ODE Cheat Sheet

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Revision 3

Fundamentals

Classification

$\frac{d^n y}{dx^n} = f(x,y)$ denotes an ODE of order n. Note that $(\frac{dy}{dx})^n \neq \frac{d^n y}{dx^n}$. ODEs of order n will have n constants in their general form solutions.
A linear ODE is one which can be written in the form $a_n(x)\frac{d^n y}{d^n x} + a_{n-1}\frac{d^{n-1}y}{d^{n-1}x} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$.

Solutions

Given some IVP $\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0$ if f and $\frac{\partial f}{\partial y}$ are continuous in the rectangle $(x_0, y_0) \in \{(x,y) : a < x < b, c < y < d\}$ then the IVP has a unique solution $\phi(x)$ in some interval $(x_0 - h, x_0 + h),\ h \geq 0$

Solution Techniques $n = 1$

Direct Integration

Directly integrate …

Seperable

For some ODE $\frac{dy}{dx} = f(x,y) = g(x)p(y)$ the differential can be split $s.t.$ $\frac{1}{p(y)}dy = g(x)dx$ which can be solved by direct integration. Note that when dividing by some function we assume that the function is nonzero. If there is a case (e.g. in an IVP) where the function is zero, the solution is lost.

Linear

For some linear ODE of the form $\frac{dy}{dx} + P(x)y = Q(x)$ we can multiply both sides of the ODE by $\mu(x) = \exp\left(\int P(x)\,dx\right)$ to obtain $\mu\frac{dy}{dx} + \mu P(x)y = \mu Q(x)$ which is equivalent to $\mu y = \mu Q(x)$ which can be solved by direct integration.

Exact

Exact equations are ODEs of the form $M dx + N dy = 0$ where $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Then,
$f(x,y) = \int M\,dx + h(y) = C$ or
$f(x,y) = \int N\,dy + g(x) = C$ and
$\frac{d}{dy}\left(\int M\,dx + h(y)\right) = N$ or
$\frac{d}{dx}\left(\int N\,dy + g(x)\right) = M$

Non-Exact

In cases where something looks exact but $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ you can find an integrating factor

$$\mu(x) = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\,dx\right)$$

$$\mu(y) = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\,dy\right)$$

Homogeneous

If each term of the ODE is of equal order (e.g. the right hand side can be expressed as a function of only $\frac{y}{x}$) we can substitute $y = ux \implies dy = udx + xdu$. This should result in a seperable equation.

Bernoulli

If we have an equation of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$ we divide by y^n and substitute $u = y^{1-n} \implies \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ (you should know what $\frac{dy}{du}$ is here). This should result in a linear equation.

Linear Substituion

An ODE of the form $\frac{dy}{dx} = f(Ax + By + C),\ B \neq 0$ can be solved by

$$u = Ax + By + C$$
$$\Rightarrow \frac{du}{dx} = A + B\frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{dx} = (\frac{du}{dx} - A)\frac{1}{B}$$

Solution Techniques $n = 2$

Reduction of Order

If you solve a second order ODE and obtain a single solution y_1, $y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}dx$ where $P(x)$ is found in $y'' + P(x)y' + Q(x)y = g(x)$

Constant Coefficients

An equation of the form $ay'' + by' + cy = g(x)$ can be solved through the characteristic equation obtained by substituting $y = e^{mt}$ and solving for m. This gives a solution of the form $y_h = C_1e^{m_1t} + C_2e^{m_2t}$.

Undetermined Coefficients

To obtain the particular solution of $ay'' + by' + cy = g(x)$ we try

$g(x)$	$y_p(x)$
$Ce^{\alpha x}$	$x^s\left(Ae^{\alpha x}\right)$
$C_nx^n + \cdots + C_1x + C_0$	$x^s\left(A_nx^n + \cdots + A_1x + A_0\right)$
$C\cos(\beta x)$, $\quad C\sin(\beta x)$	$x^s\left(A\cos(\beta x) + A_1\sin(\beta x)\right)$
$\left(C_nx^n + \cdots + C_1x + C_0\right)e^{\alpha x}$	$x^s\left(A_nx^n + \cdots + A_1x + A_0\right)e^{\alpha x}$

Variation of Parameters

For $y'' + P(x)y' + Q(x)y = g(x)$ if you have the homogeneous solutions $y_1(x)$ and $y_2(x)$, the particular solution $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ where

$$u_1(x) = -\int \frac{g(x)y_2(x)}{W\left[y_1, y_2\right]}dx$$

$$u_2(x) = \int \frac{g(x)y_1(x)}{W\left[y_1, y_2\right]}dx$$

where $W\left[y_1, y_2\right]$ is the Wronskian,

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Cauchy-Euler

An equation of the form $ax^2y'' + bxy' + cy = g(x)$ can be solved through the characteristic equation $am^2 + (b-a)m + c = 0$ obtained by substituting $y = x^{mt}$ and solving for m. In the case where m is complex here you end up with trig functions of logarithms.

Laplace Transform

If $f(t)$ has period T and is piecewise continuous on $[0, T]$ then $\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st}f(t)dt}{1-e^{sT}}$

Properties of the Laplace Transform

$\mathcal{L}\{f_1 + f_2\} = \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}$
$\mathcal{L}\{cf_1\} = c\mathcal{L}\{f_1\}$
$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$
$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$
$\mathcal{L}\{t^nf(t)\} = (-1)^n\frac{d^nF(s)}{ds^n} \implies f(t) = \frac{(-1)^n}{t^n}\mathcal{L}^{-1}\left\{\frac{d^nF(s)}{ds^n}\right\}$
$\mathcal{L}\{f(t-a)\mu(t-a)\} = e^{-as}F(s)$
$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mu(t-a)$

Solving Discontinuous IVPs with Laplace Transforms

For some ODE $ay'' + by' + cy = g(t)$
$\mathcal{L}\{g(t)\mu(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$

$$\mu(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$

Applications

Newton's Cooling

$$\frac{dT}{dt} = k(T - T_m) \implies T(t) = T_m + Ce^{kt}$$

where T is the temperature of an object, T_m the temperature of the medium in which the object sits, and k some cooling constant determined by initial/boundary conditions. C comes about as a result of solving the ODE and can also be determined using initial conditions.

Circuit Theory

$$V_{Resistor} = RI = R\frac{dQ}{dt}$$

$$V_{Inductor} = L\frac{dI}{dt} = L\frac{d^2Q}{dt^2}$$

$$V_{Capacitor} = \frac{Q}{C}$$

Miscellaneous

Partial Fractions

$\frac{px+q}{(x-a)(x-b)} \to \frac{A}{x-a} + \frac{B}{x-b}$
$\frac{px+q}{(x-a)^2} \to \frac{A}{x-a} + \frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} \to \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)} \to \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)(x^2-bx+c)} \to \frac{A}{x-a} + \frac{B}{(x^2-bx+c)}$

Systems

$x' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = Ax$ Then guess that $x = e^{\lambda t} \implies A\vec{v} = \lambda\vec{v}$. To solve for eigenvalues find $det\left(A - \lambda I\right) = 0$. Then solve for eigenvectors $\vec{v_n}$ for each λ_n with $\left(A - \lambda_n I\right)\vec{v_n} = 0$. For real, distinct eigenvector-value pairs write $y(t) = C_1 e^{\lambda_1 t}\vec{v_1} + C_2 e^{\lambda_2 t}\vec{v_2}$. For a repeated eigenvalue write $y(t) = C_1 e^{\lambda_1 t}\vec{v_1} + C_2 e^{\lambda_1 t}(\vec{v_2} + t\vec{v_1})$ where $A\vec{v_2} - \lambda_1\vec{v_2} = \vec{v_1}$