Optimal valuation of American callable credit default swaps under drawdown

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Zbigniew Palmowski
Wroclaw University of Science and Technology

Budhi Arta Surya
Victoria University of Wellington

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Z. Palmowski [*†], B.A. Surya [‡§]

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Abstract

This paper discusses the valuation of credit default swaps, where default is announced when the reference asset price has gone below certain level from the last record maximum, also known as the high-water mark or drawdown. We assume that the protection buyer pays premium at fixed rate when the asset price is above a pre-specified level and continuously pays whenever the price increases. This payment scheme is in favour of the buyer as she only pays the premium when the market is in good condition for the protection against financial downturn. Under this framework, we look at an embedded option which gives the issuer an opportunity to call back the contract to a new one with reduced premium payment rate and slightly lower default coverage subject to paying a certain cost. We assume that the buyer is risk neutral investor trying to maximize the expected monetary value of the option over a class of stopping time. We discuss optimal solution to the stopping problem when the source of uncertainty of the asset price is modelled by Lévy process with only downward jumps. Using recent development in excursion theory of Lévy process, the results are given explicitly in terms of scale function of the Lévy process. Furthermore, the value function of the stopping problem is shown to satisfy continuous and smooth pasting conditions regardless of regularity of the sample paths of the Lévy process. Optimality and uniqueness of the solution are established using martingale approach for drawdown process and convexity of the scale function under Esscher transform of measure. Some numerical examples are discussed to illustrate the main results.

Keywords: Lévy process; drawdown; credit risk; credit default swaps

[*]Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, ul. Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland, email: zbigniew.palmowski@gmail.com

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[‡]School of Mathematics and Statistics, Victoria University of Wellington, New Zealand, email: budhi.surya@msor.vuw.ac.nz

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1 Credit default swaps

Over the past decades, some discussions have been developed on risk protection mechanism against financial asset's outperformance over its last record maximum, or high-water mark also known as the drawdown, may affect towards fund managers' compensation; see, among others, Agarwal et al. [1] and Goetzmann et al. [5] for details. Such risk may be protected against using an insurance. In their recent works, Zhang et al. [18] and Palmowski and Tumilewicz [11] discussed fair valuation and design of such insurance. Analysis of the Parisian type of default with reference to the last record maximum is given in Surya [14].

In the framework of default by drawdown, we extend the credit default swaps (CDS) problem, see e.g. Leung and Yamazaki [10], by allowing the protection buyer to pay the premium whenever the price of reference asset is above certain threshold and continues to pay when the price is increasing, and pays nothing otherwise. This payment scheme is in favors of the protection buyer whereby she only pays when economy is doing good and stop paying in reverse condition.

The source of uncertainty in the reference asset is modeled by exponential Lévy process. For this purpose, let $X=\{X_t:t\geq 0\}$ be a Lévy process with downward jumps defined on filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t:t\geq 0\},\mathbb{P})$, where $\mathcal{F}_t$ is the natural filtration of X satisfying the usual conditions of right-continuity and completeness. Denote by $\{\mathbb{P}_x, x\in\mathbb{R}\}$ the family of probability measure corresponding to a translation of X such that $X_0=x$, with $\mathbb{P}=\mathbb{P}_0$.

The Lévy-Itô sample paths decomposition of the Lévy process is given by

$$\begin{aligned} X_t = \mu t + \sigma B_t &+ \int_0^t \int_{\{x<-1\}} x\nu(dx,ds) \\ &+ \int_0^t \int_{\{-1\leq x<0\}} x\big(\nu(dx,ds)-\Pi(dx)ds\big), \end{aligned} \tag{1.1}$$

where $\mu\in\mathbb{R}$, $\sigma\geq 0$ and $(B_t)_{t\geq 0}$ is standard Brownian motion, whilst $\nu(dx,dt)$ denotes the Poisson random measure associated with the jumps process $\Delta X_t := X_t - X_{t-}$ of X. This Poisson random measure has compensator given by $\Pi(dx)dt$, where Π is the Lévy measure satisfying the integrability condition:

$$\int_{-\infty}^0 (1\wedge x^2)\Pi(dx) < \infty. \tag{1.2}$$

Due to the absence of positive jumps, it is therefore sensible to define

$$\psi(\lambda) = \frac{1}{t}\log \mathbb{E}\big\{e^{\lambda X_t}\big\} = \mu\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} \big(e^{\lambda x} - 1 - \lambda x \mathbf{1}_{\{x>-1\}}\big)\Pi(dx), \tag{1.3}$$

the Laplace exponent of X, which is analytic on $(\mathfrak{Im}(\lambda)\leq 0)$. It is easily shown that ψ is zero at the origin, tends to infinity at infinity and is strictly convex. We denote by $\Phi:[0,\infty)\to[0,\infty)$ the right continuous inverse of $\psi(\lambda)$, so that

$$\Phi(\theta) = \sup\{p>0:\psi(p)=\theta\} \quad \text{and} \quad \psi(\Phi(\lambda))=\lambda \quad \text{for all} \quad \lambda\geq 0.$$

We refer to Ch. VI in Bertoin [4] or Ch. 2 in Kyprianou [7] for details.

Next, we denote by $\overline{X}_t = \sup_{0 \le s \le t} X_s$ the running maximum of X up to time t and assume that from some arbitrary prior point of reference in time X has the current maximum $y \ge x$. Define $S_t = \overline{X}_t \vee y$, where $a \vee b = \max\{a, b\}$ and the reflected process $Y_t = S_t - X_t$. Recall that the process $Y = \{Y_t : t \ge 0\}$ possesses strong Markov property. Furthermore, we alter slightly our notation for the probability measure $\mathbb{P}_{x,y}$ under which at time zero X has the current maximum $y \ge x$ and position $x \in \mathbb{R}$, and we simply write $\mathbb{P}_{|y} := \mathbb{P}_{0,y}$ to denote the law of Y under which $Y_0 = y$, and use the notation $\mathbb{E}_x$, $\mathbb{E}_{x,y}$ and $\mathbb{E}_{|y}$ to define the corresponding expectation operator to the above probability measures.

Following [11, 14, 18], default is announced as soon as the underlying Lévy process has gone below a fixed level $b > 0$ from its last record maximum,

$$\tau_b^+ = \inf\{t > 0 : Y_t > b\}.$$

We assume that an optimal default level b has been chosen endogenously by optimizing the CDS issuing firm's equity/capital structure such as discussed, among others, in Leland and Toft [9] and Kyprianou and Surya [8]. Under a $T-$year CDS on a unit face value, the protection buyer continuously pays a fixed coupon p, whenever the reference asset price is above current maximum and increasing over time until default occurs or maturity T, whichever is sooner. If default occurs prior to T, the buyer will receive the default payment $\alpha := 1-R$ at default time τ_b^+, where R is the assumed constant recovery rate (typically 40%). From the buyer's perspective, the expected discounted payoff of CDS is

$$\overline{C}_T(x, y, b; p, \alpha) = \mathbb{E}_{x,y}\Big[-\int_0^{\tau_b^+ \wedge T} e^{-rt} p \mathbf{1}_{\{Y_t = 0\}} dt + \alpha e^{-r\tau_b^+} \mathbf{1}_{\{\tau_b^+ \le T\}} \Big], \qquad (1.4)$$

where $r > 0$ is a fixed risk-free interest rate. The quantity $\overline{C}_T(x, y, b; p, \alpha)$ can be viewed as the market price for the buyer to enter (or long) a CDS contract with agreed premium $p > 0$, promised payment upon default α and the maturity T.

On the opposite side of the trade, the protection seller's expected cash flow is $-\overline{C}_T(x, y, b; p, \alpha) = \overline{C}_T(x, y, b; -p, -\alpha)$. In practice, the CDS premium $\overline{p}$ is set as such that $\overline{C}_T(x, y, b; \overline{p}, \alpha) = 0$, yielding zero expected cash flows for both parties. Following (1.4) it is straightforward to show that the credit spreads $\overline{p}$ is

$$\overline{p} = \frac{\alpha \mathbb{E}_{x,y}\big[e^{-r\tau_b^+} \mathbf{1}_{\{\tau_b^+ \le T\}}\big]}{\mathbb{E}_{x,y}\big[\int_0^{\tau_b^+ \wedge T} e^{-rt} \mathbf{1}_{\{Y_t = 0\}} dt\big]}. \qquad (1.5)$$

Recall that the two quantities $\mathbb{E}_{x,y}\big[\int_0^{\tau_b^+ \wedge T} e^{-rt} \mathbf{1}_{\{Y_t=0\}} dt\big]$ and $\mathbb{E}_{x,y}\big[e^{-r\tau_b^+} \mathbf{1}_{\{\tau_b^+ \le T\}}\big]$ are not in general available in explicit form. However, following Theorem 1 in Avram et al. [3] their Laplace transforms on maturity T are given in terms of the so-called scale function $W^{(u)}(x)$ whose Laplace transform is defined by

$$\int_0^\infty e^{-\lambda x} W^{(u)}(x) dx = \frac{1}{\psi(\lambda) - u}, \quad \text{for } \lambda > \Phi(u),\ u > 0, \qquad (1.6)$$

with $W^{(u)}(x) = 0$ for $x < 0$, which is increasing and is continuously differentiable for $x > 0$ when X has paths of unbounded variation and bounded variation with the Levy measure Π has no atoms. See [7] for details. We write $W(x) = W^{(0)}(x)$.

Define the function $Z^{(u)}(x) := 1 + u\int_0^x W^{(u)}(z)dz$. Then, following [3],

$$\mathbb{E}_{x,y}\big[e^{-\lambda\tau_b^+}\big] = Z^{(\lambda)}(b-z) - \lambda\frac{W^{(\lambda)}(b)}{W^{(\lambda)\prime}(b)}W^{(\lambda)}(b-z), \tag{1.7}$$

where $z = y - x \geq 0$. Similarly, the Laplace transform of the discounted payoff $\mathbb{E}_{x,y}\big[\int_0^{\tau_b^+\wedge T} e^{-rt}\mathbf{1}_{\{Y_t=0\}}dt\big]$ with respect to T as a function β could be derived from the identity (2.10) applied for the exponentially killed Lévy process X with intensity β, that is by taking $q = r + \beta$ there. Recall that due to spatial homogeneity of the sample paths of the Lévy process X, it is therefore sufficient to consider the valuation (1.4) under the measure $\mathbb{P}_{|y}$. To simplify analysis from now on we focus only on the perpetual counterpart of (1.4):

$$\overline{C}_\infty(y,b;p,\alpha) = \mathbb{E}_{|y}\Big[-\int_0^{\tau_b^+} e^{-rt}p\mathbf{1}_{\{Y_t=0\}}dt + \alpha e^{-r\tau_b^+}\mathbf{1}_{\{\tau_b^+<\infty\}}\Big]. \tag{1.8}$$

The main goal of this work is pricing American callable credit default swaps.

The organization of this paper is as follows. Section 2 discusses American callable credit default swaps in further details and presents some preliminary results required to solve the corresponding optimal stopping problem. Solution of the stopping problem is presented in Section 3. Optimality and uniqueness of the solution are discussed in Section 4. Some numerical examples are given in Section 5 to exemplify the main results. Section 6 concludes this paper.

2 American callable credit default swaps

Following [10] we consider a credit default swap contract under drawdown which allows the issuer to call/replace the CDS contract once with reduced premium payment rate and lower default coverage. By doing so, the issuer is subject to paying a certain amount of fee γ. For convenient, we consider perpetual case.

At any time prior to default, the buyer can select a stopping time θ to switch to a new contract with a new premium payment rate $\widehat{p} < p$ and default coverage $\widehat{\alpha} < \alpha$ subject to a fee payment γ. The default payment then changes from α to $\widehat{\alpha} = q\alpha$, $q < 1$, after the exercise time θ of the new contract. Given that the buyer is risk neutral, she/he is interested in finding optimal stopping time θ to switch the CDS contract so as to maximizes the net expected cash flow:

$$\begin{aligned}V_b(y;p,\widehat{p},\alpha,\widehat{\alpha},\gamma) = \sup_{\theta\in\mathcal{T}_{[0,\infty)}}\mathbb{E}_{|y}\Big[-\int_0^{\theta\wedge\tau_b^+} e^{-rt}p\mathbf{1}_{\{Y_t=0\}}dt + e^{-r\tau_b^+}\big(\widehat{\alpha}\mathbf{1}_{\{\theta\leq\tau_b^+\}} + \alpha\mathbf{1}_{\{\theta>\tau_b^+\}}\big)\\ - \mathbf{1}_{\{\theta\leq\tau_b^+\}}\Big(\int_\theta^{\tau_b^+} e^{-rt}\widehat{p}\mathbf{1}_{\{Y_t=0\}}dt + e^{-r\theta}\gamma\Big)\Big],\end{aligned} \tag{2.1}$$

where $\mathcal{T}_{[a,b)}$ denotes the class of $\mathcal{F}_t-$stopping times taking values in $[a,b), a \geq 0$. Notice that the pay-off structure (2.1) is slightly different from that of [10].

Proposition 2.1 *Define $\widetilde{p} = \widehat{p} - p$ and $\widetilde{\alpha} = \widehat{\alpha} - \alpha$. Then, following (2.1),*

$$V_b(y; p, \widehat{p}, \alpha, \widehat{\alpha}, \gamma) = \overline{C}_\infty(y, b; p, \alpha) + \mathcal{V}_b(y; \widetilde{p}, \widetilde{\alpha}, \gamma), \tag{2.2}$$

where $\mathcal{V}_b(y; \widetilde{p}, \widetilde{\alpha}, \gamma)$, for $y \in (0, b)$, is the value function of the stopping problem,

$$\mathcal{V}_b(y; \widetilde{p}, \widetilde{\alpha}, \gamma) = \sup_{\theta \in \mathcal{T}_{[0,\infty)}} \mathbb{E}_{|y}\Big[\mathbf{1}_{\{\theta \leq \tau_b^+\}}\Big(-\int_\theta^{\tau_b^+} e^{-rt}\widetilde{p}\mathbf{1}_{\{Y_t=0\}}dt + e^{-r\tau_b^+}\widetilde{\alpha} - e^{-r\theta}\gamma\Big)\Big]. \tag{2.3}$$

Proof By rearranging the payoff structure of the value function (2.1),

$$\begin{aligned}
& -\int_0^{\theta \wedge \tau_b^+} e^{-rt}p\mathbf{1}_{\{Y_t=0\}}dt + e^{-r\tau_b^+}\big(\widehat{\alpha}\mathbf{1}_{\{\theta \leq \tau_b^+\}} + \alpha\mathbf{1}_{\{\theta > \tau_b^+\}}\big) \\
& \qquad\qquad -\mathbf{1}_{\{\theta \leq \tau_b^+\}}\Big(\int_\theta^{\tau_b^+} e^{-rt}\widehat{p}\mathbf{1}_{\{Y_t=0\}}dt + e^{-r\theta}\gamma\Big) \\
= \; & \mathbf{1}_{\{\theta \leq \tau_b^+\}}\Big[-\int_\theta^{\tau_b^+} e^{-rt}\widetilde{p}\mathbf{1}_{\{Y_t=0\}}dt + e^{-r\tau_b^+}\widetilde{\alpha} - e^{-r\theta}\gamma\Big] \\
& \qquad\qquad -\int_0^{\tau_b^+} e^{-rt}p\mathbf{1}_{\{Y_t=0\}}dt + e^{-r\tau_b^+}\alpha. \qquad \square
\end{aligned}$$

This produces the assertion of the proposition. □

We observe following (2.2) that the introduction of the embedded option increases the buyer's expected monetary value by the value function $\mathcal{V}_b(y; \widetilde{p}, \widetilde{\alpha}, \gamma)$.

We will show in the next section that an optimal stopping to the problem (2.3) takes the form of first passage below a level of the drawdown process Y

$$\tau_h^- = \inf\{t > 0 : Y_t < h\}, \text{ with } h > 0, \text{ under } \mathbb{P}_{|y}. \tag{2.4}$$

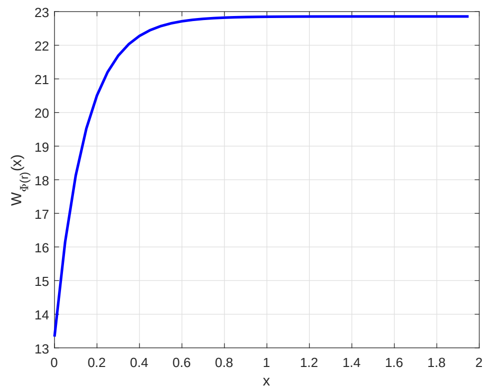
To this end, we denote by $h^\star \in (0, b)$ the largest root, whenever it exists, of

$$\widetilde{\alpha}Z^{(r)}(b-h) - r\widetilde{\alpha}\frac{\big(W^{(r)}(b-h)\big)^2}{W^{(r)\prime}(b-h)} - \gamma = 0. \tag{2.5}$$

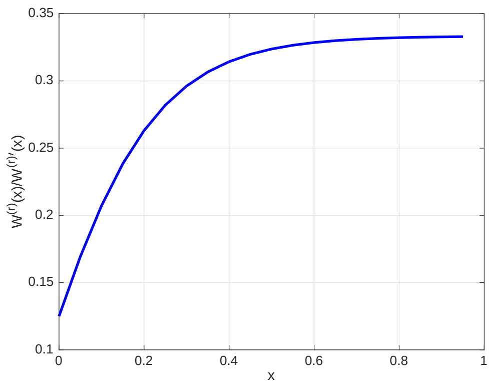
In the section below we give some preliminary results to solve (2.3).

2.1 Preliminaries

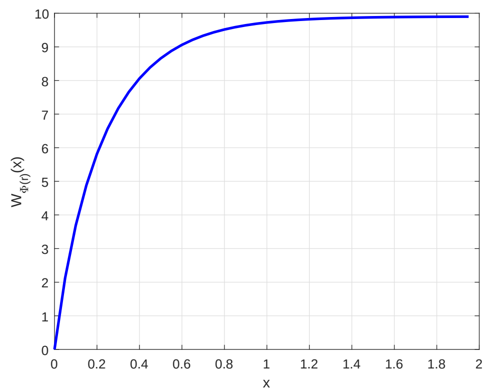
The decomposition of the value function $V_b(y; p, \widehat{p}, \alpha, \widehat{\alpha}, \gamma)$(2.2)-(2.3) can be represented in terms of the scale function $W^{(u)}(x)$ (1.6). The scale function $W^{(u)}(x)$ plays an important role in getting semi-explicit solution to the two-sided exit



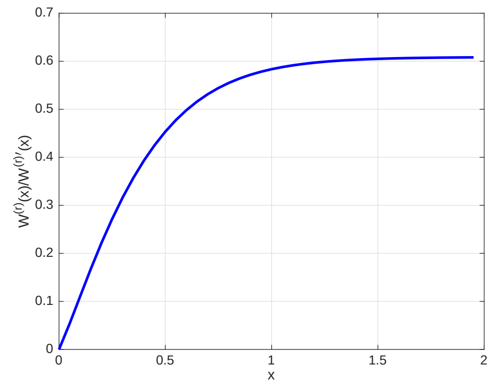
the case $\mu = 0.075$ and $\sigma = 0$.



the case $\mu = 0.075$ and $\sigma = 0$.



the case $\mu = 0.075$ and $\sigma = 0.2$.



the case $\mu = 0.075$ and $\sigma = 0.2$.

Figure 1: Plots of $W_{\Phi(r)}(x)$ (2.9) and $W^{(r)}(x)/W^{(r)\prime}(x)$, with $r = 0.1$, for jump-diffusion process with $\psi(\lambda) = \mu\lambda + \frac{\sigma^2}{2}\lambda^2 - \frac{a\lambda}{\lambda+c}$ for $a = 0.5$ and $c = 9$.

problem of Lévy process X as shown by the following identity. Let T_b^+ and T_b^- be respectively the first entrance time of X into (b,∞) and $(-\infty,-b)$, for $b > 0$, defined by the $\mathcal{F}_t-$stopping times $T_{-b}^- = \inf\{t \geq 0 : X_t < -b\}$ and $T_b^+ = \inf\{t \geq 0 : X_t > b\}$. Under the measure $\mathbb{P}_x$, the identity concerning the first exit of X above level $b \geq x$ before first passage of X below zero is given by

$$\mathbb{E}_x\Big[e^{-uT_b^+}\mathbf{1}_{\{T_b^+ < T_0^-\}}\Big] = \frac{W^{(u)}(x)}{W^{(u)}(b)}. \tag{2.6}$$

The result below plays important role in establishing the main results.

Lemma 2.2 *For $u \geq 0$, $W^{(u)}(x)/W^{(u)\prime}(x)$ is monotone increasing in x, i.e.,*

$$\frac{d}{dx}\Big(\frac{W^{(u)}(x)}{W^{(u)\prime}(x)}\Big) > 0, \ \forall\, x \geq 0, \ \textit{and is bounded above by } 1/\Phi(u). \tag{2.7}$$

Proof It is worth noting that under a new change of measure $\mathbb{P}_x^\nu$ defined by the Esscher transform $d\mathbb{P}_x^\nu/d\mathbb{P}_x = e^{\nu(X_t - x)-\psi(\nu)t}$, $(X,\mathbb{P}_x^\nu)$ is a spectrally negative

Lévy process with Laplace exponent $\psi_\nu(\lambda) = \psi(\lambda+\nu) - \psi(\nu)$. Under the new measure, it is straightforward to check by taking Laplace transform on both sides that $W^{(u)}(x) = e^{\Phi(u)x} W_{\Phi(u)}(x)$, where $W_{\Phi(u)}(x)$ is the scale function under $\mathbb{P}^{\Phi(u)}$. In terms of the distribution $\mathbb{P}\{-\underline{X}_{\mathbf{e}_u} \leq x\}$ of the running infimum $\inf_{0\leq s\leq \mathbf{e}_u} X_s$, with $\mathbf{e}_u$ an independent exponential random time, $W_{\Phi(u)}(x)$ is represented by

$$W_{\Phi(u)}(x) = \frac{\Phi(u)}{u} e^{-\Phi(u)x} \mathbb{P}\{-\underline{X}_{\mathbf{e}_u} \leq x\} + \frac{\Phi(u)^2}{u} \int_0^x e^{-\Phi(u)y} \mathbb{P}\{-\underline{X}_{\mathbf{e}_u} \leq y\} dy,$$

which is continuously differentiable increasing function bounded from above by $1/\psi'(\Phi(u))$. The latter is established on account of the fact that for $\lambda > 0$,

$$\int_0^\infty e^{-\lambda x} \mathbb{P}\{-\underline{X}_{\mathbf{e}_u} \leq x\} dx = \frac{u(\lambda - \Phi(u))}{\lambda \Phi(u)(\psi(\lambda) - u)}.$$

See for e.g. eqn. (7.4.2) and (7.4.3) in [16]. By the above representation and boundednes, it is clear $\lim_{x\to\infty} W'_{\Phi(u)}(x) = 0$. Furthermore, we assume without loss of generality that $\mathbb{P}\{-\underline{X}_{\mathbf{e}_u} \leq x\}$ is concave. It follows $W_{\Phi(u)}(x)$ is concave. See [15] and Ch. 7. in [16] for various numerical examples of $W_{\Phi(u)}(x)$. Hence, $W'_{\Phi(u)}(x)/W_{\Phi(u)}(x)$ is monotone decreasing in x by which (2.7) is proved. $\square$

Example 2.3 *Consider one-sided jump-diffusion process X with $\psi(\lambda) = \mu\lambda + \frac{\sigma^2}{2}\lambda^2 - \frac{a\lambda}{\lambda + c}$ for all $\lambda \in \mathbb{R}$ s.t. $\lambda \neq -c$. It is known, see e.g. [7], that for $u > 0$,*

$$W^{(u)}(x) = \frac{e^{-\xi_2 x}}{\psi'(-\xi_2)} + \frac{e^{-\xi_1 x}}{\psi'(-\xi_1)} + \frac{e^{\Phi(u)x}}{\psi'(\Phi(u))}, \quad \forall x \geq 0, \tag{2.8}$$

where $-\xi_1$, $-\xi_2$, and $\Phi(u)$ denotes three roots of $\psi(\lambda) = u$ s.t. $-\xi_2 < -c < -\xi_1 < 0 < \Phi(u)$. It is straightforward to check that $W_{\Phi(u)}(x)$ is given by

$$W_{\Phi(u)}(x) = \frac{e^{-(\xi_2 + \Phi(u))x}}{\psi'(-\xi_2)} + \frac{e^{-(\xi_1 + \Phi(u))x}}{\psi'(-\xi_1)} + \frac{1}{\psi'(\Phi(u))}, \quad \forall x \geq 0. \tag{2.9}$$

The convexity of $\psi(\lambda)$ implies $\psi'(-\xi_2) < 0$, $\psi'(-\xi_1) < 0$ and $\psi'(\Phi(u)) > 0$. Hence, $W_{\Phi(u)}(x)$ is increasing, concave and bounded from above by $1/\psi'(\Phi(u))$.

The scale functions $W_{\Phi(r)}(x)$ (2.9) and $W^{(r)}(x)/W^{(r)\prime}(x)$ are displayed in Figure 1. Both functions are increasing in x and have non-zero and zero values at $x = 0$ when $\sigma = 0$ (X has bounded variation) and $\sigma \neq 0$ (X has unbounded variation) respectively. Notice that $W^{(r)}(x)/W^{(r)\prime}(x)$ is bounded above by $1/\Phi(r)$.

The key to obtaining solution to (2.2)-(2.3) is given by following identities.

Proposition 2.4 *For given $q, b > 0$ the following identities hold $\forall y \in (0, b)$:*

$$\mathbb{E}_{|y}\Big[\int_0^{\tau_b^+} e^{-qt} \mathbf{1}_{\{Y_t = 0\}} dt\Big] = \frac{W^{(q)}(b-y)}{W^{(q)\prime}(b)}, \tag{2.10}$$

$$\mathbb{E}_{|y}\big[e^{-q\tau_b^+}\big] = Z^{(q)}(b-y) - q\frac{W^{(q)}(b)}{W^{(q)\prime}(b)} W^{(q)}(b-y). \tag{2.11}$$

Proof The proof of (2.11) follows from (1.7) under the probability measure $\mathbb{P}_{|y}$, whereas the proof of (2.10) goes as follows. First, we define the stopping time $\tau_{\{0\}} = \inf\{t \geq 0 : Y_t = 0\}$. Next, notice that $Y_t > 0$ a.s. for all $t < \tau_{\{0\}}$. Therefore, by strong Markov property of Y and iterated law of conditional expectation,

$$\begin{aligned}\mathbb{E}_{|y}\Big[\int_0^{\tau_b^+} e^{-qt}\mathbf{1}_{\{Y_t=0\}}dt\Big] &= \mathbb{E}_{|y}\Big[\mathbf{1}_{\{\tau_{\{0\}}<\tau_b^+\}}\int_{\tau_{\{0\}}}^{\tau_b^+} e^{-qt}\mathbf{1}_{\{Y_t=0\}}dt\Big]\\ &= \mathbb{E}_{|y}\Big[e^{-q\tau_{\{0\}}}\mathbf{1}_{\{\tau_{\{0\}}<\tau_b^+\}}\mathbb{E}_{|Y_{\tau_{\{0\}}}}\Big[\int_0^{\tau_b^+} e^{-qt}\mathbf{1}_{\{Y_t=0\}}dt\Big]\Big]\\ &= \mathbb{E}_{|y}\big[e^{-q\tau_{\{0\}}}\mathbf{1}_{\{\tau_{\{0\}}<\tau_b^+\}}\big]\mathbb{E}_{|0}\Big[\int_0^{\tau_b^+} e^{-qt}\mathbf{1}_{\{Y_t=0\}}dt\Big]. \quad (2.12)\end{aligned}$$

The first expectation on the r.h.s can be worked out using the fact that the two events $\{Y_t, t < \tau_{\{0\}}, \mathbb{P}_{|y}\}$ and $\{-X_t, t < T_0^+, \mathbb{P}_{-y}\}$ are equivalent as such that

$$\mathbb{E}_{|y}\big[e^{-q\tau_{\{0\}}}\mathbf{1}_{\{\tau_{\{0\}}<\tau_b^+\}}\big] = \mathbb{E}_{-y}\big[e^{-qT_0^+}\mathbf{1}_{\{T_0^+<T_{-b}^-\}}\big] = \frac{W^{(q)}(b-y)}{W^{(q)}(b)},$$

where the second equality follows from the identity (2.6). To deal with the second expectation in (2.12), recall that under the measure $\mathbb{P}_{|0}$ the closure of the set of random time $\{t > 0 : Y_t = 0\}$ corresponds to the support of the Stieltjes measure dL_t, where L_t denotes the local time at the running maximum of the Lévy process X. It is known that for spectrally negative Lévy process, L_t is given by the running supremum S_t under $\mathbb{P}_{|0}$, see for instance Ch. VII [4], s.t.,

$$\mathbb{E}_{|0}\Big[\int_0^{\tau_b^+} e^{-qt}\mathbf{1}_{\{Y_t=0\}}dt\Big] = \mathbb{E}_{|0}\Big[\int_0^{\tau_b^+} e^{-qt}dS_t\Big] = \frac{W^{(q)}(b)}{W^{(q)\prime}(b)}, \quad (2.13)$$

where in the last equality we have used the result in eqn. (3.13) of [2]. This completes the proof. □

Remark 2.5 *The identity (2.13) can be used to justify the claim (2.7) on taking account of the fact that for* $0 < b_1 < b_2$, $\tau_{b_1}^+ < \tau_{b_2}^+$ $\mathbb{P}_{|y}-$*almost surely.*

Proposition 2.6 *For a given* $0 < a < b$ *and* $q \geq 0$*, we have for all* $y \in [a, b]$*,*

$$\begin{aligned}\mathbb{E}_{|y}\big[e^{-q\tau_a^-}\mathbf{1}_{\{\tau_a^-\leq\tau_b^+\}}\big] =& \frac{W^{(q)}(b-y)}{W^{(q)}(b-a)}, \quad (2.14)\\ \mathbb{E}_{|y}\big[e^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+\leq\tau_a^-\}}\big] =& Z^{(q)}(b-y) - \frac{Z^{(q)}(b-a)}{W^{(q)}(b-a)}W^{(q)}(b-y). \quad (2.15)\end{aligned}$$

Proof The proof of (2.14) follows from the observation that $\tau_a^- < \tau_{\{0\}}$ a.s. and the equivalent between the two events $\{Y_t, t < \tau_{\{0\}}, \mathbb{P}_{|y}\}$ and $\{-X_t, t < T_0^+, \mathbb{P}_{-y}\}$.

The identity (2.15) is established using the strong Markov property of Y, (2.14), (2.11) along with applying the tower property of conditional expectation, i.e.

$$\mathbb{E}_{|y}\left[e^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+\le\tau_a^-\}}\right]=\mathbb{E}_{|y}\left[e^{-q\tau_b^+}\right]-\mathbb{E}_{|y}\left[e^{-q\tau_a^-}\mathbf{1}_{\{\tau_a^-<\tau_b^+\}}\right]\mathbb{E}_{|a}\left[e^{-q\tau_b^+}\right]. \qquad \square$$

Proposition 2.7 *Define* $G_b(y;\widetilde{p},\widetilde{\alpha}):=\overline{C}_\infty(y,b;\widetilde{p},\widetilde{\alpha})$. *Following (1.8),*

$$G_b(y;\widetilde{p},\widetilde{\alpha}) \;=\; \widetilde{\alpha}Z^{(r)}(b-y)-\frac{\left(\widetilde{p}+r\widetilde{\alpha}W^{(r)}(b)\right)}{W^{(r)\prime}(b)}W^{(r)}(b-y). \tag{2.16}$$

Next, define $\mathcal{G}_b(y;p,\alpha,\gamma)=G_b(y;p,\alpha)-\gamma$. *Then (2.3) becomes*

$$\mathcal{V}_b(y;\widetilde{p},\widetilde{\alpha},\gamma) \;=\; \sup_{\theta\in\mathcal{T}_{[0,\infty)}}\mathbb{E}_{|y}\left[e^{-r\theta}\mathcal{G}_b(Y_\theta;\widetilde{p},\widetilde{\alpha},\gamma);\theta\le\tau_b^+\right]. \tag{2.17}$$

Note that we have used the notational convention: $\mathbb{E}[\cdot;A]=\mathbb{E}[\cdot\mathbf{1}_A]$.

Proof The proof of (2.16) follows from applying Proposition 2.4 to (1.8). The expression for $\mathcal{V}_b$ is obtained by the strong Markov property. First, recall that

$$\mathbb{E}_{|y}\left[e^{-r\tau_b^+}\mathbf{1}_{\{\theta<\tau_b^+\}}\right]=\mathbb{E}_{|y}\left[\mathbb{E}\left[e^{-r\tau_b^+}\mathbf{1}_{\{\theta<\tau_b^+\}}\big|\mathcal{F}_\theta\right]\right]=\mathbb{E}_{|y}\left[e^{-r\theta}\mathbf{1}_{\{\theta<\tau_b^+\}}\mathbb{E}_{|Y_\theta}\left[e^{-r\tau_b^+}\right]\right],$$

where the inner expectation $\mathbb{E}_{|Y_\theta}\left[e^{-r\tau_b^+}\right]$ is given using (2.11) by

$$\mathbb{E}_{|Y_\theta}\left[e^{-r\tau_b^+}\right]=Z^{(r)}(b-Y_\theta)-r\frac{W^{(r)}(b)}{W^{(r)\prime}(b)}W^{(r)}(b-Y_\theta). \tag{2.18}$$

Again, by iterated law of conditional expectation and strong Markov property,

$$\mathbb{E}_{|y}\left[\mathbf{1}_{\{\theta<\tau_b^+\}}\int_\theta^{\tau_b^+}e^{-rt}\mathbf{1}_{\{Y_t=0\}}dt\right]=\mathbb{E}_{|y}\left[e^{-r\theta}\mathbf{1}_{\{\theta<\tau_b^+\}}\mathbb{E}_{|Y_\theta}\left[\int_0^{\tau_b^+}e^{-rt}\mathbf{1}_{\{Y_t=0\}}dt\right]\right].$$

Following the identity (2.10), the inner expectation is given by

$$\mathbb{E}_{|Y_\theta}\left[\int_0^{\tau_b^+}e^{-rt}\mathbf{1}_{\{Y_t=0\}}dt\right]=\frac{W^{(r)}(b-Y_\theta)}{W^{(r)\prime}(b)}. \tag{2.19}$$

Putting the two pieces (2.19) and (2.18) together leads to $\mathcal{G}_b(Y_\theta;\widetilde{p},\widetilde{\alpha},\gamma)$. $\square$

In the sequel below we use the shorthand notation $\mathcal{V}_b(y)$ and $\mathcal{G}_b(y)$ for the value function $\mathcal{V}_b(y;\widetilde{p},\widetilde{\alpha},\gamma)$ and the payoff function $\mathcal{G}_b(y;\widetilde{p},\widetilde{\alpha},\gamma)$, respectively.

3 Solution of the stopping problem (2.17)

In this section we discuss method of solution to the stopping problem (2.17). We will show that the stopping problem can be reduced to first-passage of drawdown process Y below a fixed level. Our approach is similar to that of proposed by van Moerbeke in [17]. Denote by $\mathcal{L}_Y$ an infinitesimal generator of reflected process $Y = S - X$ defined by

$$\begin{aligned} \mathcal{L}_Y F(z) &= -\mu F'(z) + \frac{\sigma^2}{2} F''(z) \\ &+ \int_{-\infty}^{0} \big[F(z-w) - F(z) + w\mathbf{1}_{\{-1\leq w<0\}} F'(z)\big] \Pi(dw), \end{aligned} \tag{3.1}$$

for bounded continuous function F, which is twice continuously differentiable, i.e., $F \in C_b^0(\mathbb{R}_+) \cap C^2(\mathbb{R}_+)$, where F' and F'' denote the first and second derivative of F. Note that the above generator corresponds to the case where X has paths of unbounded variation with $\sigma > 0$. However, when X has paths of bounded variation, we set $\sigma = 0$ and the total jumps in the integral is replaced by $\int_{\{w<-1\}} \big[F(z-w) - F(z)\big] \Pi(dw)$ followed by adjusting the drift of X (1.1).

To solve the problem (2.17), we reduce the optimal stopping rule to the first-passage below a level of drawdown process Y. That is, we will show the value function of the optimal stopping (2.17) coincides with the function

$$\widetilde{\mathcal{V}}_b(y) = \begin{cases} \mathcal{G}_b(y), & \text{for } y \in [0, h^\star] \\ \mathcal{G}_b(h^\star) \frac{W^{(r)}(b-y)}{W^{(r)}(b-h^\star)}, & \text{for } y \in [h^\star, b], \end{cases} \tag{3.2}$$

where $0 < h^\star < b$ is defined as the largest root, when exists, of equation (2.5).

The result below gives a condition on the switching cost γ for which the equation (2.5) has a unique positively valued solution $h^\star < b$.

Proposition 3.1 *There exists a unique solution $h^\star$ to the equation (2.5) for*

$$\widetilde{\alpha}\Big(Z^{(r)}(b) - r\frac{W^{(r)}(b)^2}{W^{(r)\prime}(b)}\Big) > \gamma > \widetilde{\alpha}\Big(1 - r\frac{W^{(r)}(0)^2}{W^{(r)\prime}(0)}\Big). \tag{3.3}$$

See Lemma 4.3 and Lemma 4.4 in [8] for the values of $W^{(r)}(0)$ and $W^{(r)\prime}(0)$.

Proof The proof is established in two parts. First, we show for a given $r > 0$,

$$Z^{(r)}(b) - r\frac{W^{(r)}(b)^2}{W^{(r)\prime}(b)} < 1 - r\frac{W^{(r)}(0)^2}{W^{(r)\prime}(0)}, \quad \forall\, b \geq 0, \tag{3.4}$$

leading to the existence of such γ. For this purpose, consider the function

$$f_r(b) := Z^{(r)}(b) - r\Big(\frac{W^{(r)}(b)^2}{W^{(r)\prime}(b)} - \frac{W^{(r)}(0)^2}{W^{(r)\prime}(0)}\Big) - 1. \tag{3.5}$$

Taking derivative w.r.t b of $f_r(b)$, we obtain after some calculations that

$$\frac{d}{db}f_r(b) = -rW^{(r)}(b)\frac{d}{db}\Big(\frac{W^{(r)}(b)}{W^{(r)\prime}(b)}\Big) < 0,$$

where the inequality follows on account of (2.7), which in turn leading to (3.4) given that $f_r(0) = 0$ and subsequently to (3.3) as $\widetilde{\alpha} < 0$. Next, from (2.5),

$$\begin{aligned}\frac{d}{dh}\Big(f(h) := \widetilde{\alpha}Z^{(r)}(b-h) - r\widetilde{\alpha}\frac{\big(W^{(r)}(b-h)\big)^2}{W^{(r)\prime}(b-h)} - \gamma\Big) \qquad (3.6)\\ = r\widetilde{\alpha}W^{(r)}(b-h)\frac{d}{dx}\Big(\frac{W^{(r)}(x)}{W^{(r)\prime}(x)}\Big)\Big|_{x=b-h} < 0, \quad \text{for } 0 \le h \le b\end{aligned}$$

where the inequality sign is due to $\widetilde{\alpha} < 0$ and (2.7). Uniqueness of solution to (2.5) follows on account of (3.3) by which we have $f(0) > 0$ and $f(b) < 0$. $\square$

Proposition 3.2 *Let τ_h^-, with $h > 0$, be the stopping time (2.4). Then,*

$$\widetilde{\mathcal{V}}_b(y) = \sup_h \mathbb{E}_{|y}\big[e^{-r\tau_h^-}\mathcal{G}_b\big(Y_{\tau_h^-}\big); \tau_h^- \le \tau_b^+\big]. \qquad (3.7)$$

Proof Recall that in the absence of positive jumps, $Y_{\tau_h^-} = h$ a.s. under $\mathbb{P}_{|y}$. Thus, on account of Proposition 2.6, we have

$$J_h(y) := \mathbb{E}_{|y}\big[e^{-r\tau_h^-}\mathcal{G}_b(Y_{\tau_h^-}); \tau_h^- \le \tau_b^+\big] = \mathcal{G}_b(h)\frac{W^{(r)}(b-y)}{W^{(r)}(b-h)}. \qquad (3.8)$$

By applying first order Euler condition to the function $h \to J_h(y)$, we have

$$\begin{aligned}0 = \frac{\partial}{\partial h}J_h(y) &= W^{(r)}(b-y)\frac{\big(\mathcal{G}_b'(h)W^{(r)}(b-h) + \mathcal{G}_b(h)W^{(r)\prime}(b-h)\big)}{\big[W^{(r)}(b-h)\big]^2}\\ &= W^{(r)}(b-y)\frac{\big[-r\widetilde{\alpha}\big[W^{(r)}(b-h)\big]^2 + \widetilde{\alpha}Z^{(r)}(b-h)W^{(r)\prime}(b-h) - \gamma W^{(r)\prime}(b-h)\big]}{\big[W^{(r)}(b-h)\big]^2},\end{aligned}$$

from which we deduce following Proposition 3.1 that $h^\star$ uniquely solves the equation (2.5). Further calculation shows that

$$\begin{aligned}\frac{\partial^2}{\partial h^2}J_h(y)\Big|_{h=h^\star} =& r\widetilde{\alpha}W^{(r)}(b-h^\star)W^{(r)\prime}(b-h^\star)\\ &\times\frac{\big([W^{(r)\prime}(b-h^\star)]^2 - W^{(r)}(b-h^\star)W^{(r)\prime\prime}(b-h^\star)\big)}{(W^{(r)\prime}(b-h^\star))^2}\\ =& r\widetilde{\alpha}W^{(r)}(b-h^\star)W^{(r)\prime}(b-h^\star)\frac{d}{dx}\Big(\frac{W^{(r)}(x)}{W^{(r)\prime}(x)}\Big)\Big|_{x=b-h^\star},\end{aligned}$$

which by (2.7) confirming that $h^\star$ maximizes the function $h \to J_h(y)$.

Furthermore, for $0 \le y \le h^\star$, $\tau_{h^\star}^- = 0$ a.s. under $\mathbb{P}_{|y}$ leading to $\widetilde{\mathcal{V}}_b(y) = \mathcal{G}_b(y)$ on account of $\mathbb{P}_{|y}\{\tau_b^+ \ge 0\} = 1$, which in turn establishes (3.7) and (3.2). $\square$

We prove the main result on account of the following fact. Necessarily, we assume throughout the remaining that $\gamma \le 0$ satisfying the constraint (3.3).

Proposition 3.3 *The payoff function $\mathcal{G}_b(y)$ of (2.17) satisfies the equation:*

$$\big(\mathcal{L}_Y - r\big)\mathcal{G}_b(y) = r\gamma, \quad \text{for all } 0 \leq y \leq b. \tag{3.9}$$

Note that the left-hand side of inequality (3.9) is well-defined by (2.16) and the smoothness of the scale function, which is $C^1(\mathbb{R}_+)$ when X has paths of finite variation and the Lévy measure has no atom, and is $C^2(\mathbb{R}_+)$ if X has paths of unbounded variation with $\sigma > 0$.

Theorem 3.4 *The value function $\mathcal{V}_b(y;\widetilde{p},\widetilde{\alpha},\gamma)$ of the optimal stopping problem (2.17) is given by (3.2) and is obtained at $\tau_{h^\star}^- := \inf\{t>0 : Y_t < h^\star\}$, i.e.,*

$$\mathcal{V}_b(y;\widetilde{p},\widetilde{\alpha},\gamma) = \mathbb{E}_{|y}\big[e^{-r\tau_{h^\star}^-}\mathcal{G}_b\big(Y_{\tau_{h^\star}^-};\widetilde{p},\widetilde{\alpha},\gamma\big);\tau_{h^\star}^- \leq \tau_b^+\big]. \tag{3.10}$$

Furthermore, regardless of the regularity of the sample paths of X, the value function satisfies both continuous and smooth pasting conditions at the boundary,

$$\begin{aligned}\mathcal{V}_b(y;\widetilde{p},\widetilde{\alpha},\gamma) &= \mathcal{G}_b(y;\widetilde{p},\widetilde{\alpha},\gamma) \quad \text{at } y = h^\star,\\ \mathcal{V}_b'(y;\widetilde{p},\widetilde{\alpha},\gamma) &= \mathcal{G}_b'(y;\widetilde{p},\widetilde{\alpha},\gamma) \quad \text{at } y = h^\star.\end{aligned}$$

Proposition 3.5 *The function $\mathcal{V}_b(y)$ solves uniquely the variational inequality*

$$\max\big\{\mathcal{G}_b(y) - \mathcal{V}_b(y), \big(\mathcal{L}_Y - r\big)\mathcal{V}_b(y)\big\} = 0, \quad \text{for } 0 \leq y \leq b. \tag{3.11}$$

Note that the equation (3.11) may be used/extended to numerically solve the finite-maturity counter part of the optimal stopping problem (2.17).

The above theorem states an optimal solution to the credit default swaps by exercising the call option at reduced premium rate $\widehat{p}$ and lower default payment $\widehat{\alpha}$ when the reference asset is increasing subject to paying a cost γ.

4 Optimality and uniqueness of the solution

The following results are required to establish the main results of Section 3.

Lemma 4.1 *By the strong Markov property, for any $0 \leq h < b$ the processes*

$$\big\{e^{-u(t\wedge\tau_h^-\wedge\tau_b^+)}W^{(u)}(b - Y_{t\wedge\tau_h^-\wedge\tau_b^+})\big\}\ ,\ \big\{e^{-u(t\wedge\tau_h^-\wedge\tau_b^+)}Z^{(u)}(b - Y_{t\wedge\tau_h^-\wedge\tau_b^+})\big\}, \tag{4.1}$$

are $\mathcal{F}_t-$martingale under the probability measure $\mathbb{P}_{|y}$, for $h \leq y < b$.

Proof The proof follows by adapting the approach of [3] for drawdown Lévy process. To be more precise, to show the martingale property of the process $\big\{e^{-u(t\wedge\tau_h^-\wedge\tau_b^+)}W^{(u)}(b - Y_{t\wedge\tau_h^-\wedge\tau_b^+})\big\}$, recall that $W^{(u)}(x) = 0$ for $x < 0$ and the following $\mathbb{P}_{|y}-$almost surely equivalence

$$\mathbf{1}_{\{\tau_h^- \leq \tau_b^+\}} = W^{(u)}(b - Y_{\tau_h^-\wedge\tau_b^+})/W^{(u)}(b-h). \tag{4.2}$$

Thus, following the identity (2.14), (4.2) and the strong Markov property,

$$
\begin{aligned}
&\mathbb{E}_{|y}\left[e^{-u(\tau_h^-\wedge\tau_b^+)}\frac{W^{(u)}(b-Y_{\tau_h^-\wedge\tau_b^+})}{W^{(u)}(b-h)}\Big|\mathcal{F}_t\right]\\
&\qquad= \mathbf{1}_{\{\tau_h^-\wedge\tau_b^+\geq t\}}e^{-ut}\mathbb{E}_{|Y_t}\left[e^{-u(\tau_h^-\wedge\tau_b^+)}\frac{W^{(u)}(b-Y_{\tau_h^-\wedge\tau_b^+})}{W^{(u)}(b-h)}\right]\\
&\qquad\quad+\mathbf{1}_{\{\tau_h^-\wedge\tau_b^+<t\}}e^{-u(\tau_h^-\wedge\tau_b^+)}\frac{W^{(u)}(b-Y_{\tau_h^-\wedge\tau_b^+})}{W^{(u)}(b-h)}\\
&\qquad= \mathbf{1}_{\{\tau_h^-\wedge\tau_b^+\geq t\}}e^{-ut}\frac{W^{(u)}(b-Y_t)}{W^{(u)}(b-h)}\\
&\qquad\quad+\mathbf{1}_{\{\tau_h^-\wedge\tau_b^+<t\}}e^{-u(\tau_h^-\wedge\tau_b^+)}\frac{W^{(u)}(b-Y_{\tau_h^-\wedge\tau_b^+})}{W^{(u)}(b-h)}\\
&\qquad= e^{-u(t\wedge\tau_h^-\wedge\tau_b^+)}\frac{W^{(u)}(b-Y_{t\wedge\tau_h^-\wedge\tau_b^+})}{W^{(u)}(b-h)}.
\end{aligned}
$$

Hence, $\{e^{-u(t\wedge\tau_h^-\wedge\tau_b^+)}W^{(u)}(b-Y_{t\wedge\tau_h^-\wedge\tau_b^+})\}$ is $\mathbb{P}_{|y}$ $\mathcal{F}_t-$martingale. Given that

$$
\mathbf{1}_{\{\tau_b^+\leq\tau_h^-\}} = Z^{(u)}(b-Y_{\tau_h^-\wedge\tau_b^+}) - \frac{Z^{(u)}(b-h)}{W^{(u)}(b-h)}W^{(u)}(b-Y_{\tau_h^-\wedge\tau_b^+}),
$$

one can show using the identity (2.15) and the strong Markov property that

$$
\left\{e^{-u(t\wedge\tau_h^-\wedge\tau_b^+)}\Big(Z^{(u)}(b-Y_{t\wedge\tau_h^-\wedge\tau_b^+}) - \frac{Z^{(u)}(b-h)}{W^{(u)}(b-h)}W^{(u)}(b-Y_{t\wedge\tau_h^-\wedge\tau_b^+})\Big)\right\},
$$

is $\mathbb{P}_{|y}$ $\mathcal{F}_t-$martingale, and hence so is $\big\{e^{-u(t\wedge\tau_h^-\wedge\tau_b^+)}Z^{(u)}(b-Y_{t\wedge\tau_h^-\wedge\tau_b^+})\big\}$. □

Proposition 4.2 *For any $0\leq h<b$, the process $\{e^{-r(t\wedge\tau_h^-\wedge\tau_b^+)}G_b(Y_{t\wedge\tau_h^-\wedge\tau_b^+})\}$ is $\mathcal{F}_t-$martingale under the measure $\mathbb{P}_{|y}$, with $h\leq y\leq b$.*

Proof The proof is straightforward from applying Lemma 4.1 to (2.16). □

Proposition 4.3 *For all $y\in[h^\star,b]$, the function $\widetilde{\mathcal{V}}_b(y)$ satisfies:*

(i) $\widetilde{\mathcal{V}}_b'(y)\leq 0$ *and* $\widetilde{\mathcal{V}}_b(y)\geq 0$ *(for all $0\leq y\leq b$),*

(ii) $(\mathcal{L}_Y-r)\widetilde{\mathcal{V}}_b(y)=0$,

(iii) $\widetilde{\mathcal{V}}_b(y)\geq\mathcal{G}_b(y)$.

Proof

(i) The proof is straightforward following the definition of $\widetilde{\mathcal{V}}_b(y)$ (3.2), (2.7), and the fact that $0 < W^{(r)}(x)$, increasing $\forall x \geq 0$ and for $\widetilde{\alpha}, \widetilde{p} < 0$,

$$\mathcal{G}_b(h^\star) = r\widetilde{\alpha}W^{(r)}(b-h^\star)\Big[\frac{W^{(r)}(b-h^\star)}{W^{(r)\prime}(b-h^\star)} - \frac{W^{(r)}(b)}{W^{(r)\prime}(b)}\Big] - \widetilde{p}\frac{W^{(r)}(b-h^\star)}{W^{(r)\prime}(b)} > 0$$

and the payoff function $\mathcal{G}_b(y)$ is monotone decreasing for all $0 \leq y \leq b$ as

$$\mathcal{G}_b'(y) = \widetilde{p}\frac{W^{(r)\prime}(b-y)}{W^{(r)\prime}(b)} + r\widetilde{\alpha}\frac{\big(W^{(r)}(b)\big)^2}{W^{(r)\prime}(b)}\frac{d}{db}\Big(\frac{W^{(r)}(b-y)}{W^{(r)}(b)}\Big) \leq 0. \qquad \square \quad (4.3)$$

(ii) By Lemma 4.1, the process $\{e^{-r(t\wedge\tau_{h^\star}^-\wedge\tau_b^+)}\widetilde{\mathcal{V}}_b(Y_{t\wedge\tau_{h^\star}^-\wedge\tau_b^+})\}$ is $\mathcal{F}_t-$martingale. Hence, on account that the event $\{t : t < \tau_{\{0\}}\}$ has zero Stieltjes measure dS_t under $\mathbb{P}_{|y}$, it implies that $(\mathcal{L}_Y - r)\widetilde{\mathcal{V}}_b(y) = 0$ for all $y \in [h^\star, b]$, see (4.4).

(iii) The proof follows from definition of $h^\star$ (2.5) and (2.7) by which we have

$$\widetilde{\mathcal{V}}_b'(y) - \mathcal{G}_b'(y) = r\widetilde{\alpha}W^{(r)\prime}(b-y)\Big[\frac{W^{(r)}(b-y)}{W^{(r)\prime}(b-y)} - \frac{W^{(r)}(b-h^\star)}{W^{(r)\prime}(b-h^\star)}\Big] \geq 0.$$

The claim on the majorant property follows as $\widetilde{\mathcal{V}}_b(h^\star) - \mathcal{G}_b(h^\star) = 0$. $\square$

Proposition 4.4 *The process* $\{e^{-r(t\wedge\tau_b^+)}\widetilde{\mathcal{V}}_b(Y_{t\wedge\tau_b^+})\}$ *is* $\mathcal{F}_t-$*supermartingale.*

Proof Given the smoothness of the scale function $W^{(r)}(x)$, we have by applying the change-of-variable formula for the trivariate process (t, S_t, X_t), see Theorem 33 in Protter [13], applied to the discounted process $e^{-r(t\wedge\tau_b^+)}\widetilde{\mathcal{V}}_b(Y_{t\wedge\tau_b^+})$, the Lévy-Itô sample paths decomposition of the discounted process given for $t \geq 0$ by

$$\begin{aligned} e^{-r(t\wedge\tau_b^+)}\widetilde{\mathcal{V}}_b(Y_{t\wedge\tau_b^+}) =& \widetilde{\mathcal{V}}_b(y) + \int_0^{t\wedge\tau_b^+} e^{-ru}\widetilde{\mathcal{V}}_b'(0)\mathbf{1}_{\{Y_u=0\}}dS_u \\ &+ \int_0^{t\wedge\tau_b^+} e^{-ru}\big(\mathcal{L}_Y - r\big)\widetilde{\mathcal{V}}_b(Y_u)du + M_{t\wedge\tau_b^+}, \end{aligned} \qquad (4.4)$$

under $\mathbb{P}_{|y}$, with $0 \leq y \leq b$, where by Doob's optional stopping theorem, $M_{t\wedge\tau_b^+}$ is $\mathcal{F}_t-$martingale with $\mathbb{E}_{|y}\big[M_{t\wedge\tau_b^+}\big] = 0$. By *(ii)* of Proposition 4.3 and that $\widetilde{\mathcal{V}}_b(y) = \mathcal{G}_b(y)$ for $0 \leq y \leq h^\star$, the claim is established on account of (3.9) and (4.3) by which it follows by definition (3.2) of $\widetilde{\mathcal{V}}_b(y)$ that $\widetilde{\mathcal{V}}_b'(0) = \mathcal{G}_b'(0) \leq 0$. $\square$

4.1 Proof of Proposition 3.3

On account of the fact that the event $\{t < \tau_h^- \wedge \tau_b^+\}$ has zero Stieltjes measure dS_t under $\mathbb{P}_{|y}$, with $0 \leq h \leq y \leq b$, it follows from Proposition 4.2 and the paths decomposition (4.4) for the process $\big\{e^{-r(t\wedge\tau_h^-\wedge\tau_b^+)}G_b(Y_{t\wedge\tau_h^-\wedge\tau_b^+})\big\}$ that $(\mathcal{L}_Y - r)G_b(y) = 0$ for all $y \in [h, b]$. Since $\mathcal{G}_b(y) = G_b(y) - \gamma$, it follows that $(\mathcal{L}_Y - r)\mathcal{G}_b(y) = r\gamma$. The claim (3.9) is established given that h is arbitrary. $\square$

4.2 Proof of Theorem 3.4

Recall following (2.17) that the value function $\mathcal{V}_b(y)$ satisfies the majorant property over the payoff function $\mathcal{G}_b(y)$, i.e., $\mathcal{V}_b(y) \geq \mathcal{G}_b(y)$ for all $y \in [0, b]$. More precisely, following (2.17) we have $\mathcal{V}_b(y) \geq \mathbb{E}_{|y}\big[e^{-r\theta}\mathcal{G}_b(Y_\theta); \theta \leq \tau_b^+\big]$ for all stopping time $\theta \in \mathcal{T}_{[0,\infty)}$. Since $0 \in \mathcal{T}_{[0,\infty)}$ and $\mathbb{P}_{|y}\{\tau_b^+ \geq 0\} = 1$, the claim follows for $\theta = 0$. Moreover, $\mathcal{V}_b(y) = \mathcal{G}_b(y)$ holds for some $0 \leq y \leq b$ such that $\mathbb{P}_{|y}\{\theta = 0\} = 1$. The set $\mathcal{S} = \{0 \leq y \leq b : \mathcal{V}_b(y) = \mathcal{G}_b(y)\}$ corresponds to the stopping region of the problem (2.17). If the value function $\mathcal{V}_b(y)$ is continuous, $\mathcal{S}$ is a closed set. The complement $\mathcal{C}$ of the set $\mathcal{S}$ refers to the continuation region of (2.17), i.e., $\mathcal{C} = \{0 \leq y \leq b : \mathcal{V}_b(y) > \mathcal{G}_b(y)\}$. To show that the stopping problem (2.17) can be reduced under (3.9) to the first-passage below a level of drawdown process Y, let us rewrite without loss of generality the problem (2.17) as follows:

$$\mathcal{V}_b(y) = \sup_{\theta \in \mathcal{T}_{[0,\tau_b^+)}} \mathbb{E}_{|y}\big[e^{-r\theta}\mathcal{G}_b(Y_\theta)\big].$$

Note following (2.17) that $\mathcal{V}_b(y) \geq \widetilde{\mathcal{V}}_b(y)$ for all $y \in [0, b)$. Moreover, we have $Y_t \in [0, b]$ for $t \leq \tau_b^+$. To show the reverse inequality we will use Optional Stopping Theorem.

First, we check that the continuous pasting is satisfied at the boundary $y = h^\star$. To show the smooth pasting condition, recall that the right derivative of the function $\mathcal{V}_b(y) = \mathcal{G}_b(h^\star)W^{(r)}(b-y)/W^{(r)}(b-h^\star)$ at the point $y = h^\star$ is given by $\mathcal{V}_b'(h^\star) = -\mathcal{G}_b(h^\star)W^{(r)\prime}(b-h^\star)/W^{(r)}(b-h^\star)$. By evaluating the latter on account of the fact that $h^\star$ solves the equation (2.5) leads to

$$\mathcal{V}_b'(h^\star) = -r\widetilde{\alpha}W^{(r)}(b-h^\star) + \frac{\big(\widetilde{p} + r\widetilde{\alpha}W^{(r)}(b)\big)}{W^{(r)\prime}(b)}W^{(r)\prime}(b-h^\star) = \mathcal{G}_b'(h^\star).$$

As a result, we see that the continuous and smooth pasting conditions are satisfied regardless of the regularity condition on the sample paths of the Lévy process.

Recall following Proposition 4.4 that the process $\{e^{-r(t\wedge\tau_b^+)}\widetilde{\mathcal{V}}_b(Y_{t\wedge\tau_b^+}), t \geq 0\}$ is supermartingale. Hence, by Lemma 7 of Palmowski and Tumilewicz [11], positivity of $\widetilde{\mathcal{V}}_b(y)$, and (iii) of Proposition 4.3 we have for any stopping time θ,

$$\widetilde{\mathcal{V}}_b(y) \geq \mathbb{E}_{|y}\big[e^{-r(\theta\wedge\tau_b^+)}\widetilde{\mathcal{V}}_b(Y_{\theta\wedge\tau_b^+})\big] \geq \mathbb{E}_{|y}\big[e^{-r\theta}\mathcal{G}_b(Y_\theta); \theta \leq \tau_b^+\big]. \tag{4.5}$$

We used above the majorant property $\widetilde{\mathcal{V}}_b(y) \geq \mathcal{G}_b(y)$ that holds $\forall y \in [0, b]$. Taking supremum over all stopping times on the right hand side of (4.5) completes the proof of the first assertion. □

4.3 Proof of Proposition 3.5

It is straightforward to check following Propositions 4.3 and 3.3 that the value function $\mathcal{V}_b(y)$ of the optimal stopping (2.17) satisfies the variational inequality

(3.11). Let (U,d) be a pair solution to (3.11) such that $U(y)\geq 0$ for all $0\leq y\leq b$, $U(y)=\mathcal{G}_b(y)$ for all $0\leq y\leq d$ and $U(y)\geq \mathcal{G}_b(y)$, otherwise. Assume that U has degree of smoothness such that the Lévy-Itô decomposition (4.4) applies, i.e.,

$$\begin{aligned} e^{-r(t\wedge\tau_b^+)}U(Y_{t\wedge\tau_b^+}) =&U(y)+\int_0^{t\wedge\tau_b^+} e^{-ru}U'(0)\mathbf{1}_{\{Y_u=0\}}dS_u \\ &+r\gamma\int_0^{t\wedge\tau_b^+} e^{-ru}\mathbf{1}_{\{0\leq Y_u\leq d\}}du + M_{t\wedge\tau_b^+}. \end{aligned} \tag{4.6}$$

Notice that we have applied the result of Proposition 3.3. Since $(\mathcal{V}_b, h^\star)$ is a pair of optimal solution to the stopping problem (2.17), we have for all $0\leq y\leq b$,

$$\mathcal{V}_b(y)\geq U(y), \tag{4.7}$$

which in turn implies that $h^\star\leq d$. Next, for a given $y\in[h^\star,d]$, we have after replacing t by $\tau_{h^\star}^-$ in the decomposition (4.6) and taking expectation $\mathbb{E}_{|y}$ that

$$\mathbb{E}_{|y}\Big[e^{-r(\tau_{h^\star}^-\wedge\tau_b^+)}U(Y_{\tau_{h^\star}^-\wedge\tau_b^+})\Big] = U(y)+r\gamma\mathbb{E}_{|y}\Big[\int_0^{\tau_{h^\star}^-\wedge\tau_b^+} e^{-ru}\mathbf{1}_{\{0\leq Y_u\leq d\}}du\Big]. \tag{4.8}$$

By positivity of $U(y)$ on $0\leq y\leq b$ and that $U(y)=\mathcal{G}_b(y)$ for $0\leq y\leq d$ along with the fact that $\mathbb{E}_{|y}\big[e^{-r\tau_{h^\star}^-}\mathcal{G}_b(Y_{\tau_{h^\star}^-});\tau_{h^\star}^-\leq\tau_b^+\big]=\mathcal{V}_b(y)$, we then obtain

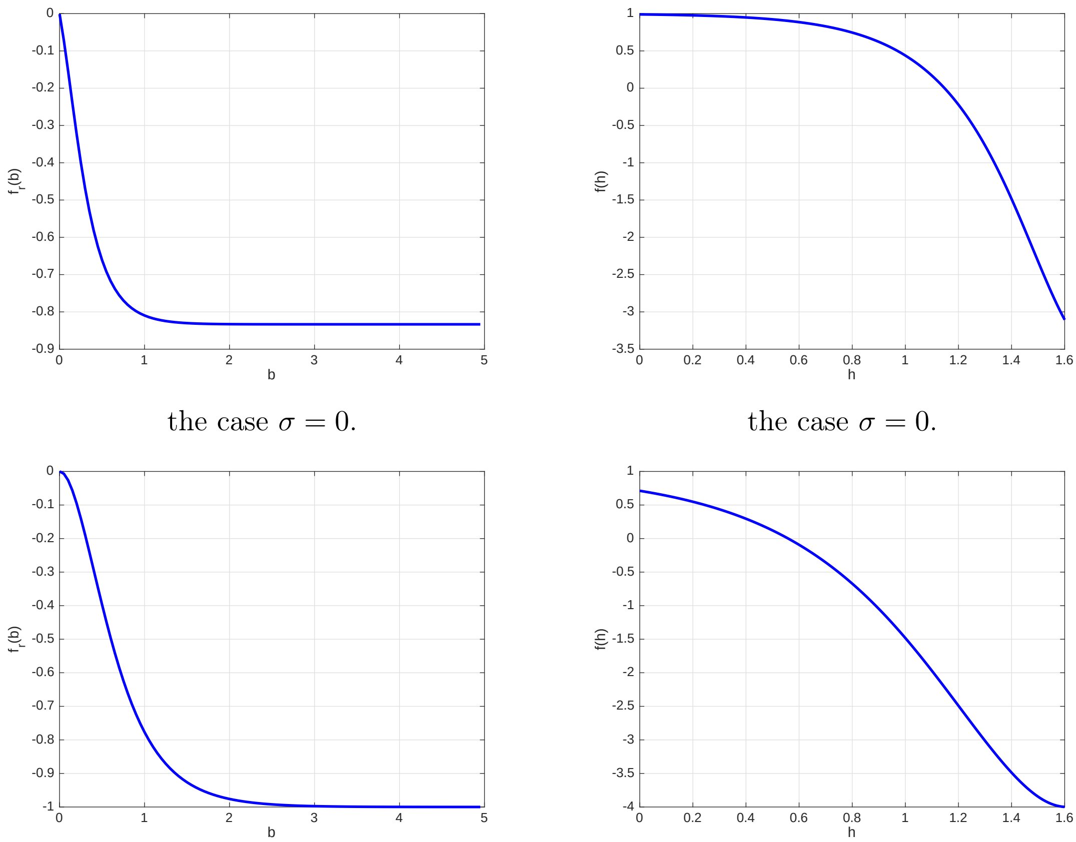
$$U(y)+r\gamma\mathbb{E}_{|y}\Big[\int_0^{\tau_{h^\star}^-\wedge\tau_b^+} e^{-ru}\mathbf{1}_{\{0\leq Y_u\leq d\}}du\Big]\geq \mathcal{V}_b(y),$$

which by (4.7) leads to contrary given that $\gamma\leq 0$. Hence, $\{h^\star\leq d\}$ is an empty set which in turn it follows that $h^\star=d$ and $U(y)=\mathcal{V}_b(y)$ for all $0\leq y\leq b$.

Similar arguments may be adapted to deal with a finite maturity counterpart of the stopping problem (2.17) for which case the proof of unique solution to (3.11) is reduced to showing uniqueness of curved stopping boundary $h^\star(t)$ solving nonlinear integral equation (4.8). This approach was used in Jacka [6], Peskir [12] and Surya [16] for the case of pricing American put option. □

5 Numerical examples

To exemplify the main results, we discuss some numerical examples for one-sided jump-diffusion process X with Laplace exponent $\psi(\lambda)=\mu\lambda+\frac{\sigma^2}{2}\lambda^2-\frac{a\lambda}{\lambda+c}$ for all $\lambda\in\mathbb{R}$ s.t. $\lambda\neq -c$. See Example 2.3 for the corresponding scale function. We set $\mu=0.075$, $a=0.5$ and $c=9$ (on average once every two years the firm suffers an instantaneous loss of 10% of its value). We assume that the firm's default level is $b=\log(5)$ and $r=10\%$. The issuer calls the existing contract with a new one offering $\widetilde{\alpha}=-\$5$ less default coverage with lower premium rate $\widetilde{p}=-2.5\%$ than the existing credit default swap, subject to the switching cost $\gamma=-\$1$.



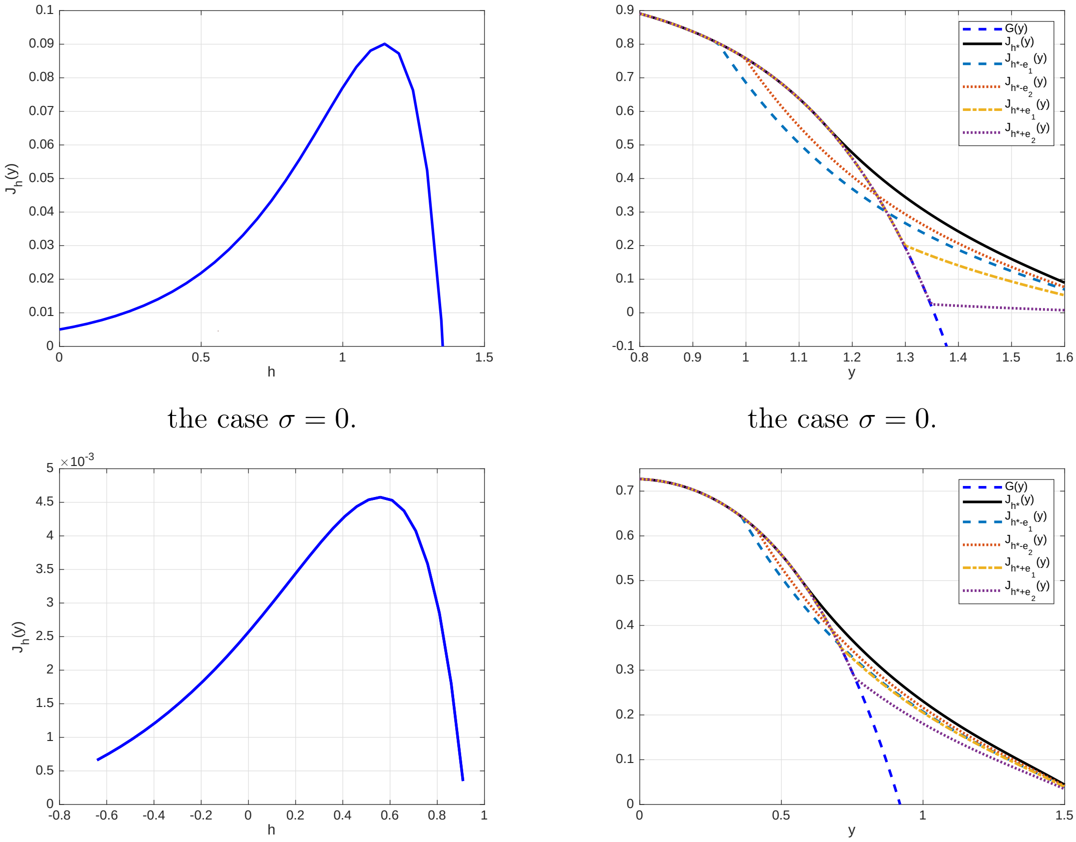
the case $\sigma = 0$. the case $\sigma = 0$.

the case $\sigma = 0.2$. the case $\sigma = 0.2$.

Figure 2: Plots of the functions $f_r(b)$ (3.5) and $f(h)$, the left hand side of (2.5).

We consider two cases: $\sigma = 0$ and $\sigma = 0.2$. The first case corresponds to the underlying process X of the firm value having paths of bounded variation, whereas the other with unbounded variation. Figure 2 displays the function $f_r(b)$ (3.5) introduced in the proof of uniqueness of the solution to eqn. (2.5), and the function $f(h)$, the left hand side of (2.5). In both cases we notice that the two functions exhibit decreasing property which is required in the proof, in particular the function $f(h)$ has a unique root $h = h^\star$ below which $f(h)$ is negative.

Figure 3 presents the shape of the function $J_h(y)$ (3.8) for various values of $h < y$ and the value function $\widetilde{\mathcal{V}}_b(y) = J_{h^\star}(y)$. The optimal stopping level for the case $\sigma = 0$ is $h^\star = 1.1476$, whereas $h^\star = 0.5590$ for $\sigma = 0.2$. In both cases, we see that the function $h \to J_h(y)$ achieves the maximum value at $h = h^\star$. As a result, the value function $\mathcal{V}_b(y)$ of the optimal stopping (2.17) dominates the pay-off function $\mathcal{G}_b(y)$ and sub-optimal solution $J_{h^\star \pm \varepsilon}(y)$, for $\varepsilon > 0$, of the stopping problem (2.17) for all values of $0 \leq y \leq b$. The value function $\widetilde{\mathcal{V}}_b(y)$ is positively valued and is decreasing. Unlike sub-optimal solutions $J_{h^\star \pm \varepsilon}(y)$, $\widetilde{\mathcal{V}}_b(y)$ satisfies both continuous and smooth pasting conditions at the optimal boundary $h^\star$ for



the case $\sigma = 0$. the case $\sigma = 0$.

the case $\sigma = 0.2$. the case $\sigma = 0.2$.

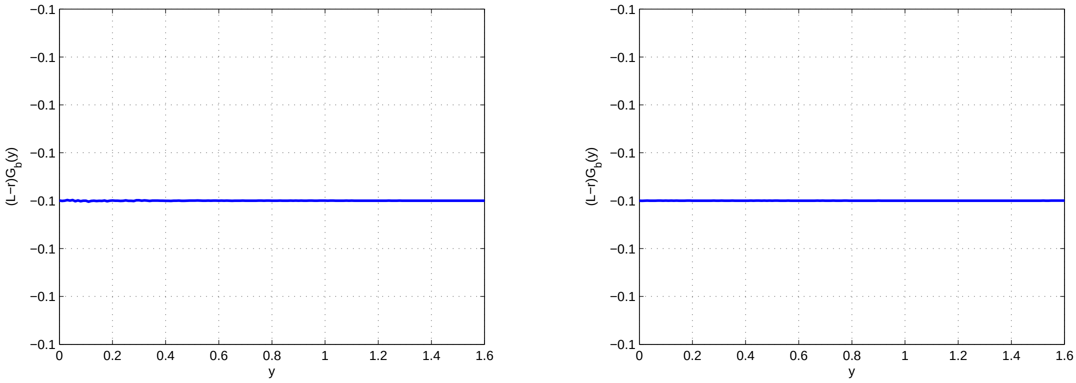
Figure 3: The function $h \to J_h(y)$ and the value function of (2.17).

both cases $\sigma = 0$ and $\sigma = 0.2$, all of which confirm the main results.

Applying the infinitesimal generator $\mathcal{L}_Y$ (3.1) to the payoff function $\mathcal{G}_b(y)$, expressed in terms of the scale function (2.8), the function $(\mathcal{L}_Y - r)\mathcal{G}_b(y)$ is plotted for all $0 \leq y \leq b$ in Figure 4. The graph shows that the function has the same value -0.1 for all $y \in [0, b]$, which is indeed equal to $r\gamma$ (3.9).

6 Conclusion

This paper presents optimal valuation of American call option for credit default swaps under drawdown of Lévy process with only downward jumps. The option gives an opportunity for the issuer to call back the existing swaps contract by replacing it with a new one at reduced premium payment rate with slightly lower default coverage subject to paying some costs. The valuation is formulated in terms of optimal stopping which a risk-neutral protection buyer solves over a class of stopping times adapted to the natural filtration of the asset price. Solution to the optimal stopping problem exists under some constraints imposed



the case $\sigma = 0$. the case $\sigma = 0.2$.

Figure 4: The function $(\mathcal{L}_Y - r)\mathcal{G}_b(y)$ for $0 \le y \le b$.

on the new premium rate, default coverage and the costs to call the contract. The solution is given explicitly in terms of the scale function of the Lévy process. Optimality and uniqueness of the solution are established using martingale approach for drawdown and convexity of the scale function under Esscher transform of measure. Numerical examples are presented to confirm the main results that the solution of the stopping problem (the fair value of the call option) is positively valued, decreasing and has majorant property over the payoff function. Furthermore, it satisfies both continuous and smooth pasting conditions which holds regardless of the regularity of the sample paths of the Lévy process.

References

[1] Agarwal, V., Daniel, N. and Naik, N. (2009). Role of managerial incentives and discretion in hedge fund performance. *J. Finance*, **64(5)**, 2221-2256.

[2] Avram, F., Palmowski, Z. and Pistorius, M.R. (2007). On the optimal dividend problem for a spectrally negative Lévy process. *Ann. Appl. Probab.*, **17(1)**, 156–180.

[3] Avram, F., Kyprianou, A.E. and Pistorius, M.R. (2004). Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian Options. *Ann. Appl. Probab.*, **14**, 215–38.

[4] Bertoin, J. (1996). *Lévy Processes.* Cambridge Tracts in Mathematics **121**. Cambridge: Cambridge Univ. Press.

[5] Goetzmann, W.N., Ingersoll Jr., J.E. and Ross, S.A. (2003). High-water marks and hedge fund management contracts. *J. Finance*, **58(4)**, 1685–1717.

[6] Jacka, S. D. (1991). Optimal stopping and the American put. *Math. Finance*, **1**, 1-14.

[7] Kyprianou, A.E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications.* Berlin: Springer.

[8] Kyprianou, A.E. and Surya, B.A. (2007). Principles of smooth and continuous fit in the determination of endogenous bankruptcy levels. *Finance Stoch.*, **11**, 131–152.

[9] Leland, H.E. and Toft, K.B. (1996). Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *J. Finance*, **51(3)**, 987–1019.

[10] Leung, T. and Yamazaki, K. (2013). American step-up and step-down credit default swaps under Lévy models. *Quant. Finance*, **13(1)**, 137–157.

[11] Palmowski, Z. and Tumilewicz, J. (2019). Fair valuation of Lévy-type drawdown-drawup contracts with general insured and penalty functions. *Appl. Math. Optim.*.

[12] Peskir, G. (2005). On the American option problem. *Math. Finance*, **15**, 169-181.

[13] Protter, P. (2004). *Stochastic Integration and Differential Equations*, Springer-Verlag, Berlin.

[14] Surya, B.A. (2019). Parisian excursion below a fixed level from the last record maximum of Lévy insurance risk process. *In: Wood D., de Gier J., Praeger C., Tao T. (eds) 2017 MATRIX Annals. MATRIX Book Series, Vol 2, Springer Nature, 311-326.*

[15] Surya, B.A. (2008). Evaluating scale functions of spectrally negative Lévy processes. *J. Appl. Probab.*, **45**, 135-149.

[16] Surya, B.A. (2007). *Optimal Stopping Problems Driven by Lévy Processes and Pasting Principles*, PhD dissertation, University of Utrecht.

[17] van Moerbeke, P. (1976). On optimal stopping and free boundary problem. *Arch. Rational Mech. Anal.*, **60**, 101–148.

[18] Zhang, H., Leung, T. and Hadjiliadis, O. (2013). Stochastic modeling and fair valuation of drawdown insurance. *Insur. Math. Econ.*, **53**, 840–850.