Calculus II: Assignment 3

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Consider the region below the curve $y = \frac{1}{x}$ and above the x-axis for $1 \leq x < \infty$.

Problem 1. Compute the area of the given region, both by hand and using SageMath.

Solution 1.

$$
\begin{aligned}
&= \int_1^\infty \frac{1}{x}\,dx \\
&= \lim_{n\to\infty}\left[\int_1^n \frac{1}{x}\,dx\right] \\
&= \lim_{n\to\infty}\left[\ln(x)\Big|_1^n\right] \\
&= \lim_{n\to\infty}\left[\ln(n) - \ln(1)\right] \\
&= \lim_{n\to\infty} \ln(n) \\
&= +\infty
\end{aligned}
$$

Using sage to evaluate:

```
[1]: clear_vars()
     from sage.symbolic.integration.integral import definite_integral # Necessary in some versions
     ↪  for definite_integral()

     x = var('x')
     t = var('n')

     f = 1/x

     assume(n-1>0)

     limit(definite_integral(f, x, 1, n), n=oo)
```

```
[1]: +Infinity
```

$$\therefore \text{ the area of } \frac{1}{x} \text{ on } [1, \infty) \text{ is } \infty$$

Problem 2. Compute the volume of the solid obtained by revolving the given region about the x-axis, both by hand and using SageMath.

Solution 2.

$$
\begin{aligned}
&= \pi \int_1^\infty \frac{1}{x^2}\,dx \\
&= \lim_{n\to\infty} \left[\pi \int_1^n \frac{1}{x^2}\,dx\right] \\
&= \lim_{n\to\infty} \left[-\pi\, \frac{1}{x}\bigg|_n^1\right] \\
&= -\pi \lim_{n\to\infty} \left[\frac{1}{x}\bigg|_n^1\right] \\
&= -\pi \lim_{n\to\infty} \left[\frac{1}{n} - \frac{1}{1}\right] \\
&= -\pi \lim_{n\to\infty} \frac{1}{n} - 1 \\
&= \pi
\end{aligned}
$$

Using sage to evaluate:

[2]:
```
clear_vars()
from sage.symbolic.integration.integral import definite_integral # Necessary in some versions
↪ for definite_integral()

x = var('x')
n = var('n')

f = 1/x

assume(n-1>0)

limit(definite_integral(pi*f^2, x, 1, n), n=oo)
```

[2]:
```
pi
```

$$\therefore \text{ the volume of } \frac{1}{x} \text{ on } [1,\infty) \text{ rotated about the } x\text{-axis is } \pi$$

Problem 3. There is something a little paradoxical about the (correct :-) answers to 1 and 2. What is the paradox? Explain what's going on as best you can.

Solution 3. The area under $\frac{1}{x}$ (on $[1,\infty)$, implied) is infinite, but the volume of the solid of revolution about the x-axis is finite, π in this case. This seems paradoxical as rotating an infinite area about the x-axis should give you an infinite volume. I think the reason this doesn't occur is because $\int \frac{1}{x}\,dx = \ln(x)$, which is strictly increasing (though very slowly, as $x \to \infty$, $\ln(x) \to \infty$), whereas $\int \frac{1}{x^2}\,dx = -\frac{1}{x}$, which is strictly *decreasing* (again though very slowly, as $x \to \infty$, $-\frac{1}{x} \to \infty$, though $-\frac{1}{x}$ will never equal 0). This means that though the area is infinite, the amount added to the volume decays ("falls off") quick enough that it approaches a finite value, which is a little weird sounding but I think it makes sense.

Woohoo, finally figured out how to do nice embedding of sage code in LaTeX, now to figure out plotting solids of revolution