

Physics 2605H: Assignment I

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Problem 1.

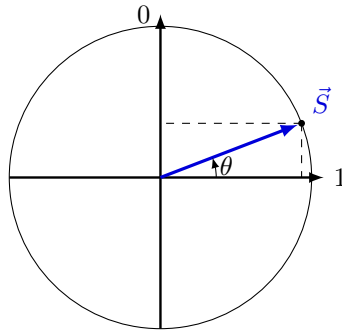
- (a) Find the real part and imaginary of the complex number $\frac{5+3i}{1+5i}$ and express your answer in polar form.
- (b) Consider the product of two complex numbers $(1+3i)$ and $(5+4i)$. Find the complex conjugate in two ways:
- (i) Take the complex conjugates before and after the multiplication and show that they are the same. **what does this mean?? why is there only (i)??**

Solution 1.

$$\frac{5+3i}{1+5i} = \frac{5+3i}{1+5i} \frac{1-5i}{1-5i} = \frac{10}{13} - \frac{11}{13}i \approx \frac{\sqrt{221}}{13} e^{-0.83i}$$

Consider the product of two complex numbers $(1+3i)$ and $(5+4i)$. Find the complex conjugate in two ways: (i) Take the complex conjugates before and after the multiplication and show that they are the same. **what does this mean?? why is there only (i)??**

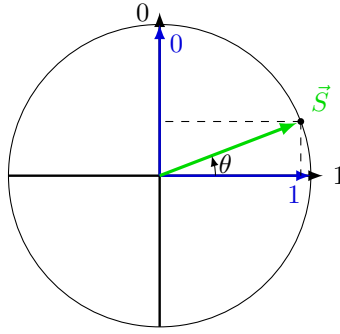
Problem 2. The figure shows the general representation of a qubit



- (a) How many states are possible for the qubit to be in?
- (b) If it is classical bit where would you draw your vector that represents classical state?
- (c) If the vector makes an angle 45 degree with the x -axis, how would you represent the qubit state mathematically?
- (d) If the vector makes an angle of 135 degrees with x -axis, how would your response to (c) change?

Solution 2.

- (a) There are an infinite number of possible states as the cardinality of $[0, 1]$ —the set of possible values for the components of the state vector—is the same as the cardinality of \mathbb{R} . This can be shown several ways, one of which is through the bijection established somewhat informally by the fact that $[0, 1]$ clearly injects into \mathbb{R} as $[0, 1] \subset \mathbb{R}$ and $x \mapsto \frac{2[\arctan(x) + \frac{\pi}{2}]}{\pi}$ which injects \mathbb{R} into $[0, 1]$ by virtue of the periodic nature of \arctan .
- (b) Here the blue vectors represent classical states 0 and 1 and the green vector represents some arbitrary quantum state



$$(c) \vec{S} = \left(\cos\left(\frac{\pi}{4}\right), \sin\left(\frac{\pi}{4}\right) \right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$$

$$(d) \vec{S} = \left(\cos\left(\frac{3\pi}{4}\right), \sin\left(\frac{3\pi}{4}\right) \right) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right) \text{ which is perfectly fine as we end up squaring the magnitudes of the components when talking about probability.}$$

Problem 3.

- (a) Can a scalar be a complex number?
- (b) Given entries i , $-i$, $1+i$ create a column vector A and find
- A^\dagger
 - $A^\dagger A$
- (c) Which of the following matrices are unitary matrices?

$$\begin{bmatrix} 1 & 1-i \\ 1+i & 4 \end{bmatrix} \quad \frac{1}{5} \begin{bmatrix} -1+2i & -4-2i \\ 2-4i & -2-i \end{bmatrix}$$

- (d) Can the following matrices be Hermitian? Why?

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \frac{1}{5} \begin{bmatrix} 1 & 2-i & 3-i \\ 2+i & 3 & 1+2i \\ 3+i & 1-2i & 7 \end{bmatrix}$$

Solution 3.

- (a) Yes. Just as any element of $\mathbb{Z}, \mathbb{Q}, \mathbb{R} \dots$ is a scalar, so is any element of \mathbb{C} .

$$(b) A = \begin{bmatrix} i \\ -i \\ 1+i \end{bmatrix}$$

$$(i) A^\dagger = (A^T)^* = [-i \quad i \quad 1-i]$$

$$(ii) A^\dagger A = [-i \quad i \quad 1-i] \begin{bmatrix} i \\ -i \\ 1+i \end{bmatrix} = [1 \quad 1 \quad 2]$$

- (c) Which of the following matrices are unitary matrices?

$$\begin{bmatrix} 1 & 1-i \\ 1+i & 4 \end{bmatrix} \begin{bmatrix} 1 & 1+i \\ 1-i & 4 \end{bmatrix} = \begin{bmatrix} 1-2i & 5-3i \\ 3+i & 16+2i \end{bmatrix} \neq I_2 \therefore \text{not unitary}$$

$$\begin{bmatrix} -1+2i & -4-2i \\ 2-4i & -2-i \end{bmatrix} \begin{bmatrix} -1-2i & -4+2i \\ 2+4i & -2+i \end{bmatrix} = \begin{bmatrix} 5-20i & 10-10i \\ -10-10i & 5+20i \end{bmatrix} \neq I_2 \therefore \text{not unitary}$$

- (d) Both matrices are, by inspection, Hermitian as both are square and have mirrored-conjugate non-diagonal elements which means $A = A^\dagger$.

Problem 4. Find the eigenvalues and eigenvectors of

(a)

$$\begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Solution 4.

(a) Eigenvalues:

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det\left(\begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \begin{vmatrix} -\lambda & i \\ 1 & -\lambda \end{vmatrix} \\ &= \lambda^2 - i \\ \implies \lambda &= \pm\sqrt{i} \end{aligned}$$

Eigenvectors:

$$\begin{aligned} \begin{bmatrix} -\sqrt{i} & i & 0 \\ 1 & -\sqrt{i} & 0 \end{bmatrix} &\sim \begin{bmatrix} -1 & \sqrt{i} & 0 \\ 1 & -\sqrt{i} & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & \sqrt{i} & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \vec{v}_1 = \begin{bmatrix} \sqrt{i} \\ 1 \end{bmatrix} \\ \begin{bmatrix} \sqrt{i} & i & 0 \\ 1 & \sqrt{i} & 0 \end{bmatrix} &\sim \begin{bmatrix} 1 & \sqrt{i} & 0 \\ 1 & -\sqrt{i} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \sqrt{i} & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \vec{v}_2 = \begin{bmatrix} -\sqrt{i} \\ 1 \end{bmatrix} \end{aligned}$$

(b) Eigenvalues:

$$\begin{aligned} 0 &= \det\left(\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \begin{vmatrix} -\lambda & i \\ i & -\lambda \end{vmatrix} \\ &= \lambda^2 + 1 \\ \implies \lambda &= \pm i \end{aligned}$$

Eigenvectors

$$\begin{aligned} \begin{bmatrix} -i & i & 0 \\ i & -i & 0 \end{bmatrix} &\sim \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} i & i & 0 \\ i & i & 0 \end{bmatrix} &\sim \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$