

Math 2350H: Assignment I

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Problem 1. Demonstrate that there does not exist $\lambda \in \mathbb{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + i) = (12 - 5i, 7 + 22i, -32 - 9i)$$

Solution 1. Let $\lambda = a + bi$ then

$$\lambda(2 - 3i, 5 + 4i, -6 + i) = (12 - 5i, 7 + 22i, -32 - 9i)$$

$$\implies (2a + 3b) + (-3a + 2b)i = 12 - 5i \wedge (5a + 4b) + (4a + 5b)i = 7 + 22i \wedge (-6a - 1b) + (1a - 6b)i = -32 - 9i$$

Here the determinant of the matrix of the coefficients of i is

$$\begin{vmatrix} -3 & 2 & -5 \\ 4 & 5 & 2 \\ 1 & -6 & -9 \end{vmatrix} = -3(5 \cdot (-9) - 22 \cdot (-6)) + (-1)2(4 \cdot (-9) - 1 \cdot 22) - 5(4 \cdot (-6) - 1 \cdot 5) = 0$$

so there exist no a, b which satisfy all three coefficients of i and therefore there exists no λ that satisfies the original equation.

Problem 2. Let $V = \mathbb{R}^2$. If (x_1, x_2) and (y_1, y_2) are elements of V , and $\alpha \in \mathbb{R}$, define

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 y_2),$$

and

$$\alpha \cdot (x_1, x_2) = (\alpha x_1, x_2).$$

Is V a vector space over \mathbb{R} with these operations? Justify your answer.

Solution 2. For V to be a vector space over \mathbb{R} with these operations the following must hold $\forall (x_1, y_1) \in V, \alpha \in \mathbb{R}$:

- (i) $+$ must be commutative and associative
- (ii) \cdot must be associative
- (iii) $0 \in V$
- (iv) There must exist a multiplicative identity for scalar multiplication
- (v) There must exist an additive inverse
- (vi) The additive and multiplicative distributive law must hold
- (i) Commutativity: Let $(x_1, x_2), (y_1, y_2) \in V$.
Given that $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 y_2)$,

$$(y_1, y_2) + (x_1, x_2) = (y_1 + x_1, y_2 x_2) \text{ (by definition)}$$

$$(y_1, y_2) + (x_1, x_2) = (x_1 + y_1, x_2 y_2) \text{ (by commutativity in } \mathbb{R}) \therefore + \text{ under } V \text{ is commutative}$$

Associativity: Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in V$.

$$\text{Given that } (x_1, x_2) + ((y_1, y_2) + (z_1, z_2)) = (x_1, x_2) + (y_1 + z_1, y_2 z_2) = (x_1 + y_1 + z_1, x_2 y_2 z_2)$$

$$= ((x_1, x_2) + (y_1, y_2)) + (z_1, z_2)$$

$$= (x_1 + y_1, x_2 y_2) + (z_1, z_2) \text{ (by definition)}$$

$$= (z_1, z_2) + (x_1 + y_1, x_2 y_2) \text{ (by previously demonstrated commutativity)}$$

$$= (z_1 + x_1 + y_1, z_2 x_2 y_2)$$

$$= (x_1 + y_1 + z_1, x_2 y_2 z_2) \text{ (by commutativity in } \mathbb{R}) \therefore + \text{ under } V \text{ is associative}$$

(ii) Associativity: Let $(x_1, x_2) \in V$ and $\alpha, \beta \in \mathbb{R}$

Given that, by definition, $(\alpha \cdot \beta) \cdot (x_1, x_2) = (\alpha \cdot \beta \cdot x_1, x_2)$

$$\begin{aligned} &= \alpha \cdot (\beta \cdot (x_1, x_2)) \\ &= \alpha \cdot (\beta \cdot x_1, x_2) \\ &= (\alpha \cdot \beta \cdot x_1, x_2) \therefore \cdot \text{ under } V \text{ is associative} \end{aligned}$$

(iii) Let $(x_1, x_2) \in V$

$$\begin{aligned} &= (x_1, x_2) + (0, 1) \text{ (by handwaving)} \\ &= (x_1 + 0, x_2 \cdot 1) \text{ (by definition)} \\ &= (x_1, x_2) \therefore (0, 1) = 0_V \end{aligned}$$

(iv) Let $(x_1, x_2) \in V$

$$\begin{aligned} &= 1 \cdot (x_1, x_2) \\ &= (1 \cdot x_1, x_2) \text{ (by definition)} \\ &= (x_1, x_2) \therefore 1 \text{ is the multiplicative identity} \end{aligned}$$

(v) Let $v, v' \in V$. We are looking to show that $v + v' = 0_V$. By handwaving suppose that the additive inverse exists and is $v' = \left(-x_1, \frac{1}{y_1}\right)$ then,

$$\begin{aligned} v + v' &= (x_1, y_1) + \left(-x_1, \frac{1}{y_1}\right) \\ &= \left(x_1 - x_1, y_1 \frac{1}{y_1}\right) \text{ by definition} \\ &= (0, 1) \therefore \text{ by rules for addition and multiplication under } \mathbb{R}, \left(-x_1, \frac{1}{y_1}\right) \text{ is the additive inverse} \end{aligned}$$

(vi) Additive: Let $v_1, v_2 \in V, \alpha \in \mathbb{R}$

$$\begin{aligned} &= \alpha(v_1 + v_2) \\ &= \alpha((x_1, y_1) + (x_2, y_2)) \\ &= \alpha((x_1 + x_2, y_1 y_2)) \text{ (by definition)} \\ &= (\alpha(x_1 + x_2), y_1 y_2) \text{ (by definition)} \end{aligned}$$

Then,

$$\begin{aligned} &= \alpha v_1 + \alpha v_2 \\ &= \alpha(x_1, y_1) + \alpha(x_2, y_2) \\ &= (\alpha x_1, y_1) + (\alpha x_2, y_2) \text{ (by definition)} \\ &= (\alpha x_1 + \alpha x_2, y_1 y_2) \text{ (by definition)} \\ &= (\alpha(x_1 + x_2), y_1 y_2) \therefore \text{ the distributive law holds} \end{aligned}$$

Multiplicative: Let $v \in V, \alpha, \beta \in \mathbb{R}$

$$\begin{aligned} &= \alpha v + \beta v \\ &= (\alpha x_1, y_1) + (\beta x_1, y_1) \text{ (by definition)} \\ &= (\alpha x_1 + \beta x_1, y_1 y_1) \text{ (by definition)} \\ &= ((\alpha + \beta)x_1, y_1 y_1) \end{aligned}$$

Then,

$$\begin{aligned} &= (\alpha + \beta)v \\ &= ((\alpha + \beta)x_1, y_1) \text{ (by definition)} \\ &= (\alpha x_1 + \beta x_1, y_1) \therefore \text{ the distributive law does not hold} \end{aligned}$$

$\therefore V$ is not a vector space over \mathbb{R} with the given operations.

Problem 3. Let $V = \mathbb{R}^2$. If (x_1, y_1) and (x_2, y_2) are elements of V , and $\alpha \in \mathbb{R}$, define

$$(x_1, y_1) + (x_2, y_2) = (x_1 + 2x_2, y_1 + 3y_2),$$

and

$$\alpha \cdot (x_1, y_1) = (\alpha x_1, \alpha y_1).$$

Is V a vector space over \mathbb{R} with these operations? Justify your answer.

Solution 3. Nicely, this one fails on the first one I checked, additive commutativity: Let $(x_1, x_2), (y_1, y_2) \in V$. Given that $(x_1, x_2) + (y_1, y_2) = (x_1 + 2x_2, y_1 + 3y_2)$,

$$(y_1, y_2) + (x_1, x_2) = (x_2 + 2x_1, y_2 + 3y_1) \therefore + \text{ is not commutative}$$

$\therefore V$ is not a vector space over \mathbb{R} with the given operations.

Problem 4. Which of the following sets are subspaces of \mathbb{R}^3 under the usual operations of addition and scalar multiplication in \mathbb{R}^3 .

(a) $W_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 = x_2 \text{ and } x_3 = -x_2\}.$

(b) $W_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 - 4x_2 - x_3 = 0\}.$

(c) $W_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | 5x_1^2 - 3x_2^2 + 6x_3^2 = 0\}.$

Solution 4. The subspace criterion state that a non-empty subset S is a subspace of V (with scalars F) iff $\forall v_1, v_2 \in S, \alpha \in F$

(i) $0_V \in S$

(ii) $v_1 + v_2 \in S$

(iii) $\alpha v_1 \in S$

Now,

(a) (i) $0_{\mathbb{R}^3} \in W_1$ because $x_1 = 0 \implies x_2 = 0 \implies x_3 = 0$

(ii) Let $v_1, v_2 \in W_1$,

$$\begin{aligned} &= v_1 + v_2 \\ &= (x_1, x_2, x_3) + (y_1, y_2, y_3) \\ &= (x_2, x_2, -x_2) + (y_2, y_2, -y_2) \text{ (by definition)} \\ &= (x_2 + y_2, x_2 + y_2, -x_2 - y_2) \end{aligned}$$

Then, by the definition of W_1 addition here is closed as $x_2 + y_2 = x_2 + y_2$ and $-(x_2 + y_2) = -x_2 - y_2$

(iii) Let $v_1 \in W_1, \alpha \in \mathbb{R}$,

$$\begin{aligned} &= \alpha v_1 \\ &= \alpha(x_1, x_2, x_3) \\ &= (\alpha x_1, \alpha x_2, \alpha x_3) \\ &= (\alpha x_2, \alpha x_2, -\alpha x_2) \end{aligned}$$

Which satisfies the definition of W_1 so W_1 is a subspace of \mathbb{R}^3 as it satisfies the subspace criterion.

(b) (i) $0_{\mathbb{R}^3} \in W_2$ because $x_1 = x_2 = x_3 = 0 \implies 0 - 4 \cdot 0 - 0 = 0$

(ii) Let $v_1, v_2 \in W_2$,

$$\begin{aligned} &= v_1 + v_2 \\ &= (x_1, x_2, x_3) + (y_1, y_2, y_3) \\ &= (x_1 + y_1, x_2 + y_2, x_3 + y_3) \end{aligned}$$

For this to be in W_2 it must satisfy the requirement $(x_1 + y_1) - 4(x_2 + y_2) - (x_3 + y_3) = 0$,

$$\begin{aligned} &= (x_1 + y_1) - 4(x_2 + y_2) - (x_3 + y_3) \\ &= x_1 + y_1 - 4x_2 - 4y_2 - x_3 - y_3 \\ &= (x_1 - 4x_2 - x_3) + (y_1 - 4y_2 - y_3) \\ &= (0) + (0) = 0 \quad (v_1, v_2 \in W_2) \end{aligned}$$

So this set is closed under addition.

(iii) Let $v_1 \in W_2$, $\alpha \in \mathbb{R}$,

$$\begin{aligned} &= \alpha v_1 \\ &= (\alpha x_1, \alpha x_2, \alpha x_3) \end{aligned}$$

Then, again, we must satisfy $\alpha x_1 - 4\alpha x_2 - \alpha x_3 = 0$,

$$\begin{aligned} &= \alpha x_1 - 4\alpha x_2 - \alpha x_3 \\ &= \alpha(x_1 - 4x_2 - x_3) \\ &= \alpha(0) = 0 \end{aligned}$$

So W_2 is a subspace of \mathbb{R}^3 as it satisfies the subspace criterion.

(c) (i) $0_{\mathbb{R}^3} \in W_3$ because $x_1 = x_2 = x_3 = 0 \implies 5 \cdot 0^2 - 3 \cdot 0^2 + 6 \cdot 0^2 = 0$

(ii) Let $v_1, v_2 \in W_2$,

$$\begin{aligned} &= v_1 + v_2 \\ &= (x_1, x_2, x_3) + (y_1, y_2, y_3) \\ &= (x_1 + y_1, x_2 + y_2, x_3 + y_3) \end{aligned}$$

For this to be in W_3 it must satisfy the requirement $5(x_1 + y_1)^2 - 3(x_2 + y_2)^2 + 6(x_3 + y_3)^2 = 0$,

$$\begin{aligned} &= 5(x_1 + y_1)^2 - 3(x_2 + y_2)^2 + 6(x_3 + y_3)^2 \\ &= 5(x_1^2 + 2x_1y_1 + y_1^2) - 3(x_2^2 + 2x_2y_2 + y_2^2) + 6(x_3^2 + 2x_3y_3 + y_3^2) \\ &= 5x_1^2 + 10x_1y_1 + 5y_1^2 - 3x_2^2 - 6x_2y_2 - 3y_2^2 + 6x_3^2 + 12x_3y_3 + 6y_3^2 \\ &= (5x_1^2 - 3x_2^2 + 6x_3^2) + (5y_1^2 - 3y_2^2 + 6y_3^2) + 10x_1y_1 - 6x_2y_2 + 12x_3y_3 \\ &= 10x_1y_1 - 6x_2y_2 + 12x_3y_3 \end{aligned}$$

Which is not necessarily equal to 0 (e.g. $v_1 = v_2 = (1, 1, 1)$) so this set is not closed under addition and therefore not a subspace of \mathbb{R}^3 .