Physics 3610H: Assignment X

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November 28, 2025

Problem 1. In class we showed that if $|E_n\rangle$ is normalized, then $|E_{n+1}\rangle = \hat{a}_+ |E_n\rangle / \sqrt{n+1}$ is also normalized. Assume again that $|E_n\rangle$ is normalized and show that for $|E_{n-1}\rangle$ to be normalized it must equal $\hat{a}_- |E_n\rangle / \sqrt{n}$.

Solution 1. Well, we know that $|E_{n-1}\rangle \propto \hat{a}_- |E_n\rangle$ by the definition of $\hat{a}_-$. So

$$|E_{n-1}\rangle = C_{n-1}\hat{a}_- |E_n\rangle .$$

And so

$$\begin{aligned}
\langle E_{n-1}|E_{n-1}\rangle &= (C_{n-1}\hat{a}_- |E_n\rangle)^\dagger C_{n-1}\hat{a}_- |E_n\rangle \\
&= |C_{n-1}|^2 \langle E_n| \hat{a}_-^\dagger \hat{a}_- |E_n\rangle \\
&= |C_{n-1}|^2 \langle E_n| \hat{a}_+ \hat{a}_- |E_n\rangle .
\end{aligned}$$

We found in class that

$$\hat{a}_+\hat{a}_- = \frac{m\omega}{2\hbar}\hat{x}^2 + \frac{1}{2m\omega\hbar}\hat{p}_x^2 - \frac{1}{2}$$

and from this that

$$\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right) \implies \hat{a}_+\hat{a}_- = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2}$$

and so our previous inner product becomes

$$\begin{aligned}
|C_{n-1}|^2 \langle E_n| \hat{a}_+\hat{a}_- |E_n\rangle &= |C_{n-1}|^2 \langle E_n| \left(\frac{\hat{H}}{\hbar\omega} - \frac{1}{2}\right) |E_n\rangle \\
&= |C_{n-1}|^2 \langle E_n| \left(\frac{\hat{H}}{\hbar\omega} |E_n\rangle - \frac{1}{2} |E_n\rangle\right) \\
&= |C_{n-1}|^2 \left(\frac{1}{\hbar\omega} \langle E_n| \hat{H} |E_n\rangle - \frac{1}{2} \langle E_n|E_n\rangle\right) \\
&= |C_{n-1}|^2 \left(\frac{1}{\hbar\omega} E_n - \frac{1}{2}\right) \\
&= |C_{n-1}|^2 \left(n + \frac{1}{2} - \frac{1}{2}\right) \\
&= |C_{n-1}|^2 n
\end{aligned}$$

we want this expression to be normalized (recall with originally began with $\langle E_{n-1}|E_{n-1}\rangle$) and so we set it equal to 1,

$$|C_{n-1}|^2 n = 1 \implies C_{n-1} = \frac{1}{\sqrt{n}}$$

as we wanted.

Problem 2. In class, we used $\hat{a}_- |E_0\rangle = 0$ to show that $\psi_0(\xi) \propto e^{-\xi^2/2}$. In fact, when you normalize this you find

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}.$$

Use the raising operator to find $\psi_1(x)$.

Solution 2. The raising operator, $\hat{a}_+$ is, in position representation, given by

$$\hat{a}_+ = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}\hat{x} - i\frac{\hat{p}_x}{\sqrt{m\omega\hbar}}\right] = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}x - \sqrt{\frac{\hbar}{m\omega}}\frac{\partial}{\partial x}\right].$$

Applying this to ψ_0,

$$\begin{aligned}
\hat{a}_+\psi_0(x) &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}x - \sqrt{\frac{\hbar}{m\omega}}\frac{\partial}{\partial x}\right]\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} \\
&= \left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left[\sqrt{\frac{m\omega}{\hbar}}xe^{-m\omega x^2/2\hbar} - \sqrt{\frac{\hbar}{m\omega}}\frac{\partial}{\partial x}e^{-m\omega x^2/2\hbar}\right] \\
&= \left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left[\sqrt{\frac{m\omega}{\hbar}}xe^{-m\omega x^2/2\hbar} + \sqrt{\frac{\hbar}{m\omega}}\frac{m\omega}{\hbar}xe^{-m\omega x^2/2\hbar}\right] \\
&= \left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left[\sqrt{\frac{m\omega}{\hbar}}xe^{-m\omega x^2/2\hbar} + \sqrt{\frac{m\omega}{\hbar}}xe^{-m\omega x^2/2\hbar}\right] \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}xe^{-m\omega x^2/2\hbar}.
\end{aligned}$$

Problem 3. The state of a system is described by the vector $\left(1/3, 1/3, 1/\sqrt{3}, 2/3, 0, 0, 0, \ldots\right)$ in the basis of the eigenfunctions of the infinite square well. What is the wavefunction for this system in position representation?

Solution 3. Recall that the eigenfunction of the infinite square well are of the form

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right).$$

The given state vector represents the coefficients of the eigenfunction expansion of a full solution in these eigenfunctions,

$$\Psi(x) = \sum_n c_n\psi_n(x).$$

So the full expansion is

$$\Psi(x) = \sqrt{\frac{2}{a}}\left[\frac{1}{3}\sin\left(\frac{\pi x}{a}\right) + \frac{1}{3}\sin\left(\frac{2\pi x}{a}\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{3\pi x}{a}\right) + \frac{2}{3}\sin\left(\frac{4\pi x}{a}\right)\right].$$

Problem 4. Consider the matrix $\underline{\underline{M}}$ corresponding to the operator $\hat{x}^4$ in the basis of eigenstates of the harmonic oscillator, i.e. $\{|E_n\rangle\}$. Using

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left[\hat{a}_+ + \hat{a}_-\right]$$

(a) Find M_{54}.

(b) Find M_{53}.

(c) Find $M_{n+2,n}$.

Solution 4. (a) To obtain elements of the form $n+1, n$ we would need an odd power of $\hat{x}$. With an even power we can only obtain states corresponding to even energy shifts. Hence $M_{54} = 0$.

(b) Writing $\hat{x}^4$ as a matrix we obtain i, j elements

$$\begin{aligned}
M_ij &= \langle E_i| \hat{x}^4 |E_j\rangle \\
&= \left(\frac{\hbar}{2m\omega}\right)^2 \langle E_i| [\hat{a}_+ + \hat{a}_-]^4 |E_j\rangle \\
&= \left(\frac{\hbar}{2m\omega}\right)^2 \langle E_i| \left[\hat{a}_+^2 \left[\hat{a}_+^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-^2\right]\right. \\
&\quad + \hat{a}_+\hat{a}_- \left[\hat{a}_+^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-^2\right] \\
&\quad + \hat{a}_-\hat{a}_+ \left[\hat{a}_+^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-^2\right] \\
&\quad \left. + \hat{a}_-^2 \left[\hat{a}_+^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-^2\right]\right] |E_j\rangle \\
&= \left(\frac{\hbar}{2m\omega}\right)^2 \langle E_i| \left[\left[\hat{a}_+^4 + \hat{a}_+^2\hat{a}_+\hat{a}_- + \hat{a}_+^2\hat{a}_-\hat{a}_+ + \hat{a}_+^2\hat{a}_-^2\right]\right. \\
&\quad + \left[\hat{a}_+\hat{a}_-\hat{a}_+^2 + \hat{a}_+\hat{a}_-\hat{a}_+\hat{a}_- + \hat{a}_+\hat{a}_-\hat{a}_-\hat{a}_+ + \hat{a}_+\hat{a}_-\hat{a}_-^2\right] \\
&\quad + \left[\hat{a}_-\hat{a}_+\hat{a}_+^2 + \hat{a}_-\hat{a}_+\hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+\hat{a}_-\hat{a}_+ + \hat{a}_-\hat{a}_+\hat{a}_-^2\right] \\
&\quad \left. + \left[\hat{a}_-^2\hat{a}_+^2 + \hat{a}_-^2\hat{a}_+\hat{a}_- + \hat{a}_-^2\hat{a}_-\hat{a}_+ + \hat{a}_-^2\hat{a}_-^2\right]\right] |E_j\rangle \\
&= \left(\frac{\hbar}{2m\omega}\right)^2 \left[\left[\langle E_i| \hat{a}_+^4 |E_j\rangle + \langle E_i| \hat{a}_+^2\hat{a}_+\hat{a}_- |E_j\rangle + \langle E_i| \hat{a}_+^2\hat{a}_-\hat{a}_+ |E_j\rangle + \langle E_i| \hat{a}_+^2\hat{a}_-^2 |E_j\rangle\right]\right. \\
&\quad + \left[\langle E_i| \hat{a}_+\hat{a}_-\hat{a}_+^2 |E_j\rangle + \langle E_i| \hat{a}_+\hat{a}_-\hat{a}_+\hat{a}_- |E_j\rangle + \langle E_i| \hat{a}_+\hat{a}_-\hat{a}_-\hat{a}_+ |E_j\rangle + \langle E_i| \hat{a}_+\hat{a}_-\hat{a}_-^2 |E_j\rangle\right] \\
&\quad + \left[\langle E_i| \hat{a}_-\hat{a}_+\hat{a}_+^2 |E_j\rangle + \langle E_i| \hat{a}_-\hat{a}_+\hat{a}_+\hat{a}_- |E_j\rangle + \langle E_i| \hat{a}_-\hat{a}_+\hat{a}_-\hat{a}_+ |E_j\rangle + \langle E_i| \hat{a}_-\hat{a}_+\hat{a}_-^2 |E_j\rangle\right] \\
&\quad \left. + \left[\langle E_i| \hat{a}_-^2\hat{a}_+^2 |E_j\rangle + \langle E_i| \hat{a}_-^2\hat{a}_+\hat{a}_- |E_j\rangle + \langle E_i| \hat{a}_-^2\hat{a}_-\hat{a}_+ |E_j\rangle + \langle E_i| \hat{a}_-^2\hat{a}_-^2 |E_j\rangle\right]\right].
\end{aligned}$$

We can then skip ahead a bit to part (c) as it asks for a general formula for the $n+2, n$ elements to which M_{53} belongs. Here $n = 3$ and so

$$M_{53} = \left(\frac{\hbar}{2m\omega}\right)^2 (4\cdot 3 - 2)\sqrt{3}\sqrt{3-1} = \left(\frac{\hbar}{2m\omega}\right)^2 10\sqrt{6}.$$

(c) $M_{n+2,n}$ will be the result of all the terms in

$$\begin{aligned}
&\left(\frac{\hbar}{2m\omega}\right)^2 \left[\left[\langle E_n| \hat{a}_+^4 |E_n\rangle + \langle E_n| \hat{a}_+^2\hat{a}_+\hat{a}_- |E_n\rangle + \langle E_n| \hat{a}_+^2\hat{a}_-\hat{a}_+ |E_n\rangle + \langle E_n| \hat{a}_+^2\hat{a}_-^2 |E_n\rangle\right]\right. \\
&\quad + \left[\langle E_n| \hat{a}_+\hat{a}_-\hat{a}_+^2 |E_n\rangle + \langle E_n| \hat{a}_+\hat{a}_-\hat{a}_+\hat{a}_- |E_n\rangle + \langle E_n| \hat{a}_+\hat{a}_-\hat{a}_-\hat{a}_+ |E_n\rangle + \langle E_n| \hat{a}_+\hat{a}_-\hat{a}_-^2 |E_n\rangle\right] \\
&\quad + \left[\langle E_n| \hat{a}_-\hat{a}_+\hat{a}_+^2 |E_n\rangle + \langle E_n| \hat{a}_-\hat{a}_+\hat{a}_+\hat{a}_- |E_n\rangle + \langle E_n| \hat{a}_-\hat{a}_+\hat{a}_-\hat{a}_+ |E_n\rangle + \langle E_n| \hat{a}_-\hat{a}_+\hat{a}_-^2 |E_n\rangle\right] \\
&\quad \left. + \left[\langle E_n| \hat{a}_-^2\hat{a}_+^2 |E_n\rangle + \langle E_n| \hat{a}_-^2\hat{a}_+\hat{a}_- |E_n\rangle + \langle E_n| \hat{a}_-^2\hat{a}_-\hat{a}_+ |E_n\rangle + \langle E_n| \hat{a}_-^2\hat{a}_-^2 |E_n\rangle\right]\right]
\end{aligned}$$

which yield a net -2 change in energy (because the operators apply to the left). All others will result in inner products which evaluate to zero. Removing all the terms we know will be zero then we obtain

$$\left(\frac{\hbar}{2m\omega}\right)^2 \left[\langle E_n| \hat{a}_+\hat{a}_-\hat{a}_-^2 |E_n\rangle + braE_n\hat{a}_-\hat{a}_+\hat{a}_-^2 |E_n\rangle + \langle E_n| \hat{a}_-^2\hat{a}_+\hat{a}_- |E_n\rangle + \langle E_n| \hat{a}_-^2\hat{a}_-\hat{a}_+ |E_n\rangle\right].$$

Now we simplify using the relations

$$\hat{a}_+ |E_n\rangle = \sqrt{n+1}\, |E_{n+1}\rangle; \qquad \hat{a}_- |E_n\rangle = \sqrt{n}\, |E_{n-1}\rangle,$$

$$
\begin{aligned}
M_{n+2,n} &= \left(\frac{\hbar}{2m\omega}\right)^2 \left[\langle E_n|\,\hat{a}_+\hat{a}_-\hat{a}_-^2\,|E_n\rangle + braE_n\hat{a}_-\hat{a}_+\hat{a}_-^2\,|E_n\rangle + \langle E_n|\,\hat{a}_-^2\hat{a}_+\hat{a}_-\,|E_n\rangle + \langle E_n|\,\hat{a}_-^2\hat{a}_-\hat{a}_+\,|E_n\rangle\right] \\
&= \left(\frac{\hbar}{2m\omega}\right)^2 \left[\sqrt{n}\sqrt{n-1}(n-2)\,\langle E_n|E_{n-2}\rangle\right. \\
&\quad + \sqrt{n}\sqrt{n-1}(n-1)\,\langle E_n|E_{n-2}\rangle \\
&\quad + n\sqrt{n}\sqrt{n-1}\,\langle E_n|E_{n-2}\rangle \\
&\quad \left.+\sqrt{n+1}\sqrt{n+1}\sqrt{n}\sqrt{n-1}\,\langle E_n|E_{n-2}\rangle\right] \\
&= \left(\frac{\hbar}{2m\omega}\right)^2 (4n-2)\sqrt{n}\sqrt{n-1}
\end{aligned}
$$