Math 3310H: Assignment V

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December 6, 2025

Problem 1. Find all abelian groups (up to isomorphism) of the given order.

(a) 2025

(b) 1234

Solution 1. (a) Note that $2025 = 3^4 \cdot 5^2$. So we get

$$\mathbb{Z}_{2025}$$
$$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$$
$$\mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$$
$$\mathbb{Z}_{27} \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$$
$$\mathbb{Z}_{81} \times \mathbb{Z}_5 \times \mathbb{Z}_5$$
$$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$$
$$\mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$$
$$\mathbb{Z}_{27} \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$$
$$\mathbb{Z}_9 \times \mathbb{Z}_9 \times \mathbb{Z}_{25}$$
$$\mathbb{Z}_9 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \times \mathbb{Z}_5$$

(b) Note that $1234 = 2 \cdot 617$. So we get just

$$\mathbb{Z}_{1234}.$$

Problem 2. Determine whether or not $U(8)$ is isomorphic to $U(5)$.

Solution 2. Well,

$$U(8) = \{1, 3, 5, 7\}$$

and

$$U(5) = \{1, 2, 3, 4\}$$

so they are both of order 4. They are also both abelian and so must both be isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ respectively. If one is cyclic then it will be $\mathbb{Z}_4$, otherwise it will be $\mathbb{Z}_2 \times \mathbb{Z}_2$. $U(8)$ is not cyclic as none of its elements are of order 8. $U(5)$ is cyclic with generator 2. Since these two are not isomorphic to the same groups (and those groups are not isomorphic) the groups themselves are not isomorphic.

Problem 3. Find all homomorphisms ϕ between the given groups.

(a) $\phi : \mathbb{Z}_6 \to \mathbb{Z}_9$

(b) $\phi : \mathbb{Z}_{18} \to \mathbb{Z}_{12}$

(c) $\phi : \mathbb{Z}_{30} \to \mathbb{Z}_{13}$

(d) $\phi : \mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2$

Solution 3.

(a) We know that a map $\phi : \mathbb{Z}_n \to \mathbb{Z}_m$ where $\phi(x) = kx$ will only be a homomorphism iff $|k|$ in $\mathbb{Z}_n$ is a divisor of both n and m. Here the divisors of the first group are $1, 2, 3, 6$ and the divisors of the second group are $1, 3, 9$. The common divisors are $1, 3$ so we need to find elements of $\mathbb{Z}_6$ with orders 1 or 3. We know that elements k of order 3 will satisfy

$$3 = \frac{6}{\gcd(k, 6)} \implies 2 = \gcd(k, 6).$$

The elements in $\mathbb{Z}_6$ which satisfy this are $2, 4$. Now for 1 we need elements which satisfy $6 = \gcd(k, 6)$ which is just 0. So all the homomorphisms are $\phi_0(x) = 0$, $\phi_2(x) = 2x$, $\phi_4(x) = 4x$.

(b) Following same steps as previously we have common divisors of the order of both groups $1, 2, 3, 6$ and hence we need all elements of these orders in $\mathbb{Z}_{18}$ so

$$6 = \frac{18}{\gcd(k, 18)} \implies k = 3, 15$$

$$3 = \frac{18}{\gcd(k, 18)} \implies k = 6, 12$$

$$2 = \frac{18}{\gcd(k, 18)} \implies k = 9$$

$$1 = \frac{18}{\gcd(k, 18)} \implies k = 0$$

so we have all homomorphisms being all $\phi_k(x) = kx$ where $k \in \{0, 3, 6, 9, 12, 15\}$.

(c) As 13 is prime, the only common divisor here is 1 and the only element of order 1 is zero, hence the only homomorphism here is the trivial one.

(d) Since all the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are of order 1 (the identity) or 2, both of which divide the order of $\mathbb{Z}_4$, all the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ form homomorphisms. So we have $\phi_k(x) = kx$ where $k \in \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$.

Problem 4. Suppose $\phi : \mathbb{Z}_{50} \to \mathbb{Z}_{15}$ is a group homomorphism with $\phi(7) = 6$.

(a) Determine $\phi(x)$ where $x \in \mathbb{Z}_{50}$.

(b) Find $\phi[\mathbb{Z}_{50}]$.

(c) Find $\ker \phi$.

(d) Find $\phi^{-1}(3)$.

Solution 4.

(a) Because $\gcd(7, 50) = 1$, $\langle 7 \rangle = \mathbb{Z}_{50}$ so $\phi(7 + 7 + \dots) = \phi(7) + \phi(7) + \cdots = 6 + 6 + \cdots \implies \phi(7n) = 6n$, mod their respective group orders of course. Hence for $k \in \mathbb{Z}_{50}$ where $k = 7n$, $\phi(k) = 6n$. Note that n for 1 is 43 and so $\phi(1) = 6 \cdot_{15} 43 = 3 \implies \phi(x) = 3 \cdot_{15} x$.

(b) Under the definition above this is just

$$\phi[\mathbb{Z}_{50}] = \langle 3 \rangle = \{0, 3, 6, 9, 12\}.$$

(c) By definition,

$$\begin{aligned}\ker \phi &= \{x \in \mathbb{Z}_{50} | 3x \equiv_{15} 0\} \\ &= \{0, 5, 10, 15, 20, 25, 30, 35, 40, 45\}\end{aligned}$$

(d) Recalling that if $\phi(g) = h$ then $\phi^{-1}(h) = g \ker \phi$ we have that because $\phi(1) = 3$, $\phi^{-1}(3) = \ker \phi$.

Problem 5. Let G be a group of order 24, and suppose $\phi : \mathbb{Z}_{36} \to G$ is a group homomorphism.

(a) Find all possible images $\phi[\mathbb{Z}_{36}]$ for such a map ϕ.

(b) What is $\ker \phi$ for each of the maps in part (a)?

Solution 5.

(a) Since $\phi(1) = k$ defines $\phi(x) = kx$ and we know that $|k|$ must divide the order of the input and output groups, k must take on one of the values $1, 2, 3, 4, 6, 12$ as these are the common divisors of 36 and 24. Additionally, since homomorphisms preserve the cyclic and abelian nature of a group in its image we know that the image must be both cyclic and abelian, as $\mathbb{Z}_{36}$ is both of these. As all cyclic groups of order n are isomorphic to $\mathbb{Z}_n$, the image $\mathbb{Z}_{36}$ under ϕ could be $\mathbb{Z}_k$ for $k \in \{1, 2, 3, 4, 6, 12\}$.

(b) For $\phi_1 : \mathbb{Z}_{36} \to \mathbb{Z}_1$ we know that $\ker \phi_1 < \mathbb{Z}_{36}$ and hence that $\mathbb{Z}_{36}/\ker \phi_1 \cong \mathbb{Z}_1$. The order of $\mathbb{Z}_1$ is 1 as it contains only the identity and hence $|\mathbb{Z}_{36}/\ker \phi| = 1 \implies |\ker \phi_1| = 36$. We also know that this kernel group should be cyclic and generated by an element of order 36. We know that for $k \in \mathbb{Z}_{36}$ k must satisfy $|\langle k \rangle| = 36 = \frac{36}{\gcd(k,36)} \implies \gcd(k, 36) = 1$ so we can pick anything relatively prime to 36. 1 will work and so $\ker \phi_1 = \mathbb{Z}_{36}$.

Now for $\phi_2 : \mathbb{Z}_{36} \to \mathbb{Z}_2$ we repeat the same process: $|\mathbb{Z}_2| = 2 \implies |\mathbb{Z}_{36}/\ker \phi| = 2 \implies |\ker \phi_2| = 18$, so we seek a k of order 18 in $\mathbb{Z}_{36}$. k will satisfy $18 = \frac{36}{\gcd(k,36)} \implies 2 = \gcd(k, 36)$ so we can pick 2 as a generator which gives

$$\ker \phi_2 = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34\}.$$

Again we go by the same process for $\phi_3 : \mathbb{Z}_{36} \to \mathbb{Z}_3$: $|\mathbb{Z}_3| = 3 \implies |\mathbb{Z}_{36}/\ker \phi| = 3 \implies |\ker \phi_3| = 12$, so want a k of order 12 which will satisfy $12 = \frac{36}{\gcd(k,36)} \implies 3 = \gcd(k, 36)$ so we can pick 3 as a generator which gives

$$\ker \phi_3 = \{0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33\}.$$

Again for $\phi_4 : \mathbb{Z}_{36} \to \mathbb{Z}_4$: $|\mathbb{Z}_4| = 4 \implies |\mathbb{Z}_{36}/\ker \phi| = 4 \implies |\ker \phi_4| = 9$ so we want k such that $9 = \frac{36}{\gcd(k,36)} \implies 4 = \gcd(k, 36) \implies k = 4$ so

$$\ker \phi_4 = \{0, 4, 8, 12, 16, 20, 24, 28, 32\}.$$

Again for $\phi_6 : \mathbb{Z}_{36} \to \mathbb{Z}_6$: $|\mathbb{Z}_6| = 6 \implies |\mathbb{Z}_{36}/\ker \phi| = 6 \implies |\ker \phi_6| = 6$ so we want k such that $6 = \frac{36}{\gcd(k,36)} \implies 6 = \gcd(k, 36) \implies k = 6$ so

$$\ker \phi_6 = \{0, 6, 12, 18, 24, 30\}.$$

And one last time for $\phi_6 : \mathbb{Z}_{36} \to \mathbb{Z}_{12}$: $|\mathbb{Z}_{12}| = 12 \implies |\mathbb{Z}_{36}/\ker \phi| = 12 \implies |\ker \phi_{12}| = 3$ so we want k such that $3 = \frac{36}{\gcd(k,36)} \implies 12 = \gcd(k, 36) \implies k = 12$ so

$$\ker \phi_{12} = \{0, 12, 24\}.$$

Problem 6. Find the isomorphism class of the following factor groups:

(a) $U(32)/H$ where $H = \{1, 15\}$.

(b) $\mathbb{Z}_{18} \times \mathbb{Z}_{24}/\langle(2, 12)\rangle$.

Solution 6.

(a) We know that $|U(32)/\{1, 15\}| = 16/2 = 8$. Since this group is abelian it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathbb{Z}_2 \times \mathbb{Z}_4$, or $\mathbb{Z}_8$. We can look at the orders of the elements of the factor group to determine which these it is. Consider the $3 \cdot \{1, 15\}$ coset where we'll check the orders that would determine which of the previous groups this is isomorphic to

$$\begin{aligned} 3 \cdot \{1, 15\} &= \{3, 13\} \\ 3^2 \cdot \{1, 15\} &= \{9, 4\} \\ 3^4 \cdot \{1, 15\} &= \{17, 31\} \\ 3^8 \cdot \{1, 15\} &= \{1, 15\}. \end{aligned}$$

So $U(32)/\{1, 15\}$ contains an element of order 8 and can therefore only be isomorphic to $\mathbb{Z}_8$

(b) We know that $|\mathbb{Z}_{18} \times \mathbb{Z}_{24}/\langle(2,12)\rangle| = 432/9 = 48$ since

$$\langle(2,12)\rangle = \{(0,0),(2,12),(4,0),(6,12),(8,0),(10,12),(12,0),(14,12),(16,12)\}\,.$$

This could be isomorphic to quite a few things however we can see that the element $(1,0) + \langle(2,12)\rangle$ has order 18 and so we can drop all potential groups that could not produce an element of such an order. This cuts us down to just $\mathbb{Z}_{48}$ as $48 = 2^4 \cdot 3$ so there is no other way we can construct a group containing elements of order 18 that is abelian.