Calculus II: Assignment 10

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Problem 1. For what values of x does the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ converge?

Solution 1.

$$
\begin{aligned}
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!} \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \frac{x^{2n+3}}{x^{2n+1}} \frac{(2n+1)!}{(2n+3)!} \right| \\
&= \lim_{n\to\infty} \left| (-1)(x^2) \frac{(2n+1)!}{(2n+3)!} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)(x^2)}{(2n+3)(2n+2)} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)(x^2)}{4n^2+10n+6} \right| \\
&= \lim_{n\to\infty} \left| \frac{\frac{(-1)(x^2)}{n^2}}{\frac{4n^2}{n^2}+\frac{10n}{n^2}+\frac{6}{n^2}} \right| \\
&= 0
\end{aligned}
$$

$\therefore$ By the ratio test the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ converges for all x

Problem 2. What function does the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ equal when it converges?

Solution 2. The series should converge to some alternating function, so it's probably a trig function of some kind. To determine which I used SageMath so I didn't have to go through all of them, but I'll also work through determining properly. Knowing which function the series equals, I look at the n^{th} derivative pattern for the Taylor Series of $\sin(x)$ about 0:

$f^{(0)}(a=0)$	$\sin(a)$	0
$f^{(1)}(a)$	$\cos(a)$	1
$f^{(2)}(a)$	$-\sin(a)$	0
$f^{(3)}(a)$	$-\cos(a)$	-1
$f^{(4)}(a)$	$\sin(a)$	0

So the series expansion should look like

$$
\begin{aligned}
&= \frac{0}{0!}x^0 + \frac{1}{1!}x^1 + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \dots \\
&= \frac{1}{1!}x^1 + \frac{-1}{3!}x^3 + \frac{1}{5!}x^5 + \dots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\end{aligned}
$$

$\therefore \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin(x)$ when it converges

Problem 3. For what values of x does the series $\sum_{n=0}^{\infty}(n+1)x^n$ converge?

Solution 3.

$$\begin{aligned}
&= \lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right| \\
&= \lim_{n\to\infty}\left|x\frac{(n+2)}{(n+1)}\right| \\
&= \lim_{n\to\infty}\left|x\frac{1+\frac{2}{n}}{(1+\frac{1}{n})}\right| \\
&= \lim_{n\to\infty}|x|
\end{aligned}$$

Testing endpoints as well

$$\begin{aligned}
&= \lim_{n\to\infty}|(n+1)(\pm 1)^n| \\
&= \lim_{n\to\infty}(n+1) \\
&= \infty
\end{aligned}$$

$\therefore$ By the ratio test the series $\sum_{n=0}^{\infty}(n+1)x^n$ converges for all $|x|<1$

Problem 4. What function does the series $\sum_{n=0}^{\infty}(n+1)x^n$ equal when it converges?

Solution 4. The easiest way I can think of to approach this is to try and get $(n+1)x^n$ into the form ax^n which can then be simplified using the formula for geometric series.

$$\begin{aligned}
&= \int\sum_{n=0}^{\infty}(n+1)x^n\,dx \\
&= \sum_{n=0}^{\infty}\int(n+1)x^n\,dx \\
&= \sum_{n=0}^{\infty}x^{n+1} \\
&= \sum_{n=0}^{\infty}xx^n
\end{aligned}$$

Which is a geometric series with common ratio x and first term x, which can be rewritten as $\frac{x}{1-x}$ (for all $x<1$) which we can then differentiate to cancel out the previous integration and arrive at a function for the original series.

$$\begin{aligned}
&= \frac{d}{dx}\frac{x}{1-x} \\
&= \frac{\left[\frac{d}{dx}(x)\right](1-x)-\left[\frac{d}{dx}(1-x)\right](x)}{(1-x)^2} \\
&= \frac{1-x+x}{(1-x)^2} \\
&= \frac{1}{(1-x)^2}
\end{aligned}$$

$\therefore \sum_{n=0}^{\infty}(n+1)x^n = \dfrac{1}{(1-x)^2}$ when it converges

$$\{\} = 0,\ n+1 = n\cup\{n\} \implies \{\{\}\} = 1 \therefore \{\{\}\}\cup\{\{\{\}\}\} = \{\{\},\{\{\}\}\} = 2$$