Math 3310H: Assignment I

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Problem 1. Define a relation $\mathbb{R} \times \mathbb{R}$ by $(a, b) \sim (c, d)$ if $2(a-c) - 3(b-d) = 0$

(a) Show that $\sim$ is an equivalence relation on $\mathbb{R}$.

(b) Give an example of two pairs $(a, b), (c, d) \in \mathbb{R} \times \mathbb{R}$, which lie in the same equivalence class, and two pairs that don't.

(c) This equivalence relation partitions the 2D plane $\mathbb{R} \times \mathbb{R}$ into subregions. What does the equivalence class (a, b) look like as a region of the plane?

Solution 1. (a) For $\sim$ to be an equivalence relation it must satisfy the following properties for a set S (proofs included)

(i) Reflexivity: $x \sim x \, \forall x \in S$.

Proof. Let $(a, b) \in \mathbb{R} \times \mathbb{R}$, then

$$\begin{aligned} &(a, b) \overset{?}{\sim} (a, b) \\ \implies &2(a-a) - 3(b-b) = 0 \end{aligned}$$

Which satisfies our relation as defined. Therefore the relation is reflexive. □

(ii) Symmetry: $x \sim y \implies y \sim x \, \forall x, y \in S$

Proof. Let $(a, b), (c, d) \in \mathbb{R} \times \mathbb{R}$, then

$$\begin{aligned} &(a, b) \sim (c, d) \\ \implies &2(a-c) - 3(b-d) = 0 \\ \implies &2(a-c) = 3(b-d) \\ \implies &-2(a-c) = -3(b-d) \\ \implies &2(c-a) = 3(d-b) \\ \implies &2(c-a) - 3(d-b) = 0 \\ \implies &(c, d) \sim (a, b) \end{aligned}$$ □

(iii) Transitivity: $x \sim y \sim z \implies x \sim z \, \forall x, y, z \in S$

Proof. Let $(a, b), (c, d), (e, f) \in \mathbb{R} \times \mathbb{R}$, then

$$\begin{aligned} &(a, b) \sim (c, d) \\ \implies &2(a-c) - 3(b-d) = 0 \end{aligned}$$

and

$$\begin{aligned} &(c, d) \sim (e, f) \\ \implies &2(c-e) - 3(d-f) = 0 \end{aligned}$$

so

$$
\begin{aligned}
&2(a-c)-3(c-d)+2(c-e)-3(d-f)=0\\
\implies &2(a-c+c-e)-3(b-d+d-f)=0\\
\implies &2(a-e)-3(b-f)=0\\
\implies &(a,b)\sim(e,f)
\end{aligned}
\qquad \square
$$

Therefore $\sim$ is an equivalence relation on $\mathbb{R}\times\mathbb{R}$.

(b) For representative element $(1,1)$ we get that for an element $(a,b)\in\mathbb{R}\times\mathbb{R}$ to belong to the associated equivalence class we must have

$$2(1-a)-3(1-b)=0$$

which can be rearranged to obtain

$$a=-\frac{1-3b}{2}$$

so for $b=\pm1$ we get two members of the equivalence class represented by $(1,1)$ under $\sim$, $(1,1)$ and $(-2,1)$. The elements (π,e) and (ϕ,i^i) where π,e take on their usual definitions, ϕ is the golden ratio and i^i is, interestingly, both transcendental *and* real!

(c) The equivalence class with representative (a,b) is the set $E=\{(x,y)\in\mathbb{R}\times\mathbb{R}|x\sim(a,b)\}$. This gives the equation

$$2(a-x)-3(b-y)=0 \implies y=\frac{2(a-x)-3b}{-3}$$

so the class looks like a line with slope $2/3$ and y-intercept $b-2a/3$

Problem 2. For each of the following sets S, determine whether S is closed under addition modulo n, or multiplication modulo n, or both or neither. (Addition and multiplication modulo n are defined in Exercise Set 2).

(a) $S=\{0,4,8,12\}, n=16$.

(b) $S=\{0,3,6,9,12\}, n=15$.

(c) $S=\{1,2,3,4\}, n=5$.

(d) $S=\{0,2,3,4,6,8,9,10\}, n=12$.

(e) $S=\{1,5,7,11\}, n=12$.

Solution 2.

(a)

$+_{16}$	0	4	8	12
0	0	4	8	12
4	4	8	12	0
8	8	12	0	4
12	12	0	4	8

$\cdot_{16}$	0	4	8	12
0	0	0	0	0
4	0	0	0	0
8	0	0	0	0
12	0	0	0	0

That these tables, being every possible combination of elements on each set with their respective operations contain no elements not members of S means that both are closed under $+_{16}$ and $\cdot_{16}$.

(b)

$+_{15}$	0	3	6	9	12
0	0	3	6	9	12
3	3	6	9	12	0
6	6	9	12	0	3
9	9	12	0	3	6
12	12	0	3	6	9

$\cdot_{15}$	0	3	6	9	12
0	0	0	0	0	0
3	0	9	3	12	6
6	0	3	6	9	12
9	0	12	9	6	3
12	0	6	12	3	9

Again because these tables contain only elements of S S is closed under both of their respective operations.

(c)

$+_5$	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

$\cdot_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Here because $0 \notin S \implies S$ is not closed under $+_5$ but is closed under $\cdot_5$ for the same reasons as previously.

(d)

$+_{12}$	0	2	3	4	6	8	9	10
0	0	2	3	4	6	8	9	10
2	2	4	5	6	8	10	11	0
3	3	5	6	7	9	11	0	1
4	4	6	7	8	10	0	1	2
6	6	8	9	10	0	2	3	4
8	8	10	11	0	2	4	5	6
9	9	11	0	1	3	5	6	7
10	10	0	1	2	4	6	7	8

$\cdot_{12}$	0	2	3	4	6	8	9	10
0	0	0	0	0	0	0	0	0
2	0	4	6	8	0	4	6	8
3	0	6	9	0	6	0	3	6
4	0	8	0	4	0	8	0	4
6	0	0	6	0	0	0	6	0
8	0	4	0	8	0	4	0	8
9	0	6	3	0	6	0	9	6
10	0	8	6	4	0	8	6	4

Here because $1, 5, 7, 11 \notin S$ S is not closed under $+_12$ but is closed under $\cdot_12$ for the same reasons as previously.

(e)

$+_{12}$	1	5	7	11
1	2	6	8	0
5	6	10	0	4
7	8	0	2	6
11	0	4	6	10

$\cdot_{12}$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

Here because $0, 2, 4, 6, 8, 10 \notin S$ S is not closed under $+_12$ but is closed under $\cdot_12$ for the same reasons as previously.

Problem 3. Determine whether the given binary operation $*$ is commutative, associative, both or neither. Justify your answers with proof.

(a) The operation $*$ on $\mathbb{Z}$ given by $a * b = a + b + ab$

(b) The operation $*$ on $\mathbb{R}$ given by $a * b = a + b - ab$

(c) The operation $*$ on $\mathbb{R}$ given by $a * b = a + 2ab$

(d) The operation $*$ on $\mathbb{Z} \times \mathbb{Z}$ given by $(a, b) * (c, d) = (ad + bc, bd)$

(e) The operation $*$ on $\mathbb{Z} \times \mathbb{Z}$ given by $(a, b) * (c, d) = (ad, bc)$

Solution 3.

Counterexamples for the failures

(a) For commutativity,

Proof. Let $a, b \in \mathbb{Z}$, then

$$\begin{aligned} a * b &= a + b + ab \\ &= b + a + ba && \text{Commutativity of } + \text{ and } - \text{ on } \mathbb{Z} \\ &= b * a && \text{Definition of } * \end{aligned}$$ ☐

For associativity,

Proof. Let $a, b, c \in \mathbb{Z}$, then

$$\begin{aligned} a * (b * c) &= a * (b + c + bc) \\ &= a + b + c + bc + a(b + c + bc) \\ &= a + b + c + bc + ab + ac + abc && \cdot \text{ distributive on } \mathbb{Z} \end{aligned}$$

and

$$\begin{aligned} (a * b) * c &= (a + b + ab) * c \\ &= a + b + ab + c + (a + b + ab)c \\ &= a + b + ab + c + ac + bc + abc && \cdot \text{ distributive on } \mathbb{Z} \\ &= a + b + c + bc + ab + ac + abc && + \text{ commutative on } \mathbb{Z} \end{aligned}$$

because the two are equal we have associativity. ☐

(b) For commutativity,

Proof. Let $a, b \in \mathbb{Z}$, then

$$\begin{aligned} a * b &= a + b - ab \\ &= b + a - ba && \text{Commutativity of } + \text{ and } - \text{ on } \mathbb{Z} \\ &= b * a && \text{Definition of } * \end{aligned}$$ ☐

For associativity,

Proof. Let $a, b, c \in \mathbb{Z}$, then

$$\begin{aligned} a * (b * c) &= a * (b + c - bc) \\ &= a + b + c - bc + a(b + c - bc) \\ &= a + b + c - bc + ab + ac - abc && \cdot \text{ distributive on } \mathbb{Z} \end{aligned}$$

and

$$\begin{aligned} (a * b) * c &= (a + b - ab) * c \\ &= a + b - ab + c + (a + b - ab)c \\ &= a + b - ab + c + ac + bc - abc && \cdot \text{ distributive on } \mathbb{Z} \end{aligned}$$

because of the difference in sign on the ab terms these two cannot be made to be equal, therefore $*$ is not associative. ☐

(c) For commutativity,

Proof. Let $a, b \in \mathbb{Z}$, then

$$a * b = a + 2ab; \qquad b * a = b + 2ba$$

which cannot be manipulated to be equal, therefore $*$ is not commutative here. $\square$

For associativity,

Proof. Let $a, b, c \in \mathbb{Z}$, then

$$\begin{aligned} a * (b * c) &= a * (b + 2bc) \\ &= a + 2a(b + 2bc) \\ &= a + 2ab + 4abc && \cdot \text{ distributive on } \mathbb{Z} \end{aligned}$$

and

$$\begin{aligned} (a * b) * c &= (a + 2ab) * c \\ &= c + 2c(a + 2ab) \\ &= c + 2ca + 4cab && \cdot \text{ distributive on } \mathbb{Z} \end{aligned}$$

which cannot be manipulated to be equal, therefore $*$ is not associative here. $\square$

(d) The operation $*$ on $\mathbb{Z} \times \mathbb{Z}$ given by $(a, b) * (c, d) = (ad + bc, bd)$

(e) The operation $*$ on $\mathbb{Z} \times \mathbb{Z}$ given by $(a, b) * (c, d) = (ad, bc)$

Problem 4. Let S be a nonempty set. A binary algebraic structure $(S, *)$ is called a semigroup if $*$ is associative.

(a) Let S be the set of positive rational numbers. Show that $(S, *)$ is a commutative semigroup if

$$a * b = \frac{ab}{a + b}$$

(the usual operations on the right) for all $a, b \in S$

(b) Let S be a set containing more than one element. Define

$$a * b = b$$

for all $a, b \in S$. Show that $(S, *)$ is a noncommutative semigroup with no identity element.

Solution 4.