Math 3150H: Assignment II

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My student number is 0805980 so $p = 9$, $q = 5$, and $r = 22$.

Problem 1. Solve for $u(x,t)$.

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} - 8\frac{\partial u}{\partial x}, \qquad 0 < x < 2, \quad t > 0,$$
$$u(0,t) = 0, \quad u(2,t) = 0, \quad t > 0, \quad \text{and} \quad u(x,0) = (x^2 - 2x)e^{2x}$$

Solution 1. Since this problem is already homogenous we can go directly to separation of variables. Let

$$u(x,t) = \phi(x)T(t).$$

Then our PDE becomes

$$\frac{\partial \phi(x)T(t)}{\partial t} = 2\frac{\partial^2 \phi(x)T(t)}{\partial x^2} - 8\frac{\partial \phi(x)T(t)}{\partial x}$$

which we divide through by $\phi(x)T(t)$ to obtain

$$\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = \frac{2}{\phi(x)}\frac{\partial^2 \phi(x)}{\partial x^2} - \frac{8}{\phi(x)}\frac{\partial \phi(x)}{\partial x} = P.$$

Which gives us two ODEs to solve,

$$2\frac{\partial^2 \phi(x)}{\partial x^2} - 8\frac{\partial \phi(x)}{\partial x} = P\phi(x), \qquad \phi(0) = \phi(2) = 0$$

and

$$\frac{\partial T(t)}{\partial t} = PT(t).$$

Let's start with the ϕ equation. We have three cases to check:

(1) $P = 0$: Here our ODE becomes

$$\frac{\partial^2 \phi(x)}{\partial x^2} - 4\frac{\partial \phi(x)}{\partial x} = 0$$

which is constant coefficient with characteristic polynomial

$$m^2 - 4m = 0 \implies m = 0, 4$$

and so our solution is

$$\phi(x) = C_1 e^{4x} + C_2.$$

Applying initial conditions,

$$\phi(0) = C_1 + C_2 = 0 \implies C_1 = -C_2$$

and

$$\phi(2) = C_1 e^8 - C_1 = 0 \implies C_1 = C_2 = 0$$

and so this is a trivial case and $P = 0$ is not an eigenvalue.

(2) $P > 0 \implies P = \lambda^2$: Here our ODE becomes

$$2\frac{\partial^2\phi(x)}{\partial x^2} - 8\frac{\partial\phi(x)}{\partial x} - \phi\lambda^2 = 0$$

which is again constant coefficient now with characteristic polynomial

$$2m^2 - 8m - \lambda^2 = 0$$

which has roots

$$m = \frac{8 \pm \sqrt{64 - 4(2)(-\lambda^2)}}{4} = \frac{4 \pm \sqrt{16 + 2\lambda^2}}{2}$$

and so

$$\phi(x) = C_1 e^{(\frac{4+\sqrt{16+2\lambda^2}}{2})x} + C_2 e^{(\frac{4-\sqrt{16+2\lambda^2}}{2})x}.$$

Applying initial conditions then,

$$\phi(0) = C_1 + C_2 = 0 \implies C_1 = -C_2$$

and

$$\phi(2) = C_1 e^{4+\sqrt{16+2\lambda^2}} - C_1 e^{4-\sqrt{16+2\lambda^2}} = 0 \implies C_1 e^{4+\sqrt{16+2\lambda^2}} = C_1 e^{4-\sqrt{16+2\lambda^2}} \implies C_1 = C_2 = 0$$

and so this is a trivial case and $P = \lambda^2$ is not an eigenvalue.

(3) $P < 0 \implies P = -\lambda^2$: Here our ODE becomes

$$2\frac{\partial^2\phi(x)}{\partial x^2} - 8\frac{\partial\phi(x)}{\partial x} + \phi\lambda^2 = 0$$

which is again constant coefficient with characteristic polynomial

$$2m^2 - 8m + \lambda^2 = 0$$

which has roots

$$m = \frac{8 \pm \sqrt{64 - 8\lambda^2}}{4} = \frac{4 \pm \sqrt{16 - 2\lambda^2}}{2}$$

which only produces something unique from the previous two cases when

$$16 - 2\lambda^2 \leq 0 \implies \lambda \geq \sqrt{8}.$$

We must again check cases for λ in this range then,

(i) $\lambda = \sqrt{8}$ gives us one value for m, $m = 2$ and hence

$$\phi(x) = C_1 e^{2x} + C_2 x e^{2x}.$$

Applying initial conditions then,

$$\phi(0) = C_1 = 0$$

and

$$\phi(2) = 2C_2 e^4 = 0 \implies C_2 = 0$$

and so this is a trivial case.

(ii) $\lambda > \sqrt{8}$ means we obtain

$$m = \frac{4 \pm i\sqrt{2\lambda^2 - 16}}{2} = 2 \pm \frac{\sqrt{2\lambda^2 - 16}}{2} i$$

because the discriminant becomes negative. This gives us

$$\phi(x) = e^{2x}\left(C_1 \cos\left(\frac{\sqrt{2\lambda^2 - 16}}{2} x\right) + C_2 \sin\left(\frac{\sqrt{2\lambda^2 - 16}}{2} x\right)\right).$$

Applying initial conditions,

$$\phi(0) = C_1 = 0$$

and

$$\phi(2) = C_2 e^4 \sin\left(\sqrt{2\lambda^2 - 16}\right) = 0$$

which means that either C_2 is zero and we obtain another trivial case, or

$$\sqrt{2\lambda^2 - 16} = n\pi \implies \lambda = \sqrt{\frac{(n\pi)^2 + 16}{2}}$$

a non trivial solution!

So we obtain, finally, a nontrivial solution:

$$\phi_n(x) = Ce^{2x} \sin\left(\frac{n\pi}{2} x\right)$$

we've also obtained that

$$P = -\lambda^2 = -\frac{(n\pi)^2 + 16}{2}.$$

We can also now write our full ϕ as a sum of the ϕ_ns:

$$\phi(x) = \sum_{n=1}^{\infty} C_n e^{2x} \sin\left(\frac{n\pi}{2} x\right)$$

Now we have to solve

$$\frac{\partial T(t)}{\partial t} = PT(t)$$

which is easy,

$$T(t) = Ce^{Pt} = C \exp\left(-\frac{(n\pi)^2 + 16}{2} t\right).$$

We can now recombine our solutions to obtain

$$u = \phi(x)T(t) = \sum_{n=1}^{\infty} C_n e^{2x} \sin\left(\frac{n\pi}{2} x\right) \exp\left(-\frac{(n\pi)^2 + 16}{2} t\right)$$

and use our last initial condition to determine C:

$$u(x, 0) = (x^2 - 2x)\cancel{e^{2x}} = \sum_{n=1}^{\infty} C_n \cancel{e^{2x}} \sin\left(\frac{n\pi}{2} x\right)$$

which gives

$$C_n = \int_0^2 (x^2 - 2x) \sin\left(\frac{n\pi}{2} x\right) dx$$

which I threw into Sage:

[1]:
```
x, n = var('x n')
assume(n, "integer")

fn = (x^2-2*x)*sin((n*pi*x)/2)

show(integral(fn, x, 0, 2).full_simplify())
```

[1]: $\frac{16\left((-1)^n - 1\right)}{\pi^3 n^3}$

So the problem is solved,

$$u(x, t) = \sum_{n=1}^{\infty} \frac{16((-1)^n - 1)}{\pi^3 n^3} e^{2x} \sin\left(\frac{n\pi}{2} x\right) \exp\left(-\frac{(n\pi)^2 + 16}{2} t\right)$$

Problem 2. Solve for $u(x,t)$.

$$\frac{\partial u}{\partial t} = 2q^2 t \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0,$$
$$u(0,t) = t^2, \quad u(1,t) = 1 + t^2, \quad t > 0, \quad \text{and} \quad u(x,0) = 1$$

Solution 2. Here we start with

$$u(x,t) = v(x,t) + w(x,t)$$

with

$$w(x,t) = t^2 + x$$

so that

$$w(0,t) = u(0,t) = t^2 \implies v(0,t) = 0$$

and

$$w(1,t) = u(1,t) = t^2 + 1 \implies v(1,t) = 0.$$

Now we are solving

$$\frac{\partial}{\partial t}\left(v(x,t) + w(x,t)\right) = 2q^2 t \frac{\partial^2}{\partial x^2}\left(v(x,t) + w(x,t)\right)$$

with boundary conditions

$$v(0,t) = v(1,t) = 0, \quad v(x,0) = u(x,0) - w(x,0) = 1 - x.$$

We begin by simplifying the PDE to

$$\frac{\partial v}{\partial t} = 2q^2 t \frac{\partial^2 v}{\partial x^2} - 2t$$

and we write

$$v(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin(n\pi x)$$

which when plugged into our PDE gives

$$\sum_{n=1}^{\infty} \frac{db_n}{dt} \sin(n\pi x) = \left(-\frac{2q^2 t}{n^2\pi^2} \sum_{n=1}^{\infty} \frac{db_n}{dt} \sin(n\pi x)\right) - 2t$$

so

$$\sum_{n=1}^{\infty} \left[\frac{db_n}{dt} + 2(qn\pi)^2 t b_n\right] \sin(n\pi x) = -2t$$

which is a Fourier sine series for $-2t$ and hence the coefficient in the series, a differential equation here, is given by the usual integral,

$$\frac{db_n}{dt} + 2(qn\pi)^2 t b_n = 2\int_0^1 (-2t)\sin(n\pi x)\,dx = -4t\left[\frac{(-1)^n - 1}{n\pi}\right]$$

which is linear with integrating factor

$$\mu = \exp\left(\int 2(qn\pi)^2 t\,dt\right) = \exp\bigl((qn\pi)^2 t^2\bigr)$$

and therefore has solution

$$b_n = \exp\bigl(-(qn\pi)^2 t^2\bigr) \int -4t \exp\bigl((qn\pi)^2 t^2\bigr)\left[\frac{(-1)^n - 1}{n\pi}\right] dt + C\exp\bigl(-(qn\pi)^2 t^2\bigr)$$

which I solved with Sage to obtain

$$b_n = -\frac{2((-1)^n - 1)}{\pi^3 n^3 q^2} + C\exp\bigl(-(qn\pi)^2 t^2\bigr)$$

so we have now

$$v(x,t) = \sum_{n=1}^{\infty} \left(-\frac{2((-1)^n - 1)}{\pi^3 n^3 q^2} + C\exp\bigl(-(qn\pi)^2 t^2\bigr)\right)\sin(n\pi x).$$

We can now apply our initial conditions:

$$v(0,t)=\sum_{n=1}^{\infty}\left(-\frac{2((-1)^n-1)}{\pi^3n^3q^2}+C\exp\bigl(-(qn\pi)^2t^2\bigr)\right)\cdot\sin(0)=0$$

and

$$v(1,t)=\sum_{n=1}^{\infty}\left(-\frac{2((-1)^n-1)}{\pi^3n^3q^2}+C\exp\bigl(-(qn\pi)^2t^2\bigr)\right)\cdot\sin(n\pi)=0$$

and our boundary condition:

$$v(x,0)=\sum_{n=1}^{\infty}\left(-\frac{2((-1)^n-1)}{\pi^3n^3q^2}+C\right)\sin(n\pi x)=1-x$$

which is another Fourier sine series and must satisfy

$$-\frac{2((-1)^n-1)}{\pi^3n^3q^2}+C=2\int_0^1(1-x)\sin(n\pi x)\,dx=\frac{2}{n\pi}\implies C=\frac{2}{n\pi}+\frac{2((-1)^n-1)}{\pi^3n^3q^2}=\frac{2n^2\pi^2q^2+2((-1)^n-1)}{n^3\pi^3q^2}$$

so

$$v(x,t)=\sum_{n=1}^{\infty}\frac{2}{n\pi}\exp\bigl(-(qn\pi)^2t^2\bigr)\sin(n\pi x).$$

So we have our full solution (with my $q=5$),

$$u(x,t)=v(x,t)+w(x,t)=t^2+x+\sum_{n=1}^{\infty}\frac{2}{n\pi}\exp\bigl(-(5n\pi)^2t^2\bigr)\sin(n\pi x)$$

Problem 3. Solve for $u(x,t)$.

$$\frac{\partial^2 u}{\partial t^2} = q^2 \frac{\partial^2 u}{\partial x^2} + 6q^2(x - x^2), \qquad 0 < x < 1, \quad t > 0,$$

$$u(0,t) = 2, \quad \frac{\partial u}{\partial x}(1,t) = 1, \quad t > 0, \quad \text{and} \quad u(x,0) = \frac{1}{2}x^4 - x^3 + x^2 + (2+p)x, \quad \frac{\partial u}{\partial t}(x,0) = 0$$

Solution 3. Let

$$u(x,t) = w(x,t) + v(x)$$

so we get that

$$w_{tt} = q^2(w_{xx} + v_{xx}) + 6q^2(x - x^2).$$

We want

$$q^2 v_{xx} + 6q^2(x - x^2) = 0$$

so

$$v(x) = -x^3 + \frac{1}{2}x^4 - Cx - C_1.$$

Our boundary conditions on $u(x,t)$ are that

$$u(0,t) = w(0,t) + v(0) = 2$$

and we want $w(0,t) = 0$ so we say that $v(0) = 2$. We do the same for $u_x(1,t) = w_x(1,t) + v_x(1) = 1$ so $v_x(1) = 1$. Applying these, $v(0) = -C_1 = 2$ and $v_x(1) = -3 + 2 - C = 1 \implies C = 2$ so $v(x) = \frac{1}{2}x^4 - x^3 + 2x - 2$. Now our pde becomes

$$w_{tt} = q^2 w_{xx} + 6q^2(x^2 - x) + 6q^2(x - x^2) = q^2 w_{xx}$$

with conditions

$$w(0,t) = 0, \quad \frac{\partial w}{\partial x}(1,t) = 0, \quad t > 0, \quad \text{and} \quad w(x,0) = x^2 + px + 2, \quad \frac{\partial w}{\partial t}(x,0) = 0.$$

Now we can do separation of variables with

$$w(x,t) = \phi(x)T(t)$$

which gives us

$$\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = \frac{q^2}{\phi}\frac{\partial^2 \phi}{\partial x^2} = P$$

which gives us two equations,

$$\phi_{xx} - \frac{P}{q^2}\phi(x) = 0$$

and

$$\frac{\partial^2 T}{\partial t^2} - PT = 0.$$

Solving the position equation first we look at the usual cases:

(1) $P = 0$: this gives us $\phi_{xx} = 0 \implies \phi = Cx$ and $\phi(1) = 0 \implies C = 0$.

(2) $P > 0 \implies P = \lambda^2$: $\phi_{xx} - \frac{\lambda^2}{q^2}\phi(x) = 0$ which gives $\phi(x) = C_1 e^{\lambda x/q} + C_2 e^{-\lambda x/q}$ and $\phi(0) = C_1 + C_2 \implies C_1 = -C_2$ and $\phi(1) = C_1 e^{\lambda/q} - C_1 e^{-\lambda/q} \implies C_1 = C_2 = 0$.

(3) $P < 0 \implies P = -\lambda^2$: $\phi_{xx} + \frac{\lambda^2}{q^2}\phi(x) = 0$ which gives $\phi(x) = C_1 \cos\left(\frac{\lambda}{q}x\right) + C_2 \sin\left(\frac{\lambda}{q}x\right)$ and $\phi(0) = C_2 = 0$ and $\phi(1) = C_1 \sin\left(\frac{\lambda}{q}\right) = 0 \implies \lambda = qn\pi$

Now solving

$$\frac{\partial^2 T}{\partial t^2} + q^2 n^2 \pi^2 T = 0$$

which gives

$$T = C_3 \cos(qn\pi t) + C_4 \sin(qn\pi t)$$

so our full solution is

$$w(x,t) = \sum_{n=1}^{\infty} (C_1 \cos(qn\pi t) + C_2 \sin(qn\pi t)) \sin(n\pi x)\,.$$

Now we apply our initial conditions:

$$w(x,0) = \sum_{n=1}^{\infty} C_1 \sin(n\pi x) = x^2 + px + 2$$

so

$$C_1 = 2\int_0^1 (x^2 + px + 2)\sin(n\pi x)\, dx$$

which Sage evaluates as

$$-\frac{2(\pi^2(-1)^n n^2 p - 2\pi^2 n^2 + (3\pi^2 n^2 - 2)(-1)^n + 2)}{\pi^3 n^3}$$

and the other condition on

$$\frac{\partial}{\partial t}\sum_{n=1}^{\infty} qn\pi(-C_1 \sin(qn\pi t) + C_2 \cos(qn\pi t)) \sin(n\pi x)$$

gives

$$w_t(x,0) = \sum_{n=1}^{\infty} C_2 qn\pi \cos(qn\pi t) \sin(n\pi x) \implies C_2 = 0$$

so

$$w(x,t) = -2\sum_{n=1}^{\infty} \frac{\pi^2(-1)^n n^2 p - 2\pi^2 n^2 + (3\pi^2 n^2 - 2)(-1)^n + 2}{\pi^3 n^3} \cos(qn\pi t)\sin(n\pi x)$$

so

$$u(x,t) = w(x,t) + v(x) = \frac{1}{2}x^4 - x^3 + 2x - 2 - 2\sum_{n=1}^{\infty} \frac{\pi^2(-1)^n n^2 p - 2\pi^2 n^2 + (3\pi^2 n^2 - 2)(-1)^n + 2}{\pi^3 n^3} \cos(qn\pi t)\sin(n\pi x)$$

Problem 4. Consider your own homogeneous convection problem with:

$$u(x,0) = 10x(L-x)\sin\left(\frac{\pi x}{L}\right)$$

$$h = q, \quad \kappa = p, \quad L = qp, \quad k = \frac{p}{q+p}$$

where p is the largest digit and q is the smallest nonzero digit of the last four digits of your student I.D. number. Using SAGE to carry out an investigation of the temperature distribution in the bar:

(a) Find the first 20 eigenvalues of the problem

(b) Obtain the temperature distribution as a function of x and t.

(c) Plot the temperature of the right end of the bar for the first 120 seconds.

(d) Plot the temperature of the right end of the bar for the first 120 seconds and determine the maximum temperature that the right of the bar.

(e) Obtain the time(s) at which the middle of the bar temperature is 25 degrees.

(f) Plot the temperature distribution in the bar at time(s) found in part (e).

Solution 4.

q4

December 2, 2025

[23]:
```
# a
clear_vars()
p = 9
q = 5
r = 22

x, n, t = var("x n t")
h = q
kappa = p
L = q*p
k = p/(q+p)
T1 = 0 # Assumed because this is in the sample code
T0 = 0
f = 10*x*(L-x)*sin(pi*x/L)
bn(s) = integral(f*sin(s*x), x, 0, L)/integral(sin(s*x)^2, x, 0, L)

w = 0
for n in [1..20]:
    wn=w
    EQ=tan(x*L)+kappa*x/h
    lambn=EQ.find_root((2*n-1)*pi/(2*L),n*pi/L)
    bnn=N(bn(lambn))
    w=wn+bnn*sin(lambn*x)*e^(-lambn^2*k*t)
    show(rf"lambda_{n}: {lambn}")
```

```
lambda_1: 0.06714051111739618

lambda_2: 0.13435344136504954

lambda_3: 0.2017003456182572

lambda_4: 0.26922477792503907

lambda_5: 0.33695044796079876

lambda_6: 0.4048837225601791

lambda_7: 0.473018293558356

lambda_8: 0.5413400467651973

lambda_9: 0.6098310313607336
```

```
lambda_10: 0.6784722169145844

lambda_11: 0.7472451706647032

lambda_12: 0.8161329398175129

lambda_13: 0.8851204159390972

lambda_14: 0.9541943936557794

lambda_15: 1.0233434667116108

lambda_16: 1.092557849800966

lambda_17: 1.161829177050288

lambda_18: 1.2311503042395802

lambda_19: 1.3005151276895273

lambda_20: 1.3699184247385792
```

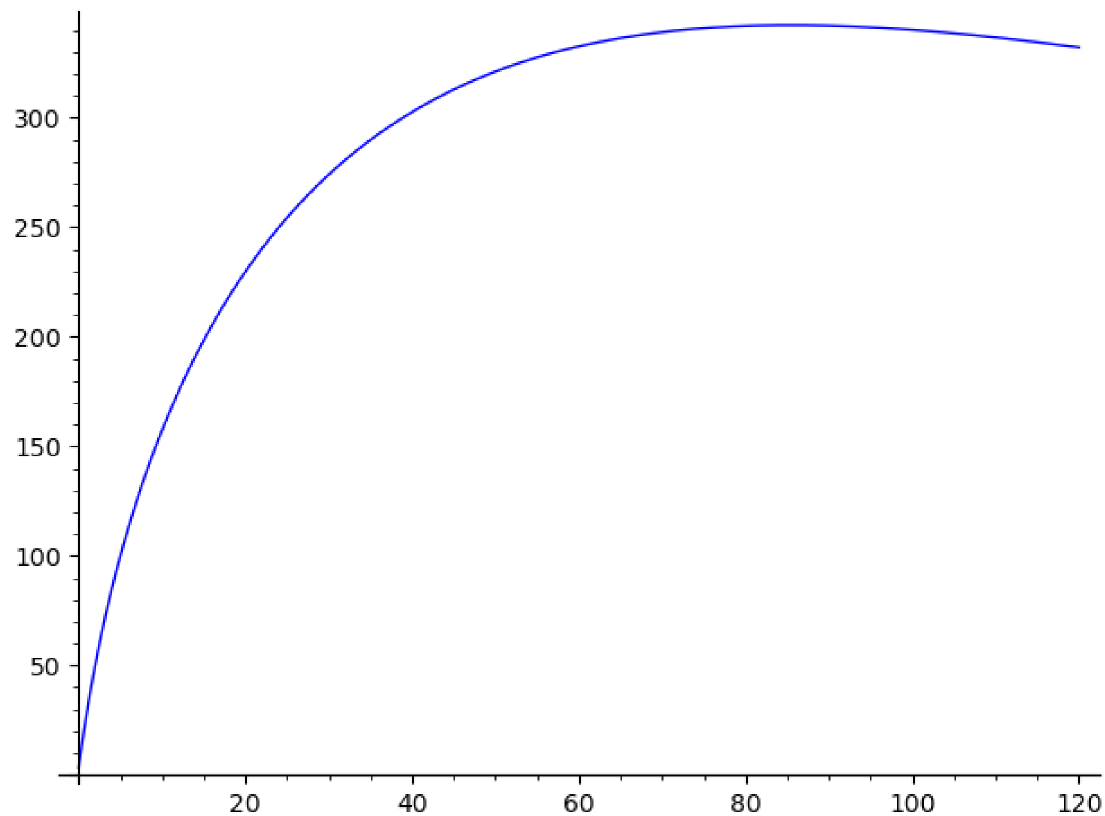
[24]:

```
# b
v=T0+x*h*(T1-T0)/(kappa+h*L)
u(x,t)=w+v
```

[25]:

```
# c
EQ1 = u(L,t)
plot(EQ1,t,0,120)
```

[25]:



[26]:
```
# d
maxtime, maxtemp  = find_local_maximum(EQ1, 60, 120) # Guessed range based on␣
↪graph

show(rf"max temp occurs at {maxtime}s and is {maxtemp} units")
```

max temp occurs at 342.2351971425617s and is 85.36838592140077 units

[27]:
```
# e
fun = EQ1 - 25 # will cross at T=25 units
plot(fun, t, 0, 120)
tft = fun.find_root(0,10)
show(rf"25 deg occurs at {tft}s")
```

25 deg occurs at 0.8255692025802429s

[28]:
```
# f
plot(u(x, tft), x, 0, L)
```

[28]: