EEE-391 Section 2

Homework 2

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(1)

1)

a) Determine the z-transforms of the following two signals. Note that the z-transforms for both have the same algebraic expression and differ only in the ROC.

i) $x_1[n] = (1/2)^n u[n]$

ii) $x_2[n] = -(1/2)^n u[-n-1]$

b) Sketch the pole-zero plot and ROC for each signal in part (a).

c) Repeat parts (a) and (b) for the following two signals:

i) $x_3[n] = 2u[n]$

ii) $x_4[n] = -(2)^n u[-n-1]$

a) i) $X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} = \sum_{n=0}^{\infty} (\frac{1}{2} z^{-1})^n = \frac{1}{1-\frac{1}{2}z^{-1}}$,

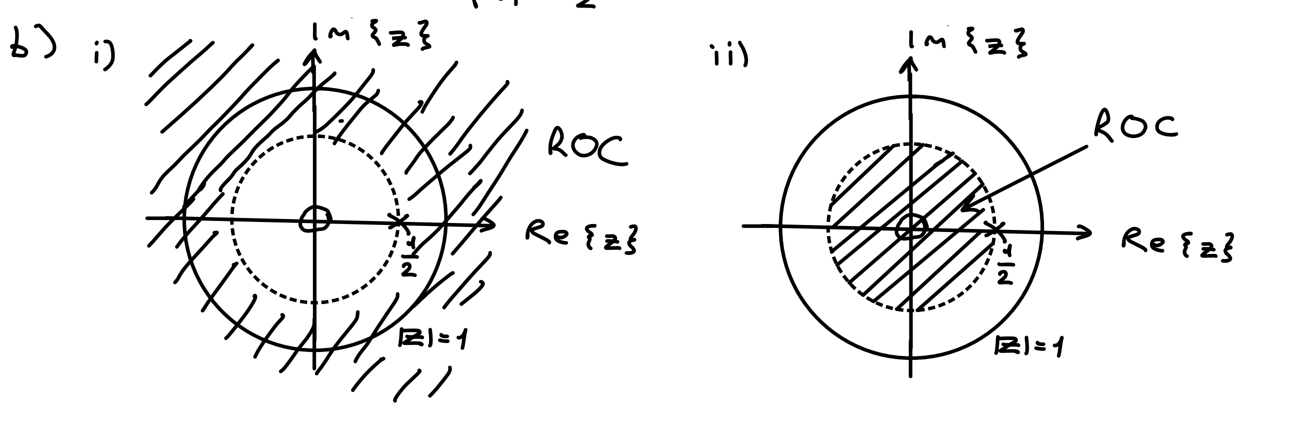
ROC $= \left|\frac{1}{2z}\right| < 1$, or $|z| > \frac{1}{2}$

ii) $X_2(z) = \sum_{n=-\infty}^{-1} (\frac{1}{2})^n z^{-n}$ converting it to positive coefficients by letting $n=-m$, we get

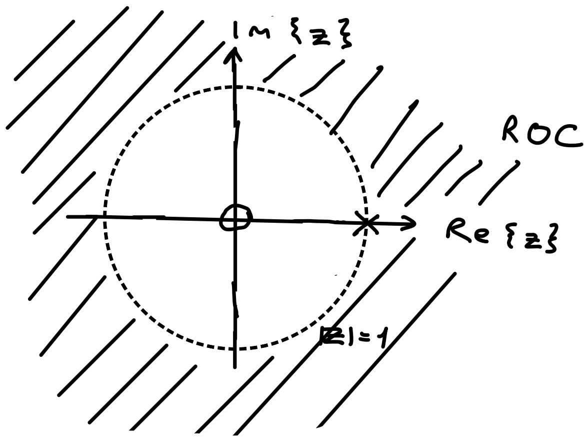
$$X_2(z) = -\sum_{m=1}^{\infty} (\tfrac{1}{2})^{-m} z^m = -\sum_{m=1}^{\infty} (2z)^m = -\frac{2z}{1-2z} = \frac{1}{1-\frac{1}{2}z^{-1}},$$

ROC $= |2z| < 1$, or $|z| < \frac{1}{2}$.

b)

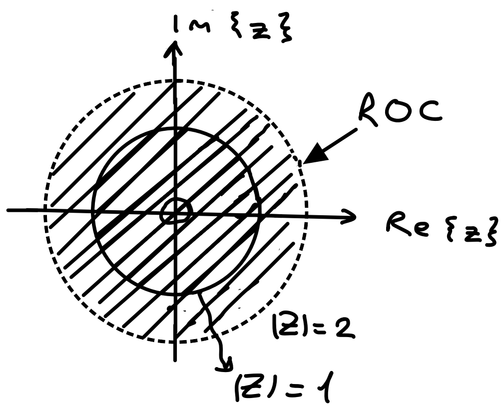


c) i) $X_3(z) = 2\sum_{n=0}^{\infty} z^{-n} = 2\left(\frac{1}{1-z^{-1}}\right) = \frac{2z}{z-1}$. The ROC is $|z|>1$.



ii) $X_4(z) = -\sum_{n=-\infty}^{-1} 2^n z^{-n} = -\sum_{n=1}^{\infty} 2^{-n} z^n = -\sum_{n=1}^{\infty}\left(\frac{z}{2}\right)^n = -\frac{z/2}{1-z/2} = \frac{z}{z-2}$,

ROC = $|z/2| < 1 \Rightarrow |z| < 2$.



2) The frequency response of a linear time-invariant filter is given by the formula

$$H(e^{jw}) = (1 + e^{-jw})(1 - e^{j2\pi/3}e^{-jw})(1 + e^{-j2\pi/3}e^{-jw})$$

a) Write the difference equation that gives the relation between the input $x[n]$ and the output $y[n]$.
b) What is the output if the input is $x[n] = \delta[n]$?
c) If the input is of the form $Ae^{j\theta}e^{jwn}$, for what values of $-\pi \leq w \leq \pi$ will $y[n] = 0$ for all n?

a)

$$H(e^{jw}) = (1 + e^{-jw})\left(1 + e^{-j2\pi/3}e^{-jw} - e^{j2\pi/3}e^{-jw} - e^{-2jw}\right)$$

$$= (1 + e^{-jw})\left(1 + \left(e^{-j2\pi/3} - e^{j2\pi/3}\right)e^{-jw} - e^{-2jw}\right)$$

$$= 1 + \left(e^{-j2\pi/3} - e^{j2\pi/3} + 1\right)e^{-jw} + \left(e^{-j2\pi/3} - e^{j2\pi/3} - 1\right)e^{-2jw} - e^{-3jw}$$

Recall that $e^{j2\pi/3} - e^{-j2\pi/3} = 2\sin(2\pi/3) \cdot j = \sqrt{3}\, j$

$$\Rightarrow H(e^{jw}) = 1 + (1 - \sqrt{3}j)e^{-jw} + (-\sqrt{3}j - 1)e^{-2jw} - e^{-3jw}$$

$\Rightarrow$ Difference Equation:

$$y[n] = x[n] + (1 - \sqrt{3}j)\,x[n-1] - (1 + \sqrt{3}j)\,x[n-2] - x[n-3]$$

b) when $x[n] = \delta[n]$, $y[n] = h[n]$ impulse response:

$$h[n] = \delta[n] + (1 - \sqrt{3}j)\,\delta[n-1] - (1 + \sqrt{3}j)\,\delta[n-2] - \delta[n-3]$$

c) For $y[n] = 0$ for all n, the input $x[n]$ must have frequencies at $w = \pm 2\pi/3$ (where the zeros of $H(e^{jw})$ are located). Thus, the values of w satisfying this condition will result in $y[n] = 0$ for all n.

3) For each of the following z-transforms determine the inverse z-transform.

a) $X(z) = \frac{1}{1+\frac{1}{2}z^{-1}},$ $|z|>\frac{1}{2}$

b) $X(z) = \frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-2}},$ $|z|>\frac{1}{2}$

c) $X(z) = \frac{1-az^{-1}}{z^{-1}-a},$ $|z|>|\frac{1}{a}|$

a)
Recall that geometric sum formula is equal to $S_\infty = \frac{a}{1-r}$
and its open form is $S_\infty = a + ar + ar^2 + \dots$
Then using geometric sum, we have

$$X(z) = 1 - \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2} - \frac{1}{8} z^{-3} + \dots$$

we recognize that

$$x[n] = \left(-\frac{1}{2}\right)^n u[n]$$

b) $X_b(z) = \frac{1-r}{1-r^2} = \frac{1}{1+r} = \frac{1}{1+\frac{1}{2}z^{-1}} = X_a(z) \Rightarrow x[n] = \left(-\frac{1}{2}\right)^n u[n]$ from part A

c) $$X(z) = \frac{1-\frac{a}{z}}{\frac{1}{z}-a} = \frac{\frac{z-a}{z}}{\frac{1-az}{z}} = \frac{z-a}{1-az} = -\frac{1}{a} + \frac{\left(\frac{1-a^2}{a}\right)}{1-az}$$

$$= -\frac{1}{a} - \frac{\left(\frac{1-a^2}{a^2}\right) z^{-1}}{1-a^{-1}z^{-1}} \Longrightarrow x[n] = -\frac{1}{a}\delta[n] - \frac{(1-a^2)}{a^{n+1}} u[n-1]$$

4) Consider a signal $y[n]$ which is related to two signals $x_1[n]$ and $x_2[n]$ by

$$y[n] = x_1[n+3] * x_2[-n+1]$$

where

$$x_1[n] = (1/2)^n u[n] \text{ and } x_2[n] = (1/3)^n u[n]$$

Given that

$$\text{(z-transform) } a^n u[n] \leftrightarrow \frac{1}{1+az^{-1}}, \quad |z|>|a|$$

use properties of z-transform to determine the z-transform $Y(z)$ of $y[n]$.

From the given information, we have

$$x_1[n] \overset{z}{\longleftrightarrow} X_1(z) = \frac{1}{1+\frac{1}{2}z^{-1}}, |z| > \frac{1}{2} \quad \text{and}$$

$$x_2[n] \overset{z}{\longleftrightarrow} X_2(z) = \frac{1}{1+\frac{1}{3}z^{-1}}, |z| > \frac{1}{3}.$$

using the time shifting property, we get

$$x_1[n+3] \overset{z}{\longleftrightarrow} z^3 X_1(z), \quad |z| > \frac{1}{2}$$

using the time reversal and shift properties, we get

$$x_2[-n+1] \overset{z}{\longleftrightarrow} z^{-1} X_2(z^{-1}), \quad |z| < 3$$

using the convolution property:

$$y[n] = x_1[n+3] * x_2[-n+1] \overset{z}{\longleftrightarrow} Y(z) = z^2 X_1(z) X_2(z^{-1}), \quad \frac{1}{2} < |z| < 3$$

Then,

$$Y(z) = \frac{z^2}{(1+\frac{1}{2}z^{-1})(1+\frac{1}{3}z)}.$$

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5) A particular causal LTI system is described by the difference equation

$$y[n] - \frac{\sqrt{2}}{2} y[n-1] + \frac{1}{4} y[n-2] = x[n] - x[n-1]$$

a) Find the impulse response of this system.
b) Sketch the log magnitude and the phase of the frequency response of the system.

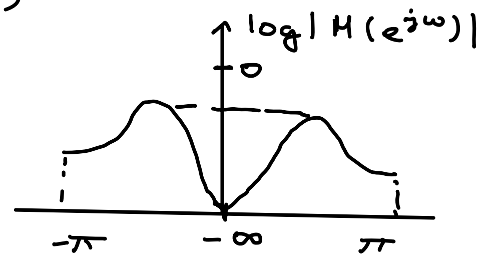
a) Take Fourier transform of given equation:

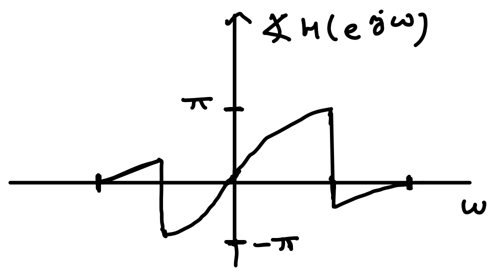
$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - e^{-j\omega}}{1 - \frac{\sqrt{2}}{2} e^{-j\omega} + \frac{1}{4} e^{-2j\omega}}.$$

$\Rightarrow$ Taking inverse fourier transform of $H(e^{j\omega})$, we get:

$$h[n] = \left(\frac{1}{2}\right)^n \cos(\pi n/4)\, u[n] - (2\sqrt{2}-1)\left(\frac{1}{2}\right)^n \sin(\pi n/4)\, u[n].$$

b)





6) Let $x(t)$ be a signal with Nyquist rate w_0. Determine the Nyquist rate for each of the following signals:

a) $x(t) + x(t-1)$

b) $\frac{dx(t)}{dt}$

c) $x^2(t)$

d) $x(t)cosw_0t$

a) If the signal $x(t)$ has Nyquist rate of w_0, then its Fourier transform $X(\omega)=0$ for $|\omega| > \omega_0/2$

Let $x(t) \xleftrightarrow{FT} X(\omega)$

using the properties of Fourier transform,

$$y(t) = x(t) + x(t-1) \xleftrightarrow{FT} Y(\omega)$$

$$Y(\omega) = X(\omega) + e^{-j\omega} X(\omega) = X(\omega)(1+e^{-j\omega})$$

we can only guarantee that $Y(\omega)=0$ for $|\omega| > \frac{\omega_0}{2}$. Therefore, the Nyquist rate for $y(t)$ is also ω_0. //

b) $$y(t) = \frac{d(x(t))}{d(t)} \xleftrightarrow{FT} Y(\omega)$$

$$Y(\omega) = j\omega X(\omega)$$

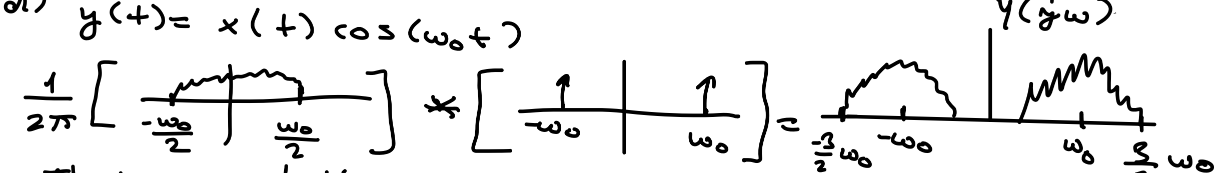
As before, we can only guarantee that $Y(\omega)=0$ for $|\omega| > \frac{\omega_0}{2}$. Therefore, the Nyquist rate for $y(t)$ is also ω_0. //

c) $$y(t) = x^2 \xleftrightarrow{FT} Y(\omega)$$

$$Y(\omega) = \frac{1}{2\pi}\left[X(\omega) * X(\omega)\right]$$

we can only guarantee that $Y(\omega)=0$ for $|\omega| > \omega_0$. Therefore, the Nyquist rate for $y(t)$ is $2\omega_0$. //

d) $y(t) = x(t)\cos(\omega_0 t)$



$\Rightarrow$ That convolution will triple the bandwidth, $\boxed{3\omega_0}$ //

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7) Given that $x(t)$ has the Fourier transform $X(jw)$, express the Fourier transforms of the signals listed below in terms of $X(jw)$.

a) $x_1(t) = x(1-t) + x(-1-t)$

b) $x_2(t) = x(3t-6)$

Let $x(t) \xleftrightarrow{FT} X_1(jw)$.

a) using the time reversal property, we get

$$x(-t) \xleftrightarrow{FT} X(-jw)$$

using the time shifting property on this, we have

$$x(-t+1) \xleftrightarrow{FT} e^{-jwt} X(-jw) \quad \text{and} \quad x(-t-1) \xleftrightarrow{FT} e^{jwt} X(-jw)$$

$$\Rightarrow x_1(t) = x(-t+1) + x(-t-1) \xleftrightarrow{FT} e^{-jwt} X(-jw) + e^{jwt} X(-jw)$$

$$\xleftrightarrow{FT} 2X(-jw)\cos w$$

b) using the time scaling property, we get

$$x(3t) \xleftrightarrow{FT} \frac{1}{3} X\left(j\frac{w}{3}\right)$$

using the time shifting property on it, we get

$$x_2(t) = x(3(t-2)) \xleftrightarrow{FT} e^{-2jw} \frac{1}{3} X\left(j\frac{w}{3}\right)$$

8) Given an IIR filter defined by the difference equation

$$y[n] = \frac{1}{2}y[n-1] + x[n]$$

a) When the input to the system is a unit step sequence, $u[n]$ determine the functional form of the output signal $y[n]$. Use the inverse z- transform method. Assume that the output signal $y[n]$ is zero for n<0.

b) Find the output when $x[n]$ is a complex exponential that starts at n = 0:

$$x[n] = e^{j\left(\frac{\pi}{4}\right)n}u[n]$$

c) From (b),identify the steady state component of the response, and compare its magnitude and phase to the frequency response at w = $\frac{\pi}{4}$.

a) According to this question, $x[n] = u[n]$. Taking z-transform:

$$X(z) = \frac{1}{1-z^{-1}}.$$

Taking z-transform of $y[n]$ gives, $Y(z) = \frac{1}{2}Y(z)z^{-1} + X(z)$

Thus, $\frac{Y(z)}{X(z)} = H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$.

Hence, $Y(z) = H(z)X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \cdot \frac{1}{1-z^{-1}} = \frac{-1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1-z^{-1}}$

with partial fraction formula, $Y(z)$ can be expressed as:

$$Y(z) = \frac{A}{(1-\frac{1}{2}z^{-1})} + \frac{B}{(1-z^{-1})}, \text{ where } A = -1 \text{ and } B = 2.$$

Hence, $y[n] = 2u[n] - \left(\frac{1}{2}\right)^n u[n] = \left(2 - \left(\frac{1}{2}\right)^n\right)u[n]$. //

b) To determine the output of the IIR filter when the input is a complex exponential that starts at $n=0$, we can use a partial fraction expansion:

$$Y(z) = H(z)X(z) = \left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1-e^{j\pi/4}z^{-1}}\right)$$

$$= \frac{\frac{1}{1-2e^{j\pi/4}}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{1-\frac{1}{2}e^{-j\pi/4}}}{1-e^{j\pi/4}z^{-1}}$$

Therefore, $y[n] = \left(\frac{1}{1-2e^{j\pi/4}}\right)\left(\frac{1}{2}\right)^n u[n] + \underbrace{\left(\frac{1}{1-\frac{1}{2}e^{-j\pi/4}}\right)}_{H(e^{j\pi/4})} e^{j(\pi/4)n} u[n]$

c) Notice that $H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} \Rightarrow$ second term is $y[n]$ at part (b) is steady-state component. That is:

$$y[n] = \left(\frac{1}{1-2e^{j\pi/4}}\right)\left(\frac{1}{2}\right)^n u[n] + \underbrace{H(e^{j\pi/4})e^{j(\pi/4)n} u[n]}_{\text{steady state}}$$

The frequency response of the IIR filter at $w = \frac{\pi}{4}$ is the magnitude of the transfer function evaluated at $z = e^{j(\frac{\pi}{4})}$;

$$|H(e^{j(\frac{\pi}{4})})| = \left| \frac{e^{j(\frac{\pi}{4})} - e^{j(\frac{\pi}{2})}}{(\frac{1}{2})e^{j(\frac{\pi}{4})} + (e^{j(\frac{\pi}{2})} - 1)} \right| = 1$$

The phase of the frequency response at $w = \frac{\pi}{4}$ is the angle of the transfer function evaluated at $z = e^{j(\frac{\pi}{4})}$:

$$\arg(H(e^{j(\frac{\pi}{4})})) = \frac{\pi}{4}$$