Physics 2250: Problem Set III

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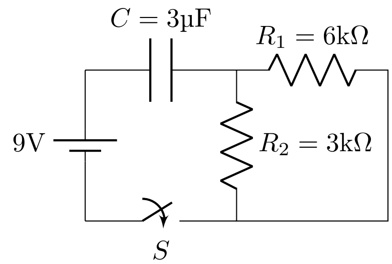
Problem 1 (BONUS). Consider the circuit below.



Solution 1.

a) Because resistors in parallel share a voltage drop we can simplify R_1 and R_2 here $R_{eq} = \left(\frac{1}{6\text{k}\Omega} + \frac{1}{3\text{k}\Omega}\right)^{-1} = 2\text{k}\Omega$
Using the formula for voltage across a resistor as a function of time in an RC circuit given in the course text we get that

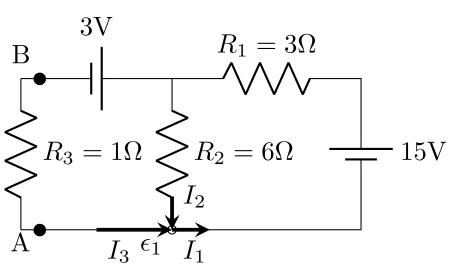
$$\begin{aligned}
4.5\text{V} &= 9\text{V}e^{\frac{-t}{R_{eq}C}} \\
4.5\text{V} &= 9\text{V}e^{\frac{-t}{2\times10^{3}\Omega\cdot3\times10^{-6}\text{F}}} \\
4.5\text{V} &= 9\text{V}e^{\frac{-t}{6\times10^{-3}\Omega\,\text{F}}} \\
\ln\left(\frac{1}{2}\right) &= \frac{-t}{6\times10^{-3}\Omega\,\text{F}} \\
-6\times10^{-3}\Omega\,\text{F}\ln\left(\frac{1}{2}\right) &= t \approxeq 4.159\times10^{-3}\text{s} = 4.159\text{ms}
\end{aligned}$$

b) Again using the course text to obtain a formula for the voltage across a capacitor versus time, which, when equal to the source voltage will indicate a fully charged capacitor

$$v_c(t) = V_s\left[1 - e^{\frac{-t}{R_{eq}C}}\right]$$

without doing any rearranging for t it can be seen that for $v_c = 9\text{V}$ $1 - e^{\frac{-t}{R_{eq}C}}$ must be 1, thus the exponential term must be zero which only occurs at an infinitely distant time. So the capacitor can never be "fully" charged in a mathematical sense. However (I think this makes sense) because the universe isn't infinitely granular (right?) when $9\text{V}C - v_cC < e$ (the electron charge) the capacitor can be considered effectively fully charged as there exists no smaller charge carrier to further equalize the charge at that instant. The aformentioned instant occurs at 0.0132s. See "I think this makes sense" for rationalization.

Problem 2. Find the value of current I flowing through the 1Ω resistor.



Solution 2.

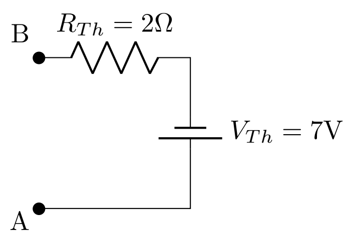
a) Going around both loops clockwise starting at ϵ_1 (and looking at the junction at ϵ_1) we get

$$0 = -I_3R_3 + 3\text{V} - I_2R_2$$
$$0 = 15\text{V} - I_1R_1 - I_2R_2$$
$$0 = I_3 + I_2 - I_1$$

which when solved yields $I_3 = \frac{7}{3}\text{A}$

b) Removing the 3V battery and replacing it with a wire we get a simple three resistor circuit where $R_{eq} = \left(\frac{1}{1\Omega} + \frac{1}{6\Omega}\right)^{-1} + 3\Omega = \frac{27}{7}\Omega$ which therefore has an $I_{tot} = \frac{15\text{V}}{\frac{27}{7}\Omega} = \frac{35}{9}\text{A}$ which is divided into the R_3 branch per the current divider rule $I_3 = \frac{35}{9}\text{A} \cdot \frac{6}{7} = \frac{10}{3}\text{A}$ flowing "downwards" (negative). Then removing the 15V resistor we get a circuit with $R_{eq} = \left(\frac{1}{3\Omega} + \frac{1}{6\Omega}\right)^{-1} + 1\Omega = 3\Omega$ which has an $I_{tot} = \frac{3\text{V}}{3\Omega} = 1\text{A}$ all of which flows "upwards" through R_3. Therefore the total current when these two are superimposed is $I_3 = 1\text{A} - \frac{10}{3}\text{A} = -\frac{7}{3}\text{A}$ upwards, which is actually downwards because of the sign.

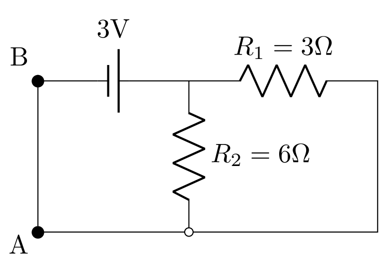
c) We can solve the circuit created by removing R_3 using superposition to determine that the 3V-only circuit will have a potential drop across port AB of 3V and the 15V-only circuit will have one of 10V. This means that $V_{Th} = V_{15AB} - V_{3AB} = 10\text{V} - 3\text{V} = 7\text{V}$. R_{Th} is obtained by shorting both voltage sources and looking at the current path from terminal A to terminal B so $R_{Th} = \left(\frac{1}{3\Omega} + \frac{1}{6\Omega}\right)^{-1} = 2\Omega$ so the circuit's Thévenin equivalent looks like



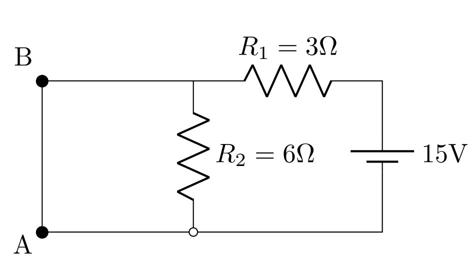
So when a 1Ω load is placed across port AB it will experience a current of $I_{load} = \frac{V_{Th}}{R_{tot}} = \frac{7\text{V}}{2\Omega + 1\Omega} = \frac{7}{3}\text{A}$

d) Using superposition to solve for I_N we get two circuits

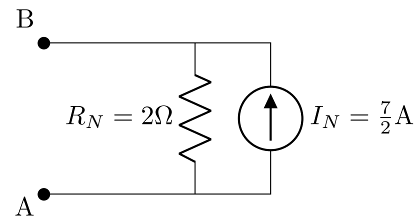
Circuit A



Circuit B

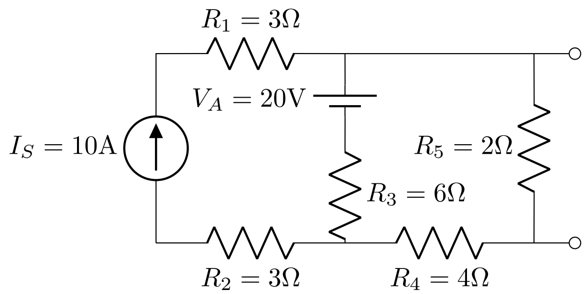


In circuit A we get a current through port AB of $I_{AB} = 3\text{V}\left(3^{-1} + 6^{-1}\right) = \frac{3}{2}\text{A}$. In circuit B we've shorted R_2 so the current through port AB is $I_{AB} = \frac{15\text{V}}{3\Omega} = 5\text{A}$. So when superimposing these currents we get that $I_N = 5\text{A} - \frac{3}{2}\text{A} = \frac{7}{2}\text{A}$. R_N can be found by shorting both voltage sources and looking in from port AB at how current will travel so $R_N = \left(6^{-1}\Omega^{-1} + 3^{-1}\Omega^{-1}\right)^{-1} = 2\Omega$. Therefore the Norton equivalent looks like



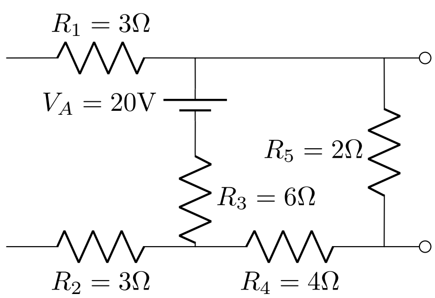
So when a 1Ω load is placed between terminals A and B it will experience a current given by the associated current divider $I_{load} = I_N \cdot \dfrac{R_N}{R_N + 1\Omega} = \dfrac{7}{2}\text{A} \cdot \dfrac{2\Omega}{2\Omega + 1\Omega} = \dfrac{7}{3}\text{A}$

Problem 3. Determine the Norton **and** Thévenin equivalents of the circuit below.

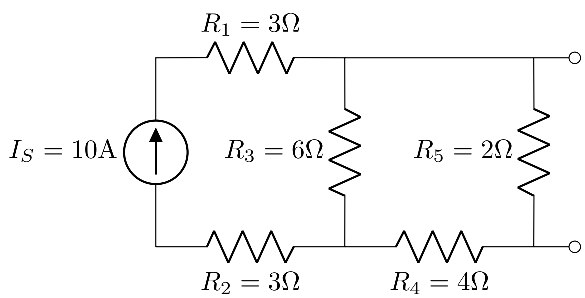


Solution 3.
Thévenin: Solving the circuit using superposition yields two sub circuits

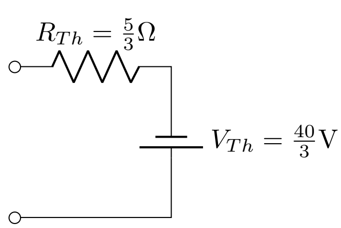
Circuit A



Circuit B

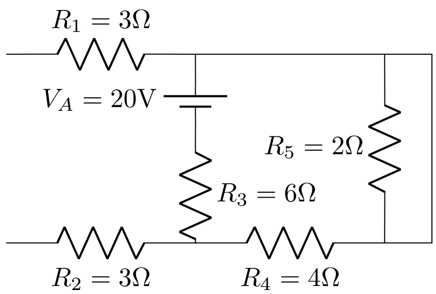


In this case we are trying to solve for the voltage across R_5 in both circuits. In circuit A R_1 and R_2 don't have any effect on the circuit as they have no current passing through them so they can be ignored. To determine the voltage across R_{5A} we can find the equivalent resistance $R_{eq} = R_3 + R_4 + R_5 = 12\Omega$ and thus $I_{tot} = \dfrac{20\text{V}}{12\Omega} = \dfrac{5}{3}\text{A}$ therefore $V_{5A} = \frac{5}{3}\text{A} \cdot 2\Omega = \frac{10}{3}\text{V}$. In circuit B the voltage across R_5 is determined by the current divider created by the R_3-R_5 branch, $I_5 = I_S \dfrac{R_3}{R_5 + R_4 + R_3} = 5\text{A} \implies V_{5B} = 5\text{A} \cdot 2\Omega = 10\text{V}$. Superimposing these two voltages gives us the voltage across the port $V_{port} = V_{5A} + V_{5B} = \dfrac{40}{3}\text{V} = V_{Th}$. R_{Th} can be found by eliminating the active components which removes R_1 and R_2 as the current source is replaced with an open. From there $R_{Th} = \left((R_3 + R_4)^{-1} + R_5^{-1}\right)^{-1} = \dfrac{5}{3}\Omega$. Therefore the Thévenin equivalent becomes

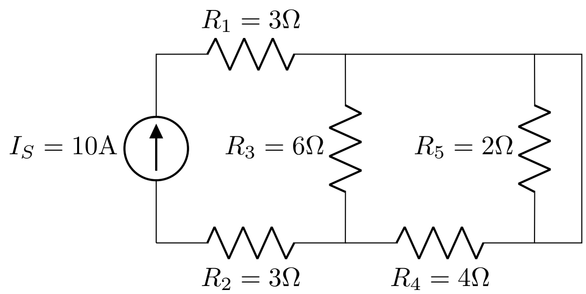


Norton: Solving the circuit using superposition and shorting across the port

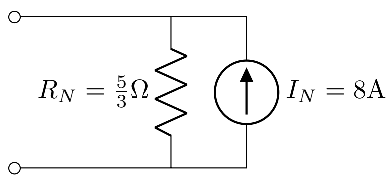
Circuit A



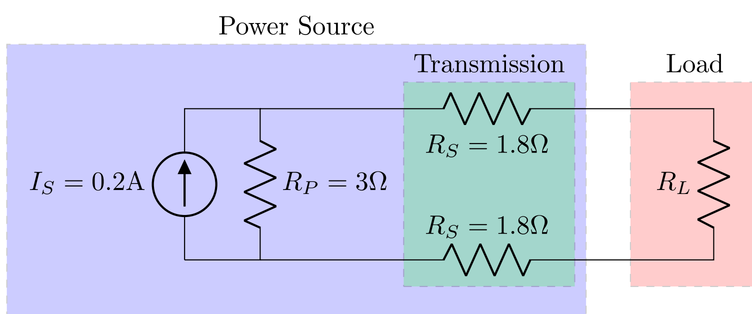
Circuit B



In circuit A R_1 and R_2 experience no current and therefore have no effect on the circuit. R_5 is shorted and therefore also has no effect on the circuit, so the current through the shorted port is $I_{tot} = I_B = \dfrac{20\text{V}}{6\Omega + 4\Omega} = 2\text{A}$. In circuit B R_5 is again shorted and therefore has no effect on the circuit so the current through the shorted port is the current through R_4 which is determined by the R_3-R_4 current divider $I_{port} = I_B = I_S \cdot \dfrac{R_3}{R_3 + R_4} = 10\text{A} \cdot \dfrac{6\Omega}{6\Omega + 4\Omega} = 6\text{A}$. So, $I_N = I_A + I_B = 8\text{A}$. R_N is determined through the same method as Thévenin so I will spare the details I went through previously and just state that $R_N = \dfrac{5}{3}\Omega$. Therefore the Norton equivalent looks like



Problem 4 (**BONUS**)**.** A simple model of a photovoltaic solar cell is a current source with the current proportional to the power of sunlight falling on it. (A more accurate model includes a diode, which is a nonlinear element we will learn about later in the course). There is some leakage current in the source, which can be modelled with a parallel resistor, Rp. There is also a voltage drop in the transmission of the power that can be modelled by resistors in series, Rs, connecting to the load. The load resistance is given by RL. This crude model is represented in the figure below



Solution 4.
a) Knowing that $P = IV = I^2R$ we can set up an equation for P_L in terms of R_L which can be differentiated to

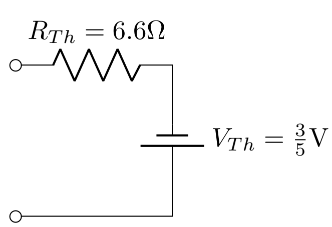
determine maxima and minima. $I_L = I_S \cdot \dfrac{R_P}{R_P + R_L + 2R_S} \implies P_L = I_L^2 R_L = \left(I_S \cdot \dfrac{R_P}{R_P + R_L + 2R_S}\right)^2 \cdot R_L$

$$
\begin{aligned}
&\frac{d}{dR_L}\left[\left(I_S \cdot \frac{R_P}{R_P + R_L + 2R_S}\right)^2 \cdot R_L\right] \\
&\frac{d}{dR_L}\left[\frac{I_S^2 R_P^2 R_L}{(R_P + R_L + 2R_S)^2}\right] \\
&\frac{\frac{d}{dR_L}\left[I_S^2 R_P^2 R_L\right](R_P + R_L + 2R_S)^2 - \frac{d}{dR_L}\left[(R_P + R_L + 2R_S)^2\right] I_S^2 R_P^2 R_L}{(R_P + R_L + 2R_S)^4} \\
&I_S^2 R_P^2 \cdot \frac{\frac{d}{dR_L}\left[R_L\right](R_P + R_L + 2R_S)^2 - \frac{d}{dR_L}\left[(R_P + R_L + 2R_S)^2\right] R_L}{(R_P + R_L + 2R_S)^4} \\
&I_S^2 R_P^2 \cdot \frac{(R_P + R_L + 2R_S)^2 - 2(R_P + R_L + 2R_S)\frac{d}{dR_L}\left[(R_P + R_L + 2R_S)\right] R_L}{(R_P + R_L + 2R_S)^4} \\
&I_S^2 R_P^2 \cdot \frac{(R_P + R_L + 2R_S)^2 - 2(R_P + R_L + 2R_S) R_L}{(R_P + R_L + 2R_S)^4}
\end{aligned}
$$

Wow, great, I think I did that correctly. Now we solve $P_L' = 0$

$$
\begin{aligned}
0 &= \frac{I_S^2 R_P^2 (R_P + R_L + 2R_S)^2}{(R_P + R_L + 2R_S)^4} - \frac{2I_S^2 R_P^2 (R_P + R_L + 2R_S) R_L}{(R_P + R_L + 2R_S)^4} \\
0 &= \frac{I_S^2 R_P^2}{(R_P + R_L + 2R_S)^2} - \frac{2I_S^2 R_P^2 R_L}{(R_P + R_L + 2R_S)^3} \\
\frac{2I_S^2 R_P^2 R_L}{(R_P + R_L + 2R_S)} &= I_S^2 R_P^2 \\
\frac{1}{2R_L} &= \frac{1}{(R_P + R_L + 2R_S)} \\
2R_L &= R_P + R_L + 2R_S \\
R_L &= R_P + 2R_S
\end{aligned}
$$

So P_L' has only one zero at $R_L = 3 + 2(1.8) = 6.6\Omega$. Looking at the function on a graph which I will omit here to save myself the paper and ink reveals this point to be a maximum. The same conclusion could also be arrived at through testing the second derivative.
b) Removing R_L also takes both transmission resistors out of the equation whichs means that $V_{Th} = I_S R_P = 0.2\text{A} \cdot 3\Omega = \dfrac{3}{5}\text{V}$. R_{Th} is the combined resistance of both transmission resistors and the parallel resistor as when I_S is opened that is the only path through which current flows from the port so $R_{Th} = 2R_S + R_P = 6.6\Omega$. Therefore the Thévenin equivalent looks like



R_{Th} is the same as the optimal R_L because the optimal R_L will shift to match R_{Th} per $P_L = I_S V_L = I_S^2 R_L$. Because the current is constant if $R_L < R_{Th}$ then R_L is no longer optimal and must increase up to R_{Th} to create the greatest possible voltage drop across itself. In reverse, increasing R_L past this happy point will mean more current travels through R_P which defeats the purpose of increasing R_L. c) The maximum power delivery occurs when R_L is optimal so is $P_L = I_L^2 R_L = \left(0.2\text{A}\frac{3\Omega}{2 \cdot 1.8\Omega + 6.6\Omega + 3\Omega}\right)^2 6.6\Omega \approxeq 13.64\text{mW}$. $P_{tot} = V_{tot} I_S = (V_P + V_S + V_L) I_S \approxeq 92.73\text{mW}$, therefore the resistor uses $\frac{13.64\text{mW}}{92.73\text{mW}} \cdot 100 \approxeq 14.71\%$ of the total expendable power.