Physics 3200Y: Assignment IV

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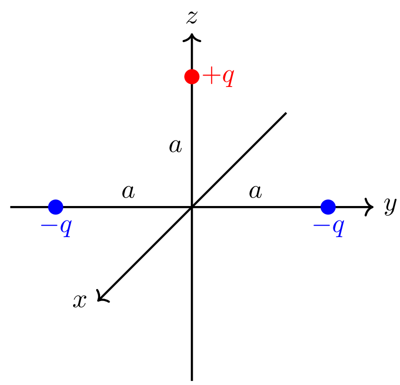
Problem 1. Three point charges are located as shown, each a distance a from the origin. Find the approximate electric field at points far from the origin. Express your answer in spherical coordinates, and include the two lowest orders in the multipole expansion.



Solution 1. This configuration has net charge $-q$ so the monopole term will be nonzero. Explicitly for large distances from the charge configuration,

$$\mathbf{E}_m \approx \frac{-q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}.$$

We then need the second order term for which we need to determine the dipole moment,

$$\begin{aligned}
\mathbf{p} &= \sum_i q_i \mathbf{r'}_i \\
&= (-q)(-a)\hat{\mathbf{y}} + (-q)(a)\hat{\mathbf{y}} + (q)(a)\hat{\mathbf{z}} \\
&= qa\hat{\mathbf{z}} = qa\left(\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right).
\end{aligned}$$

Then in order to determine the electric field we need the dipole potential,

$$V_d(\hat{\boldsymbol{r}}) = \frac{\mathbf{p}\cdot\hat{\boldsymbol{\mathscr{r}}}}{4\pi\epsilon_0\mathscr{r}^2} \approx \frac{qa\left(\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right)\cdot\hat{\boldsymbol{r}}}{4\pi\epsilon_0 r^2} = \frac{qa\cos\theta}{4\pi\epsilon_0 r^2}.$$

Then the electric field is

$$\mathbf{E}_d = -\nabla V_d = -\frac{\partial}{\partial r}\frac{qa\cos\theta}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial}{\partial\theta}\frac{qa\cos\theta}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{\theta}} = -\frac{qa\cos\theta}{2\pi\epsilon_0 r^3}\hat{\boldsymbol{r}} + \frac{qa\sin\theta}{4\pi\epsilon_0 r^3}\hat{\boldsymbol{\theta}}.$$

Now because electric fields obey linear superposition we can just find the total field as

$$\mathbf{E} = \mathbf{E}_m + \mathbf{E}_d \approx \frac{-q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}} - \frac{qa\cos\theta}{2\pi\epsilon_0 r^3}\hat{\boldsymbol{r}} + \frac{qa\sin\theta}{4\pi\epsilon_0 r^3}\hat{\boldsymbol{\theta}}$$

Problem 2. The goal of this problem is to find the potential outside a long metal pipe of radius a. Put the axis of the pipe along the z axis, and let there be an external field $\mathbf{E}_0 = E_0\hat{\mathbf{x}}$. The total field is the sum of the external field and the induced field due to the pipe. To solve this problem,

(a) List the boundary conditions.

(b) Solve Laplace's equation in cylindrical coordinates using separation of variables.

(c) Apply the boundary conditions to find the coefficients in your solution.

(d) Your final answer will look like

$$V(s,\theta) = \sum_n A_n f_n(s,\phi)$$

where A_n are your coefficients and $f_n(s,\phi)$ are the orthogonal functions you obtained from Laplace's equation.

i. Write a python function that takes as input n, s, and ϕ and returns the value of $f_n(s,\phi)$.

ii. Write a python code that uses this function to make a plot of the potential in the xy plane for $s > a$.

Solution 2.

(a)

(1) Since the pipe is a conductor it is also an equipotential. I'll say that the potential at which the pipe sits is zero. Formally, $V(s = a, \theta) = 0$.

(2) At large distances from the pipe the potential will be dominated by the external electric field. Formally, $V(s \gg a, \theta) = -E_0 x = -E_0 s \cos\phi$.

(3) Since we want a physical (e.g. continuous) potential the potential must meet back up with itself when we rotate about z.

(b) For a potential which does not depend on ϕ as we have here, Laplace's equation looks like

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}V\right) + \frac{1}{s^2}\frac{\partial^2}{\partial\phi^2}V = 0.$$

Using separation of variables we are looking for solutions of the form

$$V(s,\phi) = S(s)\Phi(\phi)$$

and so our Laplacian becomes

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}S(s)\Phi(\phi)\right) + \frac{1}{s^2}\frac{\partial^2}{\partial\phi^2}S(s)\Phi(\phi) = 0$$

which we can rearrange to

$$\frac{s}{S(s)}\frac{d}{ds}\left(s\frac{d}{ds}S(s)\right) + \frac{1}{\Phi(\phi)}\frac{d^2}{d\phi^2}\Phi(\phi) = 0.$$

Now in order to actually separate the variables we must choose a side to be negative. We want solutions which are continuous in all variables and so must choose the ϕ term to be the negative one else we would obtain a differential equation which would yield (real valued) exponentials which do not circle back around to their original value when adding 2π to their argument. So we have now

$$\frac{s}{S(s)}\frac{d}{ds}\left(s\frac{d}{ds}S(s)\right) = C; \qquad -\frac{1}{\Phi(\phi)}\frac{d^2}{d\phi^2}\Phi(\phi) = C.$$

Solving these two ODEs now,

$$\frac{d^2}{d\phi^2}\Phi(\phi) = -C\Phi(\phi) \implies \Phi(\phi) = A\cos(k\phi) + B\sin(k\phi), \quad C = k^2$$

and

$$\begin{aligned}
0 &= s\frac{d}{ds}\left(s\frac{d}{ds}S(s)\right) - k^2 S(s) \\
&= s\left(\frac{d}{ds}S(s) + s\frac{d^2}{ds^2}S(s)\right) - k^2 S(s) \\
&= \left(s\frac{d}{ds}S(s) + s^2\frac{d^2}{ds^2}S(s)\right) - k^2 S(s) \\
&= s^2\frac{d^2}{ds^2}S(s) + s\frac{d}{ds}S(s) - k^2 S(s)
\end{aligned}$$

which is a Cauchy-Euler equation and can be solved by letting $S(s) = s^n$ and solving the quadratic in n that results,

$$n^2 + (1-1)n - k^2 = 0 \implies n = \frac{0 \pm \sqrt{0 - 4(1)(-k^2)}}{2} = \pm k$$

(assuming $k \geq 0$). So our solution is of the form

$$S(s) = Ds^k + Es^{-k}.$$

Now applying our boundary conditions we first note that our requirement that $\Phi(\phi) = \Phi(\phi + 2\pi)$ means that

$$\begin{aligned}
A\cos(k\phi) + B\sin(k\phi) &= A\cos(k\phi + 2k\pi) + B\sin(k\phi + 2k\pi) \\
&= A(\cos(k\phi)\cos(2k\pi) - \sin(k\phi)\sin(2k\pi)) + B(\sin(k\phi)\cos(2k\pi) + \cos(k\phi)\sin(2k\pi))
\end{aligned}$$

which means that k must be an integer so that the terms with a $2k\pi$ argument drop out. $k = 0$ is also allowed as $\Phi(\phi) = A$ is a valid solution. $k = 0$ also gives us an entirely new ODE for both $\Phi(\phi)$ and $S(s)$ aside from the constant solutions,

$$\frac{d^2}{d\phi^2}\Phi(\phi) = 0 \implies \Phi(\phi) = F\phi + G$$

and

$$s^2\frac{d^2}{ds^2}S(s) + s\frac{d}{ds}S(s) = 0$$

which is still Cauchy-Euler it just now has a repeated root and so

$$S(s) = H\ln(s) + I.$$

Note that in our solution

$$\Phi(\phi) = F\phi + G$$

$F = 0$ as otherwise the solution does not obey boundary condition (3). With this we have enough information to write the full solution in a general form,

$$\begin{aligned}
V(s,\phi) &= A + B\ln(s) + \sum_{k=1}^{\infty} s^k(C\cos(k\phi) + D\sin(k\phi)) + s^{-k}(E\cos(k\phi) + F\sin(k\phi)) \\
&= A + B\ln(s) + \sum_{k=1}^{\infty} Cs^k\cos(k\phi) + Ds^k\sin(k\phi) + Es^{-k}\cos(k\phi) + Fs^{-k}\sin(k\phi) \\
&= A + B\ln(s) + \sum_{k=1}^{\infty}\left(Cs^k + \frac{E}{s^k}\right)\cos(k\phi) + \left(Ds^k + \frac{F}{s^k}\right)\sin(k\phi)
\end{aligned}$$

where we've merged all the constant terms out front into one, A and multiplied through by the constants in from of the $S(s)$ solution terms. Now we can apply our boundary conditions. Beginning with (2),

$$\begin{aligned}
-E_0 s\cos\phi &= \lim_{s\to\infty} A + B\ln(s) + \sum_{k=1}^{\infty}\left(Cs^k + \frac{E}{s^k}\right)\cos(k\phi) + \left(Ds^k + \frac{F}{s^k}\right)\sin(k\phi) \\
&= \lim_{s\to\infty} A + B\ln(s) + \sum_{k=1}^{\infty}\left(Cs^k\right)\cos(k\phi) + \left(Ds^k\right)\sin(k\phi)
\end{aligned}$$

For this to not blow up we must have that $B = 0$. Additionally, by inspection, to match the LHS and RHS the only k value we can accept is $k = 1$ so $D = C = 0$ (and $A = 0$) for $k \neq 1$ and in fact $D = 0$ always as there is no $\sin\phi$ term on the LHS. By the same logic $C = -E_0$. Hence,

$$V(s, \phi) = \left(-E_0 s + \frac{E}{s}\right)\cos(\phi).$$

Now applying boundary condition (1) we obtain that

$$V(a, \phi) = \left(-E_0 a + \frac{E}{a}\right)\cos(\phi) = 0 \implies E\cos(\phi) = E_0 a^2 \cos(\phi).$$

We have our full solution now,

$$V(s, \phi) = E_0\left(-s + \frac{a^2}{s}\right)\cos(\phi)$$

(c) Apply the boundary conditions to find the coefficients in your solution.

(d) Your final answer will look like

$$V(s, \theta) = \sum_n A_n f_n(s, \phi)$$

where A_n are your coefficients and $f_n(s, \phi)$ are the orthogonal functions you obtained from Laplace's equation.

i. Write a python function that takes as input n, s, and ϕ and returns the value of $f_n(s, \phi)$.

ii. Write a python code that uses this function to make a plot of the potential in the xy plane for $s > a$.

Problem 3. line of charge extends along the z-axis from $z = -a$ to $z = a$. The linear charge density is

$$\lambda(z) = \begin{cases} -\lambda_0 & z < 0 \\ \lambda_0 & z > 0 \end{cases}$$

(a) Find, by direct integration of Griffiths Eq. (2.30), the potential $V(z)$ for $|z| > a$. Note that you have to handle positive and negative values of z separately.

(b) This problem has azimuthal symmetry, which means that the potential can also be represented by Eq. (3.65). Find the unknown coefficients by matching the potential to the one you found in part (a). Make sure your solution works for both $z > 0$ and $z < 0$.

(c) Bonus. Plot your potential in the xz plane. You will find the `scipy.special.legendre` module to be useful.

Solution 3.

(a) Griffiths Eq. (2.30) is

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}}\, dl'.$$

In our case this integral splits up over the interval $[-a, a]$ as

$$V = \frac{\lambda_0}{4\pi\epsilon_0}\left[-\int_{-a}^{0} \frac{1}{|z - z'|}\, dz' + \int_0^a \frac{1}{|z - z'|}\, dz'\right]$$

which for $z < 0$ becomes

$$V = \frac{\lambda_0}{4\pi\epsilon_0}\left[-\int_{-a}^{0} \frac{1}{z' - z}\, dz' + \int_0^a \frac{1}{z' - z}\, dz'\right] = \frac{\lambda_0}{4\pi\epsilon_0}\left[-\ln(z' - z)\Big|_{-a}^{0} + \ln(z' - z)\Big|_0^a\right] = \frac{\lambda_0}{4\pi\epsilon_0}\left[-(\ln(-z) - \ln(-a - z)) + \ln\right.$$

(b) Eq. (3.65) is

$$V = \sum_\ell A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}$$

This problem has azimuthal symmetry, which means that the potential can also be represented by Eq. (3.65). Find the unknown coefficients by matching the potential to the one you found in part (a). Make sure your solution works for both $z > 0$ and $z < 0$.

(c) Bonus. Plot your potential in the xz plane. You will find the `scipy.special.legendre` module to be useful.