Physics 2130: Assignment II

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Problem 1. A particle of charge q enters with a velocity $\vec{v}$ a region of space where there are both an electric, $\vec{E}$, and a magnetic field, $\vec{B}$. It therefore experiences Lorentz electric and magnetic force, which is given by:

$$\vec{F}_L = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

The magnetic field, $\vec{B}$, is oriented along the z-axis, whilst the electric field, $\vec{E}$, lies in the yz-plane:

$$\vec{B} = B_z\vec{k}$$

$$\vec{E} = E_y\vec{j} + E_z\vec{k}$$

Solution 1. I'm setting an origin, not that it really matters here until we're getting constants, such that $x_0, y_0, z_0 = 0$
a) b) c) d) (Sorry, I only read question a and interpreted it as find $x, y, z(t)$)

$$\begin{aligned}
\vec{F} &= q\left(\vec{E} + \vec{v} \times \vec{B}\right) \\
\vec{F} &= q\left(\vec{E} + \begin{vmatrix} i & j & k \\ v_x & v_y & v_z \\ B_x & B_y & B_z \end{vmatrix}\right) \\
\vec{F} &= q\left(\vec{E} + \begin{vmatrix} i & j & k \\ v_x & v_y & v_z \\ 0 & 0 & B_z \end{vmatrix}\right) \\
\vec{F} &= q\left(\vec{E} + \left\langle v_yB_z\vec{i} - v_xB_z\vec{j} + 0\vec{k}\right\rangle\right)
\end{aligned}$$

Then, using Newton's equations

$$\begin{aligned}
F_x &= m\ddot{x} = qB\dot{y} \rightsquigarrow E_x = 0 && (1) \\
F_y &= m\ddot{y} = qE_y - qB\dot{x} && (2) \\
F_z &= m\ddot{z} = E_z && (3)
\end{aligned}$$

(3) can be integrated twice to obtain

$$z(t) = \frac{E_z}{2m}t^2 + v_{z0}t + z_0$$

Then, simplifying (1) and (2) a bit ($\alpha \equiv \dfrac{qB}{m}$) and differentiating both we obtain

$$\dddot{x} = \alpha\ddot{y}$$

$$\dddot{y} = -\alpha\ddot{x}$$

Which means that

$$\dddot{x} = \frac{q^2BE_y}{m^2} - \alpha^2\dot{x} = \alpha^2\frac{E_y}{B} - \alpha^2\dot{x}$$

$$\dddot{y} = -\alpha^2\dot{y}$$

Starting with x

$$\dddot{x} = \alpha^2 \frac{E_y}{B} - \alpha^2 \dot{x}$$
$$\ddot{v_x} = \alpha^2 \frac{E_y}{B} - \alpha^2 v_x$$
$$\ddot{v_x} + \alpha^2 v_x = \alpha^2 \frac{E_y}{B}$$

Which has a solution of the form $v_x = v_{x,\,h} + v_{x,\,p}$ where $v_{x,\,h}$ is the solution to the above equation in homogenous form (rhs is zero) and $v_{x,\,p}$ is the particular solution to the above equation.

$$\begin{aligned}
&\ddot{v_x} + \alpha^2 v_x = 0 \\
&\frac{d^2 v_x}{dt^2} + \alpha^2 v_x = 0 \rightsquigarrow v_x = e^{rt},\, v_x' = re^{rt},\, v_x'' = r^2 e^{rt} \\
&r^2 e^{rt} + \alpha^2 e^{rt} = 0 \\
&r = \pm i\alpha \implies v_{x,\,h} = A\cos(\alpha t) + K\sin(\alpha t)
\end{aligned}$$

Then,

$$\begin{aligned}
&0 + \alpha^2 C = \alpha^2 \frac{E_y}{B} \rightsquigarrow v_x = C \implies \ddot{v_x} = 0 \\
&C = \frac{E_y}{B} = v_{x,\,p}
\end{aligned}$$

$\therefore v_x(t) = A\cos(\alpha t) + K\sin(\alpha t) + \dfrac{E_y}{B} \implies x(t) = \dfrac{A}{\alpha}\sin(\alpha t) - \dfrac{K}{\alpha}\cos(\alpha t) + \dfrac{E_y t}{B}$
Then $v_x(t=0) = v_{x0} = A + \frac{E_y}{B} \implies A = v_{x0} - \frac{E_y}{B}$ and $x(t=0) = 0 = \frac{K}{\alpha} \implies K = 0$

$$\therefore x(t) = \frac{v_{x0} - \dfrac{E_y}{B}}{\alpha}\sin(\alpha t) + \frac{E_y t}{B}$$

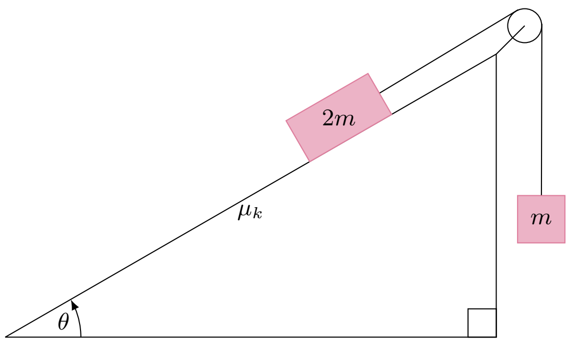
Now for y

$$\begin{aligned}
&\dddot{y} = -\alpha^2 \dot{y} \\
&\ddot{v_y} + \alpha^2 v_y = 0 \rightsquigarrow v_y = e^{rt},\, v_y' = re^{rt},\, v_y'' = r^2 e^{rt} \\
&r^2 e^{rt} + \alpha^2 e^{rt} = 0 \\
&r = \pm i\alpha \implies v_y = A\cos(\alpha t) + K\sin(\alpha t)
\end{aligned}$$

$v_y(t=0) = v_{y0} = A,\, y(t=0) = 0 = -\frac{K}{\alpha} \implies K = 0$

$$\therefore y(t) = \frac{v_{y0}}{\alpha}\sin(\alpha t)$$

Problem 2. Two blocks of mass $2m$ and m are connected by a massless and inextensible string over a smooth and massless pully (figure below). The coefficient of kinetic friction is μ_k. Find the angle θ the incline should have for the masses to move with constant velocity.



This took way longer than it should have

Solution 2. For constant velocity both masses must experience no net force so their equations of motion can be equated.

$$\begin{aligned}
&mg = 2mg\mu_k \cos\theta + 2mg\sin\theta \\
&1 = 2\left(\mu_k \cos\theta + \sin\theta\right) \\
&\frac{1}{2} = \mu_k\sqrt{1-\sin^2\theta} + \sin\theta \rightsquigarrow \cos\theta = \sqrt{1-\sin^2\theta} \\
&\frac{1}{\mu_k}\left(\frac{1}{2} - \sin\theta\right) = \sqrt{1-\sin^2\theta} \\
&\frac{1}{\mu_k^2}\left(\frac{1}{4} - \sin\theta + \sin^2\theta\right) = 1 - \sin^2\theta \\
&\frac{1}{4} - \sin\theta = \mu_k^2 - \mu_k^2\sin^2\theta - \sin^2\theta \\
&\sin\theta - \frac{1}{4} = \left(\mu_k^2 + 1\right)\sin^2\theta - \mu_k^2 \\
&0 = \left(\mu_k^2 + 1\right)\sin^2\theta - \sin\theta - \mu_k^2 + \frac{1}{4}
\end{aligned}$$

Then, using the quadratic formula ($x = sin\theta$),

$$\begin{aligned}
&\sin\theta = \frac{1 \pm \sqrt{1 - 4\left(\mu_k^2+1\right)\left(\frac{1}{4} - \mu_k^2\right)}}{2\left(\mu_k^2+1\right)} \\
&\therefore \arcsin\left(\frac{1 \pm \sqrt{1 - 4\left(\mu_k^2+1\right)\left(\frac{1}{4} - \mu_k^2\right)}}{2\left(\mu_k^2+1\right)}\right) = \theta
\end{aligned}$$

Problem 3. A particle of mass $m = 1.5\text{kg}$ is subjected only to a one-dimensional force $F(t) = kte^{-\alpha t}$ where $k = 2\text{N}\,\text{s}^{-1}$ and $\alpha = 0.7\text{s}^{-1}$. The particle is initially at rest. Calculate analytically and plot using a python program the position, $x(t)$, velocity, $v(t)$, and acceleration, $a(t)$, of the particle as a function of time.

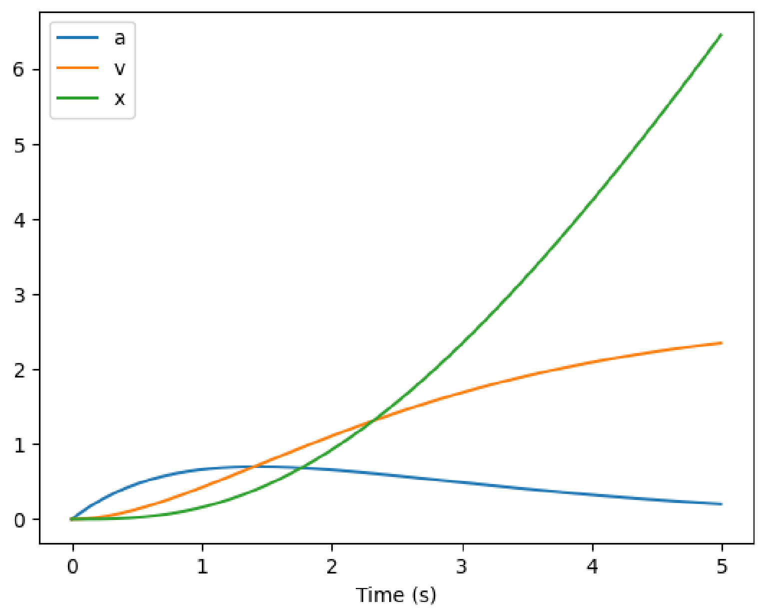
Solution 3.

$$
\begin{aligned}
&F = m\ddot{x} = kte^{-\alpha t} \\
&\frac{d^2x}{dt^2} = \frac{kt}{m}e^{-\alpha t} \\
&\frac{d\frac{dx}{dt}}{dt} = \frac{kt}{m}e^{-\alpha t} \\
&\int d\frac{dx}{dt} = \int \frac{kt^2}{m}e^{-\alpha t}\,dt \\
&\frac{dx}{dt} = \int \frac{kt}{m}e^{-\alpha t}\,dt \rightsquigarrow \text{integration by parts} \\
&\frac{dx}{dt} = -\frac{ke^{-\alpha t}}{m}\left[\frac{t}{\alpha} + \frac{1}{\alpha^2}\right] + C \rightsquigarrow v(t=0) = 0 = -\frac{k}{m\alpha^2} + C \implies C = \frac{k}{m\alpha^2}\left(\frac{\text{N}\,\text{s}^{-1}}{\text{kg}\,\text{s}^{-2}} = \text{m}\,\text{s}^{-1}\right) \\
&\frac{dx}{dt} = -\frac{ke^{-\alpha t}}{m}\left[\frac{t}{\alpha} + \frac{1}{\alpha^2}\right] + \frac{k}{m\alpha^2} \\
&x = -\frac{k}{m}\int\left[\frac{t}{\alpha} + \frac{1}{\alpha^2}\right]e^{-\alpha t}\,dt + \frac{k}{m\alpha^2}\int dt \\
&x = \frac{k}{m}\left(\frac{t}{\alpha^2}e^{-\alpha t} + \frac{1}{\alpha^3}e^{-\alpha t} + \frac{1}{\alpha^3}e^{-\alpha t}\right) + \frac{kt}{m\alpha^2} + C \\
&x = \frac{k}{m}\left(\frac{t}{\alpha^2}e^{-\alpha t} + \frac{2}{\alpha^3}e^{-\alpha t}\right) + \frac{kt}{m\alpha^2} + C \rightsquigarrow x(t=0) = 0 = \frac{k}{m}\frac{2}{\alpha^3} + C \implies C = -\frac{k}{m}\frac{2}{\alpha^3}\left(\text{kg}\,\text{m}\,\text{s}^{-3}\,\text{kg}^{-1}\,\text{s}^3 = \text{m}\right)
\end{aligned}
$$

So, with some simplfication,

$$
\begin{aligned}
a(t) &= \frac{kt}{m}e^{-\alpha t} \\
v(t) &= -\frac{ke^{-\alpha t}}{m}\left[\frac{t}{\alpha} + \frac{1}{\alpha^2}\right] + \frac{k}{m\alpha^2} \\
x(t) &= \frac{\alpha kt + 2k}{m\alpha^3 e^{\alpha t}} + \frac{kt}{m\alpha^2} - \frac{2k}{m\alpha^3}
\end{aligned}
$$

```python
import numpy as np
import matplotlib.pyplot as pyplot

m = 1.5
k = 2
a = 0.7
t = np.arange(0, 5, 0.01)

def ac(t):
    return ((k*t)/m)*np.e**(-a*t)

def v(t):
    return -(k*t*np.e**(-a*t))/(m*a)-(np.e**(-a*t))/(a**2)*(k/m)+k/(a**2*m)

def x(t):
    return (k*t)/(a**2*m*np.e**(a*t))+(2*k)/(a**3*m*np.e**(a*t))+(k/(a**2*m))*t-((2*k)/(m*a**3))

pyplot.plot(t, ac(t), label="a")
pyplot.plot(t, v(t), label="v")
pyplot.plot(t, x(t), label="x")
pyplot.legend()
pyplot.xlabel("Time (s)")
pyplot.show()
```



Problem 4. A particle of mass m moving in one dimension has potential

$$U(x) = U_0 \left[2\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^4\right]$$

where U_0 and a are positive constants

Solution 4.

a)

$$\begin{aligned}
F(x) &= -\frac{dU(x)}{dx} \\
F(x) &= -U_0 \frac{d}{dx}\left[2\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^4\right] \\
F(x) &= -U_0 \left[2\frac{d}{dx}\left(\frac{x}{a}\right)^2 - \frac{d}{dx}\left(\frac{x}{a}\right)^4\right] \\
F(x) &= -U_0 \left[\frac{4}{a}\left(\frac{x}{a}\right) - \frac{4}{a}\left(\frac{x}{a}\right)^3\right] \\
F(x) &= \frac{4U_0}{a}\left[\left(\frac{x}{a}\right)^3 - \left(\frac{x}{a}\right)\right]
\end{aligned}$$

b) Analytically,

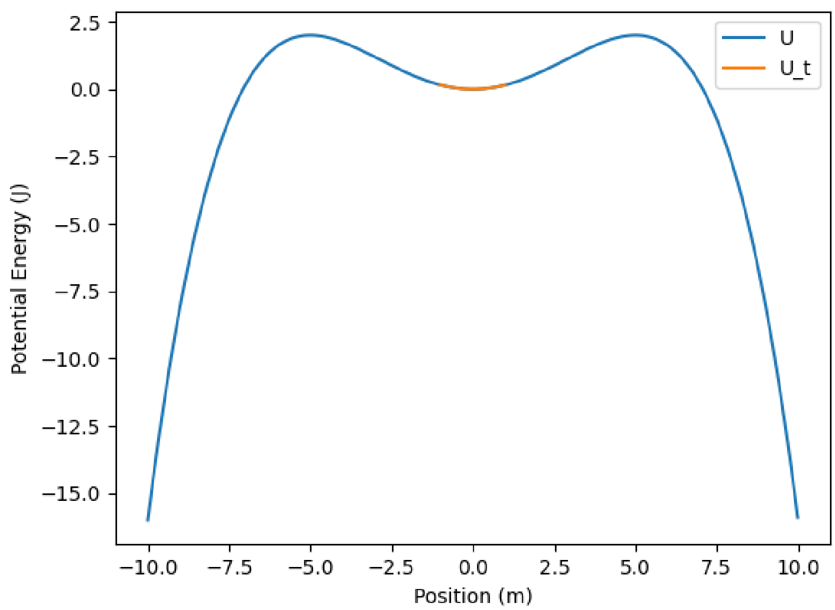
$$\begin{aligned}
&F(x) = 0 = \frac{4U_0}{a}\left[\left(\frac{x}{a}\right)^3 - \left(\frac{x}{a}\right)\right] \\
&0 = \left(\frac{x}{a}\right)^3 - \left(\frac{x}{a}\right) \\
&0 = \frac{x^2}{a^2} - 1 \rightsquigarrow x = 0 \text{ is lost here, but it is still a valid solution} \\
&x^2 = a^2 \\
&x = \pm a \implies x = 0, \pm 5
\end{aligned}$$

± 5 is unstable and 0 is stable by inspection.

Numerically (Includes code and graph for part c)),

```python
import numpy as np
import matplotlib.pyplot as pyplot
import math

U0 = 2
a = 5
x = np.arange(-10, 10, 0.01)
x_t = np.arange(-1, 1, 0.01)

mins = []
def F(x):
    return ((4*U0)/(a))*((x/a)**3-(x/a))

for x_val in x:
    if (math.isclose(F(x_val), 0, abs_tol = 1e-9)): # Hacky but works
        mins.append(x_val)
```

```python
def U(x):
    return U0*(2*(x/a)**2-(x/a)**4)

def U_t(x):
    return (2*U0*x**2)/(a**2)

print(f"Points of equilibrium {np.ceil(mins)}") # Floating point approx needs to be rounded

pyplot.plot(x, U(x), label="U")
pyplot.plot(x_t, U_t(x_t), label="U_t")
pyplot.legend()
pyplot.xlabel("Position (m)")
pyplot.ylabel("Potential Energy (J)")
pyplot.show()
```



c) $\sum_{n=0}^{2} \frac{f^{(n)}(x)}{n!} x^n$ So,

$$f(x) = U_0 \left[2\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^4 \right]$$
$$f'(x) = \frac{4U_0}{a} \left[\frac{x}{a} - \left(\frac{x}{a}\right)^3 \right]$$
$$f''(x) = \frac{4U_0}{a^2} \left[1 - 3\left(\frac{x}{a}\right)^2 \right]$$

Then,

$$= \frac{U_0 \left[2\left(\frac{x_p}{a}\right)^2 - \left(\frac{x_p}{a}\right)^4 \right]}{0!} x^0 + \frac{\frac{4U_0}{a} \left[\frac{x_p}{a} - \left(\frac{x_p}{a}\right)^3 \right]}{1!} x^1 + \frac{\frac{4U_0}{a^2} \left[1 - 3\left(\frac{x_p}{a}\right)^2 \right]}{2!} x^2$$
$$= \frac{2U_0}{a^2} x^2 \text{ because } x_p = 0$$

This is still a potential and $F(x) = -\frac{dU(x)}{dx}$ holds.

$$\therefore F(x) = m\ddot{x} = -\frac{d}{dx}\frac{2U_0}{a^2}x^2 = -\frac{4U_0}{a^2}x$$

Which is an equation in the form of a harmonic oscillator, which makes sense as an object with some energy placed at $x \approx 0$ would oscillate back and forth. The equation can be solved to determine $x(t)$ as follows

$$\begin{aligned}
&m\ddot{x} = -\frac{4U_0}{a^2}x \rightsquigarrow \omega_0 = \sqrt{\frac{4U_0}{a^2 m}} \\
&\ddot{x} = -\omega_0^2 x \\
&0 = \ddot{x} + \omega_0^2 x \rightsquigarrow x = e^{rt} \\
&0 = r^2 e^{rt} + \omega_0^2 e^{rt} \\
&0 = r^2 + \omega_0^2 \\
&r = \pm i\omega_0 \implies x = e^{\pm i\omega_0 t}
\end{aligned}$$

So $x(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t} = C_1 e^{i\sqrt{\frac{4U_0}{a^2 m}}t} + C_2 e^{-i\sqrt{\frac{4U_0}{a^2 m}}t}$ which can be equated to $(C_1 + C_2)\cos(\omega_0 t) + i(C_1 - C_2)\sin(\omega_0 t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$ which implies the oscillatory motion that we expect of a particle placed around the origin

d) To escape from the origin the particle must reach either of the other points of equilibrium and fall over the side. To do this, it must have the kinetic energy equivalent (really slightly greater but only infinitesimally so) to the potential energy it would have at either point of unstable equilibrium ($\pm$5m)

$$\begin{aligned}
&E_t = E_t' \\
&\frac{1}{2}mv_0^2 = U(\pm 5) \\
&v_0 = \sqrt{\frac{2U(\pm 5)}{m}} \\
&v_0 = \sqrt{\frac{2U_0\left[2\left(\frac{\pm 5}{a}\right)^2 - \left(\frac{\pm 5}{a}\right)^4\right]}{m}}
\end{aligned}$$