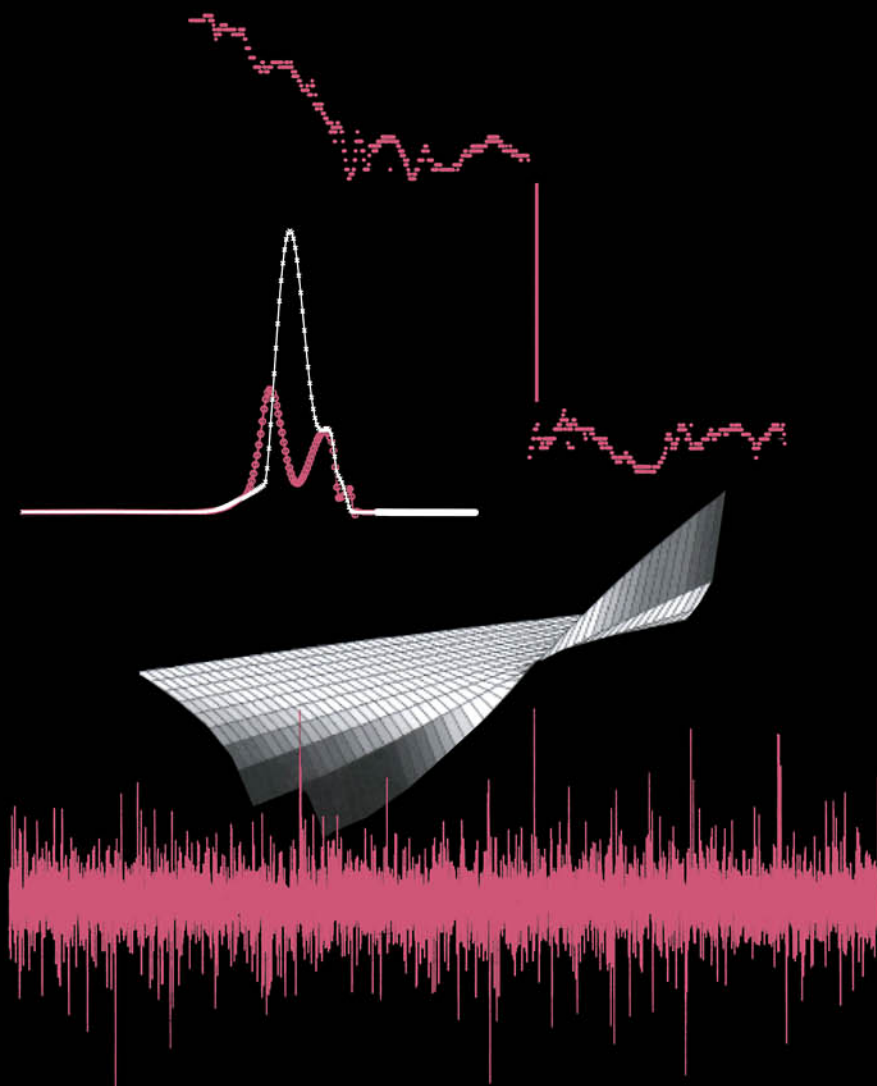


Rama Cont and Peter Tankov

Financial Modelling With Jump Processes



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**Rama Cont
Peter Tankov**

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To Rokhsaneh, for her patience and constant encouragement

To Maria

Preface

During the last decade Lévy processes and other stochastic processes with jumps have become increasingly popular for modelling market fluctuations, both for risk management and option pricing purposes. More than a hundred research papers related to this topic have been published to this date in various finance and applied mathematics journals, leading to a considerable literature which is difficult to master for the nonspecialist. The time seems therefore ripe for a book which can give a self-contained overview of the important aspects of this body of research in a way that can be relevant for applications.

While there exists a considerable volume of mathematical literature related to processes with jumps and Lévy processes in particular, this literature is quite technical and difficult to access for readers not specialized in stochastic analysis. On the other hand many of the applications of jump processes in financial modelling use fairly sophisticated analytical and probabilistic tools which are only explained in advanced mathematical texts. As a result, the recent body of research on the use of jump processes in financial modelling has been difficult to access for end users, who are sometimes under the impression that jump processes and Lévy processes are complicated notions beyond their reach.

We believe that it is not so; the concepts and tools necessary for understanding and implementing these models can be explained in simple terms and in fact are sometimes much more simple and intuitive than the ones involved in the Black-Scholes model and diffusion models.

The motivation for our manuscript is precisely to provide a self-contained overview of theoretical, numerical and empirical research on the use of jump processes in financial modelling, understandable by students, researchers and quants familiar with quantitative methods in finance at the level of classical Black-Scholes option pricing theory.

Our goal has been to:

- explain the motivation for using Lévy processes in financial modelling in terms understandable for nonspecialists
- motivate, through intuitive explanations, the necessity of introducing the various mathematical tools used in the modelling process
- provide precise mathematical statements of results while trying to avoid unnecessary technicalities
- emphasize clarity of exposition over the generality of results in order to maximize their ease of use in applications
- illustrate the mathematical concepts by many numerical and empirical examples
- provide details of numerical implementation for pricing and calibration algorithms
- provide real examples of uses of jump processes in option pricing and risk management
- provide a pedagogical exposition which can be used as teaching material for a graduate course for students in applied mathematics or quantitative finance.

The goal of the present work is *not*:

- to be a comprehensive treatise on mathematical properties of Lévy processes: such treatises already exist [49, 215, 345]. Here we intend to focus on the mathematical tools

necessary in the context of financial modelling, thus omitting other mathematically interesting topics related to Lévy processes.

- to give the most general statement of mathematical results: we have preferred to explain a concept on an example relevant for financial modelling instead of giving abstract theorems.

- to provide an exhaustive survey of the literature on Lévy processes in finance: rather than presenting a catalogue of models we have emphasized common aspects of models in order to provide *modelling tools* and tried to illustrate their use through examples.

The first part of the book (Chapters 2, 3, 4 and 5) presents a concise introduction to the mathematical theory of Lévy processes—processes with independent stationary increments which are building blocks for constructing models with jumps. Chapter 2 presents some preliminary notions on probability spaces, random variables and the Poisson process. Lévy processes are defined in Chapter 3 and their main properties are discussed: behavior of sample paths, distributional properties, the Markov property and their relation to martingales. Examples of one-dimensional Lévy processes frequently used in mathematical finance are presented and studied in Chapter 4. Chapter 5 presents some multidimensional models and tools for building them.

The second part (Chapters 6 and 7) deals with simulation and estimation of models with jumps. Chapter 6 presents various methods for Monte Carlo simulation of Lévy processes in one or several dimensions. Chapter 7 discusses statistical properties of Lévy processes, their advantages and their drawbacks for modelling financial time series.

The third and longest part of the book (Chapters 8 to 13) focuses on option pricing models based on jump processes. After a short introduction to stochastic calculus for jump processes in Chapter 8, we study in Chapter 9 the concept of equivalent change of measure, its relevance in arbitrage pricing theory and its application to Lévy processes. This allows us to show that the models with jumps we consider correspond to arbitrage-free, *incomplete* markets. These notions are further developed in Chapter 10, where we review different approaches to option pricing and hedging in incomplete markets. We then focus on a tractable class of models with jumps: exponential-Lévy models, in which the price of the underlying asset is modelled by the exponential of a Lévy process. Chapter 11 explores properties of option prices in these models, using Fourier-based pricing methods. Option prices in exponential-Lévy models can also be expressed as solutions of certain integro-differential equations: these equations are derived in Chapter 12 and numerical algorithms for solving them are presented. Chapter 13 discusses the problem of model calibration: retrieving parameters of an option pricing model from market prices of options. Several algorithms for solving this problem are presented in the context of exponential-Lévy models and their implementation and empirical performance is discussed.

The last part of the book deals with models with jumps which are not in the exponential-Lévy class. The simplest extensions of exponential-Lévy models are models in which log-returns are independent but not stationary. These time-inhomogeneous models are discussed in Chapter 14. Stochastic volatility models are another important class of models which have been the focus of a lot of recent research. Chapter 15 discusses stochastic volatility models based on Lévy processes.

Some of the results in the book are standard and well known in the literature. In this case our effort has focused on presenting them in a pedagogical way, avoiding unnecessary technicalities, giving appropriate references for further reading when

necessary. Some other parts—in particular the material in Chapters 5, 12, 13, 14—are based on research work done by the authors and collaborators.

One important question was the level of mathematical detail used to treat all these topics. Many research papers dealing with financial applications of jump processes are so technical that they are inaccessible to readers without a graduate degree in probability theory. While technical content is unavoidable, we believe that an alternative exposition is possible, provided that generality is sacrificed in order to gain in clarity. In particular we have chosen to explain the main ideas using Poisson processes and Lévy processes which are tractable examples of models with jumps, mentioning semimartingales briefly in Chapter 8. Accordingly, we have adopted the approach to stochastic integration proposed by P. Protter [324], which is more amenable to the applications considered here. Mathematical definitions and proofs are given in detail when we believe that they are important in the context of financial modelling: this is the case, for example, for the construction of stochastic integrals in Chapter 8. For results of purely “technical” nature we have given appropriate references. Sections with higher technical content are signaled by a (*) and can be skipped at first reading.

Another issue was the level of generality. What classes of models should be considered? Here the approaches in the financial modelling literature tend to be extreme; while some books are entirely focused on diffusion models and Brownian motion, others consider a knowledge of semimartingale theory more or less as a prerequisite. While semimartingales provide the general framework for stochastic integration and theoretical developments in arbitrage theory, financial modelling is focused on *computing* quantities so model building in quantitative finance has almost *exclusively* focused on Markovian models and more precisely *tractable* Markovian models, which are the only ones used in option pricing and risk management. We have therefore chosen to develop the main ideas using Lévy processes, which form a tractable subclass of jump processes for which the theory can be explained in reasonably simple terms. Extensions beyond Lévy processes are considered in the last part of the book, where time-inhomogeneous models and stochastic volatility models with jumps are considered.

We have assumed that the reader is typically familiar with the Black-Scholes model and the machinery behind it—Brownian motion, the Itô formula for continuous processes—but have tried to explain in detail notions specific to jump processes. In particular the Poisson process seems not to be known to many students trained in Black-Scholes theory!

We have tried not to give a catalog of models: since research papers typically focus on specific models we have chosen here a complementary approach namely to provide tools for building, understanding and using models with jumps and referring the reader to appropriate references for details of a particular model. The reader will judge whether we have succeeded in attaining these objectives!

This book grew out of a graduate course on “Lévy processes and applications in finance” given by R. Cont at ENSAE in 2000. R. Cont thanks Griselda Deelstra and Jean-Michel Grandmont for the opportunity to teach this course and the graduate students of ENSAE and Université de Paris IX (DEA MASE) for their participation and interest in this topic which encouraged us to write this book. The material in Chapter 12 resulted from joint work with Ekaterina Voltchkova, who deserves special thanks for the numerous discussions we had on the subject and suggestions for improving this chapter. We also thank Yann Braouezec, Andreas Kyprianou, Cecilia Mancini and Olivier Pantz

for their comments on a preliminary version of the manuscript. Finally we are grateful to Dilip Madan and the editor Sunil Nair for encouraging this project and to the CRC editorial staff who helped us during the final stages: Jasmin Naim, Helena Redshaw, Jamie Sigal and especially Andrea Demby for her careful reading of the manuscript.

Though we have done our best to avoid mistakes, they are unavoidable and we will be grateful to the readers who take their time to inform us of the errors and omissions they might remark. An updated list of corrections, as well as other additional material, will be made available on the website:

<http://www.cmap.polytechnique.fr/~rama/Jumps/>

We hope that this volume will stimulate the interest of students and researchers in applied mathematics and quantitative finance and make the realm of discontinuous stochastic models more accessible to those interested in using them.

Rama CONT and Peter TANKOV

Palaiseau (France), July 2003.

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Chapter 1

Financial modelling beyond Brownian motion

In the end, a theory is accepted not because it is confirmed by conventional empirical tests, but because researchers persuade one another that the theory is correct and relevant.

Fischer Black (1986)

In the galaxy of stochastic processes used to model price fluctuations, Brownian motion is undoubtedly the brightest star. A Brownian motion is a random process W_t with independent, stationary increments that follow a Gaussian distribution. Brownian motion is the most widely studied stochastic process and the mother of the modern stochastic analysis. Brownian motion and financial modelling have been tied together from the very beginning of the latter, when Louis Bachelier [17] proposed to model the price S_t of an asset at the Paris Bourse as:

$$S_t = S_0 + \sigma W_t. \quad (1.1)$$

The multiplicative version of Bachelier's model led to the commonly used Black-Scholes model [60] where the log-price $\ln S_t$ follows a Brownian motion:

$$S_t = S_0 \exp[\mu t + \sigma W_t]$$

or, in local form:

$$\frac{dS_t}{S_t} = \sigma dW_t + \left(\mu + \frac{\sigma^2}{2}\right)dt. \quad (1.2)$$

The process S is sometimes called a *geometric Brownian motion*. Figure 1.1 represents two curves: the evolution of (the logarithm of) the stock price for SLM Corporation (NYSE:SLM) between January 1993 and December 1996 and a sample path of Brownian motion, with the same average volatility as the stock over the three-year period considered. For the untrained eye, it may be difficult to tell which is which: the evolution of the stock does look like a sample path of Brownian motion and examples such as Figure 1.1 are given in many texts on quantitative finance to motivate the use of Brownian motion for modelling price movements.

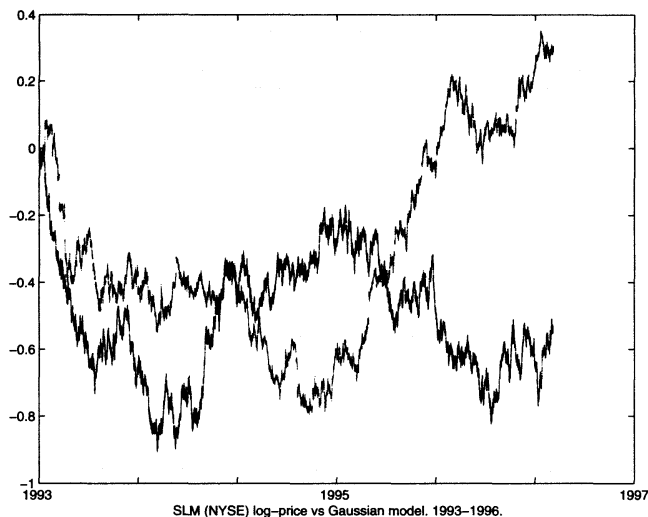


FIGURE 1.1: Evolution of the log price for SLM (NYSE), 1993–1996, compared with a sample path of Brownian motion with same annualized return and volatility. Which is which?

An important property of Brownian motion is the *continuity* of its sample paths: a typical path $t \mapsto B_t$ is a continuous function of time. This remark already allows to distinguish the two curves seen on Figure 1.1: a closer look shows that, unlike Brownian motion, the SLM stock price undergoes several abrupt downward *jumps* during this period, which appear as discontinuities in the price trajectory.

Another property of Brownian motion is its *scale invariance*: the statistical properties of Brownian motion are the same at all time resolutions. Figure 1.2 shows a zoom on the preceding figure, with only the first three months of the three-year period considered above. Clearly, the Brownian path in Figure 1.2 (left) resembles the one in Figure 1.1 and, if the scales were removed from the vertical axis one could not tell them apart. But the evolution of stock price (Figure 1.2, right) does not seem to verify this scale invariance property: the jumps become more visible and now account for more than half of the downward moves in the three-month period! The difference becomes more obvious when we zoom in closer on the price behavior: Figure 1.3 shows the evolution of SLM over a one-month period (February 1993), compared to the simulated sample path of the Brownian motion over the same period. While the Brownian path looks the same as over three years or three months, the price behavior over this period is clearly dominated by a large downward jump, which accounts for half of the monthly return. Finally, if we go down to an *intraday* scale, shown in Figure 1.4, we see that the price moves essentially through jumps while the Brownian model retains the same continuous behavior as over the long horizons.

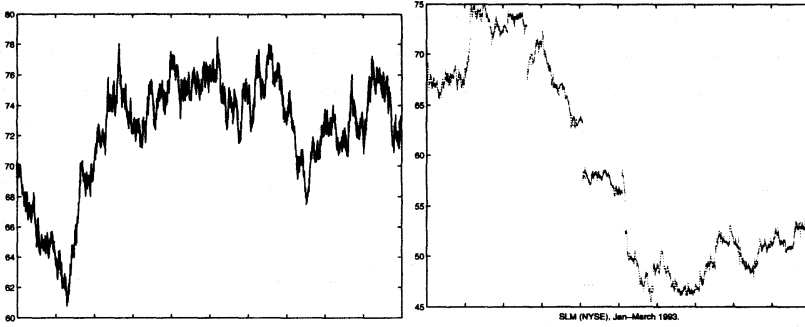


FIGURE 1.2: Evolution of SLM (NYSE), January-March 1993, compared with a scenario simulated from a Black-Scholes model with same annualized return and volatility.

These examples show that while Brownian motion does not distinguish between time scales, price behavior does: prices move essentially by jumps at intraday scales, they still manifest discontinuous behavior at the scale of months and only after coarse-graining their behavior over longer time scales do we obtain something that resembles Brownian motion. Even though a Black-Scholes model can be chosen to give the right variance of returns at a given time horizon, it does not behave properly under *time aggregation*, i.e., across time scales. Since it is difficult to model the behavior of asset returns equally well across all time scales, ranging from several minutes to several years, it is crucial to specify from the onset which time scales are relevant for applications. The perspective of this book being oriented towards option pricing models, the relevant time scales for our purpose range between several days and several months. At these time scales, as seen in Figures 1.2 and 1.3, discontinuities cannot be ignored.

Of course, the Black-Scholes model is not the only continuous time model built on Brownian motion: nonlinear Markov diffusions where instantaneous volatility can depend on the price and time via a *local volatility* function have been proposed by Dupire [122], Derman and Kani [112]:

$$\frac{dS_t}{S_t} = \sigma(t, S_t)dW_t + \mu dt. \quad (1.3)$$

Another possibility is given by stochastic volatility models [196, 203] where the price S is the component of a bivariate diffusion (S_t, σ_t) driven by a two-dimensional Brownian motion (W_t^1, W_t^2) :

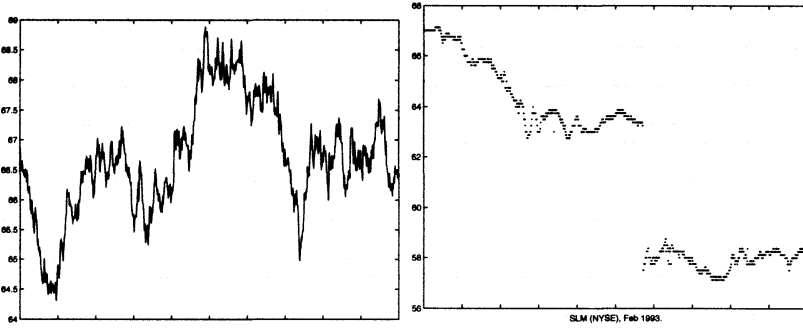


FIGURE 1.3: Price behavior of SLM (NYSE), February 1993, compared with a scenario simulated from a Black-Scholes model with same annualized return and volatility.

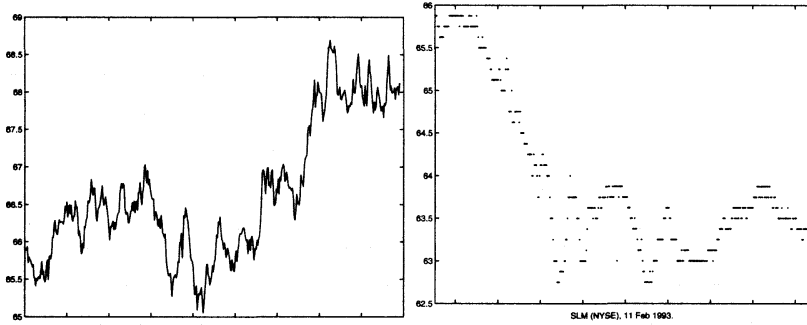


FIGURE 1.4: Price behavior of SLM (NYSE) on February 11, 1993, compared with a scenario simulated from a Black-Scholes model with same annualized return and volatility.

$$\frac{dS_t}{S_t} = \sigma_t dW_t^1 + \mu dt, \quad (1.4)$$

$$\sigma_t = f(Y_t) \quad dY_t = \alpha_t dt + \gamma_t dW_t^2. \quad (1.5)$$

While these models have more flexible statistical properties, they share with Brownian motion the property of *continuity*, which does not seem to be shared by real prices over the time scales of interest. Assuming that prices move in a continuous manner amounts to neglecting the abrupt movements in which most of the *risk* is concentrated.

Since the continuity of paths plays a crucial role in the properties of diffusion models, one would like to know whether results obtained in such models are robust to the removal of the continuity hypothesis. This book presents various stochastic models in which prices are allowed to display a *discontinuous* behavior, similar to that of market prices at the time scales of interest. By examining some of the main issues studied in quantitative finance in the framework of models with *jumps*, we will observe that many results obtained in diffusion models are actually *not* robust to the presence of jumps in prices and thus deserve to be reconsidered anew when jumps are taken into account.

A common approach to promote the use of models with jumps, has been to compare them systematically with the Black-Scholes model, given by Equation (1.2), and conclude that the alternative model is superior in describing empirical observations and its modelling flexibility. Since most classes of models with jumps include the Black-Scholes model as a particular instance, this approach is not serious and we shall not adopt it here. The universe of diffusion models extends far beyond the Black-Scholes framework and the full spectrum of diffusion models, including local volatility models and stochastic volatility models has to be considered in a comparative evaluation of modelling approaches. Our objective is not so much to promote the use of discontinuous models as to provide the reader with the necessary background to understand, explore and compare models with jumps with the more well-known diffusion models.

In the rest of this introductory chapter we will review some of the strengths and weaknesses of diffusion models in three contexts: capturing the empirical properties of asset returns, representing the main features of option prices and providing appropriate tools and insights for hedging and risk management. We will see that, while diffusion models offer a high flexibility and can be *fine-tuned* to obtain various properties, these properties appear as *generic* in models with jumps.

1.1 Models in the light of empirical facts

More striking than the comparison of price trajectories to those of Brownian paths is the comparison of *returns*, i.e., increments of the log-price, which are the relevant quantities for an investor. Figure 1.5 compares the five-minute returns on the Yen/Deutschemark exchange rate to increments of a Brownian motion with the same average volatility. While both return series have the same variance, the Brownian model achieves it by generating returns which always have roughly the same amplitude whereas the Yen/DM returns are widely dispersed in their amplitude and manifest frequent large peaks corresponding to “jumps” in the price. This high variability is a constantly observed feature of financial asset returns. In statistical terms this results in *heavy tails* in the empirical distribution of returns: the tail of the distribution decays slowly at infinity and very large moves have a significant probability of occurring. This well-known fact leads to a poor representation of the distribution of returns by a normal distribution. And no book on financial risk is nowadays

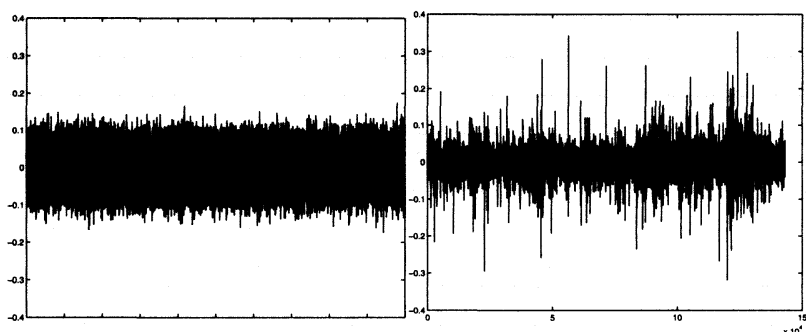


FIGURE 1.5: Five-minute log-returns for Yen/Deutschemark exchange rate, 1992–1995, compared with log-returns of a Black-Scholes model with same annualized mean and variance.

complete without a reference to the traditional six-standard deviation market moves which are commonly observed on all markets, even the largest and the most liquid ones. Since for a normal random variable the probability of occurrence of a value six times the standard deviation is less than 10^{-8} , in a Gaussian model a daily return of such magnitude occurs less than once in a million years! Saying that such a model underestimates risk is a polite understatement. Isn't this an overwhelming argument against diffusion models based on Brownian motion?

Well, not really. Let us immediately dissipate a frequently encountered misconception: nonlinear diffusion processes such as (1.3) or (1.4) are *not* Gaussian processes, even though the driving noise is Gaussian. In fact, as pointed out by Bibby and Sorensen [367], an appropriate choice of a nonlinear diffusion coefficient (along with a linear drift) can generate diffusion processes with arbitrary heavy tails. This observation discards some casual arguments that attempt to dismiss diffusion models simply by pointing to the heavy tails of returns. But, since the only degree of freedom for tuning the local behavior of a diffusion process is the diffusion coefficient, these heavy tails are produced at the price of obtaining highly varying (nonstationary) diffusion coefficients in local volatility models or unrealistically high values of “volatility of volatility” in diffusion-based stochastic volatility models.

By contrast, we will observe that the simplest Markovian models with jumps—Lévy processes—generically lead to highly variable returns with realistic tail behavior without the need for introducing nonstationarity, choosing extreme parameter values or adding unobservable random factors.

But the strongest argument for using discontinuous models is not a statistical one: it is the presence of jumps in the price! While diffusion models can generate heavy tails in returns, they *cannot* generate sudden, discontinuous moves in prices. In a diffusion model tail events are the result of the accumulation of many small moves. Even in diffusion-based stochastic volatility models where market volatility can fluctuate autonomously, it cannot change suddenly. As a result, short-term market movements are approximately Gaussian and their size is predictable. A purist would argue that one cannot tell whether a given large price move is a true discontinuity since observations are made in discrete time. Though true, this remark misses a point: the question is not really to identify whether the price

trajectory is *objectively* discontinuous (if this means anything at all), but rather to propose a model which reproduces the realistic properties of price behavior at the time scales of interest in a generic manner, i.e., without the need to fine-tune parameters to extreme values. While large sudden moves are *generic* properties of models with jumps, they are only obtainable in diffusion processes at the price of fine tuning parameters to extreme values. In a diffusion model the notion of a *sudden*, unpredictable market move, which corresponds to our perception of risk, is difficult to capture and this is where jumps are helpful. We will review the statistical properties of market prices in more detail in Chapter 7 but it should be clear from the onset that the question of using continuous or discontinuous models has important consequences for the representation of risk and is not a purely statistical issue.

1.2 Evidence from option markets

Although an outsider could imagine that the main objective of a stochastic model is to capture the empirical properties of prices, the driving force behind the introduction of continuous-time stochastic models in finance has been the development of *option pricing models*, which serve a somewhat different purpose. Here the logic is different from the traditional time series models in econometrics: an option pricing model is used as a device for capturing the features of option prices quoted on the market, relating prices of market instruments in an arbitrage-free manner (pricing of “vanilla” options consistently with the market) and extrapolating the notion of value to instruments not priced on the market (pricing of exotic options). In short, an option pricing model is an arbitrage-free interpolation and extrapolation tool. Option pricing models are also used to compute hedging strategies and to quantify the risk associated with a given position. Given these remarks, a particular class of models may do a good job in representing time series of returns, but a poor one as a model for pricing and hedging.

1.2.1 Implied volatility smiles and skews

A first requirement for an option pricing model is to capture the state of the options market at a given instant. To achieve this, the parameters of the model are chosen to “fit” the market prices of options or at least to reproduce the main features of these prices, a procedure known as the “calibration” of the model to the market prices. The need for models which can calibrate market prices has been one of the main thrusts behind the generalization of the Black-Scholes model.

The market prices of options are usually represented in terms of their Black-Scholes implied volatilities of the corresponding options. Recall that a European call option on an asset S_t paying no dividends, with maturity date T and strike price K is defined as a contingent claim with payoff $(S_T - K)^+$ at maturity. Denoting by $\tau = T - t$ the time remaining to maturity, the Black-Scholes formula for the value of this call option is:

$$C^{BS}(S_t, K, \tau, \sigma) = S_t N(d_1) - K e^{-r\tau} N(d_2), \quad (1.6)$$

$$d_1 = \frac{-\ln m + \tau(r + \frac{\sigma^2}{2})}{\sigma\sqrt{\tau}}, \quad d_2 = \frac{-\ln m + \tau(r - \frac{\sigma^2}{2})}{\sigma\sqrt{\tau}}. \quad (1.7)$$

where $m = K/S_t$ is the moneyness and $N(u) = (2\pi)^{-1/2} \int_{-\infty}^u \exp(-\frac{z^2}{2}) dz$. Let us now consider, in a market where the hypotheses of the Black-Scholes model do not necessarily hold, a call option whose (observed) market price is denoted by $C_t^*(T, K)$. Since the Black-Scholes value of a call option, as a function of the volatility parameter, is strictly increasing from $]0, +\infty[$ to $](S_t - Ke^{-r\tau})^+, S_0[$, given any observed market price within this range, one can find a value of the volatility parameter $\Sigma_t(T, K)$ such that the corresponding Black-Scholes price matches the market price:

$$\exists! \quad \Sigma_t(T, K) > 0, \quad C^{BS}(S_t, K, \tau, \Sigma_t(T, K)) = C_t^*(T, K). \quad (1.8)$$

In Rebonato's terms [330] the implied volatility is thus a “wrong number which, plugged into the wrong formula, gives the right answer.” Prices in option markets are commonly quoted in terms of Black-Scholes implied volatility. This does not mean that market participants believe in the hypotheses of the Black-Scholes model—they do not: the Black-Scholes formula is not used as a pricing model for vanilla options but as a tool for *translating* market prices into a *representation* in terms of implied volatility.

For fixed t , the implied volatility $\Sigma_t(T, K)$ depends on the characteristics of the option such as the maturity T and the strike level K : the function

$$\Sigma_t : (T, K) \rightarrow \Sigma_t(T, K) \quad (1.9)$$

is called the *implied volatility surface* at date t . A typical implied volatility surface is displayed in Figure 1.6. A large body of empirical and theoretical literature deals with the profile of the implied volatility surface for various markets as a function of (T, K) -or (m, τ) —at a given date, i.e., with (t, S_t) fixed. While the Black-Scholes model predicts a flat profile for the implied volatility surface:

$$\Sigma_t(T, K) = \sigma$$

it is a well documented empirical fact that the implied volatility is not constant as a function of strike nor as a function of time to maturity [95, 96, 121, 330]. This phenomenon can be seen in Figure 1.6 in the case of DAX index options and in Figure 1.7 for S&P 500 index options. The following properties of implied volatility surfaces have been empirically observed [95, 96, 330]:

1. Smiles and skews: for equity and foreign exchange options, implied volatilities $\Sigma_t(T, K)$ display a strong dependence with respect to the strike price: this dependence may be decreasing (“skew”) or U-shaped (“smile”) and has greatly increased since the 1987 crash.
2. Flattening of the smile: the dependence of $\Sigma_t(T, K)$ with respect to K decreases with maturity; the smile/skew flattens out as maturity increases.
3. Floating smiles: if expressed in terms of relative strikes (moneyness $m = K/S_t$), implied volatility patterns vary less in time than when expressed as a function of the strike K .

Coming up with a pricing model which can reproduce these features has become known as the “smile problem” and, sure enough, a plethora of generalizations of the Black-Scholes model have been proposed to deal with it.

How do diffusion models fare with the smile problem? Well, at the level of “fitting” the shape of the implied volatility surface, they do fairly well: as shown by Dupire [122] for any arbitrage-free profile $C_0(T, K), T \in [0, T^*], K > 0$ of call option prices observed at $t=0$, there is a unique “local volatility function” $\sigma(t, S)$ given by

$$\sigma(T, K) = \sqrt{2 \frac{\frac{\partial C_0}{\partial T}(T, K) + Kr \frac{\partial C_0}{\partial K}(T, K)}{K^2 \frac{\partial^2 C_0}{\partial K^2}(T, K)}}, \quad (1.10)$$

which is consistent with these option prices, in the sense that the model (1.3) with $\sigma(\dots)$ given by (1.10) gives back the market prices $C_t(T, K)$ for the call options.

For long maturities, this leads to local volatilities which are roughly constant, predicting a future smile that is much flatter than current smiles which is, in the words of E.Derman, “an uncomfortable and unrealistic forecast that contradicts the omnipresent nature of the skew.” More generally, though local volatility models can fit practically any cross section of prices they give rise to non-intuitive profiles of local volatility which, to this day, have received no interpretation in terms of market dynamics. This means that local volatility models, while providing an elegant solution to the “calibration” problem, do not give an *explanation* of the smile phenomenon.

Diffusion-based stochastic volatility models can also reproduce the profile of implied volatilities at a given maturity fairly well [196, 150]. However, they have more trouble across maturities, i.e., they cannot yield a realistic term structure of implied volatilities [330, 42]. In particular the “at-the-money skew”, which is the slope of implied volatility when plotted against $\ln(K/S_t)$, decays as $1/T$ in most stochastic volatility models [150], at odds with market skews which decay more slowly. In addition, stochastic volatility models require a negative correlation between movements in stock and movements in volatility for the presence of a skew. While this can be reasonably interpreted in terms of a “leverage” effect, it does not explain why in some markets such as options on major foreign exchange rates the “skew” becomes a smile: does the nature of the leverage effect vary with the underlying asset? Nor does this interpretation explain why the smile/skew patterns increased right after the 1987 crash: did the “leverage” effect change its nature

after the crash? Since the instantaneous volatility is unobservable, assertions about its (instantaneous) correlation with the returns are difficult to test but it should be clear from these remarks that the explanation of the implied volatility skew offered by stochastic volatility models is no more “structural” than the explanation offered by local volatility models.

Models with jumps, by contrast, not only lead to a variety of smile/skew patterns but also propose a simple explanation in terms of market anticipations: the presence of a skew is attributed to the fear of large negative jumps by market participants. This is clearly consistent with the fact that the skew/smile features in implied volatility patterns have greatly increased since the 1987 crash; they reflect the “jump fear” of the market participants having experienced the crash [42, 40]. Jump processes also allow to explain the distinction between skew and smile in terms of asymmetry of jumps anticipated by the market: for index options, the fear of a large downward jump leads to a downward skew as in Figure 1.6 while in foreign exchange markets such as USD/EUR where the market moves are symmetric, jumps are expected to be symmetric thus giving rise to smiles.

1.2.2 Short-term options

The shortcomings discussed above are exacerbated when we look at options with short maturities. The very *existence* of a market for short-term options is evidence that jumps in the price are not only present but also recognized as being present by participants in the options market. How else could the underlying asset move 10% out of the money in a few days?

Not only are short-term options traded at significant prices but their market implied volatilities also exhibit a significant skew, as shown for S&P 500 options in Figure 1.7. This feature is unattainable in diffusion-based stochastic volatility models: in these models, the volatility and the price are both continuous and their movements are conditionally Gaussian so one would require ridiculously high values of “volatility of volatility” to obtain realistic short-term skews. In a local volatility model, one can always obtain an arbitrary short-term skew but at the price of a very high variability in the local volatility surface, which is difficult to use or interpret. By contrast, we will see in Chapter 11 that models with jumps generically lead

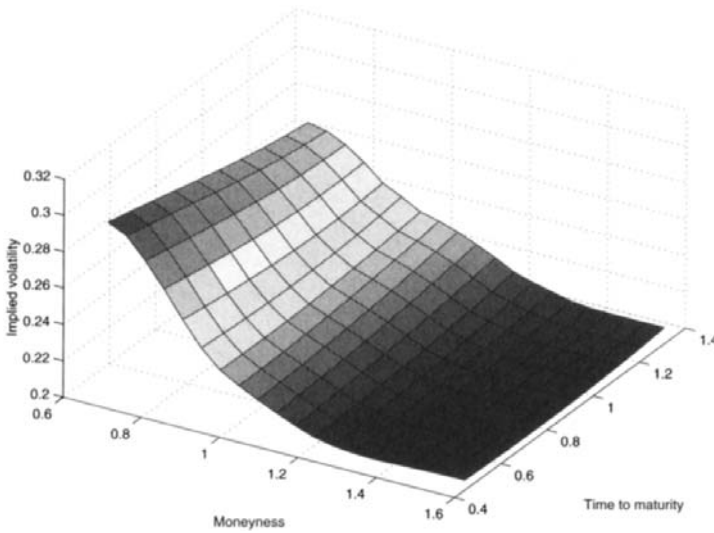


FIGURE 1.6: Implied volatilities of DAX index options, 2001.

to significant skews for short maturities and this behavior can be used to test for the presence of jumps using short-term option prices [85, 314]. More generally, we will see in Chapter 15 that by adding jumps to returns in a stochastic volatility model as in [41] one can easily enhance the empirical performance for short maturities of stochastic volatility models which have an otherwise reasonable behavior for long maturities.

1.3 Hedging and risk management

In the language of financial theory, one-dimensional diffusion models (“local volatility” models) are examples of complete markets: any option can be perfectly replicated by a self-financing strategy involving the underlying and cash. In such markets, options are redundant; they are perfectly substitutable by trading in the underlying so the very existence of an options market becomes a mystery. Of course, this mystery is easily solved: in real markets, perfect hedging is not possible and options enable market participants to hedge risks that cannot be hedged by trading in the underlying only. Options thus allow a better allocation and

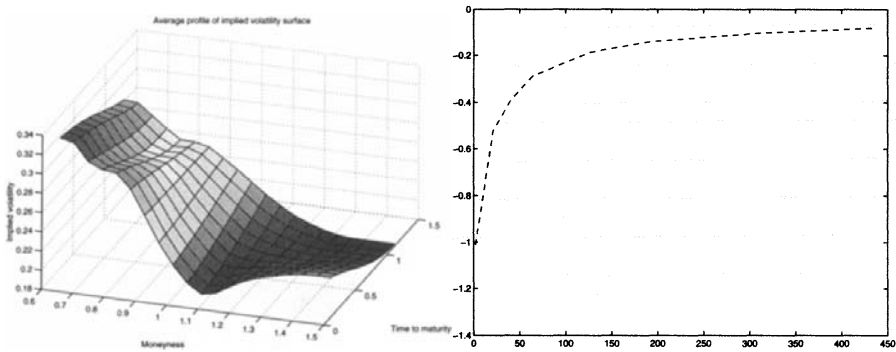


FIGURE 1.7: Left: The implied volatility surface for S&P 500 options.

Right: At-the-money skew $\frac{\partial \Sigma}{\partial K}(K = S, T)$ as a function of the maturity T (in days).

transfer of risk among market participants, which was the purpose for the creation of derivatives markets in the first place [339].

While these facts are readily recognized by most users of option pricing models, the usage has been to twist the complete market framework of diffusion models to adapt it to market realities. On the practical side one complements delta hedging (hedging with the underlying) with gamma and vega hedging. These strategies—while clearly enhancing the performance of “replication” strategies proposed in such models—appear clearly at odds with the model: indeed, in a complete market diffusion model vega and gamma hedges are redundant with respect to delta hedging. On the theoretical side, it has been shown [129] that the Black-Scholes delta-hedging strategy is valid outside the lognormal framework if one uses upper bounds for volatility to price and hedge contingent claims: this property is known as the robustness of the Black-Scholes formula. However, as we will see in Chapter 10, the upper bound for “volatility” in a model with jumps is...infinity! In other words, the only way to perfectly hedge a call option against jumps is to buy and hold the underlying asset. This remark shows that, when moving from diffusion-based complete market models to more realistic models, the concept of “replication,” which is central in diffusion models, does not provide the right framework for hedging and risk management.

Complete market models where every claim can be perfectly hedged by the underlying also fail to explain the common practice of quasi-static hedging of exotic options with vanilla options [4]. Again, this is a natural thing to do in a model with jumps since in such incomplete markets options are not redundant assets and such static (vega) hedges may be used to reduce the residual risk associated with the jumps. Also, as we will see in Chapter 10, the hedge ratio in models with jumps takes into account the possibility of large moves in the underlying asset and therefore partly captures the gamma risk.

Stochastic volatility models do recognize the impossibility of perfectly hedging options with the underlying. However in diffusion-based stochastic volatility models completeness can be restored by adding a single option to the set of available hedging instruments: stochastic volatility models then recommend setting up a perfect hedge by

trading dynamically in the underlying and one option. While options are available and used for hedging, this is often done in a static framework for liquidity reasons: dynamic hedging with options remains a challenge both in theory and practice.

By contrast, from the point of view of the discontinuous price models considered in this book, the nonexistence of a perfect hedge is not a market imperfection but an imperfection of complete market models! We will see that in models with jumps, “riskless replication” is an exception rather than the rule: any hedging strategy has a residual risk which cannot be hedged away to zero and should be taken into account in the exposure of the portfolio. This offers a more realistic picture of risk management of option portfolios. Unlike what is suggested by complete market models, option trading is a risky business!

In the models that we will consider—exponential-Lévy models, jumpdiffusion models, stochastic volatility models with jumps—one has to recognize from the onset the impossibility of perfect hedges and to distinguish the theoretical concept of replication from the practical concept of hedging: the hedging problem is concerned with *approximating* a target future payoff by a trading strategy and involves some risks which need to be quantified and minimized by an appropriate choice of hedging strategy, instead of simply being ignored. These points will be discussed in more detail in Chapter 10.

1.4 Objectives

Table 1.1 lists some of the main messages coming out of more than three decades of financial modelling and risk management and compares them with the messages conveyed by diffusion models and models with jumps. This brief comparison shows that, aside from having various empirical, computational and statistical features that have motivated their use in the first place, discontinuous models deliver *qualitatively different* messages about the key issues of hedging, replication and risk.

Our point, which will be stressed again in Chapter 10, is not so much that diffusion models such as (1.3), (1.4) or even (1.2) do not give good “fits” of empirical data: in fact, they do quite well in some circumstances. The point is that they have the wrong *qualitative* properties and therefore can convey erroneous intuitions about price fluctuations

TABLE 1.1: Modelling market moves: diffusion models vs. models with jumps.

Empirical facts	Diffusion models	Models with jumps
Large, sudden movements in prices.	Difficult: need very large volatilities.	Generic property.
Heavy tails.	Possible by choosing nonlinear volatility structures.	Generic property.
Options are risky investments.	Options can be hedged in a risk-free manner.	Perfect hedges do not exist: options are risky investments.
Markets are incomplete; some risks cannot be hedged.	Markets are complete.	Markets are incomplete.

Concentration: losses are concentrated in a few large downward moves.	Continuity: price movements are conditionally Gaussian; large sudden moves do not occur.	Discontinuity: jumps/discontinuities in prices can give rise to large losses.
Some hedging strategies are better than others.	All hedging strategies lead to the zero residual risk, regardless of the risk measure used.	Hedging strategy is obtained by solving portfolio optimization problem.
Exotic options are hedged using vanilla (call/put) options.	Options are redundant: any payoff can be replicated by dynamic hedging with the underlying.	Options are not redundant: using vanilla options can allow to reduce hedging error.

and the risk resulting from them. We will argue that, when viewed as a subset of the larger family of jump-diffusion models, which are the object of this book, diffusion models should be considered as singularities: while they should certainly be included in all finance textbooks as pedagogical examples, their conclusions for risk measurement and management cannot be taken seriously.

The points outlined above should have convinced the reader that in the models considered in this book we are not merely speaking about a generalization of the classical Black-Scholes model in view of “fitting” the distribution of returns or implied volatility curves with some additional parameters. In addition to matching empirical observations, these models will force us to critically reconsider some of the main concepts which are the backbone of the realm of diffusion models: arbitrage pricing, riskless hedges, market completeness and even the Itô formula!

Our goal has been to provide the reader with the necessary tools for understanding these models and the concepts behind them. Instead of heading for the full generality of “semimartingale” theory we have chosen to focus on tractable families of models with jumps—Lévy processes, additive processes and stochastic volatility models with jumps. The main ideas and modelling tools can be introduced in these models without falling into excessive abstraction.

Exponential Lévy models, introduced in Chapters 3 and 4, offer analytically tractable examples of positive jump processes and are the main focus of the book. They are simple enough to allow a detailed study both in terms of statistical properties (Chapter 7) and as models for risk-neutral dynamics, i.e., option pricing models (Chapter 11). The availability of closed-form expressions for characteristic function of Lévy processes (Chapter 3) enables us to use Fourier transform methods for option pricing. Also, the Markov property of the price will allow us to express option prices as solutions of partial integro-differential equations (Chapter 12). The flexibility of choice of the Lévy measure allows us to calibrate the model to market prices of options and reproduce implied volatility skews/smiles (Chapter 13).

We will see nevertheless that time-homogeneous models such as Lévy processes do not allow for a flexible representation of the term structure of implied volatility and imply empirically undesirable features for forward smiles/skews. In the last part of the book, we will introduce extensions of these models allowing to correct these shortcomings while preserving the mathematical tractability: additive processes (Chapter 14) and stochastic volatility models with jumps (Chapter 15).

Finally, let us stress that we are not striving to promote the systematic use of the models studied in this book. In the course of the exposition we will point out their shortcomings as well as their advantages. We simply aim at providing the necessary background so that jump processes and models built using them will, hopefully, hold no mystery for the reader by the time (s)he has gone through the material proposed here. Table 1.2 provides an

TABLE 1.2: Topics presented in this book

Concepts	Mathematical tools	Chapter
Constructing models with jumps	Poisson random measures	2
	Lévy processes	3,4
Multivariate models	Lévy copulas	5
Time series modelling	Statistical methods	7
Arbitrage pricing	Changes of measure	9
Hedging in incomplete markets	Stochastic calculus	8
Model calibration	Inverse problems and regularization methods	13
Numerical methods for option pricing	Monte Carlo simulation	6
	Fourier transform methods	11
	Finite difference methods	12
Time-inhomogeneous models	Additive processes	14
Stochastic volatility models with jumps	Ornstein-Uhlenbeck processes	15

outline of the different topics presented in this book and the chapters where they are discussed. The chapters have been designed to be as self-contained as possible. However, to learn the necessary mathematical tools, the reader should go through Chapters 2 and 3 before passing to the rest of the book. In addition, it is recommended to read Chapter 8 before passing to Chapters 10 and 12 and to read Chapter 9 before continuing with Chapter 13.

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Symbol Description

$\sigma(A)$	The smallest σ -algebra, with respect to which is measurable. may be a collection of subsets or a collection of functions	JX	Jump measure of a cadlag process X
		$X \stackrel{d}{=} Y$	X and Y have the same distribution
$\#A$	Number of elements in A	$X_n \xrightarrow{d} X$	(X_n) converges to X in distribution
$\mathbf{P}(A)$	Probability of event A	$\mu_n \Rightarrow \mu$	(μ_n) converges weakly to μ
$E[X]$	Expectation of random variable X	$X_n \xrightarrow{P} X$	(X_n) converges to X in probability
Φ_X	Characteristic function of a random variable X		Independent and identically distributed
Ψ_X	Cumulant generating function) of a random variable X i.i.d.		
$\phi_t(\cdot)$	Characteristic function of a Lévy process	a.s.	Almost surely
		a.e.	Almost everywhere
$\psi(\cdot)$	Characteristic exponent of a Lévy process	$B(E)$	Borel σ -algebra of E
		$\omega \in \Omega$	scenario of randomness
M_X	Moment generating function of X		Scalar product of vectors a and b