Math 2350H: Assignment IV

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April 1, 2025

Problem 1. Consider the subspace

$$U = \operatorname{span}\left((2,-1,-2,4),(-2,1,-5,5),(-1,3,7,11)\right)$$

(a) Apply the Gram-Schmidt process (with normalization) to find an orthonormal basis of U.

(b) Find a basis for $U^{\perp}$

(c) Express the vector $v = (-11, 8, -4, 18)$ as $v = x + y$ where $x \in$ and $y \in U^{\perp}$.

Solution 1.

Problem 2. The dot product is defined on $\mathcal{M}_{n\times 1}(\mathbb{R})$ and $\mathcal{M}_{n\times 1}(\mathbb{C})$ just as it is for $\mathbb{R}^n$ and $\mathbb{C}^n$. For $u, v \in \mathcal{M}_{n\times 1}(\mathbb{C})$, with

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

the dot product (or standard inner product) is

$$\langle u|v\rangle = \sum_{k=1}^{n} u_k v_k^*$$

This can be written more compactly with matrix multiplication as

$$\langle u|v\rangle = v^\dagger u$$

where $v^\dagger = \left(v^T\right)^*$. We use this inner product below.

(a) Let $P \in \mathcal{M}_{n\times n}(\mathbb{R})$. Relative to the dot product on column vectors, show that the following are equivalent:

  (i) The columns $u_1, \ldots, u_n$ of P form an orthonormal basis $\mathcal{M}_{n\times 1}(\mathbb{R})$.

  (ii) $P^T = P^{-1}$.

  (iii) The rows of P form an orthonormal basis for $\mathbb{R}^n$.

(b) A matrix $P \in \mathcal{M}_{n\times n}(\mathbb{R})$ is called an *orthogonal matrix* if $P^T = P^{-1}$. Determine which of the following matrices are orthogonal.

  (i) $\begin{pmatrix} 3/5 & 4/5 \\ -4/5 & -3/5 \end{pmatrix}$

  (ii) $\begin{pmatrix} 5/13 & 12/13 \\ 12/13 & -5/13 \end{pmatrix}$

  (iii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

  (iv) $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

(v) $\begin{pmatrix} 1/3 & 2/3 & -2/3 \\ 1/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \end{pmatrix}$

(vi) $\begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \end{pmatrix}$

(c) Let $P, Q \in \mathcal{M}_{n\times n}(\mathbb{R})$ be orthogonal matrices and $u, v \in \mathcal{M}_{n\times 1}(\mathbb{R})$. Prove that

(i) $\langle Pu|Pv\rangle = \langle u|v\rangle$

(ii) $\|Pu\| = \|u\|$

(iii) PQ is an orthogonal matrix.

Solution 2.

Problem 3. Let V be the vector space of continuous, real-valued functions defined on the interval $[0, 1]$. Then V is an inner product space with inner product

$$\langle f|g\rangle = \int_0^1 f(x)g(x)dx,$$

for $f, g \in V$. Consider the subspace U of V spanned by the functions $f(x) = \sqrt{x}$, $g(x) = x$, $h(x) = x^2$.

(a) Show that f, g, h is linearly independent.

(b) Find an orthonormal basis for U

(c) Let $p(x) = x^3$. Find the closest approximation of p in U.

Solution 3.