Math 2150: Assignment I

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Note: I've kept p, q, and r throughout my solutions and only substituted the actual numbers in at the end. This is because I find it easier, especially when dealing with things that might cancel nicely, to deal with variables rather than the numbers they represent. In my case my student number is 0805980 so $p = 9$, $q = 5$, and $r = 22$.

Problem 1.

Solution 1.

Problem 2. Solve the following differential equation using the method of undetermined coefficients.

$$\frac{d^2y}{dx^2} - 2q\frac{dy}{dx} + q^2y = 2q^2\left(q^2+4\right)^2\cos^2(x) + 6pxe^{qx} + q^3x + (q^2+1)^2p\cos(x)$$

Solution 2. First we solve the homogeneous portion

$$\begin{aligned}
&\frac{d^2y}{dx^2} - 2q\frac{dy}{dx} + q^2y = 0 \rightsquigarrow y = e^{mx} \\
&m^2 - 2qm + q^2 = 0 \implies m = \frac{2q \pm \sqrt{4q^2 - 4q^2}}{2} = q \\
&\therefore y_h(x) = C_1e^{qx} + C_2xe^{qx} \text{ where } xe^{qx} \text{ is the result of reduction of order on } e^{qx}
\end{aligned}$$

Then, using undetermined coefficients

$$\begin{aligned}
y_p =& A\cos(2x) + B\sin(2x) + Cx + D + x^2(Ex+F)e^{qx} + G\cos(x) + H\sin(x) \\
y_p' =& -2A\sin(2x) + 2B\cos(2x) + C + 3Ex^2e^{qx} + qEx^3e^{qx} + 2Fxe^{qx} + Fqx^2e^{qx} - G\sin(x) + H\cos(x) \\
y_p'' =& -4A\cos(2x) - 4B\sin(2x) + 6Exe^{qx} + 3Eqx^2e^{qx} + 3Eqx^2e^{qx} + Eq^2x^3e^{qx} + 2Fe^{qx} + \\
& 2Fqxe^qx + 2Fxe^{qx} + Fq^2x^2e^{qx} - G\cos(x) - H\sin(x)
\end{aligned}$$

Which can then be plugged into the original differential equation in place of y, y', and y''

$$\begin{aligned}
&-4A\cos(2x) - 4B\sin(2x) + 6Exe^{qx} + 3Eqx^2e^{qx} + 3Eqx^2e^{qx} + Eq^2x^3e^{qx} + 2Fe^{qx} + 2Fqxe^{qx} + 2Fxe^{qx} + Fq^2x^2e^{qx} \\
&- G\cos(x) - H\sin(x) \\
&- 2q\left[-2A\sin(2x) + 2B\cos(2x) + C + 3Ex^2e^{qx} + qEx^3e^{qx} + 2Fxe^{qx} + Fqx^2e^{qx} - G\sin(x) + H\cos(x)\right] \\
&+ q^2\left[A\cos(2x) + B\sin(2x) + Cx + D + x^2(Ex+F)e^{qx} + G\cos(x) + H\sin(x)\right] \\
&= q^2\left(q^2+4\right)^2 + q^2\left(q^2+4\right)^2\cos(2x) + 6pxe^{qx} + q^3x + (q^2+1)^2p\cos(x)
\end{aligned}$$

Which when multiplied out yields

$$\begin{aligned}
&-4A\cos(2x) - 4B\sin(2x) + 6Exe^{qx} + 3Eqx^2e^{qx} + 3Eqx^2e^{qx} + Eq^2x^3e^{qx} + 2Fe^{qx} + 2Fqxe^{qx} + 2Fxe^{qx} + Fq^2x^2e^{qx} \\
&- G\cos(x) - H\sin(x) + 4qA\sin(2x) - 4qB\cos(2x) - 2qC - 6qEx^2e^{qx} - 2q^2Ex^3e^{qx} - 4qFxe^{qx} - 2Fq^2x^2e^{qx} \\
&+ 2qG\sin(x) - 2qH\cos(x) + Aq^2\cos(2x) + Bq^2\sin(2x) + Cq^2x + D + Eq^2x^3e^{qx} + Fq^2x^2e^{qx} + q^2G\cos(x) + q^2H\sin(x) \\
&= q^2\left(q^2+4\right)^2 + q^2\left(q^2+4\right)^2\cos(2x) + 6pxe^{qx} + q^3x + (q^2+1)^2p\cos(x)
\end{aligned}$$

Which can be solved to determine the unknown coefficients

$$\begin{aligned}
&-4B + 4qA + Bq^2 = 0 \implies \frac{-4qA}{q^2-4} = B \\
&-4A - 4qB + Aq^2 = q^2\left(q^2+4\right)^2 \implies \frac{q^2\left(q^2+4\right)^2}{q^2 - 4 + \frac{16q^2}{q^2-4}} = A = 525 \implies B = -500 \\
&Cq^2 = q^3 \implies C = q = 5 \\
&-2qC + D = q^2\left(q^2+4\right)^2 \implies D = q^2\left(q^2+4\right)^2 + 2qC = 21075 \\
&2F = 0 \implies F = 0 \\
&6E + 2Fq + 2F - 4qF = 6E = 6p \implies E = p = 9 \\
&-H + 2qG + q^2H = 0 \implies \frac{-2qG}{q^2-1} = H \\
&-G - 2qH + q^2G = \left(q^2+1\right)^2p \implies G = \frac{\left(q^2+1\right)^2p}{\frac{4q^2}{q^2-1} - 1 + q^2} = 216 \implies H = -90
\end{aligned}$$

$$
\begin{aligned}
A &= 525 \\
B &= -500 \\
C &= 5 \\
D &= 21075 \\
E &= 9 \\
F &= 0 \\
G &= 216 \\
H &= -90
\end{aligned}
$$

So,

$$
y_p = 525\cos(2x) - 500\sin(2x) + 5x + 21075 + 9x^3e^{5x} + 216\cos(x) - 90\sin(x)
$$

$$
\therefore y = C_1e^{qx} + C_2xe^{qx} + 525\cos(2x) - 500\sin(2x) + 5x + 21075 + 9x^3e^{5x} + 216\cos(x) - 90\sin(x)
$$

Problem 3. Solve the following differential equation using variation of parameters

$$qx^2\frac{d^2y}{dx^2} - 9qx\frac{dy}{dx} + 26qy = px^5\ln^2(x), \qquad x > 0$$

Solution 3. First we solve the homogeneous portion

$$\begin{aligned}
&qx^2\frac{d^2y}{dx^2} - 9qx\frac{dy}{dx} + 26qy = 0 \rightsquigarrow y = x^m \\
&qx^2m^2x^{m-2} - 9qxmx^{m-1} + 26qx^m = 0 \\
&qm^2 - 10qm + 26q = 0 \implies m = \frac{10q \pm \sqrt{100q^2 - 104q^2}}{2q} = 5 \pm i \\
&\therefore y_h(x) = C_1x^{5+i} + C_2x^{5-i} = x^5\left(C_3\cos(\ln(x)) + C_4\sin(\ln(x))\right)
\end{aligned}$$

Then, using variation of parameters

$$y_p(x) = -y_1(x)\int\frac{g(x)y_2(x)}{W[y_1,y_2]}\,dx + y_2(x)\int\frac{g(x)y_1(x)}{W[y_1,y_2]}\,dx$$

Where $g(x) = \frac{p}{q}x^3\ln^2(x)$ (the left hand side divided by qx^2).

$$\begin{aligned}
W[y_1,y_2] &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^5\cos(\ln(x)) & x^5\sin(\ln(x)) \\ 5x^4\cos(\ln(x)) - x^4\sin(\ln(x)) & 5x^4\sin(\ln(x)) + x^4\cos(\ln(x)) \end{vmatrix} \\
&= \left[5x^4\sin(\ln(x)) + x^4\cos(\ln(x))\right]x^5\cos(\ln(x)) - \left[5x^4\cos(\ln(x)) - x^4\sin(\ln(x))\right]x^5\sin(\ln(x)) \\
&= \cancel{5x^9\sin(\ln(x))\cos(\ln(x))} + x^9\cos^2(\ln(x)) \cancel{-5x^9\cos(\ln(x))\sin(\ln(x))} + x^9\sin^2(\ln(x)) \\
&= x^9\cos^2(\ln(x)) + x^9\sin^2(\ln(x)) = x^9
\end{aligned}$$

So,

$$\begin{aligned}
y_p(x) &= -x^5\cos(\ln(x))\int\frac{px^8\ln^2(x)\sin(\ln(x))}{qx^9}\,dx \\
&+ x^5\sin(\ln(x))\int\frac{px^8\ln^2(x)\cos(\ln(x))}{qx^9}\,dx \\
&= -\frac{p}{q}x^5\cos(\ln(x))\left[2\ln(x)\sin(\ln(x)) + \left(2 - \ln^2(x)\right)\cos(\ln(x))\right] \\
&+ \frac{p}{q}x^5\sin(\ln(x))\left[\left(\ln^2(x) - 2\right)\sin(\ln(x)) + 2\ln(x)\cos(\ln(x))\right] \\
&= -\frac{p}{q}x^5\cos(\ln(x))\,2\ln(x)\sin(\ln(x)) - \frac{p}{q}x^5\cos(\ln(x))\left(2 - \ln^2(x)\right)\cos(\ln(x)) \\
&+ \frac{p}{q}x^5\sin(\ln(x))\left(\ln^2(x) - 2\right)\sin(\ln(x)) + \frac{p}{q}x^5\sin(\ln(x))\,2\ln(x)\cos(\ln(x)) \\
&= -\frac{p}{q}x^5\cos(\ln(x))\left(2 - \ln^2(x)\right)\cos(\ln(x)) + \frac{p}{q}\sin(\ln(x))\left(\ln^2(x) - 2\right)\sin(\ln(x)) \\
&- \frac{p}{q}x^5\left(2 - \ln^2(x)\right)\cos^2(\ln(x)) + \frac{p}{q}\left(\ln^2(x) - 2\right)\sin^2(\ln(x)) \\
&= \frac{p}{q}x^5\left[-2\cos^2(\ln(x)) + \ln^2(x)\cos^2(\ln(x)) + \ln^2(x)\sin^2(\ln(x)) - 2\sin^2(\ln(x))\right] \\
&= \frac{p}{q}x^5\left[\ln^2(x) - 2\right]
\end{aligned}$$

$$\therefore y(x) = x^5\left(C_3\cos(\ln(x)) + C_4\sin(\ln(x)) + \frac{9}{5}\left[\ln^2(x) - 2\right]\right)$$