Physics 3610H: Assignment IV

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Problem 1. In class we found the general form of the wavefunction in each region of a finite well to be

$$\psi(x) = \begin{cases} x < -a & De^{+\kappa x} \\ -a < x < a & A\cos kx + B\sin kx \\ x > a & Ce^{-\kappa x} \end{cases}$$

Using the continuity of the wavefunction and its first derivative at both $x = -a$ and $x = +a$, we arrived at the following four equations.

$$\psi(-a) = A\cos ka - B\sin ka = De^{-\kappa a} \tag{1}$$

$$\psi(+a) = A\cos ka + B\sin ka = Ce^{-\kappa a} \tag{2}$$

$$\left.\frac{d\psi}{dx}\right|_{-a} = kA\sin ka + kB\cos ka = D\kappa e^{-\kappa a} \tag{3}$$

$$\left.\frac{d\psi}{dx}\right|_{+a} = -kA\sin ka + kB\cos ka = -C\kappa e^{-\kappa a} \tag{4}$$

Together with normalization these determine A, B, C, D and E. In particular, by considering $(2) - (4)/\kappa$ and $(3) + (5)/\kappa$ we showed that

$$A\left(1 - \frac{k}{\kappa}\tan ka\right) = B\left(\tan ka + \frac{k}{\kappa}\right) = 0$$

In class we considered the even solutions by setting $B = 0$. Here, consider the odd solutions by setting $A = 0$.

(a) What two equations connect k and κ in this case?

(b) Let $x \equiv ka$ and $y \equiv \kappa a$ and plot both functions on a single plot for $2mV_o a^2/\hbar^2 = 25$.

(c) How many allowed values of energy are there in this case?

(d) Give an approximate value of κa which is allowed.

(e) Use $(1)+(3)/\kappa$ and $(2)-(4)/\kappa$ to determine the values of C and D, and write the form of the odd wavefunctions in each region in terms of B, k and κ.

Solution 1.

(a) We have

$$A\left(1 - \frac{k}{\kappa}\tan ka\right) = 0 \implies 1 - \frac{k}{\kappa}\tan ka = 0 \implies k\tan ka = \kappa$$

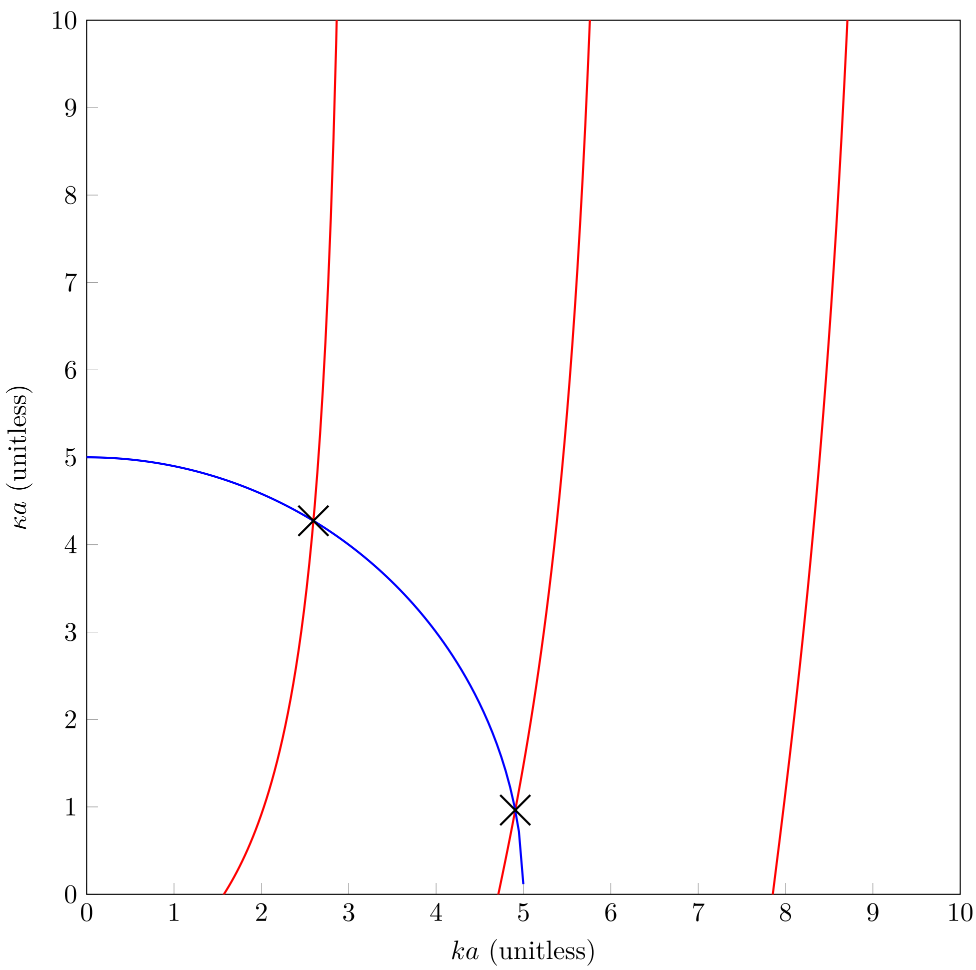
and

$$k^2 + \kappa^2 = 2mV_o a^2/\hbar^2.$$

(b) With our previous two expressions relating k and κ this gets us

$$y = -\frac{x}{\tan x} \quad \text{and} \quad y = \sqrt{25 - x^2}.$$

which are plotted in red and blue respectively in the plot below.



Points of intersection were found using SageMath after first plotting to determine approximate intervals for a numerical solution

[1]:
```
x = var('x')

f(x) = - x / tan(x)
f1(x) = sqrt(25 - x^2)

r1 = find_root(f1-f, 0, 3)
r2 = find_root(f1-f, 3, 5)

show((r1, f(r1)))
```

```
show((r2, f(r2)))
```

[1]: $(2.5957390796498125, 4.27342235572148)$
$(4.906295150856183, 0.963466601746654)$

(c) Two, as indicated by the plot in the previous part which is a representation of values of E, the only parameter which is changing in the function, which give consistent κa values.

(d) The two values here are ≈ 4.27 and ≈ 0.96, determined as above using SageMath.

(e) With $A = 0$ the equations first simplify to

$$\psi(-a) = -B\sin ka = De^{-\kappa a} \tag{5}$$

$$\psi(+a) = +B\sin ka = Ce^{-\kappa a} \tag{6}$$

$$\left.\frac{d\psi}{dx}\right|_{-a} = +kB\cos ka = D\kappa e^{-\kappa a} \tag{7}$$

$$\left.\frac{d\psi}{dx}\right|_{+a} = +kB\cos ka = -C\kappa e^{-\kappa a}. \tag{8}$$

Which gives

$$\begin{aligned}
(5) + (7)/\kappa \implies & -B\sin ka + \frac{kB}{\kappa}\cos ka = De^{-\kappa a} + De^{-\kappa a} \\
\implies & B\left(-\sin ka + \frac{k}{\kappa}\cos ka\right) = 2De^{-\kappa a} \\
\implies & D = \frac{B}{2}e^{\kappa a}\left(-\sin ka + \frac{k}{\kappa}\cos ka\right)
\end{aligned}$$

and

$$\begin{aligned}
(6) - (8)/\kappa \implies & B\sin ka - \frac{kB}{\kappa}\cos ka = Ce^{-\kappa a} + Ce^{-\kappa a} \\
\implies & B\left(\sin ka - \frac{k}{\kappa}\cos ka\right) = 2Ce^{-\kappa a} \\
\implies & C = \frac{B}{2}e^{\kappa a}\left(\sin ka - \frac{k}{\kappa}\cos ka\right)
\end{aligned}$$