Calculus II: Assignment 5

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Consider the Gamma function, the function of x defined by using x as a constant in an integral as follows:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt = \lim_{k\to\infty}\int_0^k t^{x-1}e^{-t}\,dt$$

This definition turns out to make sense whenever $x > 0$.

Problem 1. Use SageMath to compute $\Gamma\left(\frac{1}{2}\right), \Gamma\left(1\right), \Gamma\left(\frac{3}{2}\right), \Gamma\left(2\right), \Gamma\left(\frac{5}{2}\right), \Gamma\left(3\right), \Gamma\left(\frac{7}{2}\right), \Gamma\left(4\right)$

Solution 1.

[1]:
```
clear_vars()
from IPython.core.interactiveshell import InteractiveShell
from sage.symbolic.integration.integral import definite_integral

InteractiveShell.ast_node_interactivity = "all"

nums = {1/2: 0, 1: 0, 3/2: 0, 5/2: 0, 3: 0, 7/2: 0, 4: 0}

x = var('x')
t = var('t')

assume(x>0)

gamma(x) = definite_integral((t^(x-1)*e^(-t)), t, 0 , oo)

for num in nums:
    nums[num] = gamma(num)
nums
```

[1]:
```
{1/2: sqrt(pi),
1: 1,
3/2: 1/2*sqrt(pi),
5/2: 3/4*sqrt(pi),
3: 2,
7/2: 15/8*sqrt(pi),
4: 6}
```

Problem 2. By hand, show that $\Gamma(x+1) = x\Gamma(x)$.

Solution 2. Assuming that $\Gamma(x+1) = x\Gamma(x)$ is true, we should get the same integral as part of our answer, which we can use to check our work.

$$
\begin{aligned}
\Gamma(x+1) &= \int_0^\infty t^x e^{-t}\, dt \\
&= \int_0^\infty t^x e^{-t}\, dt \implies \text{ Apply integration by parts, } u = t^x,\ u' = xt^{x-1},\ v' = e^{-t},\ v = -e^{-t} \\
&= \left[-t^x e^{-t}\right]\Big|_0^\infty - \int_0^\infty -xt^{x-1}e^{-t}\, dt \\
&= \lim_{t\to\infty}\left[-t^x e^{-t}\right] - \left[-0^x e^0\right] - \int_0^\infty -xt^{x-1}e^{-t}\, dt \\
&= \lim_{t\to\infty}\left[-t^x e^{-t}\right] - 0 - \int_0^\infty -xt^{x-1}e^{-t}\, dt \\
&= \int_0^\infty xt^{x-1}e^{-t}\, dt \implies \text{ x can be taken out because it's constant} \\
&= x\int_0^\infty t^{x-1}e^{-t}\, dt
\end{aligned}
$$

$$\therefore \Gamma(x+1) = x\Gamma(x)$$

Problem 3. Using the results of questions 1 and 2, explain why $\Gamma(n+1) = n!$ for any integer $n \geq 0$.

Solution 3. Question 1 demonstrated that $\Gamma(n) = (n-1)!$ (At least for 1, 2, 3, and 4). Question 2 showed that $\Gamma(n+1) = n\Gamma(n)$. If we extend the result from question 1 to all $n \in \mathbb{Z}$ where $n \geq 0$, and combine it with the result from question 2:

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)!$$

Which, because the factorial $n!$ is defined as

$$
\begin{aligned}
&n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 3 \times 2 \times 1 \\
&n! = n(n-1)!
\end{aligned}
$$

Simplifies to

$$\Gamma(n+1) = n!$$