Advanced Scientific Engineering
Homework 3

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1 Exact Solution

Given the initial value problem:

$$\frac{dy}{dt} = -\alpha y, \quad y(0) = 1, \quad \alpha > 0, \quad 0 \le t \le T$$

Step 1: Separate variables and integrate:

$$\int \frac{1}{y}\,dy = \int -\alpha\,dt$$

$$\ln(y) = -\alpha t + C$$

Exponentiating both sides:

$$y = e^{-\alpha t + C} = e^{C} e^{-\alpha t}$$

Using the initial condition $y(0) = 1$, we find that $C_1 = 1$:

$$y = e^{-\alpha t}$$

2 Derive the Update Equation Using Forward Euler

We know that for the Forward Euler method:

$$u^{n+1} = u^n + \Delta t\, f(u^n, t^n)$$

For our equation $\frac{dy}{dt} = -\alpha y$, this becomes:

$$\frac{y^{n+1} - y^n}{\Delta t} \approx -\alpha y^n$$

Rearranging to find y^{n+1}:

$$y^{n+1} = y^n(1 - \alpha \Delta t)$$

$$\boxed{y^{n+1} = y^n(1 - \alpha \Delta t)}$$

3 Stability Condition for Forward Euler

We derived that:

$$y^{n+1} = y^n(1 - \alpha\Delta t)$$

For stability, we require $|1 - \alpha\Delta t| \leq 1$.
Upper bound:

$$1 - \alpha\Delta t \leq 1 \quad \Rightarrow \quad -\alpha\Delta t \leq 0 \quad \Rightarrow \text{always passes since } -\alpha\Delta t \text{ is always negative.}$$

Lower bound:

$$1 - \alpha\Delta t \geq -1 \quad \Rightarrow 2 \geq \alpha\Delta t \quad \Rightarrow \Delta t \leq \frac{2}{\alpha}$$

Thus, the stability condition is:

$$\Delta t \leq \frac{2}{\alpha}$$

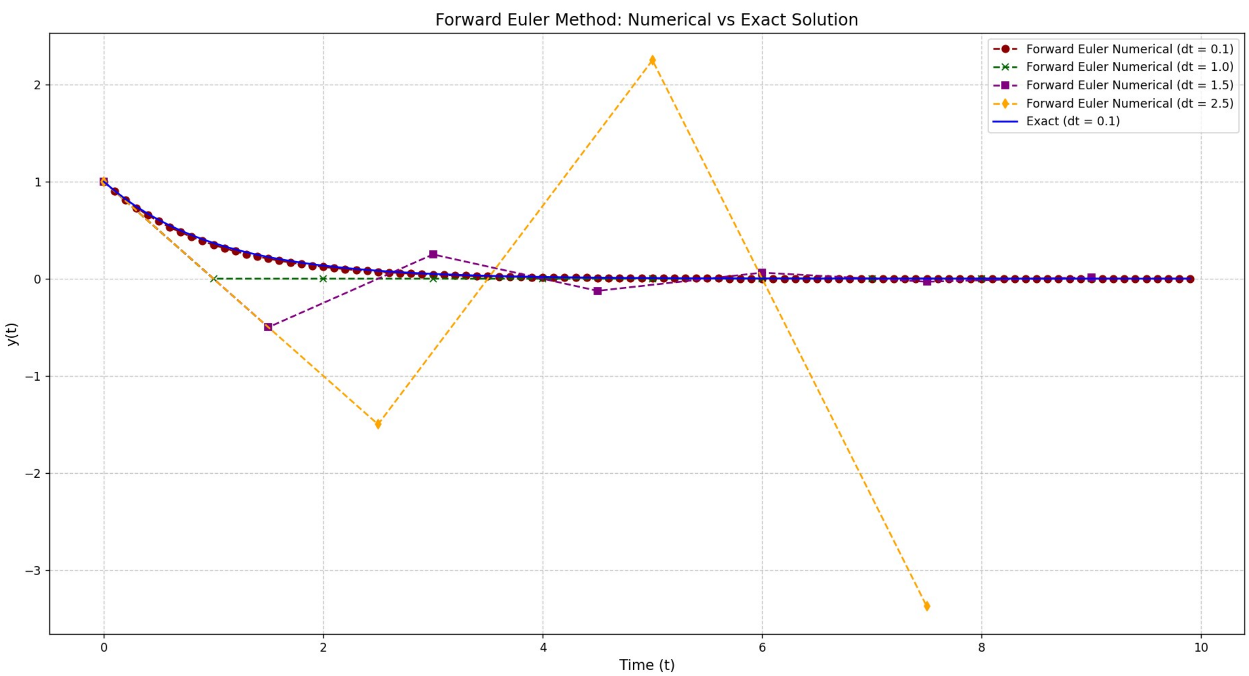

Figure 1: Forward Euler Numerical vs Exact Solution

4 Derive the Update Equation Using Backward Euler

For Backward Euler, we update using:

$$y^{n+1} = y^n + \Delta t\, f(u^{n+1})$$

For the equation $\frac{dy}{dt} = -\alpha y$:

$$\frac{y^{n+1} - y^n}{\Delta t} = -\alpha y^{n+1}$$

Rearranging:

$$y^{n+1} - y^n = -\Delta t \alpha y^{n+1}$$

$$y^{n+1}(1 + \Delta t \alpha) = y^n$$

Finally, solving for y^{n+1}:

$$y^{n+1} = \frac{y^n}{1 + \Delta t \alpha}$$

$$\boxed{y^{n+1} = \frac{y^n}{1 + \Delta t \alpha}}$$

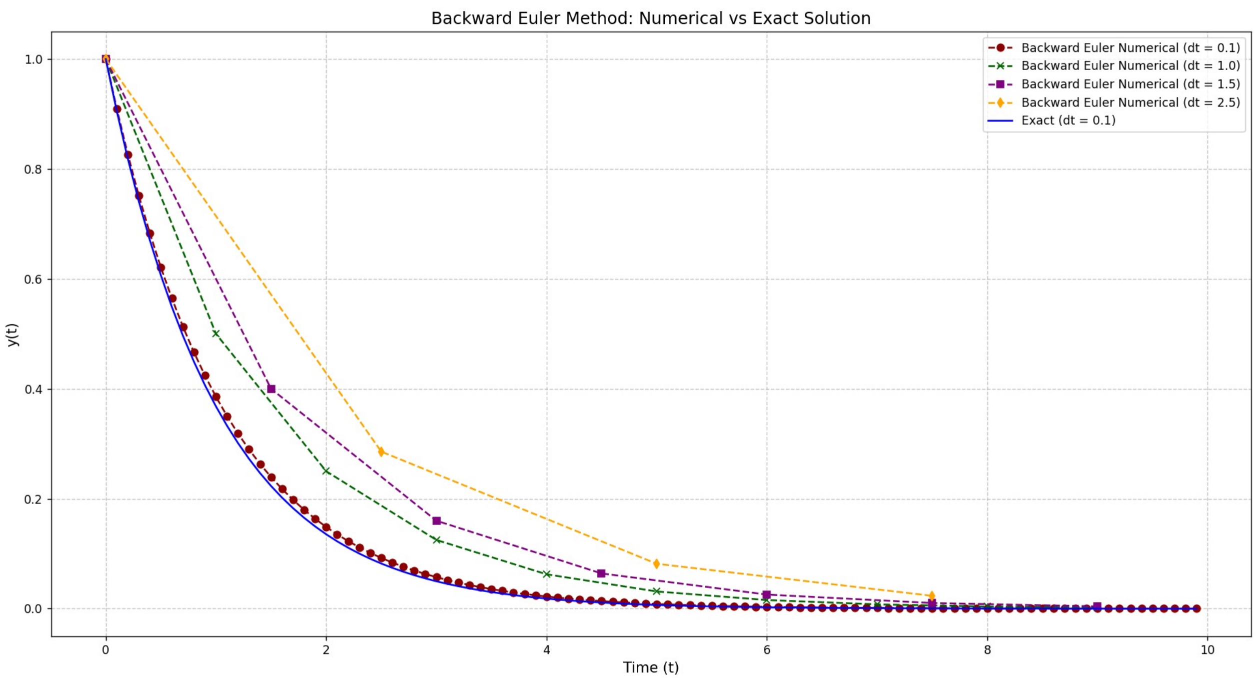

Figure 2: Backward Euler Numerical vs Exact Solution

5 Derive the Update Equation Using Trapezoidal Method

For the Trapezoidal method, we use the average of the function at the previous and next time steps:

$$y^{n+1} = y^n + \frac{\Delta t}{2} \left(f(y^n) + f(y^{n+1}) \right)$$

For the equation $\frac{dy}{dt} = -\alpha y$:

$$y^{n+1} - y^n = \frac{-\alpha \Delta t}{2} (y^n + y^{n+1})$$

Rearranging:

$$y^{n+1}(1 + \frac{\alpha \Delta t}{2}) = y^n \left(1 - \frac{\alpha \Delta t}{2} \right)$$

Finally, solving for y^{n+1}:

$$y^{n+1} = y^n \left(\frac{1 - \frac{\alpha \Delta t}{2}}{1 + \frac{\alpha \Delta t}{2}} \right)$$

$$\boxed{y^{n+1} = y^n \left(\frac{1 - \frac{\alpha \Delta t}{2}}{1 + \frac{\alpha \Delta t}{2}} \right)}$$

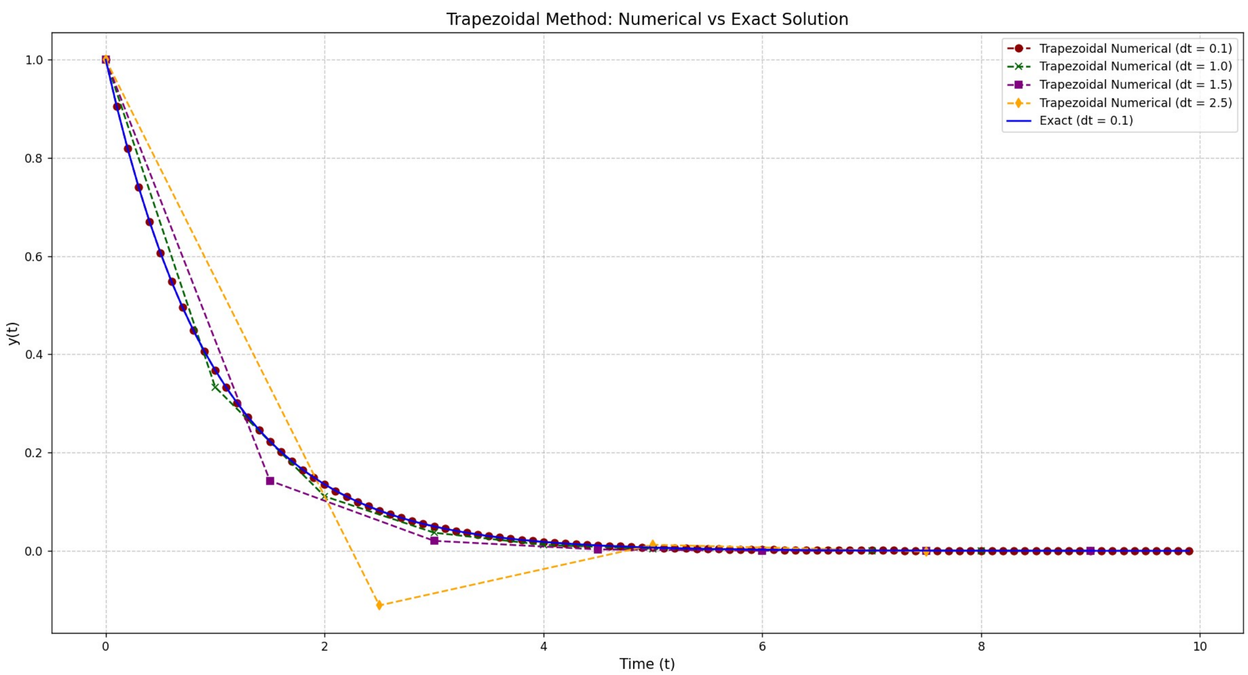

Figure 3: Trapezoidal Method Numerical vs Exact Solution