Calculus Cheat Sheet

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Revision 3

Limits

Existence

$\lim\limits_{x \to a} f(x) = L$ exists if $\forall \epsilon > 0 \exists \delta > 0$ s.t. $|x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$.

Properties

$\lim\limits_{x \to a} [cf(x)] = c \lim\limits_{x \to a} [f(x)]$

$\lim\limits_{x \to a} [f(x) \pm g(x)] = \lim\limits_{x \to a} [f(x)] \pm \lim\limits_{x \to a} [g(x)]$

$\lim\limits_{x \to a} [f(x)g(x)] = \lim\limits_{x \to a} [f(x)] \lim\limits_{x \to a} [g(x)]$

$\lim\limits_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim\limits_{x \to a} [f(x)]}{\lim\limits_{x \to a} [g(x)]}, \lim\limits_{x \to a} [g(x)] \neq 0$

Evaluation Techniques

Basics

$\lim\limits_{x \to a} [f(x)] = f(a)$ if f exists at a

L'Hôpital's Rule

If $\lim\limits_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{0}{0}$ **or** $\frac{\pm\infty}{\pm\infty}$ **then**

$\lim\limits_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \lim\limits_{x \to a} \left[\frac{f'(x)}{g'(x)} \right]$

Factoring at Infinity

If $p(x)$ and $q(x)$ are polynomials, to evaluate $\lim\limits_{x \to \pm\infty} \left[\frac{p(x)}{q(x)} \right]$ factor the greatest power of x in $q(x)$ (the denominator) out of poth $p(x)$ and $q(x)$ then compute the limit,

e.g. $\lim\limits_{x \to -\infty} \left[\frac{3x^2 - 4}{5x - 2x^2} \right] = \lim\limits_{x \to -\infty} \left[\frac{\cancel{x^2}(3 - \frac{4}{x^2})}{\cancel{x^2}(\frac{5}{x} - 2)} \right]$

$\frac{3-0}{0-2} = -\frac{3}{2}$

Derivatives

Definition

$\frac{d}{dx} f(x) = f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Techniques

Sum Rule

$\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$

Product Rule

$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

Quotient Rule

$\frac{d}{dx}\left(\frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Power Rule

$\frac{d}{dx}(x^n) = nx^{n-1}$

Chain Rule

$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

Parametric

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ recurse y for nth derivative $(x, y) + (x', y')t$ for parametric tangent

Common Derivatives

$\frac{d}{dx}(x) = 1 \quad \frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x \quad \frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \quad \sin^{-1} x \neq \frac{1}{\sin x}$

$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \quad \frac{d}{dx}(a^x) = a^x \ln a$

$\frac{d}{dx}(e^x) = e^x \quad \frac{d}{dx}(\ln x) = \frac{1}{x}, x > 0$

$\frac{d}{dx}(\ln |x|) = \frac{1}{x}, x \neq 0 \quad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

Common Chain Rule Derivatives

$\frac{d}{dx}([f(x)]^n) = n[f(x)]^{n-1} f'(x)$

$\frac{d}{dx}(e^{f(x)}) = e^{f(x)} f'(x)$

$\frac{d}{dx}(\ln[f(x)]) = \frac{f'(x)}{f(x)}$

$\frac{d}{dx}(\sin[f(x)]) = f'(x) \cos[f(x)]$

$\frac{d}{dx}(\cos[f(x)]) = -f'(x) \sin[f(x)]$

$\frac{d}{dx}(\tan[f(x)]) = f'(x) \sec^2[f(x)]$

Trig derivatives, same old same old

$\frac{d}{dx}(f(x)^{g(x)}) = \frac{g(x)f'(x)}{f(x)} + \ln[f(x)]g'(x)$

Integrals

Definition

The integral of some function $f(x)$ is a function $f^\star(x)$ s.t. $f^{\star\prime}(x) = f(x)$. **Don't forget your constant!** The integral can be done using the Right Riemann Sum:

$\int_a^b f(x)\, dx =$

$\lim\limits_{n \to \infty} \sum_{i=1}^{n} \left[\frac{b-a}{n} \cdot f(a + i \cdot \frac{b-a}{n}) \right]$

U-Substitution

$\int_a^b f(g(x)) \cdot g'\, dx = \int_{g(a)}^{g(b)} f(u)\, du$. Using U-Substitution, $u = g(x)$ and $du = g'(x)dx$ ($dx = \frac{du}{g'}$).

Integration By Parts

$\int u(x)v'(x)\, dx =$
$u(x)v(x) - \int u'(x)v(x)\, dx$ and
$\int_a^b u(x)v'(x)\, dx = uv|_a^b - \int_a^b u'(x)v(x)\, dx$.

Common Integrals

$\int x^n\, dx = \frac{1}{n+1} x^{n+1} + C$

$\int \frac{1}{x}\, dx = \ln |x| + C$

$\int \frac{1}{ax+b}\, dx = \frac{1}{a} \ln |ax + b| + C$

$\int \ln x\, dx = x \ln x - x + C$

$\int e^x\, dx = e^x + C$

$\int a^x\, dx = \frac{a^x}{\ln a} + C$

$\int \sin x\, dx = -\cos x + C$

$\int \cos x\, dx = \sin x + C$

$\int \tan x\, dx = \ln |\sec x| + C$

$\int \sec^2 x\, dx = \tan x + C$

$\int \sec x \tan x\, dx = \sec x + C$

$\int \csc x \cot x\, dx = -\csc x + C$

$\int \csc^2 x\, dx = -\cot x + C$

$\int \sec x\, dx = \ln |\sec x + \tan x| + C$

$\int \frac{1}{a^2 + u^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a} \right) + C$

$\int \frac{1}{\sqrt{a^2 + u^2}}\, dx = \sin^{-1}\left(\frac{x}{a} \right) + C$

Trig Reduction Formulae

(For $n \geq 2$)

$\int \sin^n(x)\, dx =$
$-\frac{1}{n} \sin^{n-1}(x) \cos(x) + \frac{n-1}{n} \int \sin^{n-2}(x)\, dx$

$\int \cos^n(x)\, dx =$
$\frac{1}{n} \cos^{n-1}(x) \sin(x) + \frac{n-1}{n} \int \cos^{n-2}(x)\, dx$

$\int \tan^n(x)\, dx =$
$\frac{1}{n-1} \tan^{n-1}(x) + \frac{n-1}{n} \int \tan^{n-2}(x)\, dx$

$\int \sec^n(x)\, dx = \frac{1}{n-1} \sec^{n-2}(x) \tan(x) + \frac{n-2}{n-1} \int \sec^{n-2}(x)\, dx$

Trig Identities

$\sin^2(x) + \cos^2(x) = 1$

$1 + \tan^2(x) = \sec^2(x)$

$\sin(2x) = 2 \sin(x) \cos(x)$

$\cos(2x) = \cos^2(x) - \sin^2(x)$

$= 2 \cos^2(x) - 1$

$= 1 - 2 \sin^2(x)$

Trig Substitutions

$\sqrt{a^2 - x^2} \rightsquigarrow x = a \sin(\theta)$

$\sqrt{a^2 + x^2} \rightsquigarrow x = a \tan(\theta)$

$\sqrt{x^2 - a^2} \rightsquigarrow x = a \sec(\theta)$

Partial Fractions & Polynomial Division

If the degree of the numerator is greater than that of a the denominator, use polynomial division to divide the denominator into the numerator ($denom\,)numer$), then integrate the result. If the degree of the denominator is greater than that of the numerator, use partial fractions to decompose the integral as follows:

If the denominator contains different linear terms, break it down to $\frac{A}{ax+b}$

If it contains a repeated linear term $((ax + b)^2)$, break it down to

$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$

If it contains an irreducible quadratic $(x^2 + bx + c)$, break it down to

$\frac{A}{dx+o} + \frac{Bx+C}{x^2+bx+c}$

Then, set up equations for the coefficients of the powers of x in the numerator and solve them to determine A, B, C and so on.

Applications

Centroids

$C = (\hat{x}, \hat{y}), \hat{x} = \frac{\int x f(x)\, dx}{\int f(x)\, dx}$ & $\hat{y} = \frac{\int y f(y)\, dy}{\int f(y)\, dy}$

Arc Length

$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$

for parametric. Area is just this times $2\pi x$ or y for opposite of axis.

Volume

Use the variable parallel to the axis of revolution. $V = \int_a^b A(x)\, dx$ where $A(x)$ is a function which gives the area of one "slice" of the solid. Slices are generally

circular and may have holes in them, area of a circle is πr^2, and the area of a washer is $\pi(r_{outer}^2 - r_{inner}^2)$

Sequences and Series

Series With Nice Formulae

The geometric series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ provided that $|r| < 1$ The telescoping series where the "inbetween" terms cancel. Take the limit as $k \to \infty$

Basic Divergence Test

If for some series $\sum_{n=0}^{\infty} a_n$ the limit $\lim_{n\to\infty} a_n \neq 0$ the series diverges. However, if the limit is 0, no information can be determined from this test.

Integral Test

If some function $f(n) = a_n$ is decreasing on $[c, \infty)$ then the series $\sum_{n=c}^{\infty} a_n$ converges exactly as $\int_c^{\infty} f(x)dx$ does / does not. This works because $\int_c^{\infty} f(x)dx \leq \sum_{n=c}^{\infty} a_n \leq f(c) + \int_c^{\infty} f(x)dx$

P-Test

Generally, $\sum_{n=c}^{\infty} \frac{an^k + \cdots + a_0}{bn^l + \cdots + b_0}$ converges if $p = l - k < 1$ and diverges if $p \geq 1$

Basic Comparsion Test

If two sequences $\{a_n\}$ and $\{b_n\}$ exist, are comprised of positive terms, and satisfy $0 < a_n \leq b_n$ past some point then

(a) if $\sum_{n=0}^{\infty} b_n$ converges, so does $\sum_{n=0}^{\infty} a_n$

(b) if $\sum_{n=0}^{\infty} a_n$ diverges, so does $\sum_{n=0}^{\infty} b_n$

Limit Comparison Test

If two sequences $\{a_n\}$ and $\{b_n\}$ exist and are comprised of positive terms past some point then $\lim_{n\to\infty} \frac{a_n}{b_n} = c$

(a) if $c > 0$ both series either converge or diverge (one converging means the other does and vice versa)

(b) if $c = 0$ then a_n diverges so does b_n and if b_n converges so does a_n

(c) if $c = \infty$ then a_n converges so does b_n and if b_n diverges so does a_n

Alternating Series Test

If each a_n in the series $\sum_{n=0}^{\infty} a_n$ and

(1) $|a_{n+1}| < |a_n|$ (The series is decreasing)

(2) $a_{n+1} < 0$ & $a_n > 0$ (The series is alternating)

(3) $\lim_{n\to\infty} |a_n| = 0$

then the series converges

Ratio Test

If past some point $a_n \neq 0$ then for the series $\sum_{n=0}^{\infty} a_n$ if the limit $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ then

(a) if $L < 1$ the series converges absolutely

(b) if $L = 1$ no information can be obtained through this test

(c) if $L > 1$ the series diverges

Root Test

For the series $\sum_{n=0}^{\infty} a_n$ if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ then

(a) if $L < 1$ the series converges absolutely

(b) if $L = 1$ no information can be obtained through this test

(c) if $L > 1$ the series diverges

Taylor Series

Taylor's formula states that if some function $f(x)$ can be expanded as a power series around $x = a$ its representation is given by

$$\sum_{n=0}^{\infty} \frac{\frac{d^n}{dx^n} f(a)}{n!} (x - a)^n$$

Remainder Terms

If there exists some function $f(x)$ whose Taylor Series is defined as $T_n(x)$ then the remainder term is defined as $R_n(x) = f(x) - T_n(x)$, which is how "far off" the Taylor Series $T_n(x)$ is from the function $f(x)$ for some n.

Tidbits

$\vec{v}$ here is the velocity vector (duh)
Unit tangent vector $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$

Curvature $\kappa = \left| \frac{d\vec{T}}{d\vec{s}} \right| = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$

Normal vector $\vec{N} = \frac{1}{\kappa} \frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$

For conservative fields
$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}, \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}$.