Homework 7

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1. (20pts) Approximate $\int_0^\pi \sin x\,dx$ using the 4-point quadrature rule on a parent domain of $-1 \le \xi \le 1$.

From the notes, it says to look up point weights and locations, so that's what I did.

$$N_q = 4, \quad q_i : \{-0.861, -0.348, 0.348, 0.861\}, \quad w_i : \{0.348, 0.652, 0.652, 0.348\}.$$

$$\int_a^b f(x)\,dx = \int_{-1}^1 f(x(\xi))\frac{dx}{d\xi}\,d\xi = \sum_{i=1}^{N_q} w_i f(x(q_i)).$$

We have to do element mapping because the grid is unstructured.

$$\frac{d\xi}{dx} = \frac{2}{h}, \quad dx = \frac{h}{2}\,d\xi, \quad h = b - a \implies dx = \frac{\pi - 0}{2}\,d\xi = \frac{\pi}{2}\,d\xi.$$

Now,

$$x(\xi) = \frac{\pi}{2}\frac{\xi+1}{2} + 0 \implies x(\xi) = \frac{\pi}{2}\frac{\xi+1}{2}.$$

$$\int_a^b f(x)\,dx = \int_{-1}^1 f(x(\xi))\frac{dx}{d\xi}\,d\xi = \sum_{i=1}^{N_q} w_i f(x(\xi_i))\frac{dx}{d\xi}.$$

$$\frac{dx}{d\xi} = \frac{\pi}{2}.$$

Evaluations:

$$x_1 = f\left(\frac{\pi}{2}(-0.861+1)\right) = 0.34025,$$
$$x_2 = f\left(\frac{\pi}{2}(-0.348+1)\right) = 1.34190,$$
$$x_3 = f\left(\frac{\pi}{2}(0.348+1)\right) = 1.34190,$$
$$x_4 = f\left(\frac{\pi}{2}(0.861+1)\right) = 0.34025.$$

$$\sum_{i=1}^{N_q} w_i f(x_i) = (0.652)(1.34190) + (0.652)(1.34190) + (0.348)(0.34025) + (0.348)(0.34025).$$

Final result:

$$\approx 1.984.$$

2. (20pts) Find the constants c_0, c_1, and x_1 so that the quadrature formula

$$\int_0^1 f(x)\,dx = c_0 f(0) + c_1 f(x_1)$$

has the highest possible degree of precision.

- **Highest degree of precision:** For $2N_q - 1 = 2(2) - 1 = 3$, we need the formula to be exact up to x^2.
- **Independent parameters:** c_1 and x_1.

The conditions are:

$$\int_0^1 1\,dx = c_0 f(0) + c_1 f(x_1) = 1,$$

$$\int_0^1 x\,dx = c_0(0) + c_1 x_1 = \frac{1}{2},$$

$$\int_0^1 x^2\,dx = c_0(0) + c_1 x_1^2 = \frac{1}{3}.$$

Step 1: Solve for c_0 and c_1

$$c_0 + c_1 = 1 \quad \Rightarrow \quad c_0 = 1 - c_1.$$

Step 2: Solve for c_1 and x_1 from the second condition

$$c_1 x_1 = \frac{1}{2}.$$

$$c_1 = \frac{1}{x_1}.$$

Step 3: Use the third condition to solve for x_1

$$c_1 x_1^2 = \frac{1}{3}.$$

Substituting $c_1 = \frac{1}{x_1}$,

$$\frac{x_1}{x_1} \cdot x_1^2 = \frac{1}{3},$$

$$x_1 = \frac{2}{3}.$$

Step 4: Solve for c_1 and c_0

$$c_1 = \frac{1}{x_1} = \frac{1}{\frac{2}{3}} = \frac{3}{2}.$$

$$c_0 = 1 - c_1 = 1 - \frac{3}{2} = \frac{1}{4}.$$

Final Results:

$$c_0 = \frac{1}{4}, \quad c_1 = \frac{3}{2}, \quad x_1 = \frac{2}{3}.$$

3. (20pts) Using 2D Gaussian quadrature, compute the integral of the 2D function

$$f(x, y) = x^2y^2$$

defined on the reference quadrilateral on the domain $[-1, 1] \times [-1, 1]$.

Solution:

If 1D Gaussian quadrature is:

$$\int f(x)\, dx \approx \sum_{i=1}^{N} w_i f(x_i),$$

then 2D Gaussian quadrature is:

$$\int\int f(x, y)\, dx\, dy \approx \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j f(x_i, y_j).$$

Here, $N = 2$ because the highest degree polynomial is x^2.

$$N = 2, \quad q_i = \{-0.58, 0.58\}, \quad w_i = \{1, 1\}.$$

The formula is:

$$w_1w_1f(x_1, y_1) + w_1w_2f(x_1, y_2) + w_2w_1f(x_2, y_1) + w_2w_2f(x_2, y_2).$$

Substituting the values:

$$1 \cdot 1 \cdot (-0.58)^2(-0.58)^2 + 1 \cdot 1 \cdot (-0.58)^2(0.58)^2 + 1 \cdot 1 \cdot (0.58)^2(-0.58)^2 + 1 \cdot 1 \cdot (0.58)^2(0.58)^2.$$

Evaluating:

$$(1)(0.58^2)(0.58^2) + (1)(0.58^2)(0.58^2) + (1)(0.58^2)(0.58^2) + (1)(0.58^2)(0.58^2).$$

Answer is approximately: 0.444.

4. (20pts) Define the 2D linear Lagrange polynomials on a unit quadrilateral $[-1, 1] \times [-1, 1]$ by *taking the product of the two 1D polynomials.* Use the polynomials to numerically integrate the function

$$f(x, y) = \frac{1}{4}(1 - x - y + x^2y^2).$$

Plot the original function versus your interpolate for comparison.

Solution:

- $N = 2$ because the highest polynomial is x^2.
- The 2D basis polynomials are obtained as the product of the 1D polynomials.

$$L_1(x) = \frac{1-x}{2}, \quad L_2(x) = \frac{1+x}{2}, \quad L_1(y) = \frac{1-y}{2}, \quad L_2(y) = \frac{1+y}{2}.$$

The 2D polynomials are:

$$\begin{aligned}
\phi_1(x, y) &= L_1(x)L_1(y), \\
\phi_2(x, y) &= L_1(x)L_2(y), \\
\phi_3(x, y) &= L_2(x)L_1(y), \\
\phi_4(x, y) &= L_2(x)L_2(y).
\end{aligned}$$

$$\phi_1(x, y) = \frac{(1-x)(1-y)}{4}, \quad \phi_2(x, y) = \frac{(1-x)(1+y)}{4},$$
$$\phi_3(x, y) = \frac{(1+x)(1-y)}{4}, \quad \phi_4(x, y) = \frac{(1+x)(1+y)}{4}.$$

Numerical Integration:
The 2D integral is:

$$\int_{-1}^{1} \int_{-1}^{1} f(x, y)\, dx\, dy \approx \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j f(x_i, y_j).$$

Function Evaluation:

$$f(-1, -1) = \frac{1}{4}(1 - (-1) - (-1) + (-1)^2(-1)^2) = \frac{1}{4} \cdot 4 = 1,$$

$$f(-1, 1) = \frac{1}{4}(1 - (-1) - 1 + (-1)^2(1)^2) = \frac{1}{4} \cdot 2 = \frac{1}{2},$$

$$f(1, -1) = \frac{1}{4}(1 - 1 - (-1) + (1)^2(-1)^2) = \frac{1}{4} \cdot 2 = \frac{1}{2},$$

$$f(1, 1) = \frac{1}{4}(1 - 1 - 1 + (1)^2(1)^2) = \frac{1}{4} \cdot 0 = 0.$$

Interpolating Polynomial:

$$f(x, y) \approx \phi_1(x, y)f(-1, -1) + \phi_2(x, y)f(-1, 1) + \phi_3(x, y)f(1, -1) + \phi_4(x, y)f(1, 1).$$

Substitute the values of $\phi_i(x, y)$ and $f(x_i, y_i)$:

$$f(x, y) = \frac{(1-x)(1-y)}{4}(1) + \frac{(1-x)(1+y)}{4}\left(\frac{1}{2}\right) + \frac{(1+x)(1-y)}{4}\left(\frac{1}{2}\right) + \frac{(1+x)(1+y)}{4}(0).$$

Simplify:

$$f(x, y) = \frac{1-x}{4}.$$

Now integrate:

$$\int_{-1}^{1} \int_{-1}^{1} f(x, y)\, dx\, dy = \int_{0}^{1} \int_{0}^{1} \left(1 - \frac{x}{4}\right) dx\, dy.$$

Perform the integration step by step:

$$\int_0^1 \int_0^1 \left(1 - \frac{x}{4}\right) \, dx \, dy = \int_0^1 \left[\int_0^1 \left(1 - \frac{x}{4}\right) dx \right] dy.$$

First, integrate with respect to x:

$$\int_0^1 \left(1 - \frac{x}{4}\right) dx = \left[x - \frac{x^2}{8} \right]_0^1 = \left(1 - \frac{1}{8}\right) = \frac{7}{8}.$$

Now, integrate with respect to y:

$$\int_0^1 \frac{7}{8} \, dy = \frac{7}{8} \cdot 1 = \frac{7}{8}.$$

Correct the domain back to $[-1, 1]$:

$$\int_{-1}^1 \int_{-1}^1 f(x, y) \, dx \, dy = \int_{-1}^1 \frac{7}{8} \, dy = 1.$$

Final Answer:

$$\boxed{1}.$$

5. (20pts) Find the Jacobian matrix,

$$\begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix},$$

using 2D Lagrange interpolation mapping from a quadrilateral with the nodal coordinates $(0, 0)$, $(1, 0)$, $(2, 2)$, and $(0, 1)$.

Solution:

We start by defining the 2D Lagrange polynomials as derived earlier:

$$\begin{aligned} N_1(\xi, \eta) &= \frac{1}{4}(1 - \xi)(1 - \eta), \\ N_2(\xi, \eta) &= \frac{1}{4}(1 + \xi)(1 - \eta), \\ N_3(\xi, \eta) &= \frac{1}{4}(1 + \xi)(1 + \eta), \\ N_4(\xi, \eta) &= \frac{1}{4}(1 - \xi)(1 + \eta). \end{aligned}$$

The coordinates $x(\xi, \eta)$ and $y(\xi, \eta)$ are expressed as:

$$x(\xi, \eta) = \sum_{i=1}^4 N_i(\xi, \eta) x_i,$$

$$y(\xi, \eta) = \sum_{i=1}^4 N_i(\xi, \eta) y_i,$$

where the nodal coordinates are:

$$(x_1, y_1) = (0, 0), \quad (x_2, y_2) = (1, 0), \quad (x_3, y_3) = (2, 2), \quad (x_4, y_4) = (0, 1).$$

Step 1: Compute derivatives of N_i with respect to ξ and η:

$$\begin{aligned}
\frac{\partial N_1}{\partial \xi} &= -\frac{1}{4}(1-\eta), & \frac{\partial N_1}{\partial \eta} &= -\frac{1}{4}(1-\xi), \\
\frac{\partial N_2}{\partial \xi} &= \frac{1}{4}(1-\eta), & \frac{\partial N_2}{\partial \eta} &= -\frac{1}{4}(1+\xi), \\
\frac{\partial N_3}{\partial \xi} &= \frac{1}{4}(1+\eta), & \frac{\partial N_3}{\partial \eta} &= \frac{1}{4}(1+\xi), \\
\frac{\partial N_4}{\partial \xi} &= -\frac{1}{4}(1+\eta), & \frac{\partial N_4}{\partial \eta} &= \frac{1}{4}(1-\xi).
\end{aligned}$$

Step 2: Compute $\frac{\partial x}{\partial \xi}, \frac{\partial x}{\partial \eta}, \frac{\partial y}{\partial \xi}, \frac{\partial y}{\partial \eta}$:

$$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} x_i,$$

$$\frac{\partial y}{\partial \xi} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} y_i,$$

$$\frac{\partial x}{\partial \eta} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} x_i,$$

$$\frac{\partial y}{\partial \eta} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} y_i.$$

Substituting the nodal values into the expressions:

$$\frac{\partial x}{\partial \xi} = -\frac{1}{4}(1-\eta)(0) + \frac{1}{4}(1-\eta)(1) + \frac{1}{4}(1+\eta)(2) + -\frac{1}{4}(1+\eta)(0),$$

$$\frac{\partial x}{\partial \xi} = \frac{1}{4}(1-\eta) + \frac{1}{2}(1+\eta),$$

$$\frac{\partial x}{\partial \xi} = \frac{1}{4}(2\eta + 3).$$

Similarly:

$$\frac{\partial y}{\partial \xi} = -\frac{1}{4}(1-\eta)(0) + \frac{1}{4}(1-\eta)(0) + \frac{1}{4}(1+\eta)(2) + -\frac{1}{4}(1+\eta)(1),$$

$$\frac{\partial y}{\partial \xi} = \frac{1}{4}(1+\eta).$$

$$\frac{\partial x}{\partial \eta} = -\frac{1}{4}(1-\xi)(0) + -\frac{1}{4}(1+\xi)(1) + \frac{1}{4}(1+\xi)(2) + \frac{1}{4}(1-\xi)(0),$$

$$\frac{\partial x}{\partial \eta} = \frac{1}{4}(1+\xi).$$

$$\frac{\partial y}{\partial \eta} = -\frac{1}{4}(1-\xi)(0) + -\frac{1}{4}(1+\xi)(0) + \frac{1}{4}(1+\xi)(2) + \frac{1}{4}(1-\xi)(1),$$

$$\frac{\partial y}{\partial \eta} = \frac{1}{4}(1+\xi).$$

Step 3: Write the Jacobian matrix:

$$\mathbf{J} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}(2\eta+3) & \frac{1}{4}(1+\eta) \\ \frac{1}{4}(1+\xi) & \frac{1}{4}(1+\xi) \end{pmatrix}.$$