

Physics 3200Y: Assignment I

Jeremy Favro (0805980)
Trent University, Peterborough, ON, Canada

September 23, 2025

Problem 1. Let $\mathbf{z} = \mathbf{r} - \mathbf{r}'$ be the separation between \mathbf{r} and \mathbf{r}' , where $\mathbf{r}' = (x', y', z')$ is a fixed point and $\mathbf{r} = (x, y, z)$. Let $z = |\mathbf{z}|$ be the magnitude of the separation.

- (a) Show that $\nabla (z^2) = 2\mathbf{z}$.
- (b) Show that $\nabla \exp(\vec{k} \cdot \vec{z}) = \vec{k} \exp(\vec{k} \cdot \vec{z})$, where \vec{k} is a vector constant.
- (c) Show that $\nabla \exp(kz) = k\hat{\mathbf{z}} \exp(kz)$.
- (d) Show that $\nabla (1/z) = -\hat{\mathbf{z}}/z^2$.

Solution 1.

(a) *Proof.*

$$\begin{aligned}
 &= \nabla (z^2) \\
 &= \left[\frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right] \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}^2 \\
 &= \left[\frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right] \left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right] \\
 &= \frac{\partial}{\partial x} (x-x')^2 \hat{x} + \frac{\partial}{\partial y} (y-y')^2 \hat{y} + \frac{\partial}{\partial z} (z-z')^2 \hat{z} \quad \text{Note}^1 \\
 &= 2(x-x') \hat{x} + 2(y-y') \hat{y} + 2(z-z') \hat{z} \quad \text{By chain rule} \\
 &= 2((x-x'), (y-y'), (z-z')) = 2\mathbf{z}
 \end{aligned}$$

□

(b) *Proof.*

$$\begin{aligned}
 &= \nabla \exp(\vec{k} \cdot \vec{z}) \\
 &= \nabla \exp(k_x(x-x') + k_y(y-y') + k_z(z-z')) \\
 &= k_x \exp(k_x(x-x') + k_y(y-y') + k_z(z-z')) \hat{x} \\
 &\quad + k_y \exp(k_x(x-x') + k_y(y-y') + k_z(z-z')) \hat{y} \\
 &\quad + k_z \exp(k_x(x-x') + k_y(y-y') + k_z(z-z')) \hat{z} \\
 &= (k_x, k_y, k_z) \exp(k_x(x-x') + k_y(y-y') + k_z(z-z')) \hat{\mathbf{z}} \\
 &= \vec{k} \exp(\vec{k} \cdot \vec{z})
 \end{aligned}$$

□

¹The partials kill the terms that don't contain their variable of differentiation, omitted for brevity

(c) *Proof.*

$$\begin{aligned}
&= \nabla \exp(kz) \\
&= \nabla \exp\left(\sqrt{k_x^2 + k_y^2 + k_z^2} \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\right) \\
&= \nabla \exp\left(\sqrt{k_x^2 + k_y^2 + k_z^2} \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\right) \\
&= \exp\left(\sqrt{k_x^2 + k_y^2 + k_z^2} \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\right) \\
&\quad \vec{k} \nabla \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \quad \text{By chain rule} \\
&= \frac{\frac{1}{2} \vec{k} \exp\left(\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\right) [\hat{x}(x-x') + \hat{y}(y-y') + \hat{z}(z-z')]}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \\
&= \vec{k} \exp(kz) \frac{[(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}]}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \\
&= \vec{k} \hat{\mathbf{z}} \exp(kz)
\end{aligned}$$

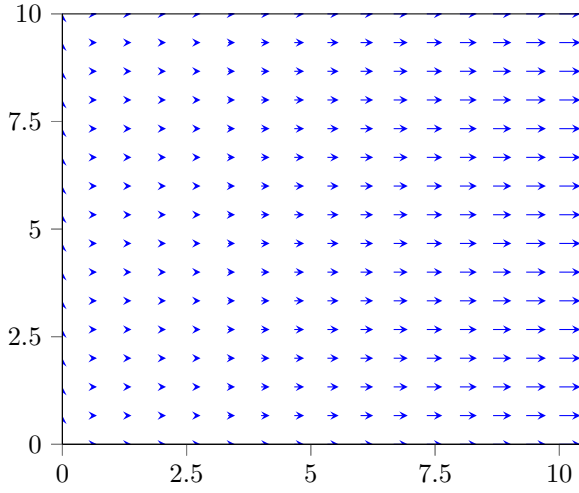
□

(d) *Proof.*

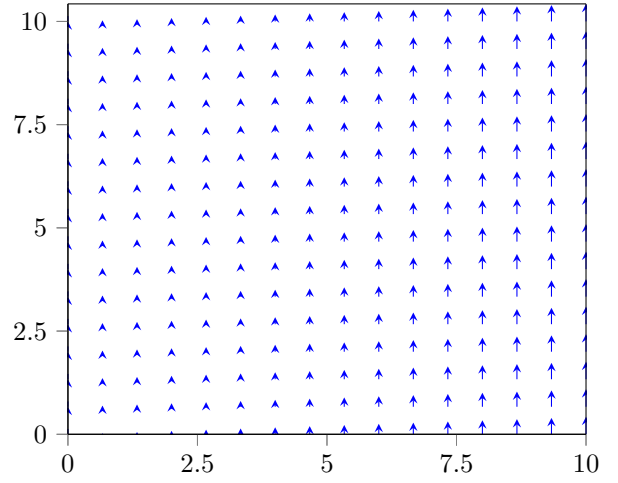
$$\begin{aligned}
&= \nabla (1/z) \\
&= \nabla \left((x-x')^2 + (y-y')^2 + (z-z')^2 \right)^{-1/2} \\
&= -\frac{1}{2} \left((x-x')^2 + (y-y')^2 + (z-z')^2 \right)^{-3/2} (\hat{x}(x-x') + \hat{y}(y-y') + \hat{z}(z-z')) \quad \text{By chain rule...} \\
&= -\frac{\mathbf{r}}{z^3} = -\frac{\hat{\mathbf{z}}}{z^2}
\end{aligned}$$

□

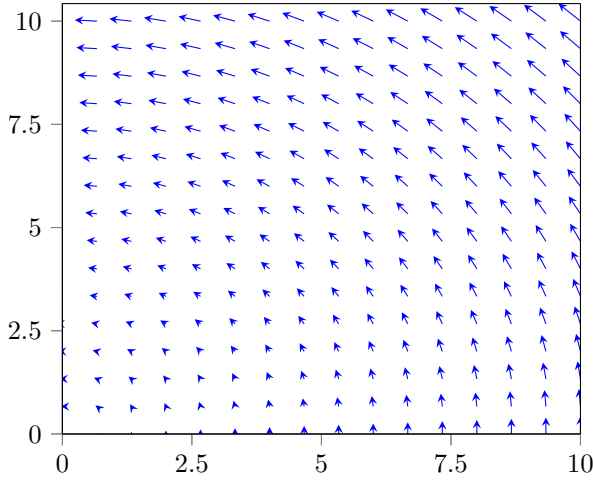
Problem 2. For each of the vector fields illustrated below, find an equation that *could* describe the field and calculate its divergence and curl.



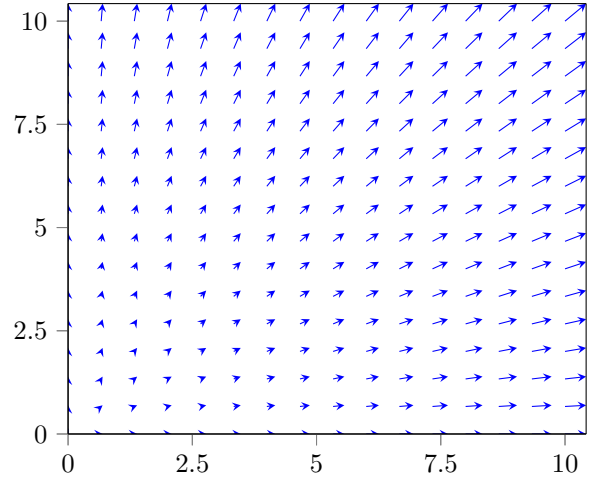
(a)



(b)



(c)



(d)

Figure 1: Plots generated with TikZ

Solution 2. (a) $\mathbf{F} = x\hat{x} + 0\hat{y} + 0\hat{z}$. $\text{div}(\mathbf{F}) = \vec{\nabla} \cdot \mathbf{F} = 1$.

$$\begin{aligned} \text{curl}(\mathbf{F}) &= \vec{\nabla} \times \mathbf{F} \\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 0 & 0 \end{vmatrix} \\ &= 0\hat{x} + 0\hat{y} + 0\hat{z} \end{aligned}$$

(b) $\mathbf{F} = 0\hat{x} + x\hat{y} + 0\hat{z}$. $\text{div}(\mathbf{F}) = \vec{\nabla} \cdot \mathbf{F} = 0$

$$\begin{aligned} \text{curl}(\mathbf{F}) &= \vec{\nabla} \times \mathbf{F} \\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix} \\ &= 0\hat{x} + 0\hat{y} + 1\hat{z} \end{aligned}$$

$$(c) \quad \mathbf{F} = -y\hat{x} + x\hat{y} + 0\hat{z}. \quad \text{div}(\mathbf{F}) = \vec{\nabla} \cdot \mathbf{F} = 0$$

$$\begin{aligned} \text{curl}(\mathbf{F}) &= \vec{\nabla} \times \mathbf{F} \\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} \\ &= 0\hat{x} + 0\hat{y} + 2\hat{z} \end{aligned}$$

$$(d) \quad \mathbf{F} = x\hat{x} + y\hat{y} + 0\hat{z}. \quad \text{div}(\mathbf{F}) = \vec{\nabla} \cdot \mathbf{F} = 2$$

$$\begin{aligned} \text{curl}(\mathbf{F}) &= \vec{\nabla} \times \mathbf{F} \\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} \\ &= 0\hat{x} + 0\hat{y} + 0\hat{z} \end{aligned}$$

Problem 3. Find the electric field a distance z above the centre of a flat circular disk of radius R that carries a uniform charge density σ . Complete all the steps we discussed in class:

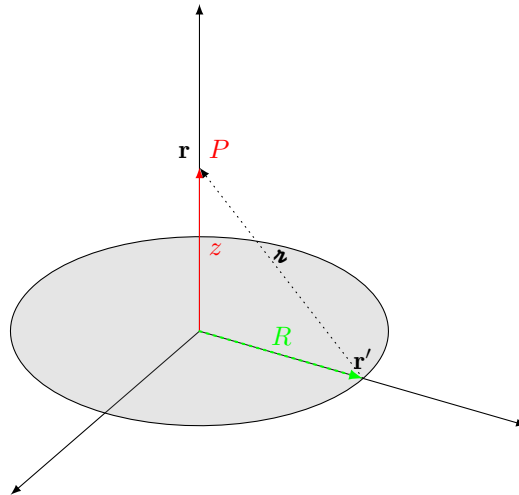
- (a) State which equation you will use and draw a diagram that illustrates the variables used in the equation.
- (b) Find expressions for each of the variables (\mathbf{r} , \mathbf{r}' , etc.) that are needed to evaluate the equation for the field.
- (c) Solve the equation for the field.
 - Be careful to distinguish between $z > 0$ and $z < 0$.
 - Check that your answer satisfies the boundary conditions given by Eq. (2.33) in Griffiths.
- (d) Check that your answer makes sense in the limit $z \gg R$, where it should look like a point charge.

Solution 3.

- (a) For this we will use the surface charge equation,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\mathbf{r}')}{r^2} \hat{\mathbf{r}} da'.$$

We will integrate over the disk which will probably require a change to polar coordinates but means that da' represents the area of the circle.



- (b) r is the vector to the field “test point”, labeled P in the figure. It’s fairly obvious that it is $(0, 0, z)$. r' is also fairly obviously $(x, y, 0)$ as it is every point s.t. $x^2 + y^2 \leq R^2$ (this is also our bound of integration until we make a change to polar coordinates). This means that $\mathbf{z} = (-x, -y, z)$

(c)

$$\begin{aligned}
&= \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\mathbf{r}')}{z^2} \hat{\mathbf{z}} da' \\
&= \frac{\sigma}{4\pi\epsilon_0} \int \frac{\hat{\mathbf{z}}}{z^3} da' \\
&= \frac{\sigma}{4\pi\epsilon_0} \int \frac{(-x, -y, z)}{(x^2 + y^2 + z^2)^{3/2}} da' \\
&= \frac{\sigma}{4\pi\epsilon_0} \int \frac{(-x, -y, z)}{(x^2 + y^2 + z^2)^{3/2}} da' \rightsquigarrow x = r' \cos \theta, y = r' \sin \theta \\
&= \frac{\sigma}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^R \frac{(-r' \cos \theta, -r' \sin \theta, z)}{(r'^2 \cos^2 \theta + r'^2 \sin^2 \theta + z^2)^{3/2}} r' dr' d\theta \rightsquigarrow x = r' \cos \theta, y = r' \sin \theta \\
&= \frac{\sigma}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^R \frac{(-r' \cos \theta, -r' \sin \theta, z)}{(r'^2 + z^2)^{3/2}} r' dr' d\theta \quad \text{Integrating w.r.t } \theta \text{ kills the sin and cos by symmetry} \\
&= \frac{\sigma}{4\pi\epsilon_0} \int_{0=r'}^R \frac{(0, 0, 2\pi z)}{2u^{3/2}} du \rightsquigarrow u = r'^2 + z^2 \implies du/2r' = dr' \\
&= \frac{\sigma}{4\epsilon_0} \int_{0=r'}^R \frac{(0, 0, z)}{u^{3/2}} du \\
&= -\frac{z\sigma}{2\epsilon_0} \left[(r'^2 + z^2)^{-1/2} \right]_0^R \hat{\mathbf{z}} = \frac{z\sigma}{2\epsilon_0} \left[\frac{1}{|z|} - \frac{1}{\sqrt{R^2 + z^2}} \right] \hat{\mathbf{z}}
\end{aligned}$$

For boundary conditions

Why do I have to discard the abs

$$\begin{aligned}
\mathbf{E}_{above} - \mathbf{E}_{below} &= \frac{z\sigma}{2\epsilon_0} \left[\frac{1}{z} - \frac{1}{\sqrt{R^2 + z^2}} + \frac{1}{z} + \frac{1}{\sqrt{R^2 + z^2}} \right] \hat{\mathbf{z}} \\
&= \frac{z\sigma}{2\epsilon_0} \left[\frac{2}{z} \right] \hat{\mathbf{z}} \\
&= \frac{\sigma}{\epsilon_0} \hat{\mathbf{z}}
\end{aligned}$$

Which satisfies equation 2.33 as $\hat{\mathbf{z}}$ here is our normal vector.

(d) We first begin with some reworking using the binomial approximation given by $(1+x)^n \approx 1+nx$

$$\begin{aligned}
&= \frac{z\sigma}{2\epsilon_0} \left[\frac{1}{z} - \frac{1}{\sqrt{R^2 + z^2}} \right] \hat{\mathbf{z}} \\
&= \frac{\sigma}{2\epsilon_0} \left[1 - 1 - \left[1 + \left(\frac{R}{z} \right)^2 \right]^{-1/2} \right] \hat{\mathbf{z}} \\
&\approx \frac{\sigma}{2\epsilon_0} \left[\frac{R^2}{2z^2} \right] \hat{\mathbf{z}} \\
&= \frac{\sigma R^2}{4\epsilon_0 z^2} \hat{\mathbf{z}}
\end{aligned}$$

We then recognize that $\sigma = Q/\pi R^2$ because we have a uniform charge distribution of net charge Q spread over a disk of area πR^2 . Substituting this in,

$$\frac{QR^2}{4\epsilon_0 \pi R^2 z^2} \hat{\mathbf{z}} = \frac{Q}{4\pi\epsilon_0 z^2} \hat{\mathbf{z}}$$

Which is the equation for a point charge, as we would hope.

Problem 4. For a solid sphere of radius R and charge density $\rho(\mathbf{r}) = \kappa r$, use Gauss' law to find

- (i) The electric field.
- (ii) The potential inside and outside the sphere.

Solution 4.

- (i) Gauss' law states that

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enc}}{\epsilon_0}$$

Where Q_{enc} is given by

$$Q_{enc} = \int_V \rho d\tau.$$

Because we have a sphere of charge we can transform this to use spherical coordinates,

$$\begin{aligned} Q_{enc} &= \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi k r^3 \sin \theta \\ &= 4\pi k \int_0^R r^3 dr \\ &= \pi k R^4 \end{aligned}$$

We can now return to our original statement involving \mathbf{E} and solve. We can also observe that because \mathbf{E} points radially outwards from our sphere and \mathbf{a} does as well we can reduce the dot product to a product of their magnitudes.

$$\frac{\pi k R^4}{\epsilon_0} = \oint_S |\mathbf{E}| da$$

Now because $|\mathbf{E}|$ is constant over the *surface* because the charge density is spherically symmetric, only depending on r , we can extract $|\mathbf{E}|$ from the integral which then just becomes the area of a sphere,

$$\begin{aligned} \frac{\pi k R^4}{\epsilon_0} &= \oint_S |\mathbf{E}| da \\ \frac{\pi k R^4}{\epsilon_0} &= |\mathbf{E}| \oint_S da \\ \frac{\pi k R^4}{\epsilon_0} &= 4\pi r^2 |\mathbf{E}| \\ \frac{k R^4}{4\epsilon_0 r^2} &= |\mathbf{E}| \end{aligned}$$

(For $r > R$)

- (ii) To start here we need the electric field inside the sphere which is given by the same process as above except we integrate from 0 to $r \leq R$ because we are now inside the sphere. This gives $Q_{enc} = \pi k r^4 \implies \mathbf{E} = \frac{k r^2}{4\epsilon_0} \hat{\mathbf{r}}$. We can now evaluate for the outside of the sphere,

$$\begin{aligned} V_{r>R}(r) &= - \int_\infty^r \frac{k R^4}{4\epsilon_0 r'^2} dr' \\ &= - \int_\infty^r \frac{k R^4}{4\epsilon_0 r'^2} dr' \\ &= \frac{k R^4}{4\epsilon_0} \left[\frac{1}{r'} \right]_\infty^r = \frac{k R^4}{4\epsilon_0 r} \end{aligned}$$

Then for the potential inside the sphere we split the integral.

I'm not sure why we do this but it's done in the text

$$\begin{aligned}
V_{R < r}(r) &= - \int_{\infty}^R \frac{kR^4}{4\epsilon_0 r'^2} dr' - \int_R^r \frac{kr^2}{4\epsilon_0} dr' \\
&= \frac{kR^4}{4\epsilon_0 R} - \frac{k}{12\epsilon_0} \left[r'^3 \right]_R^r \\
&= \frac{k}{12\epsilon_0} (4R^3 - r^3)
\end{aligned}$$

Check both of these w/ $E = -\nabla V$

Problem 5. Consider the electric field

$$\mathbf{E}(\mathbf{r}) = E_0 \left[y^2 \hat{x} + (2xy + z^2) \hat{y} + 2yz \hat{z} \right].$$

Find, by integrating $-\int \mathbf{E} \cdot d\boldsymbol{\ell}$, the potential at an arbitrary point \mathbf{r} , taking the origin as the reference point. Note that the most direct integration path may not be simplest.

Solution 5. Here we have that $d\boldsymbol{\ell} = dx\hat{x} + dy\hat{y} + dz\hat{z}$ which means that $\mathbf{E} \cdot d\boldsymbol{\ell} = E_0 [y^2 dx + (2xy + z^2) dy + 2yz dz]$ so,

$$\begin{aligned}
V(\mathbf{r}) &= - \int_0^{\mathbf{r}} E_0 [y^2 dx + (2xy + z^2) dy + 2yz dz] \\
&= -E_0 \left[\int_0^{r_x} y^2 dx + \int_0^{r_y} (2xy + z^2) dy + \int_0^{r_z} 2yz dz \right]
\end{aligned}$$

Which simplifies nicely along several paths (though it's already fairly simple). I'll go along the x , the y , then the z . This means that for our first integral along x we are moving from $\langle 0, 0, 0 \rangle$ to $\langle r_x, 0, 0 \rangle$ so y is zero along the whole range which means that the integral just becomes zero as its integrand is y^2 . Next along y we are moving from $\langle r_x, 0, 0 \rangle$ to $\langle r_x, r_y, 0 \rangle$ so

$$\int_0^{r_y} (2xy + z^2) dy = r_x r_y^2.$$

Finally along z we are moving from $\langle r_x, r_y, 0 \rangle$ to $\langle r_x, r_y, r_z \rangle$ so

$$\int_0^{r_z} 2yz dz = r_y r_z^2.$$

This yields

$$V(\mathbf{r}) = -E_0 [r_x r_y^2 + r_y r_z^2].$$