Math 2150: Assignment I

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Note: I've kept p, q, and r throughout my solutions and only substituted the actual numbers in at the end. This is because I find it easier, especially when dealing with things that might cancel nicely, to deal with variables rather than the numbers they represent. In my case my student number is 0805980 so $p = 9$, $q = 5$, and $r = 22$. Additionally, this document is the distilled form of my full process for solving all of these so there may at points be some lacking detail as I've transcribed from my work on paper.

Problem 1. Solve each of the following differential equations.

(a) $2qy\sin(x)\cos(x) - py + 2ry^2e^{xy^2} + \sec^2(qx) = \left(px + \cos(py) - q\sin^2(x) - 4rxye^{xy^2}\right)\dfrac{dy}{dx}$

(b) $\dfrac{dy}{dx} = \dfrac{yx + px - qy - pq}{xy + ry - px - rp}$

Solution 1.

(a) This is probably exact with $M = 2qy\sin(x)\cos(x) - py + 2ry^2e^{xy^2} + \sec^2(qx)$ and $N = -px - \cos(py) + q\sin^2(x) + 4rxye^{xy^2}$. To proceed with solving it as an exact equation we need to check that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.
$\dfrac{\partial M}{\partial y} = 2q\sin(x)\cos(x) - p + 4rye^{xy^2} + 4rxy^3e^{xy^2}$
$\dfrac{\partial N}{\partial x} = -p + 2q\sin(x)\cos(x) + 4rye^{xy^2} + 4rxy^3e^{xy^2}$ $\therefore$ this differential equation is exact.

$$\begin{aligned}
f(x,y) = C &= \int M\,dx + h(y)\\
&= \int 2qy\sin(x)\cos(x) - py + 2ry^2e^{xy^2} + \sec^2(qx)\,dx + h(y)\\
&= 2qy\int \sin(x)\cos(x)\,dx - py\int dx + 2ry^2\int e^{xy^2}\,dx + \int \sec^2(qx)\,dx + h(y)\\
&= qy\sin^2(x) - pyx + 2re^{xy^2} + \frac{1}{q}\tan(qx) + h(y)
\end{aligned}$$

Then, $\dfrac{\partial f(x,y)}{\partial y} = N$

$$\begin{aligned}
&\frac{\partial}{\partial y}\left[qy\sin^2(x) - pyx + 2re^{xy^2} + \frac{1}{q}\tan(qx) + h(y)\right] = N\\
&\cancel{q\sin^2(x)} - \cancel{px} + \cancel{4rxye^{xy^2}} + \frac{dh(y)}{dy} = \cancel{-px} - \cos(py) + \cancel{q\sin^2(x)} + \cancel{4rxye^{xy^2}}\\
&\frac{dh(y)}{dy} = -\cos(py) \implies h(y) = -\frac{1}{p}\sin(py)
\end{aligned}$$

$\therefore f(x,y) = yx + 2re^{xy^2} + \dfrac{1}{q}\tan(qx) + \dfrac{1}{p}\sin(py) = 5y\sin^2(x) - 9yx + 44e^{xy^2} + \dfrac{1}{5}\tan(5x) - \dfrac{1}{9}\sin(9y) = C$ where C is some constant that can only be determined given initial conditions, which we do not have in this case.

(b) With some factoring this can be made more obviously seperable

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x(y+p)-q(y+p)}{y(x+r)-p(x+r)} \\
\frac{dy}{dx} &= \frac{(x-q)(y+p)}{(y-p)(x+r)} \\
\frac{y-p}{y+p}dy &= \frac{x-q}{x+r}dx
\end{aligned}$$

I'll solve both integrals seperately so I can better show what I'm doing. Starting with y:

$$\begin{aligned}
&= \int \frac{y-p}{y+p}\,dy \\
&= \int \frac{y}{y+p}\,dy - p\int \frac{1}{y+p}\,dy \\
&= \int \frac{y}{y+p}\,dy - p\ln(y+p) \rightsquigarrow u = y+p \implies du = dy \\
&= \int \frac{u-p}{u}\,dy - p\ln(y+p) \\
&= \int \frac{u}{u}\,du - p\int \frac{1}{u}\,du - p\ln(y+p) \\
&= y+p-p\ln(y+p)-p\ln(y+p)+C \\
&= y+p(1-2\ln(y+p))+C
\end{aligned}$$

Now for x:

$$\begin{aligned}
&= \int \frac{x-q}{x+r}\,dy \\
&= \int \frac{x}{x+r}\,dy - q\int \frac{1}{x+r}\,dy \\
&= \int \frac{x}{x+r}\,dy - q\ln(x+r) \rightsquigarrow u = x+r \implies du = dy \\
&= \int \frac{u-r}{u}\,du - q\ln(x+r) \\
&= \int \frac{u}{u}\,du - r\int \frac{1}{u}\,du - q\ln(x+r) \\
&= x+r-r\ln(x+r)-q\ln(x+r)+C \\
&= x+r-(\ln(x+r))(r+q)+C
\end{aligned}$$

So the solution is $x+22-(\ln(x+22))(27) = y+9(1-2\ln(y+9))+C$

Problem 2. Solve each of the following initial value problems. Give your answers **explicitly** as a function of x.

(a) $\left(2pxy^2 + 2rxy\right) dx + \left(px^2y + rx^2 + 2py + 2r\right) dy = 0,\ y(1) = q$

(b) $(x+p)\dfrac{dy}{dx} + (qx + qp + 1)y = 3prx^2e^{px^3 - qx},\ y(0) = \dfrac{qr}{p}$

(c) $pyx\dfrac{dy}{dx} = py^2 + x\sqrt{r^2x^2 + y^2 - q^2x^2},\ y(1) = -q$

(d) $(p-1)(x+1)e^{rx}\dfrac{dy}{dx} + y^p = ry(1+x)e^{rx},\ y(0) = -1$

Solution 2.

(a) This looks exact, and almost is $\left(\dfrac{\partial M}{\partial y} = 4pxy + 2rx,\ \dfrac{\partial N}{\partial x} = 2pxy + 2rx\right)$. However, it's actually seperable

$$
\begin{aligned}
&2xy\cancel{(py+r)}dx + \left[x^2\cancel{(py+r)} + 2\cancel{(py+r)}\right] dy = 0\\
&2xydx + \left(x^2+2\right) dy = 0\\
&2xydx = -\left(x^2+2\right) dy\\
&-\frac{2x}{(x^2+2)}dx = \frac{1}{y}dy\\
&-2\int \frac{x}{(x^2+2)}\,dx = \int \frac{1}{y}\,dy\\
&-2\int \frac{x}{(x^2+2)}\,dx = \ln(y) \rightsquigarrow u = x^2+2 \implies \frac{1}{2x}du = dx\\
&-\int \frac{1}{u}\,du = \ln(y) + C\\
&-ln\left(x^2+2\right) = \ln(y) + C
\end{aligned}
$$

Then we rearrange for y and use the initial conditions to determine C

$$
\begin{aligned}
&-ln\left(x^2+2\right) + C = \ln(y)\\
&\frac{C}{x^2+2} = y\\
&y(1) = 5 = \frac{C}{x^2+2} \implies C = 15
\end{aligned}
$$

So the solution is $y = \dfrac{15}{x^2+2}$

(b) This one is linear

$$
\begin{aligned}
&(x+p)\frac{dy}{dx} + (qx+qp+1)y = 3prx^2e^{px^3-qx}\\
&\frac{dy}{dx} + \frac{qx+qp+1}{x+p}y = \frac{3prx^2e^{px^3-qx}}{x+p}
\end{aligned}
$$

Our integrating factor is therefore

$$
\begin{aligned}
&e^{\int \frac{qx+qp+1}{x+p}\,dx}\\
&e^{q\int \frac{x+p}{x+p}\,dx + \int \frac{1}{x+p}\,dx}\\
&e^{qx+\ln(x+p)} = e^{qx}e^{\ln(x+p)} = (x+p)e^{qx}
\end{aligned}
$$

So,

$$
\begin{aligned}
y(x+p)e^{qx} &= \int (x+p)e^{qx}\frac{3prx^2e^{px^3-qx}}{x+p}\,dx \\
y(x+p)e^{qx} &= 3pr\int x^2e^{px^3}\,dx \\
y(x+p)e^{qx} &= 3pr\int x^2e^{px^3}\,dx \rightsquigarrow u = px^3 \implies \frac{1}{3px^2}du = dx \\
y(x+p)e^{qx} &= r\int e^u\,du \\
y(x+p)e^{qx} &= re^{px^3}+C \\
y &= \frac{re^{px^3-qx}}{(x+p)}+\frac{C}{e^{qx}(x+p)} \\
y(0) &= \frac{22e^{9(0)^3-5(0)}}{((0)+9)}+\frac{C}{e^{5(0)}((0)+9)} = \left(\frac{110}{9}\right) \implies C = 88
\end{aligned}
$$

$\therefore y(x) = \dfrac{22e^{9x^3-5x}}{(x+9)}+\dfrac{88}{e^{5x}(x+9)}$

(c) This one is homogenous

$$
\begin{aligned}
&pyx\frac{dy}{dx} = py^2+x^2\sqrt{r^2+\frac{y^2}{x^2}-q^2} \\
&pux^2(udx+xdu) = (pu^2x^2+x^2\sqrt{r^2+u^2-q^2})dx \rightsquigarrow u=\frac{y}{x} \implies dy = udx+xdu \\
&\cancel{pu^2x^2dx}+pux^3du = \cancel{pu^2x^2dx}+x^2\sqrt{r^2+u^2-q^2}dx \\
&pux^3du = x^2\sqrt{r^2+u^2-q^2}dx \\
&\frac{pu}{\sqrt{r^2+u^2-q^2}}du = \frac{1}{x}dx \\
&p\int\frac{u}{\sqrt{r^2+u^2-q^2}}\,du = \ln(x) \rightsquigarrow s = r^2+u^2-q^2 \implies \frac{1}{2u}ds = du \\
&\frac{p}{2}\int\frac{1}{\sqrt{u}}\,du = \ln(x) \\
&p\sqrt{r^2+u^2-q^2} = \ln(x)+C \\
&\frac{y^2}{x^2}-q^2 = \frac{\ln^2(x)}{p^2}+C-r^2 \\
&y = \pm x\sqrt{\frac{\ln^2(x)}{p^2}+C+p^2-r^2+q^2} \\
&y(1) = -1\sqrt{\frac{\ln^2(1)}{9^2}+C+9^2-22^2+5^2} = -5 \implies C = 403
\end{aligned}
$$

$\therefore y(x) = -x\sqrt{\frac{\ln^2(x)}{81}+25}$

(d) My first thought with this one was that the y^p probably means this one is a Bernoulli equation, which it is.

$$
\begin{aligned}
&(p-1)(x+1)e^{rx}\frac{dy}{dx} + y^p = ry(1+x)e^{rx} \\
&\frac{dy}{dx} + \frac{y^p}{(p-1)(x+1)e^{rx}} = \frac{ry}{p-1} \\
&\frac{dy}{dx} - \frac{ry}{p-1} = -\frac{y^p}{(p-1)(x+1)e^{rx}} \rightsquigarrow \frac{dy}{dx} + P(x)y = Q(x)y^n
\end{aligned}
$$

Now we divide the whole thing by y^p and substitute for $z = y^{1-p}$

$$
\begin{aligned}
&\frac{1}{\cancel{y^p}}\frac{\cancel{y^p}}{1-p}\frac{dz}{dx} - \frac{ry^{1-p}}{p-1} = -\frac{1}{(p-1)(x+1)e^{rx}} \rightsquigarrow z = y^{1-p} \implies \frac{y^p}{(1-p)}dz = dy \\
&\frac{dz}{dx} - rz = -\frac{1}{(x+1)e^{rx}}
\end{aligned}
$$

Now that we've made the original equation linear we find the integrating factor

$$\mu = e^{\int r\, dx} = e^{rx}$$

So our linear ODE becomes

$$
\begin{aligned}
&rze^{rx} = \ln(x+1) + C \\
&ry^{1-p}e^{rx} = \ln(x+1) + C \\
&y^{1-p} = \frac{\frac{\ln(x+1)}{r} + C}{e^{rx}} \\
&y = \pm\sqrt[1-p]{\frac{\frac{\ln(x+1)}{r} + C}{e^{rx}}} \\
&y(0) = \pm\sqrt[1-9]{\frac{\frac{\ln(0+1)}{22} + C}{e^{22(0)}}} = -1 \implies C = 1
\end{aligned}
$$

$$\therefore y(x) = -\sqrt[-8]{\frac{\frac{\ln(x+1)}{22} + 1}{e^{22x}}}$$

Problem 3. SAGE

(a) Consider the equation

$$y^q \sin\left(y^p + x^q + x^r\right) = e^{(p-q)xy} - 1$$

i. Use the **diff** command to find the implicit derivative to obtain the differential equation for which this is an implicit solution.

ii. Plot the curve of the implicit solution on the rectangle using $-4 \le x \le 4$ and $-40 \le y \le 40$ the command **implicit_plot**.

iii. Describe the plot.

(b) Consider an indoor temperature T_m that has oscillations that is driven by the outdoor temperature according to

$$T_m = T_i = pe^{-\frac{1}{rq}t} \cos\left(\frac{\pi}{12}t\right) - re^{-\frac{1}{rq}t} \sin\left(\frac{\pi}{12}t\right)$$

where $T_i = 70 + p - q$. Note that, that the oscillation has a 24 hour period and the cycle of the outdoor temperature repeats daily.

i. Use the command **desolve** to find the general solution of the corresponding Newton's cooling equation.

ii. Apply the initial condition $T(0) = T_i + r$ to determine the particular solution.

iii. Apply the condition $T(1) = T_i + r - p$ to determine the constant k in the Newton's cooling equation.

iv. Plot the particular solution the first 72 hours.

v. Describe the plot

vi. Compute the average indoor temperature over the first 72 hours.

(c) Consider an indoor temperature T_m that has oscillations that is driven by the outdoor temperature according to

$$\frac{dy}{dx} = (y+p)^q \sin\left(qxy^p + 1\right) \cos\left(pyx^q\right) e^{-\frac{1}{r}yx}$$

i. Draw the associated direction field using **plot_slope_field**.

ii. Obtain the numerical solution of the differential equation subject to each of the initial conditions $y(0) = -4, -2, 0, 2, 4$ using **desolve_rk4**. (ie. 5 IVP solution plots)

iii. Put the plots of each of the IVPs as well as the direction field plot on the same graph.

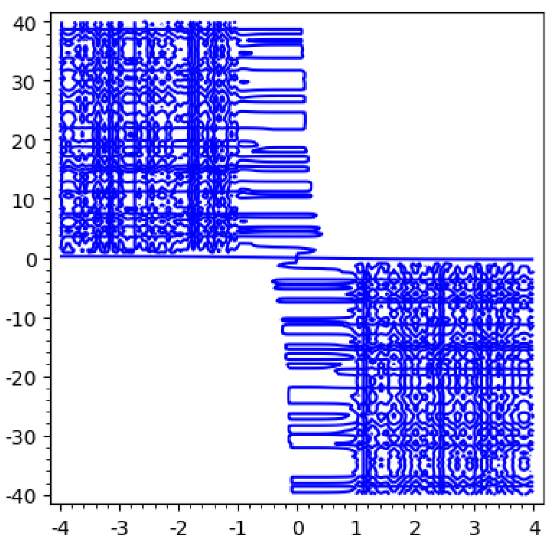
Solution 3.

(a) The graph looks garbled and messy but I believe that's how it's supposed to be.

[1]:
```
clear_vars()
from IPython.core.interactiveshell import InteractiveShell
InteractiveShell.ast_node_interactivity = "all"
x = var('x')
y = var('y')

q = 5
p = 9
r = 22

x = var('x')
y = function('y')(x)
eq = y^q*sin(y^p+x^q+x^r)==e^((p-q)*x*y)-1
d = solve(diff(eq, x), diff(y,x))
show(d)

y = var('y')
eq = y^q*sin(y^p+x^q+x^r)==e^((p-q)*x*y)-1
implicit_plot(eq, (x,-4,4), (y,-40,40), aspect_ratio=0.1)
```

[1]:

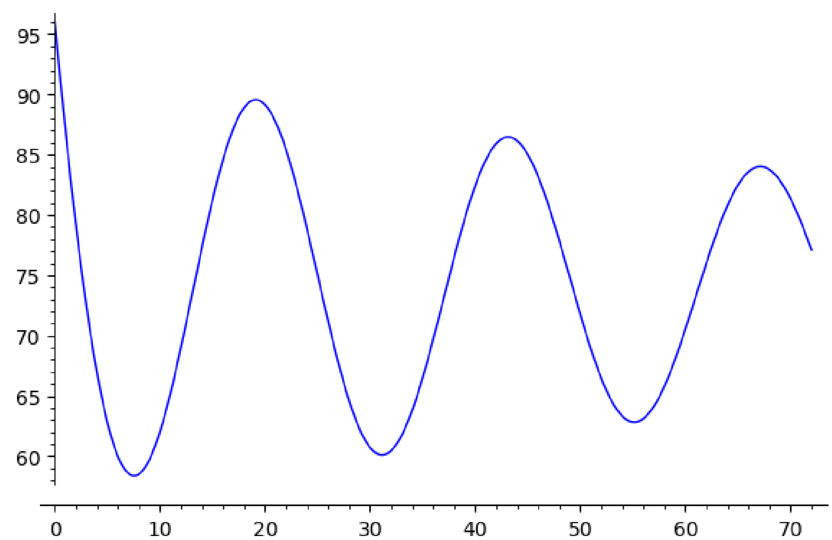
$$\left[\frac{\partial}{\partial x}y\left(x\right) = -\frac{\left(22\,x^{21} + 5\,x^{4}\right)\cos\left(x^{22} + y\left(x\right)^{9} + x^{5}\right)y\left(x\right)^{5} - 4\,e^{\left(4\,xy\left(x\right)\right)}y\left(x\right)}{9\,\cos\left(x^{22} + y\left(x\right)^{9} + x^{5}\right)y\left(x\right)^{13} + 5\,\sin\left(x^{22} + y\left(x\right)^{9} + x^{5}\right)y\left(x\right)^{4} - 4\,xe^{\left(4\,xy\left(x\right)\right)}}\right]$$



(b) The graph looks like a damped oscillation, which is what we'd expect as $T \to T_{avg}$ as $t \to \infty$

[1]:
```
clear_vars()
from IPython.core.interactiveshell import InteractiveShell
from sage.symbolic.integration.integral import definite_integral
InteractiveShell.ast_node_interactivity = "all"
t, k = var('t, k')
T = function('T')(t)
dTdt = diff(T, t)

q = 5
p = 9
r = 22

T_i = 70 + p - q
A = T_i - p*e^(-1/(r*q)*t)*cos(pi/12*t)-r*e^(-1/(r*q)*t)*sin(pi/12*t)
show(A)

sol = desolve(dTdt==k*(T-A), T, ivar=t) # i
usol = desolve(dTdt==k*(T-A), T, ivar=t, ics=[0,T_i+r]) # ii
usolsub = (usol==T_i+r-p).subs(t==1)
kval = find_root(usolsub, -1, 0) # iii
plot(usol.subs(k==kval), 0, 72) # iv
tavg = definite_integral(usol.subs(k==kval), t, 0, 72)/72 #vi
show(N(tavg))
```

[1]: $-9 \cos\left(\frac{1}{12}\pi t\right) e^{\left(-\frac{1}{110}t\right)} - 22\, e^{\left(-\frac{1}{110}t\right)} \sin\left(\frac{1}{12}\pi t\right) + 74$



74.2811410454468

(c)

[1]:

```
clear_vars()
x = var('x')
y = var('y')

# I changed these numbers because the actual values created a graph that did not
↪ demonstrate the slope field and function. This was done on the advice of Maya Peters.
q = 1
p = 4
r = 14

eqn = (y+p)^q*sin(q*x*y^p+1)*cos(p*y*x^q)*e^(-(1/r)*y*x)
sf = plot_slope_field(eqn, (x,-5,5), (y,-5,5))

p1=desolve_rk4(eqn, y, ics=[0,-4], output='plot', end_points=[-5,5], color="green")
p2=desolve_rk4(eqn, y, ics=[0,-2], output='plot', end_points=[-5,5], color="red")
p3=desolve_rk4(eqn, y, ics=[0,0], output='plot', end_points=[-5,5], color="blue")
p4=desolve_rk4(eqn, y, ics=[0,2], output='plot', end_points=[-5,5], color="orange")
p5=desolve_rk4(eqn, y, ics=[0,4], output='plot', end_points=[-5,5], color="purple")

show(p1+p2+p3+p4+p5+sf)
```

[1]: