Gauss–Newton Fitting of Circles and Spheres with a Known Radius

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1 Problem Statement

Let $\mathbf{x}_i \in \mathbb{R}^d$ ($d \in \{2, 3\}$) denote noisy measurements sampled from the surface of a circle or sphere with a known radius R. The unknown centre $\mathbf{c} \in \mathbb{R}^d$ is estimated by minimising the orthogonal distance residuals

$$r_i(\mathbf{c}) = \|\mathbf{x}_i - \mathbf{c}\|_2 - R. \tag{1}$$

The weighted nonlinear least-squares problem reads

$$\min_{\mathbf{c}\in\mathbb{R}^d} \frac{1}{2}\sum_{i=1}^{n} w_i\, r_i(\mathbf{c})^2, \tag{2}$$

where $w_i > 0$ modulate the confidence assigned to each observation (set $w_i = 1$ for uniform weighting).

2 Gauss–Newton Iterations

Equation (2) is smooth, so Gauss–Newton iterations provide an efficient solver. Let $\mathbf{c}^{(k)}$ denote the current centre estimate. Linearising the residuals around this iterate gives

$$\mathbf{r}(\mathbf{c}^{(k)} + \Delta\mathbf{c}) \approx \mathbf{r}(\mathbf{c}^{(k)}) + \mathbf{J}(\mathbf{c}^{(k)})\,\Delta\mathbf{c}, \tag{3}$$

where the Jacobian rows are

$$\mathbf{J}_i(\mathbf{c}^{(k)}) = \frac{\partial r_i}{\partial \mathbf{c}}\Big|_{\mathbf{c}=\mathbf{c}^{(k)}} = \frac{\mathbf{c}^{(k)} - \mathbf{x}_i}{\|\mathbf{x}_i - \mathbf{c}^{(k)}\|_2}. \tag{4}$$

Substituting into (2) produces the normal equations

$$\mathbf{J}(\mathbf{c}^{(k)})^\top \mathbf{W}\mathbf{J}(\mathbf{c}^{(k)})\,\Delta\mathbf{c} = -\mathbf{J}(\mathbf{c}^{(k)})^\top \mathbf{W}\mathbf{r}(\mathbf{c}^{(k)}), \tag{5}$$

with $\mathbf{W} = \operatorname{diag}(w_1, \ldots, w_n)$. Solving (5) yields the update direction $\Delta\mathbf{c}$ and the iterate is advanced as $\mathbf{c}^{(k+1)} = \mathbf{c}^{(k)} + \Delta\mathbf{c}$.

3 Initialization and Convergence

A practical scheme requires an initial guess $\mathbf{c}^{(0)}$. Two simple choices are the sample centroid (uniform weights) or the weighted centroid $\sum_i w_i\mathbf{x}_i / \sum_i w_i$. The iterations terminate when either $\|\Delta\mathbf{c}\|_2$ or the weighted root-mean-square error (RMSE)

$$\mathrm{RMSE} = \sqrt{\frac{\sum_i w_i\, r_i(\mathbf{c}^{(k)})^2}{\sum_i w_i}} \tag{6}$$

falls below a prescribed tolerance. The method converges quadratically when the residuals are small and the Jacobian has full column rank.

Robustness can be improved by discarding points with $\|\mathbf{x}_i - \mathbf{c}^{(k)}\|_2 \approx 0$ (which otherwise destabilise the Jacobian) and by capping the number of iterations. In practice, a handful of steps suffices when the initial guess is close to the true centre. The iterative refinement enforces geometric distances, aligning with standard statistically consistent orthogonal fitting practices.