Math 2150: Assignment II

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Note: I've kept p, q, and r throughout my solutions and only substituted the actual numbers in at the end. This is because I find it easier, especially when dealing with things that might cancel nicely, to deal with variables rather than the numbers they represent. In my case my student number is 0805980 so $p = 9$, $q = 5$, and $r = 22$.

Problem 1.

(a) Determine $\mathcal{L}^{-1}\left\{\frac{(s-2)}{(s^2+1)(s^2-qs)}\right\}$

(b) Determine $\mathcal{L}^{-1}\left\{\frac{2s+1}{(s^2+4)(s^2+q^2)}\right\}$

(c) Determine $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s-q)}\right\}$

(d) Determine $\mathcal{L}\left\{t\sin^2(qt)\right\}$

(e) Determine $\mathcal{L}\left\{e^{-qt}\sin(pt)\sin(qt)\right\}$

(f) Determine $\mathcal{L}\left\{e^{-qt}t\sin\left(t-\frac{\pi}{6}\right)\right\}$

(g) Determine $\mathcal{L}^{-1}\left\{\ln\left(\frac{7+ps}{9+qs}\right)\right\}$

Solution 1.

(a)

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left\{\frac{(s-2)}{s(s^2+1)(s-q)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s-q)}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)(s-q)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{A_1s+B_1}{(s^2+1)} + \frac{C_1}{(s-q)}\right\} - 2\mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{Bs+C}{(s^2+1)} + \frac{D}{(s-q)}\right\}
\end{aligned}
$$

Which I'll solve seperately and then recombine

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left\{\frac{A_1s+B_1}{(s^2+1)} + \frac{C_1}{(s-q)}\right\} \\
&\Rightarrow A_1s(s-q) + B_1(s-q) + C_1\left(s^2+1\right) = 1 \\
&\Rightarrow s = q \implies C_1\left(q^2+1\right) = 1 \implies C_1 = \frac{1}{q^2+1} \\
&\Rightarrow s = 0 \implies B_1(-q) + \frac{1}{q^2+1} = 1 \implies B_1 = \frac{1-\frac{1}{q^2+1}}{-q} \\
&\Rightarrow s = 1 \implies A_1(1-q) + \frac{(1-q)\left(1-\frac{1}{q^2+1}\right)}{-q} + \frac{2}{q^2+1} = 1 \implies A_1 = \frac{1-\frac{(1-q)\left(1-\frac{1}{q^2+1}\right)}{-q} - \frac{2}{q^2+1}}{(1-q)} \\
&\Rightarrow F_1(s) = A_1\cos(t) + B_1\sin(t) + C_1e^{qt}
\end{aligned}
$$

And

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left\{\frac{A}{s}+\frac{Bs+C}{(s^2+1)}+\frac{D}{(s-q)}\right\} \\
&\Rightarrow A\left(s^2+1\right)(s-q)+Bs(s)(s-q)+C(s)(s-q)+D(s)\left(s^2+1\right)=1 \\
&\Rightarrow s=q \implies D(q)\left(q^2+1\right)=1 \implies D=\frac{1}{q\left(q^2+1\right)} \\
&\Rightarrow s=0 \implies A(-q)=1 \implies A=\frac{1}{-q} \\
&\Rightarrow A\left(s^2+1\right)(s-q)+Bs(s)(s-q)+C(s)(s-q)+D(s)\left(s^2+1\right)=1 \\
&\implies As^3-Aqs^2+As-Aq+Bs^3-Bqs^2+Cs^2-Cqs+Ds^3+Ds=1 \\
&\implies A+B+D=0,\ -Aq-Bq+C=0,\ A-Cq+D=0,\ -Aq=1 \\
&\implies B=\frac{1}{q}-\frac{1}{q\left(q^2+1\right)},\ C=-1+q\left(\frac{1}{q}-\frac{1}{q\left(q^2+1\right)}\right) \\
&\Rightarrow F_2(s)=A+De^{qt}+C\sin(t)+B\cos(t)
\end{aligned}
$$

So,

$$
\begin{aligned}
&F(s)=F_1(s)-2(F_2(s)) \\
&=A_1\cos(t)+B_1\sin(t)+C_1e^{qt}-2A-2De^{qt}-2C\sin(t)-2B\cos(t) \\
&=(A_1-2B)\cos(t)+(B_1-2C)\sin(t)+(C_1-2D)\,e^{qt}-2A \\
&=\left(\frac{1-\frac{(1-q)\left(1-\frac{1}{q^2+1}\right)}{-q}-\frac{2}{q^2+1}}{(1-q)}-2\frac{1}{q}+2\frac{1}{q\left(q^2+1\right)}\right)\cos(t) \\
&+\left(\frac{1-\frac{1}{q^2+1}}{-q}+2-2q\left(\frac{1}{q}-\frac{1}{q\left(q^2+1\right)}\right)\right)\sin(t) \\
&+\left(\frac{1}{q^2+1}-2\frac{1}{q\left(q^2+1\right)}\right)e^{qt}+\frac{2}{q} \\
&=-\frac{11}{26}\cos(t)-\frac{3}{26}\sin(t)+\frac{3}{130}e^{5t}+\frac{2}{5}
\end{aligned}
$$

(b)

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left\{\frac{2s+1}{\left(s^2+4\right)\left(s^2+q^2\right)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{2s}{\left(s^2+4\right)\left(s^2+q^2\right)}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2+4\right)\left(s^2+q^2\right)}\right\}
\end{aligned}
$$

Which I will again solve seperately and then add

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left\{\frac{2s}{(s^2+4)(s^2+q^2)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{A_1 s + B_1}{(s^2+4)} + \frac{C_1 s + D_1}{(s^2+q^2)}\right\} \\
&\Rightarrow (A_1 s + B_1)\left(s^2+q^2\right) + (C_1 s + D_1)\left(s^2+4\right) = 2s \\
&\implies A_1 s^3 + A_1 q^2 s + B_1 s^2 + B_1 q^2 + C_1 s^3 + 4C_1 s + D_1 s^2 + 4D_1 = 2s \\
&\implies A_1 + C_1 = 0 \implies A_1 = -C_1 \\
&\implies B_1 + D_1 = 0 \implies B_1 = -D_1 \\
&\implies A_1 q^2 + 4C_1 = 2 \implies A_1 = \frac{2}{q^2-4} \implies C_1 = -\frac{2}{q^2-4} \\
&\implies B_1 q^2 + 4D_1 = 0 \implies B_1, D_1 = 0 \\
&\Rightarrow F_1(s) = A_1 \cos(2t) + C_1 \cos(qt)
\end{aligned}
$$

And

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)(s^2+q^2)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{As + B}{(s^2+4)} + \frac{Cs + D}{(s^2+q^2)}\right\} \\
&\Rightarrow (As + B)\left(s^2+q^2\right) + (Cs + D)\left(s^2+4\right) = 1 \\
&\implies As^3 + Aq^2 s + Bs^2 + Bq^2 + Cs^3 + 4Cs + Ds^2 + 4D = 1 \\
&\implies A + C = 0 \implies A = -C \\
&\implies B + D = 0 \implies B = -D \\
&\implies Aq^2 + 4C = 0 \implies A, C = 0 \\
&\implies Bq^2 + 4D = 1 \implies B = \frac{1}{q^2-4} \implies D = -\frac{1}{q^2-4} \\
&\Rightarrow \mathcal{L}^{-1}\left\{\frac{B}{(s^2+4)} + \frac{D}{(s^2+q^2)}\right\} \\
&\Rightarrow \frac{B}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s^2+4)}\right\} + \frac{D}{q}\mathcal{L}^{-1}\left\{\frac{q}{(s^2+q^2)}\right\} \\
&\Rightarrow F_2(s) = \frac{B}{2}\sin(2t) + \frac{D}{q}\sin(qt)
\end{aligned}
$$

So,

$$
\begin{aligned}
F(s) &= F_1(s) + F_2(s) \\
&= A_1 \cos(2t) + C_1 \cos(qt) + \frac{B}{2}\sin(2t) + \frac{D}{q}\sin(qt) \\
&= \frac{2}{q^2-4}\cos(2t) - \frac{2}{q^2-4}\cos(qt) + \frac{\frac{1}{q^2-4}}{2}\sin(2t) + \frac{-\frac{1}{q^2-4}}{q}\sin(qt) \\
&= \frac{2}{21}\cos(2t) - \frac{2}{21}\cos(5t) + \frac{1}{42}\sin(2t) - \frac{1}{105}\sin(5t)
\end{aligned}
$$

(c) Determine $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s-q)}\right\}$

(d) Determine $\mathcal{L}\left\{t\sin^2(qt)\right\}$

(e) Determine $\mathcal{L}\left\{e^{-qt}\sin(pt)\sin(qt)\right\}$

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