Math 2150: Assignment III

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Note: I've kept p, q, and r throughout my solutions and only substituted the actual numbers in at the end. This is because I find it easier, especially when dealing with things that might cancel nicely, to deal with variables rather than the numbers they represent. In my case my student number is 0805980 so $p = 9$, $q = 5$, and $r = 22$.

Problem 1.

(a) Determine $\mathcal{L}^{-1}\left\{\frac{(s-2)}{(s^2+1)(s^2-qs)}\right\}$

(b) Determine $\mathcal{L}^{-1}\left\{\frac{2s+1}{(s^2+4)(s^2+q^2)}\right\}$

(c) Determine $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s-q)}\right\}$

(d) Determine $\mathcal{L}\left\{t\sin^2(qt)\right\}$

(e) Determine $\mathcal{L}\left\{e^{-qt}\sin(pt)\sin(qt)\right\}$

(f) Determine $\mathcal{L}\left\{e^{-qt}t\sin\left(t-\frac{\pi}{6}\right)\right\}$

(g) Determine $\mathcal{L}^{-1}\left\{\ln\left(\frac{7+ps}{9+qs}\right)\right\}$

Solution 1.

(a)

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left\{\frac{(s-2)}{s(s^2+1)(s-q)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s-q)}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)(s-q)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{A_1s+B_1}{(s^2+1)} + \frac{C_1}{(s-q)}\right\} - 2\mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{Bs+C}{(s^2+1)} + \frac{D}{(s-q)}\right\}
\end{aligned}
$$

Which I'll solve seperately and then recombine

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left\{\frac{A_1s+B_1}{(s^2+1)} + \frac{C_1}{(s-q)}\right\} \\
&\Rightarrow A_1s(s-q) + B_1(s-q) + C_1\left(s^2+1\right) = 1 \\
&\Rightarrow s = q \implies C_1\left(q^2+1\right) = 1 \implies C_1 = \frac{1}{q^2+1} \\
&\Rightarrow s = 0 \implies B_1(-q) + \frac{1}{q^2+1} = 1 \implies B_1 = \frac{1-\frac{1}{q^2+1}}{-q} \\
&\Rightarrow s = 1 \implies A_1(1-q) + \frac{(1-q)\left(1-\frac{1}{q^2+1}\right)}{-q} + \frac{2}{q^2+1} = 1 \implies A_1 = \frac{1 - \frac{(1-q)\left(1-\frac{1}{q^2+1}\right)}{-q} - \frac{2}{q^2+1}}{(1-q)} \\
&\Rightarrow f_1(t) = A_1\cos(t) + B_1\sin(t) + C_1e^{qt}
\end{aligned}
$$

And

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left\{\frac{A}{s}+\frac{Bs+C}{(s^2+1)}+\frac{D}{(s-q)}\right\} \\
&\Rightarrow A\left(s^2+1\right)(s-q)+Bs(s)(s-q)+C(s)(s-q)+D(s)\left(s^2+1\right)=1 \\
&\Rightarrow s=q \implies D(q)\left(q^2+1\right)=1 \implies D=\frac{1}{q\left(q^2+1\right)} \\
&\Rightarrow s=0 \implies A(-q)=1 \implies A=\frac{1}{-q} \\
&\Rightarrow A\left(s^2+1\right)(s-q)+Bs(s)(s-q)+C(s)(s-q)+D(s)\left(s^2+1\right)=1 \\
&\implies As^3-Aqs^2+As-Aq+Bs^3-Bqs^2+Cs^2-Cqs+Ds^3+Ds=1 \\
&\implies A+B+D=0,\ -Aq-Bq+C=0,\ A-Cq+D=0,\ -Aq=1 \\
&\implies B=\frac{1}{q}-\frac{1}{q\left(q^2+1\right)},\ C=-1+q\left(\frac{1}{q}-\frac{1}{q\left(q^2+1\right)}\right) \\
&\Rightarrow f_2(t)=A+De^{qt}+C\sin(t)+B\cos(t)
\end{aligned}
$$

So,

$$
\begin{aligned}
&f(t)=f_1(s)-2(f_2(s)) \\
&=A_1\cos(t)+B_1\sin(t)+C_1e^{qt}-2A-2De^{qt}-2C\sin(t)-2B\cos(t) \\
&=(A_1-2B)\cos(t)+(B_1-2C)\sin(t)+(C_1-2D)e^{qt}-2A \\
&=\left(\frac{1-\frac{(1-q)\left(1-\frac{1}{q^2+1}\right)}{-q}-\frac{2}{q^2+1}}{(1-q)}-2\frac{1}{q}+2\frac{1}{q\left(q^2+1\right)}\right)\cos(t) \\
&+\left(\frac{1-\frac{1}{q^2+1}}{-q}+2-2q\left(\frac{1}{q}-\frac{1}{q\left(q^2+1\right)}\right)\right)\sin(t) \\
&+\left(\frac{1}{q^2+1}-2\frac{1}{q\left(q^2+1\right)}\right)e^{qt}+\frac{2}{q} \\
&=-\frac{11}{26}\cos(t)-\frac{3}{26}\sin(t)+\frac{3}{130}e^{5t}+\frac{2}{5}
\end{aligned}
$$

(b)

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left\{\frac{2s+1}{\left(s^2+4\right)\left(s^2+q^2\right)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{2s}{\left(s^2+4\right)\left(s^2+q^2\right)}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2+4\right)\left(s^2+q^2\right)}\right\}
\end{aligned}
$$

Which I will again solve seperately and then add

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left\{\frac{2s}{(s^2+4)(s^2+q^2)}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{A_1s+B_1}{(s^2+4)}+\frac{C_1s+D_1}{(s^2+q^2)}\right\}\\
&\Rightarrow (A_1s+B_1)\left(s^2+q^2\right)+(C_1s+D_1)\left(s^2+4\right)=2s\\
&\implies A_1s^3+A_1q^2s+B_1s^2+B_1q^2+C_1s^3+4C_1s+D_1s^2+4D_1=2s\\
&\implies A_1+C_1=0 \implies A_1=-C_1\\
&\implies B_1+D_1=0 \implies B_1=-D_1\\
&\implies A_1q^2+4C_1=2 \implies A_1=\frac{2}{q^2-4} \implies C_1=-\frac{2}{q^2-4}\\
&\implies B_1q^2+4D_1=0 \implies B_1, D_1=0\\
&\Rightarrow f_1(t)=A_1\cos(2t)+C_1\cos(qt)
\end{aligned}
$$

And

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)(s^2+q^2)}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{As+B}{(s^2+4)}+\frac{Cs+D}{(s^2+q^2)}\right\}\\
&\Rightarrow (As+B)\left(s^2+q^2\right)+(Cs+D)\left(s^2+4\right)=1\\
&\implies As^3+Aq^2s+Bs^2+Bq^2+Cs^3+4Cs+Ds^2+4D=1\\
&\implies A+C=0 \implies A=-C\\
&\implies B+D=0 \implies B=-D\\
&\implies Aq^2+4C=0 \implies A, C=0\\
&\implies Bq^2+4D=1 \implies B=\frac{1}{q^2-4} \implies D=-\frac{1}{q^2-4}\\
&\Rightarrow \mathcal{L}^{-1}\left\{\frac{B}{(s^2+4)}+\frac{D}{(s^2+q^2)}\right\}\\
&\Rightarrow \frac{B}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s^2+4)}\right\}+\frac{D}{q}\mathcal{L}^{-1}\left\{\frac{q}{(s^2+q^2)}\right\}\\
&\Rightarrow f_2(t)=\frac{B}{2}\sin(2t)+\frac{D}{q}\sin(qt)
\end{aligned}
$$

So,

$$
\begin{aligned}
&f(t)=f_1(t)+f_2(t)\\
&=A_1\cos(2t)+C_1\cos(qt)+\frac{B}{2}\sin(2t)+\frac{D}{q}\sin(qt)\\
&=\frac{2}{q^2-4}\cos(2t)-\frac{2}{q^2-4}\cos(qt)+\frac{\frac{1}{q^2-4}}{2}\sin(2t)+\frac{-\frac{1}{q^2-4}}{q}\sin(qt)\\
&=\frac{2}{21}\cos(2t)-\frac{2}{21}\cos(5t)+\frac{1}{42}\sin(2t)-\frac{1}{105}\sin(5t)
\end{aligned}
$$

(c) Here e^{-2s} can be "eliminated" using the theorem

$$
\mathcal{L}^{-1}\left\{e^{-\alpha s}F(s)\right\}=f(t-\alpha)\mu(t-\alpha)
$$

where $\mu(t)$ is the Heaviside function.

$$\begin{aligned}
&= \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2\,(s-q)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s^2\,(s-q)}\right\}\mu(t-2) \\
&= \mathcal{L}^{-1}\left\{\frac{As+B}{s^2}+\frac{C}{s-q}\right\}\mu(t-2)
\end{aligned}$$

Then

$$\begin{aligned}
&Cs^2 + As\,(s-q) + B\,(s-q) = 1 \\
&\Rightarrow s=-q \implies Cq^2 = 1 \implies C = \frac{1}{q^2} \\
&\Rightarrow s=0 \implies B\,(-q) = 1 \implies B = \frac{1}{-q} \\
&\Rightarrow s=1 \implies \frac{1}{q^2} + A\,(1-q) - \frac{1}{q}\,(1-q) = 1 \\
&\implies A = \frac{1-\frac{1}{q^2}+\frac{1}{q}\,(1-q)}{1-q}
\end{aligned}$$

Then

$$\begin{aligned}
f(t) &= A + Bt + Ce^{qt} \\
&= \frac{1-\frac{1}{q^2}+\frac{1}{q}\,(1-q)}{1-q} + Bt + Ce^{qt} \\
&= -\frac{1}{25} - \frac{1}{5}t + \frac{1}{25}Ce^{5t}
\end{aligned}$$

So the full solution is

$$\left(-\frac{1}{25} - \frac{1}{5}\,(t-2) + \frac{1}{25}Ce^{5(t-2)}\right)\mu(t-2)$$

(d) Here the t can be "eliminated" through the theorem $\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n\,\dfrac{d^nF(s)}{ds^n}$ so,

$$\begin{aligned}
&= \mathcal{L}\left\{\sin^2(qt)\right\} \\
F(s) &= \frac{2q^2}{s\,(s^2+4q^2)} \rightsquigarrow \text{ by table}
\end{aligned}$$

Now because $n=1$ here we must differentiate $F(s)$ once and switch its sign,

$$\begin{aligned}
&= -\frac{d}{ds}\left[\frac{2q^2}{s\,(s^2+4q^2)}\right] \\
&= \frac{2q^2\left(3s^2+4q^2\right)}{s^2\,(s^2+4q^2)^2} \\
&= \frac{50\left(3s^2+100\right)}{s^2\,(s^2+100)^2}
\end{aligned}$$

(e) Here the e^{-qt} can be "eliminated" through the theorem $\mathcal{L}\left\{f(t)e^{at}\right\} = F(s-a)$ so,

$$\begin{aligned}
&= \mathcal{L}\left\{\sin(pt)\sin(qt)\right\} \\
&= \frac{1}{2}\mathcal{L}\left\{\cos((p-q)\,t) - \cos((p+q)\,t)\right\} \\
&= \frac{1}{2}\mathcal{L}\left\{\cos((p-q)\,t) - \cos((p+q)\,t)\right\} \\
&= \frac{1}{2}\left[\frac{s}{s^2+(p-q)^2} - \frac{s}{s^2+(p+q)^2}\right]
\end{aligned}$$

Then, applying the shift,

$$
\begin{aligned}
F(s) &= \frac{1}{2}\left[\frac{(s+q)}{(s+q)^2+(p-q)^2} - \frac{(s+q)}{(s+q)^2+(p+q)^2}\right] \\
&= \frac{1}{2}\left[\frac{(s+5)}{(s+5)^2+16} - \frac{(s+5)}{(s+5)^2+196}\right]
\end{aligned}
$$

(f) The expression here can be simplified using the two theorems from the previous questions, $\mathcal{L}\left\{f(t)e^{at}\right\} = F(s-a)$ and $\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n \dfrac{d^n F(s)}{ds^n}$

$$
\begin{aligned}
&= \mathcal{L}\left\{e^{-qt} t \sin\left(t - \frac{\pi}{6}\right)\right\} \\
&= \mathcal{L}\left\{\sin\left(t - \frac{\pi}{6}\right)\right\} \\
&= \mathcal{L}\left\{\sin(t)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)\cos(t)\right\} \\
&= \mathcal{L}\left\{\frac{\sqrt{3}}{2}\sin(t) - \frac{1}{2}\cos(t)\right\} \\
&= \frac{\sqrt{3}}{2}\mathcal{L}\left\{\sin(t)\right\} - \frac{1}{2}\mathcal{L}\left\{\cos(t)\right\} \\
&= \frac{\sqrt{3}}{2\left(s^2+1\right)} - \frac{s}{2\left(s^2+1\right)}
\end{aligned}
$$

Then we need to apply the derivative and shift (I don't think order should matter here),

$$
\begin{aligned}
&= -\frac{d}{ds}\left[\frac{\sqrt{3}}{2\left(s^2+1\right)} - \frac{s}{2\left(s^2+1\right)}\right] \\
&= \frac{s^2 - 2\sqrt{3}s - 1}{2\left(s^2+1\right)^2} \\
\Rightarrow F(s) &= \frac{(s+q)^2 - 2\sqrt{3}(s+q) - 1}{2\left((s+q)^2+1\right)^2} \\
&= \frac{(s+5)^2 - 2\sqrt{3}(s+5) - 1}{2\left((s+5)^2+1\right)^2}
\end{aligned}
$$

(g)

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left\{\ln\left(\frac{7+ps}{9+qs}\right)\right\} \\
&= \mathcal{L}^{-1}\left\{\ln(7+ps) - \ln(9+qs)\right\} \\
&= \mathcal{L}^{-1}\left\{\ln(7+ps)\right\} - \mathcal{L}^{-1}\left\{\ln(9+qs)\right\}
\end{aligned}
$$

Here we can use the theorem $\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n \dfrac{d^n F(s)}{ds^n} \implies f(t) = \dfrac{(-1)^n}{t^n}\mathcal{L}^{-1}\left\{\dfrac{d^n F(s)}{ds^n}\right\}$

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left\{\frac{1}{7+ps}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{9+qs}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{\frac{7}{p}+s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{\frac{9}{q}+s}\right\} \\
&= e^{-\frac{7}{p}t} - e^{-\frac{9}{q}t}
\end{aligned}
$$

Then, using the theorem

$$= \frac{1}{t}\left[-e^{-\frac{9}{q}t} - e^{-\frac{7}{p}t}\right]$$
$$f(t) = \frac{1}{t}\left[-e^{-\frac{9}{5}t} - e^{-\frac{7}{9}t}\right]$$

Problem 2. Use the **Laplace Transform** method to solve the following initial value problems:

(a) $y'' + 2qy' + q^2 y = t^p e^{-qt}$ $\quad y(0) = y'(0) = 0$

(b) $y'' - 3qy' + 2q^2 y = e^{qt}\cos(pt)$ $\quad y(0) = y'(0) = 0$

Solution 2.

(a)

$$\begin{aligned}
&\mathcal{L}\left\{y'' + 2qy' + q^2 y\right\} = \mathcal{L}\left\{t^p e^{-qt}\right\}\\
&\mathcal{L}\left\{y''\right\} + \mathcal{L}\left\{2qy'\right\} + \mathcal{L}\left\{q^2 y\right\} = \mathcal{L}\left\{t^p e^{-qt}\right\}\\
&s^2 Y(s) + 2qsY(s) + q^2 Y(s) = \frac{p!}{(s+q)^{p+1}}\\
&Y(s) = \frac{p!}{(s^2 + 2qs + q^2)(s+q)^{p+1}}
\end{aligned}$$

Then

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{p!}{(s^2 + 2qs + q^2)(s+q)^{p+1}}\right\}\\
&= p!\mathcal{L}^{-1}\left\{\frac{1}{(s+q)^2 (s+q)^{p+1}}\right\}\\
&= p!\mathcal{L}^{-1}\left\{\frac{1}{(s+q)^{p+3}}\right\}\\
&= p!e^{-qt}\mathcal{L}^{-1}\left\{\frac{1}{s^{p+3}}\right\}\\
&= \frac{p!e^{-qt}}{(p+2)!}\mathcal{L}^{-1}\left\{\frac{(p+2)!}{s^{p+3}}\right\}\\
&= \frac{p!e^{-qt}}{(p+2)!}t^{p+2}\\
&= \frac{e^{-qt}}{(p+1)(p+2)}t^{p+2}\\
&= \frac{e^{-5t}}{110}t^{11}
\end{aligned}$$

(b)

$$\begin{aligned}
&\mathcal{L}\left\{y'' - 3qy' + 2q^2 y\right\} = \mathcal{L}\left\{e^{qt}\cos(pt)\right\}\\
&\mathcal{L}\left\{y''\right\} - \mathcal{L}\left\{3qy'\right\} + \mathcal{L}\left\{2q^2 y\right\} = \mathcal{L}\left\{e^{qt}\cos(pt)\right\}\\
&Y(s)\left(s^2 - 3qs + 2q^2\right) = \frac{(s-q)}{(s-q)^2 + k^2}\\
&Y(s) = \frac{(s-q)}{(s^2 - 3qs + 2q^2)\left((s-q)^2 + p^2\right)}
\end{aligned}$$

Then,

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left\{\frac{1}{(s-2q)\left((s-q)^2+p^2\right)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{A}{s-2q}+\frac{Bs+C}{s^2-2qs+q^2+p^2}\right\} \\
&\Rightarrow A\left(s^2-2qs+q^2+p^2\right)+Bs\left(s-2q\right)+C\left(s-2q\right)=1 \\
&\Rightarrow s=2q \implies A\left(q^2+p^2\right)=1 \implies A=\frac{1}{q^2+p^2} \\
&\Rightarrow s=0 \implies 1+C\left(-2q\right)=1 \implies C=0 \\
&\Rightarrow s=1 \implies \frac{1}{q^2+p^2}\left(1-2q+q^2+p^2\right)+B\left(1-2q\right)=1 \implies B=\frac{1-\frac{1}{q^2+p^2}\left(1-2q+q^2+p^2\right)}{\left(1-2q\right)} \\
&= \mathcal{L}^{-1}\left\{\frac{A}{s-2q}+\frac{Bs}{(s-q)^2+p^2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{A}{s-2q}+\frac{B\left(s-q\right)}{(s-q)^2+p^2}+\frac{Bq}{(s-q)^2+p^2}\right\} \\
&= Ae^{2qt}+Be^{qt}\cos\left(pt\right)+\frac{Bq}{p}e^{qt}\sin\left(pt\right) \\
&= \frac{1}{106}e^{10t}-\frac{1}{106}e^{5t}\cos\left(9t\right)-\frac{5}{954}e^{5t}\sin\left(9t\right)
\end{aligned}
$$