Calculus II: Assignment 9

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Problem 1.
For each of the following series, determine whether it converges or diverges. If it converges, find or approximate the sum as best you can. (Series in solution)

Solution 1.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!}$ Converges by the ratio test to e^3

$$\begin{aligned}
&= \lim_{n\to\infty} \left| \frac{3^{n+1}}{(n+1)!} \frac{n!}{3^n} \right| \\
&= \lim_{n\to\infty} \left| -\frac{3}{n+1} \right| \\
&= 0 < 1
\end{aligned}$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{n!}$ Converges by the ratio test to $\frac{1}{e^3}$

$$\begin{aligned}
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} 3^{n+1}}{(n+1)!} \frac{n!}{3^n (-1)^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{3}{n+1} \right| \\
&= 0 < 1
\end{aligned}$$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n^n}{n!}$ Diverges by the ratio test

$$\begin{aligned}
&= \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \frac{n!}{n^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)(n+1)^n}{(n+1)} \frac{1}{n^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)^n}{n^n} \right| \\
&= \infty > 1
\end{aligned}$$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n^n}{n!}$ Diverges by the ratio test

$$\begin{aligned}
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)^{n+1}}{(n+1)!} \frac{n!}{(-1)^n n^n} \right| \\
&= \lim_{n\to\infty} \left| -\frac{(n+1)(n+1)^n}{(n+1)} \frac{1}{n^n} \right| \\
&= \lim_{n\to\infty} \left| -\frac{(n+1)^n}{n^n} \right| \\
&= \infty > 1
\end{aligned}$$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n^n}$ Converges by the ratio test to 7.66289531150130

$$\begin{aligned}
&= \lim_{n\to\infty} \left| \frac{3^{n+1}}{(n+1)^{n+1}} \frac{n^n}{3^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{3}{(n+1)^{n+1}} (n^n) \right| \\
&= 0 < 1
\end{aligned}$$

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{n^n}$ Converges by the alternating series test to -0.498038749478000.

$$\begin{aligned}
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}3^{n+1}}{(n+1)^{n+1}} \frac{n^n}{(-1)^n 3^n} \right| \\
&= \lim_{n\to\infty} \left| -\frac{3}{(n+1)^{n+1}} (n^n) \right| \\
&= 0 < 1
\end{aligned}$$

(g) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$ Diverges by the ratio test

$$\begin{aligned}
&= \lim_{n\to\infty} \left| \frac{(n+1)!}{3^{n+1}} \frac{3^n}{n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)!}{3(n!)} \right| \\
&= \infty > 1
\end{aligned}$$

(h) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{(-1)^n 3^n}$ Diverges by the ratio test

$$\begin{aligned}
&= \lim_{n\to\infty} \left| \frac{(n+1)!}{(-1)^{n+1}3^{n+1}} \frac{(-1)^n 3^n}{n!} \right| \\
&= \lim_{n\to\infty} \left| -\frac{(n+1)!}{3(n!)} \right| \\
&= \infty > 1
\end{aligned}$$

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^n}$ Converges by the ratio test to 1.879853862175258

$$
\begin{aligned}
&= \lim_{n\to\infty} \left| \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{n^n}{(n+1)^n} \right| \\
&= 0 < 1
\end{aligned}
$$

(j) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!}{n^n}$ Converges by the ratio test to 0.344168399133

$$
\begin{aligned}
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)!}{(n+1)^{n+1}} \frac{n^n}{(-1)^n n!} \right| \\
&= \lim_{n\to\infty} \left| -\frac{n^n}{(n+1)^n} \right| \\
&= 0 < 1
\end{aligned}
$$

(k) $\displaystyle\sum_{n=0}^{\infty} \frac{n^n}{3^n}$ Diverges by the divergence test

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{n^n}{3^n} \\
&= \infty \neq 0
\end{aligned}
$$

(l) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n^n}{3^n}$ Diverges by the divergence test

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{(-1)^n n^n}{3^n} \\
&= \infty \neq 0
\end{aligned}
$$

A couple notes are warranted here, first of all I got the summation results in a) and b) using the Maclaurin series (Taylor series about 0), which is something I learned very recently in MATH-1550H, the other results were gotten through messing around with Sage, which was being remarkably frustrating for these sums for some reason. I tried to find nicer ways to express the sums, both by hand and with Sage, to little success so apologies for the really accurate decimal expansions. Here's Sage being petulant despite my best attempts to get it to behave. Note the sum starting at 1 instead of 0 because starting at 0 causes Sage to throw an non-descriptive error which ends with what I think is an empty string explaining the failure reason, very helpful!

[1]:
```
from IPython.core.interactiveshell import InteractiveShell
InteractiveShell.ast_node_interactivity = "all"

n = var('n')
k = var('k')

sum(((3^n)/(n^n)), n, 1, oo)
limit(sum(((3^n)/(n^n)), n, 1, k), k=oo)
```

```
[1]: sum(3^n/n^n, n, 1, +Infinity)
```

```
[1]: limit(sum(3^n/n^n, n, 1, k), k, +Infinity)
```

Problem 2. Does the series $\sum_{n=0}^{\infty}\left[\frac{1}{3n+1}-\frac{1}{3n+2}+\frac{1}{3n+3}\right]$ converge or diverge? If it converges, does it do so conditionally or absolutely?

Solution 2.

$$\sum_{n=0}^{\infty}\left[\frac{1}{3n+1}-\frac{1}{3n+2}+\frac{1}{3n+3}\right]$$
$$\sum_{n=0}^{\infty}\frac{1}{3n+1}-\sum_{n=0}^{\infty}\frac{1}{3n+2}+\sum_{n=0}^{\infty}\frac{1}{3n+3}$$

Each of those diverge, so the original series will be some variation of divergent as well. This can be verified with the integral test as follows

$$\begin{aligned}
&= \lim_{k\to\infty}\int_0^k \frac{1}{3x+1}-\frac{1}{3x+2}+\frac{1}{3x+3}\,dx\\
&= \lim_{k\to\infty}\left[\int_0^k \frac{1}{3x+1}\,dx-\int_0^k \frac{1}{3x+2}\,dx+\int_0^k \frac{1}{3x+3}\,dx\right]\\
&= \lim_{k\to\infty}\left[\left.\frac{1}{3}\ln(3x+1)\right|_0^k-\left.\frac{1}{3}\ln(3x+2)\right|_0^k+\left.\frac{1}{3}\ln(3x+3)\right|_0^k\right]\\
&= \lim_{k\to\infty}\frac{1}{3}\left.\left[\ln(3x+1)-\ln(3x+2)+\ln(3x+3)\right]\right|_0^k\\
&= \lim_{k\to\infty}\frac{1}{3}\left.\left[\ln\left(\frac{3x+1}{3x+2}(3x+3)\right)\right]\right|_0^k\\
&= \lim_{k\to\infty}\frac{1}{3}\left[\ln\left(\frac{(3k+1)(3k+3)}{3k+2}\right)\right]\\
&= \infty
\end{aligned}$$

$\therefore$ Because the integral diverges per the integral test for convergence the series must also diverge.