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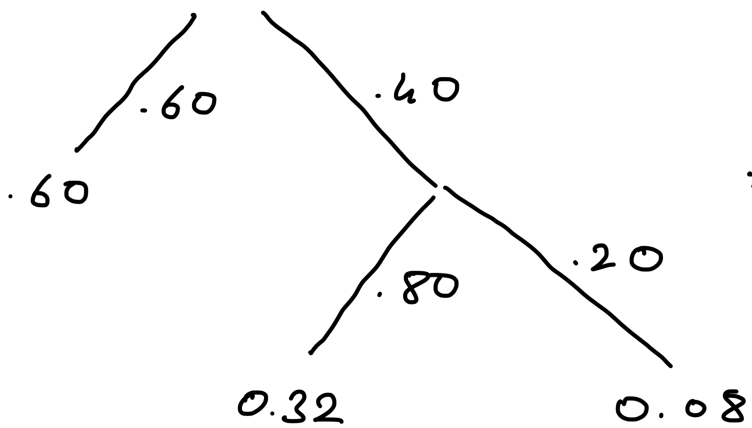
1. Each student is allowed two attempts to pass an exam. Experience shows that 60% of the students pass on the first try and that, for those who don't, 80% pass on the second try.
a) What is the probability that a student passes the exam?
b) If a student passed, what is the probability that he or she passed on the first attempt?

a)



Passes 1st try + Passes 2nd try

$= 0.60 + 0.40 \times 0.80$

$= 0.60 + 0.32 = 0.92$

$\boxed{\text{A student passes the exam with 92 \% probability.}}$

b) Let A = Student passed the exam on the first attempt.

Let B = Student passed the exam

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.60}{0.92} = 0.652$$

$\boxed{\text{If a student passed, the propability of passing on the first attempt is 65,2\%}}$

2. Probability distribution function of X is given by

$$f(x) = k(1+x)^2 \qquad x = -2,0,1,2$$

a) Find the value of k.

b) Find expected value of X.

c) Find standard deviation of X.

a) PDF $f(x)$ must satisfies following properties:

1) $f(x) \geq 0$

2) $\sum_x f(x) = 1$

From first property,

$k(1+x)^2 \geq 0 \Rightarrow \boxed{k \geq 0}$

From second property,

$k.(-1)^2 + k\,1^2 + k.\,2^2 + k.\,3^2 = 1$

$k + k + 4k + 9k = 1 \Rightarrow 15k = 1 \Rightarrow \boxed{k = 1/15}$

b) Expected value $= \mu = E(x) = \sum_x x\,f(x)$

$\Rightarrow E(x) = -2.\,k + 0.\,k + 1.\,4k + 2.\,9k$

$= 20k = 20.\,1/15$

$\Rightarrow \boxed{E(x) = 4/3 = \mu}$

c) $\sigma^2 = Var(x) = E(x-\mu)^2$ where $\mu = E(x)$

$= \sum_x (x-\mu)^2 f(x)$

standard deviation of x, $\sigma = \sqrt{Var(x)}$

$\sum_x (x-\mu)^2 f(x) = \left(-2-4/3\right)^2 k + \left(0-4/3\right)^2 k + \left(1-4/3\right)^2 4k$

$+\left(2-4/3\right)^2 9k = 1.156$ by calculator $= \sigma^2$

$\boxed{\sigma = \sqrt{1.156} = 1.075 = \text{std. dev.}}$

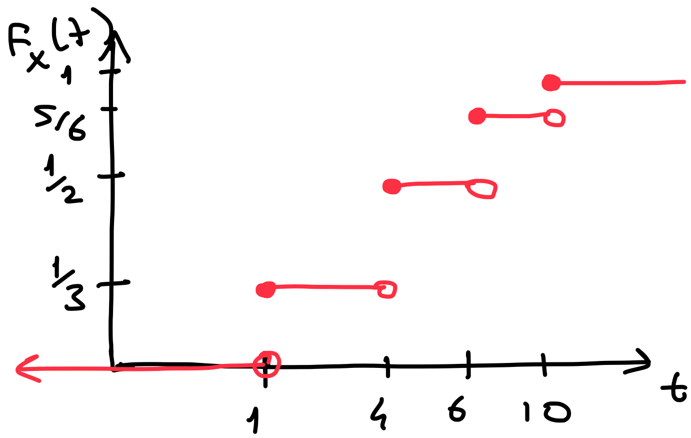
3. An investment firm offers its customers special bonds that mature after varying number of years. Let X be the number of years to maturity for a randomly selected bond, cumulative distribution of X is given below:

$$F(x)=\begin{cases}0 & x<1\\ \frac{1}{3} & 1\le x<4\\ \frac{1}{2} & 4\le x<6\\ \frac{5}{6} & 6\le x<10\\ 1 & x\ge 10\end{cases}$$

find a) P(X= 4) , b) $P(1.5 < X < 7)$, c) the expected value of X.

a) $F_X(t) = P(X \le t)$ where t is a real number

CDF is calculated for discrete rv; therefore, X is a discrete rv.



$$\boxed{P(X=4) = \frac{1}{2} - \frac{1}{3} = 1/6 \text{ from the graph.}}$$

b) $P(1.5 < X < 7) = P(2) + P(3) + P(4) + P(5) + P(6)$

$= 0 + 0 + 1/6 + 0 + \left(\frac{5}{6} - \frac{1}{2}\right) = \boxed{\frac{1}{6} + \frac{2}{6} = \frac{1}{2} \text{ from the graph.}}$

c) Expected value $= \mu = E(X) = \sum_x x\, p(x)$

Note that $P(1) = 1/3 - 0 = \frac{1}{3}$, $P(4) = 1/6$, $P(6) = \frac{1}{3}$,

$P(10) = 1 - \frac{5}{6} = \frac{1}{6}$ and $P(x) = 0$ for other x values. Then,

$E(X) = 1 \cdot P(1) + 4 \cdot P(4) + 6 \cdot P(6) + 10 \cdot P(10) = \frac{1}{3} + \frac{4}{6} + \frac{6}{3} + \frac{10}{6} = \frac{28}{6} = \frac{14}{3}$

$\boxed{\text{The expected value of } X \text{ is } 14/3.}$

4. An engineering firm is faced with the task of preparing a proposal for a research contract. The cost of preparing the proposal is $5000 and if the proposal is accepted the probabilities of potential incomes of $50000, $30000, $10000 or $0 are 0.20, 0.50, 0.20, and 0.10, respectively..
The firm's proposal will be accepted with probability of 0.30. Find the expected net profit for the firm.

Let X be a discrete rv, which defines potential incomes of proposal.

Expected value of $X = \mu = E(x) = \sum_{x} x\,p(x)$ where $p(x)$ is pdf of X. Then,

$E(x) = 50.000 \times P(50.000) + 30.000 \times P(30.000)$
$+ 10.000 \times P(10.000) + 0 \cdot P(0)$
$= 50.000 \times 0.2 + 30.000 \times 0.5 + 10.000 \times 0.2 + 0$
$= 27.000 \Rightarrow$ if the proposal is accepted, the expected potential is \$27.000.

Let y be a Bernoulli rv, which defines the probability of accepted proposal. Then, PDF of y

$$p(y) = \begin{cases} 0.3 & y=1 \\ 0.7 & y=0 \end{cases}$$

Then the firm gets \$27.000 by 0.3 probability and gets \$0 by 0.7 probability.

$\Rightarrow 27.000 \times 0.3 + 0 \cdot 0.7 = 8100$

Expected income is \$8100. Since prob. of paying \$5000 is 1, expected net profit is $8100 - 5000 = 3100$.

$\boxed{\text{Expected net profit is \$3100.}}$

5. Keeping an adequate supply of spare parts on hand is important for a large electronics firm. The monthly demand for microcomputer printer boards was studied for some months and found to average 28, with standard deviation 4. How many printer boards should be stocked at the beginning of each month to ensure that demand will exceed the supply with probability less than 0.10?

Let d = monthly demand for boards be a rv with mean $\mu = 28$ and std. dev. σ 4.

Also, let s = stocked amount $s = ?$

$$P(s < d) < 0.10 = P(d \leq s) \geq 0.90$$

$$P(d \leq s) = P(d - 28 \leq s - 28)$$

$$P(d-28 \leq s-28) \geq P(-(s-28) \leq d-28 \leq s-28)$$

$$\geq P(-(s-28) \leq d-28 \leq s-28)$$

$$\geq P(|d-28| < s-28)$$

Using Chebyshev's Inequality:

$$P(|d-28| < s-28) \geq 1 - \frac{1}{k^2} = 0.90$$

$$1 - \frac{1}{k^2} = 0.90 \Rightarrow \frac{1}{k^2} = 0.1 \Rightarrow \frac{1}{0.1} = k^2$$

$$\Rightarrow 10 = k^2 \Rightarrow k = \sqrt{10}$$

$s - 28 = k\sigma$ from Chebyshev's Inequality

$$s - 28 = \sqrt{10} \cdot 4 \Rightarrow s = 28 + 4\sqrt{10} \cong 40.65 \cong 41$$

$\boxed{\text{Approximately 41 printer board should be stocked each month to ensure demand will exceed the supply with prob} < 0.10.}$

6. A chip manufacturer knows that 5% of his production is defective. He gives a guarantee on his
shipment of 1000 chips by promising to refund the money if more than c chips are defective.
Using Chebyshev's inequality, determine the smallest value of c, so that he will refund the
money at most 1% of time.

Let D be the # of defective products

since all products can be defective or not defective

independently, D is a Binomial rv $\Rightarrow D \sim Bin(n, p)$

where $n = 1000$ and $p = 0.05$. By properties of Bin. rv,

$E(D) = \mu = np = 1000 \times 0.05 = 50$

$Var(D) = np(1-p) = 1000 \times 0.05 \times 0.95 = 47.5$

Question ask c for the eq. $P(D \leq C) \geq 0.99$

$$\Rightarrow P(D \leq C) \geq P(-(c-50) \leq D-50 \leq c-50)$$

$$\geq P(-(c-50) < D-50 < c-50)$$

$$\geq P(|D-50| < c-50)$$

By chebyshev's inequality,

$$P(|D-50| < c-50) \geq 1 - \frac{1}{k^2} = 0.99$$

$$\Rightarrow 1 - \frac{1}{k^2} = 0.99 \Rightarrow \frac{1}{k^2} = 0.01 \Rightarrow k = 10$$

$$\Rightarrow C - 50 = k\sigma$$

$$Var(D) = \sigma^2 = 47.5 \Rightarrow C - 50 = 10 \times \sqrt{47.5} \Rightarrow C = 118.92$$

The smallest value of C is 119.

7. The probability a new driver will pass a driving test is 0.8. Suppose 3 students take the driving test until each has passed it. Assuming independency, find the probability that exactly 1 of the 3 students will take at least two attempts to pass the test. Define random variable, determine its type and parameters to find the requested probability.

Let X: ~~#~~ # of students pass the test after at least two attempts

$X \sim Bin(n=3, p)$ where

$p = P($passed in at least two attemps$)$

$= 1 - P($passed in the first attempt$)$

$= 1 - 0.8 = 0.2$

$q = P($passed in the first attempt$) = 0.8$

Desired prob. (Prop. of Bin Dist is used)

$P(X=1) = \binom{3}{1} p^1 q^{3-1} = 3\,p\,q^2 = 3 \times (0.2) \times (0.8)^2$

$\boxed{= 0.384 = \text{The prob. of exactly 1 of the 3 students will take at least two attempts to pass the test.}}$

8. A box contains k red marbles and 12- k green marbles. Five times, a marble is chosen randomly and then replaced. What value of k maximizes the probability that the number of red marbles among the five obtained is exactly 2?

Replace $\Rightarrow$ Independent trials $\Rightarrow$ Binomial Distribution

Let X: # of red marbles obtained among the five

$X \sim Bin(n=5, p)$ where

$p = P($choosing red marble$) = \frac{k}{12}$

$q = P($choosing green marble$) = \frac{12-k}{12}$

Desired prob.

$P(X=2) = \binom{5}{2} \cdot \left(\frac{k}{12}\right)^2 \cdot \left(\frac{12-k}{12}\right)^3 \Rightarrow$ To maximize derivative must be zero $\Rightarrow P'(x=2) = 0$

To maximize $P(X=2)$, $P'(X=2)$ must be zero.

$$\Rightarrow P'(X=2) = 0 = \frac{d\left(10\left(\frac{k}{12}\right)^2\left(\frac{12-k}{12}\right)^3\right)}{dk}$$

$$= 10 \cdot 2 \cdot \frac{k}{12} \cdot \frac{1}{12}\left(\frac{12-k}{12}\right)^3 + 10 \cdot \left(\frac{k}{12}\right)^2 \cdot 3 \cdot \left(\frac{12-k}{12}\right)^2 \cdot \left(-\frac{1}{12}\right)$$

$$= 10 \cdot \frac{1}{12} \cdot \frac{k}{12} \cdot \left(\frac{12-k}{12}\right)^2 \left(2\left(\frac{12-k}{12}\right) - 3\left(\frac{k}{12}\right)\right)$$

$$= \frac{10k}{144} \cdot \left(\frac{12-k}{12}\right)^2 \left(\frac{24-2k-3k}{12}\right) = \frac{10k}{144}\left(\frac{12-k}{12}\right)^2\left(\frac{24-5k}{12}\right)$$

$$= 0 \Rightarrow k = 0 \text{ or } k = 12 \text{ or } k = 4.8.$$

To be able to select exactly two red marbles among the five obtained, there must be at least 2 red and 3 green marble $\Rightarrow k \geq 2 \cup 12-k \geq 3 \Rightarrow 2 \leq k \leq 9$

$\Rightarrow k = 4.8$ //

Since the # of red marbles the box contains should be an integer, $\boxed{k=5}$ meaning that 5 red marble maximizes the given probability.