Calculus II: Assignment 2

Jeremy Favro

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A tank has the shape of the solid of revolution obtained by revolving the curve $x = \sin\left(\frac{\pi y}{3}\right)$, for $0 \le y \le 3$, about the y-axis, with both axes being measured in metres. The tank is completely filled with water which is then drained from the tank at a contant rate of 100 litres per minute. Suppose that a given instant the water in the tank is w metres deep.

Problem 1. What is the volume (in litres) of the water in the tank at the given instant? Work it out both by hand and by using SageMath.

Solution 1.

$$V(w) = \pi \int_0^w f^2(x)\,dx, \text{ where f(x) is the function of the solid of revolution} \tag{1}$$

So,

$$\begin{aligned}
&= \pi \int_0^w \sin^2(\frac{\pi y}{3})\,dy \\
&= \pi \int_0^w \sin^2(\frac{\pi y}{3})\,dy \implies \text{ let } u = \frac{\pi y}{3},\, dy = \frac{3du}{\pi} \\
&= \pi \int_0^w \sin^2(u)\,\frac{3}{\pi}du \\
&= \cancel{\pi}\frac{3}{\cancel{\pi}} \int_0^w \sin^2(u)\,du \implies \text{ use reduction formula} \\
&= 3\int_0^w \left[-\frac{1}{2}\sin(u)\cos(u) + \frac{1}{2}\int_0^w \sin^0(u)\,du\right] du \\
&= 3\int_0^w \left[-\frac{1}{2}\sin(u)\cos(u) + \frac{w}{2}\right] du \\
&= 3\left[-\frac{1}{2}\sin(u)\cos(u) + \frac{u}{2}\right]\Bigg|_0^w \\
&= 3\left[-\frac{1}{2}\sin\left(\frac{\pi y}{3}\right)\cos\left(\frac{\pi y}{3}\right) + \frac{\pi y}{6}\right]\Bigg|_0^w \\
&= -\frac{3}{2}\sin\left(\frac{\pi w}{3}\right)\cos\left(\frac{\pi w}{3}\right) + \frac{\pi w}{2}
\end{aligned}$$

$$\therefore V(w) = \left[-\frac{3}{2}\sin\left(\frac{\pi w}{3}\right)\cos\left(\frac{\pi w}{3}\right) + \frac{\pi w}{2}\right]\cdot 1000 \leftarrow \text{convert to liters}$$

Using sage to evaluate:

```
clear_vars()

y = var('y')
w = var('w')

f = sin((pi*y)/3)
```

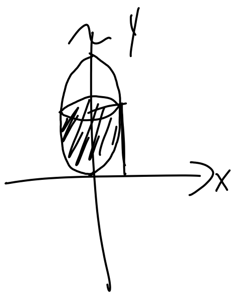
```
assume(w>0) # Has to be done to not have sage throw an error, true anyways as depth will
↪  never be negative and if it's zero volume is zero as well

definite_integral(pi*f^2, y, 0, w)
```

Problem 2. How is the depth of the water in the tank changing at the instant that the depth is 2 metres? Work it out without implicitly or explicitly using your final answer to question 1. You may use SageMath, or do it by hand, or mix these up.

Solution 2. I wasn't able to figure this out without using my answer to q1, and was unfortunately not able to make it to any of the office hours as they overlapped with my classes in the latter half of the week. Here's the couple pages of work (ramblings) I did in trying to solve it in case that counts:

$x = \text{Sin}\left(\frac{\pi y}{3}\right)$, $[0, 3]$ revolved about y-axis



A of one circular cross-section $= \pi \, \text{Sin}^2\left(\frac{\pi y}{3}\right)$

$$= \int_0^3 \pi \, \text{Sin}^2\left(\frac{\pi y}{3}\right) dy$$

$$= \pi \int_0^3 \text{Sin}^2\left(\frac{\pi y}{3}\right) dy \qquad \text{let } w = \frac{1}{3}\pi y$$

$$= \pi \int_0^3 \text{Sin}^2(w) \frac{dw}{\frac{1}{3}\pi} \qquad \frac{dw}{dy} = \frac{1}{3}\pi$$

$$dw = \frac{1}{3}\pi \, dy$$

$$= 3 \int_0^3 \text{Sin}^2(w) \, dw \qquad \frac{dw}{\frac{1}{3}\pi} = dy$$

$$= 3\left(-\frac{1}{2} \text{Sin}(w) \text{Cos} w + \frac{1}{2}\int \text{Sin}^0 w \cdot dw\right)$$

$$= 3\left(-\frac{1}{2} \text{Sin} w \, \text{Cos} w + \frac{1}{2} w\right)$$

$$= -\frac{3}{2} \text{Sin}\left(\frac{\pi y}{3}\right) \text{Cos}\left(\frac{\pi y}{3}\right) + \frac{\pi y}{2}$$

Convert to liters

1) $$Z\Big|_0^d = \left(-\frac{3}{2} \text{Sin} \frac{\pi d}{3} \text{Cos}\left(\frac{\pi d}{3}\right) + \frac{\pi d}{2}\right) \cdot 1000$$

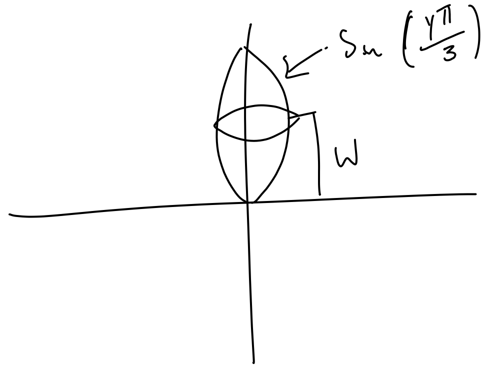
2) $$\frac{d(\text{Volume})}{d(\text{time})} = \frac{d(\text{Volume})}{d(\cancel{\text{depth}})} \times \frac{d(\cancel{\text{depth}})}{d(\text{time})} = \frac{\Delta \text{Volume}}{\Delta \text{depth}} \times \frac{\Delta \text{depth}}{\Delta \text{time}}$$

$$\frac{3\pi}{2} - 0.1t = \frac{dV}{dw} \times \frac{dw}{dt}$$

$$\frac{\Delta V_{olume}}{\Delta T_{ime}} = -0.1 \ m^3/min = \frac{\Delta V_{olume}}{\Delta D_{epth}} \times \frac{\Delta D_{epth}}{\Delta T_{ime}} \rightarrow \left.\frac{\Delta D}{\Delta t}\right|_w$$

$$\frac{\Delta D_{epth}}{\Delta T_{ime}} = \frac{\Delta D_{epth}}{\Delta V_{olume}} \times \frac{\Delta V_{olume}}{\Delta T_{ime}} \qquad \frac{\Delta V}{\Delta t} = -0.1$$

$$\frac{\Delta D}{\Delta T} = \frac{\Delta D}{\Delta V} \times 0.1 \ m^3/min$$



$$\frac{\Delta w}{\Delta t} = \frac{\Delta w}{\Delta V} \times -0.1 = \frac{\Delta V}{\Delta w}^{-1} \times 0.1$$

$$= \left(\frac{\pi}{2} + \right.$$

$$\frac{dV}{dw} = \left(-\frac{3}{2} \sin\left(\frac{\pi d}{3}\right) \cos\left(\frac{\pi d}{3}\right)\right)' + \left(\frac{\pi d}{2}\right)'$$

$$= -\frac{\pi}{2} \cos\left(\frac{2\pi d}{3}\right) + \frac{\pi}{2}$$

$$\frac{\Delta V}{\Delta t} = \frac{3\pi}{2} - 100t$$

$$\frac{dw}{dt} = \frac{dw}{dV} \cdot \frac{dV}{dt}$$

$$\frac{dw}{dt} = \frac{dw}{dV} \cdot (-0.1) \longrightarrow V(w) = -\frac{3}{4} \sin\left(\frac{2\pi d}{3}\right) + \frac{\pi d}{2} \qquad V_i = \frac{3\pi}{2}$$

$$\frac{dw}{dt} = (V'(w))^{-1} \cdot -0.1 \qquad V_t = \frac{3\pi}{2} - 0.1(t)$$

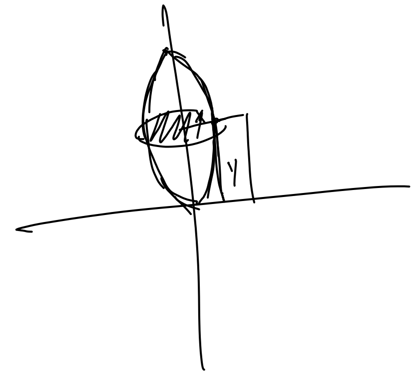
$$= \left(-\pi \cos\left(\frac{2\pi d}{3}\right) + \frac{\pi}{2}\right)^{-1} \cdot -0.1$$

$$= \frac{-0.1}{\left(-\pi \cos\left(\frac{2\pi d}{3}\right) + \frac{\pi}{2}\right)}$$

$$= \pi \int_0^3 \sin^2\left(\frac{\pi y}{3}\right) dy = \frac{3\pi}{2} \ m^3$$

at 2m, depth is changing at -0.032

W at any given point is y



How can I relate volume to depth

Total volume when depth = 3 is $\frac{3\pi}{2}$

$V(t) = \frac{3\pi}{2} L - 100 \cdot t \ (L/m)$

$\frac{\Delta V}{\Delta t} = -100 \ L/min$

$t(d) =$

$V_{t=0} = \frac{3\pi}{2} \ m^3$

$V(t) = \frac{3\pi}{2} - 0.1t$, t in minutes

$x = \sin\left(\frac{\pi y}{3}\right)$

$\frac{dV}{dt} = \frac{dV}{dw}\frac{dw}{dt}$

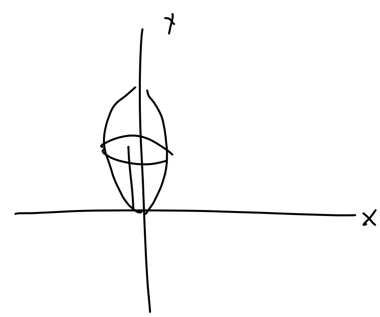
$\frac{dw}{dt} = \frac{dw}{dV}\frac{dV}{dt}$

$= \frac{dw}{dV} \cdot -0.1$

$\frac{3}{\pi}\sin^{-1}(x) = W$

$\frac{dV}{dt}$ is the rate

they have and can use.

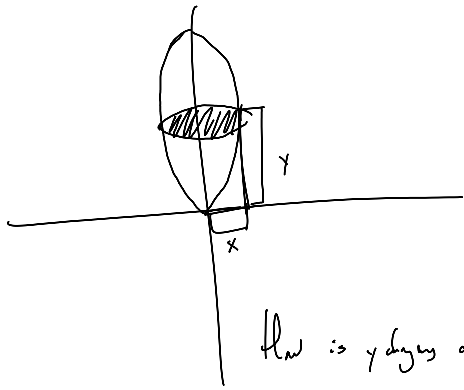


depth $= W = y = \frac{3}{\pi}\sin^{-1}(x)$

$W = 2\ m$

$\frac{dV}{dt} = -100\ L/min$

How is the depth w changing at $w = 2$



How is y changing at $y=2$

$\frac{dw}{dt} = \frac{dw}{dV} \times \frac{dV}{dt}$

$\frac{dw}{dt} = \frac{dw}{dV} \times (-100\ L/min)$

$\frac{dw}{dx} = \frac{3}{\pi} \cdot \frac{1}{\cos\left(\frac{\pi w}{13}\right)}$

Change in depth with respect to volume

$\shortparallel$

$\frac{dw}{dV}$

But how to express that without answer to q.1 ? Find an expression for depth in terms of volume... wtf ?