Math 2120H: Assignment III

Jeremy Favro (0805980)
Trent University, Peterborough, ON, Canada

February 25, 2025

Problem 1. Find the unit tangent vector $\mathbf{T}$, the principle normal vector $\mathbf{N}$ and the curvature κ for the curves below

(a)
$$\mathbf{r}(t) = (\cos t + t\sin t)\,\mathbf{i} + (\sin t - t\cos t)\,\mathbf{j}, \qquad t > 0$$

(b)
$$\mathbf{r}(t) = \left(e^t \cos t\right)\mathbf{i} + \left(e^t \sin t\right)\mathbf{j} + 2\mathbf{k}$$

Solution 1.

(a) By the formula $\mathbf{T} = \frac{d\mathbf{r}}{dt}/\left|\frac{d\mathbf{r}}{dt}\right|$, $\mathbf{N} = \frac{d\mathbf{T}}{dt}/\left|\frac{d\mathbf{T}}{dt}\right|$, $\kappa = \left|\frac{d\mathbf{T}}{dt}\right|/\left|\frac{d\mathbf{r}}{dt}\right|$ so,

$$\begin{aligned}
\mathbf{T} &= \frac{d\mathbf{r}}{dt}/\left|\frac{d\mathbf{r}}{dt}\right| \\
&= \left[(\cos t + t\sin t)'\,\mathbf{i} + (\sin t - t\cos t)'\,\mathbf{j}\right]/\sqrt{\left[(\cos t + t\sin t)'\right]^2 + \left[(\sin t - t\cos t)'\right]^2} \\
&= \left[(-\sin t + \sin t + t\cos t)\,\mathbf{i} + (\cos t - \cos t + t\sin t)\,\mathbf{j}\right]/t \qquad (t > 0) \\
&= \left[\cancel{t}\cos t\mathbf{i} + \cancel{t}\sin t\mathbf{j}\right]/\cancel{t} = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{N} &= \frac{d\mathbf{T}}{dt}/\left|\frac{d\mathbf{T}}{dt}\right| \\
&= \left[-\sin(t)\mathbf{i} + \cos(t)\mathbf{j}\right]/\sqrt{\sin^2(t) + \cos^2(t)} = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}
\end{aligned}$$

and

$$\begin{aligned}
\kappa &= \left|\frac{d\mathbf{T}}{dt}\right|/\left|\frac{d\mathbf{r}}{dt}\right| \\
&= \frac{1}{t}
\end{aligned}$$

(b) Again by the given formulae,

$$\begin{aligned}
\mathbf{T} &= \left[\left(e^t\cos t - e^t\sin t\right)\mathbf{i} + \left(e^t\sin t + e^t\cos t\right)\mathbf{j} + 0\mathbf{k}\right]/(e^t\sqrt{2}) \\
&= \frac{1}{\sqrt{2}}\left[(\cos t - \sin t)\,\mathbf{i} + (\sin t + \cos t)\,\mathbf{j}\right]
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{N} &= \frac{1}{\sqrt{2}}\left[(-\cos t - \sin t)\,\mathbf{i} + (-\sin t + \cos t)\,\mathbf{j}\right]/\frac{1}{\sqrt{2}}\sqrt{(-\cos t - \sin t)^2 + (-\sin t + \cos t)^2} \\
&= \left[(-\cos t - \sin t)\,\mathbf{i} + (-\sin t + \cos t)\,\mathbf{j}\right]/\sqrt{\cos^2 t + 2\sin t\cos t + \sin^2 t + \cos^2 t - 2\sin t\cos t + \sin^2 t} \\
&= \left[(-\cos t - \sin t)\,\mathbf{i} + (-\sin t + \cos t)\,\mathbf{j}\right]/\sqrt{2\cos^2 t + 2\sin^2 t} \\
&= \frac{1}{\sqrt{2}}\left[(-\cos t - \sin t)\,\mathbf{i} + (-\sin t + \cos t)\,\mathbf{j}\right]
\end{aligned}$$

and

$$\kappa = \frac{e^{-t}}{\sqrt{2}}$$

Problem 2. Find the curvature of the parabola $y = 4x^2$ when $x = 1$

Solution 2. We can parametrize y using the natural parametrization $x = t \implies y = 4t^2$ which gives us $\mathbf{r}(t) = t\mathbf{i} + 4t^2\mathbf{j}$. We can then apply the formula to find $\kappa = \left|\frac{d\mathbf{T}}{dt}\right| / \left|\frac{d\mathbf{r}}{dt}\right|$,

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= 1\mathbf{i} + 8t\mathbf{j} \\ \implies \left|\frac{d\mathbf{r}}{dt}\right| &= \sqrt{64t^2+1} \end{aligned}$$

then,

$$\begin{aligned} \mathbf{T} &= \frac{d\mathbf{r}}{dt} / \left|\frac{d\mathbf{r}}{dt}\right| \\ &= \frac{1}{\sqrt{64t^2+1}}\mathbf{i} + \frac{8t}{\sqrt{64t^2+1}}\mathbf{j} \\ \implies \frac{d\mathbf{T}}{dt} &= -64t\left(64t^2+1\right)^{-\frac{3}{2}}\mathbf{i} + 8\left(64t^2+1\right)^{-\frac{3}{2}}\mathbf{j} \\ \implies \left|\frac{d\mathbf{T}}{dt}\right| &= \sqrt{\left[-64t\left(64t^2+1\right)^{-\frac{3}{2}}\right]^2 + \left[8\left(64t^2+1\right)^{-\frac{3}{2}}\right]^2} \\ &= \sqrt{4096t^2\left(64t^2+1\right)^{-3} + 64\left(64t^2+1\right)^{-3}} \\ &= \left(64t^2+1\right)^{-\frac{3}{2}}\sqrt{4096t^2+64} = 8\left(64t^2+1\right)^{-\frac{3}{2}}\left(64t^2+1\right)^{\frac{1}{2}} \\ &= \frac{8}{\left(64t^2+1\right)} \end{aligned}$$

so,

$$\begin{aligned} \kappa &= \left|\frac{d\mathbf{T}}{dt}\right| / \left|\frac{d\mathbf{r}}{dt}\right| \\ &= 8\left(64t^2+1\right)^{-1}\left(64t^2+1\right)^{-\frac{1}{2}} = 8\left(64t^2+1\right)^{-\frac{3}{2}} \end{aligned}$$

So the curvature κ at $t = 1$ is $\dfrac{8}{65^{\frac{3}{2}}} \approx 0.0153$

Problem 3. Write the acceleration vector $\mathbf{a}$ in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ at the given value of t without finding $\mathbf{T}$ and $\mathbf{N}$.

$$\mathbf{r}(t) = t^2\mathbf{i} + \left(t + \frac{1}{3}t^3\right)\mathbf{j} + \left(t - \frac{1}{3}t^3\right)\mathbf{k}, \qquad t = 0$$

Solution 3. We define $a_T = \dfrac{d\left|\frac{d\mathbf{r}}{dt}\right|}{dt} \implies a_N = \sqrt{|\mathbf{a}|^2 - a_T^2}$ (because a_N and a_T are scalars). So,

$$\begin{aligned}
a_T &= \frac{d\left|\frac{d\mathbf{r}}{dt}\right|}{dt} \\
&= \frac{d}{dt}\sqrt{4t^2 + \left(1+t^2\right)^2 + \left(1-t^2\right)^2} \\
&= \frac{d}{dt}\sqrt{4t^2 + \left(1+t^2\right)^2 + \left(1-t^2\right)^2} \\
&= \frac{6t}{\sqrt{4t^2 + \left(1+t^2\right)^2 + \left(1-t^2\right)^2}} \\
&= 0 \qquad (\text{at } t = 0)
\end{aligned}$$

then,

$$\begin{aligned}
&\mathbf{a} = 2\mathbf{i} + 2t\mathbf{j} - 2t\mathbf{k} \\
\implies &|\mathbf{a}| = \sqrt{4 + 8t^2}
\end{aligned}$$

so,

$$\begin{aligned}
a_N &= \sqrt{4 + 8t^2 - \frac{6t}{\sqrt{4t^2 + \left(1+t^2\right)^2 + \left(1-t^2\right)^2}}} \\
&= \sqrt{4 + \cancelto{0}{8t^2} - \cancelto{0}{\frac{36t^2}{4t^2 + \left(1+t^2\right)^2 + \left(1-t^2\right)^2}}} \\
&= 2 \qquad (\text{at } t = 0)
\end{aligned}$$

So $\mathbf{a} = 0 \cdot \mathbf{T} + 2 \cdot \mathbf{N}$.