

Math 2120H: Assignment II

Jeremy Favro

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Problem 1. Find the given limits.

$$\lim_{t \rightarrow \pi} \left[\left(\sin \frac{t}{2} \right) \mathbf{i} + \left(\cos \frac{2t}{3} \right) \mathbf{j} + \left(\tan \frac{5t}{4} \right) \mathbf{k} \right]$$

Solution 1. By the component-wise law for limits of vector-valued functions this limit is equivalent to the sum of each of its component limits along their respective unit vectors,

$$\begin{aligned} &= \lim_{t \rightarrow \pi} \left(\sin \frac{t}{2} \right) \mathbf{i} + \lim_{t \rightarrow \pi} \left(\cos \frac{2t}{3} \right) \mathbf{j} + \lim_{t \rightarrow \pi} \left(\tan \frac{5t}{4} \right) \mathbf{k} \\ &= 1\mathbf{i} - \frac{1}{2}\mathbf{j} + 1\mathbf{k} \end{aligned}$$

Problem 2. Below $\mathbf{r}(t)$ is the position of a particle in space at time t . Find the particle's velocity and acceleration vectors. Then find the particle's speed at the given value of t .

$$\mathbf{r}(t) = (t + 1)\mathbf{i} + (t^2 - 1)\mathbf{j} + 2t\mathbf{k}, \quad t = 1$$

Solution 2.

$$\begin{aligned} \mathbf{v}(t) \Big|_{t=1} &= \frac{d}{dt} \mathbf{r}(t) \Big|_{t=1} = \frac{d}{dt} (t + 1) \Big|_{t=1} \mathbf{i} + \frac{d}{dt} (t^2 - 1) \Big|_{t=1} \mathbf{j} + \frac{d}{dt} (2t) \Big|_{t=1} \mathbf{k} \\ &= 1 \Big|_{t=1} \mathbf{i} + 2t \Big|_{t=1} \mathbf{j} + 2 \Big|_{t=1} \mathbf{k} \\ &= \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \implies \text{speed} = \sqrt{1^2 + 2^2 + 2^2} = 3 \end{aligned}$$

then,

$$\begin{aligned} \mathbf{a}(t) \Big|_{t=1} &= \frac{d}{dt} \mathbf{v}(t) \Big|_{t=1} = \frac{d}{dt} \Big|_{t=1} \mathbf{i} + \frac{d}{dt} 2t \Big|_{t=1} \mathbf{j} + \frac{d}{dt} 2 \Big|_{t=1} \mathbf{k} \\ &= 0 \Big|_{t=1} \mathbf{i} + \frac{d}{dt} 2 \Big|_{t=1} \mathbf{j} + 0 \Big|_{t=1} \mathbf{k} \\ &= 0\mathbf{i} + 2\mathbf{j} + 0\mathbf{k} \end{aligned}$$

Problem 3. Evaluate the integrals.

(a)

$$\int_0^1 t^3 \mathbf{i} + 7\mathbf{j} + (t+1)\mathbf{k} dt$$

(b)

$$\int_1^4 \frac{1}{t} \mathbf{i} + \frac{1}{5-t} \mathbf{j} + \frac{1}{2t} \mathbf{k} dt$$

Solution 3.

(a)

$$\begin{aligned} &= \int_0^1 t^3 \mathbf{i} + 7\mathbf{j} + (t+1)\mathbf{k} dt \\ &= \int_0^1 t^3 dt \mathbf{i} + \int_0^1 7 dt \mathbf{j} + \int_0^1 (t+1) dt \mathbf{k} \\ &= \frac{t^4}{4} \Big|_0^1 \mathbf{i} + 7t \Big|_0^1 \mathbf{j} + \frac{t^2}{2} + t \Big|_0^1 \mathbf{k} \\ &= \frac{1}{4} \mathbf{i} + 7\mathbf{j} + \frac{3}{2} \mathbf{k} \end{aligned}$$

(b)

$$\begin{aligned} &= \int_1^4 \frac{1}{t} \mathbf{i} + \frac{1}{5-t} \mathbf{j} + \frac{1}{2t} \mathbf{k} dt \\ &= \int_1^4 \frac{1}{t} dt \mathbf{i} + \int_1^4 \frac{1}{5-t} dt \mathbf{j} + \int_1^4 \frac{1}{2t} dt \mathbf{k} \\ &= \ln(t) \Big|_1^4 \mathbf{i} - \ln(5-t) \Big|_1^4 \mathbf{j} + 2 \ln(2t) \Big|_1^4 \mathbf{k} \\ &= \ln(4) \mathbf{i} + \ln(4) \mathbf{j} + \ln(16) \mathbf{k} \end{aligned}$$

Problem 4. Solve the initial value problem.

$$\mathbf{r}'(t) = (t^3 + 4t)\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}, \quad \mathbf{r}(0) = \mathbf{i} + \mathbf{j}$$

Solution 4.

$$\begin{aligned} \mathbf{r}(t) &= \int (t^3 + 4t)\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k} dt \\ &= \int (t^3 + 4t) dt \mathbf{i} + \int t dt \mathbf{j} + \int 2t^2 dt \mathbf{k} \\ &= \left(\frac{t^4}{4} + 2t^2 \right) \mathbf{i} + \frac{t^2}{2} \mathbf{j} + \frac{2t^3}{3} \mathbf{k} + \mathbf{C} \rightsquigarrow \mathbf{r}(t=0) = \mathbf{i} + \mathbf{j} \\ \Rightarrow \mathbf{i} + \mathbf{j} &= \left(\frac{0^4}{4} + 2 \cdot 0^2 \right) \mathbf{i} + \frac{0^2}{2} \mathbf{j} + \frac{2 \cdot 0^3}{3} \mathbf{k} + \mathbf{C} \\ &= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} + \mathbf{C} \Rightarrow \mathbf{C} = \mathbf{i} + \mathbf{j} \\ \therefore \mathbf{r}(t) &= \left(\frac{t^4}{4} + 2t^2 + 1 \right) \mathbf{i} + \left(\frac{t^2}{2} + 1 \right) \mathbf{j} + \frac{2t^3}{3} \mathbf{k} \end{aligned}$$