Stochastic Processes with Application to Finance

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Abstract

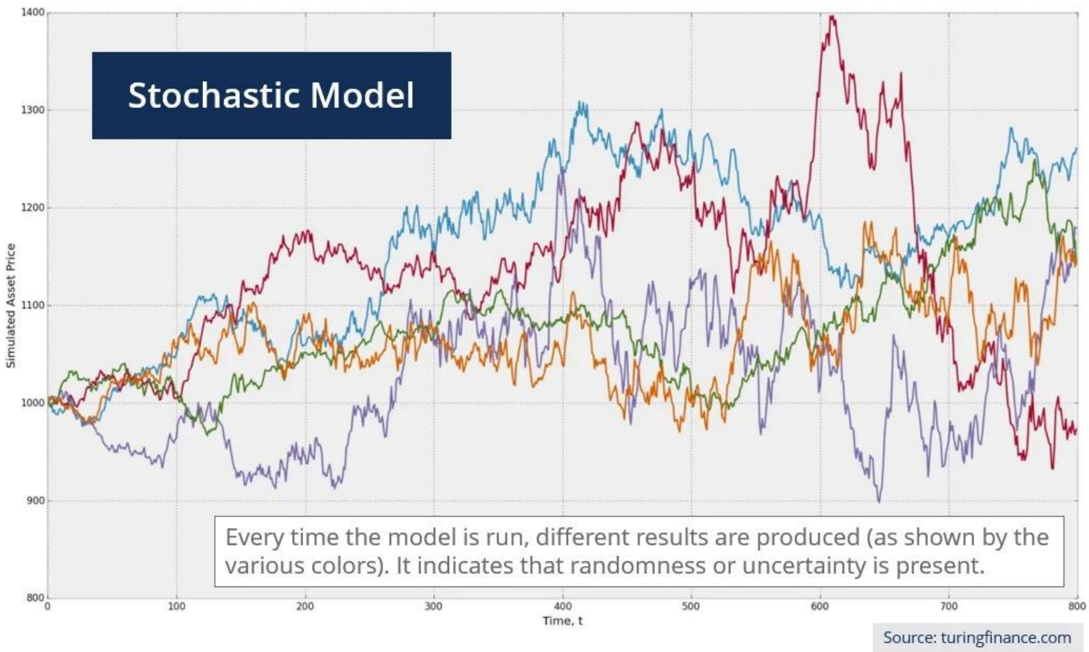
Knowledge about stochastic processes is the foundation needed in order to explore the market. These are used to model asset dynamics and their properties allow us to build model serving several purposes. One of these is pricing. In this thesis we will explore some types of stochastic processes, the theory on which they are based on and some application related to finance. More precisely, we will see diffusion and jump diffusion processes, what makes them palatable for us and how we can play with them to perform different tasks. The focus will be on the theory and the steps needed in order to obtain a valid system to price options. After some introductory, but necessary, topics we will see how pricing of barrier options can be carried out having the process at hand, and the legitimacy of this procedure.

In probability theory and related fields, a stochastic or **random process** is a mathematical object usually defined as a family of random variables in a probability space, where the index of the family often has the interpretation of time. Stochastic processes are widely used as mathematical models of systems and phenomena that appear to vary in a random manner. Examples include the growth of a bacterial population, an electrical current fluctuating due to thermal noise, or the movement of a gas molecule. Asset prices are quantities which are governed by uncertainty. Models to try to predict and forecast prices have been developed with more or less success, but the bottom line is that there is no possibility to do such thing. A reasonable starting point, tough, is to try to study the behaviour of such quantities in the most general form possible, backing up the treatment with solid mathematical ratiocination. This is the purpose of the theory of stochastic process in relation to finance. The need to identify and shape the quantities we are interested in, goes beyond the mere prediction aim, but rather an exploratory rationale. The idea of using stochastic processes to model asset price is an old one, even if the big leaps forward achieved on this matter are rather recent.

The reason why is quite straightforward: stochastic processes describe the evolution, over time in our case, of a random quantity, which, in turn, give rise to aleatory dynamics. Once the theory for the treatment is established, we can start to operate on them, within the limits of it. We will mainly make use of the theory in order to arrive to a fair price for a derivative instrument, which is dependent from a stochastic process itself. The argument could be of different nature too. For example, how to perfectly hedge a position in an option or with an option might be the logical next step.

Financial institutions use stochastic models to assess the risk of their portfolios by simulating a wide range of possible future scenarios. This allows them to estimate Value at Risk (VaR) and Conditional Value at Risk (CVaR), which are crucial metrics for understanding potential losses under adverse market conditions. By employing stochastic processes, risk managers can better prepare for extreme events and make more informed decisions about capital allocation and risk mitigation strategies.

Another significant application is in the field of asset pricing. Stochastic models enable analysts to capture the dynamic behavior of asset prices, which is essential for valuing complex financial instruments. For instance, the Cox-Ross-Rubinstein binomial model uses a discrete-time stochastic process to price options by simulating different paths an asset's price might take over time. This model provides a flexible and intuitive approach to option pricing, accommodating various market conditions and allowing for the incorporation of dividends and other factors. The ability to model the probabilistic nature of asset prices enhances the accuracy of pricing and helps investors identify mispriced securities, thereby uncovering arbitrage opportunities.



In this section, we introduce the classic Ornstein–Uhlenbeck model and its modification based on Lévy processes. The origin of the classic OU process was in 1930, when G.E. Uhlenbeck and L.S. Ornstein showed relation between velocity of a Brownian particle and the normal distribution. Four decades later, in the 1970s, Vasiček proposed a stochastic financial model, where the interest rate was modeled by the OU process. The classic Ornstein–Uhlenbeck process $\{X_t\}_{t\geq 0}$ is defined as a solution of stochastic differential equation of the following form:

$dX_t = \theta(\mu - X_t)dt + \sigma dB_t$.

Parameter μ represents a long-term mean of the OU process, θ is a value of mean-reverting speed, and σ corresponds to the deviation of stochastic factor

For the stock prices described above, there are many equivalent measures under which the discounted price process is a martingale, in contrast to the geometric Brownian model. In other words, such a market is incomplete that is, contingent claims cannot in general be hedged by a suitable portfolio. Because there does not exist a unique equivalent martingale measure, it is not possible simply to use the martingale measure to price a contingent claim in the manner just described. Instead, additional criteria must be used to select an appropriate martingale measure from among the uncountably many such measures with which to price a contingent claim.

Many different approaches to this problem have been proposed in recent years but there is as yet no definitive way of pricing contingent claims in incomplete markets which is preferable to the other possible methods in all situations. Moreover, compared to the large body of work devoted to finding new approaches to option pricing in incomplete markets, relatively little seems to have been done to compare and to investigate the relationship between the various approaches.

Concluding remarks.

We have focussed on only some possible approaches to option pricing under the framework of stock prices driven by Levy processes. There are many other approaches which have been suggested by various authors. For most of these, explicit pricing formulas of the type presented here should be obtainable. The main drawback of this idea is that the choice of utility function depends not only on the investor's risk-averseness or other aspects of the investor's preference, but also on the market itself since the Esscher parameter depends on the market parameters. In any case, for the model studied here based on, the Esscher transform does not admit such an interpretation

Bank

- **Stochastic process** (*proces stochastyczny*): A mathematical object modeling randomness over time.
- **Model** (*model*): A simplified representation of reality for analysis.
- **Assets** (*aktywa*): Resources with economic value, like stocks or bonds.
- **Palatable** (*znośny, akceptowalny*): Easy to accept or apply in a given context.
- **Price options** (*wyceniać opcje*): Determine the value of financial derivatives.
- **Random variable** (*zmienna losowa*): A variable whose value depends on chance.
- **Quantities** (*wielkości*): Measurable amounts or values.
- **Ratiocination** (*rozumowanie*): Logical reasoning or argumentation.
- **Risk** (*ryzyko*): The possibility of loss or adverse outcomes.
- **Scenarios** (*scenariusze*): Possible future situations or outcomes.
- **Pricing** (*wycena*): Assigning value to goods or financial instruments.
- **Normal distribution** (*rozkład normalny*): A symmetric probability distribution in statistics.
- **Interest rate** (*stopa procentowa*): The cost of borrowing money, expressed as a percentage.

- **Path** (*ścieżka, trajektoria*): The trajectory or progression of a random variable over time.
- **Exploration** (tu, badania): Investigating or analyzing something in detail.
- **Derivative instrument** (*instrument pochodny*): A financial contract deriving its value from an underlying asset.
- **Straightforward** (*prosty, jasny*): Easy to understand or implement.

1. True/False.

a) Stochastic models allow for perfect predictions of future asset prices. F

b) Stochastic processes can be used to price barrier options. T

c) Cox-Ross-Rubinstein models are applied only to assets that do not pay dividends. F

d) Stochastic processes enable modeling the dynamic behavior of asset prices. T

2. Fill in the Blanks (scenarios, quantity, path, risk, exploration)

A) Stochastic processes describe the evolution of a ___ (quantity) over time in a random manner.

B) Financial institutions use stochastic models to simulate future market ___ (scenarios).

C) The Cox-Ross-Rubinstein model works by simulating different ___ (paths) an asset's price might take over time.

D) The goal of using stochastic processes in finance is not only prediction but also ___ (exploration) of asset behavior.

E) Stochastic processes are essential for estimating metrics like Conditional Value at ___ (Risk) to manage extreme events.

3. Questions.

1. How do you think stochastic processes help in understanding financial markets?

2. Do you think financial institutions use stochastic models to manage risk?

3. How do you think stochastic processes truly model financial markets well, and if so, why?