Math 3310H: Assignment III

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Problem 1. Show that a group G cannot be the union of two proper subgroups, in other words, if $G = H \cup K$ where H and K are subgroups of G, then $H = G$ or $K = G$.

Solution 1. Suppose, by way of contradiction, that $G = H \cup K$ and $H \neq G \neq K$. Then there are elements $a \in H$ and $b \in K$ but $a \notin K$ and $b \notin H$. Because $G = H \cup K$ and H, K, and G are closed by definition, $ab \in H$ or $ab \in K$. First then suppose that $ab \in H \implies a^{-1}ab \in H \implies eb \in H \implies b \in H$, but we began with the assumption that $b \notin H$, so unless $H = K = G$, K cannot be a subgroup. The same argument works in the other direction: Suppose $ab \in K \implies abb^{-1} \in K \implies ae \in K \implies a \in K$, but a was created to be something only in H, not K, meaning H is not closed unless $H = K = G$.

Problem 2. Let G be a group with identity e and $e \in G$. Show that if $a^n = e$ then the order of a divides n.

Solution 2. Let $|a| = k$ be the order of a. By the division algorithm we can write $n = qk + r$ for some $q, r \in \mathbb{Z}$ with $0 \leq r < k$. So

$$\begin{aligned}
e &= a^n \\
&= a^{qk+r} \\
&= a^{qk}a^r \\
&= (a^k)^q a^r \\
&= e^q a^r && a^k = e \text{ by definition.} \\
&= a^r.
\end{aligned}$$

For the expression $e = a^r$ to hold true r must be some multiple of the order of a, k. This means that our expression using the division algorithm becomes $n = qk + sk$ for $sk = r$ which means that $n/k = q + s$ which is an integer meaning that the order of a, k, divides n.

Problem 3. Let G be a cyclic group of order n with identity e. Suppose 15 divides n. How many solutions to $x^15 = e$ are there in G?

Solution 3.

Problem 4. Show that $H = \sigma \in S_n | \sigma(1) = 1$ is a subgroup of S_n.

Solution 4. For H to be a subgroup of S_n it must satisfy the following:

(i) Closure: This is fairly obvious, constructing any $\sigma'' = \sigma \circ \sigma'$ will always satisfy $\sigma''(1) = 1$ as both σ and σ' must map $1 \to 1$ to belong to H in the first place.

(ii) Contains the identity: The identity map looks like

$$\iota = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 2 & 3 & \dots & n \end{pmatrix}$$

which satisfies $\sigma(1) = 1$

(iii) Contains inverses: All inverses for a $\sigma \in H$ will map $1 \to 1$ by the definition of σ and so will belong to H.

Problem 5. Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 4 & 1 & 7 & 5 & 8 & 9 & 6 \end{pmatrix}$$

and

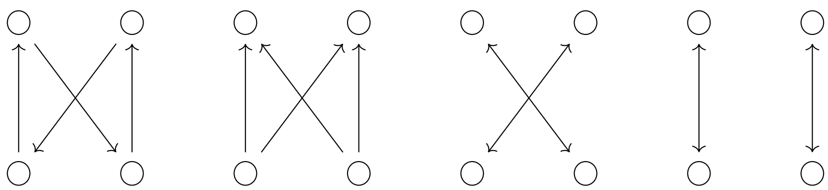
$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 3 & 7 & 9 & 1 & 8 & 2 & 4 & 5 \end{pmatrix}.$$

(a) Compute σ^2, $\sigma\tau$, $\tau\sigma$, σ^{-1}, $\sigma\tau\sigma^{-1}$, and $\tau\sigma\tau^{-1}$.

(b) Find the order of τ

Solution 5.

Problem 6. Below are four recommended car tire rotation patterns.



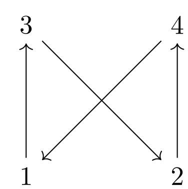
(a) Explain how these patterns can be represented as elements of S_4.

(b) Find the smallest subgroup H of S_4 that contains these four patterns.

(c) Is H abelian?

Solution 6.

(a) If we represent the "default" state of the tires as



then each rotation of the tires is a permutation of this default state. By definition S_4 is the group containing all permutations of 4 elements and so these will belong to S_4. We can express them as compositions of known permutations. The first

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}; \quad b = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$$

(b) Find the smallest subgroup H of S_4 that contains these four patterns.

(c) Is H abelian?