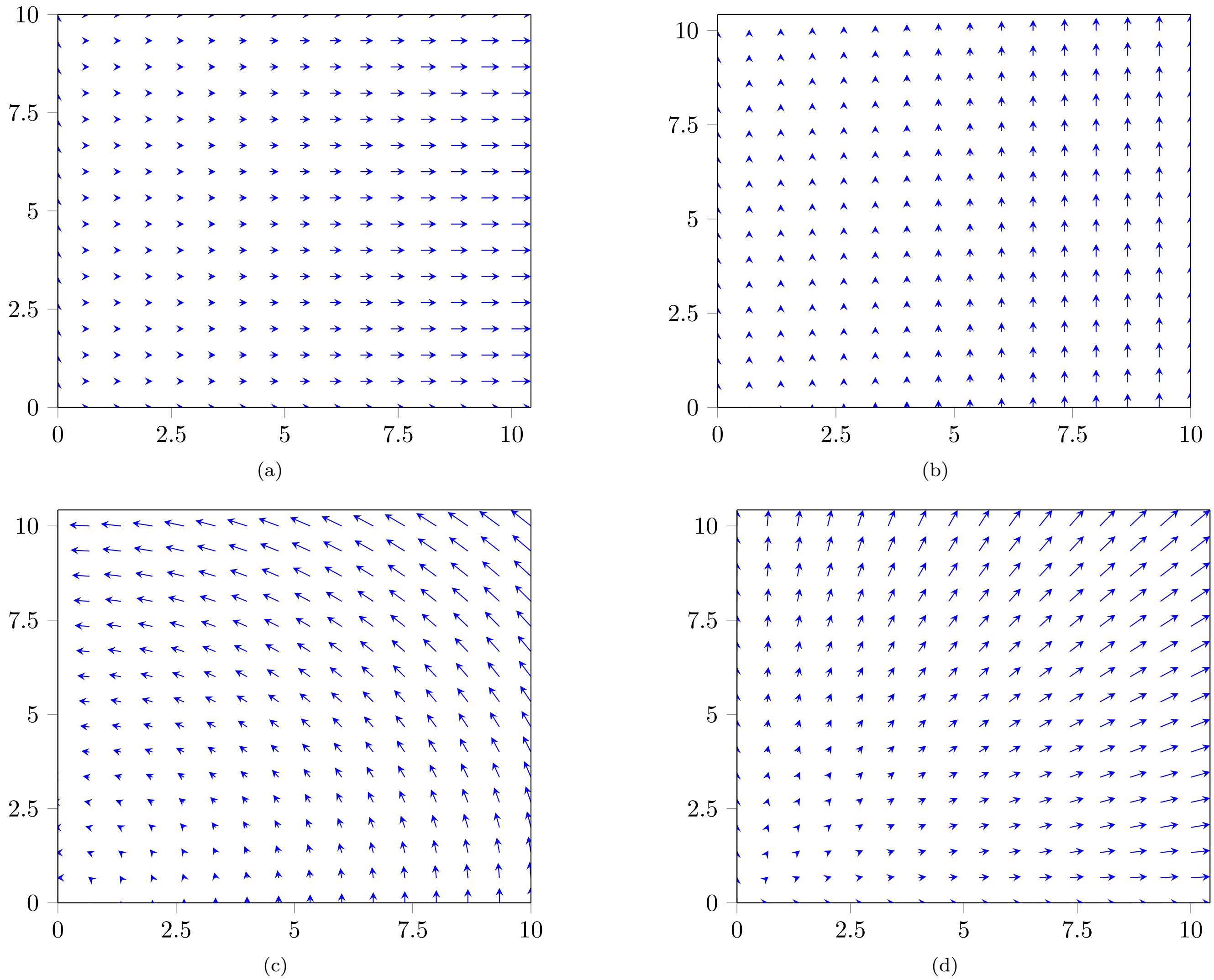
□

(d) *Proof.*

$$
\begin{aligned}
&= \nabla\left(1/\text{\textit{r}}\right)\\
&= \nabla\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{-1/2}\\
&= -\frac{\cancel{1}}{\cancel{2}}\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{-3/2}\left(\cancel{2}(x-x')\hat{x}+\cancel{2}(y-y')\hat{y}+\cancel{2}(z-z')\hat{z}\right) \quad \text{By chain rule...}\\
&= -\frac{\boldsymbol{r}}{\text{\textit{r}}^3} = -\frac{\hat{\boldsymbol{r}}}{\text{\textit{r}}^2}
\end{aligned}
$$

□

Problem 2. For each of the vector fields illustrated below, find an equation that *could* describe the field and calculate its divergence and curl.



(a) (b) (c) (d)

Figure 1: Plots generated with TikZ

Solution 2. (a) $\mathbf{F} = x\hat{x} + 0\hat{y} + 0\hat{z}$. $\mathrm{div}(\mathbf{F}) = \vec{\nabla} \cdot \mathbf{F} = 1$.

$$\begin{aligned} \mathrm{curl}(\mathbf{F}) &= \vec{\nabla} \times \mathbf{F} \\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 0 & 0 \end{vmatrix} \\ &= 0\hat{x} + 0\hat{y} + 0\hat{z} \end{aligned}$$

(b) $\mathbf{F} = 0\hat{x} + x\hat{y} + 0\hat{z}$. $\mathrm{div}(\mathbf{F}) = \vec{\nabla} \cdot \mathbf{F} = 0$

$$\begin{aligned} \mathrm{curl}(\mathbf{F}) &= \vec{\nabla} \times \mathbf{F} \\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix} \\ &= 0\hat{x} + 0\hat{y} + 1\hat{z} \end{aligned}$$

(c) $\mathbf{F} = -y\hat{x} + x\hat{y} + 0\hat{z}$. $\text{div}(\mathbf{F}) = \vec{\nabla} \cdot \mathbf{F} = 0$

$$\begin{aligned} \text{curl}(\mathbf{F}) &= \vec{\nabla} \times \mathbf{F} \\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} \\ &= 0\hat{x} + 0\hat{y} + 2\hat{z} \end{aligned}$$

(d) $\mathbf{F} = x\hat{x} + y\hat{y} + 0\hat{z}$. $\text{div}(\mathbf{F}) = \vec{\nabla} \cdot \mathbf{F} = 2$

$$\begin{aligned} \text{curl}(\mathbf{F}) &= \vec{\nabla} \times \mathbf{F} \\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} \\ &= 0\hat{x} + 0\hat{y} + 0\hat{z} \end{aligned}$$

Problem 3. Find the electric field a distance z above the centre of a flat circular disk of radius R that carries a uniform charge density σ. Complete all the steps we discussed in class:

(a) State which equation you will use and draw a diagram that illustrates the variables used in the equation.

(b) Find expressions for each of the variables ($\mathbf{r}$, $\mathbf{r}'$, etc.) that are needed to evaluate the equation for the field.

(c) Solve the equation for the field.

- Be careful to distinguish between $z > 0$ and $z < 0$.
- Check that your answer satisfies the boundary conditions given by Eq. (2.33) in Griffiths.

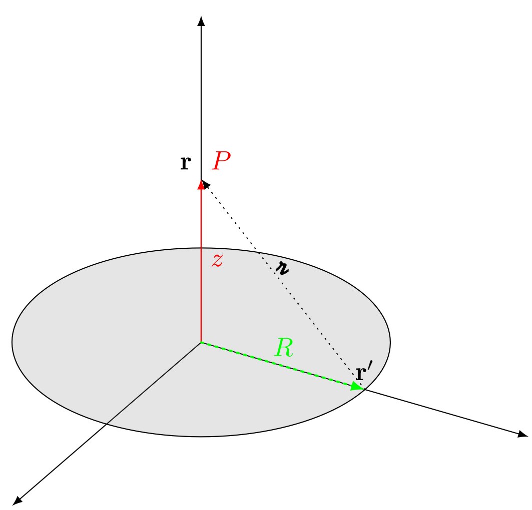
(d) Check that your answer makes sense in the limit $z \gg R$, where it should look like a point charge.

Solution 3.

(a) For this we will use the surface charge equation,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}} da'.$$

We will integrate over the disk which will probably require a change to polar coordinates but means that a' represents the area of the circle.



(b) r is the vector to the field "test point", labeled P in the figure. It's fairly obvious that it is $(0, 0, z)$. r' is also fairly obviously $(x, y, 0)$ as it is every point s.t. $x^2 + y^2 \leq R^2$ (this is also our bound of integration until we make a change to polar coordinates). This means that $\boldsymbol{\mathscr{r}} = (-x, -y, z)$

(c)

$$
\begin{aligned}
&= \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}da' \\
&= \frac{\sigma}{4\pi\epsilon_0}\int \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}da' \\
&= \frac{\sigma}{4\pi\epsilon_0}\int \frac{(-x,-y,z)}{(x^2+y^2+z^2)^{3/2}}da' \\
&= \frac{\sigma}{4\pi\epsilon_0}\int \frac{(-x,-y,z)}{(x^2+y^2+z^2)^{3/2}}da' \rightsquigarrow x = r'\cos\theta, y = r'\sin\theta \\
&= \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^R \frac{(-r'\cos\theta,-r'\sin\theta,z)}{\left(r'^2\cos^2\theta + r'^2\sin^2\theta + z^2\right)^{3/2}}r'dr'd\theta \rightsquigarrow x = r'\cos\theta, y = r'\sin\theta \\
&= \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^R \frac{(-r'\cos\theta,-r'\sin\theta,z)}{\left(r'^2 + z^2\right)^{3/2}}r'dr'd\theta \quad \text{Integrating w.r.t } \theta \text{ kills the sin and cos by symmetry} \\
&= \frac{\sigma}{4\pi\epsilon_0}\int_{0=r'}^R \frac{(0,0,2\pi z)}{2u^{3/2}}du \rightsquigarrow u = r'^2 + z^2 \implies du/2r' = dr' \\
&= \frac{\sigma}{4\epsilon_0}\int_{0=r'}^R \frac{(0,0,z)}{u^{3/2}}du \\
&= -\frac{z\sigma}{2\epsilon_0}\left[\left(r'^2+z^2\right)^{-1/2}\right]_0^R \hat{z} = \frac{z\sigma}{2\epsilon_0}\left[\frac{1}{|z|} - \frac{1}{\sqrt{R^2+z^2}}\right]\hat{z}
\end{aligned}
$$

Which, for $z < 0$, still works. We just need to note that $z = -z$ (by symmetry) and we are in the $-\hat{z}$ direction. For boundary conditions

$$
\begin{aligned}
\mathbf{E}_{above} - \mathbf{E}_{below} &= \frac{z\sigma}{2\epsilon_0}\left[\frac{1}{|z|} - \frac{1}{\sqrt{R^2+z^2}} + \frac{1}{|-z|} - \frac{1}{\sqrt{R^2+(-z)^2}}\right]\hat{z} \\
&= \frac{z\sigma}{2\epsilon_0}\left[\frac{2}{z} - \frac{2}{\sqrt{R^2+z^2}}\right]\hat{z} \\
&= \left[\frac{\sigma}{\epsilon_0} - \frac{z\sigma}{\epsilon_0\sqrt{R^2+z^2}}\right]\hat{z}
\end{aligned}
$$

We can then note that the boundary condition equation given in the text only holds when $z \to 0$ which kills the problem term and gives us what we want, satisfying equation 2.33 as $\hat{z}$ here is our normal vector.

(d) We first begin with some reworking using the binomial approximation given by $(1+x)^n \approx 1 + xn$ which is the low order terms of the Taylor series expansion of $(1+x)^n$. This works because higher order terms are very very small in the case of $z \gg R$ and so can be approximated as zero.

$$
\begin{aligned}
&= \frac{z\sigma}{2\epsilon_0}\left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right]\hat{z} \\
&= \frac{\sigma}{2\epsilon_0}\left[1 - \left[1 + \left(\frac{R}{z}\right)^2\right]^{-1/2}\right]\hat{z} \\
&= \frac{\sigma}{2\epsilon_0}\left[1 - 1 + \frac{R^2}{2z^2}\right]\hat{z} \\
&\approx \frac{\sigma}{2\epsilon_0}\left[\frac{R^2}{2z^2}\right]\hat{z} \\
&= \frac{\sigma R^2}{4\epsilon_0 z^2}\hat{z}
\end{aligned}
$$

We then recognize that $\sigma = Q/\pi R^2$ because we have a uniform charge distribution of net charge Q spread over a disk of area πR^2. Substituting this in,

$$\frac{QR^2}{4\epsilon_0 \pi R^2 z^2}\hat{z} = \frac{Q}{4\pi\epsilon_0 z^2}\hat{z}$$

Which is the equation for a point charge, as we would hope.

Problem 4. For a solid sphere of radius R and charge density $\rho(\mathbf{r}) = \kappa r$, use Gauss' law to find

(i) The electric field.

(ii) The potential inside and outside the sphere.

Solution 4.

(i) Gauss' law states that

$$\oint_S \mathbf{E}\cdot d\mathbf{a} = \frac{Q_{enc}}{\epsilon_0}$$

Where Q_{enc} is given by

$$Q_{enc} = \int_{\mathcal{V}} \rho\, d\tau.$$

Because we have a sphere of charge we can transform this to use spherical coordinates,

$$\begin{aligned} Q_{enc} &= \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi\, kr^3 \sin\theta \\ &= 4\pi k \int_0^R r^3 dr \\ &= \pi k R^4 \end{aligned}$$

We can now return to our original statement involving $\mathbf{E}$ and solve. We can also observe that because $\mathbf{E}$ points radially outwards from our sphere and $\mathbf{a}$ does as well we can reduce the dot product to a product of their magnitudes.

$$\frac{\pi k R^4}{\epsilon_0} = \oint_S |\mathbf{E}| da$$

Now because $|\mathbf{E}|$ is constant over the *surface* because the charge density is spherically symmetric, only depending on r, we can extract $|\mathbf{E}|$ from the integral which then just becomes the area of a sphere,

$$\begin{aligned} \frac{\pi k R^4}{\epsilon_0} &= \oint_S |\mathbf{E}| da \\ \frac{\pi k R^4}{\epsilon_0} &= |\mathbf{E}| \oint_S da \\ \frac{\pi k R^4}{\epsilon_0} &= 4\pi r^2 |\mathbf{E}| \\ \frac{kR^4}{4\epsilon_0 r^2} &= |\mathbf{E}| \end{aligned}$$

(For $r > R$)

(ii) To start here we need the electric field inside the sphere which is given by the same process as above except we integrate from 0 to $r \leq R$ because we are now inside the sphere. This gives $Q_{enc} = \pi k r^4 \implies \mathbf{E} = \frac{kr^2}{4\epsilon_0}\hat{\mathbf{r}}$. We can now evaluate for the outside of the sphere,

$$\begin{aligned} V_{r>R}(\mathbf{r}) &= -\int_\infty^r \frac{kR^4}{4\epsilon_0 r'^2} dr' \\ &= -\int_\infty^r \frac{kR^4}{4\epsilon_0 r'^2} dr' \\ &= \frac{kR^4}{4\epsilon_0}\left[\frac{1}{r'}\right]_\infty^r = \frac{kR^4}{4\epsilon_0 r} \end{aligned}$$

Then for the potential inside the sphere we split the integral due to the contribution of the field both inside and outside the sphere,

$$\begin{aligned}
V_{r<R}(\mathbf{r}) &= -\int_{\infty}^{R} \frac{kR^4}{4\epsilon_0 r'^2} dr' - \int_{R}^{r} \frac{kr^2}{4\epsilon_0} dr' \\
&= \frac{kR^4}{4\epsilon_0 R} - \frac{k}{12\epsilon_0} \left[r'^3 \right]_R^r \\
&= \frac{k}{12\epsilon_0} \left(4R^3 - r^3 \right)
\end{aligned}$$

Problem 5. Consider the electric field

$$\mathbf{E}(\mathbf{r}) = E_0 \left[y^2 \hat{x} + \left(2xy + z^2 \right) \hat{y} + 2yz\hat{z} \right].$$

Find, by integrating $-\int \mathbf{E} \cdot d\boldsymbol{\ell}$, the potential at an arbitrary point $\mathbf{r}$, taking the origin as the reference point. Note that the most direct integration path may not be simplest.

Solution 5. Here we have that $d\boldsymbol{\ell} = dx\hat{x} + dy\hat{y} + dz\hat{z}$ which means that $\mathbf{E} \cdot d\boldsymbol{\ell} = E_0 \left[y^2 dx + \left(2xy + z^2 \right) dy + 2yzdz \right]$ so,

$$\begin{aligned}
V(\mathbf{r}) &= -\int_0^{\mathbf{r}} E_0 \left[y^2 dx + \left(2xy + z^2 \right) dy + 2yzdz \right] \\
&= -E_0 \left[\int_0^{r_x} y^2 dx + \int_0^{r_y} \left(2xy + z^2 \right) dy + \int_0^{r_z} 2yzdz \right]
\end{aligned}$$

Which simplifies nicely along several paths (though it's already fairly simple). I'll go along the x, the y, then the z. This means that for our first integral along x we are moving from $\langle 0, 0, 0 \rangle$ to $\langle r_x, 0, 0 \rangle$ so y is zero along the whole range which means that the integral just becomes zero as its integrand is y^2. Next along y we are moving from $\langle r_x, 0, 0 \rangle$ to $\langle r_x, r_y, 0 \rangle$ so

$$\int_0^{r_y} \left(2xy + \cancelto{0}{z^2} \right) dy = r_x r_y^2.$$

Finally along z we are moving from $\langle r_x, r_y, 0 \rangle$ to $\langle r_x, r_y, r_z \rangle$ so

$$\int_0^{r_z} 2yzdz = \int_0^{r_z} 2yzdz = r_y r_z^2.$$

This yields

$$V(\mathbf{r}) = -E_0 \left[r_x r_y^2 + r_y r_z^2 \right].$$