Homework 5

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Problem 1

(20 pts) Use a uniform four-element mesh on $[0, 1]$ and piecewise Lagrangian linear finite element basis functions to construct a finite element interpolant, f_h, of the function $f(x) = \sin(\pi x)$. Set the values $f_h(x_i) = \sin(\pi x_i)$ for $i = 1, 2, 3, 4, 5$, and plot the function f and f_h.

Solution

Let the mesh points be $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, and $x_4 = 1$. For each node, we compute:

$$f_h(x_i) = f(x_i) = \sin(\pi x_i) \quad \text{for } i = 0, 1, 2, 3, 4.$$

Using linear basis functions for each element, we can construct $f_h(x)$ as a piecewise linear function interpolating these values. We then plot $f(x)$ and $f_h(x)$.

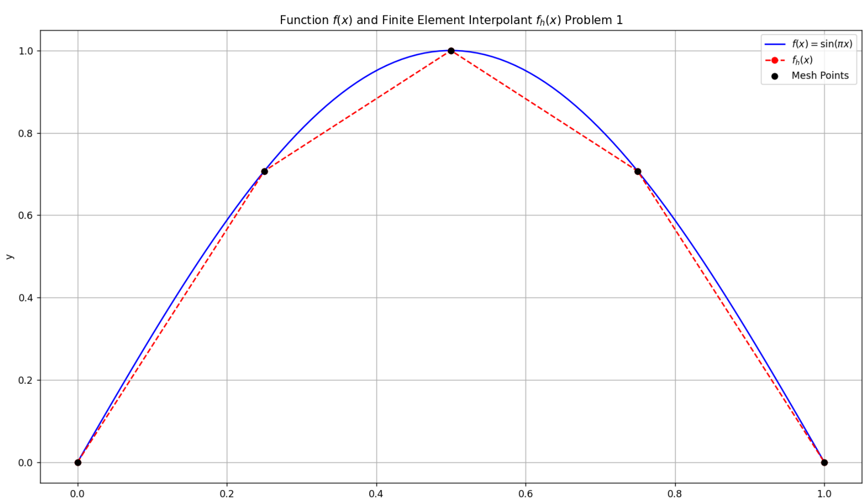

Figure 1: Plot of $f(x) = \sin(\pi x)$ and finite element interpolant $f_h(x)$

Problem 2

Consider the boundary value problem

$$-y'' = x, \quad y(0) = y(1) = 0.$$

(a) (10 pts) Find the exact solution of the problem.

To find the exact solution, we solve:

$$y'' = -x.$$

Integrating twice, we get:

$$y'(x) = -\frac{x^2}{2} + C_1,$$

$$y(x) = -\frac{x^3}{6} + C_1 x + C_2.$$

Using the boundary conditions $y(0) = 0$ and $y(1) = 0$, we find:

$$y(0) = C_2 = 0,$$

$$y(1) = -\frac{1}{6} + C_1 = 0 \Rightarrow C_1 = \frac{1}{6}.$$

Thus, the exact solution is:

$$y(x) = -\frac{x^3}{6} + \frac{x}{6}.$$

(b) (10 pts) Derive the weak form of the problem.

To derive the weak form, multiply both sides by a test function $v(x)$ and integrate over $[0, 1]$:

$$\int_0^1 -y'' v \, dx = \int_0^1 xv \, dx.$$

Integrating by parts, we get:

$$-y'v\big|_0^1 + \int_0^1 y'v' \, dx = \int_0^1 xv \, dx.$$

Applying the boundary conditions $y(0) = y(1) = 0$, the weak form becomes:

$$\int_0^1 y'v' \, dx = \int_0^1 xv \, dx.$$

(c) (20 pts) Galerkin Approximation with Basis Functions

Let $N = 3$ and choose basis functions $\phi_i = \sin(i\pi x)$ for $i = 1, 2, 3$. The Galerkin method requires us to calculate the stiffness matrix K_{ij} and load vector f_i.

1. **Stiffness Matrix**:

$$K_{ij} = \int_0^1 \phi_i' \phi_j' \, dx.$$

From the calculations in the notes, we have for each entry:

$$K_{ij} = \int_0^1 (i\pi \cos(i\pi x)) \, (j\pi \cos(j\pi x)) \, dx.$$

Each entry K_{ij} is computed and substituted in the matrix.

2. **Load Vector**:

$$f_i = \int_0^1 x \phi_i \, dx.$$

Substitute each basis function $\phi_i(x) = \sin(i\pi x)$, and evaluate each integral to find f_i.

Solving the resulting system of equations for the coefficients gives the approximate solution $y_h(x)$.

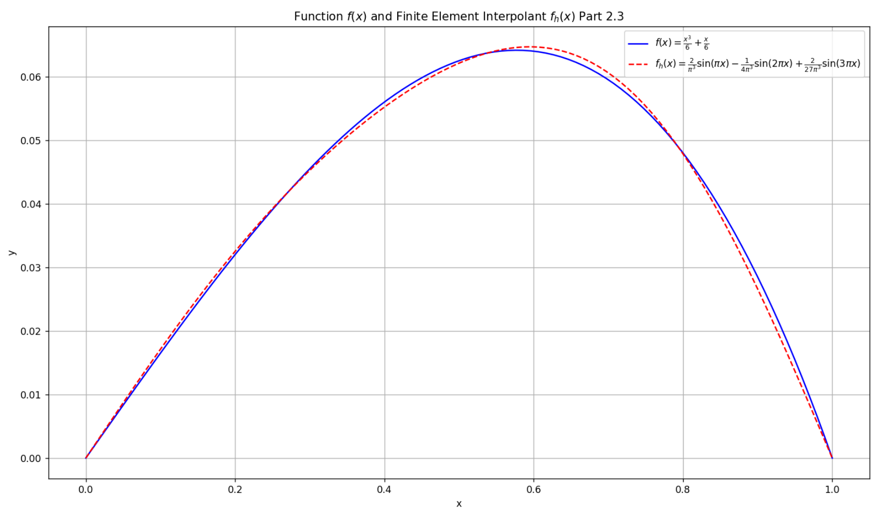

Figure 2: Plot of exact solution $y(x)$ and approximate solution $y_h(x)$

(d) (40 pts) Galerkin Piece-wise Linear Finite Element Approximation

Using a uniform mesh with mesh size $h = 0.25$, we have nodes at $x = 0, 0.25, 0.5, 0.75, 1$. Define piecewise linear basis functions corresponding to each interval.

1. **Local Stiffness Matrix for Each Element**: For each element e, the local stiffness matrix is:

$$K_{ij}^e = \int_{x_e}^{x_{e+1}} \phi_i' \phi_j' \, dx$$

Summing these contributions forms the global stiffness matrix.

2. **Local Load Vector for Each Element**: For each element, compute:

$$f_i^e = \int_{x_e}^{x_{e+1}} x\phi_i \, dx.$$

This yields the components of the global load vector, which are then assembled from each element.

3. **Assemble the Global System**: Solve the resulting system to find the approximate solution $y_h(x)$.

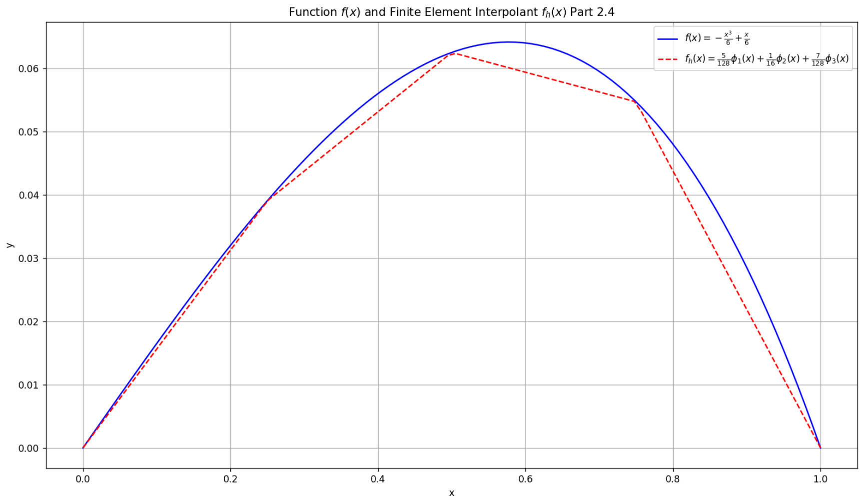

Figure 3: Plot of exact solution $y(x)$ and finite element approximation $y_h(x)$ with $h = 0.25$