Physics 2700H: Assignment II

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Problem 1. Consider a mix of N_2 and O_2, which we may treat as an ideal gas, inside a car engine's cylinder that follows the idealized Otto cycle. Assume points a, b, c, d in Fig. 4.15(b) correspond to (V, P) values of: $\{(7V_1, 1\,\text{atm}), (V_1, 15.25\,\text{atm}), (V_1, 30.50\,\text{atm}), (7V_1, 2\,\text{atm})\}$, respectively.

(a) Confirm that these (V, P) values are consistent with the gas experiencing one adiabatic compression and one adiabatic expansion over each cycle.

(b) If the car's engine size (the displacement of here, <u>four</u> cylinders) due to compression/expansion is 2.4 L, what is V_1?

(c) Find the net work done by the gas in all four cylinders of the engine over one cycle (you may use the relevant expression for work from either p. 44 or p. 77 of the textbook).

Solution 1.

(a) Here the adiabats are along $\overrightarrow{ab}$ and $\overrightarrow{cd}$. For an adiabat $P_iV_i^\gamma = P_fV_f^\gamma$ holds so

$$\begin{aligned}(7V_1)^\gamma \cdot 1\,\text{atm} &= V_1^\gamma \cdot 15.25\,\text{atm}\\ V_1^\gamma \cdot 1\,544\,734.579\,\text{Pa} &= V_1^\gamma \cdot 1\,545\,206.25\,\text{Pa}\end{aligned}$$

Which is approximately adiabatic. For $\overrightarrow{cd}$

$$\begin{aligned}V_1^\gamma \cdot 30.50\,\text{atm} &= (7V_1)^\gamma \cdot 2\,\text{atm}\\ V_1^\gamma \cdot 3\,090\,412.5\,\text{Pa} &= V_1^\gamma \cdot 3\,089\,469.158\,\text{Pa}\end{aligned}$$

Which is also approximately adiabatic.

(b) The volume difference between compression and expansion **for a single cylinder** is $7V_1 - V_1 = 6V_1$ so $4 \cdot 6V_1 = 2.4\,\text{L} \implies V_1 = 0.1\,\text{L} = 1 \times 10^{-4}\,\text{m}^3$

(c) This formula gives work done on the gas, so we must flip the sign to determine work done by the gas

$$\begin{aligned}W_{\overrightarrow{ab}} &= -\frac{P_iV_i}{\gamma - 1}\left[\left(\frac{V_i}{V_f}\right)^{\gamma-1} - 1\right]\\ &= -\frac{7\times10^{-4}\,\text{m}^3 \cdot 101\,325\,\text{Pa}}{\frac{7}{5}-1}\left[\left(\frac{7\times10^{-4}\,\text{m}^3}{1\times10^{-4}\,\text{m}^3}\right)^{\frac{7}{5}-1} - 1\right]\\ &\approx -208.9\,\text{J}\end{aligned}$$

$$\begin{aligned}W_{\overrightarrow{cd}} &= -\frac{1\times10^{-4}\,\text{m}^3 \cdot 3\,090\,412.5\,\text{Pa}}{\frac{7}{5}-1}\left[\left(\frac{1\times10^{-4}\,\text{m}^3}{7\times10^{-4}\,\text{m}^3}\right)^{\frac{7}{5}-1} - 1\right]\\ &\approx 417.9\,\text{J}\end{aligned}$$

Net work for one cylinder is $W_{net} = 417.9\,\text{J} - 208.9\,\text{J} = 209\,\text{J}$. This means that net work in all four cylinders (in lock-step e.g. not on a camshaft, I think) is 836 J

Problem 2. An inventor claims to have developed an engine that takes in 1.1×10^8 J at 400 K, rejects 5.07×10^7 J at 200 K, and delivers 16.7 kW hours of work. Would you advise investing money in this project?

Solution 2. A Carnot engine operating in this environment has $\eta = 1 - \frac{T_2}{T_1} = 1 - \frac{200\,\mathrm{K}}{400\,\mathrm{K}} = 0.5$ and this engine has efficiency $\eta = 1 - \frac{5.07\times 10^7\,\mathrm{J}}{1.1\times 10^8\,\mathrm{J}} \approx 0.54$. As the efficiency of a "real" engine cannot exceed that of a Carnot engine I would not reccomend investing in this project.

Problem 3. Suppose a house requires 4.3 GJ of heating in a winter month. The utility company charges \$0.14 per kW h.

(a) Find the cost savings of using a heat pump versus a 95%-efficient natural gas furnace. Assume a Carnot heat pump with average temperatures of 20 °C indoors and 0 °C outdoors.

(b) Repeat part (a) using a more realistic coefficient of performance of 4.0 for the heat pump.

Solution 3.

(a) $\eta_{furn} = 0.95 = \frac{W_{furn}}{Q_1} \implies 0.95Q_1 = W_{furn} = 4.085 \times 10^9$ J and $\eta_c = 1 - \frac{T_2}{T_1} = \frac{W_{HP}}{Q_1} = COP_{HP}^{-1} = 14.6575$ which means $14.6575 = \frac{Q_1}{W_{HP}} \implies W_{HP} = \frac{Q_1}{14.6575} \approx 0.2933 \times 10^9$ J so $\Delta W = W_{furn} - W_{HP} \approx 3.792 \times 10^9$ J $\approx$ 1027.778 kW h $\approx$ 143.89\$. Note that this seems very high but this is an ideal heat pump working in decent conditions so it could be "realistic".

(b) The work done by the furnace stays the same so we end up with $\Delta W = W_{furn} - W_{HP} = W_{furn} - \frac{Q_1}{COP_{HP}} =$ 3.01 J $= 836.\overline{1} \approx$ 117.06\$ which also seems quite high for a heat pump now operating with an apparently more realistic COP. Maybe the authors of this problem are just very out of touch.

Problem 4. A hypothetical engine, with an ideal gas as the working substance, operates in the cycle shown in Figure 4.17. Show that the efficiency of the engine is

$$\eta = 1 - \frac{1}{\gamma}\left(\frac{1 - \frac{P_3}{P_1}}{1 - \frac{V_1}{V_3}}\right)$$

Solution 4. Here we know that $Q_1 = nC_P\Delta T_{12}$ and $Q_2 = nC_V\Delta T_{23}$ and heat across the adiabat is by definition zero. Then,

$$Q_1 = nC_P\left[T_2 - T_1\right] = nC_P\left[\frac{P_1V_3}{nR} - \frac{P_1V_1}{nR}\right] = \frac{C_PP_1}{R}\left[V_3 - V_1\right]$$

And

$$Q_2 = nC_V\left[T_3 - T_2\right] = nC_V\left[\frac{P_3V_3}{nR} - \frac{P_1V_3}{nR}\right] = \frac{C_VV_3}{R}\left[P_3 - P_1\right]$$

Then we know that $W = Q_1 + Q_2$ because this is a cycle with $\Delta U = 0$ and $\eta = \frac{W}{Q_1} = \frac{Q_1+Q_2}{Q_1} = 1 + \frac{Q_2}{Q_1}$ so,

$$\begin{aligned}\eta &= 1 + \frac{Q_2}{Q_1} = 1 + \frac{\frac{C_VV_3}{R}\left[P_3 - P_1\right]}{\frac{C_PP_1}{R}\left[V_3 - V_1\right]} = 1 + \frac{R}{C_PP_1}\frac{C_VV_3}{R}\frac{\left[P_3 - P_1\right]}{\left[V_3 - V_1\right]} \\ &= 1 + \frac{1}{\gamma}\frac{V_3}{P_1}\frac{\left[P_3 - P_1\right]}{\left[V_3 - V_1\right]} = 1 + \frac{1}{\gamma}\frac{\left[\frac{P_3}{P_1} - \frac{P_1}{P_1}\right]}{\left[\frac{V_3}{V_3} - \frac{V_1}{V_3}\right]} = 1 + \frac{1}{\gamma}\frac{\left[\frac{P_3}{P_1} - 1\right]}{\left[1 - \frac{V_1}{V_3}\right]} \\ &= 1 - \frac{1}{\gamma}\frac{\left[1 - \frac{P_3}{P_1}\right]}{\left[1 - \frac{V_1}{V_3}\right]}\end{aligned}$$