Physics 2610H: Assignment II

Jeremy Favro

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Problem 1. Write out the total wave function $\Psi(x,t)$ for an electron in the $n = 3$ state of a $10\,\mathrm{nm}$ wide infinite well. Other than the symbols x and t, the function should include only numerical values.

Solution 1. We can split $\Psi(x,t) = \psi(x)\phi(t) = \psi(x)\exp\left(\frac{-iEt}{\hbar}\right)$ then for a particle in an infinite potential well $\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\sqrt{\frac{2mE}{\hbar^2}}x\right)$ and $E = \frac{n^2\pi^2\hbar^2}{2mL^2}$ So,

$$\begin{aligned}
\Psi(x,t) &= \sqrt{\frac{2}{L}}\sin\left(\sqrt{\frac{2m\frac{n^2\pi^2\hbar^2}{2mL^2}}{\hbar^2}}x\right)\exp\left(\frac{-i\frac{n^2\pi^2\hbar^2}{2mL^2}t}{\hbar}\right) \\
&= \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\exp\left(-i\frac{n^2\pi^2\hbar t}{2mL^2}\right) \\
&= \sqrt{\frac{2}{10\times 10^{-9}}}\sin\left(\frac{3\pi x}{10\times 10^{-9}}\right)\exp\left(-i\frac{3^2\pi^2\hbar t}{2\cdot 9.109\times 10^{-31}\cdot (10\times 10^{-9})^2}\right) \\
&\approx 1.414\times 10^3\,\mathrm{m}^{-0.5}\sin\left(9.425\times 10^{17}\,\mathrm{m}^{-1}x\right)\exp\left(-i\cdot 5.142\times 10^{13}\,\mathrm{s}^{-1}\right)
\end{aligned}$$

Problem 2. An electron in the $n = 4$ state of a $5\,\mathrm{nm}$ wide infinite well makes a transition to the ground state, giving off energy in the form of a photon. What is the photon's wavelength?

Solution 2. By equation 1.11 from the formula sheet the energy of a particle of mass m in state n in an infinite well of width L is

$$E = \frac{n^2h^2}{8mL^2}$$

Here we're looking at ΔE of an electron between $n = 4$ and $n = 1$ so,

$$\Delta E = \left.\frac{n^2h^2}{8mL^2}\right|_4^1 = \frac{15h^2}{8m_eL^2}$$

Then, $E = \dfrac{hc}{\lambda} \implies \lambda = hc\dfrac{8m_e(5\,\mathrm{nm})^2}{15h^2} = \dfrac{8cm_e(5\,\mathrm{nm})^2}{15h} \approx 5500\,\mathrm{nm}$

Problem 3. What is the probability that a particle in the first excited ($n = 2$) state of an infinite well would be found in the middle third of the well? How does this compare with the classical expectation? Why?

Solution 3. Because this is a particle in an infinite well the probability is the integral between the two given points of $\left|\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right|^2$ so,

$$
\begin{aligned}
&= \int_{\frac{L}{3}}^{\frac{2L}{3}} \left[\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right]^2 dx = \frac{2}{L}\int_{\frac{L}{3}}^{\frac{2L}{3}} \sin^2\left(\frac{n\pi x}{L}\right) dx \\
&= \frac{2}{L}\left[\frac{x}{2} - \frac{\sin\left(\frac{2n\pi x}{L}\right)}{\frac{4n\pi}{L}}\right]_{\frac{L}{3}}^{\frac{2L}{3}} \quad \text{(From formula sheet)} \\
&= \frac{2}{L}\left[\frac{x}{2} - \frac{L}{4n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]_{\frac{L}{3}}^{\frac{2L}{3}} \\
&= \frac{2}{L}\left[\frac{\frac{2L}{3}}{2} - \frac{L}{4n\pi}\sin\left(\frac{2n\pi\frac{2L}{3}}{L}\right) - \frac{\frac{L}{3}}{2} + \frac{L}{4n\pi}\sin\left(\frac{2n\pi\frac{L}{3}}{L}\right)\right] \\
&= \frac{2}{3} - \frac{2}{6} + \frac{2}{4n\pi}\sin\left(\frac{2n\pi}{3}\right) - \frac{2}{4n\pi}\sin\left(\frac{4n\pi}{3}\right) \\
&= \frac{1}{3} + \frac{1}{2n\pi}\left[\sin\left(\frac{2n\pi}{3}\right) - \sin\left(\frac{4n\pi}{3}\right)\right] \approx 0.196
\end{aligned}
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Classically I think we'd expect that the particle has an even probability density over the entire well and therefore the probability that it is inside one third of the well is $\frac{1}{3}$. The difference between these two arises from the fact that in the $n = 2$ state we have an antinode in the middle third which means that the probability is lesser than the outer two thirds.