Math 2350H: Assignment IV

Jeremy Favro (0805980)
Trent University, Peterborough, ON, Canada

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Problem 1. Consider the subspace

$$U = \operatorname{span}\left((2,-1,-2,4),(-2,1,-5,5),(-1,3,7,11)\right)$$

(a) Apply the Gram-Schmidt process (with normalization) to find an orthonormal basis of U.

(b) Find a basis for $U^\perp$

(c) Express the vector $v = (-11, 8, -4, 18)$ as $v = x + y$ where $x \in U$ and $y \in U^\perp$.

Solution 1.

(a)

$$\begin{aligned}
u_1 &= (2,-1,-2,4) \implies \hat{u}_1 = \left(\frac{2}{5}, -\frac{1}{5}, -\frac{2}{5}, \frac{4}{5}\right) \\
u_2 &= v_2 - \frac{\langle v_2|u_1\rangle}{\|u_1\|^2}u_1 \\
&= (-2,1,-5,5) - \frac{(-2,1,-5,5)\cdot(2,-1,-2,4)}{25}(2,-1,-2,4) \\
&= (-4,2,-3,1) \implies \hat{u}_2 = \left(-\frac{4}{\sqrt{30}}, \frac{2}{\sqrt{30}}, -\frac{3}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right) \\
u_3 &= v_3 - \frac{\langle v_3|u_1\rangle}{\|u_1\|^2}u_1 - \frac{\langle v_3|u_2\rangle}{\|u_2\|^2}u_2 \\
&= (-1,3,7,11) - \frac{(-1,3,7,11)\cdot(2,-1,-2,4)}{25}(2,-1,-2,4) - \frac{(-1,3,7,11)\cdot(-4,2,-3,1)}{30}(-4,2,-3,1) \\
&= (-3,4,9,7) \implies \hat{u}_3 = \left(-\frac{3}{\sqrt{155}}, \frac{4}{\sqrt{155}}, \frac{9}{\sqrt{155}}, \frac{7}{\sqrt{155}}\right)
\end{aligned}$$

(b) We are looking for the vector $v \in U^\perp$ such that $\langle v|u_n\rangle = 0$ which can be found by solving the system represented by the matrix of the basis vectors (u_n) we found previously (un-normalized because it yields the same result and is easier),

$$\begin{pmatrix} 2 & -1 & -2 & 4 \\ -4 & 2 & -3 & 1 \\ -3 & 4 & 9 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 34/7 \\ 0 & 1 & 0 & 58/7 \\ 0 & 0 & 1 & -9/7 \end{pmatrix}$$

which means that $U^\perp = \operatorname{span}(-34/7, -58/7, 9/7, 1)$.

(c) Using the fact that the coefficients of a linear combination are given by the inner products of the result of that linear combination and the vector paired with the coefficient and calling $u_4 = (-34/7, -58/7, 9/7, 1)$

$$\begin{aligned}
\langle v|u_1\rangle &= \langle x+y|u_1\rangle = \langle x|u_1\rangle + \langle y|u_1\rangle = \langle x|u_1\rangle = c_1, \\
\langle v|u_2\rangle &= \langle x+y|u_2\rangle = \langle x|u_2\rangle + \langle y|u_2\rangle = \langle x|u_2\rangle = c_2, \\
\langle v|u_3\rangle &= \langle x+y|u_3\rangle = \langle x|u_3\rangle + \langle y|u_3\rangle = \langle x|u_3\rangle = c_3, \\
\langle v|u_4\rangle &= \langle x+y|u_4\rangle = \langle x|u_4\rangle + \langle y|u_4\rangle = \langle y|u_4\rangle = c_4
\end{aligned}$$

so

$$v = \langle v|u_1\rangle u_1 + \langle v|u_2\rangle u_2 + \langle v|u_3\rangle u_3 + \langle v|u_4\rangle u_4$$

which gives $x = 2u_1 + 3u_2 + u_3$ and $y = 0u_4$

Problem 2. The dot product is defined on $\mathcal{M}_{n\times 1}(\mathbb{R})$ and $\mathcal{M}_{n\times 1}(\mathbb{C})$ just as it is for $\mathbb{R}^n$ and $\mathbb{C}^n$. For $u, v \in \mathcal{M}_{n\times 1}(\mathbb{C})$, with

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

the dot product (or standard inner product) is

$$\langle u|v\rangle = \sum_{k=1}^{n} u_k v_k^*$$

This can be written more compactly with matrix multiplication as

$$\langle u|v\rangle = v^\dagger u$$

where $v^\dagger = \left(v^T\right)^*$. We use this inner product below.

(a) Let $P \in \mathcal{M}_{n\times n}(\mathbb{R})$. Relative to the dot product on column vectors, show that the following are equivalent:

- (i) The columns $u_1, \ldots, u_n$ of P form an orthonormal basis $\mathcal{M}_{n\times 1}(\mathbb{R})$.
- (ii) $P^T = P^{-1}$.
- (iii) The rows of P form an orthonormal basis for $\mathbb{R}^n$.

(b) A matrix $P \in \mathcal{M}_{n\times n}(\mathbb{R})$ is called an *orthogonal matrix* if $P^T = P^{-1}$. Determine which of the following matrices are orthogonal.

- (i) $\begin{pmatrix} 3/5 & 4/5 \\ -4/5 & -3/5 \end{pmatrix}$
- (ii) $\begin{pmatrix} 5/13 & 12/13 \\ 12/13 & -5/13 \end{pmatrix}$
- (iii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- (iv) $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$
- (v) $\begin{pmatrix} 1/3 & 2/3 & -2/3 \\ 1/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \end{pmatrix}$
- (vi) $\begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \end{pmatrix}$

(c) Let $P, Q \in \mathcal{M}_{n\times n}(\mathbb{R})$ be orthogonal matrices and $u, v \in \mathcal{M}_{n\times 1}(\mathbb{R})$. Prove that

- (i) $\langle Pu|Pv\rangle = \langle u|v\rangle$
- (ii) $\|Pu\| = \|u\|$
- (iii) PQ is an orthogonal matrix.

Solution 2.

(a) To prove the equivalency of these statements we need to prove that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i). For (i) $\Longrightarrow$ (ii) we know that

$$\langle u_i|u_j\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

because these columns are linearly independent. By the definition of this inner product this is equal to $u_j^T u_i$ which is the ij-th entry of $P^T P$ which we know from above must be I so $P^T P = I \implies P^T = P^{-1}$.
For (ii) $\Longrightarrow$ (iii) we are trying to show that if $P^T = P^{-1}$ then the rows of P form a basis for $\mathbb{R}^n$ where $P \in \mathcal{M}_{n\times n}(\mathbb{R})$. We start by noting that $P^T = P^{-1} \implies P^T P = I$ so

$$P_{ji}P_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad .$$

The entries of $P^T P$ are given by the inner product $\left\langle P_k \middle| P_h^T \right\rangle$ where P_k is the k-th row of P and P_h the h-th column of P^T which is equivalent to the h-th *row* of P which means we have that

$$\langle P_k|P_h\rangle = \begin{cases} 1 & h = k \\ 0 & h \neq k \end{cases}$$

which means that the rows of P are by definition linearly independent and normalized and because we have n of them form an orthonormal basis for $\mathbb{R}^n$.
For $\Longrightarrow$ (iii) $\Longrightarrow$ (i) we can see that a matrix with linearly independent rows will reduce to the appropriate identity matrix which will also have independent columns because Gaussian elimination preserves linear independence between rows/columns.

(b) A matrix $P \in \mathcal{M}_{n\times n}(\mathbb{R})$ is called an *orthogonal matrix* if $P^T = P^{-1}$. Determine which of the following matrices are orthogonal.

(i) From part (a) it's enough to note that $\langle u_1|u_2\rangle = \begin{pmatrix} 3/5 & -4/5 \end{pmatrix} \begin{pmatrix} 4/5 \\ -3/5 \end{pmatrix} = 24/25 \neq 0$ so the columns of this matrix do not form an orthonormal basis and therefore $P^T \neq P^{-1}$

(ii) Here again it is sufficient to note that, by inspection (as the columns are flipped and the 22 entry is negative), the columns of this matrix form an orthonormal basis for $\mathcal{M}_{n\times 1}(\mathbb{R})$ and therefore $P^T = P^{-1}$

(iii) Here again it is sufficient to note that, by inspection, the columns of this matrix form an orthonormal basis for $\mathcal{M}_{n\times 1}(\mathbb{R})$ and therefore $P^T = P^{-1}$

(iv) Here again it is sufficient to note that, by inspection (as the columns are flipped and the 21 entry is negative), the columns of this matrix form an orthonormal basis for $\mathcal{M}_{n\times 1}(\mathbb{R})$ and therefore $P^T = P^{-1}$

(v) Here it is sufficient to note that the first column is not normalized ($u_1 = \sqrt{2\cdot(1/3)^2 + (2/3)^2} = \sqrt{6}/3 =\neq 1$) and therefore the columns cannot form an orthonormal basis so $P^T \neq P^{-1}$

(vi) Here $\langle u_1|u_2\rangle = \begin{pmatrix} 2/3 & -2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix} = 0$, $\langle u_1|u_3\rangle = \begin{pmatrix} 2/3 & 1/3 & -2/3 \end{pmatrix} \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix} = 0$, and $\langle u_2|u_3\rangle = \begin{pmatrix} 2/3 & 1/3 & -2/3 \end{pmatrix} \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix} = 0$ and u_1, u_2, and u_3 are normalized so $P^T = P^{-1}$

(c) (i) $\langle Pu|Pv\rangle = v^T P^T P u = v^T I u = v^T u = \langle u|v\rangle$

(ii) $\|Pu\| = \langle Pu|Pu\rangle = u^T P^T P u = u^T I u = \langle u|u\rangle = \|u\|$

(iii) Because P, Q are of the same size we can multiply them (and their transposes), $(PQ)^T PQ = Q^T P^T P Q = Q^T I Q = Q^T Q = I$ which satisfies the property of an orthogonal matrix A that $A^T A = I$

Problem 3. Let V be the vector space of continuous, real-valued functions defined on the interval $[0, 1]$. Then V is an inner product space with inner product

$$\langle f|g\rangle = \int_0^1 f(x)g(x)dx,$$

for $f, g \in V$. Consider the subspace U of V spanned by the functions $f(x) = \sqrt{x}$, $g(x) = x$, $h(x) = x^2$.

(a) Show that f, g, h is linearly independent.

(b) Find an orthonormal basis for U

(c) Let $p(x) = x^3$. Find the closest approximation of p in U.

Solution 3.

(a) By the definition of linear independence

$$c_1 f(x) + c_2 g(x) + c_3 h(x) = 0.$$

Using the Wronskian here for fun we get $\frac{3\sqrt{x}}{4}$ (calculated using SageMath) which gives 0 only at $x = 0$ which, by the definition of the Wronskian, implies linear independence.

(b) Here we apply the Gram-Schmidt process,

$$w_1 = \sqrt{x}$$

$$w_2 = x - \frac{\langle x | \sqrt{x} \rangle}{\|\sqrt{x}\|^2} \sqrt{x} = x - \frac{\int_0^1 x\sqrt{x}\, dx}{\int_0^1 x\, dx} \sqrt{x} = x - \frac{4}{5}\sqrt{x}$$

$$\begin{aligned} w_3 &= x^2 - \frac{\langle x^2 | \sqrt{x} \rangle}{\|\sqrt{x}\|^2} \sqrt{x} - \frac{\langle x^2 | x - \frac{4}{5}\sqrt{x} \rangle}{\|x - \frac{4}{5}\sqrt{x}\|^2} \left(x - \frac{4}{5}\sqrt{x} \right) \\ &= x^2 - \frac{\int_0^1 x^2 \sqrt{x}\, dx}{\frac{1}{2}} \sqrt{x} - \frac{\int_0^1 x^2 \left(x - \frac{4}{5}\sqrt{x} \right)\, dx}{\frac{1}{75}} \left(x - \frac{4}{5}\sqrt{x} \right) \\ &= x^2 - \frac{4}{7}\sqrt{x} - \frac{45}{28} \left(x - \frac{4}{5}\sqrt{x} \right) \\ &= x^2 - \frac{45}{28} x + \frac{5}{7}\sqrt{x} \end{aligned}$$

then we normalize,

$$\|w_1\| = \frac{1}{\sqrt{2}} \implies \hat{u}_1 = \sqrt{2x}$$

$$\|w_2\| = \frac{1}{\sqrt{75}} \implies \hat{u}_2 = \sqrt{75} \left(x - \frac{4}{5}\sqrt{x} \right)$$

$$\|w_3\| = \int_0^1 \left(x^2 - \frac{45}{28} x + \frac{5}{7}\sqrt{x} \right)^2 dx = \sqrt{\frac{9}{3920}} \implies \hat{u}_3 = \sqrt{\frac{3920}{9}} \left(x^2 - \frac{45}{28} x + \frac{5}{7}\sqrt{x} \right)$$

(c) Here we are looking for the projection map

$$\begin{aligned} P_U(x^3) = &\left\langle x^3 \middle| \sqrt{2x} \right\rangle \left(\sqrt{2x} \right) + \\ &\left\langle x^3 \middle| \sqrt{75} \left(x - \frac{4}{5}\sqrt{x} \right) \right\rangle \sqrt{75} \left(x - \frac{4}{5}\sqrt{x} \right) + \\ &\left\langle x^3 \middle| \sqrt{\frac{3920}{9}} \left(x^2 - \frac{45}{28} x + \frac{5}{7}\sqrt{x} \right) \right\rangle \sqrt{\frac{3920}{9}} \left(x^2 - \frac{45}{28} x + \frac{5}{7}\sqrt{x} \right) \\ = &\left[\int_0^1 x^3 \sqrt{2x}\, dx \right] \sqrt{2x} + \\ &\left[\int_0^1 x^3 \sqrt{75} \left(x - \frac{4}{5}\sqrt{x} \right) dx \right] \sqrt{75} \left(x - \frac{4}{5}\sqrt{x} \right) + \\ &\left[\int_0^1 x^3 \sqrt{\frac{3920}{9}} \left(x^2 - \frac{45}{28} x + \frac{5}{7}\sqrt{x} \right) dx \right] \sqrt{\frac{3920}{9}} \left(x^2 - \frac{45}{28} x + \frac{5}{7}\sqrt{x} \right) \\ = &\frac{4}{9}\sqrt{x} + \frac{5}{3} \left(x - \frac{4}{5}\sqrt{x} \right) + \frac{140}{81} \left(x^2 - \frac{45}{28} x + \frac{5}{7}\sqrt{x} \right) \end{aligned}$$