Homework 6

Jacob Hands
COE 352

November 13, 2023

1) Compute the Jacobian matrix for the following:

1. (15pts)

$$\boldsymbol{F}(x, y) = \begin{bmatrix} x^2 + \sin(x) \\ x(y-2) \\ y^2 - 3xy \end{bmatrix}$$

2. (15pts)

$$\boldsymbol{G}(x, y, z) = \begin{bmatrix} x^2 - y^2 \\ 3xyz - 5 \end{bmatrix}$$

Solution

The Jacobian matrix is defined as:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \cdots \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \cdots \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \cdots \end{bmatrix}$$

1) For $\boldsymbol{F}(x, y)$:

$$J_{\boldsymbol{F}}(x, y) = \begin{bmatrix} 2x + \cos(x) & 0 \\ y - 2 & x \\ -3y & 2y - 3x \end{bmatrix}$$

2) For $\boldsymbol{G}(x, y, z)$:

$$J_{\boldsymbol{G}}(x, y, z) = \begin{bmatrix} 2x & -2y & 0 \\ 3yz & 3xz & 3xy \end{bmatrix}$$

Problem 2: Jacobian of Polar Coordinates Transformation

Given:

$$x = r\cos\theta, \quad y = r\sin\theta$$

The transformation can be expressed as:

$$T(r, \theta) = \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Jacobian Matrix:

The Jacobian matrix is defined as:

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Jacobian Determinant:

$$\begin{aligned} \det\left(\frac{\partial(x, y)}{\partial(r, \theta)}\right) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} &= (\cos\theta)(r\cos\theta) - (-r\sin\theta)(\sin\theta) \\ &= r\cos^2\theta + r\sin^2\theta \\ &= r(\cos^2\theta + \sin^2\theta) \\ &= r \end{aligned}$$

Conclusion:

The Jacobian determinant for the polar coordinate transformation is:

$$\boxed{r}$$

Problem 3: Derive the Weak Form of the 1D Wave Equation

Given:

The 1D wave equation is:

$$\frac{\partial^2 u(x, t)}{\partial t^2} - c^2\nabla^2 u(x, t) = f(x, t),$$

where c is a scalar-valued wave speed.

Steps:

To derive the weak form:

- Multiply by a test function $V(x)$.
- Integrate by parts (IBP) for spatial derivatives.

We Know:

The term $\frac{\partial^2 u(x,t)}{\partial t^2}$ is a temporal derivative, as $\frac{\partial u(x,t)}{\partial t}$ is with respect to time.

Start with the Weak Form:

$$\int V(x)\left(\frac{\partial^2 u(x,t)}{\partial t^2} - c^2\nabla^2 u(x,t)\right)dx = \int f(x,t)V(x)\,dx.$$

Split into two terms:

$$\int \frac{\partial^2 u(x,t)}{\partial t^2}V(x)\,dx - c^2\int \nabla^2 u(x,t)V(x)\,dx = \int f(x,t)V(x)\,dx.$$

Using integration by parts (IBP):

$$V(x)\cdot\nabla u(x,t) - \int \nabla u(x,t)\cdot\nabla V(x)\,dx,$$

where:

$$u = V(x),\quad du = \frac{dV(x)}{dx},\quad v = \frac{\partial u(x,t)}{\partial x},\quad dv = \frac{\partial^2 u(x,t)}{\partial x^2}\,dx,\quad \nabla = \frac{d}{dx}.$$

Substituting, we obtain:

$$\int \frac{\partial^2 u(x,t)}{\partial t^2}V(x)\,dx - c^2\left[V(x)\cdot\nabla u(x,t) - \int \nabla u(x,t)\cdot\nabla V(x)\,dx\right] = \int f(x,t)V(x)\,dx.$$

$$\boxed{\int \frac{\partial^2 u(x,t)}{\partial t^2}V(x)\,dx - c^2\left[V(x)\cdot\nabla u(x,t)\Big|_0^1 - \int \nabla u(x,t)\cdot\nabla V(x)\,dx\right] = \int f(x,t)V(x)\,dx}.$$

Evaluate the Boundary Condition:

$$V(x)\cdot\nabla u(x,t)\Big|_0^1 = 0,\quad \text{because the test function evaluates to 0 at the boundary.}$$

Final Weak Form:

$$\boxed{\int \frac{\partial^2 u(x,t)}{\partial t^2}V(x)\,dx + c^2\int \nabla u(x,t)\cdot\nabla V(x)\,dx = \int f(x,t)V(x)\,dx}.$$

Problem 4: Lagrangian Interpolation Approximation

Given:

Let $f(x) = e^x$ for $0 \le x \le 2$. Approximate $f(0.25)$ using Lagrangian linear interpolation with $x_0 = 0$ and $x_1 = 0.5$. Compare your answer to the real one.

Steps:

We know that Lagrangian polynomials have:

$$\frac{d^2}{dx^2} = 0 \quad \text{at desired points.}$$

The given points are:

$$f(0) = e^0 = 1, \quad f(0.5) = e^{0.5} \approx 1.6487$$

The Lagrangian interpolation formula is:

$$f(x) = y_0 \frac{(x - x_1)}{(x_0 - x_1)} + y_1 \frac{(x - x_0)}{(x_1 - x_0)}$$

Substituting the values:

$$f(x) = 1 \cdot \frac{(x - 0.5)}{(0 - 0.5)} + 1.6487 \cdot \frac{(x - 0)}{(0.5 - 0)}$$

Simplify:

$$f(x) = -2x + 1 + 1.6487(2x)$$

$$f(x) = 1.2974x + 1$$

Approximation for $f(0.25)$:

Substitute $x = 0.25$:

$$f(0.25) = 1.2974(0.25) + 1 = 1.32435$$

Exact Value:

The exact value is:

$$f(0.25) = e^{0.25} \approx 1.284025$$

Percentage Error:

$$\text{Error} = \frac{|1.32435 - 1.284025|}{1.284025} \approx 0.0314 = 3.14\%$$

Conclusion:

The approximation $f(0.25) = 1.32435$ is about 3.14% above the exact value 1.284025.

Problem 5: Lagrangian Interpolating Polynomial

Given:

Let $P_3(x)$ be the Lagrangian interpolating polynomial for the data $(0, 0)$, $(0.5, y)$, $(1, 3)$, and $(2, 2)$. Find y if the coefficient of the x^3 term in $P_3(x)$ is 6.

Steps:

The Lagrangian polynomial is expressed as:

$$f(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x)$$

Each term $f_k(x)$ is:

$$f_k(x) = y_k \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}$$

For the given points:

$$f(x) = 0 \cdot \frac{(x-0.5)(x-1)(x-2)}{(0-0.5)(0-1)(0-2)} + y \cdot \frac{(x-0)(x-1)(x-2)}{(0.5-0)(0.5-1)(0.5-2)} + 3 \cdot \frac{(x-0)(x-0.5)(x-2)}{(1-0)(1-0.5)(1-2)} + 2 \cdot \frac{(x-0)(x-0.5)(x-}{(2-0)(2-0.5)(2-}$$

Simplify each term:

$$f(x) = y \cdot \frac{(x-0)(x-1)(x-2)}{(0.5)(-0.5)(-1.5)} + 3 \cdot \frac{(x-0)(x-0.5)(x-2)}{(1)(0.5)(-1)} + 2 \cdot \frac{(x-0)(x-0.5)(x-1)}{(2)(1.5)(1)}$$

After simplifying:

$$f(x) = \frac{1.5yx^3 - 3x^3 - 4.5yx^2 + 9.5625x^2 + 3yx - 6.5625x}{0.5625}$$

Finding y:

The coefficient of x^3 is given as 6. Thus:

$$\frac{1.5y - 3}{0.5625} = 6$$

Solve for y:

$$1.5y - 3 = 6 \cdot 0.5625 = 3.375 \Rightarrow 1.5y = 3.375 + 3 = 6.375$$

$$y = \frac{6.375}{1.5} = 4.25$$

Conclusion:

The value of y is:

$$\boxed{4.25}$$