# Homework # 5

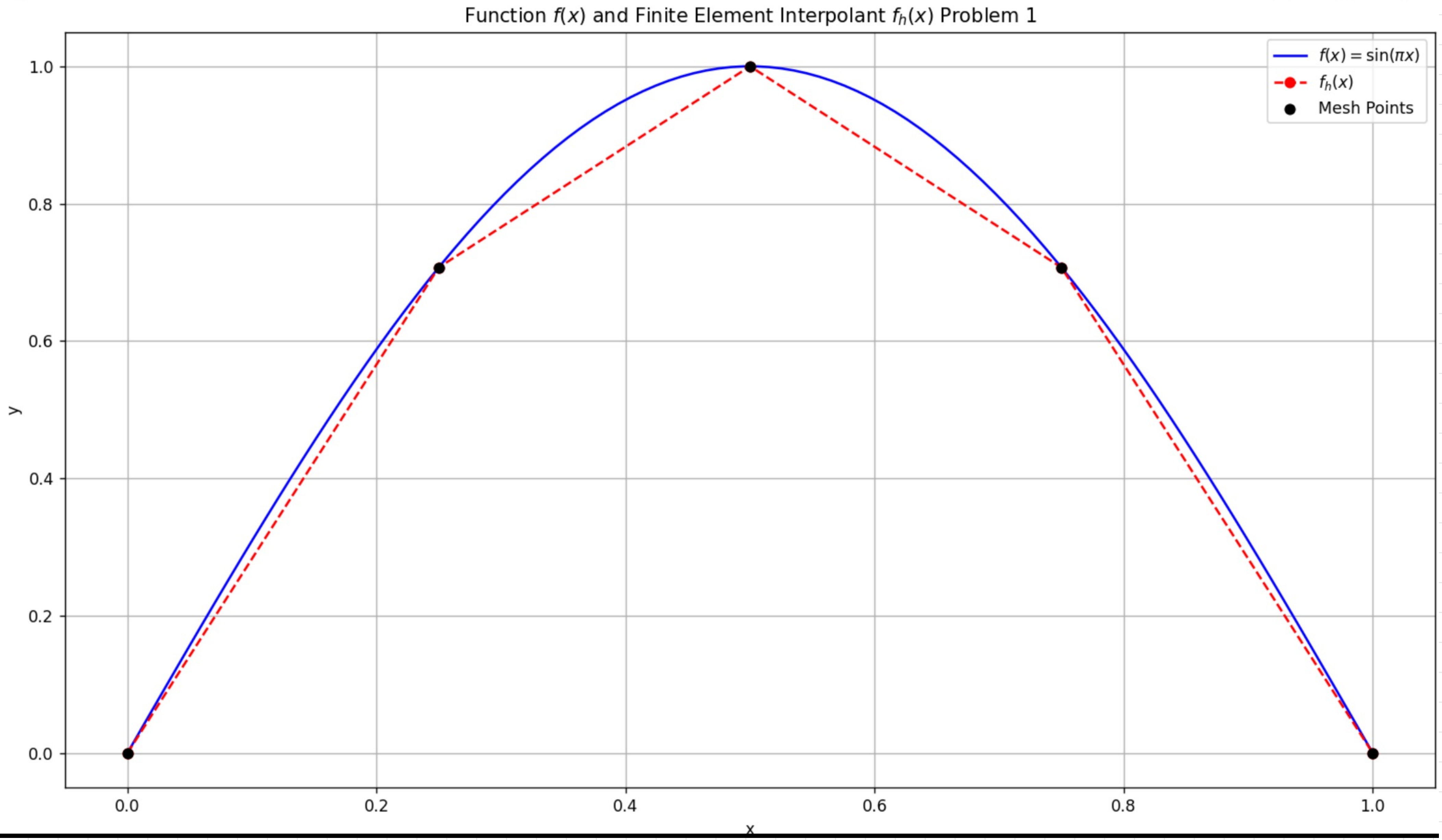
1) (20 pts) Use a uniform four-element mesh on $[0, 1]$ and piecewise Lagrangian linear finite element basis functions to construct a finite element interpolant, $f_h$ of the function $f(x) = sin(\pi x)$, that is, set the values $f_h(x_i) = sin(\pi x_i), i = 1, 2, 3, 4, 5,$ and plot the function $f$ and $f_h$.

divide boundary $0 \le x \le 1$ into 5 nodes

| | | | |
|---|---|---|---|
| $x_0$ | $0$ | $y_0$ | $\sin(\pi(0)) = 0$ |
| $x_1$ | $.25$ | $y_1$ | $\sin(\pi(.25)) = \sqrt{2}/2$ |
| $x_2$ | $.50$ | $y_2$ | $\sin(\pi(.5)) = 1$ |
| $x_3$ | $.75$ | $y_3$ | $\sin(\pi(.75)) = \sqrt{2}/2$ |
| $x_4$ | $1$ | $y_4$ | $\sin(\pi(1)) = 0$ |

$$f_h = \int_0^1 f(x)\,\phi_i(x)\,dx = \sum_{i=0}^{4} f(x_i)\,\phi_i(x)$$

basis functions: $\phi_i = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x_i \le x \le x_{i+1} \\ 0 & \text{Otherwise} \end{cases}$


Function $f(x)$ and Finite Element Interpolant $f_h(x)$ Problem 1
$f(x) = \sin(\pi x)$
$f_h(x)$
Mesh Points
1.0
0.8
0.6
0.4
0.2
0.0
y
0.0
0.2
0.4
0.6
0.8
1.0
x

2) Consider the boundary value problem

$$-y'' = x$$
$$y(0) = y(1) = 0$$

- (10 pts) Find the exact solution of the problem.
- (10 pts) Derive the weak form of the problem.
- (20 pts) ) For the Galerkin approximation of the problem, let $N = 3$ and choose the basis function $\phi_i = sin(i\pi x), i = 1, 2, 3$. Calculate the stiffness matrix, $K_{ij}$ and the load vector $f_i$. Solve for the coefficients and construct the approximate solution $y_h$. Plot the exact and approximate solutions.
- (40 pts) Using a uniform mesh with mesh size $h = 0.25$, compute the Galerkin piece-wise linear finite element approximation by hand. Plot the exact and approximate solutions.

1) $-y'' = x \Rightarrow \quad y'' = -x \quad \Rightarrow \int y''\,dy = \int -x\,dx \Rightarrow \quad y' = -\frac{x^2}{2} + C_1 \quad \Rightarrow \int y'\,dy = -\int \frac{x^2}{2} + C_1\,dx$

$$y = -\frac{x^3}{6} + C_1 x + C_2$$

$$y(1) = 0 = -\frac{x^3}{6} + C_1 x + C_2 \Rightarrow -\frac{1}{6} + C_1 + C_2 \Rightarrow C_2 = 0 \Rightarrow C_1 = \frac{1}{6}$$

$$y(0) = 0 = -\frac{0}{6} + C_1(0) + C_2 \Rightarrow \quad C_2 = 0$$

$$y = -\frac{x^3}{6} + \frac{(1)}{6}x \Rightarrow \quad -\frac{x^3}{6} + \frac{x}{6}$$

2) Weak form of the solution

$uv - \int v\,du$ $\quad dv = y'' = u''$ $\quad u = V(x)$

$-y'' = x = f(x) = u''$ V(x) is a test function $\quad V = y' = u'$ $\quad du = v'\,dx$

$$\Rightarrow \int_0^1 u''(x)V(x)\,dx \overset{I.B.P}{=} \cancel{u'v\Big|_0^1}^{\,0} + \int_0^1 u'V'\,dx = \int_0^1 f(x)V(x)\,dx = \int_0^1 -y''v(x)\,dx = \int_0^1 xV(x)\,dx$$

3) $$u(x) = \sum_{i=1}^{N} u_i \sin(j\pi x) \Rightarrow \frac{d}{dx}\sin(j\pi x) = j\pi\cos(j\pi x) = \phi_i'$$

$$K_{ij} = \int_0^1 c(x)\,\phi_i'\,\phi_j'\,dx \quad , \quad f = \int_0^1 f(x)\,\phi_i(x)\,dx = \int_0^1 x\,\phi_i(x)\,dx$$

$$K_{ij} = \int_0^1 \phi_j'(x)\,\psi_i'(x)\,dx \Rightarrow \phi = \psi \Rightarrow ij\pi^2\int_0^1 \cos(j\pi x)\cos(i\pi x)$$

$$K_{jj} = K_{ii} = \int_0^1 \phi_i'(x)\,\phi_i'(x)\,dx$$

$$K_{11} = 1^2\pi^2\int_0^1 \cos(\pi x)\cos(\pi x)\,dx = \frac{\pi^2}{2}$$

$$K_{21} = K_{12} = (1)(2)\pi^2\int_0^1 \cos(\pi x)\cos(2\pi x)\,dx = 0$$

$$K_{31} = K_{13} = (1)(3)\pi^2\int_0^1 \cos(\pi x)\cos(3\pi x)\,dx = 0$$

$$K_{22} = (2)(2)\pi^2\int_0^1 \cos(2\pi x)\cos(2\pi x)\,dx = 2\pi^2$$

$$K_{33} = (3)(3)\pi^2\int_0^1 \cos(3\pi x)\cos(3\pi x)\,dx = \frac{9}{2}\pi^2$$

$$K = \begin{bmatrix} \frac{\pi^2}{2} & 0 & 0 \\ 0 & 2\pi^2 & 0 \\ 0 & 0 & \frac{9}{2}\pi^2 \end{bmatrix}$$

force vector $f = \int f(x)\,\phi(x)\,dx$

$$f_1 = \int_0^1 x \sin(\pi x)\,dx = 1/\pi$$

$$f_2 = \int_0^1 x \sin(2\pi x)\,dx = -1/2\pi$$

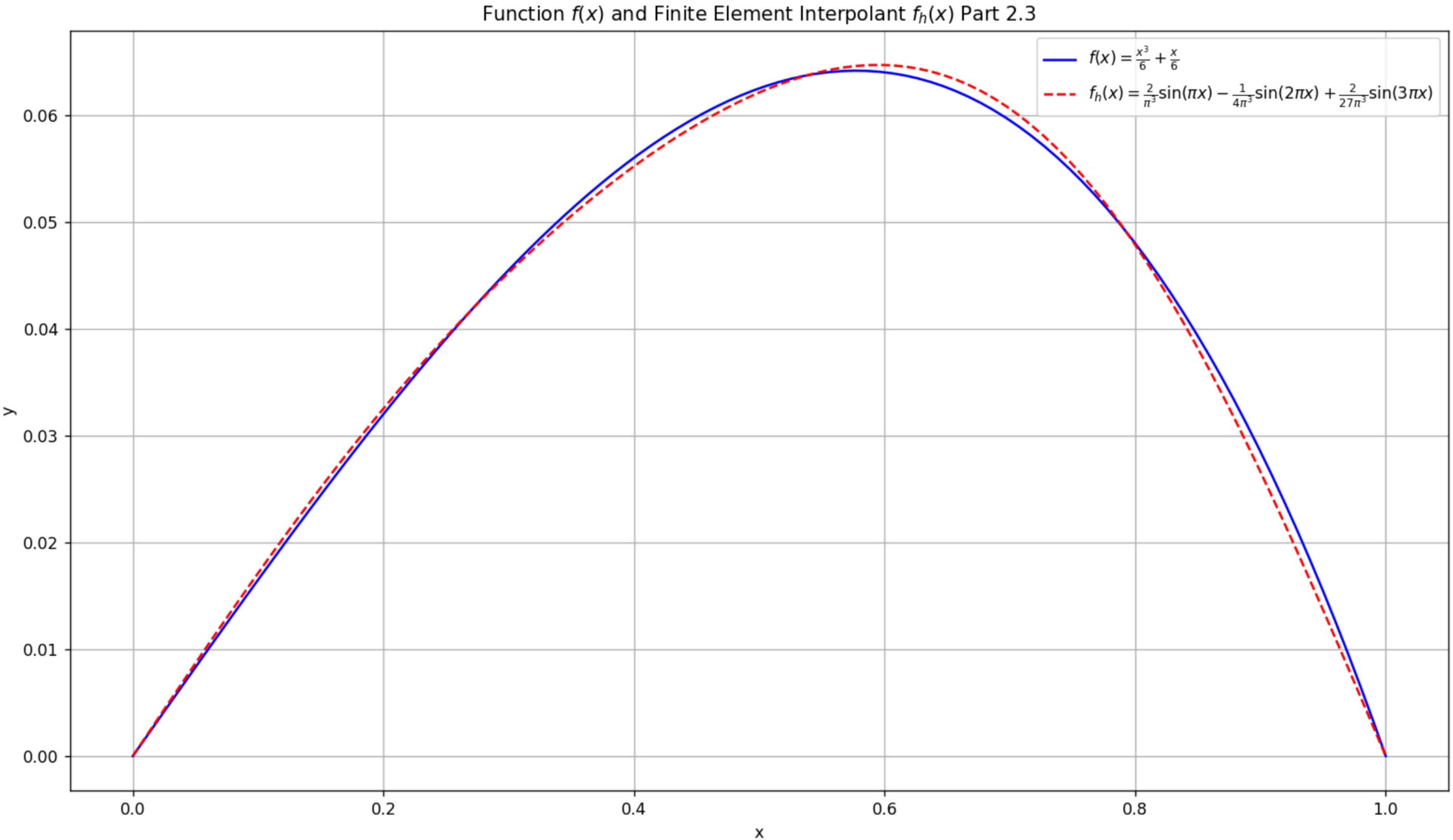
$$f_3 = \int_0^1 x \sin(3\pi x)\,dx = 1/3\pi$$

$$\vec{f} = \begin{bmatrix} 1/\pi \\ -1/2\pi \\ 1/3\pi \end{bmatrix}$$

$KU = F$

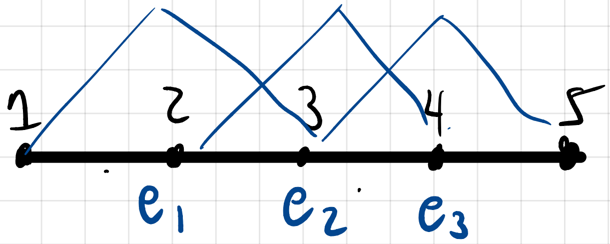
$$\begin{bmatrix} \frac{\pi^2}{2} & 0 & 0 \\ 0 & 2\pi^2 & 0 \\ 0 & 0 & \frac{9}{2}\pi^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1/\pi \\ -1/2\pi \\ 1/3\pi \end{bmatrix} \Rightarrow \begin{array}{l} u_1 = \frac{2}{\pi^3} \\ u_2 = -\frac{1}{4\pi^3} \\ u_3 = \frac{2}{27\pi^3} \end{array}$$

$$y_n(x) = \frac{2}{\pi^3}\sin(\pi x) - \frac{1}{4\pi^3}\sin(2\pi x) + \frac{2}{27\pi^3}\sin(3\pi x)$$



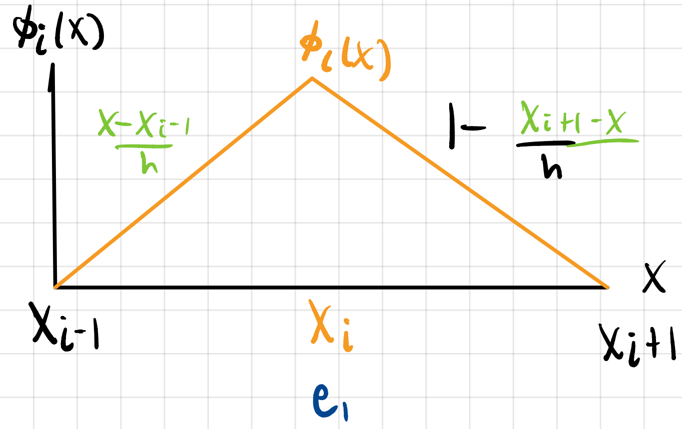
Function $f(x)$ and Finite Element Interpolant $f_h(x)$ Part 2.3
$f(x) = \frac{x^3}{6} + \frac{x}{6}$
$f_h(x) = \frac{2}{\pi^3}\sin(\pi x) - \frac{1}{4\pi^3}\sin(2\pi x) + \frac{2}{27\pi^3}\sin(3\pi x)$
0.06
0.05
0.04
0.03
0.02
0.01
0.00
y
0.0
0.2
0.4
0.6
0.8
1.0
x

4) basis functions




Each triangle is an element

$\phi_i(x)$ can be a piecewise linear




$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h} & x_{i-1} \leq x \leq x_i \\ \frac{x_{i+1} - x}{h} & x_i \leq x \leq x_{i+1} \\ 0 & \end{cases}$$

$$\phi_j'(x) = \begin{cases} 1/h & x_{j-1} \leq x \leq x_j \\ -\frac{1}{h} & x_j \leq x \leq x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

Mesh =
$x_0 = 0$
$x_1 = 0.25$
$x_2 = 0.5$
$x_3 = 0.75$
$x_4 = 1$

$$\phi_1(x) = \begin{cases} \frac{x-0}{0.25} & 0 \leq x \leq .25 & = 4x \\ \frac{0.5-x}{.25} & .25 \leq x \leq .5 & = 2-4x \\ 0 & \text{otherwise} & 0 \end{cases}$$

$$\phi_2(x) = \begin{cases} \frac{x-0.25}{0.25} & .25 \leq x \leq 0.5 & = 4x-1 \\ \frac{0.75-x}{.25} & .5 \leq x \leq .75 & = 3-4x \\ 0 & \text{otherwise} & = 0 \end{cases}$$

$$\phi_3(x) = \begin{cases} \frac{x-.5}{.25} & .5 \le x \le .75 \\ \frac{1-x}{.25} & .75 \le x \le 1 \\ 0 & \text{Otherwise} \end{cases} = \begin{matrix} 4x-2 \\ 4-4x \\ 0 \end{matrix}$$

$$\phi_i' = \begin{cases} 4 & 0 \le x \le .25,\ .25 \le x \le 0.5,\ .5 \le x \le .75 \\ -4 & .25 \le x \le .5,\ .5 \le x \le .75,\ .75 \le x \le 1 \\ 0 & \text{Otherwise} \end{cases}$$

Create K matrix

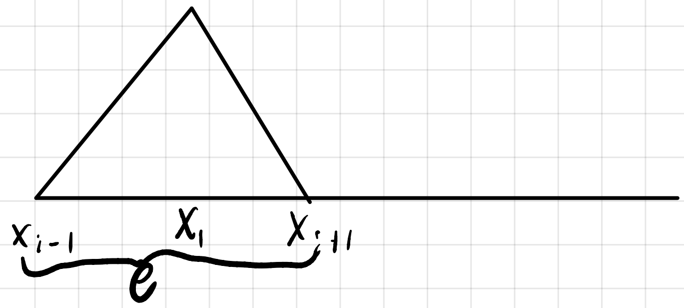
we know $K_{i,j} = \int \phi_i' \phi_j' \, dx$

$$K_{ii} = K_{jj} = \sum \underbrace{\int_0^{.25} 4(4)}_{K_{ii}} + \underbrace{\int_0^{.25} (-4)(-4)\,dx}_{K_{jj}} = 16(x|_0^{.25}) + 16(x|_0^{0.25}) = 8$$

$$K_{ij} = \int_0^{0.25} (4)(-4)\,dx \Rightarrow -16\, x|_0^{0.25} = -4$$

$$K = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$$

$y'' = x = f(x)$




Force vector $F = \int_a^b f(x)\, \phi_i \, dx$

$e_1$ $\quad F_1 = \int_0^{0.5} x\, \phi_1(x)\, dx = \int_0^{0.25} x(4x)\, dx + \int_{0.25}^{0.5} x(2-4x)\, dx = \frac{1}{16}$

$e_2$ $\quad F_2 = \int_{0.25}^{0.75} x\, \phi_2(x)\, dx = \int_{0.25}^{.5} x(4x-1)\, dx + \int_{0.5}^{0.75} x(3-4x)\, dx = \frac{1}{8}$

$e_3$ $\quad F_3 = \int_{0.5}^{1} x\, \phi_3(x)\, dx = \int_{0.5}^{0.75} x(4x-2)\, dx + \int_{0.75}^{1} x(4-4x)\, dx = \frac{3}{16}$

Force Vector $\Rightarrow$ $\vec{F} = \begin{bmatrix} 1/16 \\ 1/8 \\ 3/16 \end{bmatrix}$

We know: $K U = F \Rightarrow$

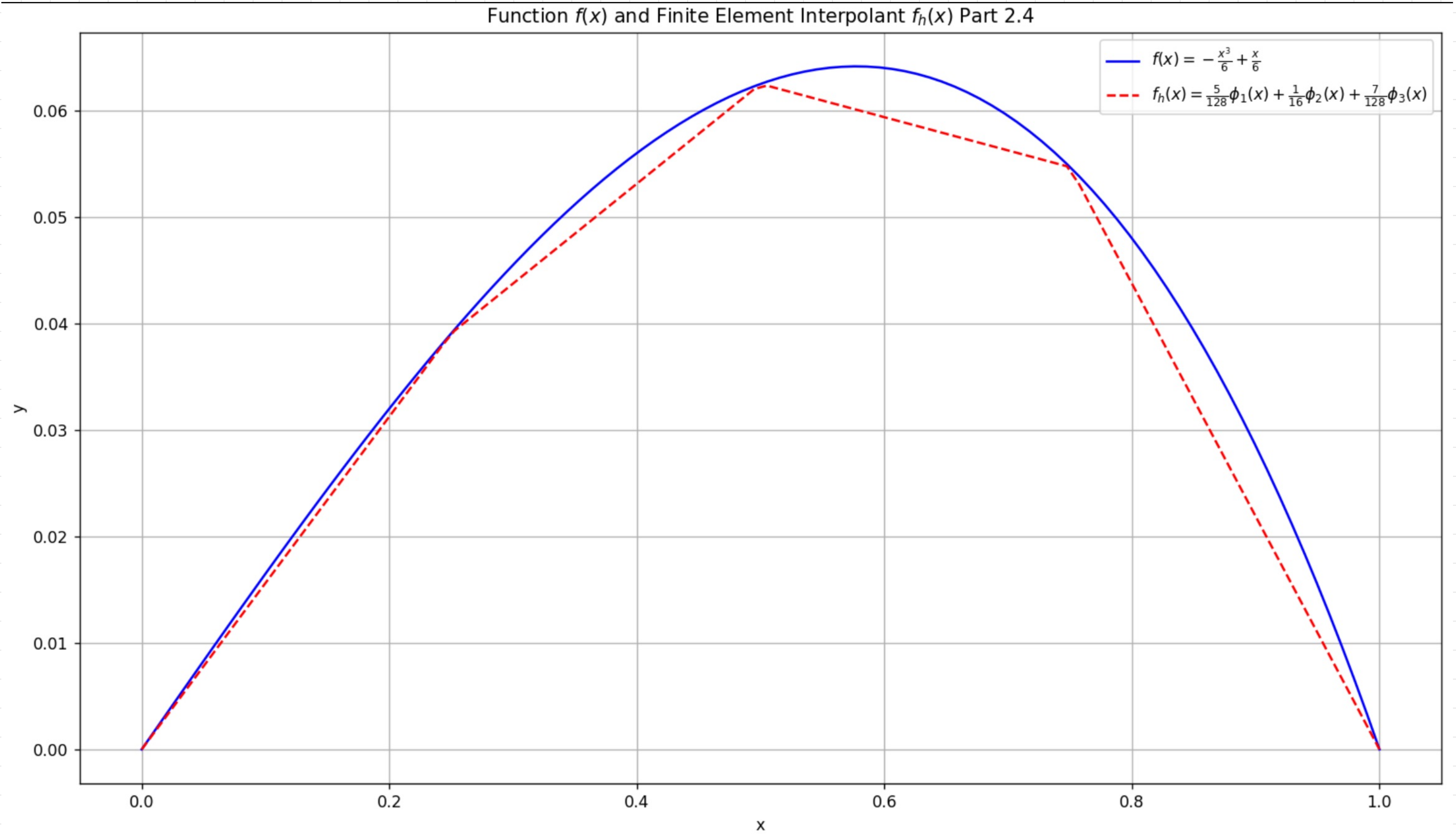
$$\begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1/16 \\ 1/8 \\ 3/16 \end{bmatrix}$$

solve using matlab

$u_1 = \frac{5}{128}$ $u_2 = \frac{1}{16}$ $u_3 = \frac{7}{128}$

we can write the weights & basis functions as

$$y_n = \frac{5}{128}\phi_1 + \frac{1}{16}\phi_2 + \frac{7}{128}\phi_3$$


Function $f(x)$ and Finite Element Interpolant $f_h(x)$ Part 2.4
$f(x) = -\frac{x^3}{6} + \frac{x}{6}$
$f_h(x) = \frac{5}{128}\phi_1(x) + \frac{1}{16}\phi_2(x) + \frac{7}{128}\phi_3(x)$
0.00
0.01
0.02
0.03
0.04
0.05
0.06
y
0.0
0.2
0.4
0.6
0.8
1.0
x