5

Stochastic Differential Equations

5.1 Examples and Some Solution Methods

We now return to the possible solutions $X_t(\omega)$ of the stochastic differential equation

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t)W_t, \quad b(t,x) \in \mathbf{R}, \sigma(t,x) \in \mathbf{R} \tag{5.1.1}$$

where W_t is 1-dimensional "white noise". As discussed in Chapter 3 the Itô interpretation of (5.1.1) is that X_t satisfies the stochastic integral equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$$

or in differential form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \ . \tag{5.1.2}$$

Therefore, to get from (5.1.1) to (5.1.2) we formally just replace the white noise W_t by $\frac{dB_t}{dt}$ in (5.1.1) and multiply by dt. It is natural to ask:

(A) Can one obtain existence and uniqueness theorems for such equations? What are the properties of the solutions?
(B) How can one solve a given such equation?

We will first consider question (B) by looking at some simple examples, and then in Section 5.2 we will discuss (A).

It is the Itô formula that is the key to the solution of many stochastic differential equations. The method is illustrated in the following examples.

B. Øksendal, *Stochastic Differential Equations*, Universitext,
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Example 5.1.1 Let us return to the population growth model in Chapter 1:

$$\frac{dN_t}{dt} = a_t N_t \ , \qquad N_0 \text{ given}$$

where $a_t = r_t + \alpha W_t, W_t =$ white noise, $\alpha =$ constant.

Let us assume that $r_t = r =$ constant. By the Itô interpretation (5.1.2) this equation is equivalent to (here $\sigma(t,x) = \alpha x$)

$$dN_t = rN_t dt + \alpha N_t dB_t \tag{5.1.3}$$

or

$$\frac{dN_t}{N_t} = rdt + \alpha dB_t \ .$$

Hence

$$\int_0^t \frac{dN_s}{N_s} = rt + \alpha B_t \quad (B_0 = 0) \ . \tag{5.1.4}$$

To evaluate the integral on the left hand side we use the Itô formula for the function

$$g(t,x) = \ln x \ ; \qquad x > 0$$

and obtain

$$\begin{aligned} d(\ln N_t) &= \frac{1}{N_t} \cdot dN_t + \tfrac{1}{2}\Big(- \frac{1}{N_t^2}\Big)(dN_t)^2 \\ &= \frac{dN_t}{N_t} - \frac{1}{2N_t^2} \cdot \alpha^2 N_t^2 dt = \frac{dN_t}{N_t} - \tfrac{1}{2}\alpha^2 dt \ . \end{aligned}$$

Hence

$$\frac{dN_t}{N_t} = d(\ln N_t) + \tfrac{1}{2}\alpha^2 dt$$

so from (5.1.4) we conclude

$$\ln \frac{N_t}{N_0} = (r - \frac{1}{2}\alpha^2)t + \alpha B_t$$

or

$$N_t = N_0 \exp((r - \tfrac{1}{2}\alpha^2)t + \alpha B_t) \ . \tag{5.1.5}$$

For comparison, referring to the discussion at the end of Chapter 3, the *Stratonovich* interpretation of (5.1.3),

$$d\overline{N}_t = r\overline{N}_t dt + \alpha \overline{N}_t \circ dB_t \ ,$$

would have given the solution

$$\overline{N}_t = N_0 \exp(rt + \alpha B_t) \ . \tag{5.1.6}$$

The solutions $N_t, \overline{N}_t$ are both processes of the type

$$X_t = X_0 \exp(\mu t + \alpha B_t) \qquad (\mu, \alpha \text{ constants}) \ .$$

Such processes are called *geometric Brownian motions*. They are important also as models for stochastic prices in economics. See Chapters 10, 11, 12.

Remark. It seems reasonable that if B_t is independent of N_0 we should have

$$E[N_t] = E[N_0]e^{rt}, \tag{$*$}$$

i.e. the same as when there is no noise in a_t. To see if this is indeed the case, we let

$$Y_t = e^{\alpha B_t}$$

and apply Itô's formula:

$$dY_t = \alpha e^{\alpha B_t} dB_t + \tfrac{1}{2}\alpha^2 e^{\alpha B_t} dt$$

or

$$Y_t = Y_0 + \alpha \int_0^t e^{\alpha B_s} dB_s + \tfrac{1}{2}\alpha^2 \int_0^t e^{\alpha B_s} ds \ .$$

Since $E[\int_0^t e^{\alpha B_s} dB_s] = 0$ (Theorem 3.2.1 (iii)), we get

$$E[Y_t] = E[Y_0] + \tfrac{1}{2}\alpha^2 \int_0^t E[Y_s]ds$$

i.e.

$$\frac{d}{dt}E[Y_t] = \tfrac{1}{2}\alpha^2 E[Y_t], E[Y_0] = 1 \ .$$

So

$$E[Y_t] = e^{\frac{1}{2}\alpha^2 t} \ ,$$

and therefore – as anticipated – we obtain

$$E[N_t] = E[N_0]e^{rt} \ .$$

For the *Stratonovich* solution, however, the same calculation gives

$$E[\overline{N}_t] = E[N_0]e^{(r+\frac{1}{2}\alpha^2)t} \ .$$

Now that we have found the explicit solutions N_t and $\overline{N}_t$ in (5.1.5), (5.1.6) we can use our knowledge about the behaviour of B_t to gain information on these solutions. For example, for the Itô solution N_t we get the following:

(i) If $r > \frac{1}{2}\alpha^2$ then $N_t \to \infty$ as $t \to \infty$, a.s.

(ii) If $r < \frac{1}{2}\alpha^2$ then $N_t \to 0$ as $t \to \infty$, a.s.

(iii) If $r = \frac{1}{2}\alpha^2$ then N_t will fluctuate between arbitrary large and arbitrary small values as $t \to \infty$, a.s.

These conclusions are direct consequences of the formula (5.1.5) for N_t together with the following basic result about 1-dimensional Brownian motion B_t:

Theorem 5.1.2 (The law of iterated logarithm)

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 1 \quad a.s.$$

For a proof we refer to Lamperti (1977), §22.

For the *Stratonovich* solution $\overline{N}_t$ we get by the same argument that $\overline{N}_t \to 0$ a.s. if $r < 0$ and $\overline{N}_t \to \infty$ a.s. if $r > 0$.

Thus the two solutions have fundamentally different properties and it is an interesting question what solution gives the best description of the situation.

Example 5.1.3 Let us return to the equation in Problem 2 of Chapter 1:

$$LQ_t'' + RQ_t' + \frac{1}{C}Q_t = F_t = G_t + \alpha W_t \ . \tag{5.1.7}$$

We introduce the vector

$$X = X(t,\omega) = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} Q_t \\ Q_t' \end{pmatrix} \qquad \text{and obtain}$$

$$\begin{cases} X_1' = X_2 \\ LX_2' = -RX_2 - \frac{1}{C}X_1 + G_t + \alpha W_t \end{cases} \tag{5.1.8}$$

or, in matrix notation,

$$dX = dX(t) = AX(t)dt + H(t)dt + KdB_t \tag{5.1.9}$$

where

$$dX = \begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix}, \ A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{CL} & -\frac{R}{L} \end{pmatrix}, \ H(t) = \begin{pmatrix} 0 \\ \frac{1}{L}G_t \end{pmatrix}, \ K = \begin{pmatrix} 0 \\ \frac{\alpha}{L} \end{pmatrix}, \tag{5.1.10}$$

and B_t is a 1-dimensional Brownian motion.

Thus we are led to a *2-dimensional stochastic differential equation.* We rewrite (5.1.9) as

$$\exp(-At)dX(t) - \exp(-At)AX(t)dt = \exp(-At)[H(t)dt + KdB_t] \ , \tag{5.1.11}$$

where for a general $n \times n$ matrix F we define $\exp(F)$ to be the $n \times n$ matrix given by $\exp(F) = \sum\limits_{n=0}^{\infty} \frac{1}{n!}F^n$. Here it is tempting to relate the left hand side to

$$d(\exp(-At)X(t)) \ .$$

To do this we use a 2-dimensional version of the Itô formula (Theorem 4.2.1).

Applying this result to the two coordinate functions g_1, g_2 of

$$g\colon [0,\infty) \times \mathbf{R}^2 \to \mathbf{R}^2 \qquad \text{given by} \quad g(t,x_1,x_2) = \exp(-At)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ,$$

we obtain that

$$d(\exp(-At)X(t)) = (-A)\exp(-At)X(t)dt + \exp(-At)dX(t) \ .$$

Substituted in (5.1.11) this gives

$$\exp(-At)X(t) - X(0) = \int_0^t \exp(-As)H(s)ds + \int_0^t \exp(-As)K dB_s$$

or

$$\begin{aligned} X(t) = \exp(At)[X(0) &+ \exp(-At)KB_t \\ &+ \int_0^t \exp(-As)[H(s) + AKB_s]ds] \ , \end{aligned} \tag{5.1.12}$$

by integration by parts (Theorem 4.1.5).

Example 5.1.4 Choose $X_t = B_t$, 1-dimensional Brownian motion, and

$$g(t,x) = e^{ix} = (\cos x, \sin x) \in \mathbf{R}^2 \qquad \text{for } x \in \mathbf{R} \ .$$

Then

$$Y(t) = g(t,X_t) = e^{iB_t} = (\cos B_t, \sin B_t)$$

is by Itô's formula again an Itô process.

Its coordinates Y_1, Y_2 satisfy

$$\begin{cases} dY_1(t) = -\sin(B_t)dB_t - \frac{1}{2}\cos(B_t)dt \\ dY_2(t) = \cos(B_t)dB_t - \frac{1}{2}\sin(B_t)dt \ . \end{cases}$$

Thus the process $Y = (Y_1, Y_2)$, which we could call *Brownian motion on the unit circle*, is the solution of the stochastic differential equations

$$\begin{cases} dY_1 = -\frac{1}{2}Y_1 dt - Y_2 dB_t \\ dY_2 = -\frac{1}{2}Y_2 dt + Y_1 dB_t \ . \end{cases} \tag{5.1.13}$$

Or, in matrix notation,

$$dY(t) = -\tfrac{1}{2}Y(t)dt + KY(t)dB_t \ , \qquad \text{where } K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ .$$

Other examples and solution methods can be found in the exercises of this chapter.

For a comprehensive description of reduction methods for 1-dimensional stochastic differential equations see Gard (1988), Chapter 4.

5.2 An Existence and Uniqueness Result

We now turn to the existence and uniqueness question (A) above.

Theorem 5.2.1 (Existence and uniqueness theorem for stochastic differential equations).

Let $T > 0$ *and* $b(\cdot,\cdot)\colon [0,T] \times \mathbf{R}^n \to \mathbf{R}^n, \sigma(\cdot,\cdot)\colon [0,T] \times \mathbf{R}^n \to \mathbf{R}^{n\times m}$ *be measurable functions satisfying*

$$|b(t,x)| + |\sigma(t,x)| \le C(1+|x|) \; ; \qquad x\in\mathbf{R}^n, \; t\in[0,T] \tag{5.2.1}$$

for some constant C, *(where* $|\sigma|^2 = \sum |\sigma_{ij}|^2$*) and such that*

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le D|x-y| \; ; \quad x,y\in\mathbf{R}^n, \; t\in[0,T] \tag{5.2.2}$$

for some constant D. *Let* Z *be a random variable which is independent of the* σ*-algebra* $\mathcal{F}_\infty^{(m)}$ *generated by* $B_s(\cdot)$, $s \ge 0$ *and such that*

$$E[|Z|^2] < \infty \,.$$

Then the stochastic differential equation

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t \,, \qquad 0 \le t \le T, \; X_0 = Z \tag{5.2.3}$$

has a unique t*-continuous solution* $X_t(\omega)$ *with the property that*

$$X_t(\omega) \text{ is adapted to the filtration } \mathcal{F}_t^Z \text{ generated by } Z \text{ and } B_s(\cdot);\; s \le t \tag{5.2.4}$$

and

$$E\Big[\int_0^T |X_t|^2 dt\Big] < \infty \,. \tag{5.2.5}$$

Remarks. Conditions (5.2.1) and (5.2.2) are natural in view of the following two simple examples from deterministic differential equations (i.e. $\sigma = 0$):

a) The equation

$$\frac{dX_t}{dt} = X_t^2, \quad X_0 = 1 \tag{5.2.6}$$

corresponding to $b(x) = x^2$ (which does not satisfy (5.2.1)) has the (unique) solution

$$X_t = \frac{1}{1-t}\ ; \qquad 0 \le t < 1\ .$$

Thus it is impossible to find a global solution (defined for all t) in this case. More generally, condition (5.2.1) ensures that the solution $X_t(\omega)$ of (5.2.3) does not *explode*, i.e. that $|X_t(\omega)|$ does not tend to ∞ in a finite time.

b) The equation

$$\frac{dX_t}{dt} = 3X_t^{2/3}\ ; \qquad X_0 = 0 \tag{5.2.7}$$

has more than one solution. In fact, for any $a > 0$ the function

$$X_t = \begin{cases} 0 & \text{for} \quad t \le a \\ (t-a)^3 & \text{for} \quad t > a \end{cases}$$

solves (5.2.7). In this case $b(x) = 3x^{2/3}$ does not satisfy the Lipschitz condition (5.2.2) at $x = 0$.

Thus condition (5.2.2) guarantees that equation (5.2.3) has a *unique* solution. Here uniqueness means that if $X_1(t,\omega)$ and $X_2(t,\omega)$ are two t-continuous processes satisfying (5.2.3), (5.2.4) and (5.2.5) then

$$X_1(t,\omega) = X_2(t,\omega) \qquad \text{for all} \ \ t \le T, \text{ a.s.} \tag{5.2.8}$$

Proof of Theorem 5.2.1. The uniqueness follows from the Itô isometry (Corollary 3.1.7) and the Lipschitz property (5.2.2): Let $X_1(t,\omega) = X_t(\omega)$ and $X_2(t,\omega) = \widehat{X}_t(\omega)$ be solutions with initial values $Z, \widehat{Z}$ respectively, i.e. $X_1(0,\omega) = Z(\omega), X_2(0,\omega) = \widehat{Z}(\omega), \omega \in \Omega$. For our purposes here we only need the case $Z = \widehat{Z}$, but the following more general estimate will be useful for us later, in connection with Feller continuity (Chapter 8).

Put $a(s,\omega) = b(s,X_s) - b(s,\widehat{X}_s)$ and $\gamma(s,\omega) = \sigma(s,X_s) - \sigma(s,\widehat{X}_s)$. Then

$$\begin{aligned}
E[|X_t - \widehat{X}_t|^2] &= E\Big[\Big(Z - \widehat{Z} + \int_0^t a\,ds + \int_0^t \gamma dB_s\Big)^2\Big] \\
&\le 3E[|Z-\widehat{Z}|^2] + 3E\Big[\Big(\int_0^t a\,ds\Big)^2\Big] + 3E\Big[\Big(\int_0^t \gamma dB_s\Big)^2\Big] \\
&\le 3E[|Z-\widehat{Z}|^2] + 3tE\Big[\int_0^t a^2 ds\Big] + 3E\Big[\int_0^t \gamma^2 ds\Big] \\
&\le 3E[|Z-\widehat{Z}|^2] + 3(1+t)D^2 \int_0^t E[|X_s - \widehat{X}_s|^2]ds\ .
\end{aligned}$$

So the function

$$v(t) = E[|X_t - \widehat{X}_t|^2] \; ; \qquad 0 \le t \le T$$

satisfies

$$v(t) \le F + A \int_0^t v(s)ds \; , \tag{5.2.9}$$

$$\text{where } \; F = 3E[|Z - \widehat{Z}|^2] \text{ and } \; A = 3(1+T)D^2 \; .$$

By the Gronwall inequality (Exercise 5.17) we conclude that

$$v(t) \le F \exp(At) \; . \tag{5.2.10}$$

Now assume that $Z = \widehat{Z}$. Then $F = 0$ and so $v(t) = 0$ for all $t \ge 0$. Hence

$$P[|X_t - \widehat{X}_t| = 0 \qquad \text{for all } \; t \in \mathbf{Q} \cap [0,T]] = 1 \; ,$$

where $\mathbf{Q}$ denotes the rational numbers.

By continuity of $t \to |X_t - \widehat{X}_t|$ it follows that

$$P[|X_1(t,\omega) - X_2(t,\omega)| = 0 \qquad \text{for all } \; t \in [0,T]] = 1 \; , \tag{5.2.11}$$

and the uniqueness is proved.

The proof of the existence is similar to the familiar existence proof for ordinary differential equations: Define $Y_t^{(0)} = X_0$ and $Y_t^{(k)} = Y_t^{(k)}(\omega)$ inductively as follows

$$Y_t^{(k+1)} = X_0 + \int_0^t b(s, Y_s^{(k)})ds + \int_0^t \sigma(s, Y_s^{(k)})dB_s \; . \tag{5.2.12}$$

Then, similar computation as for the uniqueness above gives

$$E[|Y_t^{(k+1)} - Y_t^{(k)}|^2] \le (1+T)3D^2 \int_0^t E[|Y_s^{(k)} - Y_s^{(k-1)}|^2]ds \; ,$$

for $k \ge 1$, $t \le T$ and

$$\begin{aligned} E[|Y_t^{(1)} - Y_t^{(0)}|^2] &\le 2C^2t^2E[(1+|X_0|)^2] \\ &+2C^2t(1+E[|X_0|^2]) \le A_1 t \end{aligned}$$

where the constant A_1 only depends on C, T and $E[|X_0|^2]$. So by induction on k we obtain

$$E[|Y_t^{(k+1)} - Y_t^{(k)}|^2] \le \frac{A_2^{k+1} t^{k+1}}{(k+1)!}\ ; \qquad k \ge 0,\ t \in [0,T] \tag{5.2.13}$$

for some suitable constant A_2 depending only on C, D, T and $E[|X_0|^2]$.

Hence, if λ denotes Lebesgue measure on $[0,T]$ and $m > n \ge 0$ we get

$$\begin{aligned}
\|Y_t^{(m)} - Y_t^{(n)}\|_{L^2(\lambda\times P)} &= \Big\| \sum_{k=n}^{m-1} \big(Y_t^{(k+1)} - Y_t^{(k)}\big) \Big\|_{L^2(\lambda\times P)} \\
&\le \sum_{k=n}^{m-1} \|Y_t^{(k+1)} - Y_t^{(k)}\|_{L^2(\lambda\times P)} = \sum_{k=n}^{m-1} \Big(E\Big[\int_0^T |Y_t^{(k+1)} - Y_t^{(k)}|^2 dt \Big] \Big)^{1/2} \\
&\le \sum_{k=n}^{m-1} \Big(\int_0^T \frac{A_2^{k+1} t^{k+1}}{(k+1)!} dt \Big)^{1/2} = \sum_{k=n}^{m-1} \Big(\frac{A_2^{k+1} T^{k+2}}{(k+2)!} \Big)^{1/2} \to 0
\end{aligned} \tag{5.2.14}$$

as $m, n \to \infty$.

Therefore $\{Y_t^{(n)}\}_{n=0}^\infty$ is a Cauchy sequence in $L^2(\lambda \times P)$. Hence $\{Y_t^{(n)}\}_{n=0}^\infty$ is convergent in $L^2(\lambda \times P)$. Define

$$X_t := \lim_{n\to\infty} Y_t^{(n)} \qquad (\text{limit in } L^2(\lambda \times P)).$$

Then X_t is $\mathcal{F}_t^Z$-measurable for all t, since this holds for each $Y_t^{(n)}$. We prove that X_t satisfies (5.2.3):

For all n and all $t \in [0,T]$ we have

$$Y_t^{(n+1)} = X_0 + \int_0^t b(s, Y_s^{(n)}) ds + \int_0^t \sigma(s, Y_s^{(n)}) dB_s\ .$$

Now let $n \to \infty$. Then by the Hölder inequality we get that

$$\int_0^t b(s, Y_s^{(n)}) ds \to \int_0^t b(s, X_s) ds \qquad \text{in } \ L^2(P)$$

and by the Itô isometry it follows that

$$\int_0^t \sigma(s, Y_s^{(n)}) dB_s \to \int_0^t \sigma(s, X_s) dB_s \qquad \text{in } \ L^2(P).$$

We conclude that for all $t \in [0,T]$ we have

$$X_t = X_0 + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dB_s \quad \text{a.s.} \tag{5.2.15}$$

i.e. X_t satisfies (5.2.3).

It remains to prove that X_t can be chosen to be continuous. By Theorem 3.2.5 there is a continuous version of the right hand side of (5.2.15). Denote this version by $\widetilde{X}_t$. Then $\widetilde{X}_t$ is continuous and

$$\begin{aligned}\widetilde{X}_t &= X_0 + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dB_s \qquad \text{for a.a. } \omega \\ &= \widetilde{X}_0 + \int_0^t b(s,\widetilde{X}_s)ds + \int_0^t \sigma(s,\widetilde{X}_s)dB_s \qquad \text{for a.a. } \omega\ .\end{aligned}$$

□

5.3 Weak and Strong Solutions

The solution X_t found above is called a *strong* solution, because the version B_t of Brownian motion is given in advance and the solution X_t constructed from it is $\mathcal{F}_t^Z$-adapted. If we are only given the functions $b(t,x)$ and $\sigma(t,x)$ and ask for a pair of processes $((\widetilde{X}_t, \widetilde{B}_t), \mathcal{H}_t)$ on a probability space $(\Omega, \mathcal{H}, P)$ such that (5.2.3) holds, then the solution $\widetilde{X}_t$ (or more precisely $(\widetilde{X}_t, \widetilde{B}_t)$) is called a *weak* solution. Here $\mathcal{H}_t$ is an increasing family of σ-algebras such that $\widetilde{X}_t$ is $\mathcal{H}_t$-adapted and $\widetilde{B}_t$ is an *$\mathcal{H}_t$-Brownian motion*, i.e. $\widetilde{B}_t$ is a Brownian motion, and $\widetilde{B}_t$ is a martingale w.r.t. $\mathcal{H}_t$ (and so $E[\widetilde{B}_{t+h} - \widetilde{B}_t | \mathcal{H}_t] = 0$ for all $t, h \geq 0$). Recall from Chapter 3 that this allows us to define the Itô integral on the right hand side of (5.2.3) exactly as before, even though $\widetilde{X}_t$ need *not* be $\mathcal{F}_t^Z$-adapted.

A strong solution is of course also a weak solution, but the converse is not true in general. See Example 5.3.2 below.

The uniqueness (5.2.8) that we obtain above is called *strong* or *pathwise* uniqueness, while *weak* uniqueness simply means that any two solutions (weak or strong) are identical in law, i.e. have the same finite-dimensional distributions. See Stroock and Varadhan (1979) for results about existence and uniqueness of weak solutions. A general discussion about strong and weak solutions can be found in Krylov and Zvonkin (1981).

Lemma 5.3.1 *If b and σ satisfy the conditions of Theorem 5.2.1 then we have*

A solution (weak or strong) of (5.2.3) is weakly unique .

Sketch of proof. Let $((\widetilde{X}_t, \widetilde{B}_t), \widetilde{\mathcal{H}}_t)$ and $((\widehat{X}_t, \widehat{B}_t), \widehat{\mathcal{H}}_t)$ be two weak solutions. Let X_t and Y_t be the *strong* solutions constructed from $\widetilde{B}_t$ and $\widehat{B}_t$, respectively, as above. Then the same uniqueness argument as above applies to show that $X_t = \widetilde{X}_t$ and $Y_t = \widehat{X}_t$ for all t, a.s. Therefore it suffices to show that X_t and Y_t must be identical in law. We show this by proving by induction that if $X_t^{(k)}, Y_t^{(k)}$ are the processes in the Picard iteration defined by (5.2.12) with Brownian motions $\widetilde{B}_t$ and $\widehat{B}_t$, then

$$(X_t^{(k)}, \widetilde{B}_t) \quad \text{and} \quad (Y_t^{(k)}, \widehat{B}_t)$$

have the same law for all k. $\square$

This observation will be useful for us in Chapter 7 and later, where we will investigate further the properties of processes which are solutions of stochastic differential equations (Itô diffusions).

From a modelling point of view the weak solution concept is often natural, because it does not specify beforehand the explicit representation of the white noise. Moreover, the concept is convenient for mathematical reasons, because there are stochastic differential equations which have *no strong solutions* but still a (weakly) *unique weak solution*. Here is a simple example:

Example 5.3.2 (The Tanaka equation) Consider the 1-dimensional stochastic differential equation

$$dX_t = \text{sign}(X_t)dB_t\ ; \qquad X_0 = 0\ . \tag{5.3.1}$$

where

$$\text{sign}(x) = \begin{cases} +1 \text{ if } & x \geq 0 \\ -1 \text{ if } & x < 0\ . \end{cases}$$

Note that here $\sigma(t,x) = \sigma(x) = \text{sign}(x)$ does not satisfy the Lipschitz condition (5.2.2), so Theorem 5.2.1 does not apply. Indeed, *the equation (5.3.1) has no strong solution.* To see this, let $\widehat{B}_t$ be a Brownian motion generating the filtration $\widehat{\mathcal{F}}_t$ and define

$$Y_t = \int_0^t \text{sign}(\widehat{B}_s)d\widehat{B}_s\ .$$

By the Tanaka formula (4.3.12) (Exercise 4.10) we have

$$Y_t = |\,\widehat{B}_t| - |\,\widehat{B}_0| - \widehat{L}_t(\omega)\ ,$$

where $\widehat{L}_t(\omega)$ is the local time for $\widehat{B}_t(\omega)$ at 0. It follows that Y_t is measurable w.r.t. the σ-algebra $\mathcal{G}_t$ generated by $|\,\widehat{B}_s(\cdot)|$; $s \leq t$, which is clearly strictly contained in $\widehat{\mathcal{F}}_t$. Hence the σ-algebra $\mathcal{N}_t$ generated by $Y_s(\cdot)$; $s \leq t$ is also strictly contained in $\widehat{\mathcal{F}}_t$.

Now suppose X_t is a strong solution of (5.3.1). Then by Theorem 8.4.2 it follows that X_t is a Brownian motion w.r.t. the measure P. (In case the reader is worried about the possibility of a circular argument, we point out that the proof of Theorem 8.4.2 is independent of this example!) Let $\mathcal{M}_t$ be the σ-algebra generated by $X_s(\cdot)$; $s \leq t$. Since $(\text{sign}(x))^2 = 1$ we can rewrite (5.3.1) as

$$dB_t = \text{sign}(X_t)dX_t \ .$$

By the above argument applied to $\widehat{B}_t = X_t$, $Y_t = B_t$ we conclude that $\mathcal{F}_t$ is strictly contained in $\mathcal{M}_t$.

But this contradicts that X_t is a strong solution. Hence strong solutions of (5.3.1) do not exist.

To find a weak solution of (5.3.1) we simply choose X_t to be *any* Brownian motion $\widehat{B}_t$. Then we define $\widetilde{B}_t$ by

$$\widetilde{B}_t = \int_0^t \text{sign}(\widehat{B}_s)d\widehat{B}_s = \int_0^t \text{sign}(X_s)dX_s$$

i.e.

$$d\widetilde{B}_t = \text{sign}(X_t)dX_t \ .$$

Then

$$dX_t = \text{sign}(X_t)d\widetilde{B}_t \ ,$$

so X_t *is a weak solution.*

Finally, *weak uniqueness* follows from Theorem 8.4.2, which – as noted above – implies that any weak solution X_t must be a Brownian motion w.r.t. P.

Exercises

5.1. Verify that the given processes solve the given corresponding stochastic differential equations: (B_t denotes 1-dimensional Brownian motion)

(i) $X_t = e^{B_t}$ solves $dX_t = \frac{1}{2}X_t dt + X_t dB_t$

(ii) $X_t = \frac{B_t}{1+t}$; $B_0 = 0$ solves

$$dX_t = -\frac{1}{1+t}X_t dt + \frac{1}{1+t}dB_t \ ; \qquad X_0 = 0$$

(iii) $X_t = \sin B_t$ with $B_0 = a \in (-\frac{\pi}{2}, \frac{\pi}{2})$ solves

$$dX_t = -\tfrac{1}{2}X_t dt + \sqrt{1-X_t^2}dB_t \ \text{ for } \ t < \inf\left\{s > 0; B_s \notin \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]\right\}$$

(iv) $(X_1(t), X_2(t)) = (t, e^t B_t)$ solves

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1} \end{bmatrix} dB_t$$

(v) $(X_1(t), X_2(t)) = (\cosh(B_t), \sinh(B_t))$ solves

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} dB_t \ .$$

5.2. A natural candidate for what we could call *Brownian motion on the ellipse*

$$\Big\{(x, y); \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\Big\} \qquad \text{where } \ a > 0, b > 0$$

is the process $X_t = (X_1(t), X_2(t))$ defined by

$$X_1(t) = a\cos B_t \ , \quad X_2(t) = b\sin B_t$$

where B_t is 1-dimensional Brownian motion. Show that X_t is a solution of the stochastic differential equation

$$dX_t = -\tfrac{1}{2}X_t dt + MX_t dB_t$$

where $M = \begin{bmatrix} 0 & -\frac{a}{b} \\ \frac{b}{a} & 0 \end{bmatrix}$.

5.3.* Let $(B_1, \ldots, B_n)$ be Brownian motion in $\mathbf{R}^n$, $\alpha_1, \ldots, \alpha_n$ constants. Solve the stochastic differential equation

$$dX_t = rX_t dt + X_t\Big(\sum_{k=1}^n \alpha_k dB_k(t)\Big); \qquad X_0 > 0 \ .$$

(This is a model for exponential growth with several independent white noise sources in the relative growth rate).

5.4.* Solve the following stochastic differential equations:

(i) $\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & X_1 \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix}$

(ii) $dX_t = X_t dt + dB_t$
(Hint: Multiply both sides with "the integrating factor" e^{-t} and compare with $d(e^{-t}X_t)$)

(iii) $dX_t = -X_t dt + e^{-t} dB_t$.

5.5. a) Solve the *Ornstein-Uhlenbeck equation* (or *Langevin equation*)

$$dX_t = \mu X_t dt + \sigma dB_t$$

where μ, σ are real constants, $B_t \in \mathbf{R}$.

The solution is called the *Ornstein-Uhlenbeck process*. (Hint: See Exercise 5.4 (ii).)

b) Find $E[X_t]$ and $\mathrm{Var}[X_t] := E[(X_t - E[X_t])^2]$.

5.6.* Solve the stochastic differential equation

$$dY_t = r\,dt + \alpha Y_t dB_t$$

where r, α are real constants, $B_t \in \mathbf{R}$.
(Hint: Multiply the equation by the 'integrating factor'

$$F_t = \exp\big(-\alpha B_t + \tfrac{1}{2}\alpha^2 t\big)\ .\ )$$

5.7.* The *mean-reverting Ornstein-Uhlenbeck process* is the solution X_t of the stochastic differential equation

$$dX_t = (m - X_t)dt + \sigma dB_t$$

where m, σ are real constants, $B_t \in \mathbf{R}$.

a) Solve this equation by proceeding as in Exercise 5.5 a).

b) Find $E[X_t]$ and $\mathrm{Var}[X_t] := E[(X_t - E[X_t])^2]$.

5.8.* Solve the (2-dimensional) stochastic differential equation

$$\begin{aligned} dX_1(t) &= X_2(t)dt + \alpha dB_1(t) \\ dX_2(t) &= -X_1(t)dt + \beta dB_2(t) \end{aligned}$$

where $(B_1(t), B_2(t))$ is 2-dimensional Brownian motion and α, β are constants.
This is a model for a vibrating string subject to a stochastic force. See Example 5.1.3.

5.9. Show that there is a unique strong solution X_t of the 1-dimensional stochastic differential equation

$$dX_t = \ln(1 + X_t^2)dt + \mathcal{X}_{\{X_t > 0\}} X_t dB_t\ , \qquad X_0 = a \in \mathbf{R}\ .$$

5.10. Let b, σ satisfy (5.2.1), (5.2.2) and let X_t be the unique strong solution of (5.2.3). Show that

$$E[|X_t|^2] \le K_1 \cdot \exp(K_2 t) \qquad \text{for } t \le T \tag{5.3.2}$$

where $K_1 = 3E[|Z|^2] + 6C^2T(T+1)$ and $K_2 = 6(1+T)C^2$.
(Hint: Use the argument in the proof of (5.2.10)).

Remark. With global estimates of the growth of b and σ in (5.2.1) it is possible to improve (5.3.2) to a global estimate of $E[|X_t|^2]$. See Exercise 7.5.

5.11.* (The Brownian bridge).

For fixed $a, b \in \mathbf{R}$ consider the following 1-dimensional equation

$$dY_t = \frac{b - Y_t}{1-t} dt + dB_t \ ; \qquad 0 \le t < 1 \ , \ Y_0 = a \ . \tag{5.3.3}$$

Verify that

$$Y_t = a(1-t) + bt + (1-t) \int_0^t \frac{dB_s}{1-s} \ ; \qquad 0 \le t < 1 \tag{5.3.4}$$

solves the equation and prove that $\lim_{t\to 1} Y_t = b$ a.s. The process Y_t is called *the Brownian bridge* (from a to b). For other characterizations of Y_t see Rogers and Williams (1987, pp. 86–89).

5.12.* To describe the motion of a pendulum with small, random perturbations in its environment we try an equation of the form

$$y''(t) + (1 + \epsilon W_t) y = 0 \ ; \qquad y(0), y'(0) \text{ given },$$

where $W_t = \frac{dB_t}{dt}$ is 1-dimensional white noise, $\epsilon > 0$ is constant.

a) Discuss this equation, for example by proceeding as in Example 5.1.3.

b) Show that $y(t)$ solves a *stochastic Volterra equation* of the form

$$y(t) = y(0) + y'(0) \cdot t + \int_0^t a(t,r) y(r) dr + \int_0^t \gamma(t,r) y(r) dB_r$$

where $a(t,r) = r - t$, $\gamma(t,r) = \epsilon(r-t)$.

5.13. As a model for the horizontal slow drift motions of a moored floating platform or ship responding to incoming irregular waves John Grue (1989) introduced the equation

$$x_t'' + a_0 x_t' + w^2 x_t = (T_0 - \alpha_0 x_t') \eta W_t \ , \tag{5.3.5}$$

where W_t is 1-dimensional white noise, a_0, w, T_0, α_0 and η are constants.

(i) Put $X_t = \begin{bmatrix} x_t \\ x_t' \end{bmatrix}$ and rewrite the equation in the form

$$dX_t = AX_t dt + KX_t dB_t + M dB_t \ ,$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -w^2 & -a_0 \end{bmatrix}, \quad K = \alpha_0\eta\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad M = T_0\eta\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(ii) Show that X_t satisfies the integral equation

$$X_t = \int_0^t e^{A(t-s)}KX_s dB_s + \int_0^t e^{A(t-s)}M dB_s \qquad \text{if } X_0 = 0\ .$$

(iii) Verify that

$$e^{At} = \frac{e^{-\lambda t}}{\xi}\{(\xi\cos\xi t + \lambda\sin\xi t)I + A\sin\xi t\}$$

where $\lambda = \frac{a_0}{2}, \xi = (w^2 - \frac{a_0^2}{4})^{\frac{1}{2}}$ and use this to prove that

$$x_t = \eta\int_0^t (T_0 - \alpha_0 y_s)g_{t-s}dB_s \tag{5.3.6}$$

and

$$y_t = \eta\int_0^t (T_0 - \alpha_0 y_s)h_{t-s}dB_s\ , \qquad \text{with } y_t := x_t'\ , \tag{5.3.7}$$

where

$$g_t = \frac{1}{\xi}\mathrm{Im}(e^{\zeta t})$$

$$h_t = \frac{1}{\xi}\mathrm{Im}(\zeta e^{\bar{\zeta}t})\ , \qquad \zeta = -\lambda + i\xi \quad (i = \sqrt{-1})\ .$$

So we can solve for y_t first in (5.3.7) and then substitute in (5.3.6) to find x_t.

5.14. If (B_1, B_2) denotes 2-dimensional Brownian motion we may introduce complex notation and put

$$\mathbf{B}(t) := B_1(t) + iB_2(t) \quad (i = \sqrt{-1})\ .$$

$\mathbf{B}(t)$ is called *complex Brownian motion.*

(i) If $F(z) = u(z) + iv(z)$ is an *analytic* function i.e. F satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\ ; \qquad z = x + iy$$

and we define

$$Z_t = F(\mathbf{B}(t))$$

prove that

$$dZ_t = F'(\mathbf{B}(t))d\mathbf{B}(t)\ , \tag{5.3.8}$$

where F' is the (complex) derivative of F. (Note that the usual second order terms in the (real) Itô formula are not present in (5.3.8)!)

(ii) Solve the complex stochastic differential equation

$$dZ_t = \alpha Z_t d\mathbf{B}(t) \quad (\alpha \text{ constant})\ .$$

For more information about complex stochastic calculus involving analytic functions see e.g. Ubøe (1987).

5.15. (Population growth in a stochastic, crowded environment) The nonlinear stochastic differential equation

$$dX_t = rX_t(K - X_t)dt + \beta X_t dB_t\ ; \qquad X_0 = x > 0 \tag{5.3.9}$$

is often used as a model for the growth of a population of size X_t in a stochastic, crowded environment. The constant $K > 0$ is called the *carrying capacity* of the environment, the constant $r \in \mathbf{R}$ is a measure of the quality of the environment and the constant $\beta \in \mathbf{R}$ is a measure of the size of the noise in the system.
Verify that

$$X_t = \frac{\exp\{(rK - \frac{1}{2}\beta^2)t + \beta B_t\}}{x^{-1} + r\int_0^t \exp\{(rK - \frac{1}{2}\beta^2)s + \beta B_s\}ds}\ ; \qquad t \geq 0 \tag{5.3.10}$$

is the unique (strong) solution of (5.3.9). (This solution can be found by performing a substitution (change of variables) which reduces (5.3.9) to a linear equation. See Gard (1988), Chapter 4 for details.)

5.16.* The technique used in Exercise 5.6 can be applied to more general nonlinear stochastic differential equations of the form

$$dX_t = f(t, X_t)dt + c(t)X_t dB_t\ , \qquad X_0 = x \tag{5.3.11}$$

where $f\colon \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ and $c\colon \mathbf{R} \to \mathbf{R}$ are given continuous (deterministic) functions. Proceed as follows:

a) Define the 'integrating factor'

$$F_t = F_t(\omega) = \exp\Big(- \int_0^t c(s)dB_s + \tfrac{1}{2}\int_0^t c^2(s)ds \Big) . \tag{5.3.12}$$

Show that (5.3.11) can be written

$$d(F_t X_t) = F_t \cdot f(t, X_t)dt . \tag{5.3.13}$$

b) Now define

$$Y_t(\omega) = F_t(\omega)X_t(\omega) \tag{5.3.14}$$

so that

$$X_t = F_t^{-1} Y_t . \tag{5.3.15}$$

Deduce that equation (5.3.13) gets the form

$$\frac{dY_t(\omega)}{dt} = F_t(\omega) \cdot f(t, F_t^{-1}(\omega)Y_t(\omega)) ; \qquad Y_0 = x . \tag{5.3.16}$$

Note that this is just a *deterministic* differential equation in the function $t \to Y_t(\omega)$, for each $\omega \in \Omega$. *We can therefore solve (5.3.16) with* ω *as a parameter to find* $Y_t(\omega)$ *and then obtain* $X_t(\omega)$ *from (5.3.15).*

c) Apply this method to solve the stochastic differential equation

$$dX_t = \frac{1}{X_t}dt + \alpha X_t dB_t ; \qquad X_0 = x > 0 \tag{5.3.17}$$

where α is constant.

d) Apply the method to study the solutions of the stochastic differential equation

$$dX_t = X_t^\gamma dt + \alpha X_t dB_t ; \qquad X_0 = x > 0 \tag{5.3.18}$$

where α and γ are constants.
For what values of γ do we get explosion?

5.17. (The Gronwall inequality)
Let $v(t)$ be a nonnegative function such that

$$v(t) \le C + A\int_0^t v(s)ds \qquad \text{for } 0 \le t \le T$$

for some constants C, A where $A \ge 0$. Prove that

$$v(t) \le C\exp(At) \qquad \text{for } 0 \le t \le T . \tag{5.3.19}$$

[Hint: We may assume $A \neq 0$. Define $w(t) = \int\limits_0^t v(s)ds$. Then $w'(t) \leq C + Aw(t)$. Deduce that

$$w(t) \leq \frac{C}{A}(\exp(At) - 1) \tag{5.3.20}$$

by considering $f(t) := w(t)\exp(-At)$.
Use (5.3.20) to deduce (5.3.19).)]

5.18. The *geometric mean reversion process* X_t is defined as the solution of the stochastic differential equation

$$dX_t = \kappa(\alpha - \log X_t)X_t dt + \sigma X_t dB_t \ ; \qquad X_0 = x > 0 \tag{5.3.21}$$

where κ, α, σ and x are positive constants.
This process was used by J. Tvedt (1995) to model the spot freight rate in shipping.

a) Show that the solution of (5.3.21) is given by

$$\begin{aligned} X_t := \exp\Big(e^{-\kappa t}\ln x + \Big(\alpha - \tfrac{\sigma^2}{2\kappa}\Big)(1 - e^{\kappa t}) \\ + \sigma\, e^{-\kappa t}\int\limits_0^t e^{\kappa s} dB_s\Big). \end{aligned} \tag{5.3.22}$$

[Hint: The substitution

$$Y_t = \log X_t$$

transforms (5.3.21) into a linear equation for Y_t]

b) Show that

$$E[X_t] = \exp\Big(e^{-\kappa t}\ln x + \Big(\alpha - \frac{\sigma^2}{2\kappa}\Big)(1 - e^{-\kappa t}) + \frac{\sigma^2(1 - e^{-2\kappa t})}{2\kappa}\Big).$$

5.19. Let $Y_t^{(k)}$ be the process defined inductively by (5.2.12). Show that $\{Y_t^{(n)}\}_{n=0}^{\infty}$ is uniformly convergent for $t \in [0, T]$, for a.a. ω. Since each $Y_t^{(n)}$ is continuous, this gives a direct proof that X_t can be chosen to be continuous in Theorem 5.2.1.
[Hint: Note that

$$\sup_{0 \leq t \leq T} |Y_t^{(k+1)} - Y_t^{(k)}| \leq \int\limits_0^T |b(s, Y_s^{(k)}) - b(s, Y_s^{(k-1)}|ds$$

$$+ \sup_{0 \leq t \leq T} \Big| \int\limits_0^t (\sigma(s, Y^{(k)}(s)) - \sigma(s, Y_s^{(k-1)}))dB_s \Big|$$

Hence

$$
\begin{aligned}
&P\Big[\sup_{0\le t\le T}|Y_t^{(k+1)}-Y_t^{(k)}|>2^{-k}\Big]\\
&\qquad\le P\Big[\int_0^T|b(s,Y_s^{(k)})-b(s,Y_s^{(k-1)})|ds>2^{-k-1}\Big]\\
&\qquad\quad+P\Big[\sup_{0\le t\le T}\Big|\int_0^t(\sigma(s,Y_s^{(k)})-\sigma(s,Y_s^{(k-1)}))dB_s\Big|>2^{-k-1}\Big].
\end{aligned}
$$

Now use the Chebychev inequality, the Hölder inequality and the martingale inequality (Theorem 3.2.4), respectively, combined with (5.2.13), to prove that

$$P\Big[\sup_{0\le t\le T}|Y_t^{(k+1)}-Y_t^{(k)}|>2^{-k}\Big]\le\frac{(A_3T)^{k+1}}{(k+1)!}$$

for some constant $A<\infty$. Therefore the result follows by the Borel-Cantelli lemma.]