Chapter 2
Mathematical concepts – from numbers to computational systems

Mathematical astronomy can be defined as a technique for predicting astronomical phenomena using mathematical algorithms. Procedure texts contain verbal representations of such algorithms. This chapter offers a systematic analysis of the concepts and algorithms of Babylonian mathematical astronomy and their verbal representations in the procedure texts, using the tabular texts as a complementary source of information.[93] Setting out from the most elementary concepts to more complex ones, the analysis results in a hierarchy of concepts comprising three levels (Fig. 2.1). In order of increasing complexity they are defined by (i) numbers (§2.1), arithmetical and other elementary operations (§2.2–§2.4) and coordinate systems (§2.5); (ii) algorithms and functions (§2.6) and (iii) computational systems (§2.7). §2.6 is concerned only with generic algorithms; specific implementations of these algorithms, and other algorithms that are unique to the planets or the Moon, are discussed in Ch. 3 and 4, respectively.

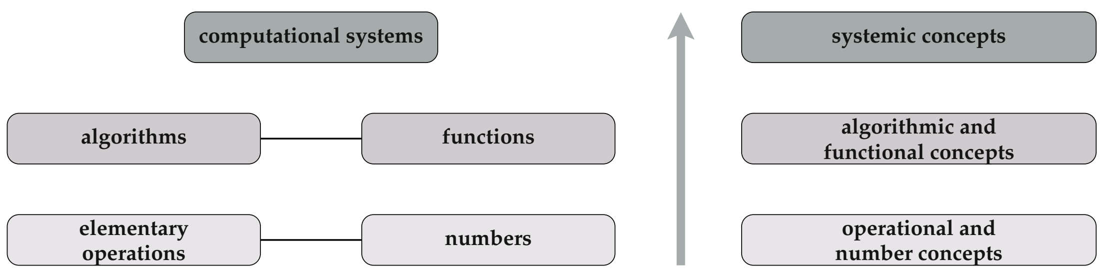

Fig. 2.1: Hierarchy of the mathematical concepts of Babylonian mathematical astronomy.

2.1 The sexagesimal place-value system

The first level of concepts to be discussed concerns numbers and arithmetic operations: addition, subtraction, multiplication and division. Babylonian mathematical astronomy is based on the sexagesimal place-value system, i.e. numbers are represented as sequences of digits using base number 60 ('sexagesimal'), such that the value of each digit depends on its position within the sequence ('place-value' or 'positional' system). The resulting number system operates in a manner entirely analogous to our own decimal system, as may be illustrated by an example:

$$\begin{aligned}10{,}0;6 &= 10\cdot 60^1 + 0\cdot 60^0 + 6\cdot 60^{-1}\\ &= 6\cdot 10^2 + 0\cdot 10^1 + 0\cdot 10^0 + 1\cdot 10^{-1} = 600.1\end{aligned}$$

In the modern notation for sexagesimal numbers, individual digits 0–59 are separated by commas, except for those pertaining to $60^0 = 1$ and $60^{-1} = 1/60$, which are separated by a semicolon (;). This is called the absolute notation, because each digit is multiplied by a unique power of 60. In cuneiform, sexagesimal numbers are written using a relative notation, because there is no sign equivalent to the semicolon that would indicate which digit pertains to 1. Hence the power of 60 corresponding to a given digit is defined only in relation to the other digits. In transliterations this feature is conserved by using the period (.) as a neutral separator between all digits. The cuneiform signs representing 10,0;6 are therefore transliterated as 10.0.6.[94] Also 10;0,6 or 0;10,0,6 are possible absolute values of that cuneiform number, whose correct

[93] New editions and analyses of the tabular texts will be presented in Volume II. Previous editions of most tabular texts can be found in *ACT*.
[94] For these and other practical aspects of transliteration and translation cf. §5.1.

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interpretation can be determined only from context.[95] In order to do basic calculations in the sexagesimal positional system it was necessary to master the elementary algorithms for adding, subtracting, multiplying and dividing multidigit sexagesimal numbers.[96] All numbers in Babylonian mathematics and mathematical astronomy are finite sexagesimal numbers, which means that they can be represented as a finite sequence of sexagesimal digits. Apart from whole numbers this includes numbers containing fractions smaller than 1, provided that they do not have any other dividers than 2, 3 and 5, the dividers of 60. Therefore, all finite sexagesimal numbers can be factorised as

$$\frac{n}{2^p \cdot 3^q \cdot 5^r}, \tag{2.1}$$

where n, p, q, r are (non-negative) whole numbers. Examples of numbers that cannot be represented in this way are 1/7, 1/11, 1/13, 1/17, 1/19, etc.

The origin of the sexagesimal place-value system has not been fully explained. Already long before its introduction the number 60 featured prominently in various Mesopotamian number systems, but these were not of the positional type. Apart from mathematical astronomy the sexagesimal place-value system is mainly known from the OB mathematical corpus, but the earliest textual evidence for it dates from the Ur-III period (2100–2000 BC).[97] The sexagesimal notation of mathematical astronomy incorporates a few minor innovations with respect to that of OB mathematics: the sign GAM now represents empty digits (cf. below), and the digit 9 is represented by a simplified sign. Outside the context of scholarship and the scribal school the sexagesimal system was rarely used in writing. In administrative and other non-scholarly documents, numbers continued to be written in traditional non-positional systems until the very end of cuneiform writing. However, the practical computations in these texts are believed to be based on sexagesimal calculus as an intermediate step. This may involve an abacus-type device on which the elementary operations for addition, subtraction and multiplication could be executed by manipulating physical objects representing the sexagesimal digits. For non-scholarly purposes the scribes would always write down the final results in the traditional number systems, which were readily understood by everyone, but there was no need for that when writing mathematical or astronomical texts, since they were read only by fellow scholars.

There is little doubt that the invention of the sexagesimal system was a major factor contributing to the development of mathematics and mathematical astronomy in Babylonia. It allowed the scribes to fully exploit the advantages of the positional system as a computational tool and also for representing and storing numerical results and algorithms in written form. Several practical advantages of the sexagesimal positional system over non-positional number systems as well as positional systems with other base numbers may be mentioned. First, in a positional system a single set of signs for the digits between 0 and the base number minus 1 (0...59) suffices for representing arbitrary numbers, as opposed to non-positional systems, in which every base number is represented by a separate sign (e.g. Sumerian numerals). Second, computations proceed more efficiently because all digits are manipulated in the same manner. Third, the sexagesimal system has a specific advantage, because 60 has many dividers (2, 3, 4, 5, 6, 10, 12, 15, 20 and 30). Hence more fractions can be represented as a finite sequence of digits (e.g. 1/3 = 0;20) than in other systems including the decimal one (where 1/3 = 0.33333...). While it is clear that these factors enhanced the ability of the Babylonian scholars to solve complex problems, their historical relevance is difficult to assess.

The zero and its manifestations

The sexagesimal system of mathematical astronomy includes a special sign (GAM) for the digit 0, which is attested for the first time during the Achaemenid era.[98] GAM is used for marking the absence of an initial or intermediate digit, but not of a final digit. This usage of GAM can be traced back to its role as a separator in literary texts ('Glossenkeil'). Also in mathematical astronomy GAM is sometimes used in order to mark a separation between signs, words or numbers (transliterated as :). In Uruk GAM is commonly used within sexagesimal numbers for separating multiples of 10 and digits 1–9 in order to prevent them from being interpreted as a single digit.[99]

A unified notation for the vanishing of a digit and the vanishing of a number was not developed. The number zero is not represented as GAM, but usually as 'it does not exist' (**nu tuk**),[100] sometimes as 'nothing' (*jānu*).[101] These representations

95 In Babylonian mathematical astronomy it is meaningful to speak of the 'correct interpretation' of numbers, because they nearly always represent quantities measured in certain empirical units (degrees of arc, time degrees, barleycorns, mean tithis, etc.). This is not the case in some mathematical texts containing numerical computations devoid of context, e.g. multiplication tables.

96 For these algorithms cf. for instance Friberg (2007), pp. 6–8.

97 E.g. HS 201, a table of reciprocals from Nippur (Oelsner 2001).

98 In OB mathematics the vanishing of a digit was sometimes marked by an empty space.

99 Cf. Neugebauer (1941). In Babylon the scribes achieved this by leaving a bit of extra space. In the OB mathematical text MS 2731 (Friberg 2007, p. 40) there is an instance where GAM separates two digits that are both less than 10, perhaps in order to mark the absence of a multiple of 10?

100 Akkadian reading probably *ul ibašši*, as proven by the occasional **nu tuk**-*ši* [*ACT* 135, Ri24'].

101 No. **52** Ri9: *bi-rit* **tab** *ana* **tab** *ia-a-nu*, 'the distance from addition to addition is zero'.

of the number zero are used only in certain contexts. They can stand by themselves like other numbers, or appear as predicates of quantities ('QN does not exist/is zero'), but are rarely attested as operands in arithmetical operations.[102] In the tabular texts the empty space also serves as a representation of the number zero, e.g. in column J of lunar system A (§4.4.14).

2.2 Arithmetical operations

Astronomical procedures are essentially sequences of arithmetical operations linked together in a verbal structure. The purpose of this section is to present a lexicographic and semantic analysis of the arithmetical operations. There are several reasons why such an analysis is necessary. First, some of the logograms representing arithmetical operations are attested only in LB astronomical and mathematical texts and their Akkadian reading is not well established. Second, there are different terms for apparently equivalent operations. As Høyrup (2002) has demonstrated for OB mathematical problem texts, this might reflect certain semantic differentiations that have hitherto remained unnoticed. A third aim is to compare the arithmetical terminology of mathematical astronomy with that of LB mathematical problem texts and OB mathematics. Apart from arithmetical operations the procedures contain other elementary operations which are discussed in §2.3.

2.2.1 Identity of quantities and symmetry of operations

A notable feature of the procedure texts is the existence of apparently synonymous words or phrases for the same arithmetical operation. It is known from OB mathematical problem texts that some of these apparently synonymous terms reflect a differentiation with regard to the *identity* of the involved quantities and/or the *symmetry* of the operation.[103] Both concepts manifest themselves most clearly in the additive operations. For instance, addition by means of 'appending' (in OB mathematics: *waṣābu*; in mathematical astronomy: *ṭepû*) implies that the sum inherits the identity of the summand to which something is 'appended', i.e. this type of addition conserves the identity of a quantity. By contrast, addition by means of 'accumulation' (*kamāru*) implies that the identity of the sum is different from that of any summand, so that there is a loss of identity. Identity-conserving addition therefore occurs in situations where a difference or increment is added to something — for instance a displacement to a position, resulting in a new position. In mathematical astronomy, most computations involve time intervals, angular distances or other empirical quantities having a well-defined identity, so that one can speak meaningfully about identity conservation.[104] The concept of identity conservation is potentially also relevant for subtractions and perhaps even multiplication and division.[105] However, in practice these other arithmetical operations do not exhibit a similarly clear differentiation with respect to identity conservation.

An arithmetical operation is called symmetric if the operands can be exchanged without a change of meaning. This feature is also mainly relevant in connection with addition. In the case of subtraction or division the question about symmetry or asymmetry is mute, because these operations are intrinsically asymmetric: $x-y$ is not equivalent with $y-x$, and a similar argument applies to division. With addition however, the sum Q_1+Q_2 is numerically the same as Q_2+Q_1, but there is a conceptual difference between 'appending' Q_1 to Q_2 and 'appending' Q_2 to Q_1, because in the former case the sum inherits the identity of Q_2, in the latter case the identity of Q_1.

2.2.2 Addition

Three different verbs are used for addition: *ṭepû*(**tab**), 'to append', *kamāru* (GAR.GAR), 'to accumulate' and *(w)aṣābu* (**daḫ**), 'to append'.[106] In the translations the distinctness of these verbs is preserved by translating *ṭepû* as 'to add', *(w)aṣābu* as "to append" (in quotation marks) and *kamāru* as 'to accumulate'.

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[102] One example, 'you accumulate nothing and 10^{nin}' (**nu tuk** *u₃* 10 **ninda** GAR.GAR), occurs in No. **25** O7'. Some interpolation algorithms contain an interval where the interpolation coefficient is zero (e.g. ΦG scheme 1 in §4.4.11). In these intervals the interpolation procedure is formulated differently from the others, such that multiplication by zero is avoided.

[103] Høyrup (2002), pp. 18–20.

[104] This is not the case in purely numerical operations as they occur for instance in multiplication tables.

[105] For instance, if one multiplies the daily displacement of a planet by a time interval expressed in days, one obtains a total displacement, so that the identity of one factor is passed on to the product.

[106] The verb *ruddû*(**uš**), 'to extend', is used for addition in OB mathematics and NB administrative documents, but it is not attested in the astronomical texts, except for its cognate noun *tardītu*, 'addition' (lit. 'extension'), which is mentioned a few times (No. **93** O7'; No. **100** O6').

2.2.2.1 'To add'

Nearly all instances of addition in the procedure texts involve the verb *ṭepû*(**tab**), literally 'to append, attach' (translated as 'to add').[107] Also the cognate noun *ṭīpu*(**tab**), 'addition' (< 'attachment'), occurs in the procedure texts. The earliest evidence for *ṭepû* as a verb for addition is found in LB astronomical and mathematical texts. The verb itself is attested in the OB period, but not with an arithmetical meaning, and not in the mathematical corpus.[108] However, there was a semantic equivalent of *ṭepû* in OB mathematics, *waṣābu*(**daḫ**), with the same basic meaning 'to append'. Additions with *ṭepû* are based on the following formulations:

Q_2 *itti*(**ki**) Q_1 *ṭepû*(**tab**)	to add Q_2 with Q_1	(TAB.1)
Q_2 *ana* [*muḫḫi*(**ugu**)] Q_1 **tab**	to add Q_2 to Q_1	(TAB.2)
Q_2 *ana muḫḫi* **tab**	to add Q_2 to it	(TAB.3)
Q_1 *u* Q_2 {*itti*|*ana muḫḫi*} *aḫāmiš* **tab**	to add Q_1 with|and Q_2 together	(TAB.4)

There appears to be no difference in meaning between 'adding with' (*itti*) or 'adding to' (*ana* [*muḫḫi*]). TAB.3 is used if something is added to the result of an immediately preceding computation. The applications of TAB.1–3 imply two basic properties of *ṭepû*. First, addition with *ṭepû* is asymmetric, i.e. the summands are not interchangeable. This is already suggested by the literal meaning, since Q_2 being 'appended' to Q_1 is not the same as Q_1 being 'appended' to Q_2. Second, the sum Q_1+Q_2 inherits the identity of Q_1. The same properties apply to *waṣābu*, the OB equivalent of *ṭepû*. The following numerical example illustrates this:

[10 **ḫun** 3 *ša*₂] ⸢*al* 10 **ḫun diri**⸣ **a.ra₂** 40 **du ki** 3 *teṭeppi*(**tab**)	[10 Ari: 3,0. (The amount) by which it] ⸢exceeds 10 Ari⸣ you multiply by 0;40, add with 3,0.	No. **53** Oia2'

The conservation of identity is consistent with the interpretation of the algorithm (§4.4.5).[109] The following quotation contains a fully algebraic example:

nisḫa(**zi**) *ša*₂ *sin issuḫu*(**zi**) **ki ki** *sin* **tab**	The distance by which the Moon moved you add with the position of the Moon.	No. **61**.**A** O19

In this case the asymmetric, identity-conserving nature of the addition is immediately obvious, because a displacement is added to the position of the Moon, which results in an updated position of the Moon (§4.4.19.4). One should not expect every single instance of TAB.1–3 to respect identity conservation and asymmetry, but the exceptions are in fact very few. In the following example of TAB.1 it appears that the identity of Q_1 is not passed on to the sum Q_1+Q_2:

[*ki*] *šaqû*(**nim**) ⸢**ki** BE **tab** *ki*⸣ *šaplu*(**sig**) **ta** BE *tanassaḫ*(**zi**-*aḫ*)	If (the Moon) is 'high' you add it (R) with the elongation (Q), if it is 'low' you 'tear' it out of the elongation.	No. **53** Rii4'

The astronomical interpretation of this algorithm (§4.4.19.11) implies that the outcome is not an 'elongation' (Q), so that the identity is not conserved.[110]

In TAB.4 *ṭepû* is qualified by the adverb *aḫāmiš*, 'together'. The procedure texts imply that these additions are also asymmetric, but the sum Q_1+Q_2 does not inherit the identity of Q_1 (nor that of Q_2). This is illustrated by the following example from a procedure for lunar system A (step 1 of the Lunar Six module; cf. §4.4.19.1):

gi₆ du *u* LA₂ *ša*₂ ⸢**me**⸣ **ki** *a-ḫa-miš*₂ **tab**	(The time by which) the night has progressed (M) and the length of ⸢daylight⸣ (C) you add together.	No. **61**.**D** O3

The use of *ṭepû* instead of *kamāru*, 'to accumulate', is prompted by the asymmetry of the summands, M and C. Moreover, the identity of the sum differs from that of either summand, which explains the presence of *aḫāmiš*, 'together'. This interpretation is confirmed by other examples.[111] The textual evidence can thus be summarised as follows: *ṭepû* in its bare form (TAB.1–3) usually expresses asymmetric, identity-conserving addition. If Q_1+Q_2 does not have the same identity as Q_1 then the addition is not a pure case of 'appending', which is made explicit by including *aḫāmiš*, 'together' (TAB.4).

[107] Besides 'to add' *ṭepû* can also mean 'to be increasing' or 'to be additive'; cf. the Glossary.

[108] *CAD* Ṭ sub *ṭepû*. Moreover, in OB mathematics **tab** represents *eṣēpu*, 'to double; to repeat'; cf. Proust (2009b), pp. 183–4.

[109] This is an interpolation rule for the duration of daylight (C) for lunar system A.

[110] However, one might justify the use of *ṭepû* in its bare form by the fact that all involved quantities are time intervals. It may also be significant that the addition and the subtraction are embedded in a conditional phrase, since this may necessitate a more concise formulation.

[111] No. **16** Xi'10', No. **18** Ri15', No. **38** Y5', No. **53** Oi31', No. **61**.**A** O3, No. **61**.**D** O3 and the mathematical problem text BM 34568 Ri8.

The concept of identity conservation also sheds light on several badly understood phrases involving TAB.2 and the 'value', *meḫertu*(**gaba.ri**), of a quantity:[112]

[10.5]3.52.42 **u₄.meš** *ana* *meḫret*(**gaba.ri**) *šatti*(**mu.an.na**) *ša₂* $^{\mathrm{d}}$**utu tab**	[10;5]3,52,42 days you add to the value of the year of the Sun.	No. **95** Oii13'

A number of days is added to the 'year of the Sun', the result being another time interval, which in subsequent lines is called the 'year of the Moon'. Hence $Q_1 + Q_2$ has a different identity than Q_1, but nevertheless *ṭepû* is not accompanied by *aḫāmiš*, 'together'. I propose that this can be explained if one assumes that the word 'value' is inserted before 'year' in order to avoid the wrong impression that the outcome is a 'year of the Sun'. This explanation assumes that 'value of QN' effectively masks the identity of QN, as if the addition is applied not to the quantity itself but to its numerical value.

2.2.2.2 'To append'

A small number of procedure texts and LB mathematical problem texts employ *(w)aṣābu*(**daḫ**), 'to append'.[113] Various features of these texts suggest that they (or their originals) predate the Seleucid period, or are at least older than the bulk of the corpus of mathematical astronomy. In OB mathematics *(w)aṣābu* was the most common verb for addition.[114] In the LB period its role has been taken over by *ṭepû*(**tab**), which has the same basic meaning.[115] The additions with *(w)aṣābu* are formulated as follows:

Q_2 *ana* Q_1 *(w)aṣābu*(**daḫ**)	to 'append' Q_2 to Q_1
Q_2 [*ana muḫḫi*] **daḫ**	to 'append' Q_2 [to it]

The second variant is used if something is 'appended' is the outcome of an immediately preceding computation. The following example occurs in an early Seleucid procedure text for the Moon:

40 **ninda** *a-na* 6.4.30 **daḫ**-*ma* 6 ⸢**uš** 5.10⸣	40$^{\mathrm{nin}}$ you 'append' to 6,4;30, it is 6 · ⸢60 (and) 5;10⸣.	No. **102** O3'

The astronomical interpretation (cf. the textual commentary) confirms the asymmetric, identity-conserving nature of *(w)aṣābu*.

2.2.2.3 'To accumulate'

'To accumulate', *kamāru*, is well known from OB mathematics, where it is used for symmetric addition of two or more quantities.[116] Basically the same applies in the astronomical procedure texts:

Q_1 *u* Q_2 [{*ana muḫḫi*|*itti*} *aḫāmiš*] *kamāru* (GAR.GAR)	to accumulate Q_1 and Q_2 [together]

Addition with *kamāru* is symmetric in the sense that the summands are always interchangeable. It is therefore applied only if the summands are in one way or another similar quantities. The 'accumulated' sum does not inherit the identity of any summand, which may explain why *kamāru* is often accompanied by *aḫāmiš*, 'together'. Most examples of *kamāru* in the procedure texts concern the averaging of the coefficients q and r in the Lunar Six module of lunar system A:

ana **ugu** *a-ḫa-miš₂ ša₂ sin u šam*[*aš₂ takam-mar*(GAR].GAR)-*ma* $\frac{1}{2}$-*šu₂ tanašši*(GIŠ)	You accumulate (the coefficients) for the Moon and the Sun together and you compute half of it.	No. **53** Rii8'

The summands are similarly defined coefficients for the Moon and the Sun ($q_{☽}$ and $q_{☉}$; cf. §4.4.19.5). There is loss of identity, because the outcome is a mean coefficient which is neither a coefficient for the Moon nor for the Sun. Another

[112] Other examples: No. **18** Ri7', No. **95** Oii14'.

[113] Mathematical astronomy: No. **102** O3',16',R4; Goal-Year astronomy: *TU* 11 O11,23,37 (Brack-Bernsen & Hunger 1999); other NMAT: BM 36414+ O4',5',11'; BM 43418 O10'; BM 77250 Xi'3' (all three unpublished); BM 45821+ R15,22 (Al-Rawi & George 1991/2); mathematical texts: W 23291 Oi23', 33' (Friberg 1997).

[114] Høyrup (2002), p. 19.

[115] I reserve the translation 'to append' (in quotation marks) for *(w)aṣābu* in order to maintain a distinction between these verbs.

[116] Besides GAR.GAR, UL.GAR and GAR are also attested as logograms for *kamāru* in OB mathematical problem texts; cf. the index in Høyrup (2002).

application of *kamāru* is encountered in the following Goal-Year procedure, in which pairs of Lunar Six intervals are added together:[117]

šu₂ *u* **na** *ša₂* **iti.du₆** *tanašši*(GIŠ)-*ma ana* **ugu** *a-ḫa-miš₂ takammarma*(GAR.GAR)-*ma*	You compute ŠU₂ and NA for month VII, and you 'accumulate' them together.

Also here *kamāru* is appropriate because the Lunar Six intervals ŠU₂ and NA have a similar definition (symmetry), and their sum is not a Lunar Six interval (loss of identity).[118] For a more complicated example in which *kamāru* is used for computing the sum of a series cf. §4.4.11.

2.2.3 Subtraction

Compared to the relatively clear semantic distinctions between the different terms for addition, subtraction offers a less clear picture. There is at best a weak tendency for semantic differentiations between some of the verbs. Their investigation is complicated by lexical difficulties. Of the four logograms that represent subtraction, **la₂**, **nim**, **e₁₁** and **zi**, only the latter two have a well-established Akkadian reading: **e₁₁** = *šūlû*, 'to deduct', and **zi** = *nasāḫu*, 'to tear out'. Moreover, the most common logogram in subtractions, **la₂**, has two different Akkadian readings: *naḫāsu*, 'to deduct', and *muṭṭû*, 'to diminish', apart from several other readings that do not represent subtractions.

2.2.3.1 'To tear out'

The least common verb for subtraction in the procedure texts is *nasāḫu*(**zi**), 'to 'tear out'' (thus translated, including quotation marks):

Q_2 *ina*|*ultu*(**ta**) Q_1 *nasāḫu*(**zi**)	to 'tear out' Q_2 from Q_1
Q_2 *ina*|*ultu*(**ta**) *libbi*(**ša₃**) **zi**	to 'tear out' Q_2 from it.

The preposition 'from' is usually written logographically (**ta**), sometimes phonetically (*ina*), which may be the usual reading of **ta**, although *ultu* (likewise meaning 'from') cannot be excluded. This kind of subtraction was very common in OB mathematics, where it served as the subtractive, identity-conserving counterpart of *(w)aṣābu*(**daḫ**), 'to append' (Høyrup 2002, p. 20). One factor that may have contributed to the decline of *nasāḫu* for subtraction is the fact that this verb is commonly used in the astronomical texts for describing motion along the zodiac ('to displace itself'). The following example of *nasāḫu* occurs in a (probably) early Seleucid lunar procedure text:

[*ki*] *šaqû*(**nim**) ⌜**ki** BE **tab** *ki*⌝ *šaplu*(**sig**) **ta** BE *tanassaḫ*(**zi**-*aḫ*)	If it (= the Moon) is 'high' you add it with the elongation, if it is 'low' you 'tear' it out from the elongation.	No. **53** Rii4'

The semantic restriction to identity-conserving subtraction known from OB mathematics is mostly confirmed by the procedure texts, but the quoted example may constitute a violation, since the outcome of the computation is not an 'elongation' (for the interpretation of this algorithm cf. the commentary and §4.4.19.11). As far as the instances of 'tearing out' can be dated, they tend to appear only in procedure texts from the early Seleucid era. Nevertheless, the subtractive use of *nasāḫu* is not merely a fossilised remainder of OB practices. The cognate noun *nisḫu*(**zi**), 'subtraction', becomes a subtractive number of undetermined magnitude, a concept unknown from OB mathematics (cf. §2.4).

2.2.3.2 'To subtract'

One of the more common verbs for subtraction in mathematical astronomy is *naḫāsu*(**la₂**), 'to subtract', which occurs in the following phrases:

Q_2 *ina*(**ta**) Q_1 *naḫāsu*(**la₂**)	to subtract Q_2 from Q_1
Q_2 *ina*(**ta**) *libbi*(**ša₃**)[-*šu₂*] **la₂**	to subtract Q_2 from it.

The subtractive sense of **la₂** in these phrases has long been known, since it is implied by the procedures in which they are embedded. However, it was not recognised that **la₂** is a form of *naḫāsu*. There are two reasons for this: first, there are no

[117] BM 42282+42294 O7 (Brack-Bernsen & Hunger 2008).

[118] In fact ŠU₂ + NA is the increment for another Lunar Six interval; cf. Brack-Bernsen & Hunger (2008).

phonetically written examples of *naḫāsu*, 'to subtract', in the procedure texts. Second, the transitive meaning 'to subtract' is a semantic innovation of the NB period deriving from the more common and original intransitive meaning 'to recede', also written **la₂**, in which sense this verb is frequently used in the planetary procedure texts for describing retrograde motion (§3.3.3.2). Phonetically written attestations of transitive *naḫāsu* were not readily available prior to the appearance in 1980 of volume N I of the *CAD*.[119] The following quotation from a procedure for J (lunar system A) proves beyond doubt that **la₂** = *naḫāsu*, 'to subtract':[120]

ta 25.7.30 **ḫun en** 1[3 **absin** xxxxxxx] /**ta** *si-man*.**meš** *tanaḫḫis*(**la₂**-*is*)	From 25;7,30 Ari until 1[3 Vir] / you subtract [xxxxxxx] (|J|) from the durations (G).	No. **53** Oii24'–25'

In this example $Q_1 - Q_2$ preserves the identity of Q_1, since J is a correction to G, the duration of the month (§4.4.14). Also in the following example the identity of Q_1 is conserved:

10 **rin₂** 3 *ša₂ al* 10 **rin₂ diri a.ra₂** 40 **du ta** 3 **la₂**	10 Lib: 3,0. (The amount) by which it exceeds 10 Lib you multiply by 0;40 und subtract from 3,0.	No. **53** Oia8'

However, in other examples the identity of Q_1 is not conserved,[121] so that this is at best a rule of thumb.

2.2.3.3 'To diminish'

The same logogram **la₂** also appears in subtractions in conjunction with the preposition *ana*, 'to', which precludes an interpretation as *naḫāsu*, 'to subtract'. As I will argue, these instances of **la₂** represent *muṭṭû*, 'to diminish'. They are formulated as follows:

Q_1 *ana* Q_2 *muṭṭû*(**la₂**)	to diminish Q_2 by Q_1

The evidence for interpreting **la₂** as *muṭṭû* is as follows. First, this phrase appears as the subtractive counterpart of *ṭepû*(**tab**) 'to add':

ki **nim** *ana* BE *tumaṭṭa*(**la₂**) *ki* [**sig**] *ana* BE **tab**	If it is 'high' you diminish the elongation by it, if [it is 'low'] you add it to the elongation.	No. **61**.**A** O11

This pair of logograms can be identified with the following pair of verb forms in a Goal-Year procedure first edited by Kugler:[122]

4 *u₄-mu*.**meš** *tu-maṭ-ṭa* **igi**-*mar*	You diminish (the date of appearance 8^{yr} earlier) by 4 days, (then) you see it (= Venus).	BM 45728 O6
⌜1⌝2 **u₄** *it-ti* **igi.du₈.a** *ta-ṭep-pi-ma* **igi.la₂**	You add 12 days with the appearance (47^{yr} earlier), then it (= Mars) appears (again).	BM 45728 O12

Here *tumaṭṭa*, 'you diminish', is the subtractive counterpart of *teṭeppi*, 'you append'. Since the latter is written **tab**, this strongly suggests that *tumaṭṭa* is the Akkadian reading of **la₂** in Q_1 *ana* Q_2 **la₂**. Kugler had already recognised that *tumaṭṭa* is a subtractive operation, but this is not mentioned in the dictionaries. In OB mathematics *muṭṭû* is not attested;[123] all instances date from the LB period. Probably *muṭṭu* also accounts for many other passages in the procedure texts where **la₂** appears as the subtractive counterpart of *ṭepû*, but a preposition is lacking because the quantity from which something is subtracted is not mentioned, as in the common phrase x **tab** *u* **la₂**, 'you add and subtract x':[124]

epūš(**du₃**-*uš*) *ša₂* **zi** *sin* **ab₂** *ana* **ab₂** 42 *teṭeppi*(**tab**) *u tumaṭṭa*(**la₂**)	Procedure for the displacement of the Moon: month by month you add and subtract ('diminish by') 0;42.	No. **53** Oi14'

[119] The *AHw* does not mention a transitive meaning of *naḫāsu*. The passage No. **53** Oii24'–25' is quoted in the *AHw*, but incorrectly translated with an intransitive form. *CAD* N I sub *naḫāsu* A4 (p. 130) does provide phonetically written examples of transitive *naḫāsu* from NB and LB administrative documents, but not the instances from the LB astronomical and mathematical corpus written with **la₂**.

[120] Apart from the phonetic complement, also the preposition *ina*(**ta**), 'from', rules out that **la₂** = *muṭṭû*, 'to diminish', since that verb requires *ana*, 'to'.

[121] E.g. No. **61**.**A** R11,13.

[122] Kugler *SSB* I (1907), p. 45; Britton (2003).

[123] However, the G stem *maṭû*, 'to be lacking', is common in OB mathematics.

[124] In order to avoid the cumbersome 'you add and diminish by x' I maintain the conventional translation 'you add and subtract' for this phrase.

2.2.3.4 'To deduct'

A third verb for subtraction in the procedure texts is *šūlû*(**e₁₁**), 'to deduct', which occurs in the following phrase:

Q_1 *ina*(**ta**) Q_2 *šūlû*(**e₁₁**)	to deduct Q_1 from Q_2

The earliest attestations of *šūlû* for subtraction occur in administrative documents from the MB era.[125] The basic meaning of the verb is 'to remove', which derives from the literal meaning 'to raise'. In the OB mathematical corpus *šūlû* is not used for subtraction, and the logogram **e₁₁** is not attested.[126] The following example is taken from a procedure text for lunar system A:

[LA₂ *ša₂* **gi₆ ta**] / ⸢**gi₆**⸣ **du** *tušelli*(**e₁₁**)	You deduct [the length of night] from / (the time by which) the night has progressed.	No. **61**.**D** O1–2

Although the interpretation of the algorithm (§4.4.19.1) implies that $Q_2 - Q_1$ does not inherit the identity of Q_2 in this example, there are as many cases where it does. Hence *šūlû* does not appear to reflect a semantic differentiation with respect to other words for subtraction.

A much more commonly used logogram for subtraction in procedure texts and mathematical texts from the LB period is **nim**, which occurs in basically the same contexts as **e₁₁**:

Q_1 *ina*(**ta**) Q_2 **nim**	to deduct Q_1 from Q_2

Very likely **nim** represents the same verb *šūlû* but a definite proof is lacking.[127] The following example is taken from a procedure text for lunar system A:

zi *ša₂ šamaš₂* **zi ta ki** *šamaš₂* **nim**	The displacement by which the Sun moved you deduct from the position of the Sun.	No. **61**.**A** O5

In this case the subtraction is obviously identity-conserving, but a wider investigation indicates that, as with **e₁₁**, no semantic differentiation with respect to the other terms for subtraction can be established. The usage of *šūlû*(**nim**, **e₁₁**) is not fully equivalent with that of *naḫāsu*(**la₂**) or *nasāḫu*(**zi**), since **nim** and **e₁₁** are not attested as markers of subtractivity in subtractive numbers, nor are they used in the sense of a subtractive number of undetermined magnitude (for these issues cf. §2.4).[128]

2.2.4 *Multiplication*

Multiplication is the third most common arithmetic operation in the procedure texts. With very few exceptions multiplication is realised by *alāku*(**du**), 'to go'. Besides that, there are a few examples of multiplication with *našû*, 'to raise'.

2.2.4.1 'To go Q_1 times Q_2'

Multiplication with *alāku* is formulated as follows:

Q_1 **a.ra₂**|GAM|GAM₀ Q_2 *alāku*(**du**)	to multiply ('go') Q_1 by ('times') Q_2

Although this phrase goes back to OB mathematics, very few OB problem texts contain the complete phrase *x* **a.ra₂** *y alāku*(**du**).[129] Virtually all instances of **a.ra₂**, 'times', in OB mathematics occur in multiplication tables in the phrase *x* **a.ra₂** *y z*, '*x* times *y* is *z*'. It thus appears that **a.ra₂** 'times' was almost exclusively used for purely numerical (context-free)

[125] *CAD* E, p. 133, *elû* 11a (Š).
[126] In OB mathematics **nim**, perhaps to be read *šūlû*, is used for multiplication.
[127] The Akkadian equivalents of **nim** mentioned in the dictionaries include *elû* and *šaqû*, both meaning 'to be/become high'. In principle the Š-stems of both verbs (lit. 'to raise') are legitimate candidates for the Akkadian reading of **nim** in subtractions, but since the usage of *šūlû* for subtraction is proven while there is no such evidence for *šušqû*, I assume that **nim** = *šūlû*.
[128] In fact a cognate noun of *šūlû* that might serve that purpose does not appear to exist.
[129] E.g. *UET* V 864: O8, R11–13 (for a recent translation of this Sumerian text from Ur cf. Høyrup (2002), p. 251). If one takes *ana* to be the phonetic reading of **a.ra₂** then *TMS* IX R21 and *TMS* XII O1,2,5 are further examples.

multiplications, and that any trace of an originally geometrical connotation of x **a.ra₂** y, if it ever existed, was lost at an early stage.[130] By contrast, usage of Q_1 **a.ra₂**|GAM|GAM₀ Q_2 *alāku* in astronomical procedure texts and LB mathematical problem texts covers all possible types of multiplication. This includes the geometric multiplicative operations for which OB mathematics employed a special terminology.[131] Another innovation of the LB period is that Q_1 **a.ra₂**|GAM|GAM₀ Q_2 *alāku* is now applied to abstract quantities of undetermined magnitude, as in the following example from a procedure text for lunar system A:

BE GAM₀ *ṣilipti*(**bar.nun**) *tallak*(**du**)	You multiply the elongation by the *ṣiliptu*-coefficient.	No. **61**.**A** O11

2.2.4.2 'To raise'

In rare cases multiplication is formulated in terms of *našû*, 'to raise':[132]

Q_1 *ana* Q_2 *našû*(**il₂**)	to 'raise' Q_1 to Q_2
Q_1 **a.ra₂** Q_2 *našû*(**il₂**)	to 'raise' Q_1 times Q_2

This type of multiplication is well known from OB mathematics. In the LB procedure texts and the mathematical problem texts *našû* is predominantly used in another sense, namely 'to compute', written GIŠ, less often **il₂**, usually with a fraction or the name of a quantity as the object. That usage of *našû* is unknown from OB mathematics and appears to be an innovation of the LB period.

2.2.5 Division and reciprocals

Explicit non-trivial divisions are rare in the procedure texts. The few known examples are formulated as follows:[133]

x *ana* y *aḫḫê*(**šeš.meš**) *zâzu*(**bar**, SE₃)	To divide x into y parts

This phrase is unknown from OB mathematics and appears to be an innovation of the LB period. The use of SE₃ instead of **bar** is attested only in No. **52**; it is assumed here that the Akkadian reading is likewise *zâzu*.[134] None of the procedures in which the phrase occurs have a practical purpose connected with the computation of synodic tables. All are unusual in one way or another, and involve concrete numbers; i.e. abstract quantities are not attested. For trivial divisions by small whole numbers there is a separate phrase involving the verb *našû*, 'to compute':

n-šu₂ tanašši(GIŠ)	You compute $1/n$ of it.

Here n is an ordinal number ('third', 'fourth', etc.), but for $n = 2$ the word *mišlu*(**bar**), 'half', is used. This type of division, rather common in the procedure texts, is not attested in OB mathematics. Even though non-trivial divisions are rare, many multiplications in the procedure texts involve factors that are actually reciprocals ($1/y$), so that they can be viewed as reformulated divisions (x/y).[135] The reciprocal operation itself has exactly the same form as in OB mathematics:

130 Sumerian **a.ra₂** has several distinct meanings, all deriving from the basic meaning 'to go' of the cognate verb DU. The *PSD* (A sub **a.ra₂**) lists A: 1. 'occasion'; 2. 'times' (in multiplication); B: *alaktu* = 'way'. There is no textual evidence for a meaning 'step' proposed by Høyrup (2002), p. 22 and Friberg (2007), p. 72 (the common word for 'step', **ki.us** = *kibsu*, is not used in multiplications). The Akkadian reading of **a.ra₂** has thus far not been established beyond doubt, but it is probably *ana*. First, two OB mathematical problem texts from Susa suggest this: *TMS* IX R21: 30 *a-na* 17 *a-li-ik-ma* 8.30 *ta-mar*, 'multiply 30 times 17, you see 8.30', and *TMS* XII O1: *ma-na-at* **uš** *a-na* **sag** *i-la-ku*, 'the number that the length goes 'to' (= times) the width...' (= length times width). Second, in the LB problem text BM 78822 (Jursa 1993/4) one multiplication is formulated as x *a-na* y *alāku*. (Conversely, in No. **18** P20 **a.ra₂** replaces *ana*, 'for'; compare the duplicate No. **46** P10', but these are not multiplications.) In cases where **a.ra₂** represents 'n-fold', 'for the nth time', it is sometimes replaced by *adi* (*CAD* A sub *adi* 4), but *adi* is not attested in multiplications $x \cdot y$. One might also consider the possibility that **a.ra₂**, GAM and GAM₀ were pronounced as /ar(a)/, since GAM has a rare phonetic value *ar₅* (e.g. in the colophon of *ACT* 18).

131 E.g. *šutakūlu*(**gu₇.gu₇**) = 'to make hold one another' for the construction of a rectangle from two unequal sides ('rectangularisation') and *šutamḫuru*(NIGIN) = 'to make encounter' for the construction of a square from two equal sides ('squaring'); cf. Høyrup (2002), pp. 23–25.

132 With *ana*: No. **102** O7',9',15',R3; with **a.ra₂**: BM 45821+ R9.

133 With SE₃: repeatedly in No. **52** but nowhere else; with **bar**: in No. **82** R11 and No. **102** R5.

134 This reading of SE₃ is not attested elsewhere in the cuneiform literature as far as I know.

135 For instance, many interpolation algorithms involve coefficients which are computed from a division (§2.6.7).

igi *x* **gal$_2$.bi** *y*	The reciprocal of *x* is *y*.

It rarely appears in the procedure texts: the only known examples occur in No. **102**, an unusual early Seleucid procedure text, but it is not uncommon in the LB mathematical problem texts.[136] The usage of this operation is the same as in OB mathematics: it is not applied to abstract quantities, only to concrete numbers for which the reciprocal exists as a finite sexagesimal number.

2.2.6 The copula u, 'and', as a placeholder for arithmetical operations

In some texts the copula *u*, 'and', is used as a placeholder for various arithmetical operations — not only additions as one might expect. In the translations these instances of 'and' are placed in quotation marks. The following example from a Goal-Year procedure concerns addition:

šu$_2$ *u* **na** *ša$_2$* **du$_6$** *tanašši*(GIŠ)	You compute ŠU$_2$ 'and' NA for month VII.	*TU* 11 O36

The algorithmic interpretation of this procedure (Brack-Bernsen & Hunger 2002) implies that ŠU$_2$ + NA is meant. Examples where *u* replaces other operations are found in No. **61**, a procedure text for the Lunar Six intervals (§4.4.19):

BE *u ṣiliptu*(**bar.nun**)	The elongation 'and' the *ṣiliptu*-coefficient.
BE *u* **bar.nun** *u* 2 ḪAB-*rat*	The elongation 'and' the *ṣiliptu*-coefficient 'and' the 2 for the disk

Both phrases appear repeatedly in the text. The algorithmic interpretation leaves no doubt which operation is hidden behind the copula *u*. In both phrases the first instance of 'and' replaces multiplication. In the second phrase there is a second instance of 'and', which represents either addition (cf. steps 10–12 in the procedure for ME), or subtraction (cf. steps 10–12 in the procedure for GI$_6$). Hence *u* is attested as a placeholder for the three main elementary arithmetical operations.

2.2.7 Diachronic overview of arithmetical terms and a comparison with mathematical texts

As demonstrated in the previous sections, the arithmetical terminology of the astronomical procedure texts represents a significant break with OB mathematics. A diachronic overview is shown in Table 2.1. Only a small subset of the terminology of OB mathematics is still regularly used in mathematical astronomy, namely *kamāru* ('to accumulate'), *x* **a.ra$_2$** *y alāku* (to 'go' *x* times *y*) and *nasāḫu* ('to tear out'). A few other OB terms have not completely disappeared but become very rare in mathematical astronomy. The most common term for addition, *ṭepû* = 'to append', is an LB innovation, but its semantic function in relation to *kamāru*, 'to accumulate', is basically the same as that of its OB counterpart *(w)aṣābu*. Various new terms for subtraction (*naḫāsu* and *šūlû*) seem to be attested for the first time in MB or NB administrative texts, suggesting an origin in accounting practices. The sole surviving term for multiplication (*alāku*) is not really new, but it has a much wider application than in OB mathematics.

This raises the question of to what extent the terminological innovations are a general phenomenon affecting both the astronomical and the mathematical texts of the LB period. One superficial distinction between astronomical procedures and mathematical problem texts is that the former contain only additions, subtractions and multiplications, with very few exceptions. Explicit divisions are virtually absent because they have been converted into multiplications by reciprocals. A comparison is obviously possible only for those operations that are common in both corpora. Indeed most tablets of the small corpus of LB mathematical problem texts[137] employ exactly the same arithmetical terms for addition, subtraction and multiplication as the astronomical procedure texts. This may be illustrated by the following problem from BM 34568, which contains a numerical example of what we know as the theorem of Pythagoras:

Oi6[4 **u**]**š** *u$_3$* 5 *ṣiliptu*(**bar.nun**) **en sag** *aššu*(**mu**) *lā*(**nu**) *tīdû*(**zu**-*u$_2$*) 4 GAM 4 **7**[1]6 5 GAM 5 25 16 **ta** 25 **nim**-*ma re-ḫi* 9 **8***mi-nu-u$_2$* GAM *mi-ni-i lu*-**du**-*ma lu* 9 3 GAM 3 9 **8a**3 **sag**	**Oi6**The length is 4 and the diagonal is 5, what is the width? Since you do not know it: 4 times 4 is **7**16. 5 times 5 is 25. You deduct 16 from 25, 9 remains. **8**How much times how much should I multiply so that it is 9? 3 times 3 is 9: **8a**the width is 3.

[136] AO 6484 (*MKT* I, pp. 96–107); AO 7848 (*MCT* Text Y); BM 34568 (*MKT* III); W 23291–x (Friberg, Hunger & al-Rawi 1990).

[137] For a list of tablets of this corpus cf. Friberg (1997), pp. 356–357.

addition		OB	LB
to ‘append’	*(w)aṣābu*(**daḫ**)		
to add	*ṭepû*(**tab**)		
to accumulate	*kamāru*(GAR.GAR; OB: GAR, UL.GAR)		
subtraction			
to ‘tear out’	*nasāḫu*(**zi**)		
to subtract	*naḫāsu*(**la**$_2$)		
to deduct	*šūlû*(**e**$_{11}$, **nim**)		
to diminish	*muṭṭû*(**la**$_2$)		
multiplication			
to ‘go’ Q_1 times Q_2	Q_1 **a.ra**$_2$|GAM|GAM$_0$ Q_2 *alāku*(**du**)		
to ‘lift’	*našû*(**il**$_2$)		
division and reciprocals			
to divide into parts	*ana aḫḫē zâzu*(**bar**, SE$_3$)		
reciprocal of *x*	**igi** *x* **gal**$_2$**.bi**		

Table 2.1: Diachronic overview of the arithmetical terminology.

As mentioned earlier, OB mathematical problem texts of this geometrical type usually employ a special terminology including *nasāḫu*, ‘to tear out’, for subtraction, *šutakūlu*(**gu**$_7$**.gu**$_7$), ‘to make hold one another’, for the construction of a rectangle and *šutamḫuru*(NIGIN), ‘to make encounter’, for the construction of a square.[138] In the LB period virtually nothing remains of this, apart from the occasional use of *nasāḫu*, and even purely geometric problems are now formulated with the new ‘general’ terminology.[139] This may be interpreted as a manifestation of a new level of abstraction, in the sense that a uniform terminology is introduced for all arithmetical operations, irrespective of the nature of the quantities to which they are applied (geometric objects, weights, astronomical quantities, etc.). One remaining fundamental diffence between the astronomical procedures and the (extant) LB mathematical problem texts is that the latter are always formulated in terms of numerical examples, while most astronomical procedures employ an abstract formulation involving named quantities of undetermined magnitude. Hence the turn from an example-based formulation to a fully abstract formulation which can be observed in the procedure texts is not (yet) apparent in the mathematical corpus, but otherwise the similarities with regard to the terminology and the representation of algorithms between procedure texts and mathematical problem texts are striking. This suggests that the influence between Babylonian mathematics and astronomy went in both directions. Mathematical astronomy could not have been invented in the absence of the computational techniques and terminology of OB mathematics. On the other hand one may suspect that the complexity of mathematical astronomy is behind some of the developments in mathematical representation, which also found their way into the mathematical texts of the LB period.[140]

2.3 Other elementary operations

Several other, non-arithmetical elementary operations are found in the procedure texts, e.g. conditions, comparisons, coordination and storage of information. They are essential for constructing complex procedures.

2.3.1 Introducing initial data

Some procedures include a statement by which initial data are introduced into the algorithm. One formulation involves the verb ‘putting down’ (*šakānu*):

qaqqara(**ki**) *tašakkan*(**gar**)	You put down the position.	No. **14** Oi3’ (Mars)
qaqqar(**ki**) *sin u* **ki** *šamaš*$_2$ **gar**-*an*	You put down the position of the Moon and the Sun.	No. **61** P1–P4 (step 4)
šūqa(**nim**) *u šupla*(**sig**) *ša*$_2$ *sin* **gar**-*an*	You put down the Moon’s ‘height and depth’.	No. **61** P1–P4 (step 6)

[138] Høyrup (2002), pp. 23–25.

[139] Cf. for instance the late Achaemenid problem texts W 23291 (Friberg, Hunger & al-Rawi 1990) and W 23291-x (Friberg 1997).

[140] A similar conclusion is drawn by Høyrup (1996), who states in a footnote that ‘the rather few mathematical texts which we know from the late period [...] were written by and for members of the astronomical environment, a context that seems to have influenced the mathematical mode of thought’.

In all examples quoted here the quantity that is 'put down' is input for the procedure.[141] Occasionally the same thing is formulated in terms of 'holding' (*kullu*) or 'taking' (*ṣabātu*) 'in the hands':

u$_4$.meš *u* **ki.meš** *ana igi-gub-u$_2$* [x] / *ina qātē*(**šu**.2)-*ka tu-kal*	You hold the times and the positions for the *igigubbû*-coefficients [...] / in your hands.	No. **46** Rii1–2
mu-*ka ina* **šu**.2-*ka tu-kal*	You hold your year in your hands.	BM 42282+42294 O6
si-man ša$_2$ **gub** *u* **gur il$_2$.a** *ina* **šu**.2 *taṣabbat*(**dib**)	You take the duration of the full month and the hollow month in (your) hands.	No. **53** Rii17

In BM 42282+42294, a Goal-Year procedure text (Brack-Bernsen & Hunger 2008), the object of the 'holding' is the Goal Year for which the astronomer wishes to make a prediction.

2.3.2 Conditions

While some algorithms are simple, linear chains of operations, many have a more complex structure. Often there is a branching point where the algorithm chooses between two or more options, depending on a condition. Two basic types of conditions can be distinguished: those involving a comparison between a quantity and a threshold value, and those involving an evaluation of a quantity's change (increasing/decreasing or ascending/descending) or relative position (above/below the ecliptic).

2.3.2.1 Conditions involving a threshold value

The course of an algorithm may depend on whether a control quantity, say Q, is less than or exceeds a threshold value Q_0, i.e. $Q < Q_0$ or $Q > Q_0$. Sometimes the comparison is formulated explicitly as a conditional clause involving the verbs *atāru*(**diri**), 'to exceed', and *maṭû*(**la$_2$**), 'to lack', rarely *iṣû*, 'to be small(er)':

ki-i Q al-la Q_0 atru(**diri**) ...	If Q exceeds Q_0 ...
ki-i Q al-la Q_0 maṭû(**la$_2$**)|*i-ṣi* ...	If Q is less|smaller than Q_0 ...

Often the comparison is combined with an immediately following algorithm involving the difference between Q and Q_0. Since Babylonian calculus does not know negative numbers, this is computed as $Q - Q_0$ if $Q > Q_0$, as $Q_0 - Q$ if $Q < Q_0$. Most examples concern the reflection rule of the zigzag function (§2.6.8) or the transition rule of the step function (§2.6.10). They exhibit two different but equivalent formulations. The most complete version is as follows:

ša$_2$ al-la Q_0 **diri**|**gal** Q_0 **ta** **ša$_3$**-*šu$_2$* **la$_2$**|**e$_{11}$**|**zi** ...	That which {exceeds|is larger than} Q_0: you {subtract|deduct|'tear out'} Q_0 from it, ...
ša$_2$ al-la Q_0 {**la$_2$**-*u$_2$*|**tur**} **ta** Q_0 **la$_2$**|**e$_{11}$**|**zi** ...	That which is less|smaller than Q_0: you {subtract it|deduct it|'tear it out'} from Q_0, ...

Here the subtractions $Q - Q_0$ and $Q_0 - Q$ are carried out explicitly. Most procedures employ *atāru*(**diri**), 'to exceed', and *maṭû*(**la$_2$**), 'to be less, deficient', but some use *rabû*(**gal**), 'to be larger', and *ṣeḫēru*(**tur**), 'to be smaller'. In the shorter version, which begins with exactly the same phrase, the subtraction is not spelled out, but the algorithmic interpretation usually implies it, so that one has to translate as follows:

ša$_2$ al-la Q_0 **diri**|**gal** ...	(The amount by) which it {exceeds|is larger than} Q_0 ...
ša$_2$ al-la Q_0 {**la$_2$**-*u$_2$*|**tur**} ...	(The amount by) which it is less|smaller than Q_0 ...

This usage of *(w)atāru*(**diri**) and *maṭû*(**la$_2$**) is well known from OB mathematics.[142]

2.3.2.2 Conditions involving the change of a quantity or the relative position

The second type of condition involves clauses of the kind

[141] The alternative interpretation, that these statements anticipate the outcome of the procedure, is problematic because a quantity cannot be 'put down' unless it is available.

[142] In OB mathematics *alla*, 'beyond', is not used. Instead, *(w)atāru* is followed by *eli*(**ugu**), 'above', and *maṭû* by *ana*, '(compared) to'.

ki-i ṭepû(**tab**) ... *ki-i maṭû*(**la$_2$**) ...	If it is increasing ... if it is decreasing ...
ki-i šaqû(**nim**, **la$_2$**) ... *ki-i šaplu*(**sig**, **bur$_3$**) ...	If it is 'high' ... if it is 'low' ...
ki-i išaqqû(**nim**, **la$_2$**) ... *ki-i išappilu*(**sig**, **bur$_3$**) ...	If it is ascending ... if it is descending ...

That **tab** and **la$_2$** in the first phrase can be interpreted as statives (ideally in the subjunctive) is implied by rare instances where **la$_2$** is rendered phonetically as *ma-ṭi* (e.g. No. **53** Ri8), but a present tense is also possible in principle. The quantity which is either 'high' or 'low' in the second phrase, and which is ascending or descending in the third phrase, is in both cases the distance of the Moon or the planet to the ecliptic (§2.5.3.2). Sometimes the conditional clause introduced by *kī*, 'if' (or 'when') is placed in the middle of the sentence, as in the following example:[143]

ki nim *u* **sig** *ša$_2$* $^{\mathrm{d}}$*sin ki-i* **tab la$_2$** *ki-i* **la$_2$ tab**	If it is increasing you diminish the Moon's 'height and depth' (by it), if it is decreasing you add it with it.	No. **61**.**G** R11'–12'

For the interpretation of this procedure, which is part of the Lunar Six module of lunar system A, cf. §4.4.19.11.

2.3.3 Coordination

Procedures are essentially sequences of arithmetical operations, conditions and comparisons to be executed in a certain order. In the case of a linear chain of operations the order is implied by the sequence itself. The most common, trivial form of coordination in such sequences is achieved asyntactically, e.g. 'you add x to y, you multiply it by z,...', but often the operations are coordinated by the particle *-ma*, 'and then'. A related function of *-ma* in the procedure texts is to introduce the result of a computation (...'and it is'), in which case it plays a similar role as our equal sign (=). In linear sequences of operations the outcome of one operation is usually passed on to the immediately following one. However, *-ma* is also used for coordinating operations that are performed on different quantities, so that there is no flow of data between them.[144] An explicit form of coordination is achieved by the following phrases:

ša$_2$ reḫi(*re-ḫi*, **tag$_4$**)	what remains
ša$_2$ illi(**e$_{11}$**)[*-ka*]	what comes out [for you]

The first is attested only after subtractions, the second one predominantly after multiplications, rarely after subtractions. If an operation or subalgorithm within a procedure depends on previously computed quantities, this is sometimes made explicit by qualifying the name of that quantity by the phrase 'which you had put down'. Accordingly, the subalgorithm in which that quantity was computed terminates with a statement of the kind '... and you put it down', by which the result is, so to speak, stored for later usage.

2.4 Additive and subtractive numbers

In the previous sections numbers and arithmetical operations were treated as separate, complementary entities. A notable feature of mathematical astronomy is the existence of additive and subtractive numbers, a construction in which a number and an additive or subtractive operation are intricately linked. The formulation of these numbers in the procedure texts and the tabular texts exhibits several innovations compared to OB mathematics. This is not surprising, because it reflects the central role of the difference concept in Babylonian mathematical astronomy. In the modern sense of the word most algorithms of Babylonian mathematical astronomy are 'difference schemes'. Ultimately this is because Babylonian astronomy is mainly concerned with synodic phenomena, a restriction that almost automatically leads to a mathematical formalism whereby the angular and temporal coordinates of the planet, the Moon or the Sun are updated from one to the next phenomenon by applying additive or subtractive differences.

While additive numbers are usually taken for granted, the notion of subtractive numbers has attracted interest in general works on the history of mathematics on account of their possible relation to the modern concept of negative numbers. Neugebauer did not hesitate to use the term 'negative' when translating or discussing subtractive numbers in the context of Babylonian astronomy; the same applies to the Moon's distance below the ecliptic, which he refers to as 'negative

[143] According to Dietrich (1969) and Hackl (2007), this usage of the subjunction *kī* in LB Akkadian is typical for temporal clauses but not for conditional clauses, which may suggest that 'when' is a more appropriate translation.

[144] Examples of this can be found in the Lunar Six module of lunar system A (§4.4.19).

latitude'.[145] However, this usage is incorrect and misleading, because 'negative' numbers in the modern sense do not exist in Babylonian mathematical astronomy, nor in the mathematical texts. This is already apparent from the fact that subtractions $x-y$ are avoided when $y>x$, as will be confirmed again and again in the procedure texts. Therefore subtractive numbers (and the same holds for distance below the ecliptic) are, in the modern sense, not negative numbers but (positive) magnitudes accompanied by a subtractive operation. This becomes particularly clear in procedures where something is added to or subtracted from a subtractive number (cf. below).

Høyrup (1993) makes the fundamental point that the only meaningful way to address questions about the existence of negative numbers and other modern concepts in ancient mathematics is to phrase them in terms of functional equivalence.[146] With that in mind I will explore various manifestations of additive and subtractive numbers in mathematical astronomy. As will become apparent, certain constructions involving subtractive numbers and other pairs of numbers can be viewed as a step in the direction of a functional equivalence to our positive and negative numbers.

History of subtractive numbers

The history of subtractive numbers in cuneiform can be traced back to the ED III period (2600–2350 BC), when Sumerian **la$_2$**, 'to be lacking', appears in so-called subtractively formed numbers. This is a notation whereby numbers ending with a digit 7, 8 or 9 are written as multiples of 10 and a subtractive correction, e.g. 28 = 30 **la$_2$** 2 = '30 lacking 2'.[147] After the second millennium this practice becomes rare and seems to have been eventually abandoned. In subtractively formed numbers the subtraction is always applied to a concrete number. The first evidence of subtractive numbers being applied to quantities of undetermined magnitude is found in OB mathematical problem texts, e.g.[148]

igi.5.**gal$_2$** **a.ša$_3$** GAR.GAR **uš sag a.na uš u.gu$_3$** **sag diri** 1.40 **ba.la$_2$**	1/5 of the area of the 'accumulation' of length and width is 1,40 less than whatever the length exceeds the width.

In *MKT* (Nr. 45, p. 455) this problem is represented in the form of the modern equation $(x+y)^2/5-1{,}0\cdot(x-y)=-1{,}40$. Some later investigators concluded from such representations that there are negative numbers in OB mathematics.[149] As pointed out by Høyrup (1993), the examples from OB mathematics only prove the existence of subtractive numbers, and they are not equivalent to the modern concept of negative numbers. The use of modern equations for representing Babylonian algorithms always carries a risk of anachronistic interpretations since they are ill-suited for this, as will be argued in §2.6.4.

Additive and subtractive numbers as numerical differences

The most common manifestation of additive and subtractive numbers in mathematical astronomy is that of concrete numbers followed by a logogram representing addition or subtraction. All additive and subtractive numbers in the tabular texts are of this type. They are especially common in the lunar tables, but they also occur in planetary tables and in procedure texts. In additive numbers **tab** follows the number, in subtractive numbers **la$_2$**, rarely **zi**. Other logograms for addition or subtraction are not attested in additive and subtractive numbers. The essential feature of these numbers is that they appear isolated from the quantity to which they are applied, i.e. $\pm x$ is abstracted from the application in an actual operation $y\pm x$. While the purpose of additive and subtractive numbers is clear, namely to indicate how the preceding number is applied to other numbers, their correct Akkadian reading is not known, apart from the fact that **tab** is a form of *ṭepû*, 'to add', **la$_2$** a form of *maṭû*, 'to be lacking', and **zi** a form of *nasāḫu*, 'to tear out'.[150] Without this innovation it is difficult to imagine how a computational system as complex as lunar system A or B with its numerous additive and subtractive functions could have been formulated.

[145] Cf. for instance the entries 'lal' and 'Negative'(= **la$_2$**) in *ACT*'s index (pp. 481, 501).

[146] In Høyrup's words: 'Asking whether the Babylonians had discovered, or not discovered, negative numbers, is as meaningful as asking whether they had discovered the Eiffel tower or not; the meaningful questions concerning the Eiffel tower would be whether they made constructions that in one way or the other expressed similar aspirations as those of the illustrious engineer, and whether they had created the techniques of which he made use'.

[147] The earliest known examples of subtractively formed numbers are found in A 681, an ED III text from Adab listing areas of rectangles. For an edition cf. Edzard (1969), more recently Friberg (2007). Ist T 7375 (from Girsu; Ur-III period) is a table of reciprocals in sexagesimal place-value notation in which e.g. 59 is written as 1 **la$_2$** 1 = '1,0 lacking 1' (Friberg 2007).

[148] YBC 4668 Oiii30–33 (Nr. 45), *MKT*, pp. 427–466.

[149] E.g. Goetsch (1968), who claims (p. 83) that '(the Babylonians) did not succeed in unifying the [...] newly formed set of negative rational numbers with the original set in such a way, that in the combined set the arithmetic operations are equally applicable to both' (translated from the original German).

[150] Occasional phonetic writings suggest that **tab** and **la$_2$** in additive and subtractive numbers may be statives (*ṭepi* = 'additive', *maṭi* = 'subtractive', 'lacking'). Second, they may be cognate nouns (*ṭīpu* = 'addition'; *mīṭu*, *nisḫu* = 'subtraction'). Third, in tabular texts they might be present tenses (**tab** = *teṭeppe* = 'you add', **la$_2$** = *tumaṭṭa* = 'you diminish' or *tanaḫḫis* = 'you subtract', **zi** = *tanassaḫ*, 'you tear out'). A fourth possibility, which may apply in tabular texts, is that they were not read as Akkadian words, but functioned more or less as symbols.

Additive and subtractive numbers of undetermined magnitude

Another innovation of the LB period concerns the introduction of a notation for additive and subtractive numbers of undetermined magnitude. This involves the cognate nouns of the mentioned verbs, i.e. *ṭīpu*(**tab**) = 'addition' and *mīṭu*(**la**$_2$) or *nisḫu*(**zi**) = 'subtraction'. The composite term *ṭīpu*(**tab**) *u mīṭu*(**la**$_2$), 'addition and subtraction', is used in the sense of a difference of undetermined magnitude and undetermined additive or subtractive nature.[151] As far as known an equivalent notation did not exist in OB mathematics, where additive and subtractive numbers are always concrete numbers.

Arithmetical operations performed on additive and subtractive numbers

A third innovation concerns the explicit formulation of arithmetical operations involving additive or subtractive numbers.[152] In the following example from a procedure text for lunar system K an arithmetical operation is performed on a subtractive number:

32 **ta** 22 *nisḫi*(**zi**) *ša*$_2$ **ḫun** *tanassaḫ*(**zi**)*-ma* 21.28	You 'tear out' 0;32 from 22, the subtraction ('tearing out') for Ari, it is 21;28.	No. **52** Oi35

Something is subtracted from the subtractive number, resulting in a subtractive number with a smaller magnitude. The same text contains an analogous example where something is added to a subtractive number, resulting in a subtractive number with a larger magnitude (Ri15–16). This makes it abundantly clear why it is anachronistic and misleading to refer to subtractive numbers as negative numbers. Arithmetical operations on subtractive numbers marked with **la**$_2$ are found in several procedures for lunar system A, e.g. the following multiplication, which occurs in a procedure for computing J from B (for this algorithm cf. §4.4.14):

2.1.44 **la**$_2$ **a.ra**$_2^!$ 28.7.30 **ki** *ša*$_2$ *šamaš*$_2$ **du** 57.3.45 **la**$_2$	You multiply 2;1,44, subtractive, by 28;7,30, the position of the Sun, (it is) 57;3,45, subtractive.	No. **53** Oii26'

Here J is computed for an exemplary zodiacal position of the Moon (B = 25;7,30° Ari) in a section of the zodiac where J is subtractive and linearly increasing with B. The subtractive number 2;1,44 is an interpolation coefficient which is multiplied by 28;7,30°, a distance measured along the ecliptic.[153] More complex examples are found in procedures with interpolation rules, also belonging to lunar system A, for computing Λ, which assumes additive as well as subtractive values, from Φ (cf. §4.4.13). No. **79** contains two representations of the interpolation scheme: one in tabular form (T1') and one in verbal form (P3'). The tabular entry for interval 2 (Oii12') is as follows:

[2.0.20.2]2.13.20 **la**$_2$ 1⸢9.5⸣5.33.20 **la**$_2$ **a.ra**$_2$ 3 **du**	[2,0;20,2]2,13,20, decreasing: 1⸢9;5⸣5,33,20, subtractive, you multiply by 3.

The first number is the control value Φ_k ($k = 2$), the second one is the corresponding Λ_k, which is subtractive, and the third number is the interpolation coefficient c_k. The interpolation procedure for this interval is written out fully in P3' (Rii3–4), which is damaged but can be restored as follows:

^{3}⸢*ana*⸣ [*tar* 2.0.20.22].13.20 **la**$_2$ 19.55.33.20 **gar** ^{4}[*ša*$_2$ *al* 2.0.20.22.13.20] ⸢**la**$_2$⸣ **la**$_2$ *adi*(**en**) 2.0.2.3⸢5⸣.[33.20 **la**$_2$ **a.ra**$_2$ 3 **du**] ⸢**ta**⸣ 19.55.33.20 **nim gar**	^{3}⸢Op⸣[posite 2,0;20,22],13,20, decreasing, you put 19;55,33,20. ^{4}[(The amount) by which] it is less than [2,0;20,22,13,20], ⸢decreasing⸣, until 2,0;2,35,[33,20, decreasing, you multiply by 3], deduct ⸢from⸣ 19;55,33,20, put down.

This procedure is an example of the interpolation template IP.B.2 (§2.6.7). Whereas 19;55,33,20 is marked as subtractive in the table, this is not done in the procedure, perhaps because the scribe did not want to overburden the formulation with yet another instance of **la**$_2$.[154]

The examples demonstrate that additive and subtractive numbers in mathematical astronomy are subjected to addition, subtraction and multiplication — all three elementary arithmetical operations. This phenomenon is unknown from OB mathematics. One limitation that may be noted is that in all cases the other number is a bare number not marked as additive or subtractive. In particular there is no explicitly formulated 'sign rule' of the kind 'subtractive times subtractive is additive', or 'subtractive times additive is subtractive'.

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[151] A similar construction involving *nisḫu*(**zi**) or any other term for subtraction (**nim**, **e**$_{11}$) is not attested.

[152] I.e. operations other than the addition implied by an additive number, or the subtraction implied by a subtractive number.

[153] The distance between B and the control point 27° Psc; cf. the commentary and §4.4.14.

[154] Unfortunately the interpolation rule for intervals 9–10 or 53–54 where Λ changes from subtractive to additive or vice versa are not preserved. It may be assumed that the subtractive or additive nature of Λ does not remain unmentioned there.

2.5 Coordinate systems

Coordinates are elementary concepts of a special kind. On the one hand they belong to the fundamental concepts since all functions and parameters are defined in relation to certain coordinate systems. On the other hand they also belong to the second, algorithmic level, because the angular and temporal coordinates of the Moon, the Sun and the planets are computed with algorithms. All coordinate systems are discussed here, together with what I call the ‘event frame’.

2.5.1 The event frame

The ‘event frame’ defined by the successive occurrences of a synodic phenomenon (Moon: New Moon, Full Moon; planets: first appearances, stations, acronychal risings, last appearances)[155] can be viewed as the fundamental reference frame of Babylonian mathematical astronomy. All angular and temporal coordinates are defined on that frame. The only ‘coordinate’ of the event frame is the event number, i, which corresponds to the lines (rows) of a synodic table. The distinction between the event frame and the coordinate systems for positions and times is of fundamental importance for a proper understanding of Babylonian mathematical astronomy. For instance, the time between successive occurrences of a synodic phenomenon varies from one to the next occurrence, an empirical fact respected in nearly all computational systems. Therefore, even though one may be tempted to view the event frame as being primarily associated with time, it is not a fixed time frame. The event frame of the Moon, which is defined by New Moons and Full Moons, is of special significance because of its connection with the Babylonian calender. Moreover, the Babylonian astronomers constructed from this an artificial coordinate system for the time of the planetary phenomena, by defining the mean synodic month as the basic unit of time, and dividing this into 30 artificial ‘days’, which will be referred to as ‘mean tithis’ (cf. below).

2.5.2 Temporal coordinates

2.5.2.1 The calendar

Babylonian dates are expressed by year, month and day. The month names in the astronomical texts derive from the calendar of OB Nippur (p. xxv). In this study they are translated with Roman numerals (I–XII plus VI_2 and XII_2 for the intercalary months). Modern equivalents of Babylonian dates are expressed in the Julian calender.[156] Texts written before the Seleucid era are dated by regnal years of the ruling king (cf. p. xxvi for a list of rulers). In synodic tables from that period with predicted data, year numbers are obviously hypothetical future regnal years of the current ruler. If the year numbers include a historically attested change of rule the data must have been computed after the change of rule. When mathematical astronomy was invented, the 19^{yr} (‘Metonic’) intercalation cycle was already in place, so that all intercalations follow a fixed and well-known pattern involving the insertion of one extra month (either a VI_2 or a XII_2) in 7 out of 19 years, resulting in a cycle of 235 months. During the period of concern (450 BC to 50 BC) this had the effect of confining the Babylonian New Year, day 1 of the month Nisan, to the months March – April of the Julian calendar. The Babylonian day begins at sunset, a commonly used reference point for timing astronomical phenomena.

2.5.2.2 Time degrees

The fundamental unit of time in mathematical astronomy is the time degree (UŠ), which is part of the ‘degree system’:[157]

$$1\ \textit{ūmu}(\mathbf{me},\mathbf{u_4}) \xleftarrow{12} 1\ \textit{bēru}(\mathbf{danna}) \xleftarrow{30} 1\ \text{UŠ} \xleftarrow{60} 1\ \textit{nindanu}(\mathbf{ninda}) \tag{2.2}$$

or, in translation:

$$1\ \text{day}\ (^{\mathrm{d}}) \xleftarrow{12} 1\ \text{‘mile’}\ (^{\mathrm{b}}) \xleftarrow{30} 1\ \text{degree}\ (^{\circ}) \xleftarrow{60} 1\ \text{‘rod’}\ (^{\mathrm{nin}}) \tag{2.3}$$

The unit UŠ is usually omitted and must be inferred from the context.[158] The earliest evidence of the ‘degree system’ is found in Tablet 14 of the astrological omen series *Enūma Anu Enlil*, which is dated to appr. 1200 BC.[159] In that text,

[155] For the synodic phenomena cf. §3.1.2 (planets) and §4.1.2 (Moon).

[156] In modern publications on ancient astronomy the notation with BC is often replaced by negative year numbers, such that 0 corresponds to 1 BC, -1 to 2 BC, etc; i.e. $-n$ corresponds to $n+1$ BC.

[157] The Akkadian reading of UŠ has not been established.

[158] In publications by Neugebauer, Aaboe and others up to the 1970s, time intervals were expressed in ‘large hours’ ($^{\mathrm{H}}$), a hypothetical unit consisting of 60°, so that 6^{H} = 1 day. Since this is not a Babylonian unit, its usage has been largely abandoned in favour of the time degree.

[159] For an edition of this tablet cf. Al-Rawi & George (1991/2).

and others that predate the 5th c. BC, the degree system is used for time intervals, and not yet for angular distance. The time degree has a constant duration corresponding to 1/360th of 1^{d} (defined from midnight to midnight), so that its modern equivalent is 4 minutes. The traditional division of the night into three watches (**en.nun** = *maṣṣartu*), well known from astronomical diaries and astrological texts,[160] is not used in mathematical astronomy. However, the word 'watch' is occasionally attested with a meaning 'duration' (e.g. of the *entire* night in No. **52**). As mentioned, time intervals are usually defined in relation to sunset, sunrise or midnight.

2.5.2.3 Mean tithis

Kugler (1907)[161] led the way to Pannekoek's discovery (1916) that the planetary texts employ an artificial system for measuring time, in which the basic units are the *mean* synodic month and 1/30 of that. The latter unit was called 'Mondtage' ('month days') by van der Waerden (1941), but nowadays they are commonly known as tithis, a Sanskrit term from Indian astronomy. I will systematically call this unit the *mean* tithi ($^{\overline{\tau}}$) in order to distinguish it clearly from the (real) tithi ($^{\tau}$) which is used in the daily motion tables of lunar system A (cf. §4.4.2.2). The only named units of the mean-tithi system are the mean synodic month (*arḫu* = 'month') and the mean tithi (*ūmu* = 'day'). A terminological distinction between real days and (mean) tithis is not apparent in the texts. In the translations 'day' is put in quotation marks if it denotes a (mean) tithi. Smaller units such as 1/60 of the mean tithi are not explicitly named. Time intervals measured in mean tithis are usually defined in relation to the beginning of the year (1 Nisan), or the beginning of the month. The relation between days and mean tithis is not formulated explicitly in the texts. An analysis of Babylonian values of the mean synodic time ($\overline{\tau}$) for various planets,[162] which are expressed in mean tithis, and modern empirical values expressed in days, implies that the ratio of the mean tithi to the day is always close to

$$\frac{1^{\overline{\tau}}}{1^{\mathrm{d}}} = \frac{1^{\overline{\mathrm{m}}}}{30^{\mathrm{d}}} \approx \frac{29;31,50,8,20^{\mathrm{d}}}{30^{\mathrm{d}}} \approx 0;59,3,40,16,40. \tag{2.4}$$

This can be used for converting between degrees and mean tithis, the reciprocal being $1^{\mathrm{d}}/1^{\overline{\tau}} \approx 1;0,57,13,26,40$. The underlying value of the mean synodic month, $\overline{m}_{\mathrm{syn}} = 29;31,50,8,20^{\mathrm{d}}$, is also embedded in lunar system B (function *G*; cf. §4.5.12) and, to good approximation, lunar system A (cf. No. **82** P2.b). For the advantages of the mean-tithi system cf. §3.3.2.

2.5.3 Angular coordinates

The positions of the Moon, the Sun and the planets are expressed in a two-dimensional coordinate system based on the zodiac, a circular band whose centre is defined by the orbit of the Sun. The Sun's path was divided into 12 sections of 30° that were named after nearby constellations (p. xxv). The earliest textual evidence for the zodiac dates to about 450 – 400 BC.[163] Cuneiform texts do not mention the width of the zodiac, but in practice it does not extend much beyond the path of the Moon, which reaches to about 5° above or below the ecliptic. The division into 12 sections of 30° was likely conceived in analogy to the existing division of the day into 12 'miles' (*bēru*) of 30 time degrees, and the ancient schematic division of the year into 12 months of 30 days (Brack-Bernsen 2007). As in the degree system for measuring time, 1° of angle (UŠ) is divided into 60^{nin} ('rods'):

$$1^\circ\ (\text{UŠ}) \xleftarrow{60} 1^{\mathrm{nin}}\ (\textbf{ninda} = \textit{nindanu}) \tag{2.5}$$

2.5.3.1 Zodiacal position (*B*)

terminology	*qaqqaru*(**ki**) = 'position'; *lumāšu* = 'zodiacal sign'
unit	zodiacal sign and degrees within it [0–30]
astronomical meaning	zodiacal position

Zodiacal positions of the Moon, the Sun and the planets (conventional modern symbol: *B*) are expressed as a zodiacal sign and a number of degrees measured from the beginning of that sign (Fig. 2.2).[164] Babylonian astronomers usually

[160] E.g. in lunar eclipse omens (*EAE* Tablets 15–22; cf. Rochberg 1988).
[161] Kugler *SSB* I (1907), pp. 167–169
[162] Neugebauer (*HAMA*, p. 1070). For the synodic time, τ, cf. §3.3.2.1.
[163] Cf. Rochberg (1998), p. 30, and Britton (2010) who argues for a date near 400 BC.
[164] In formulaic representations the zodiacal sign and the position within it are usually combined in a single notation, thus allowing *B* to assume values between 0° and 360°.

referred to the boundary between adjacent signs as 30° of the preceding sign, thus avoiding the zero for denoting zodiacal positions.[165] Babylonian zodiacal positions are sidereal, i.e. fixed with respect to the stars, as opposed to modern ecliptical coordinates which are tropical, i.e. defined with respect to the moving vernal equinox.[166]

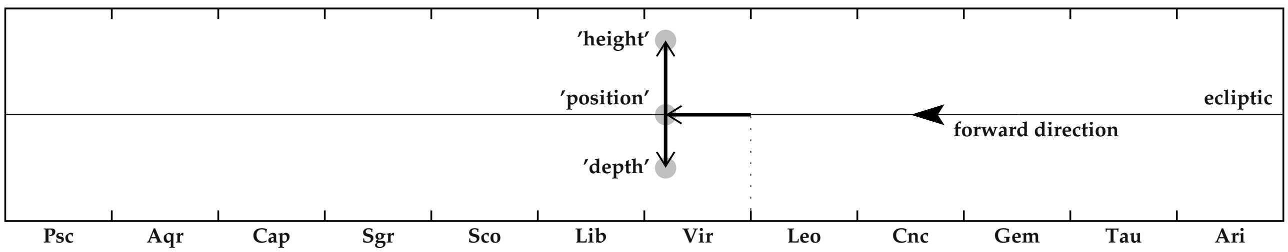

Fig. 2.2: Reference frame for the angular coordinates of the Moon, the Sun and the planets as viewed against the sky.

2.5.3.2 Distance to the ecliptic (E)

terminology	*šūqu*(**nim**, **la₂**) = 'height' ($E_\uparrow$) *šuplu*(**sig**, **bur₃**) = 'depth' ($E_\downarrow$)
units	barleycorn ($^{\text{še}}$), cubits ($^{\text{c}}$), fingers ($^{\text{f}}$) or degrees (°)
astronomical meaning	distance above or below the ecliptic

Lunar systems A, B and (probably) K, and some planetary procedure texts, include algorithms for the Moon or the planet's distance to the ecliptic (conventional modern symbol: E). The Akkadian terms for this quantity are 'height' = *šūqu*(**nim**, **la₂**) and 'depth' = *šuplu*(**sig**, **bur₃**), cognate nouns of *šaqû*, 'to be high' and *šapālu*, 'to be low' (Fig. 2.2).[167] In lunar system A the algorithms for E imply that its reference points form a circle in the middle of the zodiac similar to the modern ecliptic. Their precise location is not explicitly defined in texts, but the algorithms for eclipse magnitude and the Lunar Six intervals imply that they coincide with the annual course of the Sun's centre. The reference circles for the planets must also be near the middle of the zodiac. For lack of an alternative term, they will be called 'ecliptics', even though there are some conceptual differences with the modern ecliptic.[168] For instance, the Babylonian ecliptic is, in modern terms, sidereal, since the boundaries of the zodiacal signs are fixed with respect to the stars. The value of E is usually expressed in barleycorns ($^{\text{še}}$), sometimes in cubits ($^{\text{c}}$) and fingers ($^{\text{f}}$), all of which are units of the cubit system:

$$1\ \textit{ammatu}(\textbf{kuš}_3) \xleftarrow{24} 1\ \textit{ubānu}([\textbf{šu.}]\textbf{si}, \textit{u}) \xleftarrow{6} 1\ \textit{uṭṭatu}(\textbf{še}) \tag{2.6}$$

In translation:

$$1\ \text{cubit}\ (^{\text{c}}) \xleftarrow{24} 1\ \text{finger}\ (^{\text{f}}) \xleftarrow{6} 1\ \text{barleycorn}\ (^{\text{še}}) \tag{2.7}$$

In lunar system A the numbers in column E are usually expressed in barleycorns ($^{\text{še}}$). If E exceeds $6^{\text{še}}$ it is not converted into fingers plus a remainder in barleycorns. In the Lunar Six modules E is expressed in degrees; in the planetary procedures in fingers and/or cubits, such that $1^{\text{c}} = 24^{\text{f}}$, the same ratio known from astronomical diaries, other observational texts and NMAT procedures.[169] It will be clear from this discussion that the cubit is rarely used in mathematical astronomy. The procedures for E imply that $1^{\text{c}} = 2°$;[170] there are also traces of evidence for a 'large cubit' (*ammatu rabû*) equivalent to 2;30° and presumably containing 30^{f}.[171]

[165] However, fractional positions within the first degree of a sign are expressed as 0;...°.

[166] Due to precession (and nutation) tropical coordinates change slowly with time, a phenomenon unknown to the Babylonian astronomers (Neugebauer 1950). By analysing star positions and planetary data in the observational texts, Britton (2010) established that the difference between Babylonian (sidereal) and modern (tropical) longitudes is described by the formula $\Delta\lambda = 3.20° - 1.3828 \cdot Y$, where Y measures Julian centuries from 1 BC, a slight modification of an earlier result by Huber (1958).

[167] For **nim** and **sig** in the Sumerian tradition cf. Balke (2002).

[168] As pointed out by Steele (2003; 2007a) there is no compelling evidence that the Moon's ecliptic coincides exactly with any of the planetary ecliptics. However, evidence that they are significantly different is also lacking.

[169] E.g. BM 41004 O13 (Brack-Bernsen & Hunger 2005/6). The only other astronomical quantity measured in fingers is eclipse magnitude (Ψ etc. in lunar systems A and B). Again, if the eclipse magnitude exceeds 24^{f} it is not converted into cubits plus a remainder in fingers.

[170] In the observational texts $1^{\text{c}} \approx 2.2°$ (Fatoohi & Stephenson 1997/98; Jones 2004).

[171] The existence of the 'large cubit' is suggested by No. **9** O7 (cf. the commentary). While the 2° cubit of mathematical astronomy goes back to the NB cubit, a unit of length likewise divided into 24^{f}, the 'large cubit' may go back to the OB cubit, a unit of length containing 30^{f} (*RlA*, p. 462); less likely the Kassite 'large cubit' amounting to 1 1/2 OB cubits, which was divided into 24^{f} (*RlA*, p. 469).

Kugler (1900) did not know the Akkadian reading of the logograms for ‘height’ and ‘depth’, but he did establish the essentially correct interpretation as ‘northern latitude’ and ‘southern latitude’,[172] a terminology borrowed from the modern astronomical concept of (ecliptical) latitude. Since Neugebauer, the conventional translation has been ‘positive latitude’ and ‘negative latitude’ (e.g. *ACT*, *HAMA* and elsewhere). However, compared to Kugler’s interpretation, that translation is anachronistic and potentially misleading, since there are no negative numbers in Babylonian calculus (§2.4), and ‘height’ and ‘depth’ are positive quantities. It is difficult to avoid confusion and wrong interpretations if one translates ‘negative latitude’ instead of ‘depth’ in procedures where something is added to or subtracted from this. If something is added to the Moon’s ‘depth’ then the Moon’s distance to the ecliptic increases, but if one translates ‘negative latitude’ instead of ‘depth’ then the addition corresponds to a decrease of this distance. In order to obtain a consistent translation one would then have to go as far as translating the Babylonian additions as subtractions in all cases where something is added to ‘negative latitude’, and vice versa for ‘positive latitude’. Such problems are avoided if one adopts the semantically more accurate translations ‘height’ and ‘depth’.

2.6 Procedures and algorithms

The elementary operations and other basic concepts discussed in the previous section are the building blocks of procedures, which brings us to the next level of concepts.

2.6.1 Composite procedures and subprocedures

Individual procedures are identified by means of visual markers on the tablets, such as horizontal dividing lines, spacing and sometimes columns. However, if one uses only criteria based on visual markers this often results in composite procedures that are not strictly coherent or devoted to a single algorithm. I therefore introduce an extra layer of structure such that procedures, as defined by visual markers, may consist of subprocedures, each representing a coherent set of instructions devoted to the computation of a single quantity, or several related quantities, or a subalgorithm of a larger algorithm. In order to maintain consistency with the established numbering of procedures in previous editions, the subprocedures are labeled a,b,c,..., e.g. P2.a is subprocedure a of the (composite) procedure P2. Sometimes the correctness of a division into subprocedures is confirmed by duplicates containing the same sequence of procedures but separated by horizontal lines.

2.6.2 Initial and final statements of a procedure

Many procedures begin with a statement indicating the purpose or subject of the procedure. A common formulation of these statements also known from NMAT procedures involves the verb *epēšu*, ‘to construct’, or its cognate noun *epūšu*, ‘procedure’:

epūšu(**du$_3$**-*šu$_2$*) *ša$_2$* 2.13.20	Procedure for 2,13;20 (= Φ).	No. **53** Oi1
du$_3$-*uš ša$_2$* **zi** *sin*	Procedure for the displacement of the Moon.	No. **53** Oi14’
e-pe-šu$_2$ ša$_2$ **igi.meš** *an-ne$_2$-e-tu$_4$*	Procedure for these appearances (of Mercury).	No. **1** R1
epūšū(**du$_3$.meš**) *an-nu-tu$_2$ ša$_2$ kun-nu u tur-ru u ūm bubbuli*(**u$_4$.na$_2$.am$_3$**)	These procedures are for hollow and full months and the ‘day of disappearance’.	No. **53** Rii39–40

All but the last example appear at the beginning of the procedure. The third example makes reference to a synodic table on the obverse of the same tablet. The last example is placed at the end of a procedure (No. **53** P18’). The mentioned purposes are in fact those of the tablet as a whole.[173] In other procedures only the name of the planet is mentioned:

ša$_2$ ^{d}**gu$_4$.ud igi** *ša$_2$* **kur** *a-na* **igi** *ša$_2$* **kur**	For Mercury, eastern appearance (MF) to eastern appearance.	No. **42** O1
ša$_2$ **mul$_2$.babbar**	For Jupiter.	No. **18** Rii22’
ša$_2$ **genna**	For Saturn.	No. **42** O18

Also known from NMAT procedure texts and mathematical problem texts are statements of purpose involving an infinitive construction:

[172] ‘Nördliche Breite’ and ‘südliche Breite’ (Kugler 1900, pp. 37, 136).

[173] Similar phrases describing the purpose of an entire tablet are well known from the observational texts, where they have a highly standardised form. Diaries: ‘Regular watch from month MN$_1$ until the end of MN$_2$ of year x, king PN’; Goal-Year texts (at the end of the tablet): ‘First days, appearances, passings and eclipses which were established for year x, king PN’ (e.g. *ADRT* VI 69).

šu$_2$ *ana* **du**$_3$-*ku*	For you to 'construct' ŠU$_2$.	No. **61**.**D** O1
si-man qa-tu-u$_2$ *ana* **du**$_3$-*ka*	For you to 'construct' the final duration.	No. **53** Oii27'
[**u**$_4$.**n**]**a**$_2$.**am**$_3$ *ana* **du**$_3$-*ka*	For you to 'construct' the 'day of disappearance'.	No. **53** Rii17
nim *u* **sig** *ana* **du**$_3$-*ka*	For you to 'construct' the 'height and depth' (E).	No. **56** R17
nisḫa(**zi**) *rabû*(**gal**-*u*$_2$) *ana* **igi**-*ka*	For you to see the largest displacement (F).	No. **53** Oii16'
[**z**]**i** *ša*$_2$ **an** *ša*$_2$ *kal* **mu**.⌜**an.na**⌝ *ana* **igi**-*ka*	For you to see the displacement of Mars for the whole year.	No. **13** O24

There does not appear to be a difference between 'constructing' (*epēšu*) and 'seeing' (*amāru*) a quantity.[174] The following examples involving the verb 'to produce' (*šūṣû* = 'let come forth') concern procedures for lunar system A:

nim *u* **sig ta** ḪAB-*rat*.**meš** *ana šu-ṣu-u*$_2$	In order to produce the 'height and depth' (E) from the 'disks' (Ψ).	No. **65** O15'
nim *u* [**sig**] **ta** ḪAB-*rat ana* **e**$_3$-*u*$_2$	In order to produce the 'height and depth' from the 'disk'.	No. **53** Oii15'
[**zi** *sin* **ta** 2.13].20 *ana šu-ṣu-u*$_2$	In order to produce [the Moon's displacement (F) from 2,13];20 (Φ).	No. **65** R1

The second example is placed at the very end of the procedure (No. **53** P11'). In another procedure for G the transformation is represented by the bare preposition 'from':

si-man ina(**ta**) *nisḫi*(**zi**) *našû*(GIŠ-*u*$_2$)	The duration (G) from the computed displacement (F).	No. **96** O9'

Finally, some procedures begin with the word 'Secondly' (*šanîš*), indicating that they are alternative to another procedure, usually the preceding one.

2.6.3 Procedures as verbal representations of algorithms

In this study the term procedure is used in the sense of a verbal representation of an algorithm. Maintaining a distinction between procedures and algorithms is important for several reasons. First, the reconstruction of algorithms is an interpretative process requiring lexical, semantic, mathematical and astronomical analysis of procedure texts (and tabular texts). Second, the representation of algorithms in procedure texts is often incomplete or deficient in one way or another (cf. below). In this view procedures are actual verbal instructions, while algorithms are the underlying complete sequences of operations reconstructed from procedures, sometimes in conjunction with analysis of tabular texts. The reconstructed algorithm will be referred to as the 'algorithmic interpretation' of the procedure.

2.6.3.1 Example-based and abstract formulation

The procedure texts employ two different methods for representing algorithms: by offering numerical examples ('example-based formulation'), or general rules involving named quantities of undetermined magnitude ('abstract formulation'). In the example-based formulation an algorithm is demonstrated for specific numerical values of the involved functions. In the abstract formulation only the defining parameters of the algorithms are given numerical values, but the quantity that is computed does not assume a numerical value. With the example-based formulation, the user must first extract a general rule before being able to apply the algorithm with different initial values than the ones that happened to be used in the procedure. With the abstract formulation this interpretative step is no longer necessary, because the procedure is valid for all possible initial values of the input functions. This obviously reduces the chance that the user of the procedure makes an error. The abstract formulation necessarily involves operations on abstract quantities of undetermined magnitude, which constitutes a point of similarity with the modern concept of a variable.

The available evidence suggests that the example-based formulation occurs mainly in early procedure texts, and that it became less popular after the beginning of the Seleucid era. Perhaps No. **53**, a large procedure text for lunar system A, reflects the transition from the example-based to the abstract formulation, since it contains a mixture of both formulations. Interestingly, nearly all algorithmic interpretations derived from the procedures in No. **53** fully conform with lunar system A in its final stage. This suggests that the formulation of the procedures underwent further changes even after the algorithms themselves had reached their final form.

[174] Another verb with a similar meaning as *epēšu* is *našû* (GIŠ, **il**$_2$), 'to compute'. It is used in contexts where the focus is less on the procedural aspect of an algorithm and more on its outcome. It is not attested in statements of purpose.

2.6.3.2 Deficient procedures

That procedures are best interpreted as representations of algorithms is particularly clear from various kinds of deficiencies. If one closely examines a procedure it is usually the case that at least some step or subalgorithm is strictly speaking not completely represented, although usually clearly implied. Occasionally these omissions are so grave that it is difficult to understand how the procedure could have been used at all. For instance, No. **53** P17' is incomplete and formulated in an opaque manner, so that it is at best a 'mnemonic device' reminding the user about certain steps, without actually providing a coherent set of instructions. Moreover, in this procedure the sequence of the instructions does not agree with the order of the subalgorithms that they represent. Another reason why procedure texts do not provide a complete representation of algorithms derives from their limited purpose. Most procedure texts are mainly concerned with the practical aspects of producing astronomical tables. There may be other, more fundamental algorithms that, according to a closer analysis of the practical algorithms, underlie the computational systems. These hidden levels of mathematical structure are hardly represented in the procedure texts.

2.6.4 Representing procedures, algorithms and functions

In this section I discuss how procedures, algorithms and functions are represented in the present study. Each representation has certain limitations which must be kept in mind in order to interpret them in a proper way.

2.6.4.1 Columns, functions and parameters

The columns of the tabular texts are conventionally denoted by modern symbols, e.g. 'column Φ'.[175] I use the word function for any quantity that is determined by an algorithm, and reserve the word column for an actual column in a tabular text. This convention differs slightly from that of Neugebauer and others, who use the term column as (nearly) synonymous with function.[176] The distinction between function and column is important for two reasons. First, not all functions attested in procedure texts are represented in the tabular texts, e.g. the whole of system K. Second, some columns contain not only numerical values of a function but also other numerical or non-numerical information associated with that function.[177] Functions are also distinct from parameters, which do not have an argument, and do not vary within a computational system. The conventional modern terminology for the parameters of Babylonian mathematical astronomy is largely maintained here; e.g. m is the minimum of a function, M the maximum, d the difference for 1 synodic cycle, P the period, etc. The meaning of these and other parameters is explained in the following chapters; for a list of (modern) symbols cf. p. xxi.

2.6.4.2 Template procedures

Template procedures are essentially quotations from actual procedures in which the numerical values of functions and parameters are replaced by the corresponding modern symbols. Many actual procedures can be viewed as specific implementations of an underlying template. Templates are therefore useful in order to identify generic elements of the procedures.[178] They are derived in accordance with the following rules:

(i) Numbers are replaced by modern symbols of the corresponding quantities. These symbols always correspond to numerical values in the actual procedures (i.e. not abstract quantities of undetermined magnitude).
(ii) The Babylonian name of a quantity is either translated into English, or replaced by QN (= 'name of a quantity').
(iii) Optional elements, i.e. words or phrases that are not always present, are enclosed in square brackets, [...]
(iv) Alternative signs, words or phrases are separated by vertical bars (|) and ordered according to decreasing frequency. If necessary the alternatives are enclosed in accolades, {...|...|...}.[179]

An example may illustrate how templates are extracted from the procedure texts. No. **37** P1 is a procedure for updating the time of a synodic phenomenon (§3.3.2.1) by means of a zigzag function (§2.6.8). The algorithm includes the following 'reflection rule' near the maximum of the zigzag function:

[175] The currently used notation for the columns was established by Neugebauer (*ACT*, *HAMA*), who modified Kugler's notation in order to remove certain inconsistencies.
[176] In his early, German papers, Neugebauer did use the term function ('Funktion') much in the same sense as is done here.
[177] For instance, column M of lunar system A contains the value of M and the date of the lunation.
[178] Kugler had already reconstructed template procedures, e.g. in connection with the ΦG interpolation scheme of lunar system A (Kugler 1900, p. 171).
[179] Purely orthographic variations involving phonetic complements etc. are usually not taken into account.

ša$_2$ *al-la* 50.07.15 *atrat*(**diri**) *ina*(**ta**) 1.40.14.30 *tušelli*(**e**$_{11}$)*-ma* **gar**	That which exceeds 50;7,15 you deduct from 1,40;14,30 and put down.

Here 50.07.15 represents the maximum $M = 50;7,15^{\overline{\tau}}$. If we replace this by M and incorporate other implementations of this rule we obtain the following template:

ša$_2$ *al-la* M {**diri**|**gal**-*u*$_2$} **ta** 2M **e**$_{11}$|**nim**|**la**$_2$[-*ma* **gar**]	That which {exceeds|is larger than} M you deduct|deduct|subtract from 2M [and put down.]

Some templates are generalised one step further, in the sense that

(v) the numerical value or name of a function or parameter is replaced by the symbol Q (in the case of multiple quantities Q_1, Q_2, etc.).

This is mainly used for arithmetical operations, which are applied to both numbers as well as abstract quantities of undetermined magnitude (§2.2).

2.6.4.3 Formulaic and graphical representations

Formulaic representations

Formulaic representations have a long history in the literature on Babylonian mathematical astronomy that goes back to the very first publication by Epping and Strassmaier (1881). As in other branches of the history of mathematical astronomy it is common practice to use modern symbols for representing functions and parameters of Babylonian astronomy. This might be criticised as anachronistic, because there is no Babylonian equivalent of symbolic manipulation, neither in mathematical astronomy, nor in the mathematical corpus. In the procedure texts all manipulations are represented in verbal form. Nevertheless, formulaic representations, if interpreted with care, are useful and virtually indispensable tools in the context of Babylonian mathematical astronomy, if only because many of its quantities (for instance those in lunar system A) have a complex astronomical interpretation that is not readily captured in words. Moreover, the Babylonian names for these quantities, if known, often reveal little about their precise astronomical meaning. The modern symbols provide us with convenient labels for referring to these quantities (e.g. to speak of 'function Φ' etc.), representing their algorithms in the form of equations, and analysing their properties.

As representations of algorithms modern equations must be interpreted with care. Equations are capable of representing the outcome of algorithms, but they provide only limited information about the way in which this outcome is achieved, i.e. by which precise sequence of steps. Modern equations are thus in fact unsuitable for representing algorithms, because any modern equation may represent several different, mathematically equivalent algorithms. Another fundamental objection to the use of formulaic representations stems from the fact that modern equations imply rules, for manipulating identities, that do not always have a counterpart in Babylonian mathematical astronomy. In particular there is often no Babylonian equivalent to the operation whereby a quantity is transferred to the other side of an equation, because this involves transforming positive into negative and vice versa. Since there is no concept of negative numbers in Babylonia, this operation is possible only if the term as a whole remains positive. Modern equations are therefore not well suited for representing relations between quantities as they are formulated in the procedure texts, and the same applies to OB mathematical problem texts.

Graphical representations

For the convenience of the modern reader many algorithms are accompanied by graphical representations in which the computed function is plotted against its argument, usually the zodiacal position, event number, day number or tithi number. Also graphical representations have a long history in the field of Babylonian mathematical astronomy, and likewise they have profoundly shaped the modern discourse about its concepts. Two obvious examples are the terms zigzag function and step function, which originate from modern graphical representations of the Babylonian algorithms. However, there is no evidence that the Babylonian astronomers conceptualised these algorithms in graphical terms.

As is true for equations, mathematical graphs are mainly suitable for representing the outcome of an algorithm, but not the algorithm itself. Only for some simple algorithms such as linear interpolation (Fig. 2.4, p. 41) or the zigzag function (Fig. 2.6, p. 45), do the figures in fact represent all steps of the algorithm. Similarly, all representations of step functions (Fig. 2.10, p. 49) include the preliminary values of the step function as a dotted curve. In this connection note also that in most graphical representations the argument of a function (the coordinate) increases from left to right in agreement with the modern convention, whereas the normal apparent motion of a planet, the Moon or the Sun if viewed against the background sky proceeds from right to left for an observer in Babylonia or elsewhere in the northern hemisphere.

2.6.4.4 Flow charts

My use of flow charts is inspired by recent investigations of algorithms in Old Babylonian and ancient Egyptian mathematics.[180] Up to now flow charts have hardly been used in the context of Babylonian mathematical astronomy.[181] The main advantage of flow charts in comparison with other representations is that they can preserve all essential features of an algorithm including the temporal and logical sequence of the involved steps (subalgorithms) and the flow of information (input quantities, initial values, output quantities, parameters). Much of that information is lost when algorithms are represented as equations. Flow charts are particularly useful for representing highly complex algorithms as they occur in Babylonian mathematical astronomy. The form of the flow chart and its constitutive graphemes (Fig. 2.3) have been slightly adapted in order to cope with that complexity.[182] In order to interpret a flow chart it suffices to keep in mind the following rules, which are basically self-evident:

- a rectangular box represents an operation or subalgorithm ('computational step')
- a diamond-shaped box represents a choice between two or more options, each of which involves a condition
- a line connected to a box or diamond represents a quantity (function of parameter)
- a solid arrow indicates the flow of information (principally from left to right)
- occasionally a disconnected open arrow indicates the sequence of two operations without a transfer of information.

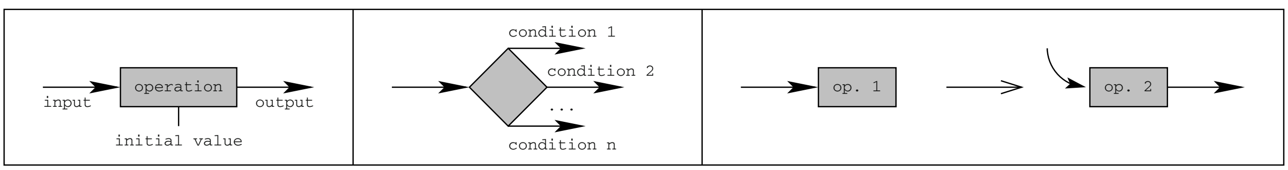

Fig. 2.3: Elementary graphemes of the flow charts. **Left**: operations and subalgorithms. The optional vertical line below the box represents an initial condition. **Middle**: conditions. **Right**: sequence of two operations without transfer of information.

2.6.5 *Purposes of the algorithms*

Having clarified the relation between procedures and algorithms, we are in a position to classify the purposes of the algorithms.

2.6.5.1 Computing or updating a function

Most of the algorithms are aimed at computing (in Akkadian: to 'construct' = *epēšu*) the columns of the tabular texts (or intermediate quantities not represented by a column). These algorithms are of two types:

(i) updating a function by applying a difference to the value at the previous synodic event, day or (mean) tithi, i.e. $f_i = f_{i-1} \pm d$. The difference d can be constant (e.g. in the case of a zigzag function) or variable (e.g. if d is modeled as a step function). Sometimes the updating involves several differences which are combined, i.e. $f_i = f_{i-1} \pm d_1 \pm d_2 \ldots$.
(ii) nested functions, i.e. $g(f)$. This concerns algorithms by which one function (g) is computed from another function (f). The updating of g then proceeds indirectly, namely by updating f_{i-1} to f_i and computing $g(f)$. Very often f is the zodiacal position; a more complex example is the interpolation algorithm for computing G from Φ in lunar system A (§4.4.11).

[180] Robson (1997); Ritter (1998, 2010); Imhausen (2002, 2003).
[181] E.g. Aaboe (2001), p. 60, and Britton (2007a), who use flow charts for representing the sequence of algorithms for computing a synodic table of lunar system A.
[182] I use a more compact notation than Imhausen (2003), such that input and output quantities are not represented by separate graphemes. Furthermore, I order the operations from left to right, not from top to bottom.

2.6.5.2 Verification

There are also procedures that can be used for verifying whether the column of a tabular text has been computed correctly. Usually this verification is achieved by deriving from the basic algorithm for computing or updating the function an algorithm for the net difference of that function for intervals longer than 1 synodic cycle. By applying the net difference, say for n cycles (corresponding to n lines in the table), to a function value in the table, say in line 1, this should ideally yield the value in line $n+1$. If not, then an error was committed somewhere in between. This kind of verification is possible only if a simple expression for the net difference can be derived, which is true for zigzag functions and step functions. However, some of the lunar functions are sufficiently complicated that there is no simple expression for the net differences (e.g. M in lunar system A). It is therefore not surprising that computational errors are especially common in these columns.

2.6.5.3 Theoretically oriented procedures

A small number of tablets contain procedures involving astronomical quantities that are known from the tabular texts or other procedures, but the algorithms do not have any obvious connection with the production or verification of tables. Some of these procedures contain rather sophisticated computations that reveal a deep insight into the astronomical significance of, or the relations between various functions or parameters (e.g. No. **82** and No. **102**). The 'theoretical' orientation and apparent lack of practical use of these procedures can be viewed as explicit evidence against the 'instrumentalist' interpretation of Babylonian mathematical astronomy that one sometimes finds in general works on the history of science.

2.6.6 The basic period relation of a function

All functions of Babylonian mathematical astronomy are periodic in the event frame, which means that they return to *exactly* the same value after a certain whole number of synodic events, days or tithis. This is known as the number period Π. A second basic parameter characterising the periodic behaviour of a Babylonian function is the corresponding number of oscillations Z, i.e. the number of times that the function assumes its minimum or maximum value in the course of Π events. They constitute the basic period relation of any function:

$$\Pi \text{ events of the synodic phenomenon } \sim Z \text{ oscillations of the function,} \tag{2.8}$$

where the symbol $\sim$ denotes exact correspondence. Several procedure texts mention period relations of this kind, using the term *palû*(**bala**), 'turn', to describe Z.[183] The ratio of Π and Z defines the 'period',

$$P = \frac{\Pi}{Z}. \tag{2.9}$$

Since Π events correspond to Z oscillations of the function, P is the (fractional) number of events after which the function completes exactly 1 oscillation. Unlike Π and Z, the period P is rarely a whole number; often it is not even a finite sexagesimal number.[184] For a proper understanding of P it is also important to recall that the event frame is not an absolute time frame. The actual time corresponding to P synodic events (or any other number of events) is determined by the algorithm for the synodic time or its lunar equivalent, the synodic month.[185]

2.6.7 Interpolation

Interpolation is one of the most common algorithms of Babylonian mathematical astronomy. It is used for computing the value of a goal function, say g, from a given value of the source function, say f. This is achieved by interpolating linearly between control values of f and g. Some algorithms for computing daily positions of the planets may be interpreted as quadratic interpolation schemes, since the planet's zodiacal position (B) assumes a quadratic dependence on the day or tithi number (i). However, this quadratic dependence is merely a consequence of the linear dependence of the daily displacement v on i. As far as known an explicit concept of quadratic interpolation is not formulated anywhere.

[183] E.g. No. **13** P11, No. **18** P22, No. **41** P4, No. **45** P1'. This usage of *palû* is not mentioned in the dictionaries.

[184] In practice this is not a problem, because P is not a fundamental parameter like Π and Z. Nevertheless some values of P are mentioned in the procedure texts.

[185] Nevertheless, one can occasionally find incorrect statements in the literature suggesting that P is expressed in multiples of the mean synodic time $\overline{\tau}$; e.g. Schmidt (1969).

The zodiacal position (B) as source function

The most common class of interpolation procedures includes those involving the zodiacal position of the planet or the Moon (B) as the source function. They are based on the following templates:

[*ana tar-ṣi*] B_k g_k [**gar**-*an*] [*mim-ma*] *ša*$_2$ *al-la* B_k **diri a.ra**$_2$|GAM|GAM$_0$ c_k **du**-*ma* **ki** g_k **tab**[-*ma* **gar**-*an*]	[Opposite] B_k [you put down] g_k. [Whatever] (the amount) by which it exceeds B_k you multiply by c_k, add to g_k [and put down].	(IP.A.1)
[*ana tar-ṣi*] B_k g_k [**gar**-*an*] [*mim-ma*] *ša*$_2$ *al-la* B_k **la**$_2$ **a.ra**$_2$|GAM|GAM$_0$ c_k **du**-*ma* **ta** g_k **la**$_2$|**nim**[-*ma* **gar**-*an*]	[Opposite] B_k [you put down] g_k. [Whatever] (the amount) by which it is less than B_k you multiply by c_k, subtract|deduct from g_k [and put down].	(IP.A.2)

A formulaic representation is easily derived. If k labels the control values (a convention used throughout this study), and B is situated between B_k and B_{k+1}, then

$$g = \begin{cases} g_k + c_k \cdot (B - B_k) & \text{if } g \text{ is increasing} \\ g_k - c_k \cdot (B - B_k) & \text{if } g \text{ is decreasing.} \end{cases} \tag{2.10}$$

For a graphical representation cf. Fig. 2.4. In templates IP.A.1,2 the subtractions $B - B_k$ are not performed explicitly but implied (§2.3.2). They have numerous applications, e.g. for computing a zigzag function (§2.6.9) or other functions of the zodiacal position such as the length of daylight (C in lunar systems A and B). For the zigzag function it would suffice to use the extrema m and M as the only control points, but the procedure texts usually employ 12 control points — one for each zodiacal sign. In the lunar systems, they are anchored at the same number of degrees within the zodiacal signs as the equinoxes in Ari and Lib, i.e. 8°, 10° or 12° for systems B, A and K, respectively.

A more elaborate, probably older formulation of the same algorithm is employed in No. **52** (lunar system K). Here several functions are computed from control values (anchored at 12° of the zodiacal signs), but a general procedure is not offered, all computations being presented as numerical examples. For the increasing branch of g the algorithm can be represented as

$$g = g_k + 0;2 \cdot (g_{k+1} - g_k) \cdot (B - B_k). \tag{2.11}$$

Hence the interpolation proceeds without any explicit reference to coefficients c_k, since these are computed separately for every instance of interpolation as $(g_{k+1} - g_k)/30$.

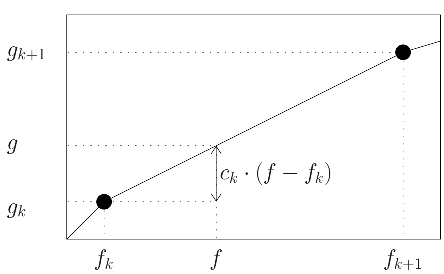

Fig. 2.4: Graphical representation of the linear interpolation algorithm for the case when f and g are both increasing.

Source functions with increasing and decreasing values

If f is a function with increasing as well as decreasing values (e.g. a zigzag function) there are four different interpolation rules:

$$g = \begin{cases} g_k + c_k \cdot (f - f_k) & \text{if } f \text{ increasing,} \quad g \text{ increasing} \\ g_k - c_k \cdot (f - f_k) & \text{if } f \text{ increasing,} \quad g \text{ decreasing} \\ g_k + c_k \cdot (f_k - f) & \text{if } f \text{ decreasing,} \quad g \text{ increasing} \\ g_k - c_k \cdot (f_k - f) & \text{if } f \text{ decreasing,} \quad g \text{ decreasing.} \end{cases} \tag{2.12}$$

A novel feature compared to the previous case is that g depends not only on the magnitude of f, but also on its direction (increasing or decreasing). The main applications are the algorithms for computing G, Λ and W from Φ in lunar system A (§4.4.11). They tend to be formulated in a highly compact form. The most common template is as follows. If f_k ($= \Phi_k$) and g_k ($= G_k$, W_k or Λ_k) are both increasing in magnitude, then

ana tar-ṣi f_k **tab**-*u*$_2$ g_k **gar**-*an* [*mim-ma*] *ša*$_2$ *al-la* f_k **tab**-*u*$_2$ **diri** [**en** f_{k+1} **tab**-*u*$_2$] **a.ra**$_2$ c_k **du**-*ma* **ki** g_k **tab**-*ma* **gar**-*an*	Opposite f_k, increasing, you put down g_k. [Whatever] (the amount) by which it exceeds f_k, increasing, [until f_{k+1}, increasing,] you multiply by c_k, and add with g_k, and put down.	(IP.B.1)

The corresponding template for the case when f_k and g_k are both decreasing is

ana tar-ṣi f_k **la**$_2$-*u*$_2$ g_k **gar**-*an* [*mim-ma*] *ša*$_2$ *al-la* f_k **la**$_2$-*u*$_2$ **la**$_2$-*u*$_2$ [**en** f_{k+1} **la**$_2$-*u*$_2$] **a.ra**$_2$ c_k **du**-*ma* **ta** g_k **nim**|**e**$_{11}$-*ma* **gar**-*an*

Opposite f_k, decreasing, you put down g_k. [Whatever] (the amount) by which it is less than f_k, decreasing, [until f_{k+1}, decreasing,] you multiply by c_k, and deduct from g_k, and put down. (IP.B.2)

Corresponding templates for the case when f_k is increasing and g_k is decreasing, or vice versa, are constructed analogously. Rarely the element ‘until f_{k+1}, increasing/decreasing’ is omitted (No. **72** P1). In one variant of these templates, attested only in No. **72** P1 and No. **97** P6’, the phrase *ša*$_2$ *al-la* ... **diri**|**la**$_2$ is followed by *tanašši*(GIŠ), resulting in ‘you compute (the amount) by which it exceeds|is less than f_k’. The following alternative but equivalent template uses a more elaborate formulation in which the subtractions $f - f_k$ or $f_k - f$ are explicitly mentioned (for applications cf. Table 4.21, p. 146):

ana tar-ṣi f_k **tab**-*u*$_2$ g_k **gar**-*an* [*mim-ma*] *ša*$_2$ *al-la* f_k **tab**-*u*$_2$ **diri en** f_{k+1} **tab**-*u*$_2$ f_k **ta ša**$_3$ **nim a.ra**$_2$ c_k **du**-*ma* **ki** g_k **tab**-*ma* **gar**-*an*

Opposite f_k, increasing, you put down g_k. What[ever] exceeds f_k, increasing, until f_{k+1}, increasing: you deduct f_k from it, you multiply it by c_k, and you add it with g_k, and you put it down. (IP.C.1)

ana tar-ṣi f_k **la**$_2$-*u*$_2$ g_k **gar**-*an* [*mim-ma*] *ša*$_2$ *al-la* f_k **la**$_2$-*u*$_2$ **la**$_2$-*u*$_2$|*i-maṭ-ṭu-u*$_2$ **en** f_{k+1} **la**$_2$-*u*$_2$ **ta** f_k **e**$_{11}$|**nim**-*ma* *ša*$_2$ *re-ḫi*|**tag**$_4$ **a.ra**$_2$|GAM c_k **du**-*ma* **ta** g_k **nim**|**e**$_{11}$-*ma* **gar**-*an*

Opposite f_k, decreasing, you put down g_k. What[ever] is less than f_k, decreasing, until f_{k+1}, decreasing, you deduct from f_k, what remains you multiply by c_k, and you deduct it from g_k, and put it down. (IP.C.2)

If f and g are zigzag-type functions (§2.6.8) that share the same period and are also exactly in phase (e.g. Φ and $F^{(1)}$ in lunar system A) then a simplified version of this template is used, with the extrema (m, M) as the only control points (No. **65** P5):

ana tar-ṣa m_f m_g **gar**-*an* *ša*$_2$ *al* m_f **diri** m_f *ina* **ša**$_3$ M_f **la**$_2$ *ša*$_2$ **tag**$_4$ GAM c **du ki** m_g **tab**-*ma* **gar**-*an*

Opposite m_f you put down m_g. That which exceeds m_f: you subtract m_f from it, what remains you multiply by c, you add it with m_g, and you put it down. (IP.D.1)

ana tar-ṣa M_f M_g **gar**-*an* *ša*$_2$ *al* M_f **la**$_2$ **ta ša**$_3$ M_f **nim** *ša*$_2$ **tag**$_4$ GAM c **du** *ša*$_2$ **tag**$_4$ **ta** M_g **nim**-*ma* **gar**-*an*

Opposite M_f you put down M_g. That which is less than M_f you deduct from M_f, what remains you multiply by c, what remains you deduct from M_g, and you put it down. (IP.D.2)

Finally, the following template is thus far attested only in No. **77** for the case where g is increasing:

ana tar-ṣi f_k g_k **gar**-*an* **ta** f_k **tab**|**la**$_2$ **en** f_{k+1} **tab**|**la**$_2$ **a.ra**$_2$ c_k **du**-*ma* **ki** g_k **tab**

Opposite f_k you put down g_k. From f_k, increasing|decreasing, until f_{k+1}, increasing|decreasing: you multiply by c_k, and add it with g_k. (IP.E)

Goal functions with additive and subtractive values

The next level of complexity of the linear interpolation algorithms involves goal functions g that assume subtractive or additive values depending on the value and direction of f. One example is the algorithm for Λ (lunar system A), which is computed from Φ (§4.4.13). The procedures employ basically the same templates.

2.6.8 Zigzag functions of the event number

Babylonian mathematical astronomy makes abundant use of zigzag functions. There are two kinds of zigzag functions, each computed with a different algorithm. Zigzag functions of the event number are updated with a constant difference from one to the next event of a synodic phenomenon. Zigzag functions of the zodiacal position (§2.6.9) are computed from the zodiacal position of the planet, the Moon or the Sun by means of linear interpolation. Both share the basic feature that the function increases and decreases linearly between a minimum and maximum. A Babylonian term for zigzag function is not attested. In the procedure texts the algorithm is formulated purely arithmetically, without any indication of an underlying graphical concept as suggested by the modern name.

2.6.8.1 Templates and algorithm

Zigzag functions of the event number appear in a wide range of computational systems, but they are a defining feature of type-B systems, where this algorithm is used for the synodic arc (for the planets cf. §3.3.1.1; for the Moon §4.1.2). Some of the procedures provide only the defining parameters of the zigzag function (Table 2.2). They employ the following templates:[186]

M šūqu(**nim**|**la₂**) *m šuplu*(**sig**) *d tašpiltu*(*taš*)	M is the 'height', m is the 'depth', d is the difference.	(ZZ.A.1)
M **la₂** : *m* **sig** : SN *ana* SN *d taš*	M is the 'height', m is the 'depth', (from) SN to SN d is the difference.	(ZZ.A.2)
B_M *M* **la₂** B_m *m* **sig** *d taš* SN *ana* SN	B_M, M: the 'height'; B_m, m: the 'depth'; d: the difference (from) SN to SN.	(ZZ.A.3)
ta *M* **en** *m* **iti** *ana* **iti** *d* **tab** *u* **la₂**	From M until m month by month you add and subtract d	(ZZ.A.4)
M rabītu(**gal**) *m ṣeḫertu*(**tur**) **ab₂** *ana* **ab₂** *d* **tab** *u* **la₂**	M is the largest one, m is the smallest one, month by month you add and subtract d	(ZZ.A.5)

Templates ZZ.A.1–3 are little more than lists of parameters. Some other procedures provide only the numerical values of m, B_m, M, B_M and d without any description.[187] The extrema are referred to either as 'height' and 'depth' or 'large one' and 'small one'. According to ZZ.A.3 the extrema occur at certain zodiacal positions, here represented as B_M and B_m. In reality the extrema of a zigzag function of the event number cluster near a certain position, but they are not fixed; this is how B_m and B_M must here be understood. ZZ.A.4–5, which are attested in lunar procedures, include a rudimentary instruction for updating the zigzag function.

symbol	meaning	Akkadian term	translation
m	minimum	*ṣeḫertu*(**tur**)	the smallest one
		šuplu(**sig**)	'depth'
M	maximum	*rabītu*(**gal**)	the largest one
		šūqu(**nim**, **la₂**)	'height'
d	difference	*tašpiltu*(*taš*) [SN *ana* SN]	difference [(from) SN to SN]
		ṭīpu(**tab**) *u mīṭu*(**la₂**)	addition and subtraction

Table 2.2: Defining parameters of a zigzag function.

The templates ZZ.A.1–5 do not explain when to add d or when to subtract d, and what to do near the extrema. More complete instructions for updating the zigzag function from event to event (or day to day) are provided by other templates, for instance:[188]

d **tab** *u* **la₂**	You add and subtract d.	(ZZ.B.1)
[**en** *M* **gal**] *ša₂ al-la M* **diri**|**gal**-*u₂* **ta** 2*M* **la₂**|**e₁₁**|**nim**[-*ma* **gar**]	[Until M, the large one;] that which exceeds|is larger than M you subtract|deduct from $2M$ [and put down].	
[**en** *m* **tur**] *ša₂ al-la m* **tur**-*er* **ta** 2*m* **la₂**|**e₁₁**|**nim**[-*ma* **gar**]	[Until m, the small one;] that which is smaller than m you subtract|deduct from $2m$ [and put down].	

Before discussing the variants we shall analyse the underlying algorithm. The function value at event i, say f_i, is obtained in two steps. All templates begin with the phrase **tab** *u* **la₂**, which usually represents *ṭeṭeppe u tumaṭṭa*, 'you add and subtract'.[189] Hence a preliminary update is achieved by adding d to, or subtracting it from f_{i-1}, depending on whether f_{i-1} is on the increasing or the decreasing branch:

$$f_i = f_{i-1} \pm d. \tag{2.13}$$

If the result lies outside the range $[m, M]$ a second step is necessary, which involves a reflection rule:

$$f_i = 2M - d - f_{i-1} \tag{2.14}$$

if f_{i-1} is on the increasing branch ('that which exceeds M' is $f_{i-1} + d$), and

$$f_i = 2m + d - f_{i-1} \tag{2.15}$$

if f_{i-1} is on the decreasing branch ('that which is less than m' is $f_{i-1} - d$). The corresponding flow chart is shown in Fig. 2.5; a graphical representation of a zigzag function is shown in Fig. 2.6. Mathematically speaking, f_i is completely determined by m, M, d and one initial value. Another version is attested in the lunar procedures (where i labels successive lunations):[190]

[186] E.g. ZZ.A.1: No. **41** P5,P6 (Saturn systems B,B"); No. **42** P7,P8; ZZ.A.2: No. **33** P1 (Jupiter system B'); ZZ.A.4: No. **83** P5' (Moon system A); ZZ.A.5: No. **52** P7.a (unknown lunar system).

[187] E.g. No. **97** P5' (lunar system B).

[188] No. **55** P1.a, No. **60** P2' (Moon system A); No. **24** P1, No. **36** P1, No. **37** P1 (Jupiter system B).

[189] An exception is No. **58** P2, which has *d* **tab** *u* **la₂**-*ka*; this implies a translation 'your addition and subtraction is d', i.e. **tab** = *ṭīpu*, **la₂** = *mīṭu*.

[190] E.g. No. **53** P1.a.

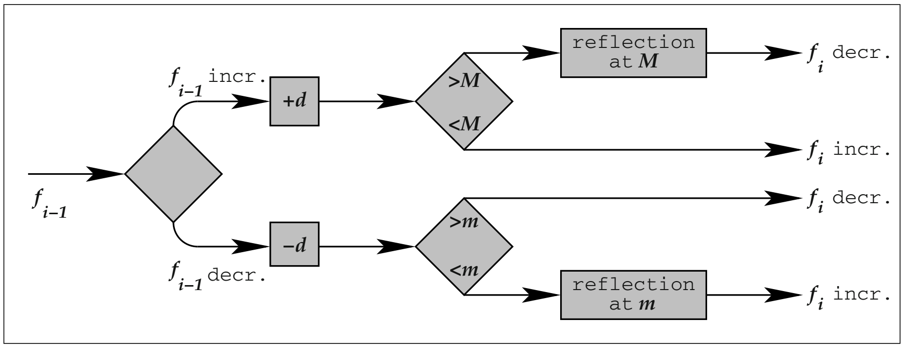

Fig. 2.5: Flow chart of the algorithm for updating a zigzag function of the event number.

d **ab₂** *ana* **ab₂** **tab** *u* **la₂**	Month by month you add and subtract *d*.	(ZZ.B.2)
ša₂ al-la M **diri** *M ina lib₃-bi-šu₂* **e₁₁** *ša₂* **tag₄** *ina M* **e₁₁**	That which exceeds *M*: you deduct *M* from it, what remains you deduct from *M*.	
ša₂ al-la m **la₂**-*u₂ ina m* **e₁₁** *ša₂* **tag₄** **ki** *m* **tab**-*ma* **gar**	That which is less than *m*: you deduct from *m*, what remains you add with *m* and put down.	

In ZZ.B.1 and ZZ.B.2 the difference between f_{i-1} and the extremum is calculated explicitly. In the following templates it is only implied:

d **ab₂** *ana* **ab₂** **tab** *u* **la₂**	Month by month you add and subtract *d*.	(ZZ.B.3)
ša₂ al-la M **diri** **ta** *M* **la₂**	(The amount) by which it exceeds *M* you subtract from *M*.	
ša₂ al-la m **la₂**-*u₂* **ki** *m* **tab**-*ma* **gar**	(The amount) by which is less than *m* you add with *m* and put down.	

This also applies to the following template, which is otherwise similar to ZZ.B.1:[191]

ab₂ *ana* **ab₂** *d* **tab** *u* **la₂**	Month by month you add and subtract *d*.	(ZZ.B.4)
lib₃-bu-u₂ ša₂ M **kur**-*ad₂ ša₂ al M* **diri** **ta** *M* **la₂**	By means of which you reach *M*; (the amount by) which it exceeds *M* you subtract from *M*.	
lib₃-bu-u₂ ša₂ m **kur**-*ad₂ ša₂ al m* **la₂**-*u₂* **ki** *m* **tab**	By means of which you reach *m*; (the amount by) which it is less than *m* you add with *m*.	

Eqs. (2.14–2.15) are retrieved by noting that ‘(the amount by) which it (the preliminary new value $f_{i-1} \pm d$) exceeds M’ is $M-(f_{i-1}+d)$, and ‘(the amount by) which it is less than m’ is $m-(f_{i-1}-d)$. Note that even templates ZZ.B.1–4 do not explicitly mention that the additive or subtractive sense of d is reversed after each reflection.

2.6.8.2 Period relations

Function values computed with the zigzag algorithm satisfy the usual period relation (Eq. 2.9), which assumes the form

$$P = \frac{\Pi}{Z} = \frac{2\Delta}{d}, \tag{2.16}$$

where

$$\Delta = M - m \tag{2.17}$$

is the amplitude of the zigzag function. Since 2Δ is the total change of f over one full oscillation (Fig. 2.6) and d the difference from event to event, their ratio P is the fractional number of synodic events after which f returns to the same value. This relation determines both Π and Z (Eq. 2.9), because they are defined as smallest possible whole numbers. Recall that Π is the whole number of synodic events after which f returns to exactly the same value (§2.6.6), as illustrated in Fig. 2.6, i.e.

$$f_{i+\Pi} = f_i, \tag{2.18}$$

and Z is the corresponding whole number of oscillations (Akk.: **bala** = *palû*, ‘turn’) of the zigzag function.

[191] No. **53** P7’ (lunar system A). For the assumed instrumental sense of *libbû ša*, which is not mentioned in the dictionaries, cf. the Glossary.

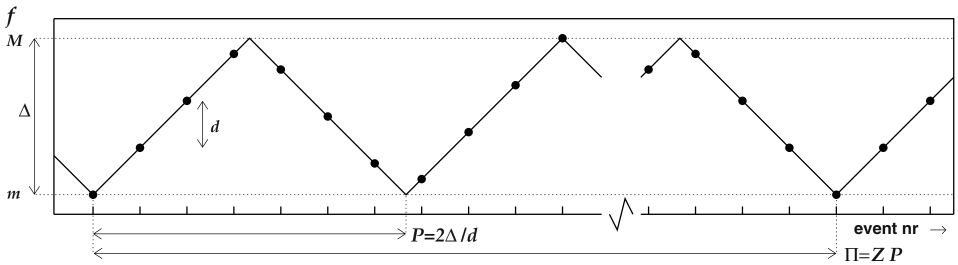

Fig. 2.6: Graphical representation of the zigzag function of the event number.

Underlying zigzag functions

For some zigzag functions it is meaningful to consider the values in the tabular texts as being sampled from a more rapidly oscillating underlying zigzag function.[192] In principle there are infinitely many such underlying zigzag functions with progressively smaller periods, but in practice it is sufficient to consider the next in sequence. This is defined by having one additional oscillation for each event, so that after Π events there have been $Z+\Pi$ oscillations as compared to Z for the tabulated function, resulting in a period

$$p = \frac{\Pi}{Z+\Pi} = \frac{P}{P+1}, \tag{2.19}$$

measured in fractional synodic events (Fig. 2.7). Conversely, the period of the tabulated function can be expressed in terms of the underlying function,

$$P = \frac{p}{1-p}. \tag{2.20}$$

This concept is relevant mainly for functions representing the lunar variation (cf. §4.1.2) such as Φ and F of lunar system

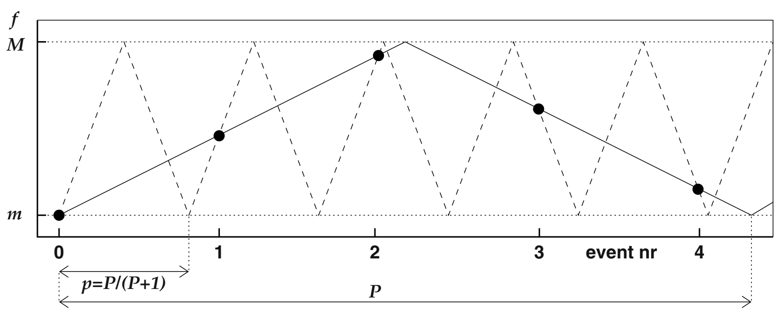

Fig. 2.7: Zigzag function of the event number (solid line), and the underlying more rapidly varying zigzag function (dashed line).

A. Whereas the period of the lunar variation is about 27.56^{d} (the anomalistic month), these functions are evaluated in the synodic tables at successive lunations separated by the synodic month, whose mean value is 29.53^{d}. Hence the tabulation interval slightly exceeds the intrinsic period of the functions, the ratio being $p \approx 0.9333$. As a result, the tabulated functions appear to vary with a much longer period $P \approx 13.99$ expressed in fractional synodic events.

2.6.8.3 Elementary steps

The period relation implies that the zigzag function assumes exactly Π different values, which are equidistantly distributed over the interval 2Δ. Hence there is a smallest possible difference separating two arbitrary values of a zigzag function. This difference, δ, referred to as the 'elementary step' of the zigzag function, equals

$$\delta = \frac{2\Delta}{\Pi} = \frac{d}{Z}. \tag{2.21}$$

The elementary step can be used for assessing whether or not two given numbers are connectible by the same zigzag function. Furthermore, the net difference of a zigzag function for intervals longer than 1 synodic cycle always corresponds to a whole number of elementary steps (cf. below). However, it is not clear to what extent, if at all, δ is a Babylonian concept.

[192] Neugebauer (1938c), pp. 215–229; *ACT*, p. 31. In modern data analysis this is known as the aliasing phenomenon, which occurs when a signal is sampled at an interval longer than the period of the signal.

2.6.8.4 Net differences for intervals longer than 1 synodic cycle

Zigzag functions of the event number are primarily used for computing quantities from one to the next occurrence of a synodic phenomenon. However, it is also possible to update a zigzag function over s synodic cycles in a single operation by adding the appropriate net difference, $df(s) = f_{i+s} - f_i$. Of particular significance are net differences for close returns to the same value, which reflect approximate period relations for the phenomenon under consideration. They are a recurrent subject of the procedure texts because they can be used for verifying whether a column with successive values of f was computed correctly. These procedures usually list numerical values of $df(s)$ for selected s, without any explanation of their origin. Values of $df(s)$ can be computed as follows. If we momentarily ignore the reflections that occur at the extrema of f, then s synodic cycles cause f to change by the amount sd. From this we obtain the net difference $df(s)$ by subtracting

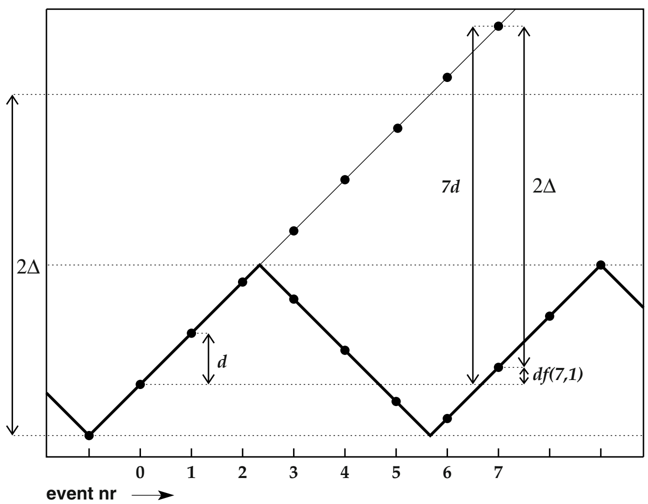

Fig. 2.8: Graphical representation of the computation of the net difference, $df(s,t)$, for a zigzag function of the event number. The example concerns $df(7,1) = 7d - 2\Delta$.

2Δ as many times, say t, as needed for obtaining a value of $sd - 2t\Delta$ within the range $[-\Delta, \Delta]$, i.e.

$$df(s,t) = sd - 2t\Delta. \tag{2.22}$$

Fig. 2.8 shows hows this works for $df(7,1)$. Note that for close returns the value of t is always uniquely defined by s. This allows us to omit the argument t from the formulaic representation, leading to $df(s)$. Further insight into the properties of $df(s)$ is obtained by expressing it in terms of elementary steps, which is achieved by inserting Eq. (2.21):

$$df(s) = (sZ - t\Pi) \cdot \delta. \tag{2.23}$$

Hence $df(s)$ always contains a whole number, $sZ - t\Pi$, of elementary steps.

2.6.8.5 Checking rule for function values on opposite branches

Some procedure texts preserve rules connecting one value of a zigzag function, say f_0, located on one branch (increasing or decreasing), to the value f_s separated from it by s events and located on an opposite branch. The latter implies that $f_0 + f_s$ is constant, i.e. independent of f_0. This can be readily seen from Fig. 2.6: if one updates $f_0 + f_s$ to $f_1 + f_{s+1}$, and assumes that both instances of f remain on the same branch, then one instance of f increases by d while the other decreases by d. The constant can be computed by updating f_0 with the net difference $df(s)$. Assuming that an odd number of extrema are passed between f_0 and f_s, so that f_s is on an opposite branch as required, and applying the appropriate reflection rule (Eqs. 2.14–2.15) to $f_0 + df(s)$, it follows that

$$f_0 + f_s = \begin{cases} 2M - df(s) & (f_0 \text{ increasing}) \\ 2m + df(s) & (f_0 \text{ decreasing}). \end{cases} \tag{2.24}$$

Hence $f_0 + f_s$ is a constant whose value depends only on whether f_0 is increasing or decreasing.

2.6.8.6 Empirical aspects and the construction of a zigzag function

Zigzag functions are obviously a coarse tool in terms of their ability to follow the temporal evolution of periodically varying empirical quantities, which tend to behave in a smooth sinusoidal fashion. It may therefore seem odd that m and

M are often defined with many sexagesimal digits. As pointed out by Neugebauer and others, this does not reflect accuracy with regard to empirical data. While the mean value of the zigzag function, $\mu = (m+M)/2$, is usually close to the mean value of the empirical quantity, the extrema tend to deviate considerably from their empirical values. It is therefore clear that the purpose of a zigzag function is not to exactly, or even closely, reproduce the extrema, or any other specific value of the empirical quantity apart from the mean value. The more important property of zigzag functions is that they satisfy an accurate period relation (P), and enable one to compute a function for arbitrary future (or past) occurrences of a synodic phenomenon without introducing round-off errors. As a result of these two features the deviations between the zigzag function and the empirical quantity vary according to a fixed pattern, and they do not accumulate on the time scale on which P is accurate. The accuracy of P depends on that of Δ/d (Eq. 2.16), and not on the number of digits to which m and M are individually defined. In practice at most the first few digits of m, M and d are empirically based, whereas the higher digits are chosen in such a way that f satisfies the given period relation. The procedure texts offer little insight about the derivation of algorithms from empirical data. However, No. **13** P11'.b presents a derivation of the defining parameters of the zigzag function for the synodic arc for Mars system B (cf. §3.6.3).

2.6.9 Zigzag functions of the zodiacal position

Zigzag functions of the zodiacal position are applied in many computational systems, both lunar and planetary (e.g. Fig. 3.15, p. 87, and Fig. 4.33, p. 168). Formally this algorithm is fully defined by the minimum m, the maximum M and the zodiacal position of m or M. The value of the zigzag function, say f, is computed through extrapolation with respect to control points B_k and f_k (§2.6.7). A common template is as follows:[193]

ša₂ al-la B_k lu-maš **diri a.ra₂**|GAM₀ *c* **du**	(The amount) by which (the position) exceeds B_k (°) of the zodiacal sign you multiply by c.	(ZZ.C.1)
ki f_k *ki-i* **tab tab** *ki-i* **la₂ la₂**	You add it with f_k if it is increasing, subtract if it is decreasing.	

A formulaic representation is

$$f = f_k \pm (B - B_k) \cdot c, \tag{2.25}$$

where c, the interpolation coefficient, equals

$$c = \frac{M - m}{3{,}0}. \tag{2.26}$$

In ZZ.C.1 the subtraction $B - B_k$ is not mentioned explicitly (§2.3.2). Usually ZZ.C.1 involves 12 control points, one for each zodiacal sign.[194] In the following equivalent templates the zigzag function is defined with only two control points, the extrema of the zigzag function:[195]

ina B_M M [*ina B_m m* **ta** *B_M* **en** *B_m*] *ana* 1 UŠ *c* **la₂** **ta** B_m **en** B_M *ana* 1 UŠ *c* **tab**	In B_M M, in B_m m. From B_M to B_m for 1° you subtract c, from B_m to B_M for 1° you add c.	(ZZ.C.2)
mim-ma ša₂ al-la B_M **diri a.ra₂** *c* **du ta** *M* QN **e₁₁** **ta** B_M **en** B_m **ta** QN **la₂ ta** B_m **en** B_M **ki** QN **tab**	(The amount) by which it (B) exceeds B_M you multiply by c, deduct from M, QN. From B_M until B_m you subtract from QN, from B_m until B_M you add with QN.	(ZZ.C.3)

2.6.10 Step functions for the synodic arc

The step function is a fundamental algorithm unique to type-A systems. It is mainly used for modeling the synodic arc σ, which is the net displacement of the planet, the Moon or the Sun along the zodiac between successive events of a synodic phenomenon, i.e.

$$B_i = B_{i-1} + \sigma(B_{i-1}). \tag{2.27}$$

In lunar system A, step functions are also used for modeling several other differences, including the monthly difference of the Moon's distance to the ecliptic (E). For these 'generalised step functions' cf. §4.4.6.

[193] Numerous examples can be found in No. **61** P1–P4 (steps 5 and 7).

[194] In the lunar systems these are always located at the same position within the zodiacal signs as the equinoxes in Ari and Lib, i.e. at 8° (system B), 10° (system A) or 12° (system K).

[195] ZZ.C.2: No. **96** P7'; ZZ.C.3: No. **53** P17' (step 5).

2.6.10.1 Updating the zodiacal position with a step function for the synodic arc

In the procedure texts, the computation of σ is not formulated as a separate algorithm, but as an integral part of the algorithm for updating the zodiacal position (B). The most common template is as follows:[196]

ta b_j **en** b_{j+1} σ_j **tab** *ša*₂ *al-la* b_{j+1} **diri** **a.ra**₂|GAM r_j **du** [**ki** b_{j+1} **tab**-*ma* **gar**-*an*]	From b_j to b_{j+1} you add σ_j. (The amount by) which it exceeds b_{j+1} you multiply by r_j, [add to b_{j+1} and put down.]	(STEP.A.1)

The modern name step function reflects that σ assumes different constant values, σ_j, in different parts of the zodiac ('zones'), resulting in a steplike appearance. Attested values of the number of zones, n, range from 1 (Venus system A_0) to 6 (Mars system A). The step function is fully defined by the boundaries of the zones, b_j (where b_j is the left or 'trailing' boundary of zone j), the preliminary values of the synodic arc, σ_j,[197] and the transition coefficients, r_j. The unnamed quantity to which σ_j is added is the previous zodiacal position B_{i-1}. If the provisional new position $B_{i-1}+\sigma_j$ is in the next zone ($j+1$), which occurs if B_{i-1} is within a distance σ_j of b_{j+1}, the 'transition rule' is invoked. This means that the distance by which b_{j+1} is transgressed is multiplied by r_j and added to the distance traveled up to b_{j+1} (cf. the flow chart in Fig. 2.9). The transition rule causes the final value of the synodic arc, σ, to differ from the preliminary value σ_j in some portions of the zodiac (Fig. 2.10 and Table 2.3). This effect is particularly strong if σ_j is not small compared to α_j (cf. below).

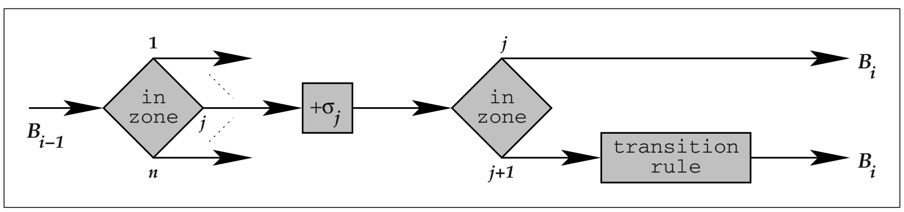

Fig. 2.9: Flow chart of the algorithm for updating B with a step function for the synodic arc.

j	1...n	index of zone
b_j	[°]	zodiacal position of 'trailing' boundary of zone j
α_j	[°]	length of zone j
σ_j	[°]	preliminary synodic arc in zone j

Table 2.3: Parameters of a step function for the synodic arc.

Formulaic representation

We now have sufficient information to derive a formulaic representation of σ (Eq. 2.27). The procedure begins with a preliminary update $B_{i-1}+\sigma_j$. If the distance from the old position to b_{j+1}, say

$$x = b_{j+1} - B_{i-1}, \tag{2.28}$$

is larger than σ_j then the new position is still in zone j, in which case $\sigma=\sigma_j$. If however $x<\sigma_j$ then the preliminary new position is in the next zone, in which case a transition rule is applied, i.e.

$$\sigma = \begin{cases} \sigma_j & \text{if } x \geq \sigma_j, \\ x + y\cdot r_j = \sigma_j + y\cdot c_j & \text{if } x < \sigma_j, \end{cases} \tag{2.29}$$

where

$$r_j = \frac{\sigma_{j+1}}{\sigma_j}, \tag{2.30}$$

$$c_j = \frac{\sigma_{j+1}-\sigma_j}{\sigma_j} = r_j - 1 \tag{2.31}$$

are transition coefficients, and y is the distance by which the *preliminary new* position exceeds b_{j+1}, i.e.

$$y = B_{i-1} + \sigma_j - b_{j+1} = \sigma_j - x. \tag{2.32}$$

[196] E.g. No. **28** P1, No. **35** P1.a, No. **53** P5', No. **55** P2, No. **62** P3'. In No. **1** P1.a,c and No. **32** P1 'you add to b_{j+1} and put down' is omitted. Some badly preserved candidates: No. **2** P2', No. **4** P1', No. **17** P1, No. **20** P1'.b.

[197] This notation differs from that of Neugebauer, who uses w_j instead of σ_j, and $\Delta\lambda$ instead of σ. I reserve w_j for the preliminary values of a 'generalised' step function, i.e. one that does not represent a synodic arc.

The representation based on r_j underlies template STEP.A.1; the one based on c_j is attested in several other templates to be discussed below. The numerical values of r_j and c_j provided in the procedure texts or derived from the tabular texts always satisfy Eqs. (2.30–2.31). Mathematically speaking σ is therefore completely defined by σ_j and b_j. The reason why r_j and c_j are defined in this way becomes clear through a further analysis of the step function algorithm (§2.6.10.3). One effect of the transition rule is that σ deviates from the preliminary values σ_j in a region of width σ_j at the end of zone j, where σ is linearly interpolated between σ_j and σ_{j+1} (Fig. 2.10).

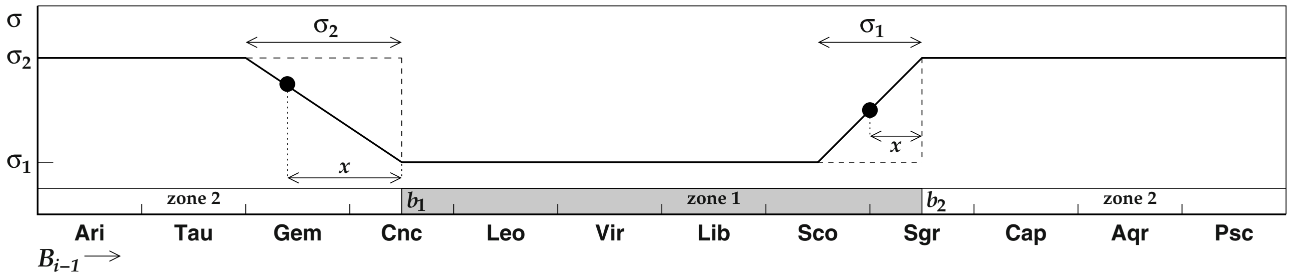

Fig. 2.10: Graphical representation of the step function for σ against the 'old' position B_{i-1}. In this example there are two zones. In the transition regions next to the zonal boundaries σ differs from the preliminary values σ_j (dashed line). Also shown is the distance x between B_{i-1} and the next zonal boundary for two exemplary positions in the transition regions.

Alternative templates

We now consider some alternative but equivalent templates. Sometimes the excess y is computed explicitly (No. **56** P2):

ta b_j **en** b_{j+1} σ_j **tab** *ša₂ al-la* b_{j+1} **diri** b_{j+1} **ta** **ša₃ nim a.ra₂** r_j **du** [**ki** b_{j+1} **tab**-*ma* **gar**-*an*]	From b_j to b_{j+1} you add σ_j. That which exceeds b_{j+1}: you deduct b_{j+1} from it, you multiply by r_j, [add it to b_{j+1} and put down.]	(STEP.A.2)

In some computational systems (e.g. Mars system A) each zone comprises a whole number of zodiacal signs. The phrase 'From b_j to b_{j+1}' is then usually modified as follows:[198]:

[*ina*] SIGN₁ SIGN₂ σ_j [**tab**] *ša₂ al* 30 SIGN₂ **diri a.ra₂** r_j **du**	[In] SIGN₁, SIGN₂ [you add] σ_j. (The amount) by which it exceeds 30 SIGN₂ you multiply by r_j.	(STEP.A.3)

A more significant group of variants employs the alternative transition coefficients c_j, e.g. (No. **42** P1.a,P2.a):

ta b_j **en** b_{j+1} σ_j **tab** b_{j+1} **dib**-*iq* GAM $\lvert c_j\rvert$ **du**-*ma* **tab**|**la₂**	From b_j to b_{j+1} you add σ_j. It passes b_{j+1}, you multiply it (= y) by $\lvert c_j\rvert$ and add|subtract it.	(STEP.B)

Here $y \cdot |c_j|$ is added or subtracted in accordance with Eq. (2.29). The use of c_j instead of r_j necessitates the consideration of 2 cases, because $y \cdot c_j$ can be additive or subtractive, whereas $y \cdot r_j$ is always additive. Another formulation (No. **31** P2'.b) involving c_j is as follows:

tur-*tu₂* σ_j **tab** b_{j+1} **dib** $\lvert c_j\rvert$ **tab murub₄**-*tu₂ ip-pal-ka*	The small one: you add σ_j. It passes b_{j+1}, you add $\lvert c_j\rvert$, you get the middle one.	(STEP.C.1)
gal-*tu₂* σ_j **tab** b_{j+1} **dib** $\lvert c_j\rvert$ **la₂ murub₄**-*tu₂ ip-pal-ka*	The large one: you add σ_j. It passes b_{j+1}, you subtract $\lvert c_j\rvert$, you get the middle one.	(STEP.C.2)

The phrase 'It passes b_{j+1}, you add|subtract $|c_j|$...', makes sense if one interprets 'you add|subtract $|c_j|$' as the same instruction to add or subtract $y \cdot |c_j|$ as in STEP.B, i.e. '(for every degree by which) it passes b_{j+1} you add|subtract $|c_j|$'. The terms 'small/middle/large one' make sense if interpreted as references to the zones (j) rather than to the synodic arcs (σ_j) themselves.[199] In the following variant of STEP.C σ_j is omitted, leaving only the transition rules:

[198] No. **15** P1, No. **43** P6'.

[199] Other instances where that meaning is obvious are discussed following STEP.F.

zi *gab-ba ša*$_2$ **tur**-*ti* **a.ra**$_2$ c_j **du**-*ma* **ki**-*šu*$_2$ **tab**-*ma* **murub**$_4$-*ti ip-pal-ka*	All displacements for the small one you multiply by c_j, and you add it with it, and you get the middle one.	(STEP.D.1)
zi *gab-ba ša*$_2$ **gal**-*ti* **a.ra**$_2$ c_j **du**-*ma* **ta ša**$_3$ **la**$_2$-*ma* **murub**$_4$-*ti ip-pal-ka*	All displacements for the large one you multiply by c_j, and you subtract from it, and you get the middle one.	(STEP.D.2)

In other procedures the transition rules are omitted,[200] and one is left with

ta b_j **en** b_{j+1} σ_j **tab**	From b_j to b_{j+1} you add σ_j.	(STEP.E)

There are indeed a few computational systems in which *B* is updated without transition rules (the ‘pseudo’ step functions discussed below), but the synodic tables imply that in all other type-A systems this involves transition rules. Finally, some procedures provide only the zonal boundaries b_j:[201]

ta b_j **en** b_{j+1} **tur**-*tu*$_2$|**murub**$_4$-*tu*$_2$|**gal**-*tu*$_2$	From b_j to b_{j+1}: the small|middle|large one.	(STEP.F)

The ‘small|middle|large one’ are references to the σ_j, but sometimes (e.g. in STEP.C) they are actually used as labels for the zones (j).[202] A general word for ‘zodiacal zone’ is not attested in the procedures for updating *B*, but a few previously unexplained procedures for Jupiter (§3.7.2.5) employ the word ‘ward; neighbourhood’ (*bābtu*), such that ‘the ‘ward’ of σ_j’ (*bābtu ša*$_2$ σ_j) denotes zone j.

Multiple transitions of the zonal boundaries during 1 synodic cycle

Until now it has been tacitly assumed that $\sigma_j < \alpha_j$, so that at most one zonal boundary is crossed between events $i-1$ and i. For Mercury and Mars this condition is not always satisfied, and two zonal boundaries may be crossed at once. The formulation of the algorithm for σ in the procedure texts is such that no special treatment is necessary in order to cope with that situation. All that is required is repeated application of the transition rules until the final zone is reached. The effects of these multiple transitions on σ are explored in Appendix C.

Constructing a step function from empirical data

The construction of a step function is obviously constrained by empirical data (the period *P* and the synodic arc’s mean value and variation along the zodiac). However, the freedom of choice with regard to the values of σ_j, the number of zones and the location of the boundaries (b_j) is also constrained by the condition that all r_j should be conveniently simple finite sexagesimal numbers. This greatly limits the number of possible implementations of a step function.[203]

‘Pseudo’ step functions

In some planetary systems (Venus system A_1/A_2) the algorithm for updating *B* is similar to the step function, but it lacks transition rules (as proven by the synodic tables). For Venus system A_1/A_2 that algorithm is also used for updating T. Moreover, in these systems the zones always comprise a whole number of zodiacal signs. The corresponding template for updating *B* or T is as follows:[204]

SIGN$_1$ SIGN$_2$ [SIGN$_3$] *x* **tab**	SIGN$_1$ SIGN$_2$ [SIGN$_3$]: you add *x*.	(PSTEP)

Here *x* is either the synodic arc σ_j or the synodic time τ_j (cf. §3.3.2.1).

[200] No. **18** P7,P8; No. **19** P1; No. **21** P2’,P3’.a; No. **44** P1.a. This appears to be common when the instruction for updating *B* occurs within a procedure for the subdivision of the synodic cycle (No. **18** P2.a, No. **23** P2’.a and No. **25** P2’a) or the synodic time (No. **18** P14–P16).

[201] E.g. No. **18** P9,P10.a,P17,P25; No. **31** P2’; No. **32** P2; No. **41** P1; No. **42** P3; No. **46** P3.

[202] This is also obvious in the phrase *ina* **tur**-*tu*$_2$ **ki** *šamaš*$_2$ *ša*$_2$ **me** 5 **zi**-*šu*$_2$, ‘In the ‘small one’ the displacement of the Sun’s position is 5 per day’ (No. **42** O20).

[203] Different strategies by which one may construct step functions for the synodic arc from empirical data have been proposed by Aaboe (1968) and Swerdlow (1998).

[204] E.g. No. **7** P1–P12.

2.6.10.2 Period relation and mean synodic arc

The algorithm for updating B with a step function for σ results in zodiacal positions B satisfying the period relation

$$P = \frac{\Pi}{Z} = \sum_j \frac{\alpha_j}{\sigma_j}. \tag{2.33}$$

Recall that Π is the smallest number of synodic events corresponding to a whole number Z of revolutions around the zodiac, and P is the fractional number of synodic events after which the synodic phenomenon performs exactly one revolution. For a given step function, defined by values of α_j and σ_j for every zone, this relation simultaneously determines Π and Z — since α_j and σ_j are finite-digit sexagesimal numbers, there is always such a pair. The actual proof of Eq. (2.33) is postponed to §2.6.10.3. It can be understood intuitively by noting that α_j/σ_j may be viewed as the 'time', expressed in fractional synodic events, that it takes a synodic phenomenon, moving a distance σ_j per synodic cycle, to cross zone j. Hence the sum of these ratios is the 'time' needed to complete one revolution around the zodiac, expressed in fractional synodic events. P need not be a finite sexagesimal number and is rarely mentioned in procedure texts. From the period relation one can derive the mean synodic arc as

$$\overline{\sigma} = \frac{6{,}0}{P} = \frac{6{,}0 \cdot Z}{\Pi}. \tag{2.34}$$

For this computation to be possible in the Babylonian sense it is usually necessary to first round off P or approximate $1/\Pi$.

2.6.10.3 Elementary steps

Our understanding of the step function for the synodic arc was greatly enhanced after van der Waerden (1957) introduced the concept of the (elementary) step, and Aaboe (1964a) explored in more detail its consequences.[205] As discovered by van der Waerden, the existence of a number period, Π, after which there is a first exact return of the synodic phenomenon, implies that there are exactly Π different positions on the zodiac that can be assumed by the planet. Therefore the zodiac can be construed as being divided into Π elementary intervals of variable length

$$\delta_j = \frac{\sigma_j}{Z}, \tag{2.35}$$

so that σ_j always contains exactly Z elementary steps. Since this applies to all zones, σ also contains Z elementary steps if a zonal boundary is crossed between B_{i-1} and B_i. By introducing elementary steps the algorithm for updating B with the step function for σ is thus summarised in a single rule, namely that the planet always proceeds by Z steps from one to the next occurrence of a synodic phenomenon. In particular, the transition coefficients

$$r_j = \frac{\sigma_{j+1}}{\sigma_j} = \frac{\delta_{j+1}}{\delta_j} \tag{2.36}$$

turn out to be a simple consequence of that single rule, because they are the ratios of the elementary steps in adjacent zones. If the elementary step is the fundamental concept underlying the step function, one may also expect α_j to contain a whole number of them. Indeed,

$$\nu_j = \frac{\alpha_j}{\delta_j} \tag{2.37}$$

is usually a whole number, with few exceptions (e.g. Jupiter system A"). By definition, the total number of steps contained in the zodiac is

$$\sum_j \nu_j = \Pi. \tag{2.38}$$

Analogous to δ for the zigzag functions, δ_j represents the smallest possible difference between two zodiacal positions in zone j. Consequently, two positions are connectible by the same step function σ if and only if they are separated by a whole number of elementary steps. As pointed out by van der Waerden (1965), p. 187,[206] the clearest evidence that the elementary step is not merely an implicit concept reconstructed from the step function algorithm but an actual Babylonian concept is found in No. **13**, a procedure text for Mars system A. Apart from that extraordinary text the elementary steps remain an implicit and largely elusive concept, and the full extent to which the Babylonian astronomers made use of them remains to be established.

[205] I use the term 'elementary step' instead of 'step' in order to distinguish it more clearly from 'step function'. This ambiguity does not exist in the original German terminology (Schritt = elementary step; Stufenfunktion = step function).

[206] English edition: Van der Waerden (1974), p. 265.

2.6.10.4 Net displacements for intervals longer than 1 synodic cycle

By repeatedly updating B with σ we obtain positions for successive events of a synodic phenomenon. Babylonian astronomers also considered net displacements for intervals longer than 1 synodic cycle. The net displacement for s events will be denoted as $dB_j(s,t)$, where t is the closest whole number of revolutions of the synodic phenomenon, and j is the zone. By definition $\sigma_j = dB_j(1,0)$. Of particular interest are values of s for which the planet or the Moon returns to nearly the same position, as illustrated in Fig. 2.11. Numerous procedures, both planetary and lunar, mention net displacements for such close returns, because they can be used for verifying the correctness of column B. Note that for close returns the argument t in $dB_j(s,t)$ can be omitted, because t is uniquely defined by s, resulting in $dB_j(s)$. The main templates are as follows:[207]

[**ta** b_j **en** b_{j+1}] [*ina*|*ana*|**a.ra**$_2$ y **mu.meš**] dB_j [**ki**-*šu*$_2$] **tab**	[From b_j until b_{j+1}] [in|for y years] you add dB_j [with it].	(DIFF.A.1)
[**ta** b_j **en** b_{j+1}] [*ina*|*ana*|**a.ra**$_2$ y **mu.meš**] dB_j **la**$_2$	[From b_j until b_{j+1}] [in|for y years] you diminish it by dB_j.	(DIFF.A.2)
ta b_j **en** b_{j+1} *ana* s **ab**$_2$ dB_j **mu.du**|**tab**|**la**$_2$	From b_j until b_{j+1} for s months {it proceeds|you add|you subtract} dB_j.	(DIFF.A.3)
ša$_2$ {s **ab**$_2$ | y **mu**} **ta** b_j **en** b_{j+1} dB_j **tab**|**la**$_2$	For {s months|y years} from b_j until b_{j+1} you add|subtract dB_j.	(DIFF.A.4)
ina y **mu.meš** dB_j [**ki**-*šu*$_2$] *ana* **nim** *illak*(**du**)	In y years [its position] proceeds dB_j to the east.	(DIFF.B.1)
ina y **mu.meš** dB_j [**ki**-*šu*$_2$ *ana* **šu**$_2$] *inaḫḫis*(**la**$_2$)	In y years [its position] recedes dB_j [to the west].	(DIFF.B.2)

The use of **a.ra**$_2$ in DIFF.A.1–2, which usually means 'times' but must represent *ana* = 'to' in this context, is quite unexpected.[208] Templates DIFF.A.3–4 are unique to lunar system A. The following template is attested in No. **5** P4,P5:

ana na-bal-kat$_2$*-tu*$_4$ *ša*$_2$ y **mu.an.na.meš** SN **ta** b_j **en** b_{j+1} dB_j **tab**|**la**$_2$	For the transgression of y years of SN, from b_j until b_{j+1} you add|subtract dB_j	(DIFF.C)

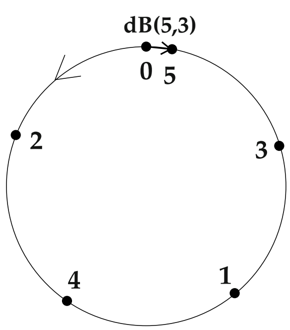

Fig. 2.11: Net displacement of a planet for a close return to the same zodiacal position. The normal, forward motion along the zodiac corresponds to the anticlockwise direction (large arrow). Dots denote successive events of the same synodic phenomenon. After $s = 5$ synodic cycles the planet completes $t = 3$ revolutions except for a small net displacement $dB(5,3)$, which is subtractive in this example.

Nearest whole number of years y corresponding to a close return

In the procedure texts $dB(s)$ is usually identified by the nearest whole number of years (y) corresponding to s synodic cycles, and not by s itself. The relation between y and s differs from planet to planet and can be established from elementary considerations. Since each occurrence of the same synodic phenomenon corresponds to approximately the same configuration of the planet with the Sun, a close return of the planet implies a close return of the Sun. The value of y is therefore obtained by counting the total number of revolutions performed by the Sun. For every revolution of the synodic phenomenon, of which there are t, the Sun also performs 1 revolution. On top of that the Sun performs m additional revolutions for each synodic cycle, where m = 0, 1 or 2 (Table 3.4). Hence

$$y = t + ms. \tag{2.39}$$

By the same token, the number of years corresponding to an *exact* return, which occurs after Z revolutions of the phenomenon, equivalent to Π synodic cycles, is $Y = Z + m\Pi$ (cf. also §3.3.1.4).

Methods for computing net displacements

How the Babylonian astronomers computed the net displacements is not revealed by the procedure texts. An easy, heuristic method would be to identify in a synodic table suitable positions B_i and B_{i+s}, separated by s lines (synodic events) and

[207] Examples: DIFF.A.1,2: No. **18** P1.b',P20; No. **23** P1'.b'; No. **25** P1'.b; No. **26** P1'.c; No. **44** P3'; No. **46** P10'. DIFF.A.3: No. **55** P2.b; DIFF.A.4: No. **65** P7.b; DIFF.B.1: No. **18** P28. DIFF.B.2: No. **44** P3'.

[208] Attested only in No. **18** P20.

both located in the same zone j, and compute the net displacement for each zone j directly as $dB_j(s) = B_{i+s} - B_i$. A somewhat more efficient approach would be to compute B for s consecutive events setting out from an initial position B_0 in zone j sufficiently far away from the zonal boundaries, so that B_s is also in zone j, compute $dB_j(s) = B_s - B_0$, and repeat this for every zone.[209] Finally, as will be shown in the next paragraph, net displacements may also be computed directly from the elementary steps δ_j.

Net displacements in terms of elementary steps

An alternative method for computing net displacements exploits that they always contain a whole number of elementary steps, analogous to the synodic arc. In order to prove this suppose that B_i and B_{i+s} are both in zone 1 and that the closest whole number of revolutions, t, is known.[210] The total distance covered by the planet is $s \cdot \sigma_j$, which contains sZ elementary steps (§2.6.10.3). Exactly $sZ - t\,(\Pi - \nu_1)$ of these steps are in zone 1, which corresponds to a distance $[sZ - t\,(\Pi - \nu_1)] \cdot \delta_1$. Since the distance traveled by the planet after passing t times through zones 2,...,n equals $t \cdot (6{,}0 - \alpha_1)$, the total distance is $t \cdot (6{,}0 - \alpha_1) + [sZ - t\,(\Pi - \nu_1)] \cdot \delta_1$. Inserting $\alpha_1 = \nu_1 \delta_1$ this reduces to $t \cdot 6{,}0 + (sZ - t\Pi) \cdot \delta_1$. Hence the planet's net displacement, defined as the excess over t full revolutions, is $dB_1(s) = (sZ - t\Pi) \cdot \delta_1$. Generalised to an arbitrary zone j this yields

$$dB_j(s,t) = (sZ - t\Pi) \cdot \delta_j. \tag{2.40}$$

It follows that $dB_j(s,t)$ contains $sZ - t\Pi$ elementary steps, regardless of the zone.[211] As expected $dB_j(1,0) = \sigma_j$ and $dB_j(\Pi, Z) = 0$. Although it was assumed that $dB_j(s,t)$ lies entirely within one zone, the number of elementary steps remains the same if the initial and final positions are in different zones. Again the formalism of elementary steps leads to a concise and convenient description of the planet's motion.

Furthermore, since σ_j and $dB_j(s,t)$ each contain a constant number of elementary steps, the ratio between σ_j and $dB_j(s,t)$ is also constant. This can be seen by combining Eqs. (2.40) and (2.35):

$$\frac{dB_j(s,t)}{\sigma_j} = \frac{sZ - t\Pi}{Z} = s - tP. \tag{2.41}$$

This property can be used for verifying whether given values of $dB_j(s,t)$ are consistent with given values of σ_j, or for reconstructing σ_j from given values $dB_j(s,t)$.[212]

Updating B with a step function for the net displacement

While the usual purpose of the net displacements appears to be verification (§2.6.5.2), they can in principle also be used for updating the zodiacal position of the planet in the same manner as is done with σ. To that end the net displacements $dB_j(s)$ computed from the synodic arc are considered as the preliminary values, say σ'_j, of a step function σ'.[213] Precisely that strategy is followed in system A₃ of Mercury, where $\sigma'_j = dB_j(3)$ (§3.4.4). In analogy to σ, the period of σ' is characterised by parameters P', Π', Z'. Their relation to P, Π and Z can be derived by reformulating Eq. (2.41) as

$$\sigma'_j = (s - tP)\,\sigma_j. \tag{2.42}$$

The period of the step function for σ' is therefore

$$P' = \sum_j \frac{\alpha_j}{\sigma'_j} = \frac{1}{s - tP} \sum_j \frac{\alpha_j}{\sigma_j} = \frac{P}{s - tP} = \frac{\Pi}{sZ - t\Pi} = \frac{\Pi'}{Z'}. \tag{2.43}$$

In this derivation it is assumed that the same values of α_j apply to σ and σ'. The original period P is reduced by a factor $s - tP$, as expected. The correct values of Π' and Z' are the smallest whole numbers Π' and Z' for which $\Pi'/Z' = P'$. In the penultimate expression P' is represented as a ratio of two whole numbers Π and $sZ - t\Pi$, but it would be premature to identify Π' with Π, and Z' with $sZ - t\Pi$, because they may still share a common factor. Conversely one may use the inverse of this expression for deriving the period P if one happens to know the step function for σ' but not that for σ:

$$P = \frac{sP'}{1 + tP'}. \tag{2.44}$$

[209] Or do this for only one zone, say j, and compute the values for the other zones as $dB_{j+1}(s) = r_j \cdot dB_j(s)$.
[210] It can be found from the requirement $s \cdot \overline{\sigma} \approx t \cdot 6{,}0°$ or, equivalently, $s/t \approx 6{,}0/\overline{\sigma} = P = \Pi/Z$.
[211] Cf. also *HAMA*, p. 433.
[212] Although $dB_j(s,t)$ and σ_j are finite sexagesimal numbers, this need not be true for their ratio $s - tP$.
[213] Cf. also *HAMA*, p. 468.

2.7 Computational systems

In this study a computational system is pragmatically defined as a distinct collection of algorithms for computing (and verifying) the columns of a synodic table or a daily motion table, or auxiliary functions for such tables. I reserve the term ‘scheme’ for algorithms concerned with interpolation or the subdivision of the synodic cycle of a planet. Most modern names of the computational systems were coined by Neugebauer, who modified and systematised the older notation handed down from Kugler, introducing new names for previously unknown systems. The resulting nomenclature is unsatisfying but widely used, so that any attempt to improve it would only add to the confusion. One problematic feature is that it does not allow for a distinction between individual computational systems and families of computational systems. In order to alleviate this problem, families of systems will be referred to as type A, type B, etc. Within these families, individual systems are named lunar system A, Mercury system A_2, Jupiter system A’, etc. The main distinction between type-A and type-B systems concerns the algorithm for the synodic arc (σ), which is a step function of the zodiacal position (A) or a zigzag function of the event number (B); cf. §3.3.1.1. In the planetary systems (Ch. 3), most of which offer only algorithms for computing zodiacal positions and times, this is the only significant difference between type-A and B systems. Lunar systems A and B (Ch. 4) are far more complex than their planetary counterparts, many lunar quantities having no parallel in the planetary systems. The differences between lunar systems A and B are also more profound and numerous than for the planetary systems of type A and B. Note that the coexistence of different computational systems for the same planet or the Moon is an interesting aspect of Babylonian mathematical astronomy whose significance is not really understood.

The procedure texts usually respect some sort of separation between different computational systems. Often all procedures on a single tablet belong to one and the same system; otherwise those belonging to one system are usually grouped together. Procedures belonging to different computational systems are sometimes introduced by the word ‘secondly’ (*šanîš*) or ‘thirdly’ (*šalšiš*). The algorithms constituting a computational system are not always coherent in the mathematical sense. For instance, many planetary systems incorporate two complementary methods for computing zodiacal positions that need not be mutually consistent: one based on the synodic arc and one based on pushes (§3.3.1).