Math 2350H: Assignment II

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February 25, 2025

Problem 1. Let $T \in \mathcal{L}(V)$. Prove that $T^2 = T_0$ iff $\operatorname{range} T \subseteq \operatorname{null} T$.

Solution 1. Assuming

$$\begin{aligned}
&\operatorname{range} T \subseteq \operatorname{null} T \\
\implies &T = T_0 \\
\implies &T(v \in V) = 0_V \\
\implies &T(T(v)) = T^2 = 0_V \ (T : V \mapsto V) \\
\implies &T^2 = T_0
\end{aligned}$$

In the other direction assuming

$$\begin{aligned}
&T^2 = T_0 \\
\implies &T(T(v \in V)) = 0_V \\
\implies &T(v' \in V) = 0_V \ (T : V \mapsto V) \\
\implies &\operatorname{range} T \subseteq \operatorname{null} T
\end{aligned}$$

$\therefore T^2 = T_0 \iff \operatorname{range} T \subseteq \operatorname{null} T$

Problem 2. Let U, V, W be vector spaces over field F, and let $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$.

(a) Show that if $T \circ S$ is injective, then S is injective.

(b) Give an example showing that if $T \circ S$ is injective then T need not be injective.

(c) Show that if $T \circ S$ is surjective, then T is surjective.

(d) Give an example showing that if $T \circ S$ is surjective then S need not be surjective.

Solution 2.

Problem 3. Let $x_0, x_1, \ldots, x_n$ be $n+1$ distinct real numbers. For $i = 0, \ldots, n$, the set of polynomials $p_0, \ldots, p_n$, defined by

$$p_i(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_n)}{(x_i - x_0)(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)}$$

are called the *Lagrange polynomials* associated to $x_0, x_1, \ldots, x_n$.

(a) Find the Lagrange polynomials associated to $-5, -1, 0, 3$.

(b) Notice that $p_i(x_j) = 0$ for $i \neq j$ and $p_i(x_i) = 1$ (verify for yourself). Use this to prove that the set of Lagrange polynomials $p_0, \ldots, p_n$ associated to $x_0, x_1, \ldots, x_n$, is a basis for $\mathcal{P}_n(\mathbb{R})$.

(c) Let $(x_0, y_0), \ldots, (x_n, y_n)$ be $n+1$ points in the plane $\mathbb{R}^2$, where $x_0, x_1, \ldots, x_n$ are distinct. Verify that the polynomial

$$f(x) = y_0 p_0(x) + \cdots + y_n p_n(x)$$

is the unique polynomial in $\mathcal{P}_n(\mathbb{R})$ which passes through these $n+1$ points.

(d) Use the Lagrange polynomials found in part (a) to find a polynomial of degree at most 3 passing through

$$(-5, 4), (-1, 1), (0, 2), (3, -3).$$

(show how you obtain your answer).

(e) Suppose $g \in \mathcal{P}_n(\mathbb{R})$ and $g(r_i) = 0$ for $n + 1$ distinct scalars $r_0, r_1, \ldots, r_n$. Use the results above to show that g is the zero polynomial. This shows that a nonzero polynomial of degree n cannot have more than n distinct roots.

Solution 3.

(a)

$$p_0(x) = \frac{(x+1)(x)(x-3)}{(-5+1)(-5)(-5-3)} = \frac{x^3 - 2x^2 - 3x}{-160}$$
$$p_1(x) = \frac{(x+5)(x)(x-3)}{(-1+1)(-1)(-1-3)} = \frac{x^3 + 2x^2 - 15x}{16}$$
$$p_2(x) = \frac{(x+1)(x+5)(x-3)}{(5)(1)(-3)} = \frac{x^3 + 3x^2 - 13x}{-15} + 1$$
$$p_3(x) = \frac{(x+1)(x+5)(x)}{(3+5)(3+1)(3)} = \frac{x^3 + 6x^2 + 5x}{96}$$

(b) For a set to form a basis it must be both linearly independent and a spanning set. Proving linear independence here is sufficient to imply the set is a spanning set as it means we have a linearly independent set of dimension $n + 1$ for $\mathcal{P}_n$. For a set to be linearly independent we have

$$c_0 p_0(x) + \ldots c_n p_n(x) = 0 \implies c_0 = \cdots = c_n = 0$$

If we evaluate the above expression for every x_i $i \leq n$ we obtain n expressions where all polynomials $p_j(x_i) = 0$ for $i \neq j$ and a single polynomial $p_i(x_i) = 1$. This single polynomial has coefficient c_i which, to satisfy the previous expression for linear independence must be 0. So, applying this to each x_i we obtain that all scalars $c_0, \ldots c_n$ must be zero. This implies linear independence which as mentioned above implies span due the nature of polynomial spaces which then implies that the set of Lagrange polynomials generated by distinct real numbers $x_0, x_1, \ldots, x_n$ is a basis for $\mathcal{P}_n(\mathbb{R})$.

(c) Again using the fact that $p_i(x_i) = 1$ it is evident that $y_i p_i(x_i) = y_i$ and so because we are also guaranteed that $p_i(x_j) = 0$ for $i \neq j$ $\sum_{i=0}^{n} y_i p_i(x_i)$ will be y_i at all corresponding x_i because only a single polynomial in the sum will be nonzero at that point.

(d) From part (c) we simply sum all of the polynomials found in part (a) and multiply them by the given y-coordinate

$$= 4 \cdot \frac{x^3 - 2x^2 - 3x}{-160} + 1 \cdot \frac{x^3 + 2x^2 - 15x}{16} + 2 \cdot \left(\frac{x^3 + 3x^2 - 13x}{-15} + 1 \right) - 3 \cdot \frac{x^3 + 6x^2 + 5x}{96}$$
$$= \frac{-61x^3 - 198x^2 + 343x}{480} + 2$$

(e) Suppose $g \in \mathcal{P}_n(\mathbb{R})$ and $g(r_i) = 0$ for $n + 1$ distinct scalars $r_0, r_1, \ldots, r_n$. Use the results above to show that g is the zero polynomial. This shows that a nonzero polynomial of degree n cannot have more than n distinct roots.