# Math 2120H: Assignment V

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**Problem 1.** Verify that the below vector field is conservative and find a potential function for $\mathbf{F}$

$$\mathbf{F} = e^{y+2z}\left(\mathbf{i} + x\mathbf{j} + 2x\mathbf{k}\right)$$

**Solution 1.** For conservative fields

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}, \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}.$$

Here $M = e^{y+2z}$, $N = xe^{y+2z}$, $P = 2xe^{y+2z}$ so,

$$\begin{aligned}
\frac{\partial M}{\partial y} &= e^{y+2z}, & \frac{\partial M}{\partial z} &= 2e^{y+2z} \\
\frac{\partial N}{\partial x} &= e^{y+2z}, & \frac{\partial N}{\partial z} &= 2xe^{y+2z} \\
\frac{\partial P}{\partial y} &= 2xe^{y+2z}, & \frac{\partial P}{\partial x} &= 2e^{y+2z}
\end{aligned}$$

which satisfies the previous set of equalities so the field is conservative. For the potential function we find $f(x, y, z) \int M\,dx = \int e^{y+2z}\,dx = xe^{y+2z} + G(y, z)$, then differentiate with respect to $y$ and $z$ to determine $G$. First with respect to $y$,

$$\frac{\partial f}{\partial y} = N = xe^{y+2z} = xe^{y+2z} + \frac{\partial G}{\partial y} \implies \frac{\partial G}{\partial y} = 0 \therefore G(y, z) = G(z)$$

Then with respect to $z$,

$$\frac{\partial f}{\partial z} = P = 2xe^{y+2z} = 2xe^{y+2z} + \frac{\partial G}{\partial z} \implies \frac{\partial G}{\partial z} = 0 \therefore G = C$$

so $f(x, y, z) = xe^{y+2z} + C$ where $C$ is a constant of integration.

**Problem 2.** Find a potential function for the field below and evaluate the integral

$$\int_{(1,1,1)}^{(1,2,3)} 3x^2\,dx + \frac{z^2}{y}\,dy + 2z\ln y\,dz$$

**Solution 2.** Here we first ensure that this field is conservative,

$$\begin{aligned}
\frac{\partial M}{\partial y} &= 0, & \frac{\partial M}{\partial z} &= 0 \\
\frac{\partial N}{\partial x} &= 0, & \frac{\partial N}{\partial z} &= \frac{2z}{y} \\
\frac{\partial P}{\partial y} &= \frac{2z}{y}, & \frac{\partial P}{\partial x} &= 0
\end{aligned}$$

which it is. Then we follow the standard procedure to determine $f(x, y, z)$,

$$f(x, y, z) = \int M\,dx = x^3 + G(y, z)$$

then we differentiate with respect to $y$,

$$\frac{\partial f}{\partial y} = \frac{z^2}{y} = 0 + \frac{\partial G}{\partial y} \implies G(y,z) = z^2 \ln y + C$$

then with respect to $z$,

$$\frac{\partial f}{\partial z} = 2z \ln y = 0 + \frac{\partial G}{\partial z} \implies G(z,y) = z^2 \ln y + C$$

So $f(x,y,z) = x^3 + z^2 \ln y + C$

**Problem 3.** Find the outward flux of the field $\mathbf{F} = xy\mathbf{i} + y^2\mathbf{j}$ over the boundary of the region enclosed by the curve $y = x^2$ and the line $y = x$.

**Solution 3.** Here the divergence of the field is $3y$. By Green's theorem we then evaluate the integral of the divergence over the given region,

$$\begin{aligned} &= \int_0^1 \int_{x^2}^{x} 3y \, dy dx \\ &= \frac{3}{2} \int_0^1 x^2 - x^4 \, dx \\ &= \frac{1}{5} \end{aligned}$$

**Problem 4.** Apply Green's Theorem to evaluate the integral below

$$\oint_C 3y \, dx + 2x \, dy, \qquad \text{Where } C \text{ is the boundary of } 0 \leq x \leq \pi,\ 0 \leq y \leq \sin x$$

**Solution 4.** Green's Theorem states that

$$\oint M \, dx + N \, dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \, dx \, dy.$$

Applying this here we get

$$\oint_C 3y \, dx + 2x \, dy = \int_0^{\pi} \int_0^{\sin x} -1 \, dy dx = \cos x \Big|_0^{\pi} = -2$$