

Homework # 6

1) Computer the Jacobian matrix for the following:

- (15pts)

$$F(x, y) = \begin{pmatrix} x^2 + \sin(x) \\ x(y - 2) \\ y^2 - 3xy \end{pmatrix} \quad xy - 2x$$

- (15pts)

$$G(x, y, z) = \begin{pmatrix} x^2 - y^2 \\ 3xyz - 5 \end{pmatrix}$$

JACOBIAN MATRIX WE KNOW

$$\begin{matrix} \frac{\partial f}{\partial x} & \frac{\partial f_1}{\partial y} & \dots \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \end{matrix}$$

I) $F(x, y) = \begin{bmatrix} 2x + \cos(x) & 0 \\ y - 2 & x \\ -3y & 2y - 3x \end{bmatrix}$

Course: Introduction to Machine Learning
Assignment 5

Important NOTE: In order to get full credit, for every question, you need to provide the details of your work on how to get to a solution or the end of a proof.

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Due date: 11:00 pm, Tuesday, November 19

All submission MUST be in pdf format, except codes. All code outputs should be reported in pdf format.

1 Question 1: (40 points) (Kernel SVM with scikit-learn)

Based on the given code below, generate a new training set of 20 samples. Using scikit-learn, plot the decision boundaries for the binary classification of the data with the following approaches:

1. Polynomial kernel (use the entire dataset): You will need to produce six plots for the six unique pairs of degree and γ (γ is an input to the scikit-learn function):

- Degree = 2, 5.
- $\gamma = 1, 2, 5$.

Discuss the significance of γ .

2. Radial basic kernel:

$$K(\mathbf{x}, \mathbf{z}) = \exp\left(-\frac{\|\mathbf{x} - \mathbf{z}\|^2}{2\sigma^2}\right).$$

You will need to produce six plots for the six unique pairs of σ and number of samples:

- $\sigma = 0.2, 1.0, 2.0$
- Training data = 10 samples, 20 samples.

2 (40 points) (Linear Regression with Kernel)

Given the model:

$$y = \boldsymbol{\theta}^T \phi(x) + \epsilon.$$

1. Derive the following kernel form (including the form/equation for β_i) for the Ridge Regression approach.

$$y_{\text{test}} = \phi(x_{\text{test}})^T \boldsymbol{\theta} = \sum_{i=1}^n \beta_i \phi(x_{\text{test}})^T \phi(x^{(i)}),$$

2. Starting from the given code below, write a short code using the derived kernel form, from part 1, with a polynomial kernel function (degree = 4). Plot the predicted function together with the true function and discuss the results.
3. Now, with the same derived kernel formula, change the basis function to the Gaussian basis function:

$$\phi_j(x) = \exp\left(-\frac{(x - u_j)^2}{2\sigma^2}\right), \quad j = 0, \dots, 10$$

Find the regression solution for the following case: $u_j = [0, 0.1, 0.2, \dots, 1]$ and $s = 1$. Plot the predicted function together with the true function and discuss the results.

3 (20 points) (Kernel basis)

Let $\mathbf{x}, \mathbf{z} \in \mathbb{R}^n$, show that the Radial Basis Function (RBF; sometimes called Gaussian) kernel,

$$K(\mathbf{x}, \mathbf{z}) = \exp\left(-\frac{\|\mathbf{x} - \mathbf{z}\|_2^2}{2\sigma^2}\right),$$

is a Mercer kernel, where $\|\cdot\|_2$ denotes ℓ_2 -norm (Euclidean norm) in \mathbb{R}^n .

2) $G(x,y,z) = \begin{bmatrix} 2x & -2y & 0 \\ 3yz & 3xz & 3xy \end{bmatrix}$

2) (15pts) Compute the Jacobian of the polar coordinates transformation

$$x = r\cos\theta, y = r\sin\theta$$

$$T(r, \theta) = \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Jacobian matrix:

$$\left| \frac{d(x,y)}{d(r,\theta)} \right| = \left| \begin{array}{cc} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{array} \right| =$$

$$\left| \begin{array}{cc} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{array} \right|$$

Jacobian determinant:

$$\cos\theta(r\cos\theta) - (-r\sin\theta)(\sin\theta) \Rightarrow r\cos^2\theta + r\sin^2\theta \Rightarrow r(\cos^2\theta + \sin^2\theta) = r$$

3) (15pts) Derive the weak form of the 1D wave equation

$$\frac{\partial^2 u(x, t)}{\partial t^2} - c^2 \nabla^2 u(x, t) = f(x, t)$$

where c is a scalar valued wave speed.

We know:

$\underline{\frac{d^2 u(x, t)}{dt^2}}$ is a temporal derivative because $\underline{\frac{du(x, t)}{dt}}$ is in respect to time

$$\int V(x) \left(\underline{\frac{d^2 u(x, t)}{dt^2}} - c^2 \nabla^2 u(x, t) \right) dx = \int f(x, t) V(x) dx \Rightarrow$$

$$\int \underline{\frac{d^2 u(x, t)}{dt^2}} V(x) dx - c^2 \int \nabla^2 u(x, t) V(x) dx = \int f(x, t) V(x) dx$$

$$U = V(x) \quad V = \underline{\frac{du(x, t)}{dx}} \quad \text{IBP} \\ dU = \underline{\frac{dV(x)}{dx}} \quad dV = \underline{\frac{d^2 u(x, t)}{dx^2}}$$

$$V(x) \cdot \underline{\frac{du(x, t)}{dx}} - \int \underline{\frac{du(x, t)}{dx}} \cdot \underline{\frac{dV(x)}{dx}} dx \\ \underline{\frac{d}{dx}} = V$$

To get weak form, multiply by test function
Integrate by spatial derivative

IBP: $uv - \int v du$

$$\int \frac{d^2 U(x,t)}{dt^2} V(x) dx - C^2 \left[V(x) \cdot \nabla U(x,t) - \int \nabla \cdot U(x,t) \cdot \nabla V(x) dx \right] = \int f(x,t) V(x) dx$$

3) (20pts) Let $f(x) = e^x$ for $0 \leq x \leq 2$. Approximate $f(0.25)$ using Lagrangian linear interpolation with $x_0 = 0$ and $x_1 = 0.5$. Compare your answer to the real one.

We know that lagrangian polynomials have a $\frac{df^2}{dx^2} = 0$ @ desired points

$$f(0) = e^{(0)} = 1 \quad f(0.5) = e^{(0.5)} \approx 1.6487$$

$$f(x) = f_1 + f_2 = y_0 \frac{(x-x_1)}{(x_0-x_1)} + y_1 \frac{(x-x_0)}{(x_1-x_0)} = 1 \frac{(x-0.5)}{(0-0.5)} + 1.6487 \frac{(x-0)}{(0.5-0)}$$

$$f(x) = -2x + 1 + 1.6487(2x) = 1.2974x + 1$$

$$f(0.25) = 1.2974(0.25) + 1 = 1.32435$$

$$\frac{1.32435 - 1.28402}{1.28402} = .0314 = 3.14\%$$

$$f(0.25) = e^{(0.25)} = 1.28402$$

$$1.32435 - 1.28402 = 0.0433 = 4 \text{ units above}$$

3) (20pts) Let $P_3(x)$ by the Lagrangian interpolating polynomial for the data $(0, 0), (0.5, y), (1, 3)$ and $(2, 2)$. Find y if the coefficient of the x^3 in $P_3(x)$ is 6.

$$x_0 \quad x_1 \quad x_2$$

x_3

$$f(x) = y_0 \frac{f_1(x)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} + y_1 \frac{f_2(x)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} + y_2 \frac{f_3(x)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}$$

$$f_3(x)$$

$$+ y_3 \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}$$

~~$$f(x) = 0 \frac{(x - 0.5)(x - 1)(x - 2)}{(0 - 0.5)(0 - 1)(0 - 2)} + y \frac{(x - 0)(x - 1)(x - 2)}{(0.5 - 0)(0.5 - 1)(0.5 - 2)} + 3 \frac{(x - 0)(x - 1)(x - 2)}{(1 - 0)(1 - 0.5)(1 - 2)}$$~~

$$+ 2 \frac{(x - 0)(x - 0.5)(x - 1)}{(2 - 0)(2 - 0.5)(2 - 1)}$$

$$f(x) = \frac{1.5yx^3 - 3x^3 - 4.5yx^2 + 9.5625x^2 + 3yx - 6.5625x}{0.5625}$$

Coefficients of $x^3 = 6$, pull out x^3 solve for y

$$\frac{1.5y x^3 - 3x^3}{0.5625} = 6x^3 \Rightarrow \frac{1.5y - 3}{0.5625} = 6 \Rightarrow 1.5y - 3 = 3.375$$

$$\Rightarrow 1.5y = 6.375 \Rightarrow y = \frac{6.375}{1.5} = 4.25$$