Math 2350H: Assignment III

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Problem 1. Let $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{M}_{2\times 2}(\mathbb{R})$ be the linear transformation given by

$$T(a+bx+cx^2+dx^3) = \begin{pmatrix} 3a+7b-2c-5d & 8a+14b-2c-11d \\ -4a-8b+2c+6d & 12a+22b-4c-17d \end{pmatrix}$$

(a) Find the matrix representation $[T]_\beta^\gamma$ for bases

$$\beta = \left\{1, x, x^2, x^3\right\}, \qquad \gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

(b) Find the matrix representation $[T]_\mathcal{B}^\mathcal{C}$ for bases

$$\mathcal{B} = \left\{1+x-x^2+2x^3, -1+2x+2x^3, 2+x-2x^2+3x^3, 1+x+2x^3\right\},$$

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \right\}.$$

(c) Let $p(x) = 3 - x + 2x^2 - 5x^3$. Find $[p(x)]_\beta$ and $[p(x)]_\mathcal{B}$.

(d) Find the image of $p(x)$ under T in the following three ways:

(i) By computing $T(p(x))$ directly

(ii) By computing $[T]_\beta^\gamma [p(x)]_\beta = [T(p(x))]_\gamma$

(iii) By computing $[T]_\mathcal{B}^\mathcal{C} [p(x)]_\mathcal{B} = [T(p(x))]_\mathcal{C}$

(e) Compute the matrix representations $[I]_\beta^\mathcal{B}$ and $[I]_\mathcal{B}^\beta$, where I is the identity map on $\mathcal{P}_3(\mathbb{R})$. Show that $\left([I]_\beta^\mathcal{B}\right)^{-1} = [I]_\mathcal{B}^\beta$

(f) Compute the matrix product $[I]_\beta^\mathcal{B} [p(x)]_\beta$. What do we notice about the result?

Solution 1.

(a) Here we start by applying T to each element of the input basis, β, and expressing the result as a linear combination of vectors in the output basis, γ,

$$\begin{aligned} T(1) &= \begin{pmatrix} 3 & 8 \\ -4 & 12 \end{pmatrix} = 3\gamma_1 + 8\gamma_2 - 4\gamma_3 + 12\gamma_4 \rightsquigarrow \text{(treating } \gamma \text{ as ordered)} \\ T(x) &= \begin{pmatrix} 7 & 14 \\ -8 & 22 \end{pmatrix} = 7\gamma_1 + 14\gamma_2 - 8\gamma_3 + 22\gamma_4 \\ T(x^2) &= \begin{pmatrix} -2 & -2 \\ 2 & -4 \end{pmatrix} = -2\gamma_1 - 2\gamma_2 + 2\gamma_3 - 4\gamma_4 \\ T(x^3) &= \begin{pmatrix} -5 & -11 \\ 6 & -17 \end{pmatrix} = -5\gamma_1 - 11\gamma_2 + 6\gamma_3 - 17\gamma_4 \end{aligned}$$

These linear combinations give us the columns of

$$[T]_\beta^\gamma = \begin{pmatrix} 3 & 7 & -2 & -5 \\ 8 & 14 & -2 & -11 \\ -4 & -8 & 2 & 6 \\ 12 & 22 & -4 & -17 \end{pmatrix}$$

(b) Applying the same process as in part (a) (again treating $\mathcal{C}$ as ordered),

$$\begin{aligned} T(1 + x - x^2 + 2x^3) &= \begin{pmatrix} 2 & 2 \\ -2 & 4 \end{pmatrix} = 2\mathcal{C}_1 + 0\mathcal{C}_2 + 0\mathcal{C}_3 + 0\mathcal{C}_4 \\ T(-1 + 2x + 2x^3) &= \begin{pmatrix} 1 & -2 \\ 0 & -2 \end{pmatrix} = 0\mathcal{C}_1 - 1\mathcal{C}_2 + 0\mathcal{C}_3 + 0\mathcal{C}_4 \\ T(2 + x - 2x^2 + 2x^3) &= \begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix} = 0\mathcal{C}_1 + 0\mathcal{C}_2 + 1\mathcal{C}_3 + 0\mathcal{C}_4 \\ T(1 + x + 2x^3) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0\mathcal{C}_1 + 0\mathcal{C}_2 + 0\mathcal{C}_3 + 0\mathcal{C}_4 \end{aligned}$$

$$\implies [T]_{\mathcal{B}}^{\mathcal{C}} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(c) Again we represent $p(x)$ as a linear combination where the coefficients give us the columns of the matrix representation. For $[p(x)]_\beta$ we get $p(x) = 3 \cdot 1 - 1 \cdot x + 2 \cdot x^2 - 5 \cdot x^3 \implies [p(x)]_\beta = \begin{pmatrix} 3 \\ -1 \\ 2 \\ -5 \end{pmatrix}$. Then for $[p(x)]_{\mathcal{B}}$ we have a less obvious solution (obtained by row-reduction of the associated matrix) of $p(x) = 32\mathcal{B}_1 - 7\mathcal{B}_2 - 17\mathcal{B}_3 - 2\mathcal{B}_4 \implies [p(x)]_{\mathcal{B}} = \begin{pmatrix} 32 \\ -7 \\ -17 \\ -2 \end{pmatrix}$

(d) Find the image of $p(x)$ under T in the following three ways:

(i) $T(p(x)) = T(3 - x + 2x^2 - 5x^3) = \begin{pmatrix} 23 & 61 \\ -30 & 91 \end{pmatrix}$

(ii)

$$
\begin{aligned}
[T]_\beta^\gamma [p(x)]_\beta &= \begin{pmatrix} 3 & 7 & -2 & -5 \\ 8 & 14 & -2 & -11 \\ -4 & -8 & 2 & 6 \\ 12 & 22 & -4 & -17 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 2 \\ -5 \end{pmatrix} \\
&= \begin{pmatrix} 3 \cdot 3 + 7 \cdot (-1) + (-2) \cdot 2 + (-5)^2 \\ 8 \cdot 3 + 14 \cdot (-1) + (-2) \cdot 2 + (-11) \cdot (-5) \\ -4 \cdot 3 + (-8) \cdot (-1) + 2 \cdot 2 + 6 \cdot (-5) \\ 12 \cdot 3 + 22 \cdot (-1) + (-4) \cdot 2 + (-17) \cdot (-5) \end{pmatrix} \\
&= \begin{pmatrix} 23 \\ 61 \\ -30 \\ 91 \end{pmatrix}
\end{aligned}
$$

(iii)

$$
\begin{aligned}
[T]_\mathcal{B}^\mathcal{C} [p(x)]_\mathcal{B} &= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 32 \\ -7 \\ -17 \\ -2 \end{pmatrix} \\
&= \begin{pmatrix} 64 \\ 7 \\ -17 \\ 0 \end{pmatrix}
\end{aligned}
$$

(e) For $[I]_\beta^\mathcal{B}$,

$$
\begin{aligned}
I(1) &= 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\
I(x) &= 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\
I(x^2) &= 0 \cdot 1 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 \\
I(x^3) &= 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 1 \cdot x^3
\end{aligned}
$$

solving the associated matrices for coefficients of the elements of $\mathcal{B}$ yields

$$
[I]_\beta^\mathcal{B} = \begin{pmatrix} 4 & 8 & -1 & -6 \\ -1 & -1 & 0 & 1 \\ -2 & -4 & 0 & 3 \\ 0 & -1 & 1 & 1 \end{pmatrix}.
$$

Then for $[I]_\mathcal{B}^\beta$,

$$
\begin{aligned}
I(1 + x - x^2 + 2x^3) &= 1 \cdot 1 + 1 \cdot x - 1 \cdot x^2 + 2 \cdot x^3 \\
I(-1 + 2x + 2x^3) &= -1 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 2 \cdot x^3 \\
I(2 + x - 2x^2 + 3x^3) &= 2 \cdot 1 + 1 \cdot x - 2 \cdot x^2 + 3 \cdot x^3 \\
I(1 + x + 2x^3) &= 1 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + 2 \cdot x^3
\end{aligned}
$$

which yields

$$
[I]_\mathcal{B}^\beta = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & -2 & 0 \\ 2 & 2 & 3 & 2 \end{pmatrix}.
$$

For $\left([I]_\beta^\mathcal{B}\right)^{-1} = [I]_\mathcal{B}^\beta \implies I_4 = [I]_\mathcal{B}^\beta\, [I]_\beta^\mathcal{B}$ (right multiplying). So,

$$= \begin{pmatrix} 1 & -1 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & -2 & 0 \\ 2 & 2 & 3 & 2 \end{pmatrix} \begin{pmatrix} 4 & 8 & -1 & -6 \\ -1 & -1 & 0 & 1 \\ -2 & -4 & 0 & 3 \\ 0 & -1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (1)\cdot(4)+(-1)\cdot(-1)+(2)\cdot(-2)+(1)\cdot(0) & (1)\cdot(8)+(-1)\cdot(-1)+(2)\cdot(-4)+(1)\cdot(-1) \\ (1)\cdot(4)+(2)\cdot(-1)+(1)\cdot(-2)+(1)\cdot(0) & (1)\cdot(8)+(2)\cdot(-1)+(1)\cdot(-4)+(1)\cdot(-1) \\ (-1)\cdot(4)+(0)\cdot(-1)+(-2)\cdot(-2)+(0)\cdot(0) & (-1)\cdot(8)+(0)\cdot(-1)+(-2)\cdot(-4)+(0)\cdot(-1) \\ (2)\cdot(4)+(2)\cdot(-1)+(3)\cdot(-2)+(2)\cdot(0) & (2)\cdot(8)+(2)\cdot(-1)+(3)\cdot(-4)+(2)\cdot(-1) \end{pmatrix.$$
$$\begin{matrix} (1)\cdot(-1)+(-1)\cdot(0)+(2)\cdot(0)+(1)\cdot(1) & (1)\cdot(-6)+(-1)\cdot(1)+(2)\cdot(3)+(1)\cdot(1) \\ (1)\cdot(-1)+(2)\cdot(0)+(1)\cdot(0)+(1)\cdot(1) & (1)\cdot(-6)+(2)\cdot(1)+(1)\cdot(3)+(1)\cdot(1) \\ (-1)\cdot(-1)+(0)\cdot(0)+(-2)\cdot(0)+(0)\cdot(1) & (-1)\cdot(-6)+(0)\cdot(1)+(-2)\cdot(3)+(0)\cdot(1) \\ (2)\cdot(-1)+(2)\cdot(0)+(3)\cdot(0)+(2)\cdot(1) & (2)\cdot(-6)+(2)\cdot(1)+(3)\cdot(3)+(2)\cdot(1) \end{matrix}\right)$$

$$\overset{\text{drumroll}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(f) $[I]_\beta^\mathcal{B}\, [p(x)]_\beta = \begin{pmatrix} 4 & 8 & -1 & -6 \\ -1 & -1 & 0 & 1 \\ -2 & -4 & 0 & 3 \\ 0 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 2 \\ -5 \end{pmatrix} = \begin{pmatrix} 32 \\ -7 \\ -17 \\ -2 \end{pmatrix} = [p(x)]_\mathcal{B}$

Problem 2. If β and γ are two bases for a finite dimensional vector space V, and I is the identity map on V, the matrix $[I]_\gamma^\beta$ is *called a change of basis matrix* (*or change of coordinates matrix*). It is always invertible because I is invertible.

(a) Let $S \in \mathcal{L}(V)$. Show that

$$[S]_\gamma = \left([I]_\gamma^\beta\right)^{-1} [S]_\beta\, [I]_\gamma^\beta$$

by using the properties (given in class) which relate matrix multiplication and composition of linear maps

(b) Let $T \in \mathcal{L}(\mathcal{M}_{2\times 2}(\mathbb{R}))$ be the map given by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -17a+11b+8c-11d & -57a+35b+24c-33d \\ -14a+10b+6c-10d & -41a+25b+16c-23d \end{pmatrix}.$$

Find the matrix representations $[T]_\beta$ and $[T]_\gamma$ for bases

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

$$\gamma = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix} \right\}.$$

(c) Find the change of basis matrix $[I]_\gamma^\beta$ and use it to verify what was shown in part (a) using the matrices computed in part (b).

Solution 2.

(a)

$$\left([I]_\gamma^\beta\right)^{-1} [S]_\beta\, [I]_\gamma^\beta = [I]_\beta^\gamma\, [S]_\beta^\beta\, [I]_\gamma^\beta$$

Because each of these are matrix representations of linear maps and composition of linear maps in matrix form is matrix multiplication the above expression can be re-written as

$$I_{\beta\gamma}(S(I_{\gamma\beta}))$$

which maps per $\gamma \to \beta \to \beta \to \gamma$, however the $\gamma \to \beta$ and $\beta \to \gamma$ steps are done using identity maps which do not change the mapped vector and can be removed from the expression yielding

$$[I]_\beta^\gamma \, [S]_\beta^\beta \, [I]_\gamma^\beta \implies I_{\beta\gamma}(S(I_{\gamma\beta})) = S \implies [S]_\gamma \, .$$

(b) For $[T]_\beta$,

$$\begin{aligned}
T(\beta_1) &= \begin{pmatrix} -17 & -57 \\ -14 & -41 \end{pmatrix} = -17\beta_1 - 57\beta_2 - 14\beta_3 - 41\beta_4 \\
T(\beta_2) &= \begin{pmatrix} 11 & 35 \\ 10 & 25 \end{pmatrix} = 11\beta_1 + 35\beta_2 + 10\beta_3 + 25\beta_4 \\
T(\beta_3) &= \begin{pmatrix} 8 & 24 \\ 6 & 16 \end{pmatrix} = 8\beta_1 + 24\beta_2 + 6\beta_3 + 16\beta_4 \\
T(\beta_4) &= \begin{pmatrix} -11 & -33 \\ -10 & -23 \end{pmatrix} = -11\beta_1 - 33\beta_2 - 10\beta_3 - 23\beta_4
\end{aligned}$$

so,

$$[T]_\beta = \begin{pmatrix} -17 & 11 & 8 & -11 \\ -57 & 35 & 24 & -33 \\ -14 & 10 & 6 & -10 \\ -41 & 25 & 16 & -23 \end{pmatrix} .$$

Then for $[T]_\gamma$ we get

$$\begin{aligned}
T(\gamma_1) &= \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = 2\gamma_1 + 0\gamma_2 + 0\gamma_3 + 0\gamma_4 \\
T(\gamma_2) &= \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} = 0\gamma_1 + 2\gamma_2 + 0\gamma_3 + 0\gamma_4 \\
T(\gamma_3) &= \begin{pmatrix} -1 & -3 \\ -2 & -3 \end{pmatrix} = 0\gamma_1 + 0\gamma_2 - 1\gamma_3 + 0\gamma_4 \\
T(\gamma_4) &= \begin{pmatrix} -4 & -12 \\ -2 & -8 \end{pmatrix} = 0\gamma_1 + 0\gamma_2 + 0\gamma_3 - 2\gamma_4
\end{aligned}$$

so,

$$[T]_\gamma = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} .$$

(c) Applying I to the input basis,

$$\begin{aligned}
I(\gamma_1) &= \gamma_1 = 0\beta_1 + 1\beta_2 + 0\beta_3 + 1\beta_4 \\
I(\gamma_2) &= \gamma_1 = 1\beta_1 + 1\beta_2 + 1\beta_3 + 0\beta_4 \\
I(\gamma_3) &= \gamma_1 = 1\beta_1 + 3\beta_2 + 2\beta_3 + 3\beta_4 \\
I(\gamma_4) &= \gamma_1 = 2\beta_1 + 6\beta_2 + 1\beta_3 + 4\beta_4
\end{aligned}$$

so $[I]_\gamma^\beta = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 6 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 4 \end{pmatrix}$. Now,

$$\begin{aligned}
&= \left([I]_\gamma^\beta\right)^{-1} [T]_\beta \, [I]_\gamma^\beta \\
&= \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 6 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 4 \end{pmatrix}^{-1} \begin{pmatrix} -17 & 11 & 8 & -11 \\ -57 & 35 & 24 & -33 \\ -14 & 10 & 6 & -10 \\ -41 & 25 & 16 & -23 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 6 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 4 \end{pmatrix} \\
&= \begin{pmatrix} -11 & 7 & 4 & -6 \\ -4 & 3 & 2 & -3 \\ 1 & -1 & 0 & 1 \\ 2 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -17 & 11 & 8 & -11 \\ -57 & 35 & 24 & -33 \\ -14 & 10 & 6 & -10 \\ -41 & 25 & 16 & -23 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 6 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 4 \end{pmatrix} \\
&= \begin{pmatrix} -22 & 14 & 8 & -12 \\ -8 & 6 & 4 & -6 \\ -1 & 1 & 0 & -1 \\ -4 & 2 & 2 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 6 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = [T]_\gamma
\end{aligned}$$

Problem 3. Let $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{M}_{2\times2}(\mathbb{R})$ be the linear transformation given by

$$T(a+bx+cx^2+dx^3) = \begin{pmatrix} a+b & a-2c \\ d & b-d \end{pmatrix}$$

(a) Find the null space of T.

(b) Show that T is invertible without giving an explicit inverse.

(c) Find the matrix representation $[T]_\beta^\gamma$ for bases

$$\beta = \{1, x, x^2, x^3\}, \qquad \gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

(d) Compute $\left([T]_\beta^\gamma\right)^{-1}$ and use this to find an expression for T^{-1}; i.e. find $T^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(e) Consider the 2×2 identity matrix

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Find its pre-image in $\mathcal{P}_3(\mathbb{R})$ under T in two ways:

(i) By computing $T^{-1}(I_2)$ is the result from (d)

(ii) By first finding $[I_2]\,\gamma$, then computing $\left([T]_\beta^\gamma\right)^{-1}[I_2]_\gamma$.

Solution 3.

(a) $\operatorname{null} T$ is all $p(x) = a+bx+cx^2+dx^3 \in \mathcal{P}_3(\mathbb{R})$ such that $a+b = a-2c = d = b-d = 0$ which has only the trivial solution, $a=b=c=d=0$ so $\operatorname{null} T = \{0\}$.

(b) Because $\operatorname{range} T = \mathcal{M}_{2\times2}(\mathbb{R})$ and $\operatorname{null} T = \{0\}$ we are guaranteed that T is both invertible and bijective.

(c) Applying T to the elements of β,

$$\begin{aligned}
T(1) &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \gamma_1 + \gamma_2 + 0\gamma_3 + 0\gamma_4 \\
T(x) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1\gamma_1 + 0\gamma_2 + 0\gamma_3 + 1\gamma_4 \\
T(x^2) &= \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} = 0\gamma_1 - 2\gamma_2 + 0\gamma_3 + 0\gamma_4 \\
T(x^3) &= \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} = 0\gamma_1 + 0\gamma_2 + 1\gamma_3 - 1\gamma_4
\end{aligned}$$

so,

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

(d) The abridged computation to save space is as follows

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

So $\left([T]_\beta^\gamma\right)^{-1} = [T]_\gamma^\beta = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}$. With

$$
\begin{aligned}
T^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= aT^{-1}(\gamma_1) + bxT^{-1}(\gamma_2) + cx^2T^{-1}(\gamma_3) + dx^3T^{-1}(\gamma_4) \\
&= a\begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} + bx\begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2} \\ 0 \end{pmatrix} + cx^2\begin{pmatrix} -1 \\ 1 \\ -\frac{1}{2} \\ 1 \end{pmatrix} + dx^2\begin{pmatrix} -1 \\ 1 \\ -\frac{1}{2} \\ 0 \end{pmatrix} \\
&\implies (a-c-d) + (c+d)x + \frac{a-b-c-d}{2}x^2 + cx^3
\end{aligned}
$$

(e) Consider the 2×2 identity matrix

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Find its pre-image in $\mathcal{P}_3(\mathbb{R})$ under T in two ways:

(i) $T^{-1}(I_2) = (1-0-1) + (0+1)x + \frac{1-0-0-1}{2}x^2 + 0x^3 = x$

(ii) First, $[I_2]_\gamma = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ then, $\left([T]_\beta^\gamma\right)^{-1}[I_2]_\gamma = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \implies x.$

Problem 4. Let $T : \mathbb{R}^3 \to \mathcal{M}_{2\times 2}(\mathbb{R})$ be the linear transformation given by

$$T(a,b,c) = \begin{pmatrix} a-b & 2a+2b+c \\ 3a+b+c & -2a-6b-2c \end{pmatrix}$$

(a) How do we know at first glance that T is not invertible?

(b) Find a basis for the range of T.

(c) Give an element of $M_{2\times 2}(\mathbb{R})$ which is not in the range of T.

(d) Compute a matrix representation M for T using any pair of bases that you like. Recall (from Linear Algebra I) that the column space of a matrix is the space spanned by its columns, and the rank of a matrix is the dimension of its column space. Find a basis for the column space of M and give its rank.

Solution 4.

(a) It can determined by inspection that the map is not surjective (4 elements in the output matrix, three in the input), which means it is not bijective and is therefore not invertible.

(b) A simple basis is the one generated by expressing the output of T in the standard basis for $\mathcal{M}_{2\times 2}(\mathbb{R})$,

$$\left\{\begin{pmatrix} 1 & 2 \\ 3 & -2 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & -6 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}\right\}.$$

However this is not actually a basis as the last element can be expressed as a linear combination of the first two (and therefore can be discarded). This means the actual basis is

$$\left\{\begin{pmatrix} 1 & 2 \\ 3 & -2 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & -6 \end{pmatrix}\right\}.$$

(c) If we add the elements of the above basis and change a single element we get a matrix in $\mathcal{M}_{2\times 2}(\mathbb{R})$ that cannot belong to range T,

$$\begin{pmatrix} 1 & 2 \\ 3 & -2 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 1 & -6 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 4 & -8 \end{pmatrix} \implies \begin{pmatrix} \mathbf{1} & 4 \\ 4 & -8 \end{pmatrix} \notin \operatorname{range} T$$

(d) For the standard bases

$$\beta = \{(1,0,0),(0,1,0),(0,0,1)\}, \qquad \gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

We apply the standard process to determine $M = [T]_\beta^\gamma$,

$$\begin{aligned} T(\beta_1) &= \begin{pmatrix} 1 & 2 \\ 3 & -2 \end{pmatrix} = \gamma_1 + 2\gamma_2 + 3\gamma_3 - 2\gamma_4 \\ T(\beta_2) &= \begin{pmatrix} -1 & 2 \\ 1 & -6 \end{pmatrix} = -1\gamma_1 + 2\gamma_2 + 1\gamma_3 - 6\gamma_4 \\ T(\beta_3) &= \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} = 0\gamma_1 + 1\gamma_2 + 1\gamma_3 - 2\gamma_4 \end{aligned}$$

$\implies [T]_\beta^\gamma = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & 1 \\ 3 & 1 & 1 \\ -2 & -6 & -2 \end{pmatrix} \implies \text{colsp } [T]_\beta^\gamma = \text{span}\left(\begin{pmatrix} 1 \\ 2 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \\ -6 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ -2 \end{pmatrix} \right)$ However the third element of the column space is redundant so the rank is actually 2.