Calculus II: Assignment 4

Jeremy Favro

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Consider the region below the curve $y = \frac{1}{x}$ and above the x-axis for $1 \leq x < \infty$.

Problem 1. Compute each of the following as best you can using SageMath.

Solution 1.

$$\int_1^\infty \frac{1}{x}\, dx$$

[1]:
```
clear_vars()
from sage.symbolic.integration.integral import definite_integral

x = var('x')
n = var('n')

f = 1/x

assume(n-1>0)

definite_integral(f, x, 1, oo)
```

[1]:
```
ValueError: Integral is divergent.
```

$$\sum_{n=1}^\infty \frac{1}{x}$$

[2]:
```
clear_vars()
from sage.symbolic.integration.integral import definite_integral

x = var('x')
n = var('n')

f = 1/x

sum(f, x, 1, oo)
```

[2]:
```
ValueError: Sum is divergent.
```

$$\int_1^\infty \frac{1}{x^2}\,dx$$

[3]:
```
clear_vars()
from sage.symbolic.integration.integral import definite_integral

x = var('x')
n = var('n')

f = 1/x^2

assume(n-1>0)

limit(definite_integral(f, x, 1, n), n=oo)
```

[3]: 1

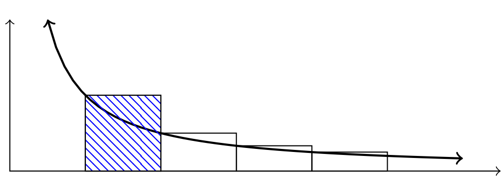
$$\sum_{n=1}^{\infty} \frac{1}{x^2}$$

[4]:
```
clear_vars()
from sage.symbolic.integration.integral import definite_integral

x = var('x')
n = var('n')

f = 1/x^2

sum(f, x, 1, oo) # Not quite sure why but this gives a nicer output than using a limit
```

[4]: 1/6*pi^2

Problem 2. Explain why the sum $\sum_{n=1}^{\infty} \frac{1}{x}$ is what it is because the integral $\int_1^\infty \frac{1}{x}\,dx$ is what it is.

Solution 2.



Looking at the above figure and focusing on a single rectangle, starting at n and ending at $n+1$ we can see that

$$\int_n^{n+1} \frac{1}{x}\,dx < \frac{1}{n}$$ [1]

Which can be extended to the whole curve

$$\sum_{n=1}^{\infty} \left[\int_n^{n+1} \frac{1}{x}\,dx\right] < \sum_{n=1}^{\infty} \frac{1}{n}$$
$$\int_1^\infty \frac{1}{x}\,dx < \sum_{n=1}^{\infty} \frac{1}{n}$$

[1] I think this could also be called less than or equal as they'd approach each other as $\frac{d}{dx}\frac{1}{x} \to 0$, but I'm not sure if it's proper to say that

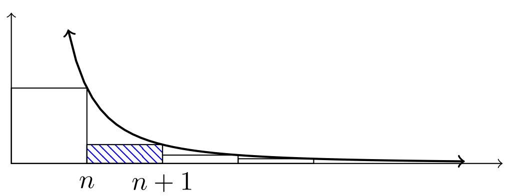
We can then see that the area under the curve given by the integral is less than the area given by the sum. Computing the integral

$$\begin{aligned}
&= \int_1^\infty \frac{1}{x}\,dx \\
&= \lim_{r\to\infty} \int_1^r \frac{1}{x}\,dx \\
&= \lim_{r\to\infty} \ln x \Big|_1^r \\
&= \lim_{r\to\infty} \ln(r) - 0 \\
&= +\infty
\end{aligned}$$

We can now see that because the integral is infinite (Sage says divergent, from my understanding of it in the textbook that's basically the same thing), and the sum is greater than the integral, the sum must also be infinite.

Problem 3. Explain why the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ has a finite value because the integral $\int_1^\infty \frac{1}{x^2}\,dx$ has a finite value.

Solution 3.



We can employ basically the same trick for this problem as the last, we just need to find a sum that gives the rectangles shown in the above figure and can somehow be equated to the given sum. The rectangles shown in the figure have an area of $\frac{1}{(n+1)^2}$, meaning that the sum $\sum_{n=1}^{k} \frac{1}{(n+1)^2}$ can be used to obtain them.

$$\sum_{n=1}^{k} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2}$$

$$\sum_{n=1}^{k} \frac{1}{(n+1)^2} = \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(k+1)^2}$$

So,

$$\sum_{n=1}^{k} \frac{1}{(n+1)^2} = \sum_{n=1}^{k} \frac{1}{n^2} - 1$$

Using the graph to identify that the integral will always be greater[1] than the sum

$$\begin{aligned}
\int_n^{n+1} \frac{1}{x^2}\,dx &> \frac{1}{(n+1)^2} \\
\sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^2}\,dx &> \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \\
\int_1^\infty \frac{1}{x^2}\,dx &> \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}
\end{aligned}$$

But we are looking to explain the finiteness of $\sum_{n=1}^{n} \frac{1}{n^2}$, not $\sum_{n=1}^{n} \frac{1}{(n+1)^2}$. So we need to use the equality determined earlier

$$\int_1^\infty \frac{1}{x^2}\,dx > \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - 1$$

$$\int_1^\infty \frac{1}{x^2}\,dx + 1 > \sum_{n=1}^{\infty} \frac{1}{n^2}$$

We can then see that the area under the curve given by the integral is greater than the area given by the sum. Computing the integral

$$\begin{aligned}
&= \int_1^\infty \frac{1}{x^2}\,dx \\
&= \lim_{r\to\infty} \int_1^r \frac{1}{x^2}\,dx \\
&= \lim_{r\to\infty} -\frac{1}{x}\bigg|_1^r \\
&= \lim_{r\to\infty} 0 + \frac{1}{1} \\
&= 1
\end{aligned}$$

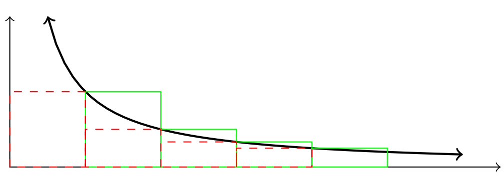
So,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$$

And the sum has an implicit lower bound of zero as it will never start adding negative terms, so, the sum must be finite and inbetween 0 and 2.

Problem 4. Explain why the limit $\lim_{k\to\infty}\left[\sum_{n=1}^{k} \frac{1}{n} - \ln(k+1)\right]$ exists and is between 0 and 1.

Solution 4.



Observing the limit and the figure it becomes relatively clear to see that the limit is just the overestimate created by the rectangles of summation. To begin with defining the bounds for that overestimate, it helps to slightly rejig the limit.

$$\begin{aligned}
&= \lim_{k\to\infty}\left[\sum_{n=1}^{k} \frac{1}{n} - \ln(k+1)\right] \\
&= \lim_{k\to\infty}\left[\sum_{n=1}^{k} \frac{1}{n} - \int_1^{k+1} \frac{1}{x}\,dx\right] \\
&= \lim_{k\to\infty}\left[\sum_{n=1}^{k} \frac{1}{n} - \sum_{n=1}^{k}\int_n^{n+1} \frac{1}{x}\,dx\right] \\
&= \lim_{k\to\infty}\left[\sum_{n=1}^{k} \left(\frac{1}{n} - \int_n^{n+1} \frac{1}{x}\,dx\right)\right]
\end{aligned}$$

We know that for a function that is decreasing on an interval, the Right Riemann Sum underestimates the area under the curve, and the Left Riemann Sum overestimates, creating bounds for the integral

$$\frac{1}{n+1} \leq \int_n^{n+1} \frac{1}{x}\, dx \leq \frac{1}{n}$$

.
The bounds for the integral can then be used to demonstrate that

$$0 < \sum_{n=1}^{k} \left[\frac{1}{n} - \int_n^{n+1} \frac{1}{x}\, dx \right]$$

which establishes our lower bound.

Then, establishing an upper bound uses a similar trick where we show that for any n, the area between the RRS and LRS rectangles ($LRS_n - RRS_n$) is greater than the integral for that interval.

$$\sum_{n=1}^{k} \left[\frac{1}{n} - \int_n^{n+1} \frac{1}{x}\, dx \right] < \sum_{n=1}^{k} \left[\frac{1}{n} - \frac{1}{n+1} \right]$$

Then simplifying and taking the limit of all three sides to determine the actual bounds

$$0 < \sum_{n=1}^{k} \left[\frac{1}{n} - \int_n^{n+1} \frac{1}{x}\, dx \right] < \frac{k}{k+1}$$

$$\lim_{k\to\infty} 0 < \lim_{k\to\infty} \sum_{n=1}^{k} \left[\frac{1}{n} - \int_n^{n+1} \frac{1}{x}\, dx \right] < \lim_{k\to\infty} \frac{k}{k+1}$$

$$0 < \lim_{k\to\infty} \sum_{n=1}^{k} \left[\frac{1}{n} - \int_n^{n+1} \frac{1}{x}\, dx \right] < 1$$

$$0 < \lim_{k\to\infty} \left[\sum_{n=1}^{k} \frac{1}{n} - \ln(k+1) \right] < 1$$

Problem 5. Use SageMath to (approximately) evaluate $\lim_{k\to\infty} \left[\sum_{n=1}^{k} \frac{1}{n} - \ln(k+1) \right]$ as best you can.

Solution 5.

[1]:
```
clear_vars()
from sage.symbolic.integration.integral import definite_integral

k = var('k')
x = var('x')
n = var('n')

N(limit((sum((1/n), n, 1, k))-ln(k+1), k=9999)) # As k -> infty this should get better and
↪ better, but Sage doesn't like that so I've just put a big number
```

[1]: 0.577165664068199