Homework Assignments COE 352

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Homework # 7

1. (20pts) Approximate $\int_0^{\pi} sinx\, dx$ using the 4-point quadrature rule on a parent domain of $-1 \leq \xi \leq 1$.

2. (20pts) Find the constants c_0, c_1, and x_1 so that the quadrature formula $\int_0^1 f(x)\, dx = c_0 f(0) + c_1 f(x_1)$ has the highest possible degree of precision.

3. (20pts) Using 2D Gaussian quadrature, compute the integral of the 2D function $f(x, y) = x^2 y^2$ defined on the reference quadrilateral on the domain $[-1, 1] \times [-1, 1]$.

4. (20pts) Define the $2D$ linear Lagrange polynomials on a unit quadrilateral $[-1, 1] \times [-1, 1]$ by taking the product of the two $1D$ polynomials. Use the polynomials to numerically integrate the function $f(x, y) = \frac{1}{4}(1 - x - y + x^2 y)$. Plot the original function versus your interpolate for comparison.

5. (20pts) Find the Jacobian matrix,

$$\begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

using 2D Lagrange interpolation mapping from a quadrilateral with the nodal coordinates $(0, 0), (1, 0), (2, 2), (0, 1)$.

1. (20pts) Approximate $\int_0^\pi sinx\,dx$ using the 4-point quadrature rule on a parent domain of $-1 \le \xi \le 1$.

From the notes it says to look up point weights & locations, so thats what I did

N_q	q_i	W_i
4	-0.861	0.348
	-0.348	0.652
	0.348	0.652
	0.861	0.348

$$\int_a^b f(x)dx = \int_0^\pi \sin(x)dx \approx \sum_{i=1}^{N_q} w_i f(q_i)$$

We have to do element mapping because the grid is unconstructed

$$\frac{d\xi}{dx} = \frac{2}{h},\ \frac{dx}{d\xi} = \frac{h}{2} \Rightarrow dx = \frac{h}{2}d\xi \quad h = b-a \Rightarrow dx = \frac{b-a}{2}d\xi$$

$$\Rightarrow [a,b] \Rightarrow [0,\pi] \Rightarrow dx = \frac{\pi-0}{2}d\xi \Rightarrow \frac{dx}{d\xi} = \frac{\pi}{2}$$

$$x(\xi) = (\xi+1)\frac{h}{2} + x_i$$

$$\int_{-1}^{1} \hat{\phi}_1(\xi)\hat{\phi}_2(\xi)\frac{dx}{d\xi}d\xi = \int_{-1}^{1} f(x(\xi))\frac{dx}{d\xi}d\xi$$

$$\xi_1 : f\left(\frac{\pi}{2}(-0.861) + \frac{\pi}{2}\right)\frac{\pi}{2} = 0.34025$$

$$\xi_2 : f\left(\frac{\pi}{2}(-0.348) + \pi/2\right)\pi/2 = 1.34190$$

$$\xi_3 : f\left(\pi/2(0.348) + \frac{\pi}{2}\right)\frac{\pi}{2} = 1.34190$$

$$\xi_4 : f\left(\pi/2(0.861) + \frac{\pi}{2}\right)\frac{\pi}{2} = 0.34025$$

$$\sum_{i=1}^{N_q} W_i f(\xi_i) = (0.652)(1.34190) + (0.65)(1.34190) + (0.348)(0.34025) + (0.348)(0.34025)$$

$$\approx 1.984$$

2. (20pts) Find the constants c_0, c_1, and x_1 so that the quadrature formula $\int_0^1 f(x)\,dx = c_0 f(0) + c_1 f(x_1)$ has the highest possible degree of precision.

highest degree of precision $2N_q - 1 = 2(2) - 1 = 3$ $O(1), O(N), O(N^2)$

$N=2$ Because the independent parameters are C_1 & X_1

$$\left.\begin{aligned} \int_0^1 1\,dx &= C_0 f(0) + C_1 f(X_1) = 1 \\ \int_0^1 x\,dx &= C_0 f(0) + C_1 f(X_1) = \tfrac{1}{2} \\ \int_0^1 x^2\,dx &= C_0 f(0) + C_1 f(X_1) = \tfrac{1}{3} \end{aligned}\right\}$$ Solve for C_0, C_1, X_1

$C_0 + C_1 = 1 \Rightarrow C_0 + \frac{3}{4} = 1 = \boxed{C_0 = \frac{1}{4}}$

$C_0(0) + C_1(X_1) = \frac{1}{2} \Rightarrow C_1 X_1 = \frac{1}{2} \Rightarrow C_1\left(\frac{2}{3}\right) = \frac{1}{2} \Rightarrow \boxed{C_1 = \frac{3}{4}}$

$\downarrow$

$C_0(0^2) + C_1(X_1^2) = \frac{1}{3}$ $C_1 X_1^2 = \frac{1}{3}$ $\frac{1}{2}(X_1) = \frac{1}{3} \Rightarrow \boxed{X_1 = \frac{2}{3}}$

3. (20pts) Using 2D Gaussian quadrature, compute the integral of the 2D function $f(x, y) = x^2y^2$ defined on the reference quadrilateral on the domain $[-1, 1] \times [-1, 1]$.

If 1D is $\sum_{i=1}^{N_a} w_i f(x_i)\,dx = \int_0^1 f(x)\,dx$

2D is just $\sum_{i=1} \sum_{j=1} w_i f(x_i)\,dx = \int_0^1\int_0^1 f(x, y)\,dx\,dy$

N=2 because the highest degree polynomial is x^2

N	q_i	w_i
2	-0.58	1
	0.58	1

$w_1 w_1 f(x_1, y_1) + w_2 w_2 f(x_2, y_2) + w_1 w_2 f(x_1, y_2) + w_2 w_1 f(x_2, y_1)$

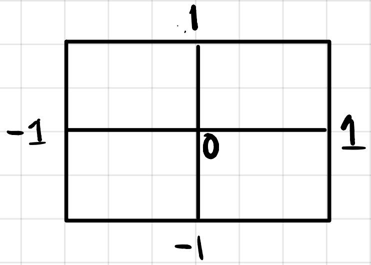
$(1)(1)(-0.58)^2(-0.58)^2 + (1)(1)(0.58)^2(0.58)^2 + (1)(1)(0.58)^2(-0.58)^2 + (1)(1)(-0.58)^2(0.58)^2$

answer is approx ≈ 0.444

4. (20pts) Define the $2D$ linear Lagrange polynomials on a unit quadrilateral $[-1,1]\times[-1,1]$ by taking the product of the two $1D$ polynomials. Use the polynomials to numerically integrate the function $f(x,y) = \frac{1}{4}(1 - x - y + x^2y)$. Plot the original function versus your interpolate for comparison.

$N=2$ because highest polynomial is x^2

It wants the product of the 1D polynomials



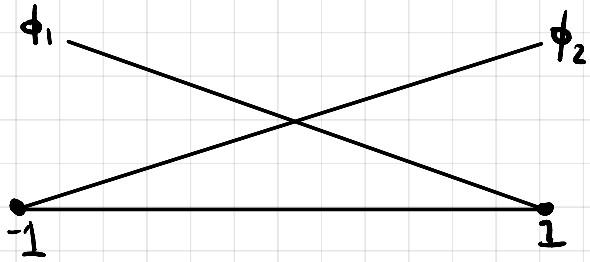
$L_{1,1}(x,y) = L_1(x)\cdot L_1(y)$

$L_{1,2}(x,y) \quad L_1(x)\cdot L_2(y)$

$L_{2,1}(x,y) = L_2(x)\cdot L_1(y)$

$L_{2,2}(x,y) = L_2(x)\cdot L_2(y)$

2D is just $\sum_{i=1}\sum_{j=1} W_i f(x_i)dx = \int_0^1\int_0^1 f(x,y)dxdy$



$\phi_1(x) = \frac{1-x}{1-(-1)} = \frac{1-x}{2} \quad \phi_2(x) = \frac{1+x}{1-(-1)} = \frac{1+x}{2}$

$\phi_1(y) = \frac{1-y}{1-(-1)} = \frac{1-y}{2} \quad \phi_2(y) = \frac{1-y}{1-(-1)} = \frac{1+y}{2}$

$f = \sum_{i=1}^{4} a_i\phi_i$

$f(x,y)$

$f(-1,1) = \frac{1}{4}(1 - (-1) - (1) + (-1)^2(1)) = \frac{1}{4}(2) = \frac{1}{2}$

$2 \quad -1 \quad +1$

$f(-1,-1) = \frac{1}{4}(1 - (-1) - (-1) + (-1)^2(-1)) = \frac{1}{4}(2) = \frac{1}{2}$

$2 \quad +1 \quad -1$

$f(1,1) = \frac{1}{4}(1 - (1) - (1) + (1)(1)) = 0$

$0 \quad -1 \quad +1$

$f(1,-1) = \frac{1}{4}(1 - (1) - (-1) + (1)^2(-1)) = 0$

$0 \quad 1 \quad -1$

Obtain the lagrange polynomial

$\phi_{11}(x,y) = \phi_1(x)\phi_1(y) = \frac{(1-x)(1-y)}{4}$

$\phi_{12}(x,y) = \phi_1(x)\phi_2(y) = \frac{(1-x)(1+y)}{4}$

$\phi_{21}(x,y) = \phi_2(x)\phi_1(y) = \frac{(1+x)(1-y)}{4}$

$\phi_{22}(x,y) = \phi_2(x)\phi_2(y) = \frac{(1+x)(1+y)}{4}$

$$\sum_{i=1}\sum_{j=1} \omega_i f(x_i)\,dx = f(-1,-1)(\phi_{11}) + f(-1,1)(\phi_{12}) + f(\cancel{1,-1})(\phi_{21}) + f(\cancel{1,1})(\phi_{22})$$

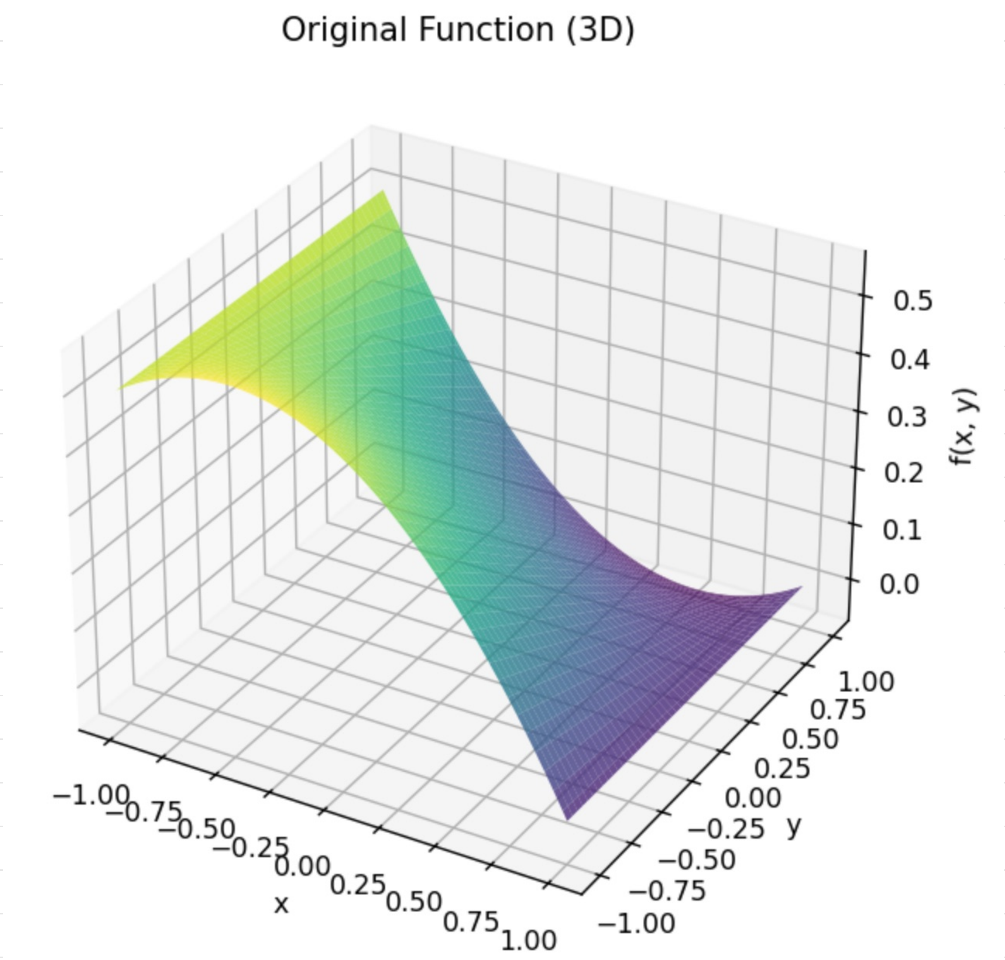
$$= \frac{1}{2}\left(\frac{(1-x)(1-y)}{4}\right) + \left(\frac{1}{2}\left(\frac{(1-x)(1+y)}{4}\right)\right)$$

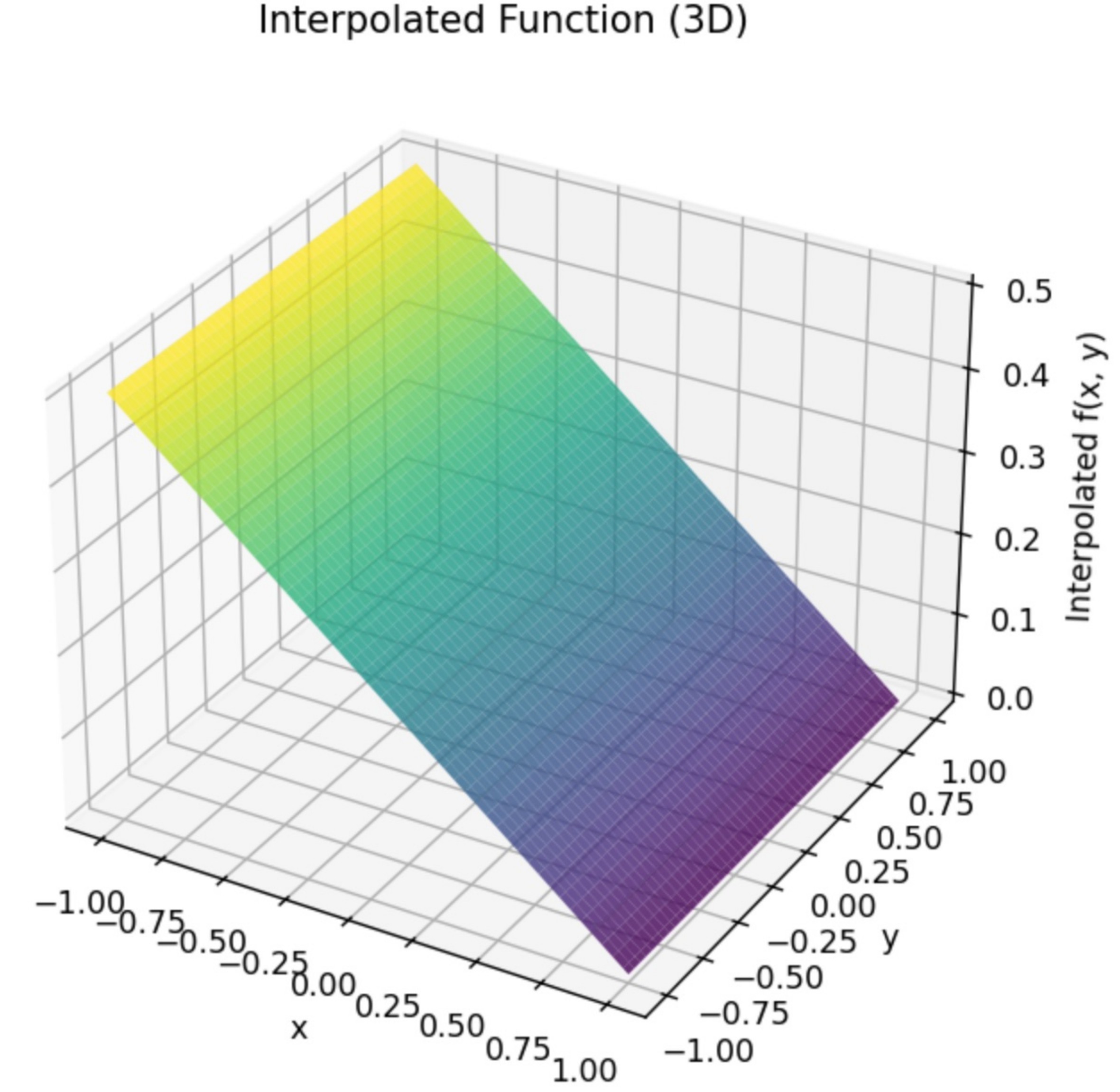
$$= \frac{1-x}{4}$$

now you integrate

$$\int_0^1\int_0^1 f(x,y)\,dx\,dy = \int_0^1\int_0^1 \frac{1-x}{4}\,dx\,dy = 1$$

Plots



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Numerical integration of original function: 1.0
Numerical integration of interpolated function: 1.0
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5. (20pts) Find the Jacobian matrix,

$$\begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

using 2D Lagrange interpolation mapping from a quadrilateral with the nodal coordinates $(0,0), (1,0), (2,2), (0,1)$.

Obtain the lagrange polynomial

$$x(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta) x_i, =$$

$$y(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta) y_i, =$$

$$N_1 = \phi_{11}(\xi,\eta) = \phi_1(x)\phi_1(y) = \frac{(1-\xi)(1-\eta)}{4}$$

$$N_2 = \phi_{12}(\xi,\eta) = \phi_1(x)\phi_2(y) = \frac{(1-\xi)(1+\eta)}{4}$$

$$N_3 = \phi_{21}(\xi,\eta) = \phi_2(x)\phi_1(y) = \frac{(1+\xi)(1-\eta)}{4}$$

$$N_4 = \phi_{22}(\xi,\eta) = \phi_2(x)\phi_2(y) = \frac{(1+\xi)(1+\eta)}{4}$$

From prob 4

Change

$x = \xi$

$y = \eta$

Take derivative:

N_1: $\frac{dN_1}{d\xi} = -\frac{1}{4}(1-\eta)$, $\frac{dN_1}{d\eta} = -\frac{1}{4}(1-\xi)$

N_2: $\frac{dN_2}{d\xi} = \frac{1}{4}(1-\eta)$ $\quad$ $\frac{dN_2}{d\eta} = -\frac{1}{4}(1+\xi)$

N_3: $\frac{dN_3}{d\xi} = \frac{1}{4}(1+\eta)$ $\quad$ $\frac{dN_3}{d\eta} = \frac{1}{4}(1+\xi)$

N_4: $\frac{dN_4}{d\xi} = -\frac{1}{4}(1+\eta)$ $\quad$ $\frac{dN_4}{d\eta} = \frac{1}{4}(1-\xi)$

$$\frac{dx_i}{d\xi} = \sum_{i=1}^{4} \frac{dN_i}{d\xi} x_i = -\frac{1}{4}(1-\eta)(0) + \frac{1}{4}(1-\eta)(1) + \frac{1}{4}(1+\eta)(2) - \frac{1}{4}(1+\eta)(0)$$

$$= \frac{(\eta+3)}{4}$$

$$\frac{dy_i}{d\xi} = \sum_{i=1}^{4} \frac{dN_i}{d\xi} y_i = -\frac{1}{4}(1-\eta)(0) + \frac{1}{4}(1-\eta)(0) + \frac{1}{4}(1+\eta)(2) - \frac{1}{4}(1+\eta)(1)$$

$$= \frac{1}{4}(1+\eta)$$

$$\frac{dx_i}{d\eta} = \sum_{i=1}^{4} \frac{dN_i}{d\eta} x_i = -\frac{1}{4}(1-\xi)(0) - \frac{1}{4}(1+\xi)(1) + \frac{1}{4}(1+\xi)(2) + \frac{1}{4}(1-\xi)(0)$$

$$= -\frac{1}{4}(1+\xi) + \frac{1}{4}(1+\xi)2 = \frac{1}{4}(1+\xi)$$

$$\frac{dy_i}{d\eta} = \sum_{i=1}^{4} \frac{dN_i}{d\eta} y_i = -\frac{1}{4}(1-\xi)(0) - \frac{1}{4}(1+\xi)(0) + \frac{1}{4}(1+\xi)(2) + \frac{1}{4}(1-\xi)(1))$$

$$= \frac{1}{2}(1+\xi) + \frac{1}{4}(-\xi) = \frac{(\xi+3)}{4}$$

$$J = \begin{bmatrix} \frac{dx}{d\xi} & \frac{dy}{d\xi} \\ \frac{dx}{d\eta} & \frac{dy}{d\eta} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}(\eta+3) & \frac{1}{4}(1+\eta) \\ \frac{1}{4}(1+\xi) & \frac{1}{4}(\xi+3) \end{bmatrix}$$