Math 2350H: Assignment I

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Problem 1. Demonstrate that there does not exist $\lambda \in \mathbb{C}$ such that

$$\lambda\,(2-3i, 5+4i, -6+i) = (12-5i, 7+22i, -32-9i)$$

Solution 1. Let $\lambda = a + bi$ then

$$\begin{aligned}
&\lambda\,(2-3i, 5+4i, -6+i) = (12-5i, 7+22i, -32-9i) \\
&\implies (2a+3b) + (-3a+2b)i = 12-5i \wedge (5a-4b) + (4a+5b)i = 7+22i \wedge (-6a-1b) + (1a-6b)i = -32-9i
\end{aligned}$$

Here the determinant of the matrix of the coefficients of i is

$$\begin{vmatrix} -3 & 2 & -5 \\ 4 & 5 & 2 \\ 1 & -6 & -9 \end{vmatrix} = -3(5\cdot(-9) - 22\cdot(-6)) + (-1)2(4\cdot(-9) - 1\cdot 22) - 5(4\cdot(-6) - 1\cdot 5) = 0$$

so there exist no a, b which satisfy all three coefficients of i and therefore there exists no λ that satisfies the original equation.

Problem 2. Let $V = \mathbb{R}^2$. If (x_1, x_2) and (y_1, y_2) are elements of V , and $\alpha \in \mathbb{R}$, define

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 y_2),$$

and

$$\alpha \cdot (x_1, x_2) = (\alpha x_1, x_2).$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Solution 2. For V to be a vector space over $\mathbb{R}$ with these operations the following must hold $\forall (x_1, y_1) \in V$, $\alpha \in \mathbb{R}$:

(i) $+$ must be commutative and associative

(ii) $\cdot$ must be associative

(iii) $0 \in V$

(iv) There must exist a multiplicative identity for scalar multiplication

(v) There must exist an additive inverse

(vi) The additive and multiplicative distributive law must hold

(i) Commutativity: Let $(x_1, x_2), (y_1, y_2) \in V$.
Given that $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 y_2)$,

$$\begin{aligned}
(y_1, y_2) + (x_1, x_2) &= (y_1 + x_1, y_2 x_2) \text{ (by definition)} \\
(y_1, y_2) + (x_1, x_2) &= (x_1 + y_1, x_2 y_2) \text{ (by commutativity in } \mathbb{R})
\end{aligned}$$

Associativity: Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in V$.
Given that $(x_1, x_2) + ((y_1, y_2) + (z_1, z_2)) = (x_1, x_2) + (y_1 + z_1, y_2 z_2) = (x_1 + y_1 + z_1, x_2 y_2 z_2)$

$$\begin{aligned}
&= ((x_1, x_2) + (y_1, y_2)) + (z_1, z_2) \\
&= (x_1 + y_1, x_2 y_2) + (z_1, z_2) \text{ (by definition)} \\
&= (z_1, z_2) + (x_1 + y_1, x_2 y_2) \text{ (by previously demonstrated commutativity)} \\
&= (z_1 + x_1 + y_1, z_2 x_2 y_2) \\
&= (x_1 + y_1 + z_1, x_2 y_2 z_2) \text{ (by commutativity in } \mathbb{R})
\end{aligned}$$

(ii) Associativity: Let $(x_1, x_2) \in V$ and $\alpha, \beta \in \mathbb{R}$
Given that, by definition, $(\alpha \cdot \beta) \cdot (x_1, x_2) = (\alpha \cdot \beta \cdot x_1, x_2)$

$$
\begin{aligned}
&= \alpha \cdot (\beta \cdot (x_1, x_2)) \\
&= \alpha \cdot (\beta \cdot x_1, x_2) \\
&= (\alpha \cdot \beta \cdot x_1, x_2)
\end{aligned}
$$

(iii) Let $(x_1, x_2) \in V$

$$
\begin{aligned}
&= (x_1, x_2) + (0, 1) \\
&= (x_1 + 0, x_2 \cdot 1) \\
&= (x_1, x_2)
\end{aligned}
$$

(iv) Let $(x_1, x_2) \in V$

$$
\begin{aligned}
&= 1 \cdot (x_1, x_2) \\
&= (1 \cdot x_1, x_2) \\
&= (x_1, x_2)
\end{aligned}
$$

(v) There must exist an additive inverse

(vi) The additive and multiplicative distributive law must hold