# Math 3770H: Assignment II

Jeremy Favro (0805980)
Trent University, Peterborough, ON, Canada

October 6, 2025

**Problem 1.** Write the function

$$f(z) = z + \frac{1}{z} \qquad (z \neq 0)$$

in the form $f(z) = u(r, \theta) + iv(r, \theta)$

**Solution 1.** Writing $f$ first in polar form to switch to a dependence on $r = |z|, \theta = \arctan(\operatorname{Im} z / \operatorname{Re} z)$,

$$z = re^{i\theta} = r\cos\theta + ir\sin\theta.$$

This is actually really nice here as it means we can easily write the inverse without using a fraction,

$$z^{-1} = r^{-1}e^{-i\theta} = r^{-1}\cos(-\theta) + ir^{-1}\sin(-\theta).$$

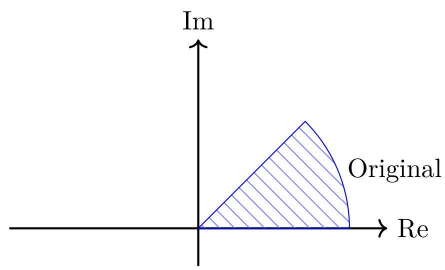
Which can be reworked further using the even/odd nature of cos and sin respectively giving:

$$\begin{aligned} f(z) &= r\cos\theta + ir\sin\theta + r^{-1}\cos\theta - ir^{-1}\sin\theta \\ &= r\cos\theta + r^{-1}\cos\theta + ir\sin\theta - ir^{-1}\sin\theta \\ &= \left(r + r^{-1}\right)\cos\theta + \left(r + r^{-1}\right)\sin\theta \end{aligned}$$

**Problem 2.** Sketch the region onto which the sector $r \leq 1$, $0 \leq \theta \leq \pi/4$ is mapped by the transformation

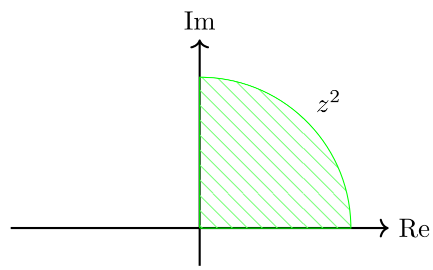
$$\text{(a) } w = z^2; \qquad \text{(b) } w = z^3; \qquad \text{(c) } w = z^4.$$

**Solution 2.** Generally the transform $w = z^n$ will give $w = r^n e^{in\theta}$, just by exponentiation laws. Our original region looks like:

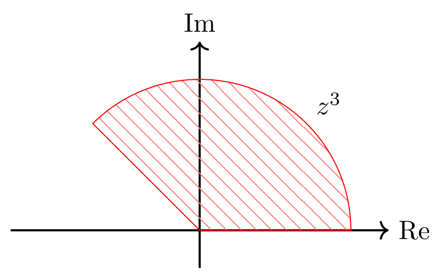



Because our region only extends out to $r = 1$ the radius won't change.

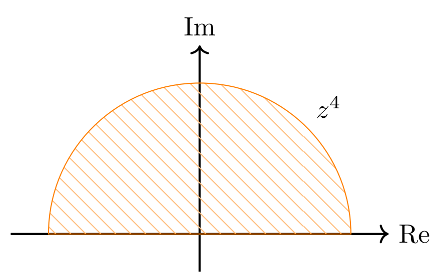
(a)




(b)




(c)



**Problem 3.** Use definition (2), Sec.15, of the limit to prove that

$$\text{(a) } \lim_{z\to z_0} \operatorname{Re} z = \operatorname{Re} z_0; \qquad \text{(b) } \lim_{z\to z_0} \overline{z} = \overline{z_0}; \qquad \text{(c) } \lim_{z\to 0} \frac{\overline{z}^2}{z} = 0.$$

**Solution 3.**

(a) Let $\epsilon > 0$. Suppose $|z - z_0| < \delta$. If we take $z = x + iy$ and $z_0 = x_0 + iy_0$ then we have

$$|\operatorname{Re} z - \operatorname{Re} z_0| = |x - x_0| \leq |z - z_0|.$$

We can make that last statement because

$$|x - x_0| \leq |x - x_0 + iy - iy_0| = |z - z_0|.$$

This means that

$$|\operatorname{Re} z - \operatorname{Re} z_0| = |z - z_0| \leq \delta$$

by the $\epsilon - \delta$ definition we are using. Having this means that provided $|z - z_0| \leq \delta = \epsilon$, then $|\operatorname{Re} z - \operatorname{Re} z_0| < \epsilon$ by transitivity.

(b) Let $\epsilon > 0$. Suppose $|z - z_0| < \delta$. We have

$$|\overline{z} - \overline{z_0}| = |\overline{z - z_0}| = |z - z_0| < \delta$$

all by properties of the complex conjugate. So provided $|z - z_0| < \delta = \epsilon$, then $|\overline{z} - \overline{z_0}| < \epsilon$.

(c) Let $\epsilon > 0$. Suppose $|z - z_0| < \delta$. We have

$$\left|\frac{\overline{z}^2}{z} - \frac{\overline{z_0}^2}{z_0}\right| = \left|\frac{\overline{z}\overline{z}}{z} - \frac{\overline{z_0}\overline{z_0}}{z_0}\right|$$

$\lim_{z\to 0} = 0$.

**Problem 4.** With the aid of the theorem in Sec. 17, show that when

$$T(z) = \frac{az+b}{cz+d} \qquad (ad - bc) \neq 0$$

(a) $\lim_{z\to\infty} T(z) = \infty \quad$ if $c = 0$;

(b) $\lim_{z\to\infty} T(z) = \frac{a}{c} \quad$ and $\quad \lim_{z\to -d/c} T(z) = \infty \quad$ if $c \neq 0$

**Solution 4.**

(a) For $c = 0$ we get

$$T(z) = \frac{az+b}{d}.$$

The referenced theorem states that

$$\lim_{z\to z_0} T(z) = \infty \quad \text{if} \quad \lim_{z\to z_0} T^{-1}(z) = 0.$$

Here we have then that

$$\lim_{z\to\infty} T^{-1}(z) = \lim_{z\to\infty} \frac{d}{az+b} = \frac{d}{\infty} = 0 \implies \lim_{z\to\infty} T(z) = \infty$$

(b)

$$\begin{aligned}
&= \lim_{z\to 0} T(z^{-1}) \\
&= \lim_{z\to 0} \frac{az^{-1}+b}{cz^{-1}+d} \\
&= \lim_{z\to 0} \frac{az^{-1}}{cz^{-1}+d} + \cancelto{0}{\frac{b}{cz^{-1}+d}} \\
&= \lim_{z\to 0} \frac{a}{\cancelto{1}{czz^{-1}} + \cancelto{0}{dz}} \\
&= \frac{a}{c}
\end{aligned}$$

Which, by the second part of the given theorem which says that

$$\lim_{z\to\infty} f(z) = w_0 \quad \text{if} \quad \lim_{z\to\infty} f(z^{-1}) = w_0$$

means that our original limit

$$\lim_{z\to\infty} T(z) = \frac{a}{c}$$

holds true. For the second part,

$$\begin{aligned} &= \lim_{z\to -d/c} T^{-1}(z) \\ &= \lim_{z\to -d/c} \frac{c(-d/c)+d}{a(-d/c)+b} \\ &= \lim_{z\to -d/c} \frac{0}{a(-d/c)+b} = 0 \end{aligned}$$

which by the first part of the theorem (as used in part (a) here) means that our original limit holds true.

**Problem 5.** Use the method in Example 2, Sec. 19, to show that $f'(z)$ does not exist at any point $z$ when

(a) $f(z) = \operatorname{Re} z$; (b) $f(z) = \operatorname{Im} z$.

**Solution 5.** (a)

$$\frac{\Delta f}{\Delta z} = \frac{\operatorname{Re}(z+\Delta z) - \operatorname{Re} z}{\Delta z} = \frac{\operatorname{Re} z + \operatorname{Re}\Delta z - \operatorname{Re} z}{\Delta z} = \frac{\operatorname{Re}\Delta z}{\Delta z}.$$

Approaching along the real axis gives $\Delta z = \Delta x + 0i$ so

$$\frac{\operatorname{Re}\Delta z}{\Delta z} = \frac{\Delta x}{\Delta x} = 1.$$

Approaching along the imaginary axis gives $Deltaz = 0 + \Delta yi$ so

$$\frac{\operatorname{Re}\Delta z}{\Delta z} = \frac{0}{\Delta yi} = 0.$$

Because these two do not agree the limit which defines the derivative cannot exist and therefore the derivative itself cannot exist.

(b)

$$\frac{\Delta f}{\Delta z} = \frac{\operatorname{Im}(z+\Delta z) - \operatorname{Im} z}{\Delta z} = \frac{\operatorname{Im} z + \operatorname{Im}\Delta z - \operatorname{Im} z}{\Delta z} = \frac{\operatorname{Im}\Delta z}{\Delta z}.$$

We can make the same argument as previously, first approaching along the real axis,

$$\frac{\operatorname{Im}\Delta z}{\Delta z} = \frac{0}{\Delta x} = 0.$$

Then the imaginary axis,

$$\frac{\operatorname{Im}\Delta z}{\Delta z} = \frac{\Delta y}{\Delta yi} = -i.$$

Again because this limit does not exist the derivative cannot exist.

**Problem 6.** Use the theorem in Sec. 24 to show that each of these functions is differentiable in the indicated domain of definition, and also to find $f'(z)$:

(a) $f(z) = 1/z^4$ $(z \neq 0)$;

(b) $f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)$ $(r > 0.0 < \theta < 2\pi)$.

Find diff

**Solution 6.**

(a) First we note that

$$f(z) = z^{-4} = r^{-4}\left(\cos(4\theta) - i\sin(4\theta)\right).$$

The partials are then

$$\begin{aligned} u_r &= -4r^{-5}\cos(4\theta) \qquad & u_\theta &= -4r^{-4}\sin(4\theta) \\ v_r &= +4r^{-5}\sin(4\theta) \qquad & v_\theta &= -4r^{-4}\cos(4\theta)\,. \end{aligned}$$

These agree with the Cauchy-Riemann equations for polar coordinates,

$$ru_r = v_\theta \qquad \text{and} \qquad u_\theta = -rv_r$$

which, alongside the fact that the partials exists for all $(r, \theta)$, satisfies the theorem. This means that the derivative will exist for all $(r, \theta)$ in the domain.

(b) Our partials here are

$$\begin{aligned} u_r &= -r^{-1}e^{-\theta}\sin(\ln r) \qquad & u_\theta &= -e^{-\theta}\cos(\ln r) \\ v_r &= r^{-1}e^{-\theta}\cos(\ln r) \qquad & v_\theta &= -e^{-\theta}\sin(\ln r)\,. \end{aligned}$$

which satisfy the Cauchy-Riemann equations. Because the trig functions are only undefined (they oscillate infinitely quickly) at $r = 0$ which is not a part of the domain the derivative exists over the domain.

**Problem 7.** With the aid of the theorem in Sec. 21, show that each of these functions is nowhere analytic: (a) $f(z) = xy + iy$; (b) $f(z) = 2xy + i(x^2 - y^2)$; (c) $f(z) = e^y e^{ix}$.

**Solution 7.**

**Problem 8.A.** Let the function $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain $D$, and consider the families of level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ , where $c_1$ and $c_2$ are arbitrary real constants. Prove that these families are orthogonal. More precisely, show that if $z_0 = (x_0, y_0)$ is a point in $D$ which is common to two particular curves $u(x, y) = c_1$ and $v(x, y) = c_2$ and if $f'(z_0) \neq 0$, then the lines tangent to those curves at $(x_0, y_0)$ are perpendicular.

**Problem 8.B.** Sketch the families of level curves of the component functions $u$ and $v$ when $f(z) = 1/z$, and note the orthogonality described in Exercise 2.

**Solution 8.A.**

**Solution 8.B.**