

# Math 3770H: Assignment II

Jeremy Favro (0805980)  
Trent University, Peterborough, ON, Canada

October 14, 2025

**Problem 1.** Write the function

$$f(z) = z + \frac{1}{z} \quad (z \neq 0)$$

in the form  $f(z) = u(r, \theta) + iv(r, \theta)$

**Solution 1.** Writing  $f$  first in polar form to switch to a dependence on  $r = |z|, \theta = \arctan(\operatorname{Im} z / \operatorname{Re} z)$ ,

$$z = re^{i\theta} = r \cos \theta + ir \sin \theta.$$

This is actually really nice here as it means we can easily write the inverse without using a fraction,

$$z^{-1} = r^{-1}e^{-i\theta} = r^{-1} \cos(-\theta) + ir^{-1} \sin(-\theta).$$

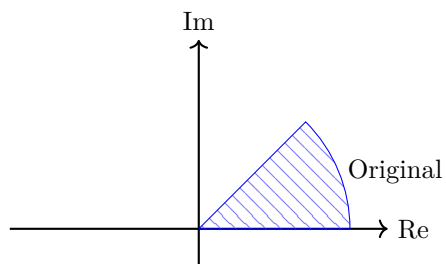
Which can be reworked further using the even/odd nature of  $\cos$  and  $\sin$  respectively giving:

$$\begin{aligned} f(z) &= r \cos \theta + ir \sin \theta + r^{-1} \cos \theta - ir^{-1} \sin \theta \\ &= r \cos \theta + r^{-1} \cos \theta + ir \sin \theta - ir^{-1} \sin \theta \\ &= (r + r^{-1}) \cos \theta + (r - r^{-1}) \sin \theta \end{aligned}$$

**Problem 2.** Sketch the region onto which the sector  $r \leq 1$ ,  $0 \leq \theta \leq \pi/4$  is mapped by the transformation

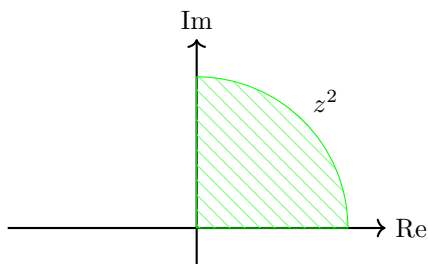
(a)  $w = z^2$ ;      (b)  $w = z^3$ ;      (c)  $w = z^4$ .

**Solution 2.** Generally the transform  $w = z^n$  will give  $w = r^n e^{in\theta}$ , just by exponentiation laws. Our original region looks like:

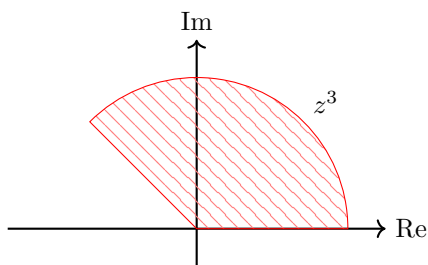


Because our region only extends out to  $r = 1$  the radius won't change.

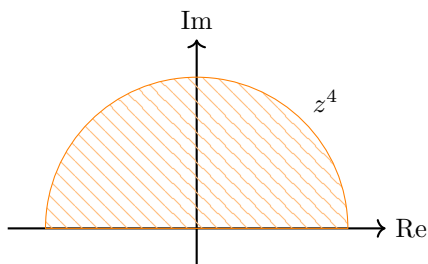
(a)



(b)



(c)



**Problem 3.** Use definition (2), Sec.15, of the limit to prove that

$$(a) \lim_{z \rightarrow z_0} \operatorname{Re} z = \operatorname{Re} z_0; \quad (b) \lim_{z \rightarrow z_0} \bar{z} = \bar{z}_0; \quad (c) \lim_{z \rightarrow 0} \frac{\bar{z}^2}{z} = 0.$$

**Solution 3.**

(a) Let  $\epsilon > 0$ . Suppose  $|z - z_0| < \delta$ . If we take  $z = x + iy$  and  $z_0 = x_0 + iy_0$  then we have

$$|\operatorname{Re} z - \operatorname{Re} z_0| = |x - x_0| \leq |z - z_0|.$$

We can make that last statement because

$$|x - x_0| \leq |x - x_0 + iy - iy_0| = |z - z_0|.$$

This means that

$$|\operatorname{Re} z - \operatorname{Re} z_0| = |z - z_0| \leq \delta$$

by the  $\epsilon - \delta$  definition we are using. Having this means that provided  $|z - z_0| \leq \delta = \epsilon$ , then  $|\operatorname{Re} z - \operatorname{Re} z_0| < \epsilon$  by transitivity.

(b) Let  $\epsilon > 0$ . Suppose  $|z - z_0| < \delta$ . We have

$$|\bar{z} - \bar{z}_0| = |\overline{z - z_0}| = |z - z_0| < \delta$$

all by properties of the complex conjugate. So provided  $|z - z_0| < \delta = \epsilon$ , then  $|\bar{z} - \bar{z}_0| < \epsilon$ .

(c) Let  $\epsilon > 0$ . Suppose  $|z - z_0| < \delta$ . We have

$$\left| \frac{\bar{z}^2}{z} - 0 \right| = \frac{|\bar{z}^2|}{|z|} = |z|.$$

This means that

$$\left| \frac{\bar{z}^2}{z} - 0 \right| = |z| < \delta = \epsilon$$

**Problem 4.** With the aid of the theorem in Sec. 17, show that when

$$T(z) = \frac{az + b}{cz + d} \quad (ad - bc) \neq 0$$

(a)  $\lim_{z \rightarrow \infty} T(z) = \infty$  if  $c = 0$ ;

(b)  $\lim_{z \rightarrow \infty} T(z) = \frac{a}{c}$  and  $\lim_{z \rightarrow -d/c} T(z) = \infty$  if  $c \neq 0$

**Solution 4.**

(a) For  $c = 0$  we get

$$T(z) = \frac{az + b}{d}.$$

The referenced theorem states that

$$\lim_{z \rightarrow z_0} T(z) = \infty \quad \text{if} \quad \lim_{z \rightarrow z_0} T^{-1}(z) = 0.$$

Here we have then that

$$\lim_{z \rightarrow \infty} T^{-1}(z) = \lim_{z \rightarrow \infty} \frac{d}{az + b} = \frac{d}{\infty} = 0 \implies \lim_{z \rightarrow \infty} T(z) = \infty$$

(b)

$$\begin{aligned}
&= \lim_{z \rightarrow 0} T(z^{-1}) \\
&= \lim_{z \rightarrow 0} \frac{az^{-1} + b}{cz^{-1} + d} \\
&= \lim_{z \rightarrow 0} \frac{az^{-1}}{cz^{-1} + d} + \frac{b}{cz^{-1} + d} \\
&= \lim_{z \rightarrow 0} \frac{a}{c + dz} + \frac{b}{c + dz} \\
&= \frac{a}{c}
\end{aligned}$$

Which, by the second part of the given theorem which says that

$$\lim_{z \rightarrow \infty} f(z) = w_0 \quad \text{if} \quad \lim_{z \rightarrow \infty} f(z^{-1}) = w_0$$

means that our original limit

$$\lim_{z \rightarrow \infty} T(z) = \frac{a}{c}$$

holds true. For the second part,

$$\begin{aligned}
&= \lim_{z \rightarrow -d/c} T^{-1}(z) \\
&= \lim_{z \rightarrow -d/c} \frac{c(-d/c) + d}{a(-d/c) + b} \\
&= \lim_{z \rightarrow -d/c} \frac{0}{a(-d/c) + b} = 0
\end{aligned}$$

which by the first part of the theorem (as used in part (a) here) means that our original limit holds true.

**Problem 5.** Use the method in Example 2, Sec. 19, to show that  $f'(z)$  does not exist at any point  $z$  when

$$(a) f(z) = \operatorname{Re} z; \quad (b) f(z) = \operatorname{Im} z.$$

**Solution 5.**

(a)

$$\frac{\Delta f}{\Delta z} = \frac{\operatorname{Re}(z + \Delta z) - \operatorname{Re} z}{\Delta z} = \frac{\operatorname{Re} z + \operatorname{Re} \Delta z - \operatorname{Re} z}{\Delta z} = \frac{\operatorname{Re} \Delta z}{\Delta z}.$$

Approaching along the real axis gives  $\Delta z = \Delta x + 0i$  so

$$\frac{\operatorname{Re} \Delta z}{\Delta z} = \frac{\Delta x}{\Delta x} = 1.$$

Approaching along the imaginary axis gives  $\Delta z = 0 + \Delta yi$  so

$$\frac{\operatorname{Re} \Delta z}{\Delta z} = \frac{0}{\Delta yi} = 0.$$

Because these two do not agree the limit which defines the derivative cannot exist and therefore the derivative itself cannot exist.

(b)

$$\frac{\Delta f}{\Delta z} = \frac{\operatorname{Im}(z + \Delta z) - \operatorname{Im} z}{\Delta z} = \frac{\operatorname{Im} z + \operatorname{Im} \Delta z - \operatorname{Im} z}{\Delta z} = \frac{\operatorname{Im} \Delta z}{\Delta z}.$$

We can make the same argument as previously, first approaching along the real axis,

$$\frac{\operatorname{Im} \Delta z}{\Delta z} = \frac{0}{\Delta x} = 0.$$

Then the imaginary axis,

$$\frac{\operatorname{Im} \Delta z}{\Delta z} = \frac{\Delta y}{\Delta yi} = -i.$$

Again because this limit does not exist the derivative cannot exist.

**Problem 6.** Use the theorem in Sec. 24 to show that each of these functions is differentiable in the indicated domain of definition, and also to find  $f'(z)$ :

- (a)  $f(z) = 1/z^4$  ( $z \neq 0$ );  
 (b)  $f(z) = e^{-\theta} \cos(\ln r) + ie^{-\theta} \sin(\ln r)$  ( $r > 0, 0 < \theta < 2\pi$ ).

**Solution 6.**

- (a) First we note that

$$f(z) = z^{-4} = r^{-4} (\cos(4\theta) - i \sin(4\theta)).$$

The partials are then

$$\begin{aligned} u_r &= -4r^{-5} \cos(4\theta) & u_\theta &= -4r^{-4} \sin(4\theta) \\ v_r &= +4r^{-5} \sin(4\theta) & v_\theta &= -4r^{-4} \cos(4\theta). \end{aligned}$$

These agree with the Cauchy-Riemann equations for polar coordinates,

$$ru_r = v_\theta \quad \text{and} \quad u_\theta = -rv_r$$

which, alongside the fact that the partials exist for all  $(r, \theta)$ , satisfies the theorem. This means that the derivative will exist for all  $(r, \theta)$  in the domain. Recalling that, as in real valued functions, the power rule applies, we have that

$$f'(x) = -4/z^5.$$

- (b) Our partials here are

$$\begin{aligned} u_r &= -r^{-1} e^{-\theta} \sin(\ln r) & u_\theta &= -e^{-\theta} \cos(\ln r) \\ v_r &= r^{-1} e^{-\theta} \cos(\ln r) & v_\theta &= -e^{-\theta} \sin(\ln r). \end{aligned}$$

which satisfy the Cauchy-Riemann equations. Because the trig functions are only undefined (they oscillate infinitely quickly) at  $r = 0$  which is not a part of the domain the derivative exists over the domain. Now using the definition of the derivative,

$$f'(x) = e^{-i\theta} [u_r + iv_r]$$

we get

$$f'(x) = e^{-i\theta} [-r^{-1} e^{-\theta} \sin(\ln r) + ir^{-1} e^{-\theta} \cos(\ln r)]$$

**Problem 7.** With the aid of the theorem in Sec. 21, show that each of these functions is nowhere analytic:

- (a)  $f(z) = xy + iy$ ;      (b)  $f(z) = 2xy + i(x^2 - y^2)$ ;      (c)  $f(z) = e^y e^{ix}$ .

**Solution 7.**

- (a) Here the partials are

$$u_x = y; \quad v_y = 1; \quad v_x = 0; \quad u_y = x.$$

This means that the Cauchy-Riemann equations are

$$y = 1; \quad x = 0.$$

As the definition for an analytic function requires that the derivative exists on a disk around a point for a function to be analytic at the center of the disk, this function cannot be analytic because it is only differentiable at  $(0, 1)$ .

- (b) Here the partials are

$$u_x = 2y; \quad v_y = -2y; \quad v_x = 2x; \quad u_y = 2x.$$

This means that the Cauchy-Riemann equations are

$$2y = -2y; \quad 2x = -2x$$

which implies the derivative only exists at  $(0, 0)$  which, by the same reasoning as above, the function is nowhere analytic.

$$(c) \ f(z) = e^y e^{ix} = e^y (\cos x + i \sin x)$$

$$u_x = -e^y \sin x; \quad v_y = e^y \sin x; \quad u_y = e^y \cos x; \quad v_x = e^y \cos x.$$

This means that the Cauchy-Riemann equations are

$$-e^y \sin x = e^y \sin x; \quad e^y \cos x = -e^y \cos x$$

Which simplifies to

$$-\sin x = \sin x; \quad \cos x = -\cos x$$

which is only true at  $x = 0$  which is, again, not a neighborhood and this is therefore nowhere analytic.

**Problem 8.A.** Let the function  $f(z) = u(x, y) + iv(x, y)$  be analytic in a domain  $D$ , and consider the families of level curves  $u(x, y) = c_1$  and  $v(x, y) = c_2$ , where  $c_1$  and  $c_2$  are arbitrary real constants. Prove that these families are orthogonal. More precisely, show that if  $z_0 = (x_0, y_0)$  is a point in  $D$  which is common to two particular curves  $u(x, y) = c_1$  and  $v(x, y) = c_2$  and if  $f'(z_0) \neq 0$ , then the lines tangent to those curves at  $(x_0, y_0)$  are perpendicular.

**Problem 8.B.** Sketch the families of level curves of the component functions  $u$  and  $v$  when  $f(z) = 1/z$ , and note the orthogonality described in Exercise 2.

**Solution 8.A.** Coming at this from a multivariate calculus standpoint we know that level curves are orthogonal if their normal vectors are orthogonal. These normal vectors are given by the gradient of the corresponding function at that point,

$$\nabla u = (u_x, u_y); \quad \nabla v = (v_x, v_y).$$

For orthogonality we need the inner product,

$$u_x v_x + u_y v_y,$$

to be zero. Now as we know that  $f(z)$  is analytic we can say that, by definition, the Cauchy-Riemann equations hold in  $D$  and therefore

$$u_x = v_y; \quad u_y = -v_x.$$

Plugging these into our previous expression for the inner product,

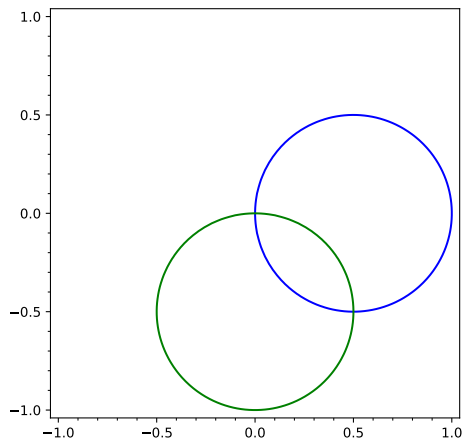
$$v_y v_x - v_x v_y = 0.$$

**Solution 8.B.** With

$$f(z) = 1/z = 1/(x + iy) = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}i$$

we have

$$u = \frac{x}{x^2 + y^2}; \quad v = -\frac{y}{x^2 + y^2}$$



Blue: level curve of  $u = 1$ , Green: level curve of  $v = 1$