# Homework 7

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## 1. (20pts) Approximate $\int_0^\pi \sin x \, dx$ using the 4-point quadrature rule on a parent domain of $-1 \leq \xi \leq 1$.

**Solution:**

From the notes, we use the point weights and locations:

$$N_q = 4, \quad q_i : \{-0.861, -0.348, 0.348, 0.861\}, \quad w_i : \{0.348, 0.652, 0.652, 0.348\}.$$

The integral is transformed using element mapping because the grid is unstructured:

$$\int_a^b f(x)\, dx = \int_{-1}^1 f(x(\xi)) \frac{dx}{d\xi}\, d\xi = \sum_{i=1}^{N_q} w_i f(x(q_i)).$$

Mapping transformation:

$$x(\xi) = \frac{\pi}{2} \frac{\xi + 1}{2}, \quad \frac{dx}{d\xi} = \frac{\pi}{2}.$$

The quadrature formula becomes:

$$\sum_{i=1}^{N_q} w_i f(x_i) \frac{\pi}{2}.$$

Evaluations:

$$f(x_1) = \sin\left(\frac{\pi}{2}(-0.861 + 1)\right) = 0.34025,$$
$$f(x_2) = \sin\left(\frac{\pi}{2}(-0.348 + 1)\right) = 1.34190,$$
$$f(x_3) = \sin\left(\frac{\pi}{2}(0.348 + 1)\right) = 1.34190,$$
$$f(x_4) = \sin\left(\frac{\pi}{2}(0.861 + 1)\right) = 0.34025.$$

Result:

$$\int_0^\pi \sin x\, dx \approx \frac{\pi}{2}\left[(0.348)(0.34025) + (0.652)(1.34190) + (0.652)(1.34190) + (0.348)(0.34025)\right] \approx 1.984.$$

**2. (20pts) Find the constants $c_0$, $c_1$, and $x_1$ so that the quadrature formula $\int_0^1 f(x)\,dx = c_0 f(0) + c_1 f(x_1)$ has the highest possible degree of precision.**

**Solution:**

For highest precision, we solve:

$$\int_0^1 1\,dx = c_0 + c_1 = 1,$$

$$\int_0^1 x\,dx = c_1 x_1 = \frac{1}{2},$$

$$\int_0^1 x^2\,dx = c_1 x_1^2 = \frac{1}{3}.$$

Solving step-by-step:

$$c_1 x_1 = \frac{1}{2}, \quad x_1 = \frac{2}{3}, \quad c_1 = \frac{3}{2}.$$

$$c_0 = 1 - c_1 = 1 - \frac{3}{2} = \frac{1}{4}.$$

Final values:

$$c_0 = \frac{1}{4}, \quad c_1 = \frac{3}{2}, \quad x_1 = \frac{2}{3}.$$

**3. (20pts) Using 2D Gaussian quadrature, compute the integral of $f(x,y) = x^2y^2$ on $[-1,1] \times [-1,1]$.**

**Solution:**

The 2D integral:

$$\int_{-1}^1 \int_{-1}^1 f(x,y)\,dx\,dy \approx \sum_{i=1}^{2} \sum_{j=1}^{2} w_i w_j f(x_i, y_j),$$

where $N = 2$, $q_i = \{-0.58, 0.58\}, w_i = \{1, 1\}$.

Substitute:

$$f(-0.58,-0.58) = (-0.58)^2(-0.58)^2, \quad f(-0.58,0.58) = (-0.58)^2(0.58)^2,$$

$$f(0.58,-0.58) = (0.58)^2(-0.58)^2, \quad f(0.58,0.58) = (0.58)^2(0.58)^2.$$

Compute:

$$\int_{-1}^1 \int_{-1}^1 f(x,y)\,dx\,dy \approx 4 \times (0.58^2)^2 = 0.444.$$

## 4. (20pts) Define the 2D linear Lagrange polynomials and integrate $f(x, y) = \frac{1}{4}(1 - x - y + x^2y^2)$.

**Solution:**

The 2D Lagrange polynomials:

$$\phi_1(x,y) = \frac{(1-x)(1-y)}{4}, \quad \phi_2(x,y) = \frac{(1-x)(1+y)}{4},$$

$$\phi_3(x,y) = \frac{(1+x)(1-y)}{4}, \quad \phi_4(x,y) = \frac{(1+x)(1+y)}{4}.$$

Interpolation:

$$f(x,y) \approx \phi_1(x,y)f(-1,-1) + \phi_2(x,y)f(-1,1) + \phi_3(x,y)f(1,-1) + \phi_4(x,y)f(1,1).$$

Substitute and simplify:

$$f(x,y) = \frac{1}{4}(1 - x - y + x^2y^2).$$

Integration:

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy = \int_{0}^{1}\int_{0}^{1} \left(1 - \frac{x}{4}\right) dx\,dy = 1.$$

## 5. (20pts) Find the Jacobian matrix.

Find the Jacobian matrix:

$$\mathbf{J} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix},$$

using 2D Lagrange interpolation mapping from a quadrilateral with the nodal coordinates $(0,0), (1,0), (2,2), (0,1)$.

**Step 1: Define the 2D Lagrange polynomials**

$$x(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)x_i, \quad y(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)y_i.$$

The shape functions $N_i(\xi,\eta)$ are:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta), \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta),$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta), \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta).$$

**Step 2: Compute partial derivatives of $N_i$:**

$$\frac{\partial N_1}{\partial \xi} = -\frac{1}{4}(1-\eta), \quad \frac{\partial N_1}{\partial \eta} = -\frac{1}{4}(1-\xi),$$

$$\frac{\partial N_2}{\partial \xi} = \frac{1}{4}(1-\eta), \quad \frac{\partial N_2}{\partial \eta} = -\frac{1}{4}(1+\xi),$$

$$\frac{\partial N_3}{\partial \xi} = \frac{1}{4}(1+\eta), \quad \frac{\partial N_3}{\partial \eta} = \frac{1}{4}(1+\xi),$$

$$\frac{\partial N_4}{\partial \xi} = -\frac{1}{4}(1+\eta), \quad \frac{\partial N_4}{\partial \eta} = \frac{1}{4}(1-\xi).$$

**Step 3: Compute the Jacobian entries**

For $x(\xi, \eta)$ and $y(\xi, \eta)$:

$$x_1 = 0,\ x_2 = 1,\ x_3 = 2,\ x_4 = 0,$$

$$y_1 = 0,\ y_2 = 0,\ y_3 = 2,\ y_4 = 1.$$

$$\frac{\partial x}{\partial \xi} = \frac{\partial N_1}{\partial \xi}x_1 + \frac{\partial N_2}{\partial \xi}x_2 + \frac{\partial N_3}{\partial \xi}x_3 + \frac{\partial N_4}{\partial \xi}x_4,$$

$$\frac{\partial x}{\partial \xi} = -\frac{1}{4}(1-\eta)(0) + \frac{1}{4}(1-\eta)(1) + \frac{1}{4}(1+\eta)(2) + -\frac{1}{4}(1+\eta)(0),$$

$$\frac{\partial x}{\partial \xi} = \frac{1}{4}(1-\eta) + \frac{1}{2}(1+\eta) = \frac{1}{4}(\eta+3).$$

Similarly:

$$\frac{\partial y}{\partial \xi} = \frac{\partial N_1}{\partial \xi}y_1 + \frac{\partial N_2}{\partial \xi}y_2 + \frac{\partial N_3}{\partial \xi}y_3 + \frac{\partial N_4}{\partial \xi}y_4,$$

$$\frac{\partial y}{\partial \xi} = -\frac{1}{4}(1-\eta)(0) + \frac{1}{4}(1-\eta)(0) + \frac{1}{4}(1+\eta)(2) + -\frac{1}{4}(1+\eta)(1),$$

$$\frac{\partial y}{\partial \xi} = \frac{1}{4}(1+\eta).$$

For $\frac{\partial x}{\partial \eta}$:

$$\frac{\partial x}{\partial \eta} = \frac{\partial N_1}{\partial \eta}x_1 + \frac{\partial N_2}{\partial \eta}x_2 + \frac{\partial N_3}{\partial \eta}x_3 + \frac{\partial N_4}{\partial \eta}x_4,$$

$$\frac{\partial x}{\partial \eta} = -\frac{1}{4}(1-\xi)(0) + -\frac{1}{4}(1+\xi)(1) + \frac{1}{4}(1+\xi)(2) + \frac{1}{4}(1-\xi)(0),$$

$$\frac{\partial x}{\partial \eta} = \frac{1}{4}(1+\xi).$$

For $\frac{\partial y}{\partial \eta}$:

$$\frac{\partial y}{\partial \eta} = \frac{\partial N_1}{\partial \eta}y_1 + \frac{\partial N_2}{\partial \eta}y_2 + \frac{\partial N_3}{\partial \eta}y_3 + \frac{\partial N_4}{\partial \eta}y_4,$$

$$\frac{\partial y}{\partial \eta} = -\frac{1}{4}(1-\xi)(0) + -\frac{1}{4}(1+\xi)(0) + \frac{1}{4}(1+\xi)(2) + \frac{1}{4}(1-\xi)(1),$$

$$\frac{\partial y}{\partial \eta} = \frac{1}{4}(1+\xi).$$

**Final Jacobian Matrix:**

$$\mathbf{J} = \begin{pmatrix} \frac{1}{4}(\eta+3) & \frac{1}{4}(1+\eta) \\ \frac{1}{4}(1+\xi) & \frac{1}{4}(1+\xi) \end{pmatrix}.$$