Calculus II: Assignment 11

Jeremy Favro

April 5, 2024

Problem 1. For what values of x does the series $\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$ converge?

Solution 1.

$$\begin{aligned}
&= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+2)x^{n+1}}{(-1)^n(n+1)x^n}\right| \\
&= \lim_{n\to\infty}\left|(-1)x\frac{(n+2)}{(n+1)}\right| \\
&= \lim_{n\to\infty}\left|x\frac{1}{1}\right| \\
&= \lim_{n\to\infty}|x|
\end{aligned}$$

$$\begin{aligned}
&= \lim_{n\to\infty}|(-1)^n(n+1)(\pm 1)^n| \\
&= \lim_{n\to\infty}|n+1| \\
&= \infty
\end{aligned}$$

$\therefore$ By the ratio test (and divergence test for endpoints) the series $\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$ converges for all $|x|<1$

Problem 2. Use Taylor's Formula to show that $\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n$ when the series converges.

Solution 2.

$$\begin{array}{r|c|c}
f^{(0)}(a=0) & \frac{1}{(1+x)^2} & 1 \\
f^{(1)}(a) & -\frac{2}{(1+x)^3} & \text{-2} \\
f^{(2)}(a) & \frac{6}{(1+x)^4} & 6 \\
f^{(3)}(a) & -\frac{24}{(1+x)^5} & \text{-24} \\
f^{(4)}(a) & \frac{120}{(1+x)^6} & 120
\end{array} \implies \frac{d^n}{dx^n} = (-1)^n\frac{(n+1)!}{(1+x)^{n+2}} = (-1)^n(n+1)! \text{ (for } x=0)$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty}\frac{(-1)^n\frac{(n+1)!}{1}}{n!}x^n \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(n+1)!}{n!}x^n \\
&= \sum_{n=0}^{\infty}(-1)^n(n+1)x^n
\end{aligned}$$

$\therefore$ By the ratio test (and divergence test for endpoints) the series $\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$ converges for all $|x|<1$

Problem 3. Use algebra to show that $\dfrac{1}{(1+x)^2} = \displaystyle\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$ when the series converges.

Solution 3. If we take $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{1+x} = \frac{1}{1-(-x)}$ then

$$
\begin{aligned}
&= \frac{d}{dx}\frac{1}{1+x} \\
&= -\frac{1}{(1+x)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
&= \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n x^n \\
&= \sum_{n=0}^{\infty}(-1)^n n x^{n-1}
\end{aligned}
$$

Which has a leading term of 0 so we can shift it up one

$$
\begin{aligned}
&= \sum_{n=1}^{\infty}(-1)^{n+1}(n+1)x^n \\
&= -\sum_{n=0}^{\infty}(-1)^n(n+1)x^n
\end{aligned}
$$

So the series for $\dfrac{1}{1+x}$ when differentiated and manipulated slightly results in the series for $\dfrac{1}{(1+x)^2}$, with the same transformations applying to the functions themselves.

$\therefore \displaystyle\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$ is indeed the power series representation of $\dfrac{1}{(1+x)^2}$ when it converges.