Homework # 3

1) (40 pts) For the IVP

$$\frac{\partial y}{\partial t} = -\alpha y \text{ with } y(0) = 1 \text{ and } \alpha > 0 \text{ and } 0 \le t \le T$$

- Calculate the exact solution ✓
- Derive the update equation using **Forward Euler time discretization**. ✓
- Derive the stability condition for the Forward Euler time discretization of $\partial y/\partial t$ for a general α and Δt. ✓
- Choose an α and a range of Δt's that includes both stable and unstable values. Write a python/Matlab code to find the numerical solution using Forward Euler time discretization. The smallest Δt value should converge to the exact solution. The largest value should be unstable. Use your stability condition to guide you here!
- Plot your numerical to exact solution for each Δt.

1) Exact solution

$y(0)=1 \quad \alpha>0 \quad 0\le t\le T$

$$\frac{dy}{dt} = -\alpha y \Rightarrow \int \frac{dy}{y} = \int -\alpha\, dt \Rightarrow e^{\ln(y)} = e^{-\alpha t + c}$$

$$\Rightarrow y = e^{-\alpha t + c} \Rightarrow y = e^{-\alpha t} e^{c} \Rightarrow e^{c} = C_1 \Rightarrow C_1 e^{-\alpha t}$$

$\Rightarrow$ Apply Initial conditions $y(0)=1 \Rightarrow 1 = C_1 e^{-\alpha(0)} \Rightarrow C_1 = 1$

$\Rightarrow$ exact solution $\Rightarrow y = e^{-\alpha t}$

2) Derive the update equation Forward Euler

equals $\left(\frac{y^{n+1} - y^n}{\Delta t}\right) = y(t) \Rightarrow \frac{dy}{dt} = \frac{du}{dt}$

We know $u^{n+1} = u^n + \Delta t \dot{u}^n = u^n + \Delta t f(u^n, t^n) \Rightarrow \frac{u^{n+1} - u^n}{\Delta t} \approx \frac{du}{dt}$ $\Rightarrow$ from the eq $\frac{dy}{dt} = -\alpha y^n$

$$\Rightarrow y^{n+1} - y^n = -\Delta t \alpha y^n \Rightarrow y^{n+1} = y^n - \Delta t \alpha y^n \Rightarrow y^{n+1} = y^n(1 - \Delta t \alpha)$$

3) $y^{n+1} = y^n(1 - \alpha \Delta t)$ $\Rightarrow$ The IVP cannot exceed 1

$1 = |1 - \alpha\Delta t| \Rightarrow$ upper: $1 \ge 1 - \alpha\Delta t \Rightarrow 0 \ge -\alpha\Delta t = \Delta t$ ✓ passes since $-\alpha\Delta t$ is always less than 1

lower: $1 \ge -(1 - \alpha\Delta t) \Rightarrow 2 \ge \alpha\Delta t \Rightarrow \frac{2}{\alpha} \ge \Delta t \rightarrow$ this is the Stability condition

2) (30 pts) For the same IVP

- Derive the update equation using **Backward Euler time discretization**.
- Write a python/Matlab code to find the numerical solution using **Backward Euler time discretization** for the same α and range of Δt's as your Forward Euler discretization.
- Plot your numerical to exact solution for each Δt. You should see stability where F. Euler was unstable!

1) Backward Euler $y^{n+1} = y^n + \Delta t \, \dot{y}^{n+1} \Rightarrow \dfrac{y^{n+1} - y^n}{\Delta t} = \dfrac{dy}{dt}^{n+1} = -\alpha y^{n+1} \Rightarrow y^{n+1} - y^n = -\Delta t \alpha y^{n+1} \Rightarrow$

$$\frac{y^{n+1} - y^n}{y^{n+1}} = -\Delta t \alpha \Rightarrow 1 - \frac{y^n}{y^{n+1}} \Rightarrow -1 - \Delta t \alpha = -\frac{y^n}{y^{n+1}} \Rightarrow y^{n+1}(1 + \Delta t \alpha) = y^n \Rightarrow y^{n+1} = \frac{y^n}{1 + \Delta t \alpha}$$

3) (30 pts) For the same IVP

- Derive the update equation using **Trapezoid Method** for time discretization.
- Write a python/Matlab code to find the numerical solution using the Trapezoid Method for the same α and range of Δt's as your Forward Euler discretization.
- Plot your numerical to exact solution for each Δt. You should see second order convergence!

1) $y^{n+1} = y^n + \frac{\Delta t}{2}\left(\dot{y}^n + \dot{y}^{n+1}\right) \Rightarrow \frac{2}{\Delta t}\left(y^{n+1} - y^n\right) = \left(\frac{dy}{dt}^n + \frac{dy}{dt}^{n+1}\right) = \left(-\alpha y^n - \alpha y^{n+1}\right) \Rightarrow -\alpha\left(y^n + y^{n+1}\right)$

$\Rightarrow y^{n+1} - y^n = \frac{-\alpha\Delta t}{2}\left(y^n + y^{n+1}\right) \Rightarrow y^{n+1} + \frac{\alpha\Delta t}{2} y^{n+1} = y^n - \frac{\alpha\Delta t}{2} y^n \Rightarrow y^{n+1}\left(1 + \frac{\alpha\Delta t}{2}\right) = y^n\left(1 - \frac{\alpha\Delta t}{2}\right)$

$$= y^{n+1} = y^n\left(\frac{1 - \frac{\alpha\Delta t}{2}}{1 + \frac{\alpha\Delta t}{2}}\right)$$