# Homework Assignments COE 352

Corey J. Trahan, Ph.D.

October 9, 2023

# Homework # 4

1) (15 pts) By hand, computer the QR Factorization of

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

2) (20 pts) By hand, find the closest line in a weighted least squares sense that approximates the following data points: $(0,0), (1,2), (2,1), (3,6)$ with point weights $2/10, 3/10, 3/10, 2/10$.
**Hint**: The inverse of a 2X2 matrix is as follows:

$$\begin{bmatrix} a & b \\ d & d \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

3) (15 pts) For the heat equation,$u_t = Du_{xx}$, derive the discrete growth term, G, using a **backward Euler** method for the time-derivative and **centered differences** for the second order spatial derivative.
**NOTE:** The identity $e^{ik\Delta x} + e^{-ik\Delta x} = 2cos(k\Delta x)$
**NOTE:** Do this by substitution of $u = Ge^{ikx}$.

4) (20 pts) Consider the function $f(x) = 4 + 8x^2 - x^4$.

- (3 pts) Find the first and second derivatives of $f(x)$.
- (5 pts) Find the y-intercept. Determine any maxima or minima and all points of inflection for $f(x)$. Give both the x and y values.
- (2 pts) Sketch the graph of $f(x)$. Is this function odd or even or neither?
- (10 pts) One of the x-intercepts is near $x = 3$. Use Newton's Method starting with $x^{(0)} = 3$ and performing two iterations to get a good approximation to this x-intercept.

5) (20 pts) Consider the function $f(x) = x^3 - 3x - 3$.

- (5 pts) Find all extrema and points of inflection, giving both the x and y values.
- (5 pts) Is this function odd, even or neither? Sketch a graph of this function.
- (10 pts) Use Newton's Method to approximate the value of the x-intercept. Start with $x^{(0)} = 2$ and perform two iterations.

---

6) (10 pts) Write a python/matlab routine that calculates the root using the **Bisection Method** of $f(x) = x^2 - 3 = 0$ for $x \in [1, 2]$.

1) (15 pts) By hand, computer the QR Factorization of

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$A = QR$ Where: $Q$ is Orthogonal

$R$ is Upper triangle

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$X = \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$

Orthogonal set

$\Rightarrow \left\| \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\| = \sqrt{3^2 + 0^2 + 1^2} = \sqrt{10}$

$\left\| \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\| = \sqrt{0^2 + 2^2 + 0^2} = \sqrt{4} = 2$

Orthonormal set

$\left\{ \begin{bmatrix} 3/\sqrt{10} \\ 0 \\ 1/\sqrt{10} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$\Rightarrow \quad Q = \begin{bmatrix} 3/\sqrt{10} & 0 \\ 0 & 1 \\ 1/\sqrt{10} & 0 \end{bmatrix}$$

Now we find R $\Rightarrow$ $A = QR \Rightarrow Q^T A = \underbrace{Q^T Q}_{I} R \Rightarrow R = Q^T A$

$$Q^T = \begin{bmatrix} 3/\sqrt{10} & 0 & 1/\sqrt{10} \\ 0 & 1 & 0 \end{bmatrix}$$

$$R = \underbrace{\begin{bmatrix} 3/\sqrt{10} & 0 & 1/\sqrt{10} \\ 0 & 1 & 0 \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}}_{3\times 2} = \underbrace{\begin{bmatrix} \frac{9}{\sqrt{10}} + \frac{1}{\sqrt{10}} & 0 \\ 0 & 2 \end{bmatrix}}_{2\times 2} = \begin{bmatrix} \frac{10}{\sqrt{10}} & 0 \\ 0 & 2 \end{bmatrix}$$

$\frac{10}{\sqrt{10}} \to \sqrt{10}$

$$Q = \begin{bmatrix} 3/\sqrt{10} & 0 \\ 0 & 1 \\ 1/\sqrt{10} & 0 \end{bmatrix} \quad R = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 2 \end{bmatrix}$$

2) (20 pts) By hand, find the closest line in a weighted least squares sense that approximates the following data points: $(0,0),(1,2),(2,1),(3,6)$ with point weights $2/10, 3/10, 3/10, 2/10$.

**Hint**: The inverse of a 2X2 matrix is as follows:

$$\begin{bmatrix} a & b \\ d & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\underbrace{\begin{array}{cc} 1 & x \\ \hline \end{array}}_{}$$

$$A \qquad \vec{u} \qquad \vec{b}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$C = \mathrm{diag}\left(\frac{1}{\sigma_n^2}\right) = \vec{u}^* = (A^T C A)^{-1} A^T C \vec{b} \Rightarrow \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} .2 & & & \\ & .3 & & \\ & & .3 & \\ & & & .2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} .2 & & & \\ & .3 & & \\ & & .3 & \\ & & & .2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

After inserting into Symbolab $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -.2571 \\ 1.5714 \end{bmatrix}$

$$y = -0.2571 + 1.5714x$$

3) (15 pts) For the heat equation, $u_t = Du_{xx}$, derive the discrete growth term, G, using a **backward Euler** method for the time-derivative and **centered differences** for the second order spatial derivative.

**NOTE:** The identity $e^{ik\Delta x} + e^{-ik\Delta x} = 2cos(k\Delta x)$

**NOTE:** Do this by substitution of $u = Ge^{ikx}$.

Centered difference = $D_0 u(x_i) = \frac{(f(x+h) - 2f(x) + f(x-h))}{h^2}$

backward Euler : $u^{n+1} = u^n + \Delta t\, f(u^{n+1}, t^{n+1}) \Rightarrow y^{n+1} = \frac{u^{n+1} - u^n}{\Delta t}$

$u_t = \frac{u^{n+1} - u^n}{\Delta t}$

$u_{xx} = \frac{u(x_i+h) - u(x_i-h)}{2h}$

$\Rightarrow \frac{u^{n+1} - u^n}{\Delta t} = D\left(\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2}\right) \Rightarrow$ Substitute $u = Ge^{ikx}$

$$\frac{Ge^{ikx} - e^{ikx}}{\Delta t} = D\left(\frac{e^{ik(x+\Delta x)} - 2e^{ikx} + e^{ik(x-\Delta x)}}{\Delta t^2}\right) \Rightarrow \frac{\cancel{e^{ikx}}(G-1)}{\cancel{\Delta t}} = D\cancel{e^{ikx}}\left(\frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{\Delta t^{\cancel{2}}}\right)$$

$$\Rightarrow G - 1 = \frac{e^{ik\Delta x} + e^{-ik\Delta x} - 2}{\Delta t} =$$

Substitute $2cos(k\Delta x)$ to $e^{ik\Delta x} + e^{-ik\Delta x}$

$$G - 1 = D\frac{2cos(k\Delta x) - 2}{\Delta t}$$

$$G = \frac{2D(\cos(k\Delta x) - 1)}{\Delta t} + 1$$

4) (20 pts) Consider the function $f(x) = 4 + 8x^2 - x^4$.

- (3 pts) Find the first and second derivatives of $f(x)$.
- (5 pts) Find the y-intercept. Determine any maxima or minima and all points of inflection for $f(x)$. Give both the x and y values.
- (2 pts) Sketch the graph of $f(x)$. Is this function odd or even or neither?
- (10 pts) One of the x-intercepts is near $x = 3$. Use Newton's Method starting with $x^{(0)} = 3$ and performing two iterations to get a good approximation to this x-intercept.

1) $f(x) = 4 + 8x^2 - x^4$

$f'(x) = 16x - 4x^3$

$f''(x) = 16 - 12x^2$

2) Find the y intercept $X=0$ $f(x=0) = 4+8(0)^2-(0)^4 = 4$

Points of inflections $f'(x) = 16x - 4x^3 = 0$ $\overset{x=0}{\Rightarrow}$ $16 - 4x^2 = 0 \Rightarrow 16 = 4x^2$

$x^2 = 4 \Rightarrow x = \pm 2$

At $X=0$ $f''(x=0) = 16 - 12(0)^2 = (+)$

$X=0$ is a local minimum

$x=2$ $f''(x=2) = 16 - 12(2)^2 = (-)$

$X=2$ is a local maximum

$X=-2$ $f''(x=-2)$ $16 - (12(-2)^2) = (-)$

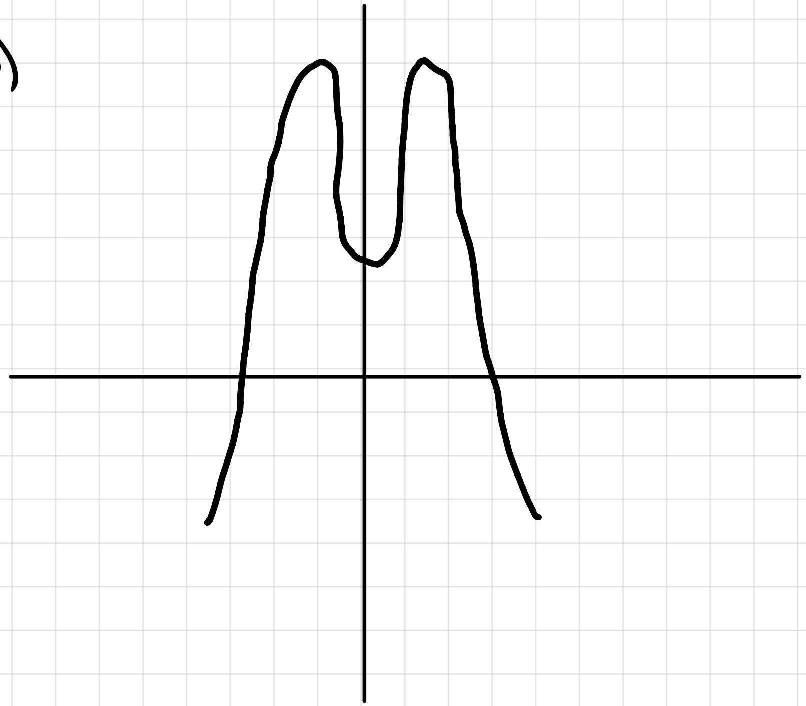
$x=-2$ is a local maximum

2) We determined

$(x=0, y=4)$ $x=0$ is the absolute minimum (minima)

$(x=\pm 2, y=20)$ $x=\pm 2$ has the absolute maxima value

3)



Because the graph is symmetric across the y-axis, this function is even.

5) (20 pts) Consider the function $f(x) = x^3 - 3x - 3$.

- (5 pts) Find all extrema and points of inflection, giving both the x and y values.
- (5 pts) Is this function odd, even or neither? Sketch a graph of this function.
- (10 pts) Use Newton's Method to approximate the value of the x-intercept. Start with $x^{(0)} = 2$ and perform two iterations.

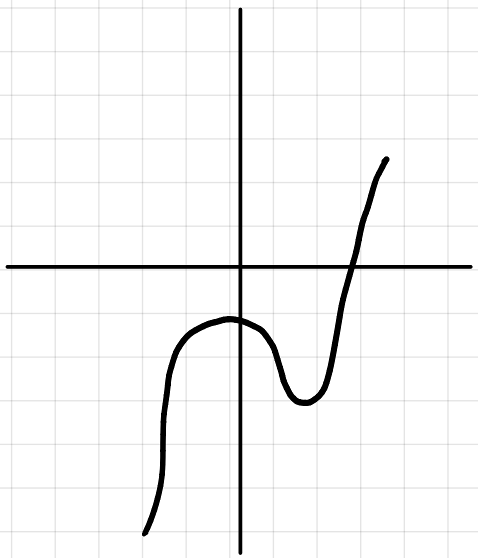
1) $f'(x) = 3x^2 - 3$ $\Rightarrow$ $f'(x) = 0 = 3x^2 - 3 \Rightarrow$ $3x^2 = 3 \Rightarrow$ $x = \pm 1$ $f(1) = -5$, $f(-1) = -1$

$f''(x) = 6x$

at point $(x=1, y=-5)$ $f''(1) = 6(1) = (+)$ absolute minimum

$(x=-1, y=-1)$ $f''(-1) = 6(-1) = (-)$ absolute maximum

2)



Checking if odd:

property: $-f(x) = f(-x)$ $x=1$

$-f(1) = f(-1)$ ? $\Rightarrow$ $-(-5) \neq -1$

Check if even:

no symmetric across y-axis

This function is neither