

Homework Assignments COE 352

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Homework # 5

1) (20 pts) Use a uniform four-element mesh on $[0, 1]$ and piecewise Lagrangian linear finite element basis functions to construct a finite element interpolant, f_h of the function $f(x) = \sin(\pi x)$, that is, set the values $f_h(x_i) = \sin(\pi x_i)$, $i = 1, 2, 3, 4, 5$, and plot the function f and f_h .

2) Consider the boundary value problem

$$\begin{aligned} -y'' &= x \\ y(0) &= y(1) = 0 \end{aligned}$$

- (10 pts) Find the exact solution of the problem.
- (10 pts) Derive the weak form of the problem.
- (20 pts) For the Galerkin approximation of the problem, let $N = 3$ and choose the basis function $\phi_i = \sin(i\pi x)$, $i = 1, 2, 3$. Calculate the stiffness matrix, K_{ij} and the load vector f_i . Solve for the coefficients and construct the approximate solution y_h . Plot the exact and approximate solutions.
- (40 pts) Using a uniform mesh with mesh size $h = 0.25$, compute the Galerkin piece-wise linear finite element approximation by hand. Plot the exact and approximate solutions.

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- 1) (20 pts) Use a uniform four-element mesh on $[0, 1]$ and piecewise Lagrangian linear finite element basis functions to construct a finite element interpolant, f_h of the function $f(x) = \sin(\pi x)$, that is, set the values $f_h(x_i) = \sin(\pi x_i)$, $i = 1, 2, 3, 4, 5$, and plot the function f and f_h .

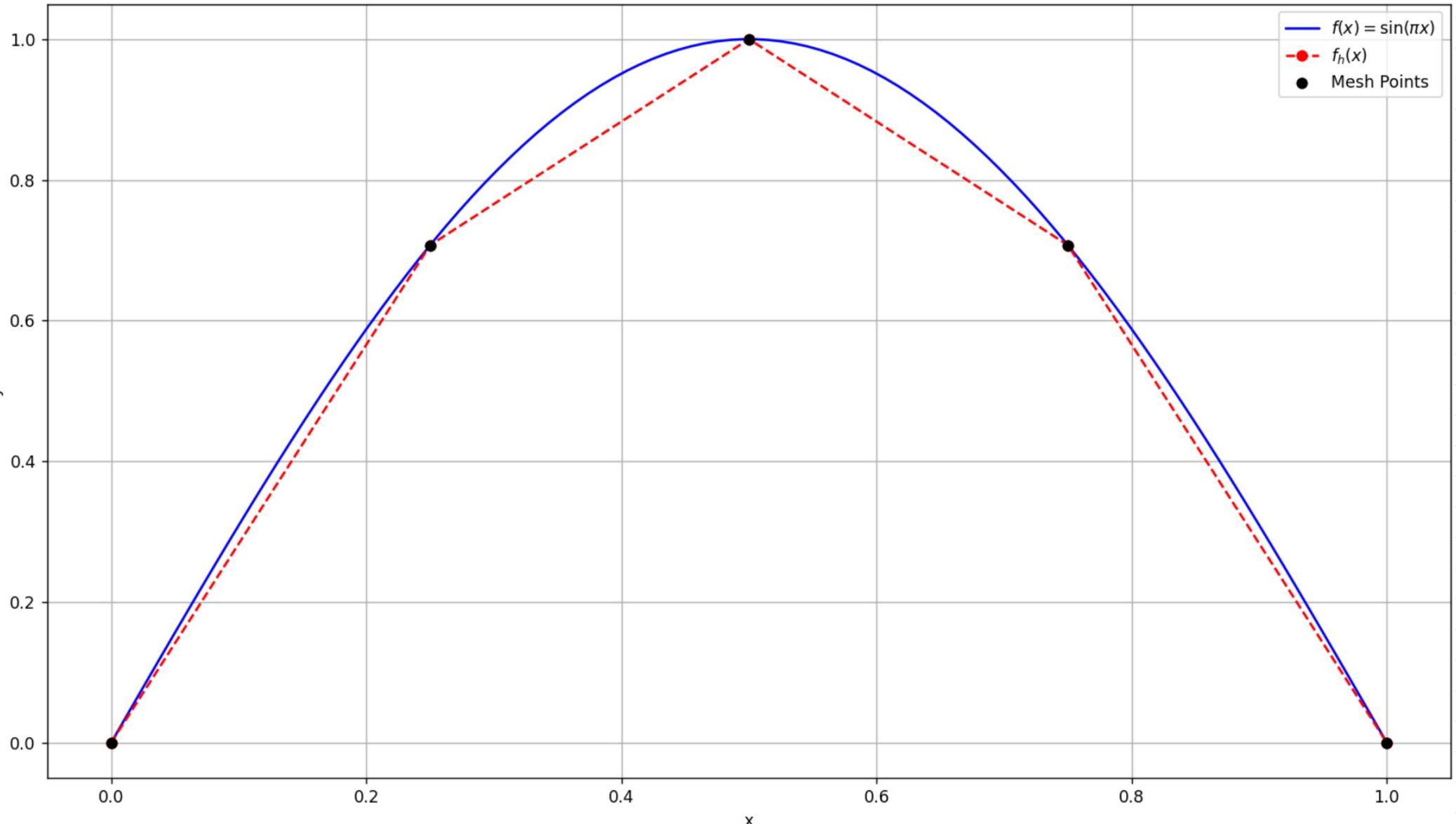
divide boundary $0 \leq x \leq 1$ into 5 nodes

x_0	0	y_0	$\sin(\pi(0)) = 0$
x_1	.25	y_1	$\sin(\pi(.25)) = \frac{\sqrt{2}}{2}$
x_2	.50	y_2	$\sin(\pi(.5)) = 1$
x_3	.75	y_3	$\sin(\pi(.75)) = \frac{\sqrt{2}}{2}$
x_4	1	y_4	$\sin(\pi(1)) = 0$

$$f_h = \int_0^1 f(x) \phi_i(x) dx = \sum_{i=0}^4 f(x_i) \phi_i(x)$$

basis functions: $\phi_i = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} \leq x \leq x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & x_i \leq x \leq x_{i+1} \\ 0 & \text{Otherwise} \end{cases}$

Function $f(x)$ and Finite Element Interpolant $f_h(x)$ Problem 1



2) Consider the boundary value problem

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- (10 pts) Find the exact solution of the problem.
- (10 pts) Derive the weak form of the problem.
- (20 pts) For the Galerkin approximation of the problem, let $N = 3$ and choose the basis function $\phi_i = \sin(i\pi x)$, $i = 1, 2, 3$. Calculate the stiffness matrix, K_{ij} and the load vector f_i . Solve for the coefficients and construct the approximate solution y_h . Plot the exact and approximate solutions.
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1) $-y'' = x \Rightarrow y'' = -x \Rightarrow \int y'' dy = \int x dx \Rightarrow y' = -\frac{x^2}{2} + C_1 \Rightarrow \int y' dy = -\int \frac{x^2}{2} + C_1 dx$

$$y = -\frac{x^3}{6} + C_1 x + C_2$$

$$y(1) = 0 = -\frac{1}{6} + C_1 + C_2 \Rightarrow -\frac{1}{6} + C_1 + C_2 = 0 \Rightarrow C_2 = \frac{1}{6}$$

$$y(0) = 0 = -\frac{0}{6} + C_1(0) + C_2 \Rightarrow C_2 = 0$$

$$y = -\frac{x^3}{6} + \frac{1}{6}x \Rightarrow -\frac{x^3}{6} + \frac{x}{6}$$

2) weak form of the solution

$$-y'' = x = f(x) = u'' \quad V(x) \text{ is a best function}$$

$UV - VdU$

$$\begin{aligned} dV = y'' &= u'' \\ V = y' &= u' \\ du &= v' dx \end{aligned}$$

$$\Rightarrow \int_0^1 u''(x)V(x)dx \stackrel{\text{I.B.P}}{=} \cancel{u'V|_0^1} + \int_0^1 u'v' dx = \boxed{\int_0^1 f(x)V(x)dx} = \boxed{\int_0^1 -y''V(x)dx = \int_0^1 xV(x)dx}$$

$$3) \quad U(x) = \sum_{i=1}^N U_i \sin(j\pi x) \Rightarrow \frac{d}{dx} \sin(j\pi x) = j\pi \cos(j\pi x) = \phi_i'$$

$$K_{ij} = \int_0^1 f(x) \phi_i'(x) \phi_j'(x) dx, \quad f = \int_0^1 f(x) \phi_i(x) dx = \int_0^1 x \phi_i(x) dx$$

$$K_{ij} = \int_0^1 \phi_j'(x) \psi_i'(x) dx \Rightarrow \phi = \psi \Rightarrow ij\pi^2 \int_0^1 \cos(j\pi x) \cos(i\pi x)$$

$$K_{jj} = K_{ii} = \int_0^1 \phi_i'(x) \phi_i'(x) dx$$

$$K_{11} = 1^2 \pi^2 \int_0^1 \cos(\pi x) \cos(\pi x) dx = \frac{\pi^2}{2}$$

$$K_{22} = (2)(2)\pi^2 \int_0^1 \cos(2\pi x) \cos(2\pi x) dx = 2\pi^2$$

$$K_{12} = K_{21} = (1)(2)\pi^2 \int_0^1 \cos(\pi x) \cos(2\pi x) dx = 0$$

$$K_{33} = (3)(3)\pi^2 \int_0^1 \cos(3\pi x) \cos(3\pi x) dx = \frac{9}{2}\pi^2$$

$$K_{13} = K_{31} = (1)(3)\pi^2 \int_0^1 \cos(\pi x) \cos(3\pi x) dx = 0$$

$$K = \begin{bmatrix} \frac{\pi^2}{2} & 0 & 0 \\ 0 & 2\pi^2 & 0 \\ 0 & 0 & \frac{9}{2}\pi^2 \end{bmatrix}$$

force vector $f = \int f(x) \phi(x) dx$

$$f_1 = \int_0^1 x \sin(\pi x) dx = 1/\pi$$

$$f_2 = \int_0^1 x \sin(2\pi x) dx = -1/2\pi$$

$$f_3 = \int_0^1 x \sin(3\pi x) dx = 1/3\pi$$

$$\vec{f} = \begin{bmatrix} 1/\pi \\ -1/2\pi \\ 1/3\pi \end{bmatrix}$$

$$KU = F$$

$$\begin{bmatrix} \frac{\pi^2}{2} & 0 & 0 \\ 0 & 2\pi^2 & 0 \\ 0 & 0 & \frac{9}{2}\pi^2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1/\pi \\ -1/2\pi \\ 1/3\pi \end{bmatrix} \Rightarrow$$

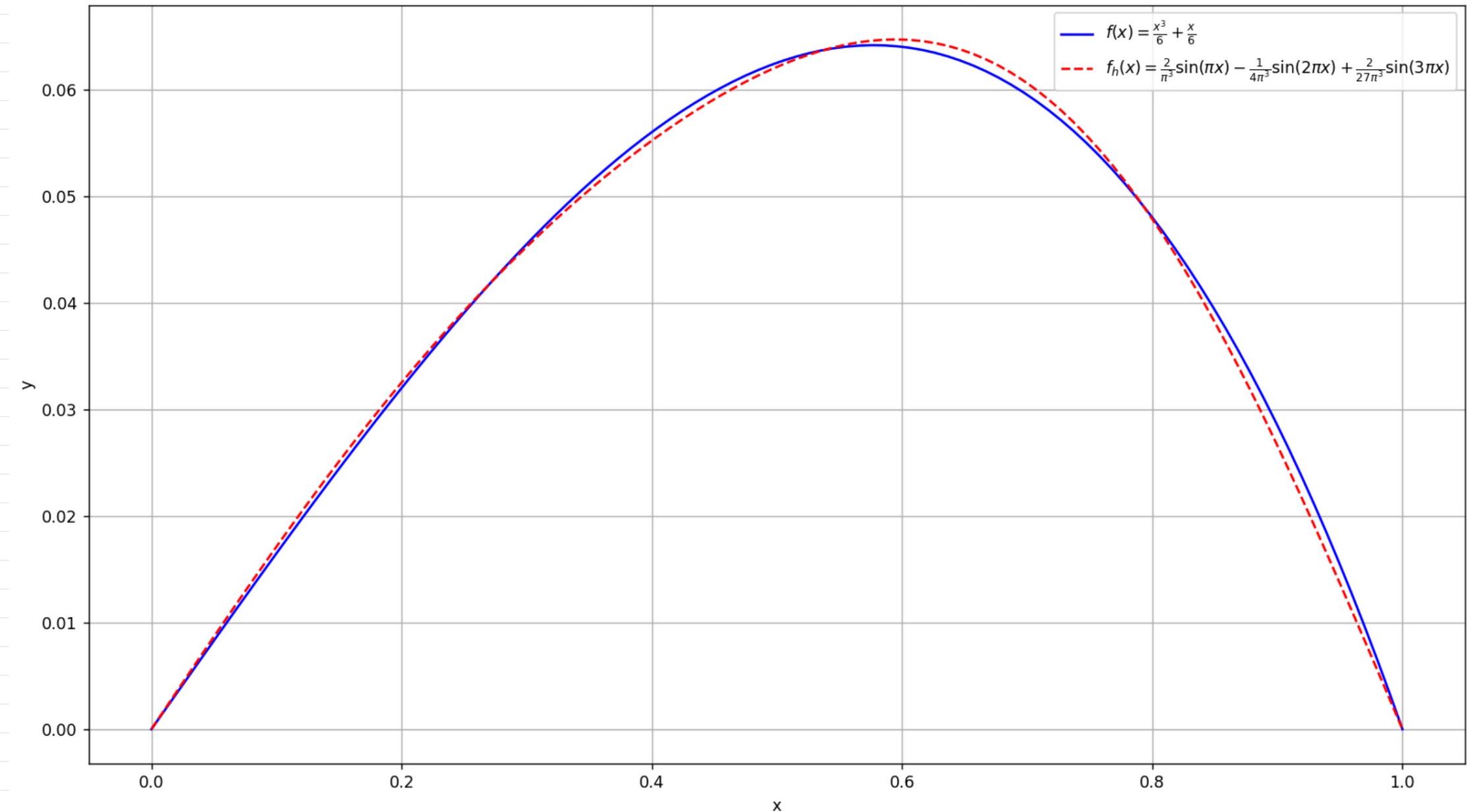
$$U_1 = \frac{2}{\pi^3}$$

$$U_2 = -\frac{1}{4\pi^3}$$

$$U_3 = \frac{2}{27\pi^3}$$

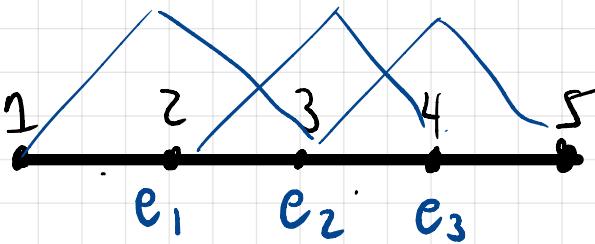
$$y_n(x) = \frac{2}{\pi^3} \sin(\pi x) - \frac{1}{4\pi^3} \sin(2\pi x) + \frac{2}{27\pi^3} \sin(3\pi x)$$

Function $f(x)$ and Finite Element Interpolant $f_h(x)$ Part 2.3



4) basis functions

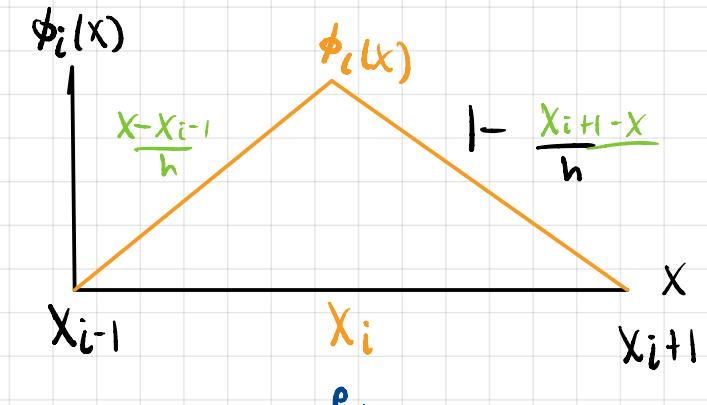
$\phi_i(x)$ can be a piecewise linear



Each triangle is an element

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h} & x_{i-1} \leq x \leq x_i \\ \frac{x_{i+1} - x}{h} & x_i \leq x \leq x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_j'(x) = \begin{cases} \frac{1}{h} & x_{j-1} \leq x \leq x_j \\ -\frac{1}{h} & x_j \leq x \leq x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$



Mesh =

$$x_0 = 0$$

$$x_1 = 0.25$$

$$x_2 = 0.5$$

$$x_3 = 0.75$$

$$x_4 = 1$$

$$\phi_1(x) = \begin{cases} \frac{x - 0}{0.25} & 0 \leq x \leq 0.25 = 4x \\ \frac{0.5 - x}{0.25} & 0.25 \leq x \leq 0.5 = 2 - 4x \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_2(x) = \begin{cases} \frac{x - 0.25}{0.25} & 0.25 \leq x \leq 0.5 = 4x - 1 \\ \frac{0.75 - x}{0.25} & 0.5 \leq x \leq 0.75 = 3 - 4x \\ 0 & \text{otherwise} = 0 \end{cases}$$

$$\phi_3(x) = \begin{cases} \frac{x-0.5}{0.25} & 0.5 \leq x \leq 0.75 \\ \frac{1-x}{0.25} & 0.75 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 4x-2 & 0.5 \leq x \leq 0.75 \\ 4-4x & 0.75 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_i^1 = \begin{cases} 4 & 0 \leq x \leq 0.25, 0.25 \leq x \leq 0.5, 0.5 \leq x \leq 0.75 \\ -4 & 0.25 \leq x \leq 0.5, 0.5 \leq x \leq 0.75, 0.75 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Create K Matrix

We know $K_{i,j} = \int \phi_i^1 \phi_j^1 dx$

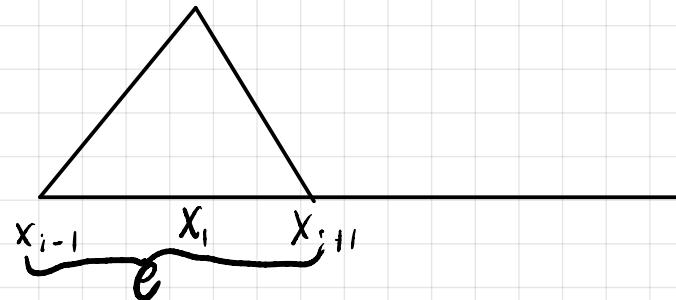
$$K_{ii} = K_{jj} = \sum \int_0^{0.25} 4(4) + \int_0^{0.25} (-4)(-4) dx = 16(x|_0^{0.25}) + 16(x|_0^{0.25}) = 8$$

$\textcolor{blue}{K_{ii}}$ $\textcolor{blue}{K_{jj}}$

$$K_{ij} = \int_0^{0.25} (4)(-4) dx \Rightarrow -16 x|_0^{0.25} = -4$$

$$K = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$$

$$y''=x=f(x)$$



$$\text{Force Vector } F = \int_a^b f(x) \phi_i dx$$

$$e_1 F_1 = \int_0^{0.5} x \phi_1(x) dx = \int_0^{0.25} x(4x) dx + \int_{0.25}^{0.5} x(2-4x) dx = \frac{1}{16}$$

$$e_2 F_2 = \int_{0.25}^{0.75} x \phi_2(x) dx = \int_{0.25}^{0.5} x(4x-1) dx + \int_{0.5}^{0.75} x(3-4x) dx = \frac{1}{8}$$

$$e_3 F_3 = \int_{0.5}^{0.75} x \phi_3(x) dx = \int_{0.5}^{0.75} x(4x-2) dx + \int_{0.75}^{1} x(4-4x) dx = \frac{3}{16}$$

$$\text{Force Vector} \Rightarrow \vec{F} = \begin{bmatrix} 1/16 \\ 1/8 \\ 3/16 \end{bmatrix}$$

$$\text{We know: } K \vec{U} = \vec{F} \Rightarrow \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1/16 \\ 1/8 \\ 3/16 \end{bmatrix}$$

SOLVE USING MATLAB

$$U_1 = \frac{5}{128} \quad U_2 = \frac{1}{16} \quad U_3 = \frac{7}{128}$$

We can write the weights & basis functions as

$$Y_n = \frac{5}{128} \phi_1 + \frac{1}{16} \phi_2 + \frac{7}{128} \phi_3$$

Function $f(x)$ and Finite Element Interpolant $f_h(x)$ Part 2.4

