Ömer Oktay Gültekin
21901413

Section 5
CS Department

1. The number of airplanes leave a busy airport according to a Poisson Process with rate of 4 per minute. What is the probability that no planes depart during a particular 30-second period? What is the probability that at least one plane departs during this time period?

a b

$\lambda = 4$ per minutes. X: # of plane departs in 30 second period.

Length of the interval 30 second $= \frac{1}{2}$ units (1 unite = 1 minute)

$\Rightarrow X \sim \text{Poisson}(\mu = \lambda t = (4).(\frac{1}{2}) = 2)$.

a) $P(X=0) = ?$ $\quad P(X=0) = \dfrac{e^{-\mu}\mu^{x}}{x!} = \dfrac{e^{-2}(2)^{0}}{0!} = \boxed{\dfrac{1}{e^{2}}}$

b) $P(X \geq 1) = ?$

$P(X \geq 1) = 1 - P(X=0) = \boxed{1 - \dfrac{1}{e^{2}}}$

2. In a daily production of a certain kind of rope, the number of defects per meter *X* is assumed to have a Poisson distribution with a mean 6 of defects. The profit per meter when the rope is sold is given by *Y*, where $Y = 50 - 2X - X^2$. Find the expected profit per meter.

From the properties of the Poisson Distribution:
(They are in lecture notes)

$E[X] = \mu = 6$

$E[X^2] = \mu^2 + \mu = Var[X] + E[X]^2 = 6^2 + 6 = 42$

$E[Y] = E[50 - 2X - X^2] = E[50] - 2E[X] - E[X^2]$
$= 50 - 2.6 - 42 = -4$

$\boxed{\text{Expected profit per meter is } -4 \text{ unit currency.}}$

3. The probability that a player wins a game at a single trial is 0.30. If the player plays until he wins, assuming the independency of these trials, find the probability that the number of trials (until he wins) is divisible by 4.

Let X denotes the number of trials until he wins.

Independent events that has 2 different possibilies
$\Rightarrow$ Bernoulli Trials. Then,

$X \sim$ Neg. Bin $(r=1, \ p=0.3)$.

when $r=1$, Neg. Bin rv $\sim$ Geometric rv $\Rightarrow X \sim \text{geometric}(p=0.3)$

$\Rightarrow P(x) = \binom{x-1}{0} p\, q^{x-1}$ where $p=0.3$, $q=0.7$.

$P(X \equiv 0 \pmod 4) = ?$

$P(X \equiv 0 \pmod 4) = P(4) + P(8) + P(12) + \dots$

$= (0.3)(0.7)^3 + (0.3)(0.7)^7 + (0.3)(0.7)^{11} + \dots$

$= (0.3)(0.7)^3 \left[1 + (0.7)^4 + (0.7)^8 + \dots\right]$

$= (0.3)(0.7)^3 \underbrace{\sum_{k=0}^{\infty} (0.7)^{4k}}_{\text{inf. geo. series}} = \boxed{(0.3)(0.7)^3 \cdot \frac{1}{1-(0.7)^4} \approx 0.135}$

4. From a lot of 10 missiles, 3 are selected at random and fired. If the lot contains 4 defective missiles that will not fire, what is the probability that
a) all 3 will fire?
b) at most 2 will not fire?

X : # of missiles will be fired in selected set.
since there is two types of missiles for selection;

$X \sim \text{Hypergeometric}(N=10,\ M=6,\ n=3)$
where M and N are non-defective and defective missiles respectively.

a) $P(X=3) = p(3) = \dfrac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} = \dfrac{\binom{6}{3}\binom{4}{0}}{\binom{10}{3}}$

$= \dfrac{\cancel{20}^{2}}{\frac{\cancel{10}\,\cancel{9}^{3}\,\cancel{8}^{2}}{\cancel{3}\,\cancel{2}\,1}} = \boxed{\dfrac{1}{6} = p(3)}$

b) P(at most 2 will not fire) = P(at least 1 will fire) = 1 − P(none will fire)

$= 1 - p(0) = 1 - \dfrac{\binom{6}{0}\binom{4}{3}}{\binom{10}{3}} = 1 - \dfrac{4\ 3\ 2}{10\ 9\ 8} = 1 - \dfrac{1}{30} = \boxed{\dfrac{29}{30}}$

5. Three people toss a fair coin and the odd one pays for coffee. If coins all turn up the same, they are tossed again. Find the probability that more than 2 tosses are needed.

Let x denotes the number of tries until odd one pays for coffee.

The results can be only success and failure for this tries.

Let success and failure are denoted by p and q respectively

$p = \{HTH, HHT, THH, HTT, THT, TTH\}$

$q = \{HHH, TTT\}$

$P(p) = \frac{6}{(\frac{1}{2})^3} = \frac{3}{4}$, $P(q) = \frac{2}{8} = \frac{1}{4} \Rightarrow X \sim \text{Neg. Bin}(r=1, p=0.75)$

when $r=1$, Neg. Bin rv $\sim$ Geometric rv

$\Rightarrow x \sim \text{geometric}(p = 0.75)$.

$\Rightarrow P(x) = \binom{x-1}{0} p q^{x-1} = p q^{x-1}$

$P(X > 2) = 1 - P(X \leq 2) = ?$

$1 - P(X \leq 2) = 1 - P(X=1) - P(X=2)$

$= 1 - (0.75)(0.25)^0 - (0.75)(0.25)^1 = 1 - \frac{3}{4} - \frac{3}{4} \cdot \frac{1}{4} = \boxed{0.0625}$

6. Let X be a discrete rv, with the pdf

$$f(x) = 3\left(\frac{1}{4}\right)^x \qquad x = 1,2,3,\ldots$$

Find moment generating function (mgf) of X and use mgf to find the expected value of X.

$f(x) = \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{x-1} \Rightarrow x$ geometric (p) where $p = \frac{3}{4}$ and $q = \frac{1}{4}$.

mgf is $M_x(t) = E(e^{tx})$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p = pe^t \sum_{x-1=0}^{\infty} e^{t(x-1)} q^{x-1}$$

$$= pe^t \left(\sum_{x-1=0}^{\infty} (qe^t)^{x-1} \right) = pe^t \frac{1}{1-qe^t}, \text{ using } \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, \ 0 < a < 1$$

$0 < qe^t < 1 \Rightarrow qe^t < 1 \Rightarrow e^t < \frac{1}{q} \Rightarrow t < -\ln q$

$$\Rightarrow M_x(t) = \frac{pe^t}{1-qe^t}, \ t < -\ln q$$

putting p and q values:

$$M_x(t) = \frac{\frac{3}{4} e^t}{1 - \frac{1}{4} e^t}, \ t < \ln 4 \Rightarrow M_x(t) = \frac{(0.75)e^t}{1-(0.25)e^t}, t < \ln 4$$

Expected value of a random variable is given by the first moment $\Rightarrow E[X] = M_x'(0) = \frac{d}{dt} M_x(t) = \frac{pe^t}{1-qe^t}$

$$= \frac{pe^t(1-qe^t) - (-qe^t)pe^t}{(1-qe^t)^2} = \frac{pe^t}{(1-qe^t)^2}$$

$$\frac{d}{dt} M_X(t) \Big|_{t=0} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \boxed{\frac{1}{p} = \frac{1}{\frac{3}{4}} = \frac{4}{3} = E[X]}$$

7. Cdf of a rv X is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 0.05x & 0 \le x < 20 \\ 1 & x \ge 20 \end{cases}$$

a) Find $E(X)$.

b) Use cdf to find $P(5 < X < 15)$.

$$F_X'(t) = \frac{d}{dt} F_X(t) = \frac{d}{dt} \int_{-\infty}^{t} f(x)\,dx = f(t) \text{ where } f \text{ is pdf}$$

$$\Rightarrow F_X'(20) = \frac{d}{dx} 0.05x = 0.05 = f(20).$$

$$\Rightarrow f(x) = \begin{cases} 0 & x < 0 \\ 0.05 & 0 \le x \le 20 \\ 0 & x > 20 \end{cases}$$

a) $$E(X) = \int_0^{20} x \cdot f(x)\,dx = \int_0^{20} x \cdot 0.05\,dx = \frac{0.05\,x^2}{2} \Big|_0^{20}$$

$$= \frac{\frac{1}{20} \cdot 20^2}{2} - \frac{\frac{1}{20} \cdot 0}{2} = \boxed{10 = E(x)}$$

b) $$P(5 < x < 15) = F_X(15) - F_X(5) = 0.05(15-5)$$

$$= \boxed{0.5 = P(5 < x < 15)}$$

8. Let X be a continuous rv with density

$$f(x) = 2e^{-2x} \qquad x > 0$$

a) Find the 75th percentile of the density.

b) Find the median.

a) $\eta_{0.75} = ?$

$$0.75 = \int_0^{\eta_{0.75}} 2e^{-2x}\,dx = \left(-e^{-2x}\right)\Big|_0^{\eta_{0.75}}$$

$$0.75 = 1 - e^{-2\eta_{0.75}} \Rightarrow e^{-2\eta_{0.75}} = 0.25$$

$$\Rightarrow -2\eta_{0.75} = \ln 4^{-1} \Rightarrow \eta_{0.75} = \frac{\ln 4}{2} \approx \boxed{0.693}$$

b) $\eta_{0.50} = ?$

$$0.50 = \left(-e^{-2x}\right)\Big|_0^{\eta_{0.50}}$$ using part a.

$$0.50 = 1 - e^{-2\eta_{0.5}} = e^{-2\eta_{0.5}} = 0.5$$

$$\Rightarrow \eta_{0.5} = \frac{\ln(0.5)}{-2} = \frac{\ln(2)}{2} \approx \boxed{0.347}$$

9. If a set of observations is normally distributed, what percent of these differ from the mean (μ) by, a) more than 1.50 σ; b) less than 0.75 σ.

a) $P(\mu + 1.5\sigma < x) + P(x < \mu - 1.5\sigma) = ?$

$P(\mu + 1.5\sigma < x) = P\left(1.5 < \frac{x-\mu}{\sigma}\right) = P(1.5 < z)$

$P(x < \mu - 1.5\sigma) = P\left(\frac{x-\mu}{\sigma} < -1.5\right) = P(z < -1.5)$

using standard normal table:

$P(1.5 < z) = 1 - F(1.5) = 1 - 0.9332 = 0.0668$

$P(z < -1.5) = F(-1.5) = 0.0668$

$P(\mu + 1.5\sigma < x) + P(x < \mu - 1.5\sigma) = 2(0.0668) = \boxed{0.1336}$

b) $P(\mu - 0.75\sigma < x < \mu + 0.75\sigma) = ?$

$P\left(-0.75 < \frac{x-\mu}{\sigma} < 0.75\right) = P(-0.75 < z < 0.75)$

using standard normal table:

$P(-0.75 < z < 0.75) = F(0.75) - F(-0.75)$

$= 0.7734 - 0.2266 = \boxed{0.5468}$