Calculus Cheat Sheet

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Revision 3

Limits

Existence

$\lim_{x\to a} f(x) = L$ exists if $\forall\epsilon > 0 \exists\delta > 0$ s.t. $|x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$.

Properties

$\lim_{x\to a}[cf(x)] = c\lim_{x\to a}[f(x)]$
$\lim_{x\to a}[f(x)\pm g(x)] = \lim_{x\to a}[f(x)] \pm \lim_{x\to a}[g(x)]$
$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a}[f(x)]\lim_{x\to a}[g(x)]$
$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a}[f(x)]}{\lim_{x\to a}[g(x)]}, \lim_{x\to a}[g(x)] \neq 0$

Squeeze Theorem

The squeeze theorem is a method for solving complex limits, such as $\lim_{x\to 0} x^4 \sin\frac{1}{x}$. You are essentially trying to find two bounding functions, such that if $f(x)$ is the function you are trying to find the limit for, $h(x) \leq f(x)$ and $f(x) \leq g(x)$ ($h(x) \leq f(x) \leq g(x)$). If you find these two bounding functions, $h(x)$ & $g(x)$ and their limits at some number a agree, then the limit of $f(x)$ at a will be equal to the limits of $h(x)$ & $g(x)$ at a. Formally, for some limit $\lim_{x\to a} f(x)$, if you find two functions such that $h(x) \leq f(x) \leq g(x)$, and $\lim_{x\to a} h(x) = \lim_{x\to a} g(x)$, then $\lim_{x\to a} h(x) = \lim_{x\to a} g(x) = \lim_{x\to a} f(x)$.

Evaluation Techniques

Basics

$\lim_{x\to a}[f(x)] = f(a)$ if f exists at a

L'Hôpital's Rule

If $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{0}{0}$ **or** $\frac{\pm\infty}{\pm\infty}$ then $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \lim_{x\to a}\left[\frac{f'(x)}{g'(x)}\right]$

Factoring at Infinity

If $p(x)$ and $q(x)$ are polynomials, to evaluate $\lim_{x\to\pm\infty}\left[\frac{p(x)}{q(x)}\right]$ factor the greatest power of x in $q(x)$ (the denominator) out of poth $p(x)$ and $q(x)$ then compute the limit, e.g. $\lim_{x\to-\infty}\left[\frac{3x^2-4}{5x-2x^2}\right] = \lim_{x\to-\infty}\left[\frac{\cancel{x^2}(3-\frac{4}{x^2})}{\cancel{x^2}(\frac{5}{x}-2)}\right]$ $\frac{3-0}{0-2} = -\frac{3}{2}$

Derivatives

Definition

$$\frac{d}{dx}f(x) = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

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Techniques

Sum Rule

$$\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$$

Product Rule

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Chain Rule

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

Common Derivatives

$\frac{d}{dx}(x) = 1$ $\frac{d}{dx}(\sin x) = \cos x$
$\frac{d}{dx}(\cos x) = -\sin x$ $\frac{d}{dx}(\tan x) = \sec^2 x$
$\frac{d}{dx}(\sec x) = \sec x \tan x$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$
$\frac{d}{dx}(\cot x) = -\csc^2 x$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\sin^{-1} x \neq \frac{1}{\sin x}$
$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\frac{d}{dx}(a^x) = a^x \ln a$
$\frac{d}{dx}(e^x) = e^x$ $\frac{d}{dx}(\ln x) = \frac{1}{x}, x > 0$
$\frac{d}{dx}(\ln|x|) = \frac{1}{x}, x \neq 0$ $\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$

Common Chain Rule Derivatives

$\frac{d}{dx}([f(x)]^n) = n[f(x)]^{n-1}f'(x)$
$\frac{d}{dx}(e^{f(x)}) = e^{f(x)}f'(x)$
$\frac{d}{dx}(\ln[f(x)]) = \frac{f'(x)}{f(x)}$
$\frac{d}{dx}(\sin[f(x)]) = f'(x)\cos[f(x)]$
$\frac{d}{dx}(\cos[f(x)]) = -f'(x)\sin[f(x)]$
$\frac{d}{dx}(\tan[f(x)]) = f'(x)\sec^2[f(x)]$
Trig derivatives, same old same old
$\frac{d}{dx}(f(x)^{g(x)}) = \frac{g(x)f'(x)}{f(x)} + \ln[f(x)]g'(x)$

Implicit Derivation

Remember $y = y(x)$ here, so products/quotients of x and y will use the product/quotient rule and derivatives of y will use the chain rule. The "trick" is to differentiate as normal and every time you differentiate a y you tack on a y' (from the chain rule). Then solve for y', probably by factoring.

Integrals

Definition

The integral of some function $f(x)$ is a function $f^\star(x)$ s.t. $f^{\star\prime}(x) = f(x)$. **Don't forget your constant!**

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

The integral can be done using the Right Riemann Sum: $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^n\left[\frac{b-a}{n}\cdot f(a+i\cdot\frac{b-a}{n})\right]$

U-Substitution

$\int_a^b f(g(x))\cdot g'\,dx = \int_{g(a)}^{g(b)} f(u)\,du$. Using U-Substitution, $u = g(x)$ and $du = g'(x)dx$ ($dx = \frac{du}{g'}$). The limits, a and b are just dropped when using indefinite integrals.

Integration By Parts

$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$ and $\int_a^b u(x)v'(x)\,dx = uv|_a^b - \int_a^b u'(x)v(x)\,dx$.

Common Integrals

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$
$\int \frac{1}{x}\,dx = \ln|x| + C$
$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$
$\int \ln x\,dx = x\ln x - x + C$
$\int e^x\,dx = e^x + C$
$\int a^x\,dx = \frac{a^x}{\ln a} + C$
$\int \sin x\,dx = -\cos x + C$
$\int \cos x\,dx = \sin x + C$
$\int \tan x\,dx = \ln|\sec x| + C$
$\int \sec^2 x\,dx = \tan x + C$
$\int \sec x\tan x\,dx = \sec x + C$
$\int \csc x\cot x\,dx = -\csc x + C$
$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x\,dx = \ln|\sec x + \tan x| + C$
$\int \frac{1}{a^2+u^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{1}{\sqrt{a^2+u^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

Trig Reduction Formulae

(For $n \geq 2$)
$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int\sin^{n-2}(x)\,dx$
$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int\cos^{n-2}(x)\,dx$
$\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) + \frac{n-1}{n}\int\tan^{n-2}(x)\,dx$
$\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int\sec^{n-2}(x)\,dx$

Trig Identities

$\sin^2(x) + \cos^2(x) = 1$
$1 + \tan^2(x) = \sec^2(x)$
$\sin(2x) = 2\sin(x)\cos(x)$
$$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x)\\ &= 2\cos^2(x) - 1\\ &= 1 - 2\sin^2(x)\end{aligned}$$

Trig Substitutions

$\sqrt{a^2 - x^2} \rightsquigarrow x = a\sin(\theta)$
$\sqrt{a^2 + x^2} \rightsquigarrow x = a\tan(\theta)$
$\sqrt{x^2 - a^2} \rightsquigarrow x = a\sec(\theta)$

Partial Fractions & Polynomial Division

If the degree of the numerator is greater than that of a the denominator, use polynomial division to divide the denominator into the numerator ($denom\,\overline{)numer}$), then integrate the result. If the degree of the denominator is greater than that of the numerator, use partial fractions to decompose the integral as follows:

If the denominator contains different linear terms, break it down to $\frac{A}{ax+b}$

If it contains a repeated linear term ($(ax+b)^2$), break it down to $\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$

If it contains an irreducible quadratic ($x^2 + bx + c$), break it down to $\frac{A}{dx+o} + \frac{Bx+C}{x^2+bx+c}$

Then, set up equations for the coefficients of the powers of x in the numerator and solve them to determine A, B, C and so on.

Applications

Centroids

$C = (\hat{x}, \hat{y})$, $\hat{x} = \frac{\int x f(x)\,dx}{\int f(x)\,dx}$ & $\hat{y} = \frac{\int y f(y)\,dy}{\int f(y)\,dy}$

Arc Length

$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$

Volume

Use the variable parallel to the axis of revolution. $V = \int_a^b A(x)\,dx$ where $A(x)$ is a function which gives the area of one "slice" of the solid. Slices are generally circular and may have holes in them, area of a circle is πr^2, and the area of a washer is $\pi(r_{outer}^2 - r_{inner}^2)$

Sequences and Series

Series With Nice Formulae

The geometric series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ provided that $|r| < 1$ The telescoping series where the "inbetween" terms cancel. Take the limit as $k \to \infty$

Basic Divergence Test

If for some series $\sum_{n=0}^{\infty} a_n$ the limit $\lim_{n\to\infty} a_n \neq 0$ the series diverges. However, if the limit is 0, no information can be determined from this test.

Integral Test

If some function $f(n) = a_n$ is decreasing on $[c, \infty)$ then the series $\sum_{n=c}^{\infty} a_n$ converges exactly as $\int_c^{\infty} f(x)dx$ does / does not. This works because $\int_c^{\infty} f(x)dx \leq \sum_{n=c}^{\infty} a_n \leq f(c) + \int_c^{\infty} f(x)dx$

P-Test

Generally, $\sum_{n=c}^{\infty} \frac{an^k + \cdots + a_0}{bn^l + \cdots + b_0}$ converges if $p = l - k < 1$ and diverges if $p \geq 1$

Basic Comparsion Test

If two sequences $\{a_n\}$ and $\{b_n\}$ exist, are comprised of positive terms, and satisfy $0 < a_n \leq b_n$ past some point then

(a) if $\sum_{n=0}^{\infty} b_n$ converges, so does $\sum_{n=0}^{\infty} a_n$

(b) if $\sum_{n=0}^{\infty} a_n$ diverges, so does $\sum_{n=0}^{\infty} b_n$

Limit Comparison Test

If two sequences $\{a_n\}$ and $\{b_n\}$ exist and are comprised of positive terms past some point then $\lim_{n\to\infty} \frac{a_n}{b_n} = c$

(a) if $c > 0$ both series either converge or diverge (one converging means the other does and vice versa)

(b) if $c = 0$ then a_n diverges so does b_n and if b_n converges so does a_n

(c) if $c = \infty$ then a_n converges so does b_n and if b_n diverges so does a_n

Alternating Series Test

If each a_n in the series $\sum_{n=0}^{\infty} a_n$ and

(1) $|a_{n+1}| < |a_n|$ (The series is decreasing)

(2) $a_{n+1} < 0$ & $a_n > 0$ (The series is alternating)

(3) $\lim_{n\to\infty} |a_n| = 0$

then the series converges

Ratio Test

If past some point $a_n \neq 0$ then for the series $\sum_{n=0}^{\infty} a_n$ if the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ then

(a) if $L < 1$ the series converges absolutely

(b) if $L = 1$ no information can be obtained through this test

(c) if $L > 1$ the series diverges

Root Test

For the series $\sum_{n=0}^{\infty} a_n$ if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ then

(a) if $L < 1$ the series converges absolutely

(b) if $L = 1$ no information can be obtained through this test

(c) if $L > 1$ the series diverges

Taylor Series

Taylor's formula states that if some function $f(x)$ can be expanded as a power series around $x = a$ its representation is given by

$$\sum_{n=0}^{\infty} \frac{\frac{d^n}{dx^n} f(a)}{n!} (x-a)^n$$

Remainder Terms

If there exists some function $f(x)$ whose Taylor Series is defined as $T_n(x)$ then the remainder term is defined as $R_n(x) = f(x) - T_n(x)$, which is how "far off" the Taylor Series $T_n(x)$ is from the function $f(x)$ for some n.

Manipulating Taylor Series

If we do something (integrating, differentiating, muliplying, etc…) to a function, then doing the same to its Taylor Series results in the Taylor Series for the new function. Ex, $f(x) \to f_k(x)$ by integrating, then $\int T_n(x)\,dx = T_{nk}(x)$ where $T_n(x)$ is the Taylor Series for $f(x)$ and $T_{nk}(x)$ is the Taylor Series for our new function, created by integrating $f(x)$, $f_k(x)$