Physics 2605H: Assignment III

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Problem 1. Solve the following problems from Chapter 6 (did you mean chapter 5? That was what I assumed.):

(a) 5.1

(b) 5.3

(c) 5.4

(d) 5.5

(e) 5.9 A and 5.9 B

(f) 5.10

Solution 1.

(a) (A) Yes, $\langle\psi|\psi\rangle = \left(\sqrt{\frac{5}{6}}\langle 0| + \sqrt{\frac{1}{6}}\langle 1|\right)\left(\sqrt{\frac{5}{6}}|0\rangle + \sqrt{\frac{1}{6}}|1\rangle\right) = \frac{5}{6} + \frac{1}{6} = 1$.

(B) $\frac{5}{6}$.

(C) $\hat{\rho} = |\psi\rangle\langle\psi| = \left(\sqrt{\frac{5}{6}}|0\rangle + \sqrt{\frac{1}{6}}|1\rangle\right)\left(\sqrt{\frac{5}{6}}\langle 0| + \sqrt{\frac{1}{6}}\langle 1|\right)$

(D) $[\hat{\rho}]_{\{|0\rangle,|1\rangle\}} = \frac{1}{6}\begin{pmatrix} 5 & \sqrt{5} \\ \sqrt{5} & 1 \end{pmatrix} \implies \mathrm{Tr}(\hat{\rho}) = \frac{5}{6} + \frac{1}{6} = 1$

(b) (A) $[\hat{\rho}]_{\{|0\rangle,|1\rangle\}} = |\psi\rangle\langle\psi| = \frac{1}{7}\begin{pmatrix} 3 & 2\sqrt{3} \\ 2\sqrt{3} & 4 \end{pmatrix}$

(B) $[\hat{\rho}]^2 = \frac{1}{49}\begin{pmatrix} 3 & 2\sqrt{3} \\ 2\sqrt{3} & 4 \end{pmatrix}\begin{pmatrix} 3 & 2\sqrt{3} \\ 2\sqrt{3} & 4 \end{pmatrix} = \frac{1}{49}\begin{pmatrix} 21 & 14\sqrt{3} \\ 14\sqrt{3} & 28 \end{pmatrix} \implies \mathrm{Tr}(\hat{\rho}^2) = \frac{21}{49} + \frac{28}{49} = 1 \therefore \mathrm{Stateispure}$

(C) Change of basis matrix here is just the Hadamard gate so,

$[\hat{\rho}]_{\{|+\rangle,|-\rangle\}} = \frac{1}{14}H\,[\hat{\rho}]_{\{|0\rangle,|1\rangle\}}\,H = \frac{1}{14}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 2\sqrt{3} \\ 2\sqrt{3} & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 7+4\sqrt{3} & -1 \\ -1 & 7-4\sqrt{3} \end{pmatrix}$

$\implies \mathrm{Tr}([\hat{\rho}]_{\{|+\rangle,|-\rangle\}}) = \frac{1}{14}\left[7+4\sqrt{3}+7-4\sqrt{3}\right] = 1$. Then,

$\left[\hat{\rho}^2\right]_{\{|+\rangle,|-\rangle\}} = \frac{1}{196}\begin{pmatrix} 7+4\sqrt{3} & -1 \\ -1 & 7-4\sqrt{3} \end{pmatrix}\begin{pmatrix} 7+4\sqrt{3} & -1 \\ -1 & 7-4\sqrt{3} \end{pmatrix} = \frac{1}{196}\begin{pmatrix} 98+56\sqrt{3} & -14 \\ -14 & 98-56\sqrt{3} \end{pmatrix}$

$\implies \mathrm{Tr}(\left[\hat{\rho}^2\right]_{\{|+\rangle,|-\rangle\}}) = \frac{1}{196}\left[98+56\sqrt{3}+98-56\sqrt{3}\right] = 1$

(c) (A) $\hat{\rho} = \frac{1}{3}\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} \implies \hat{\rho}^2 = \frac{1}{9}\begin{pmatrix} 6 & 3\sqrt{2} \\ 3\sqrt{2} & 3 \end{pmatrix} \implies \mathrm{Tr}(\hat{\rho}) = \mathrm{Tr}(\hat{\rho}^2) = 1$

(B) $\langle X\rangle = \mathrm{Tr}(\rho X) = \mathrm{Tr}\left(\frac{1}{9}\begin{pmatrix} 6 & 3\sqrt{2} \\ 3\sqrt{2} & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \mathrm{Tr}\left(\left(\begin{matrix} \frac{\sqrt{2}}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{\sqrt{2}}{3} \end{matrix}\right)\right) = \frac{2\sqrt{2}}{3}$

(d) (A) Yes. The matrix has trace 1 and is Hermitian.

(B) Because the trace of the square of the density operator is less than one this is a mixed state.

(e) (A) $\hat{\rho} = |\psi\rangle\langle\psi| = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$

(B) $[\hat{\rho}] = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\1 & 0 & 0 & 1\end{pmatrix} \implies \mathrm{Tr}(\hat{\rho}) = 1$

(f) (A) $\hat{\rho}^\dagger = \begin{pmatrix}\frac{2}{5} & -\frac{i}{8}\\ \frac{i}{8} & \frac{3}{5}\end{pmatrix}^\dagger = \begin{pmatrix}\frac{2}{5} & -\frac{i}{8}\\ \frac{i}{8} & \frac{3}{5}\end{pmatrix} = \hat{\rho}$

(B) These are not the eigenvalues. The eigenvalues for $\hat{\rho}$ are $-\frac{20+\sqrt{41}}{40}$ and $\frac{20+\sqrt{41}}{40}$.

(C) It does not as the eigenvalues are not both positive.

(D) $P(|0\rangle) = \mathrm{Tr}(\langle 0|\rho|0\rangle) = \mathrm{Tr}\left(\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\frac{2}{5} & -\frac{i}{8}\\ \frac{i}{8} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right) = \mathrm{Tr}\left(\begin{pmatrix}\frac{2}{5}\\ \frac{i}{8}\end{pmatrix}\right) = \frac{2}{5} \neq 0.66$. However this is meaningless as the matrix is not a valid density matrix.

(E) Because this matrix is not a valid density matrix it is, again, meaningless to calculate the bloch vector representation.

Problem 2. Follow the procedure discussed in the class to obtain the transformation matrix for following gates:

(i) X

(ii) Y

(iii) Z

(iv) Hadamard

Solution 2.

(i) $\left[\hat{X}\right] = \begin{pmatrix}\langle 0|\hat{X}|0\rangle & \langle 0|\hat{X}|1\rangle\\ \langle 1|\hat{X}|0\rangle & \langle 1|\hat{X}|1\rangle\end{pmatrix} = \begin{pmatrix}\langle 0|1\rangle & \langle 0|0\rangle\\ \langle 1|1\rangle & \langle 1|0\rangle\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

(ii) $\left[\hat{Y}\right] = \begin{pmatrix}\langle 0|\hat{Y}|0\rangle & \langle 0|\hat{Y}|1\rangle\\ \langle 1|\hat{Y}|0\rangle & \langle 1|\hat{Y}|1\rangle\end{pmatrix} = \begin{pmatrix}-i\langle 0|1\rangle & i\langle 0|0\rangle\\ -i\langle 1|1\rangle & i\langle 1|0\rangle\end{pmatrix} = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$

(iii) $\left[\hat{Z}\right] = \begin{pmatrix}\langle 0|\hat{Z}|0\rangle & \langle 0|\hat{Z}|1\rangle\\ \langle 1|\hat{Z}|0\rangle & \langle 1|\hat{Z}|1\rangle\end{pmatrix} = \begin{pmatrix}\langle 0|1\rangle & -\langle 0|1\rangle\\ \langle 1|1\rangle & -\langle 1|1\rangle\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

(iv) $\left[\hat{H}\right] = \begin{pmatrix}\langle 0|\hat{H}|0\rangle & \langle 0|\hat{H}|1\rangle\\ \langle 1|\hat{H}|0\rangle & \langle 1|\hat{H}|1\rangle\end{pmatrix} = \begin{pmatrix}\langle 0|+\rangle & \langle 0|-\rangle\\ \langle 1|+\rangle & \langle 1|-\rangle\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

Problem 3.

(i) In the class we learnt about T gate. Discuss the phase shift it introduces in the quantum state $|\alpha\rangle$, which is a linear combination of $|0\rangle$ and $|1\rangle$ states.

(ii) What is an S gate? Obtain its matrix form from its functionality.

(iii) How are S and T gates related to Z gate?

Solution 3.

(i) The T gate introduces a phase shift of $\frac{\pi}{4}$ rad to α.

(ii) An S gate is, like the T gate, a phase shift gate. It introduces a phase shift of $\frac{\pi}{2}$ rad. It's matrix form is given by $\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}$.

(iii) The S, T, and Z gates are all phase shift gates. The Z gate, as introduced in class, simply did not explicitly state the phase term. Because the Z gate has a bottom right entry of -1 the phase term must be $e^{ni\pi}$ where n is some odd integer.

Problem 4. In the class we discussed two qubit quantum gate C-NOT gate.

(i) Construct the input vector.

(ii) Using its truth table compute the transformation matrix.

(iii) Applying the transformation matrix from (ii) on the state (i), show that it follows the definition of C-NOT gate.

(iv) Find the output state for following input states which are passed through the CNOT gate:

(a) $|00\rangle$

(b) $|01\rangle$

(c) $|11\rangle$

(d) $\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$

(e) $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

Solution 4.

(i)

$$\begin{aligned} |00\rangle = |0\rangle \otimes |0\rangle &\to |00\rangle\,, \\ |01\rangle &\to |01\rangle\,, \\ |10\rangle &\to |11\rangle\,, \\ |11\rangle &\to |10\rangle \end{aligned}$$

(ii)

$$\begin{aligned} \left[\hat{C}\right] &= \begin{pmatrix} \langle 00|\hat{C}|00\rangle & \langle 00|\hat{C}|01\rangle & \langle 00|\hat{C}|10\rangle & \langle 00|\hat{C}|11\rangle \\ \langle 01|\hat{C}|00\rangle & \langle 01|\hat{C}|01\rangle & \langle 01|\hat{C}|10\rangle & \langle 01|\hat{C}|11\rangle \\ \langle 10|\hat{C}|00\rangle & \langle 10|\hat{C}|01\rangle & \langle 10|\hat{C}|10\rangle & \langle 10|\hat{C}|11\rangle \\ \langle 11|\hat{C}|00\rangle & \langle 11|\hat{C}|01\rangle & \langle 11|\hat{C}|10\rangle & \langle 11|\hat{C}|11\rangle \end{pmatrix} = \begin{pmatrix} \langle 00|00\rangle & \langle 00|01\rangle & \langle 00|11\rangle & \langle 00|10\rangle \\ \langle 01|00\rangle & \langle 01|01\rangle & \langle 01|11\rangle & \langle 01|10\rangle \\ \langle 10|00\rangle & \langle 10|01\rangle & \langle 10|11\rangle & \langle 10|10\rangle \\ \langle 11|00\rangle & \langle 11|01\rangle & \langle 11|11\rangle & \langle 11|10\rangle \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

(iii) I'm not sure what this question is asking. Any input vector used to construct the CNOT gate will obviously follow the definition of the CNOT gate when it is applied as it was used to construct the CNOT gate.

(iv) Find the output state for following input states which are passed through the CNOT gate:

(a) $\hat{C}|00\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle$

(b) $\hat{C}|01\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle$

(c) $\hat{C}|11\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle$

(d) $\hat{C}\frac{1}{\sqrt{2}}|01\rangle + \hat{C}\frac{1}{\sqrt{2}}|10\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$

(e) $\hat{C}\frac{1}{\sqrt{2}}|00\rangle + \hat{C}\frac{1}{2}|10\rangle - \hat{C}\frac{1}{2}|11\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|11\rangle - \frac{1}{2}|10\rangle$

Problem 5. Identify the product and entangled states in the list given below.

(a) $\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$

(b) $\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$

(c) $\frac{\sqrt{3}}{2}|00\rangle + \frac{1}{2}|11\rangle$

(d) $\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$

(e) $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

(f) $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

Solution 5.

(a) Because the system $\alpha_0\beta_0 = 0$, $\alpha_0\beta_1 = \frac{1}{\sqrt{2}}$, $\alpha_1\beta_0 = \frac{1}{\sqrt{2}}$, $\alpha_1\beta_1 = 0$ has no solution the state is entangled

(b) For the same general reason as above this state is entangled.

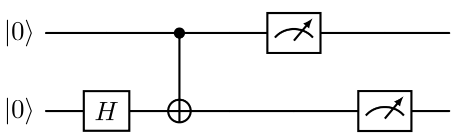
(c) For the same general reason as above this state is entangled.

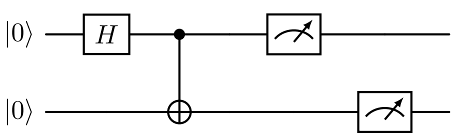
(d) Because the system $\alpha_0\beta_0 = 0$, $\alpha_0\beta_1 = \frac{1}{\sqrt{2}}$, $\alpha_1\beta_0 = 0$, $\alpha_1\beta_1 = \frac{1}{\sqrt{2}}$ has a solution for $\beta_0 = 0$, $\alpha_0 = \alpha_1 = \frac{1}{\sqrt{2}}$, and $\beta_1 = 1$ this is a product state.

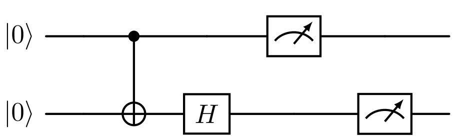
(e) Because the system $\alpha_0\beta_0 = \frac{1}{2}$, $\alpha_0\beta_1 = \frac{1}{2}$, $\alpha_1\beta_0 = \frac{1}{2}$, $\alpha_1\beta_1 = -\frac{1}{2}$ has no solution the state is entangled.

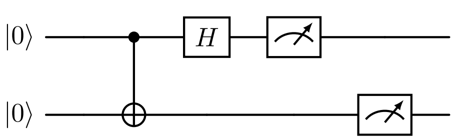
(f) Because the system $\alpha_0\beta_0 = \frac{1}{\sqrt{2}}$, $\alpha_0\beta_1 = 0$, $\alpha_1\beta_0 = \frac{1}{2}$, $\alpha_1\beta_1 = -\frac{1}{2}$ has no solution the state is entangled.

Problem 6. Predict the output states of the following quantum circuits

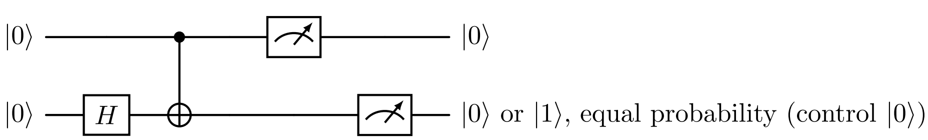
(a)


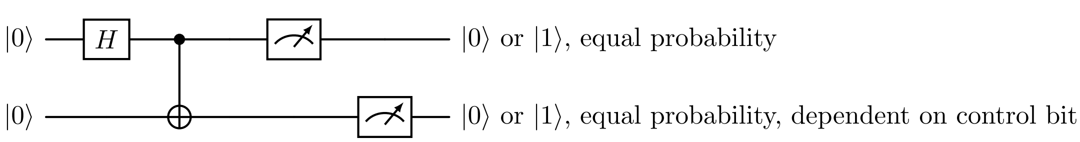
(b)


(c)


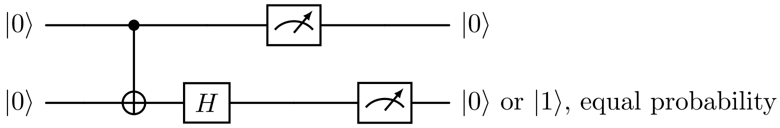
(d)


Solution 6.

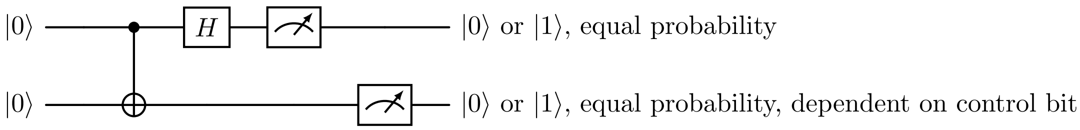
(a)


(b)


(c)


|0⟩
|0⟩
H
|0⟩
|0⟩ or |1⟩, equal probability

(d)


|0⟩
|0⟩
H
|0⟩ or |1⟩, equal probability
|0⟩ or |1⟩, equal probability, dependent on control bit