Physics 2605H: Assignment I

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Problem 1.

(a) Find the real part and imaginary of the complex number $\frac{5+3i}{1+5i}$ and express your answer in polar form.

(b) Consider the product of two complex numbers (1+3i) and (5+4i). Find the complex conjugate in two ways: (i) Take the complex conjugates before and after the multiplication and show that they are the same.

Solution 1. (a) $\frac{5+3i}{1+5i} = \frac{5+3i}{1+5i}\frac{1-5i}{1-5i} = \frac{10}{13} - \frac{11}{13}i \approx \frac{\sqrt{221}}{13}e^{-0.83i}$

(b)

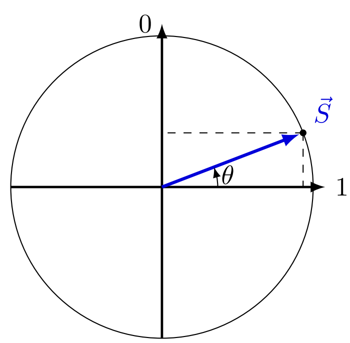
$$
\begin{aligned}
&= (1+3i)^* \cdot (5+4i)^* \\
&= (1-3i) \cdot (5-4i) \\
&= -7-19i
\end{aligned}
$$

Then,

$$
\begin{aligned}
&= ((1+3i) \cdot (5+4i))^* \\
&= (-7+19i)^* \\
&= -7-19i
\end{aligned}
$$

So conjugates are distributive over multiplication.

Problem 2. The figure shows the general representation of a qubit



(a) How many states are possible for the qubit to be in?

(b) If it is classical bit where would you draw your vector that represents classical state?

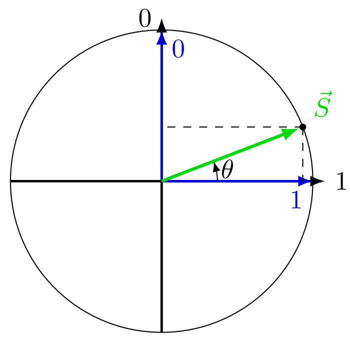
(c) If the vector makes an angle 45 degree with the x-axis, how would you represent the qubit state mathematically?

(d) If the vector makes an angle of 135 degrees with x-axis, how would your response to (c) change?

Solution 2.

(a) There are an infinite number of possible states as the cardinality of $[0,1]$—the set of possible values for the components of the state vector—is the same as the cardinality of $\mathbb{R}$. This can be shown several ways, one of which is through the bijection established somewhat informally by the fact that $[0,1]$ clearly injects into $\mathbb{R}$ as $[0,1] \subset \mathbb{R}$ and $x \mapsto \frac{2\left[\arctan(x)+\frac{\pi}{2}\right]}{\pi}$ which injects $\mathbb{R}$ into $[0,1]$ by virtue of the periodic nature of arctan.

(b) Here the blue vectors represent classical states 0 and 1 and the green vector represents some arbitrary quantum state



(c) $\vec{S} = \left(\cos\left(\frac{\pi}{4}\right), \sin\left(\frac{\pi}{4}\right)\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

(d) $\vec{S} = \left(\cos\left(\frac{3\pi}{4}\right), \sin\left(\frac{3\pi}{4}\right)\right) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ which is perfectly fine as we end up squaring the magnitudes of the components when talking about probability.

Problem 3.

(a) Can a scalar be a complex number?

(b) Given entries i , -i, 1+i create a column vector A and find

(i) $A^\dagger$

(ii) $A^\dagger A$

(c) Which of the following matrices are unitary matrices?

$$\begin{bmatrix} 1 & 1-i \\ 1+i & 4 \end{bmatrix} \qquad \frac{1}{5}\begin{bmatrix} -1+2i & -4-2i \\ 2-4i & -2-i \end{bmatrix}$$

(d) Can the following matrices be Hermitian? Why?

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \frac{1}{5}\begin{bmatrix} 1 & 2-i & 3-i \\ 2+i & 3 & 1+2i \\ 3+i & 1-2i & 7 \end{bmatrix}$$

Solution 3.

(a) Yes. Just as any element of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}\ldots$ is a scalar, so is any element of $\mathbb{C}$.

(b) $A = \begin{bmatrix} i \\ -i \\ 1+i \end{bmatrix}$

(i) $A^\dagger = (A^T)^* = \begin{bmatrix} -i & i & 1-i \end{bmatrix}$

(ii) $A^\dagger A = \begin{bmatrix} -i & i & 1-i \end{bmatrix} \begin{bmatrix} i \\ -i \\ 1+i \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$

(c) Which of the following matrices are unitary matrices?

$$\begin{bmatrix} 1 & 1-i \\ 1+i & 4 \end{bmatrix} \begin{bmatrix} 1 & 1+i \\ 1-i & 4 \end{bmatrix} = \begin{bmatrix} 1-2i & 5-3i \\ 3+i & 16+2i \end{bmatrix} \neq I_2 \therefore \text{not unitary}$$

$$\begin{bmatrix} -1+2i & -4-2i \\ 2-4i & -2-i \end{bmatrix} \begin{bmatrix} -1-2i & -4+2i \\ 2+4i & -2+i \end{bmatrix} = \begin{bmatrix} 5-20i & 10-10i \\ -10-10i & 5+20i \end{bmatrix} \neq I_2 \therefore \text{not unitary}$$

(d) Both matrices are, by inspection, Hermitian as both are square and have mirrored-conjugate non-diagonal elements which means $A = A^\dagger$.

Problem 4. Find the eigenvalues and eigenvectors of

(a)

$$\begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Solution 4.

(a) Eigenvalues:

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det\left(\begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = \begin{vmatrix} -\lambda & i \\ 1 & -\lambda \end{vmatrix} \\ &= \lambda^2 - i \\ \implies \lambda &= \pm\sqrt{i} \end{aligned}$$

Eigenvectors:

$$\begin{bmatrix} -\sqrt{i} & i & 0 \\ 1 & -\sqrt{i} & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & \sqrt{i} & 0 \\ 1 & -\sqrt{i} & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & \sqrt{i} & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \vec{v}_1 = \begin{bmatrix} \sqrt{i} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{i} & i & 0 \\ 1 & \sqrt{i} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \sqrt{i} & 0 \\ 1 & -\sqrt{i} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \sqrt{i} & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \vec{v}_2 = \begin{bmatrix} -\sqrt{i} \\ 1 \end{bmatrix}$$

(b) Eigenvalues:

$$\begin{aligned} 0 &= \det\left(\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = \begin{vmatrix} -\lambda & i \\ i & -\lambda \end{vmatrix} \\ &= \lambda^2 + 1 \\ \implies \lambda &= \pm i \end{aligned}$$

Eigenvectors

$$\begin{bmatrix} -i & i & 0 \\ i & -i & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} i & i & 0 \\ i & i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$