

Homework Assignments COE 352

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Homework # 4

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- 1) (15 pts) By hand, computer the QR Factorization of

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

- 2) (20 pts) By hand, find the closest line in a weighted least squares sense that approximates the following data points: $(0, 0), (1, 2), (2, 1), (3, 6)$ with point weights $2/10, 3/10, 3/10, 2/10$.

Hint: The inverse of a 2×2 matrix is as follows:

$$\begin{bmatrix} a & b \\ d & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- 3) (15 pts) For the heat equation, $u_t = Du_{xx}$, derive the discrete growth term, G, using a **backward Euler** method for the time-derivative and **centered differences** for the second order spatial derivative.

NOTE: The identity $e^{ik\Delta x} + e^{-ik\Delta x} = 2\cos(k\Delta x)$

NOTE: Do this by substitution of $u = Ge^{ikx}$.

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- 4) (20 pts) Consider the function $f(x) = 4 + 8x^2 - x^4$.

- (3 pts) Find the first and second derivatives of $f(x)$.
 - (5 pts) Find the y-intercept. Determine any maxima or minima and all points of inflection for $f(x)$. Give both the x and y values.
 - (2 pts) Sketch the graph of $f(x)$. Is this function odd or even or neither?
 - (10 pts) One of the x-intercepts is near $x = 3$. Use Newton's Method starting with $x^{(0)} = 3$ and performing two iterations to get a good approximation to this x-intercept.
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- 5) (20 pts) Consider the function $f(x) = x^3 - 3x - 3$.

- (5 pts) Find all extrema and points of inflection, giving both the x and y values.
- (5 pts) Is this function odd, even or neither? Sketch a graph of this function.
- (10 pts) Use Newton's Method to approximate the value of the x-intercept. Start with $x^{(0)} = 2$ and perform two iterations.

6) (10 pts) Write a python/matlab routine that calculates the root using the **Bisection Method** of $f(x) = x^2 - 3 = 0$ for $x \in [1, 2]$.

1) (15 pts) By hand, computer the QR Factorization of

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$$A = QR$$

Where: Q is Orthogonal

R is Upper triangle

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$$X = \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Orthogonal set

$$\left\| \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\| = \sqrt{3^2 + 0^2 + 1^2} = \sqrt{10}$$

$$\left\| \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\| = \sqrt{0^2 + 2^2 + 0^2} = \sqrt{4} = 2$$

Orthonormal set

$$\left\{ \begin{bmatrix} 3/\sqrt{10} \\ 0 \\ 1/\sqrt{10} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\Rightarrow Q = \begin{bmatrix} 3/\sqrt{10} & 0 \\ 0 & 1 \\ 1/\sqrt{10} & 0 \end{bmatrix}$$

NOW
we find R

$$A = QR \Rightarrow Q^T A = Q^T QR \Rightarrow R = Q^T A$$

$$Q^T = \begin{bmatrix} 3/\sqrt{10} & 0 & 1/\sqrt{10} \\ 0 & 1 & 0 \end{bmatrix}$$

$$R = \underbrace{\begin{bmatrix} 3/\sqrt{10} & 0 & 1/\sqrt{10} \\ 0 & 1 & 0 \end{bmatrix}}_{2 \times 3} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}}_{3 \times 2} =$$

$$\underbrace{\begin{bmatrix} \frac{9}{\sqrt{10}} + \frac{1}{\sqrt{10}} & 0 \\ 0 & 2 \end{bmatrix}}_{2 \times 2} =$$

$$\begin{bmatrix} \frac{10}{\sqrt{10}} & 0 \\ 0 & 2 \end{bmatrix} \quad \text{with } \sqrt{10} \text{ above the first row}$$

$$Q = \begin{bmatrix} 3/\sqrt{10} & 0 \\ 0 & 1 \\ 1/\sqrt{10} & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 2 \end{bmatrix}$$

2) (20 pts) By hand, find the closest line in a weighted least squares sense that approximates the following data points: $(0,0), (1,2), (2,1), (3,6)$ with point weights $2/10, 3/10, 3/10, 2/10$.

Hint: The inverse of a 2×2 matrix is as follows:

$$\begin{bmatrix} a & b \\ d & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^T \begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} \bar{u} \\ a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ z \\ 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$C = \text{diag}\left(\frac{1}{\delta_n^2}\right) = \bar{u}^* = (A^T C A)^{-1} A^T C \bar{b} \Rightarrow \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2/10} & \frac{1}{3/10} \\ \frac{1}{3/10} & \frac{1}{2/10} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2/10} & \frac{1}{3/10} & \frac{1}{3/10} \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

After inserting into Symbolab $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -0.24977 \\ 1.85882 \end{bmatrix}$

$$y = -0.24977x + 1.85882$$

3) (15 pts) For the heat equation, $u_t = Du_{xx}$, derive the discrete growth term, G , using a **backward Euler** method for the time-derivative and **centered differences** for the second order spatial derivative.

NOTE: The identity $e^{ik\Delta x} + e^{-ik\Delta x} = 2\cos(k\Delta x)$

NOTE: Do this by substitution of $u = Ge^{ikx}$.

$$\text{Centered difference} = D_0 u(x_i) = \frac{(f(x+h) - 2f(x) + f(x-h))}{h^2}$$

$$\text{backward Euler} : u^{n+1} = u^n + \Delta t f(u^{n+1}, t^{n+1}) \Rightarrow y^{n+1} = \frac{u^{n+1} - u^n}{\Delta t}$$

$$u_t = \frac{u^{n+1} - u^n}{\Delta t} \Rightarrow \frac{u^{n+1} - u^n}{\Delta t} = D \left(\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} \right) \Rightarrow \text{Substitute } u = G e^{ikx}$$

$$\frac{G e^{ikx} - e^{ikx}}{\Delta t} = D \left(\frac{e^{ik(x+\alpha)} - 2e^{ikx} + e^{ik(x-\alpha)}}{\Delta t^2} \right) \Rightarrow \frac{e^{ikx}(G-1)}{\Delta t} = D e^{ikx} \left(\frac{e^{ik\alpha} - 2 + e^{-ik\alpha}}{\Delta t^2} \right)$$

$$\Rightarrow G-1 = \frac{e^{ik\Delta x} + e^{-ik\Delta x} - 2}{\Delta t} = \frac{2\cos(k\Delta x) - 2}{\Delta t}$$

Substitute
 $2\cos(k\Delta x)$
for $e^{ik\Delta x} + e^{-ik\Delta x}$

$$G = \frac{2D(\cos(k\Delta x) - 1)}{\Delta t} + 1$$

4) (20 pts) Consider the function $f(x) = 4 + 8x^2 - x^4$.

- (3 pts) Find the first and second derivatives of $f(x)$.
- (5 pts) Find the y-intercept. Determine any maxima or minima and all points of inflection for $f(x)$. Give both the x and y values.
- (2 pts) Sketch the graph of $f(x)$. Is this function odd or even or neither?
- (10 pts) One of the x-intercepts is near $x = 3$. Use Newton's Method starting with $x^{(0)} = 3$ and performing two iterations to get a good approximation to this x-intercept.

1)

$$f(x) = 4 + 8x^2 - x^4$$
$$f'(x) = 16x - 4x^3$$
$$f''(x) = 16 - 12x^2$$

2) Find the y intercept $x=0$ $f(x=0) = 4 + 8(0)^2 - (0)^4 = 4$

points of inflections

$$f'(x) = 16x - 4x^3 = 0 \quad x=0 \Rightarrow 16 - 4x^2 = 0 \Rightarrow 16 = 4x^2$$

$$x^2 = 4 \Rightarrow x = \pm 2$$

At $x=0$ $f''(x=0) = 16 - 12(0)^2 = (+)$

$x=0$ is a local minimum

$$x=2 \quad f''(x=2) = 16 - 12(2)^2 = (-)$$

$x=2$ is a local maximum

$$x=-2 \quad f''(x=-2) = 16 - 12(-2)^2 = (-)$$

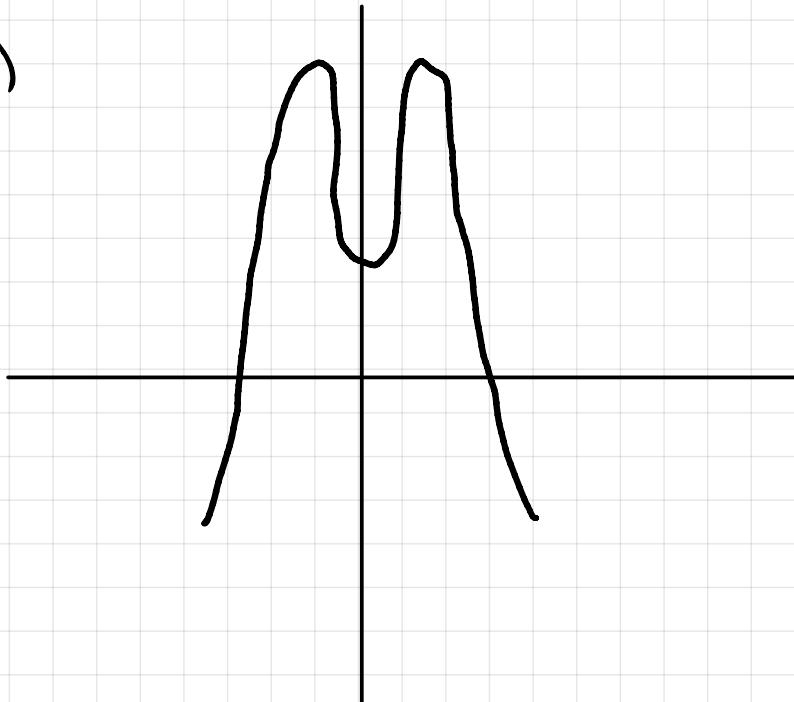
$x=-2$ is a local maximum

2) We determined

$(x=0, y=4)$ $x=0$ is the absolute minimum (minima)

$(x=\pm 2, y=20)$ $x=\pm 2$ has the absolute maxima value

3)



Because the graph is symmetric across the y-axis, this function is even.

5) (20 pts) Consider the function $f(x) = x^3 - 3x - 3$.

- (5 pts) Find all extrema and points of inflection, giving both the x and y values.
- (5 pts) Is this function odd, even or neither? Sketch a graph of this function.
- (10 pts) Use Newton's Method to approximate the value of the x-intercept. Start with $x^{(0)} = 2$ and perform two iterations.

1)

$$f'(x) = 3x^2 - 3$$

\Rightarrow

$$f'(x) = 0 = 3x^2 - 3 \Rightarrow$$

$$3x^2 = 3 \Rightarrow x = \pm 1$$

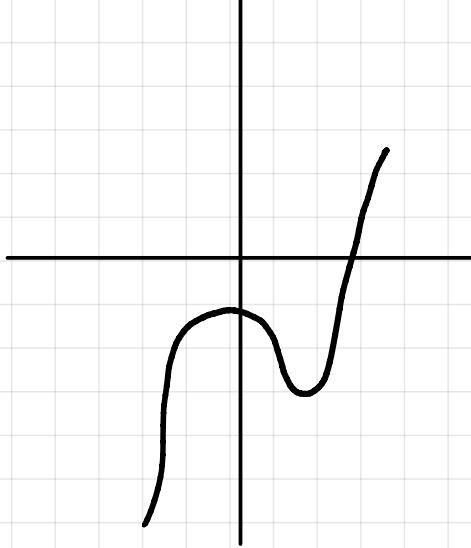
$$\begin{aligned} f(1) &= -5 \\ f(-1) &= -1 \end{aligned}$$

$$f''(x) = 6x$$

at point $(x=1, y=-5)$ $f''(1) = 6(1) = 6$ absolute minimum

$(x=-1, y=-1)$ $f''(-1) = 6(-1) = -6$ absolute maximum

2)



Checking if odd:

$$\text{property: } -f(x) = f(-x) \quad x=1$$

$$-f(1) = f(-1) ? \Rightarrow -(-5) \neq -1$$

Check if even:

NO symmetric across y-axis

This function is neither