Math 3770H: Assignment IV

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Problem 1. Let $f(z) = u(x, y) + iv(x, y)$ be a function that is continuous on a closed bounded region R and analytic and not constant throughout the interior of R. Prove that the component function $u(x, y)$ has a minimum value in R which occurs on the boundary of R and never in the interior. (See Exercise 2.)

Solution 1. By the corollary in section 59 $f(z)$ being continuous on a closed bounded region R and analytic and not constant throughout the interior of R is sufficient to guarantee us that the maximum value of $|f(z)|$ occurs on the boundary of R. Consider now the function

$$g(z) = e^{f(z)}$$

which is strictly increasing and will obtain its maximum at the same point as $f(z)$, on the boundary. Note that

$$g(z) = e^{f(z)} = g(z) = e^{u(x,y)+iv(x,y)} = e^u e^{iv} = e^u \left[\cos(v) + i \sin(v)\right]$$

and so

$$|g(z)| = \sqrt{e^{2u} \left[\cos^2(v) + \sin^2(v)\right]} = e^u.$$

We already know that $g(z)$ attains its maximum on the boundary and because the maximum is governed by e^u which has its maximum at the same point as $u(x, y)$ we know that $u(x, y)$ must have its maximum on the boundary. Hence the function

$$h(z) = \frac{1}{g(z)}$$

will obtain its *minimum* when $g(z)$ obtains its maximum which we know is only at the maximum of $u(x, y)$, on the boundary.

Problem 2. Write $z = re^{i\theta}$, where $0 < r < 1$, in the summation formula (10), Sec. 61. Then, with the aid of the theorem in Sec. 61, show that

$$\sum_{n=1}^{\infty} r^n \cos(n\theta) = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2} \quad \text{and} \quad \sum_{n=1}^{\infty} r^n \sin(n\theta) = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$$

when $0 < r < 1$. (Note that these formulas are also valid when $r = 0$.)

Solution 2. We know that

$$z = re^{i\theta} = r\cos\theta + i\sin\theta \implies z^n = r^n \cos n\theta + ir^n \sin n\theta$$

and so the formula given

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} = \frac{1}{1 - re^{i\theta}}$$

and by the theorem because this converges we can write its component sums

$$\sum_{n=0}^{\infty} z^n = \sum_{n=1}^{\infty} r^n \cos(n\theta) + i \sum_{n=1}^{\infty} r^n \sin(n\theta)$$

and we can expand

$$\begin{aligned}
\frac{1}{1 - re^{i\theta}} &= \frac{1}{1 - r(\cos\theta + i\sin\theta)} \\
&= \frac{1}{1 - r\cos\theta - ir\sin\theta} \\
&= \frac{1 - r\cos\theta + ir\sin\theta}{(1 - r\cos\theta - ir\sin\theta)(1 - r\cos\theta + ir\sin\theta)} \\
&= \frac{1 - r\cos\theta + ir\sin\theta}{(1 - r\cos\theta)^2 + (r\sin\theta)^2} \\
&= \frac{1 - r\cos\theta + ir\sin\theta}{1 - 2r\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta} \\
&= \frac{1 - r\cos\theta + ir\sin\theta}{1 - 2r\cos\theta + r^2} \\
&= \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2} + i\frac{r\sin\theta}{1 - 2r\cos\theta + r^2} \\
&= \sum_{n=1}^{\infty} r^n \cos(n\theta) + i\sum_{n=1}^{\infty} r^n \sin(n\theta)
\end{aligned}$$

Hence we obtain the desired equalities.

Problem 3. Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

Solution 3. First we want to break up

$$\frac{z}{(z-1)(z-3)}$$

with partial fractions.

$$\frac{z}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$$

where

$$\begin{aligned}
A(z-3) + B(z-1) = z& \\
&\implies Az - 3A + Bz - B = z \\
&\implies A + B = 1 \text{ and } -3A - B = 0 \\
&\implies -3A = B \\
&\implies A - 3A = 1 \\
&\implies A = -1/2, B = 3/2
\end{aligned}$$

So we have

$$\begin{aligned}
\frac{z}{(z-1)(z-3)} &= -\frac{1}{2(z-1)} + \frac{3}{2(z-3)} \\
&= -\frac{1}{2(z-1)} + \frac{3}{2}\frac{1}{z-1-2} \\
&= -\frac{1}{2(z-1)} - \frac{3}{2^2}\frac{1}{1-\frac{z-1}{2}} \\
&= -\frac{1}{2(z-1)} - \frac{3}{2^2}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n} \qquad\qquad \frac{1}{a(1-b/a)} = \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n \\
&= -\frac{1}{2(z-1)} - 3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}
\end{aligned}$$

Problem 4. Prove that if f is analytic at z_0 and $f(z_0) = f'(z_0) = \cdots = f^{(m)}(z_0) = 0$, then the function g defined by means of the equations

$$g(z) = \begin{cases} \dfrac{f(z)}{(z-z_0)^{m+1}} & \text{when } z \neq z_0 \\ \dfrac{f^{(m+1)}(z_0)}{(m+1)!} & \text{when } z = z_0 \end{cases}$$

is analytic at z_0

Solution 4. For the two cases,

(1) $z \neq z_0$: here since we already know that $f(z)$ is analytic and $1/(z-z_0)^{m+1}$ is analytic for $z \neq z_0$ $g(z \neq z_0)$ is analytic because it is the quotient of two other analytic functions.

(2) $z = z_0$: In this neighbourhood we can expand $g(z \neq z_0)$ as a Taylor series about z_0 where we start at $m+1$ because all the other derivatives at z_0 are zero:

$$\begin{aligned} g(z) &= \frac{f(z)}{(z-z_0)^{m+1}} \\ &= \frac{1}{(z-z_0)^{m+1}} \sum_{k=m+1}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \\ &= \cancel{\frac{1}{(z-z_0)^{m+1}}} \sum_{k=0}^{\infty} \frac{f^{(k+m+1)}(z_0)}{(k+m+1)!}(z-z_0)^{k\cancel{+m+1}} \\ &= \sum_{k=0}^{\infty} \frac{f^{(k+m+1)}(z_0)}{(k+m+1)!}(z-z_0)^k \end{aligned}$$

which is convergent by definition and hence $g(z)$ is analytic at $z = z_0$.

Problem 5. Use multiplication of series to show that

$$\frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \dots \quad (0 < |z| < 1).$$

Solution 5. We have that

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

and

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \dots$$

so

$$\begin{aligned}
e^z\frac{1}{1+z^2} &= \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right)\left(1 - z^2 + z^4 - z^6 + \dots\right) \\
&= \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right) - \left(z^2 + z^3 + \frac{z^4}{2!} + \frac{z^5}{3!} + \dots\right) + \left(z^4 + z^5 + \frac{z^6}{2!} + \frac{z^7}{3!} + \dots\right) - \left(z^6 + z^7 + \frac{z^8}{2!} + \frac{z^9}{3!} + \dots\right) \\
&= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} - z^2 - z^3 - \frac{z^4}{2!} - \frac{z^5}{3!} + z^4 + z^5 + \frac{z^6}{2!} + \frac{z^7}{3!} - z^6 - z^7 - \frac{z^8}{2!} - \frac{z^9}{3!} + \dots \\
&= 1 + z - \frac{z^2}{2} - \frac{5z^3}{6} + \frac{z^4}{2} - \frac{2z^5}{3} - \frac{z^6}{2} - \frac{5z^7}{6} - \frac{z^8}{2} - \frac{z^9}{6} + \dots \\
&= 1 + z - \frac{z^2}{2} - \frac{5z^3}{6} + \dots
\end{aligned}$$

which we multiply by $1/z$ to obtain

$$\frac{1}{z} + 1 - \frac{z}{2} - \frac{5z^2}{6} + \dots.$$

Problem 6. Use Cauchy's residue theorem (Sec. 76) to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:

$$\text{(a) } \frac{\exp(-z)}{z^2}; \qquad \text{(b) } \frac{\exp(-z)}{(z-1)^2}; \qquad \text{(c) } z^2 \exp\left(\frac{1}{z}\right); \qquad \text{(d) } \frac{z+1}{z^2-2z}.$$

Solution 6.

(a) We are trying to evaluate

$$\int_C \frac{\exp(-z)}{z^2}\, dz$$

which is analytic inside and on C except a $z = 0$ so we can apply Cauchy's residue theorem by expanding the integrand as a series:

$$e^{-z}\frac{1}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{(-z)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n z^{n-2}}{n!}.$$

The coefficent of the $1/z$ term in this series will be the coefficient when $n - 2 = -1 \implies n = 1$ so

$$\mathrm{Res}_{z=0} = -1/1! = -1$$

so

$$\int_C \frac{\exp(-z)}{z^2}\, dz = -2\pi i$$

(b) We are trying to evaluate

$$\int_C \frac{\exp(-z)}{(z-1)^2}\, dz$$

which is analytic inside and on C except at $z = 1$, where it has a pole of order 2. We can write the integrand as

$$\frac{\phi(z)}{(z-1)^2}, \quad \phi(z) = \exp(-z)$$

and so we can directly apply the formula

$$\mathrm{Res}_{z=1} f(z) = \frac{\phi^{(2-1)}(1)}{(2-1!)} = -\frac{1}{e}.$$

We multiply this by $2\pi i$ to obtain the value of the integral, $-2\pi i/e$.

(c)

$$\frac{z+1}{z^2-2z} = \frac{z+1}{z(z-2)}$$

which has singularities at $z = 0, 2$. We can rewrite the fraction as the only terms in its series expansion using partial fractions,

$$\frac{z+1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2}$$

where

$$A(z-2) + Bz = z + 1 \implies A = -1/2, B = 3/2$$

so

$$\frac{z+1}{z(z-2)} = -\frac{1}{2}\frac{1}{z} + \frac{3}{2}\frac{1}{z-2}$$

which immediately gives us the residues at the two singularities. By the theorem we just sum these and multiply by $2\pi i$ to obtain the result of the integral which is $2\pi i$.