

Calculus Cheat Sheet

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Revision 3

Limits

Existence

$\lim_{x \rightarrow a} f(x) = L$ exists if $\forall \epsilon > 0 \exists \delta > 0$ s.t. $|x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$.

Properties

$$\begin{aligned}\lim_{x \rightarrow a} [cf(x)] &= c \lim_{x \rightarrow a} [f(x)] \\ \lim_{x \rightarrow a} [f(x) \pm g(x)] &= \lim_{x \rightarrow a} [f(x)] \pm \lim_{x \rightarrow a} [g(x)] \\ \lim_{x \rightarrow a} [f(x)g(x)] &= \lim_{x \rightarrow a} [f(x)] \lim_{x \rightarrow a} [g(x)] \\ \lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] &= \frac{\lim_{x \rightarrow a} [f(x)]}{\lim_{x \rightarrow a} [g(x)]}, \lim_{x \rightarrow a} [g(x)] \neq 0\end{aligned}$$

Evaluation Techniques

Basics

$$\lim_{x \rightarrow a} [f(x)] = f(a) \text{ if } f \text{ exists at } a$$

L'Hôpital's Rule

If $\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ then

$$\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \lim_{x \rightarrow a} \left[\frac{f'(x)}{g'(x)} \right]$$

Factoring at Infinity

If $p(x)$ and $q(x)$ are polynomials, to evaluate $\lim_{x \rightarrow \pm\infty} \left[\frac{p(x)}{q(x)} \right]$ factor the greatest power of x in $q(x)$ (the denominator) out of both $p(x)$ and $q(x)$ then compute the limit,

$$\text{e.g. } \lim_{x \rightarrow -\infty} \left[\frac{3x^2 - 4}{5x - 2x^2} \right] = \lim_{x \rightarrow -\infty} \left[\frac{\cancel{x}^2(3 - \frac{4}{x^2})}{\cancel{x}^2(\frac{5}{x} - 2)} \right] = \frac{3 - 0}{0 - 2} = -\frac{3}{2}$$

Derivatives

Definition

$$\frac{d}{dx} f(x) = f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Techniques

Sum Rule

$$\frac{d}{dx} (f(x) \pm g(x)) = f'(x) \pm g'(x)$$

Product Rule

$$\frac{d}{dx} (f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Quotient Rule

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Power Rule

$$\frac{d}{dx} (x^n) = nx^{n-1}$$

Chain Rule

$$\frac{d}{dx} (f(g(x))) = f'(g(x))g'(x)$$

Parametric

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \text{ recurse } y \text{ for } n\text{th derivative}$$
$$(x, y) + (x', y')t \text{ for parametric tangent}$$

Common Derivatives

$$\begin{aligned}\frac{d}{dx} (x) &= 1 \quad \frac{d}{dx} (\sin x) = \cos x \\ \frac{d}{dx} (\cos x) &= -\sin x \quad \frac{d}{dx} (\tan x) = \sec^2 x \\ \frac{d}{dx} (\sec x) &= \sec x \tan x \\ \frac{d}{dx} (\csc x) &= -\csc x \cot x \\ \frac{d}{dx} (\cot x) &= -\csc^2 x \\ \frac{d}{dx} (\sin^{-1} x) &= \frac{1}{\sqrt{1-x^2}} \sin^{-1} x \neq \frac{1}{\sin x} \\ \frac{d}{dx} (\cos^{-1} x) &= -\frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx} (\tan^{-1} x) &= \frac{1}{1+x^2} \quad \frac{d}{dx} (a^x) = a^x \ln a \\ \frac{d}{dx} (e^x) &= e^x \quad \frac{d}{dx} (\ln x) = \frac{1}{x}, x > 0 \\ \frac{d}{dx} (\ln |x|) &= \frac{1}{x}, x \neq 0 \quad \frac{d}{dx} (\log_a x) = \frac{1}{x \ln a}\end{aligned}$$

Common Chain Rule Derivatives

$$\begin{aligned}\frac{d}{dx} ([f(x)]^n) &= n[f(x)]^{n-1} f'(x) \\ \frac{d}{dx} (e^{f(x)}) &= e^{f(x)} f'(x) \\ \frac{d}{dx} (\ln[f(x)]) &= \frac{f'(x)}{f(x)} \\ \frac{d}{dx} (\sin[f(x)]) &= f'(x) \cos[f(x)] \\ \frac{d}{dx} (\cos[f(x)]) &= -f'(x) \sin[f(x)] \\ \frac{d}{dx} (\tan[f(x)]) &= f'(x) \sec^2[f(x)] \\ \text{Trig derivatives, same old same old} \\ \frac{d}{dx} (f(x)^{g(x)}) &= \frac{g(x)f'(x)}{f(x)} + \ln[f(x)]g'(x)\end{aligned}$$

Integrals

Definition

The integral of some function $f(x)$ is a function $f^*(x)$ s.t. $f^{*'}(x) = f(x)$. **Don't forget your constant!** The integral can be done using the Right Riemann Sum:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\frac{b-a}{n} \cdot f\left(a + i \cdot \frac{b-a}{n}\right) \right]$$

U-Substitution

$$\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(u) du. \text{ Using}$$

U-Substitution, $u = g(x)$ and

$$du = g'(x)dx \quad (dx = \frac{du}{g'})$$

Integration By Parts

$$\begin{aligned}\int u(x)v'(x) dx &= \\ u(x)v(x) - \int u'(x)v(x) dx &\text{ and } \\ \int_a^b u(x)v'(x) dx &= uv|_a^b - \int_a^b u'(x)v(x) dx.\end{aligned}$$

Common Integrals

$$\begin{aligned}\int x^n dx &= \frac{1}{n+1} x^{n+1} + C \\ \int \frac{1}{x} dx &= \ln|x| + C \\ \int \frac{1}{ax+b} dx &= \frac{1}{a} \ln|ax+b| + C \\ \int \ln x dx &= x \ln x - x + C \\ \int e^x dx &= e^x + C \\ \int a^x dx &= \frac{a^x}{\ln a} + C \\ \int \sin x dx &= -\cos x + C \\ \int \cos x dx &= \sin x + C \\ \int \tan x dx &= \ln|\sec x| + C \\ \int \sec^2 x dx &= \tan x + C \\ \int \sec x \tan x dx &= \sec x + C \\ \int \csc x \cot x dx &= -\csc x + C \\ \int \csc^2 x dx &= -\cot x + C \\ \int \sec x dx &= \ln|\sec x + \tan x| + C \\ \int \frac{1}{a^2+u^2} dx &= \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C \\ \int \frac{1}{\sqrt{a^2+u^2}} dx &= \sin^{-1}\left(\frac{x}{a}\right) + C\end{aligned}$$

Trig Reduction Formulae

(For $n \geq 2$)

$$\begin{aligned}\int \sin^n(x) dx &= \\ -\frac{1}{n} \sin^{n-1}(x) \cos(x) &+ \frac{n-1}{n} \int \sin^{n-2}(x) dx \\ \int \cos^n(x) dx &= \\ \frac{1}{n} \cos^{n-1}(x) \sin(x) &+ \frac{n-1}{n} \int \cos^{n-2}(x) dx \\ \int \tan^n(x) dx &= \\ \frac{1}{n-1} \tan^{n-1}(x) &+ \frac{n-1}{n} \int \tan^{n-2}(x) dx \\ \int \sec^n(x) dx &= \frac{1}{n-1} \sec^{n-2}(x) \tan(x) + \\ \frac{n-2}{n-1} \int \sec^{n-2}(x) dx &\end{aligned}$$

Trig Identities

$$\begin{aligned}\sin^2(x) + \cos^2(x) &= 1 \\ 1 + \tan^2(x) &= \sec^2(x) \\ \sin(2x) &= 2 \sin(x) \cos(x) \\ \cos(2x) &= \cos^2(x) - \sin^2(x)\end{aligned}$$

$$\begin{aligned}&= 2 \cos^2(x) - 1 \\ &= 1 - 2 \sin^2(x)\end{aligned}$$

Trig Substitutions

$$\begin{aligned}\sqrt{a^2 - x^2} \rightsquigarrow x &= a \sin(\theta) \\ \sqrt{a^2 + x^2} \rightsquigarrow x &= a \tan(\theta) \\ \sqrt{x^2 - a^2} \rightsquigarrow x &= a \sec(\theta)\end{aligned}$$

Partial Fractions & Polynomial Division

If the degree of the numerator is greater than that of the denominator, use polynomial division to divide the denominator into the numerator (denom) numer), then integrate the result. If the degree of the denominator is greater than that of the numerator, use partial fractions to decompose the integral as follows:

If the denominator contains different linear terms, break it down to $\frac{A}{ax+b}$

If it contains a repeated linear term

$((ax+b)^2)$, break it down to

$$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$$

If it contains an irreducible quadratic

$(x^2 + bx + c)$, break it down to

$$\frac{A}{dx+o} + \frac{Bx+C}{x^2+bx+c}$$

Then, set up equations for the coefficients of the powers of x in the numerator and solve them to determine A, B, C and so on.

Applications

Centroids

$$C = (\hat{x}, \hat{y}), \hat{x} = \frac{\int x f(x) dx}{\int f(x) dx} \text{ \& \& } \hat{y} = \frac{\int y f(y) dy}{\int f(y) dy}$$

Arc Length

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt$$

for parametric. Area is just this times $2\pi x$ or y for opposite of axis.

Volume

Use the variable parallel to the axis of revolution. $V = \int_a^b A(x) dx$ where $A(x)$ is a function which gives the area of one "slice" of the solid. Slices are generally

circular and may have holes in them, area of a circle is πr^2 , and the area of a washer is $\pi(r_{outer}^2 - r_{inner}^2)$

Sequences and Series

Series With Nice Formulae

The geometric series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ provided that $|r| < 1$ The telescoping series where the “inbetween” terms cancel. Take the limit as $k \rightarrow \infty$

Basic Divergence Test

If for some series $\sum_{n=0}^{\infty} a_n$ the limit $\lim_{n \rightarrow \infty} a_n \neq 0$ the series diverges. However, if the limit is 0, no information can be determined from this test.

Integral Test

If some function $f(n) = a_n$ is decreasing on $[c, \infty)$ then the series $\sum_{n=c}^{\infty} a_n$ converges exactly as $\int_c^{\infty} f(x)dx$ does / does not. This works because $\int_c^{\infty} f(x)dx \leq \sum_{n=c}^{\infty} a_n \leq f(c) + \int_c^{\infty} f(x)dx$

P-Test

Generally, $\sum_{n=c}^{\infty} \frac{an^k + \dots + a_0}{bn^l + \dots + b_0}$ converges if $p = l - k < 1$ and diverges if $p \geq 1$

Basic Comparison Test

If two sequences $\{a_n\}$ and $\{b_n\}$ exist, are comprised of positive terms, and satisfy $0 < a_n \leq b_n$ past some point then

- (a) if $\sum_{n=0}^{\infty} b_n$ converges, so does $\sum_{n=0}^{\infty} a_n$
- (b) if $\sum_{n=0}^{\infty} a_n$ diverges, so does $\sum_{n=0}^{\infty} b_n$

Limit Comparison Test

If two sequences $\{a_n\}$ and $\{b_n\}$ exist and are comprised of positive terms past some point then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$

- (a) if $c > 0$ both series either converge or diverge (one converging means the other does and vice versa)
- (b) if $c = 0$ then a_n diverges so does b_n and if b_n converges so does a_n
- (c) if $c = \infty$ then a_n converges so does b_n and if b_n diverges so does a_n

Alternating Series Test

If each a_n in the series $\sum_{n=0}^{\infty} a_n$ and

- (1) $|a_{n+1}| < |a_n|$ (The series is decreasing)

- (2) $a_{n+1} < 0$ & $a_n > 0$ (The series is alternating)

- (3) $\lim_{n \rightarrow \infty} |a_n| = 0$

then the series converges

Ratio Test

If past some point $a_n \neq 0$ then for the series $\sum_{n=0}^{\infty} a_n$ if the limit $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ then

- (a) if $L < 1$ the series converges absolutely
- (b) if $L = 1$ no information can be obtained through this test
- (c) if $L > 1$ the series diverges

Root Test

For the series $\sum_{n=0}^{\infty} a_n$ if $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$ then

- (a) if $L < 1$ the series converges absolutely
- (b) if $L = 1$ no information can be obtained through this test
- (c) if $L > 1$ the series diverges

Taylor Series

Taylor's formula states that if some function $f(x)$ can be expanded as a power series around $x = a$ its representation is given by

$$\sum_{n=0}^{\infty} \frac{\frac{d^n}{dx^n} f(a)}{n!} (x - a)^n$$

Remainder Terms

If there exists some function $f(x)$ whose Taylor Series is defined as $T_n(x)$ then the remainder term is defined as $R_n(x) = f(x) - T_n(x)$, which is how “far off” the Taylor Series $T_n(x)$ is from the function $f(x)$ for some n .

Tidbits

\vec{v} here is the velocity vector (duh)

Unit tangent vector $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$

Curvature $\kappa = \left| \frac{d\vec{T}}{ds} \right| = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$

Normal vector $\vec{N} = \frac{1}{\kappa} \frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$