Olympiad Solutions

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This document features my personally written solutions to various olympiad problems I've encountered. It is updated periodically with new content.

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§1 IMO/IMO SL

Problem 1.0.1 (IMO 2019 P1)

Find all $f : \mathbb{Z} \to \mathbb{Z}$ such that,

$$f(2x) + 2f(y) = f(f(x + y))$$

Proof. Let $P(x, y)$ the condition generated by x, y. Then,

$$\begin{cases} P(x, 0) : f(2x) + 2f(0) = f(f(x)) \\ P(0, x) : f(0) + 2f(x) = f(f(x)) \end{cases}$$

Combining these two we obtain that,

$$f(2x) = 2f(x) - f(0)$$

Now let us consider the following,

$$P(x, 1) : f(2x) + 2f(1) = f(f(x + 1))$$

However from the previous observations we can say that,

$$f(f(x + 1)) = 2f(x + 1) + f(0)$$

Thus,

$$2f(x) + 2f(1) - f(0) = 2f(x + 1) + f(0)$$
$$f(x + 1) = f(x) + f(1) - f(0)$$

Thus, we see that $f(x) = f(0) + x \cdot C$ where C is some constant (specifically $f(1) - f(0)$) for all $x \in \mathbb{Z}$. Now let us consider,

$$P(x, -x) : f(2x) + 2f(-x) = f(f(0))$$

We can rewrite this as,

$$f(0) + 2xC + 2f(0) - 2xC = f(0) + f(0) \cdot C$$
$$\implies 2f(0) = f(0) \cdot C$$

This implies that either $f(0) = 0$ or $C = 2$. Let us consider both cases,

- If $f(0) = 0$, then, $f(x) = cx$ and it is simple to verify that then $f(x)$ must equal $2x$ or 0.
- If $C = 2$, then,

$$f(0) + 4x + 2(f(0) + 2y) = f(f(0) + 2(x + y)) = f(0) + 2f(0) + 4(x + y)$$
$$4x + 2f(0) + 4y = 2f(0) + 4x + 4y$$

Thus, we see the only solutions to this functional equation are $f(x) = 2x + c$ and $f(x) = 0$. ■

Problem 1.0.2 (IMO 2022 P2)

Find all $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$ there exists exactly one $y \in \mathbb{R}^+$ such that,

$$xf(y) + yf(x) \leq 2$$

Proof. Obviously $f(x) = \frac{1}{x}$ works (from AM-GM).

Let us say that $x \sim y$ if and only if the condition from the problem statement is true. Then obviously this is commutative, meaning that if $x \sim y$ then $y \sim x$.

Lemma 1.1 If $f(x) \leq \frac{1}{x}$, then, $x \sim x$.

This is simple to see due to,

$$2xf(x) \leq 2x \cdot \frac{1}{x} = 2$$

However, notice that if $x \sim y$, then,

$$xf(y) + yf(x) \leq 2$$

and in it must be that $f(x) > \frac{1}{x}$ and $f(y) > \frac{1}{y}$ which would imply that by AM-GM,

$$xf(y) + yf(x) = \frac{x}{y} + \frac{y}{x} \geq 2$$

Thus we obtain that for all $x \in \mathbb{R}^+$ it must be that $x \sim x$. Thus that,

$$f(x) \leq \frac{1}{x}$$

for all $x \in \mathbb{R}^+$.

Notice that if $f(a) < \frac{1}{a}$, then, if,

$$f(a) = \frac{1}{a} - \epsilon$$

then,

$$xf(a) + af(x) = x\left(\frac{1}{a} - \epsilon\right) + af(x) \leq x\left(\frac{1}{a} - \epsilon\right) + \frac{a}{x} \overset{?}{\leq} 2$$

However the last inequality for $\epsilon > 0$ has multiple solutions for x. Because,

$$x^2\left(\frac{1}{a} - \epsilon\right) + a - 2x \leq 0$$

The only way for this inequality to have a single solution for $x \in \mathbb{R}^+$ is if the equality has only one solution, thus the discriminant is zero,

$$\Delta = 4 - 4 \cdot a \cdot \left(\frac{1}{a} - \epsilon\right)$$

which is obviously never 0 for $\epsilon > 0$. Thus because for each x there is only one such y that $x \sim y$ it must mean that,

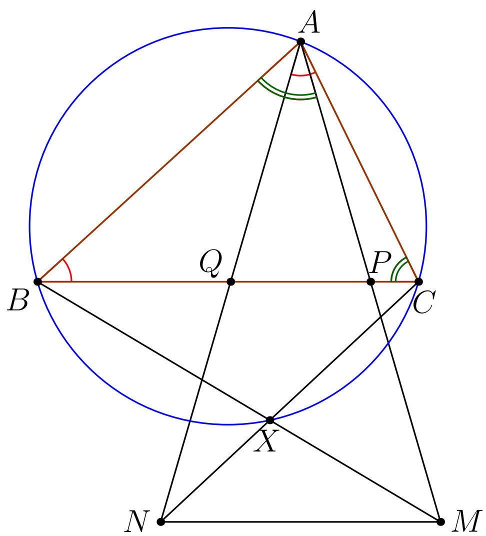
$$f(x) = \frac{1}{x}$$

for all $x \in \mathbb{R}^+$. ■

Problem 1.0.3 (IMO 2014 P4)

Let P and Q be on segment BC of an acute triangle $\triangle ABC$ such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the intersection of BM and CN is on the circumference of triangle $\triangle ABC$.

Proof. Let $\triangle ABC$ be the reference triangle in barycentric coordinates.



Then we can calculate the point P, because $\triangle ABP \sim \triangle CBA$ and $\triangle ACQ \sim \triangle BCA$. Thus,

$$\frac{BA}{BC} = \frac{BP}{BA}$$

$$\implies BP = \frac{BA^2}{BC} = \frac{c^2}{a}$$

consequently, $P = \left(0 : a - \frac{c^2}{a} : \frac{c^2}{a}\right)$, analogously we get, $Q = \left(0 : \frac{b^2}{a} : a - \frac{b^2}{a}\right)$. Now we can calculate points M and N,

$$M = \left(-a^2 : 2a^2 - 2c^2 : 2c^2\right)$$

$$N = \left(-a^2 : 2b^2 : 2a^2 - 2b^2\right)$$

Now we simply need to intersect BM and CN which is not hard to do considering that both are cevians in the triangle. It is not hard to see that the following point satisfies these conditions,

$$X = \left(-a^2 : 2b^2 : 2c^2\right)$$

Now we just need to check that this point lies on the circumcircle of $\triangle ABC$ which is given by,

$$a^2yz + b^2xz + c^2xy = 0$$

Thus,

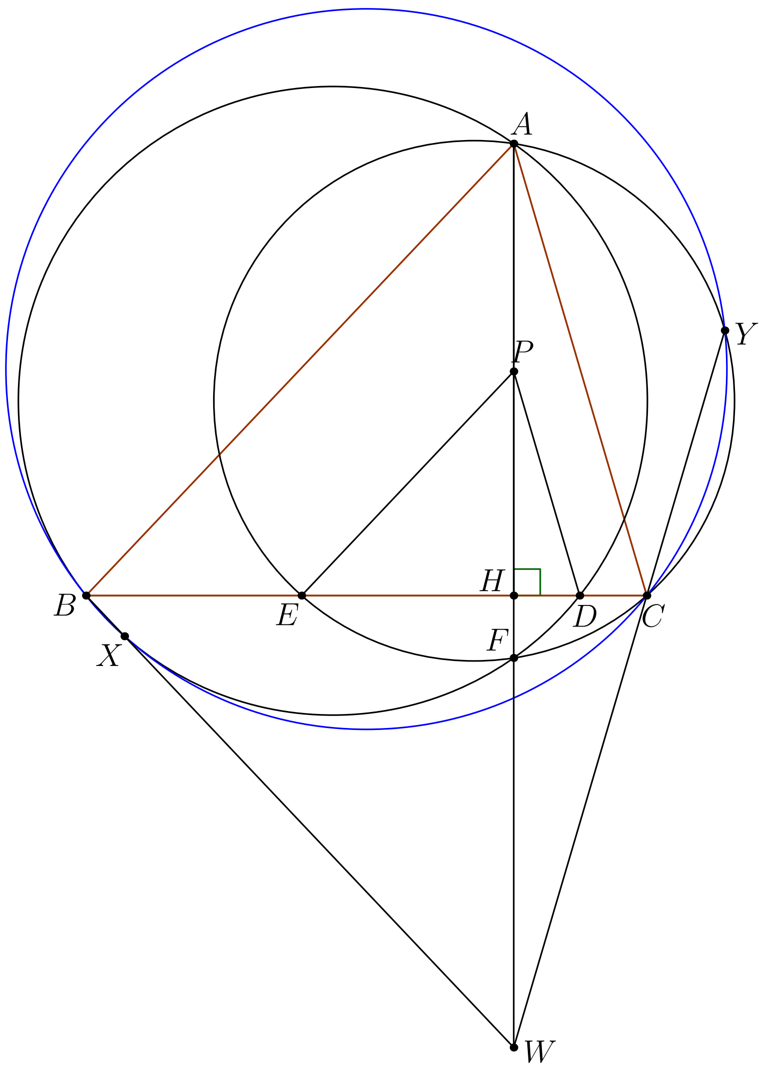
$$a^2yz + b^2xz + c^2xy = 4a^2b^2c^2 - 2a^2b^2c^2 - 2a^2b^2c^2 = 0$$

Consequently, it must be that $X \in (\triangle ABC)$. ■

Problem 1.0.4 (IMO 2022 Shortlist, G2)

In the acute-angled triangle $\triangle ABC$, the point F is the foot of the altitude from A, and P is a point on the segment AF. The lines through P parallel to AC and AB meet BC at D and E, respectively. Points $X \neq A$ and $Y \neq A$ lie on the circles ABD and ACE, respectively, such that $DA = DX$ and $EA = EY$. Prove that B, C, X and Y are concyclic.

Proof. There are a lot of observations that can be done in this problem, however this is the solution I have.



Lemma 1.2 The second intersection of (ABD) and (AEC) lies on AH.

This is true due to basic angle chase, assume that F lies on AH and (ABD), let us prove that then $AECF$ is cyclic. By Power of the Point,

$$HA \cdot HF = HD \cdot HD = HE \cdot HC$$

the last is obviously true because $PE \parallel AB$ and $PD \parallel AC$. Now let us intersect BX with AF at a point W, then the condition that $BXCY$ is cyclic by the radical center theorem is equivelent to showing that W, C and Y are colinear.

Lemma 1.3 BC is the angle bisector of $\angle ABW$

This is because,

$$\angle CBW = \angle XAD = \angle AXD = \angle ABC$$

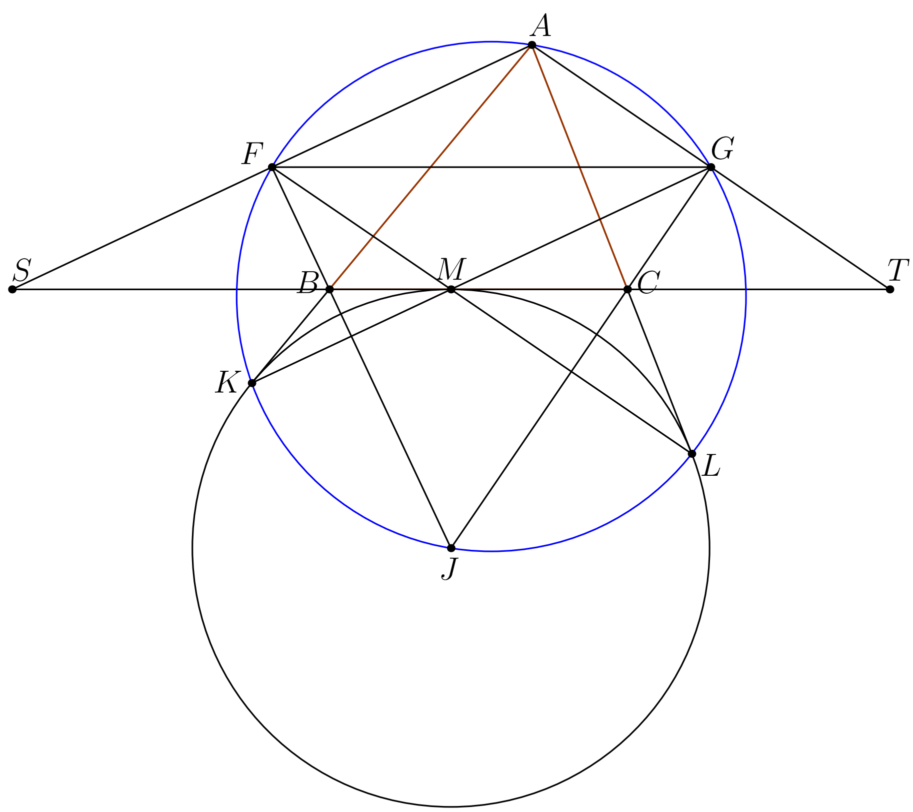
because F is the midpoint of the arc between A and X per definition. Thus, W is the reflection of A over H. The exact same argument for C shows that BC is the angle bisector of $\angle YCA$ and thus, CY must intersect AF at W.

■

Problem 1.0.5 (IMO 2012 P1)

Given triangle ABC the point J is the center of the excircle opposite the vertex A. This excircle is tangent to side BC at M, and to lines AB and AC at K and L, respectively. Lines LM and BJ meet at F, and lines KM and CJ meet at G. Let S be the point of intersection of lines AF and BC, and let T be the point of intersection of lines AG and BC. Prove that M is the midpoint of ST.

Proof. Pascal's theorem hints us towards checking whether $FAGKJL$ is cyclic.



Notice, that because $KM \perp FJ$ and $JG \perp ML$, it must be that M is the orthocenter of FGJ. Thus, $FG \parallel BC$, consequently,

$$\angle GFJ = \angle BCJ = \angle MKJ$$

Thus, $FKJG$ is cyclic, analogously we conclude that $FKJLG$ is cyclic and by Pascal's theorem it must be that A lies on this circle as well, thus $FAGLJK$ is cyclic.

Now notice,

$$\angle FGA = \angle ALF = \angle CML = \angle GFL$$

the last step comes from $FG \parallel BC$. Thus, $AT \parallel FM$, the same for $AG \parallel FM$. Consequently, $AGMF$ is a parallelogram. Because of that it must be that FG is the midline of $\triangle AST$ and M is the midpoint of the side ST. ■

Problem 1.0.6 (IMO 2022 Shortlist C1)

A ± 1-sequence is a sequence of 2022 numbers $a_1, \ldots, a_{2022}$, each equal to either $+1$ or -1. Determine the largest C such that, for any ± 1-sequence, there exists an integer k and indices $1 \leq t_1 \leq \ldots \leq t_k \leq 2022$ so that $t_{i+1} - t_i \leq 2$ for all i, and

$$\left| \sum_{i=1}^{k} a_{t_i} \right| \geq C$$

Proof. WLOG, let there be more 1 than -1 in the ± 1-sequence. Then, let us consider the following algorithm,

Algorithm 1.4 If the next element is -1, then skip to the next one and include it. If the next element is 1, then include it. Continue this process until you have went through the entire sequence.

This algorithm will garantee that we skip at least $\lceil \frac{X}{2} \rceil$, where X is the number of -1 in the sequence. Thus,

$$\left| \sum_{i=1}^{k} a_{t_i} \right| \geq (2022 - X) - \left\lfloor \frac{X}{2} \right\rfloor \geq (2022 - 1011) - \left\lfloor \frac{1011}{2} \right\rfloor = 506$$

Consequently, it must be that $C \geq 506$. Now consider the following sequence,

$$1, 1, -1, -1, 1, 1, \ldots, -1, -1, 1, -1$$

It alternates between $1, 1$ and $-1, -1$ until the very end, where it is 1 and -1.

Notice that in each consequtive pair of identical numbers at least one of the numbers is present in our sequence. Thus, no matter our choice of indices, the total sum is bounded by,

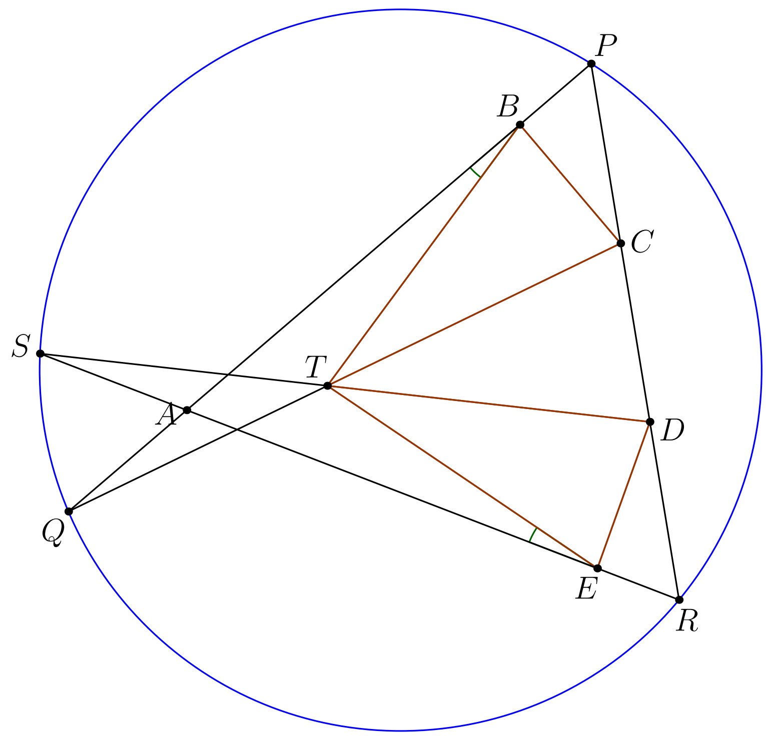
$$\left| \sum_{i=1}^{k} a_{t_i} \right| \leq 2 \cdot \frac{1010}{2} - \frac{1010}{2} + 1 = 2 \cdot 505 - 505 + 1 = 506$$

Thus, it must be that $C = 506$. ■

Problem 1.0.7 (IMO 2022 P4)

Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point T inside $ABCDE$ with $TB = TD$, $TC = TE$ and $\angle ABT = \angle TEA$. Let line AB intersect lines CD and CT at points P and Q, respectively. Assume that the points P, B, A, Q occur on their line in that order. Let line AE intersect lines CD and DT at points R and S, respectively. Assume that the points R, E, A, S occur on their line in that order. Prove that the points P, S, Q, R lie on a circle.

Proof. The first thing which one can notice is that $\triangle TDE \sim \triangle TBC$.



Now, notice that,

Lemma 1.5 $\triangle STE \sim \triangle QTB$ and $SQCD$ is cyclic.

This is because,

$$\angle STE = 180 - \angle DTE = 180 - \angle BTC = \angle QTB$$

Thus,

$$\frac{TS}{TC} = \frac{TS}{TE} = \frac{TQ}{TB} = \frac{TQ}{TD}$$
$$\implies TS \cdot TD = TC \cdot TQ$$

Consequently $SQCD$ is cyclic. Now,

$$\angle QSR = \angle QSD - \angle RSD = \angle QCD - \angle PQC = \angle QPR$$

which proves that $SQPR$ is cyclic. ■

Problem 1.0.8 (IMO 2012 P4)

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c such that $a + b + c = 0$, the following equality holds,

$$f^2(a) + f^2(b) + f^2(c) = 2f(a)f(b) + 2f(b)f(c) + 2f(a)f(c)$$

Proof. Obviously from $a = b = c = 0$ we obtain that $f(0) = 0$.

Notice, if we set into the original functional equation $b = -a$ and $c = 0$, we will obtain,

$$\begin{aligned} f^2(a) + f^2(-a) &= 2f(a)f(-a) \\ \implies (f(a) - f(-a))^2 &= 0 \\ \implies f(a) &= f(-a) \end{aligned}$$

Now, notice that a very common transformation with this type of equation is,

$$x^2 + y^2 + z^2 = 2xy + 2yz + 2xz$$

$$\begin{aligned} (x + y - z)^2 &= x^2 + y^2 + z^2 + 2xy - 2xz - 2xy \\ \implies 4xy &= (x + y - z)^2 \\ \implies xy &= \left(\frac{x + y - z}{2}\right)^2 \end{aligned}$$

Thus,

$$f(a)f(b) = \left(\frac{f(a) + f(b) - f(c)}{2}\right)^2$$

Now is the perfect time to get rid of the third variable, let $c = -a - b$ and we will obtain, (because $f(-a - b) = f(a + b)$)

$$f(a)f(b) = \left(\frac{f(a) + f(b) - f(a + b)}{2}\right)^2$$

$$f(a + b) = f(a) + f(b) \pm 2\sqrt{f(a)f(b)}$$

The nice thing about this is first of all, because $f(a)f(b)$ is always nonnegative it must be that f is either always ≥ 0 or ≤ 0. The second thing is that now we obtained an equation equivelent to the original functional equation and we can assume that f is positive (because if f is a solution, then $-f$ is a solution as well). However, we can still do some preliminary steps for example obviously we want to rewrite the equation as,

$$f(a + b) = (\sqrt{f(a)} \pm \sqrt{f(b)})^2$$

$$\implies \sqrt{f(a + b)} = \pm\sqrt{f(a)} \pm \sqrt{f(b)}$$

thus if $g(x) = \sqrt{f(x)}$, then we know that,

$$g(x+y) = \pm g(x) \pm g(y)$$

Now, if $g(1) = c$, then,

$$g(2) = \pm g(1) \pm g(1) = c \pm c = 2c \vee 0$$

Now, let us consider two cases,

Case 1) If $g(2) = 2c$, then,

$$g(3) = \pm f(1) \pm f(2) = \pm c \pm 2c = 3c \vee c$$

because f is ≥ 0. Now, we again consider two cases, right now let $g(3) = 3c$. Then,

$$\begin{cases} g(4) = \pm g(2) \pm g(2) = 4c \vee 0 \\ g(4) = \pm g(1) \pm g(3) = 4c \vee 2c \end{cases}$$

$$\implies g(4) = 4c$$

inductively it is simple to continue this logic and show that $g(n) = nc$. Thus, one of the solutions for f is $f(n) = cn^2$.

Case 2) If $g(2) = 2c$ and $g(3) = c$. Then,

$$\begin{cases} g(4) = \pm g(1) \pm g(3) = 0 \vee 2c \\ g(4) = \pm g(2) \pm g(2) = \pm 2c \pm 2c = 0 \vee 4c \end{cases}$$

$$\implies g(4) = 0$$

However, if you repeat this argument for higher value of $g(n)$ you will obtain a periodic function of the following form,

$$g(n) = \begin{cases} c, \text{ if } n \equiv 1 \pmod 2 \\ 2c, \text{ if } n \equiv 2 \pmod 4 \\ 0, \text{ if } n \equiv 0 \pmod 4 \end{cases}$$

which is equivelent to,

$$f(n) = \begin{cases} c, \text{ if } n \equiv 1 \pmod 2 \\ 4c, \text{ if } n \equiv 2 \pmod 4 \\ 0, \text{ if } n \equiv 0 \pmod 4 \end{cases}$$

Case 3) If $g(2) = 0$, then,

$$\begin{cases} g(3) = \pm g(1) \pm g(2) = c \\ g(4) = \pm g(2) \pm g(2) = 0 \\ \dots \end{cases}$$

Thus,

$$g(n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 2 \\ c, & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

$$\implies f(n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 2 \\ c, & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

Thus, we have obtained all solutions (all the solutions above have there negative counterparts as well). ■

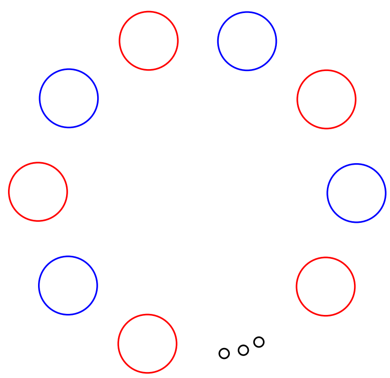
Problem 1.0.9 (IMO 2013 P2)

A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied:

1. No line passes through any point of the configuration.
2. No region contains points of both colors.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

Proof. Let us consider the following construction (alternating the colors), I claim one needs at least 2013 lines.



To prove this simply consider all the segments connecting neighboring blue/red points, then each of those segments must be split by some line (else they wouldn't be in different regions), but each line cuts no more than 2 such segments. However, there is a total of $2 \cdot 2013$ such segments, thus a minimum of 2013 lines is required.

Now, we need to prove that 2013 lines is always enough.

Lemma 1.6 Given n points blue and $k < n$ red points, where k is even, then it is possible to satisfy the conditions in the problem statement using k lines.

Consider two red points, then no blue point lies on the line between them. Thus, we can consider two close enough lines parallel to the line between the two red points and divide the plane into three parts with one of the parts containing both of the red points and no other point. Thus, if we repeat this argument for the $\frac{k}{2}$ pairs of red points we will obtain a construction satisfying the conditions of the problem statement involving only $2 \cdot \frac{k}{2} = k$ lines.

Lemma 1.7 Given n points of one color and $n-1$ points of another color, a minimum of $n-1$ lines is required to satisfy the conditions in the problem statement.

If $n-1$ is even, then we are done, thus let us assume that $n-1$ is odd.

Let us construct a convex hull given all the points. If it contains two neighboring blue points, then we separate them from the rest of the points using one line and then apply the induction hypothesis on the rest of the points and obtain $1+n-2 = n-1$ lines.

If at least one red point is present in the convex hull we can separate it from the rest of the points using one line. Then, we have $n-2$ red points left, which is an even number, thus we can satisfy the conditions of the problem statement using $1+n-2 = n-1$ lines.

Obviously the problem follows when $n = 2014$. ■

Problem 1.0.10 (IMO Shortlist 2022 N2)

Find all positive integers $n > 2$ such that,

$$n! \mid \prod_{p<q\le n} (p+q)$$

Proof. Let r be the biggest prime $\le n$. Then, obviously $v_r(n!) = 1$. However,

$$v_p\left(\prod_{p<q\le n}(p+q)\right) = \sum_{p+q=r} 1$$

because $p+q < r+r = 2r$. Thus, there must be exactly one solution to $p+q=r$ for prime p, q. However, because r is an odd prime it must be that either p or q is even, i.e. 2. WLOG $q = r-2$.

Notice, that it can't be that p, $p+2$ and $p+4$ are all prime, except for $3, 5, 7$.

Let us use the exact same logic for q, assuming that $q > 5$, which is true for $n \ge 11$. Notice, that,

$$x + y < p + p - 4 = 2(q-2)$$

where x, y are prime (due to the above). However, notice that it also can't be that $x + y = q$ where x, y are prime, also due to the argument before. Thus,

$$v_q\left(\prod_{x<y\le n}(x+y)\right) = 0 < v_q(n!)$$

contradiction! Thus, $7 \le n < 11$,

Case 1) If $n = 7$, (primes are $2, 3, 5, 7$)

$$\begin{aligned}\prod_{p<q\le n}(p+q) &= (2+3)\cdot(3+5)\cdot(5+7)\cdot(2+5)\cdot(2+7)\cdot(3+7) \\ &= 5\cdot 8\cdot 12\cdot 7\cdot 9\cdot 10 \quad (1)\end{aligned}$$

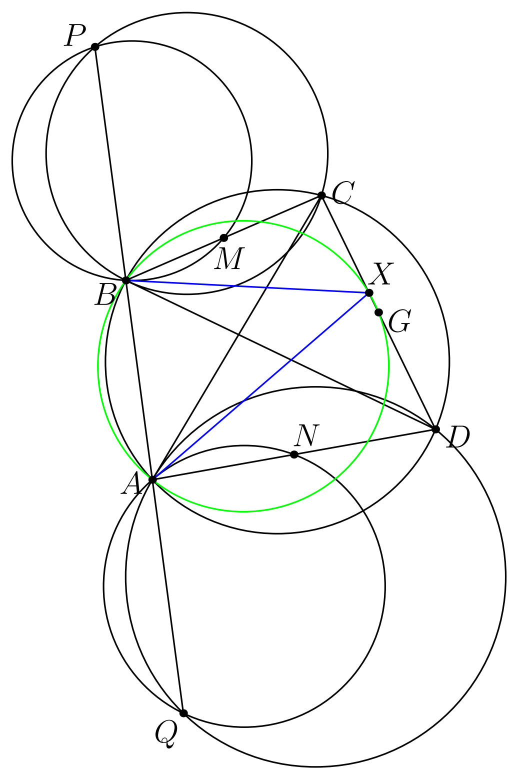
which is obviously divisible by $7!$.

Case 2) However, already for $n > 7$, there isn't enough factors in the product (because no new primes appear up until 11).

Thus, the only n satisfying the problem statement is 7. ∎

Problem 1.0.11 (IMO Shortlist 2022 G3)

Let $ABCD$ be a cyclic quadrilateral. Assume that the points Q, A, B, P are collinear in this order, in such a way that the line AC is tangent to the circle ADQ, and the line BD is tangent to the circle BCP. Let M and N be the midpoints of segments BC and AD, respectively. Prove that the following three lines are concurrent: line CD, the tangent of circle ANQ at point A, and the tangent to circle BMP at point B.



Proof. Notice that,

$$\measuredangle QDA = \measuredangle CAB = \measuredangle BDC$$

and,

$$\measuredangle QAD = \measuredangle BCD$$

Thus, $\triangle ADQ \sim \triangle CDB$, and D is the center of the spiral similarity between the triangles.

Let G be the midpoint of CD. Notice, under the spiral similarity N goes to G. Thus,

$$\measuredangle GBC = \measuredangle NQA = \measuredangle DAX$$

where X is the intersection of the tangent of (ANQ) from A with CD. Analogously we obtain that $\measuredangle DAG = \measuredangle X'BC$.

However, this means that $\measuredangle GBX' = \measuredangle GAX$, thus X and X' are the same point, the intersection of (BAG) with CD. ■

Problem 1.0.12 (IMO 2022 P1)

The Bank of Oslo issues two types of coin: aluminum (denoted A) and bronze (denoted B). Marianne has n aluminum coins and n bronze coins arranged in a row in some arbitrary initial order. A chain is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \leq 2n$, Gilberty repeatedly performs the following operation: he identifies the longest chain containing the k^{th} coin from the left and moves all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be $AABBBABA \to BBBAAABA \to AAABBBBA \to BBBBAAAA \to ...$

Find all pairs (n, k) with $1 \leq k \leq 2n$ such that for every initial ordering, at some point of the process there will be at most one aluminium coin adjacent to a copper coin.

Proof. Obviously it makes sense to split the row into consecutive blocks of coins of the same type. Then, the fact that at most one aluminium coin is adjacent to a copper coin is equivelent to there being ≤ 2 blocks. Notice,

Lemma 1.8 The number of blocks decreases or remains constant after every operation.

Thus, in order to not obtain a situation with ≤ 2 blocks, it must be that the number of blocks doesn't change. However, the number of blocks doesn't change if and only if, either the pointer is on the first block, or that the pointer is on the last block. Thus, one of these two conditions must be satisfied after a certain point in time for infinitely many operations. Thus, a situation stabalizes (i.e. doesn't change the number of blocks) if and only if the size of each block is less than the distance from the pointer to the right end of the row, or that the pointer points to the first block.

Case 1) Notice, that for $n > \lceil \frac{3n}{2} \rceil$ we can consider four blocks of alternating coin types with the sizes $\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil$, then the blocks will simply rotate and there will never be 2 blocks.

Case 2) If $k < n$, then the configuration of where the first $n - 1$ coins are of one time, and then the rest is randomly distributed will never change, thus will never achieve ≤ 2 blocks.

Case 3) If $\lceil \frac{3n}{2} \rceil \geq k \geq n$, then if there are ≥ 4 blocks the situation won't be stable (due to the above). However, a configuration with an odd number of blocks is never stable, because at some point blocks of the same type will merge. Thus, the situation will only be stable when the number of blocks is 2, which is what is required.

Thus, $n \leq k \leq \lceil \frac{3n}{2} \rceil$. ■

Problem 1.0.13 (IMO 2016 P4)

A set of positive integers is called fragrant if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2+n+1$. What is the least possible positive integer value of b such that there exists a non-negative integer a for which the set

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

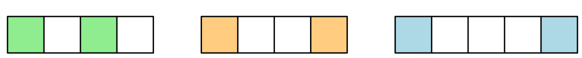
is fragrant?

Proof. It is well known that a degree N polynomial over $\mathbb{Z}_p$ has no more than N roots, thus n^2+n+1 is congruent to zero modulo p no more than twice per p consecutive values.

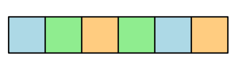
After not noticing any meaningful pattern in the problem it is reasonable to just brute force some values. Notice,

$$\begin{cases} 1+1+1=3 \\ 2^2+2+1=7 \\ 3^3+3+1=13 \\ 4^2+4+1=21=7\cdot 3 \\ 5^5+5+1=31 \\ 6^2+6+1=43 \\ 7^2+7+1=57=19\cdot 3 \\ 8^2+8+1=73 \\ 9^2+9+1=91=7\cdot 13 \\ 10^2+10+1=111=37\cdot 3 \\ 11^2+11+1=133=7\cdot 19 \end{cases}$$

Thus, it is seems apparent that the primes $3, 7$ and 19 are interesting. They have the following patterns,



The three diagrams represent $7, 3$ and 19, respectively, where the coloring represents distances between $P(x)$ and $P(y)$ uch that both are divisible by the appropriate prime. Using CRT we can essentially request 6 consecutive values with the following pattern,



Now, all we have to prove is that $b = 6$ is the smallest value for b. However, notice,

$$(n^2+n+1,(n+1)^2+n+1+1) = (n^2+n+1, n^2+3n+3) = (n^2+n+1, 2n+2)$$
$$= (n^2+n+1, n+1) = (n^2, n+1) = 1 \quad (2)$$

$$(n^2+n+1,(n+2)^2+n+2+1) = (n^2+n+1, n^2+5n+7) = (n^2+n+1, 4n+6)$$
$$= (n^2+n+1, 2n+3) \mid (2n^2+2n+2, 2n^2+3n)$$
$$= (-n+2, 2n^2+3n) = (2-n, 2n+3) = (2-n, 7) \mid 7 \quad (3)$$

$$(n^2+n+1,(n+3)^2+n+3+1) = (n^2+n+1, n^2+7n+13) = (n^2+n+1, 6n+12)$$
$$= (n^2+n+1, 3n+6) \mid (3n^2+3n+3, 3n^2+6n)$$
$$= (-3n+3, 3n^2+6n) \mid 3(-n+1, n^2+n) = 3(1-n, n+1) \mid 3 \quad (4)$$

$$(n^2+n+1,(n+4)^2+n+4+1) = (n^2+n+1, n^2+9n+21) = (n^2+n+1, 8n+20)$$
$$= (n^2+n+1, 2n+5) \mid (2n^2+2n+2, 2n^2+5n)$$
$$= (-3n+2, 2n^2+5n) = (-3n+2, 2n+5) = (3n-2, -n+7)$$
$$= (3n-2, 7-n) = (7\cdot 3-2, 7-n) \mid 19 \quad (5)$$

Thus, if $b \le 5$, then we must only use the $3, 7$ and 19 factors, however the patterns from above cannot form $b \le 5$, contradiction! ■

Problem 1.0.14 (IMO 2023 P1)

Determine all composite integers $n > 1$ that satisfy the following property: if d_1, d_2, ..., d_k are all the positive divisors of n with $1 = d_1 < d_2 < \cdots < d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \le i \le k-2$.

Proof. Let us look at the largest three divisors, then they are $\{n, \frac{n}{p}, \frac{n}{q}\}$ or $\{n, \frac{n}{p}, \frac{n}{p^2}\}$. Let us consider the first case, due to the condition in the problem statement it must be that,

$$\frac{n}{q} \mid n + \frac{n}{p}$$

$$\implies \frac{1 + \frac{1}{p}}{1/q} \in \mathbb{Z}$$

$$\implies \frac{p+1}{pq} \in \mathbb{Z}$$

However, $p+1$ is not divisible by p, thus the fraction cannot be an integer. Consequently, it must be that the three largest divisors must be $\{n, \frac{n}{p}, \frac{n}{p^2}\}$, which implies that $\{1, p, p^2\}$ are the smallest divisors. Now, let us assume that the first k divisors are of the form p^i, then, let us prove that the $k+1$-st divisor must also be of that form. Notice, otherwise,

$$p^{k-1} \mid p^k + q$$

which is impossible! Consequently all the divisors are consecutive powers of p. Thus, the only n for which the condition in the problem statement is satisfied are powers of a prime p. ■

Problem 1.0.15 (IMO 2019 P4)

Find all positive (n, k) such that,

$$k! = (2^n - 1)(2^n - 2)\dots(2^n - 2^{n-1})$$

Proof. Let us look at the 2-adic valuation of both sides,

$$v_2\Big((2^n - 1)(2^n - 2)\dots(2^n - 2^{n-1})\Big) = v_2\Big((2^n - 2)(2^n - 4)\dots(2^n - 2^{n-1})\Big)$$
$$= 1 + 2 + \dots + (n-1) = \frac{n(n-1)}{2}$$

consequently,

$$v_2(k!) = \sum_{i=1}^{\infty} \left\lfloor \frac{k}{2^i} \right\rfloor = \frac{n(n-1)}{2}$$

which implies that $k > \frac{n(n-1)}{2}$. However, notice that by LTE,

$$v_3(2^n - 1) = v_3(4^{n/2} - 1) = 1 + v_3\left(\frac{n}{2}\right)$$

Thus,

$$v_3\Big((2^n-1)(2^n-2)\dots(2^n-2^{n-1})\Big) = v_3\Big((2^n-1)\cdot 2\cdot(2^{n-1}-1)\dots\cdot 2^{n-1}\cdot(2-1)\Big)$$
$$= v_3\Big((2^n - 1)(2^{n-1} - 1)\dots(2^2 - 1)\cdot(2 - 1)\Big)$$
$$= v_3\Big((2^2 - 1)\cdot(2^4 - 1)\cdot\dots\cdot(2^n - 1)\Big)$$
$$= v_3\left(\frac{2}{2}\right) + v_3\left(\frac{4}{2}\right) + \dots + v_3\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + \left\lfloor\frac{n}{2}\right\rfloor$$
$$= \left[\sum_{w=1}^{\lfloor n/2\rfloor} v_3(w)\right] + \left\lfloor\frac{n}{2}\right\rfloor < \left\lfloor\frac{n}{2}\right\rfloor\cdot\left[\frac{1}{3}+\frac{1}{9}+\dots\right] + \left\lfloor\frac{n}{2}\right\rfloor = \frac{1}{1-\frac{1}{3}}\cdot\left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor = \frac{3}{2}\left\lfloor\frac{n}{2}\right\rfloor$$

Thus, to summarize the findings we have the following,

$$\begin{cases} v_2(k!) = \frac{n(n-1)}{2} \\ v_3(k!) < \frac{3}{2}\left\lfloor\frac{n}{2}\right\rfloor \end{cases}$$

However, obviously $v_3(k!) \geq \left\lfloor\frac{k}{3}\right\rfloor$,

$$\left\lfloor\frac{k}{3}\right\rfloor < \frac{3}{2}\left\lfloor\frac{n}{2}\right\rfloor \implies \left(\frac{k}{3} - 1\right) < \frac{3}{2}\frac{n}{2}$$

$$\implies 4k + 12 < 9n \implies k < \frac{9}{4}n + 3$$

However, then,

$$\frac{n(n-1)}{2} < k < \frac{9}{4}n + 3$$

is only true for $n \leq 6$. It is simple to verify that $(1, 1)$ and $(2, 3)$ are the only solutions. ■

Problem 1.0.16 (IMO 2022 P5)

Find all triples (a, b, p) of positive integers with p prime and

$$a^p = b! + p.$$

Proof. A relative order of variables seems to be something to look into, consequently let us consider several cases,

Case 1) If $a > b$, then we can consider the relative order between b and p. Thus, we obtain another branching in the case logic,

- Case 1.1) Notice, that if $b \geq p$, then it must be that a is divisible by p, since the RHS is divisible by p. Obviously if $b \geq 2p$, then the order of p in the RHS will be one, since $v_p(b! + p) = \min\{v_p(b!), p\} = \min\{\geq 2, 1\} = 1$, thus, $b < 2p$. However, since $p \mid a$ and $a > b$ it must be that $a \geq 2p$. To summarize the findings in a single inequality,

  $$a \geq 2p > b \geq p$$

  Notice, if $q \mid a$ and $b > q$, then the LHS will be divisible by q, however the RHS won't be divisible by q contradiction! Let us assume a prime q greater than p divides a, then,

  $$a^p \geq p^{2p} \overset{?}{>} (2p-1)! + p \geq b! + p$$

  if the inequality with the question mark is proven, then equality is impossible, contradiction! Notice,

  $$(2p-1)! = \Big(1 \cdot (2p-1)\Big) \cdot \Big(2 \cdot (2p-2)\Big) \cdot \ldots \cdot \Big((p-1) \cdot (p+1)\Big) \cdot p \leq p^{2p-1}$$

  obviously adding p won't change anything, thus the inequality is proven. If there is no other q, then $b \geq p^2$ and one will obtain exactly the same inequality.

- Case 1.2) If $b < p$, then,

  $$a^p - b! \geq a^p - p^p \geq (p+1)^p - p^p > p$$

  thus, equality can never be achieved!

Case 2) Now, let us consider the case, when $b \geq a$. Then, LHS is divisible by a and $b!$ is divisible by a, consequently it must be that $a \mid p$, consequently $a = 1 \vee p$. Since, obviously $a \neq 1$ it must be that $a = p$. Thus, we are solving the following equations,

$$p^p = b! + p$$

Obviously it must be that $v_p(b!) = 1$, else the RHS would have a p-adic valuation of 1, which is not equal to the p-adic valuation of the LHS, p. Thus, it must be that $b < 2p$. Also, since both sides need to be divisible by p it must be that $b \geq p$, consequently, $2p > b \geq p$.

$$p^p - p = p(p^{p-1} - 1) = b!$$

Notice, by LTE,

$$\begin{aligned} v_2(p^{p-1} - 1) = v_2(p-1) + v_2(p+1) + v_2(p-1) - 1 \\ = 2v_2(p-1) + v_2(p+1) - 1 \end{aligned} \tag{6}$$

Consequently, since at least one of $v_2(p-1)$ or $v_2(p+1)$ will be one for $p \geq 3$. It must be that the value above is either $2v_2(p-1)$ or $1 + v_2(p+1)$ for $p > 3$.

If $b = p$, then due to $p^p - p > p!$ for $p > 3$ we obtain a contradiction! Thus, $b \geq p + 1$, however, then, (for $p > 3$)

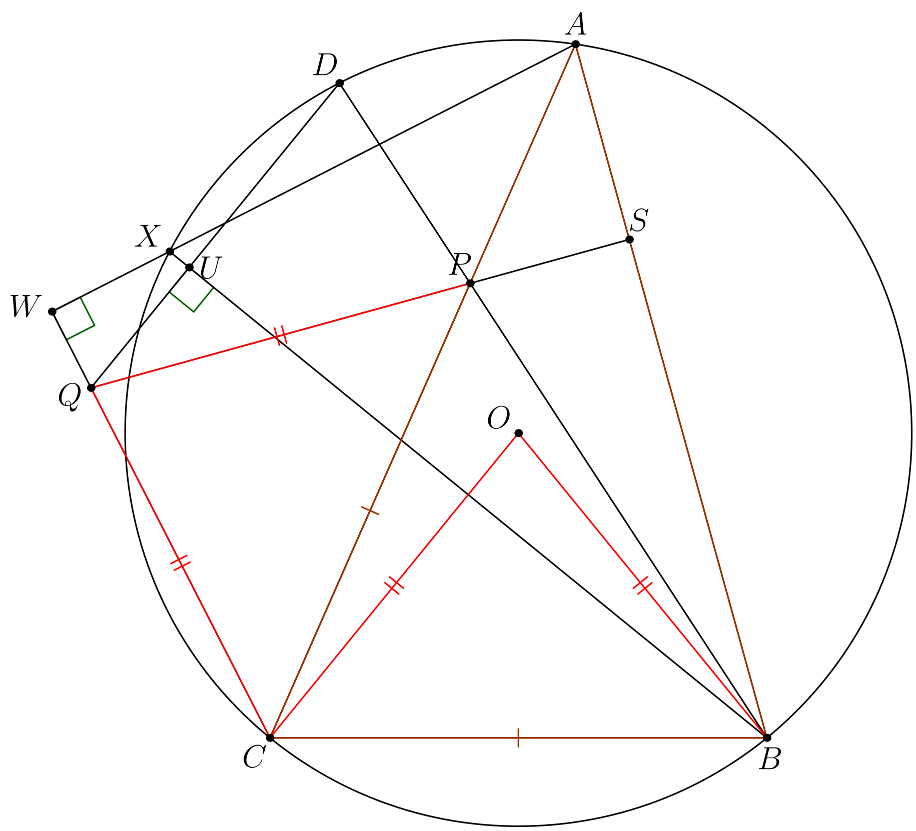
$$\begin{cases} v_2(b!) \geq v_2((p+1)!) > v_2(p+1) + v_2(2) = v_2(p+1) + 1 \\ v_2(b!) \geq v_2((p+1)!) > v_2(p-1) + v_2\left(\frac{p-1}{2}\right) + v_2(p+1) \geq 2v_2(p-1) \end{cases}$$

Thus, we just need to check $p = 2$ and $p = 3$, both of which obtain the only solutions $(2, 2, 2)$ and $(3, 4, 3)$.

$\blacksquare$

Problem 1.0.17 (IMO Shortlist 2023 G2)

Let ABC be a triangle with $AC > BC$, let ω be the circumcircle of $\triangle ABC$, and let r be its radius. Point P is chosen on $\overline{AC}$ such that $BC = CP$, and point S is the foot of the perpendicular from P to $\overline{AB}$. Ray BP mets ω again at D. Point Q is chosen on line SP such that $PQ = r$ and S, P, Q lie on a line in that order. Finally, let E be a point satisfying $\overline{AE} \perp \overline{CQ}$ and $\overline{BE} \perp \overline{DQ}$. Prove that E lies on ω.



Proof. Let U be the foot of the altitude from B onto QD and let W be the foot of the altitude from A onto CQ. Let $X = AW \cap BU$.

The condition that $PQ = r$ hints towards drawing O, the circumcircle of ω. Notice,

Lemma 1.9 $\triangle PQC \sim \triangle COB$

since,

$$\angle QPB = \angle SPA = 90 - \alpha = \angle OBC$$

thus, $\triangle PQC \sim \triangle COB$ by two sides and the angle between them. Now,

Lemma 1.10 $OD \parallel CQ$

since,

$$\begin{aligned}\angle DOC = 360 - \angle COB - \angle BOD &= 360 - 2\alpha - (180 - 2 \cdot \angle OBD) \\ &= 180 - 2\alpha + 2(\angle OBA - \angle PBA) \\ = 180 - 2\alpha + 2(90 - \gamma - (\beta - 90 + \frac{\gamma}{2})) &= 360 - 2\alpha - 2\gamma - 2\beta + 180 - \gamma \\ &= 180 - \gamma\end{aligned}$$

and $\angle QCO = \angle PCB = \gamma$, consequently $OD \parallel CQ$. However, this implies that $CQDO$ is a rhombus.

Lemma 1.11 $\angle ABX = \gamma$

since, $\angle QUX = \angle QWX = 90$, $XWQU$ is cyclic, consequently,

$$\angle ABX = \angle WQD = 180 - \angle DQC = \angle QCO = \gamma$$

This implies that $ABCX$ is cyclic, proving the problem statement. ■

Problem 1.0.18 (IMO Shortlist 2023 A2)

Let $\mathbb{R}$ be the set of real numbers. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that
$$f(x+y)f(x-y) \geqslant f(x)^2 - f(y)^2$$
for every $x, y \in \mathbb{R}$. Assume that the inequality is strict for some $x_0, y_0 \in \mathbb{R}$.

Prove that either $f(x) \geqslant 0$ for every $x \in \mathbb{R}$ or $f(x) \leqslant 0$ for every $x \in \mathbb{R}$.

Proof. Notice,

Lemma 1.12 $f(x) = \pm f(-x)$

since if we substitute $y = -x$ and the same for $x = -y$ we will obtain,

$$\begin{cases} 0 \geq f(x)^2 - f(-x)^2 \\ 0 \geq f(-x)^2 - f(x)^2 \end{cases}$$

$$\implies f(x)^2 - f(-x)^2 = 0$$

which implies that $f(x) = \pm f(-x)$.

Lemma 1.13 There exists a $a \in \mathbb{R}$ such that $f(a) = f(-a)$.

Notice, by substituting $x := y$ and $y := x$ we obtain,

$$\begin{cases} f(x+y)f(x-y) \geq f(x)^2 - f(y)^2 \\ f(x+y)f(y-x) \geq f(y)^2 - f(x)^2 \end{cases} \implies f(x)^2 - f(y)^2 \geq -f(x+y)f(y-x)$$

Consequently,
$$f(x+y)f(x-y) \geq -f(x+y)f(y-x)$$
But, since $x+y$ and $x-y$ take any values we can rewrite this inequality as,

$$f(a)f(b) \geq -f(a)f(-b)$$

$$\implies f(-b) \geq -f(b)$$

which implies that $f(x) \geq -f(-x)$.

However, the original inequality is strict for some x_0, y_0 it must be that for some $b \in \mathbb{R}$ it must be that,

$$f(-b) > -f(b)$$

which due to the lemma above implies that $f(-b) = f(b)$ for some $b \in \mathbb{R}$.

Now, simply notice that, then for any $a \in \mathbb{R}$ it must be that,

$$f(a)f(b) \geq -f(a)f(b)$$

however, if $f(a)$ didn't have the same sign as $f(b)$ this inequality would not be true, consequently all the values of $f(x)$ are either positive or negative. ■

Problem 1.0.19 (IMO 2024 P2)

Determine all pairs (a, b) of positive integers for which there exist positive integers g and N such that

$$\gcd(a^n + b, b^n + a) = g$$

holds for all integers $n \geqslant N$. (Note that $\gcd(x, y)$ denotes the greatest common divisor of integers x and y.)

Proof. Let $M = ab + 1$, then, (since $\gcd(a, ab + 1) = 1$)

$$a^n + b \equiv a^n + \frac{-1}{a} \equiv a^{n+1} - 1 \equiv 0 \pmod{M}$$

$$\implies a^{n+1} \equiv 1 \pmod{M}$$

Thus, for all $n = k\phi(M) - 1)$ the gcd will be at least M, consequently $M \mid g$.

However, notice if $n = k\phi(M) + 1$, then,

$$a^n + b \equiv a + b \pmod{M}$$

Consequently it must be that $a + b \equiv 0 \pmod{ab + 1} \implies ab + 1 \mid a + b$, which implies that at least one of the variables is one.

WLOG let $a = 1$, then,

$$\gcd(b^n + 1, b + 1) = g$$

however, for odd n the gcd is simply $b + 1$, however for even it isn't unless $b = 1$, since,

$$b^n + 1 \equiv (-1)^n + 1 \equiv 2 \pmod{b + 1}$$

Consequently the only suitable pair (a, b) is $a = b = 1$, which obviously satisfies the problem statement. ■

Problem 1.0.20 (IMO Shortlist 2021 N1)

Find all positive integers $n \geq 1$ such that there exists a pair (a, b) of positive integers, such that $a^2 + b + 3$ is not divisible by the cube of any prime, and

$$n = \frac{ab + 3b + 8}{a^2 + b + 3}.$$

Proof. Notice,

$$a^2 + b + 3 \mid ab + 3b + 8 \implies a^2 + b + 3 \mid 3\left[a^2 + b + 3\right] - \left[ab + 3b + 8\right]$$

consequently,

$$a^2 + b + 3 \mid 3a^2 - ab + 1$$

However, notice that,

$$\frac{3a^2 - ab + 1}{3} = a^2 - \frac{ab}{3} + \frac{1}{3} < a^2 + b + 3$$

Thus, it must be that,

Case 1) If,

$$3a^2 - ab + 1 = 2(a^2 + b + 3)$$

$$a^2 - ab + 1 = 2b + 6 \implies a^2 + b + 3 = 3b + 8 + ab$$

which implies that $n = 1$. However, the problem is that,

$$b(a + 2) = 2b + ab = a^2 - 5$$

but $a^2 - 4 = (a + 2)(a - 2)$, consequently it must be that 1 is divisible by $a + 2$, contradiction! Thus, this case is impossible.

Case 2) If,

$$3a^2 - ab + 1 = a^2 + b + 3 \implies 2a^2 = ab + b + 2$$

which implies that,

$$2(a^2 + b + 3) = 3\left[a^2 + b + 3\right] - \left[3a^2 - ab + 1\right] = ab + 3b + 8$$

thus $n = 2$. Indeed, it is simple to verify that $a = b = 2$ works.

Thus, only $n = 2$ satisfies the problem statement. ∎

Problem 1.0.21 (IMO 2018 P1)

For each integer $a_0 > 1$, define the sequence $a_0, a_1, \ldots$, by,

$$a_{n+1} = \begin{cases} \sqrt{a_n}, & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3, & \text{otherwise} \end{cases}$$

for each $n \geq 0$. Determine all values of a_0 for which there is a number A such that $a_n = A$ infinitely many times.

Proof. Since squares are only 0 or 1 modulo 3, it must mean that if $a_i \equiv 2 \pmod 3$ for some $i \in \mathbb{N}$, then the sequence will simply be increasing by 3 every step, consequently the statement is not true.

Lemma 1.14 If $n \equiv 0 \pmod 3$, then the statement holds.

Let us assume that the statement is true for all $n \leq x^2$ where $n \equiv 0 \pmod 3$ (I assume that $x^2 \equiv 0 \pmod 3$), then,

$$x^2 + 3 \to x^2 + 6 \to \ldots \to (x+3)^2 \to x + 3$$

however $x + 3 \leq x^2$ for all $x \geq 3$, which we can assume is true from the base case,

$$3 \to 6 \to 9 \to 3 \to \ldots$$

Consequently, for all $n \equiv 0 \pmod 3$ the statement is true.

Lemma 1.15 If $n \equiv 1 \pmod 3$, then the statement doesn't hold.

Again, let us assume that the statement is true for all $n \leq x^2$, where $n \equiv 1 \pmod 3$ and $x^2 \equiv 1 \pmod 3$. Then,

Case 1) If $x \equiv 1 \pmod 3$, then,

$$x^2 + 3 \to x^2 + 6 \to \ldots \to (x+1)^2 \to x + 1$$

however $x + 1 \equiv 2 \pmod 3$ which we know implies that the statement is false.

Case 2) If $x \equiv 2 \pmod 3$, then,

$$x^2 + 3 \to x^2 + 6 \to \ldots \to (x+2)^2 \to x + 2$$

where $x + 2 \equiv 1 \pmod 3$, however notice that $x + 2 \leq x^2$ for all $x \geq 2$, thus by the induction assumption, we may conclude that $x + 2$ is "bad", implying that the statement is false.

Consequently, we obtain that the only solutions are 1 and $3k$ where $k \in \mathbb{N}$. ∎

Problem 1.0.22 (IMO 2013 P1)

Assume that k and n are two positive integers. Prove that there exist positive integers $m_1, \ldots, m_k$ such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right) \cdots \left(1 + \frac{1}{m_k}\right).$$

Proof. Notice,

Lemma 1.16 When $n = 1$ or $k = 1$ the desired values for m_i exist.

Indeed,

Case 1) If $n = 1$, then $m_1 = m_2 = \ldots = m_k = 1$ work.

Case 2) If $k = 1$, then letting $m_1 = n$ achieves the desired result.

Assume by induction that the problem is solved for all values k up until $k - 1$, let us prove it for k.

Notice,

Lemma 1.17 If $n + 1$ is odd, then it is possible.

$$\frac{2^k - 1 + n}{n} = \prod_{i=1}^{k} \frac{m_i + 1}{m_i}$$

Let one of the variables be n, then,

$$\frac{2^k - 1 + n}{n + 1} = \prod_{i=1}^{k-1} \frac{m_i + 1}{m_i}$$

$$\frac{2^k - 1 + n}{n + 1} = 1 + \frac{2^k - 1 + n - n - 1}{n + 1} = 1 + \frac{2^k - 2}{n + 1} = 1 + \frac{2^{k-1} - 1}{(n + 1)/2}$$

$$\implies 1 + \frac{2^{k-1} - 1}{(n + 1)/2} = \prod_{i=1}^{k-1} \frac{m_i + 1}{m_i}$$

which by induction we know how to solve.

Lemma 1.18 If n is even, then it is possible as well.

Let $n = 2n_0$, then,

$$\frac{2^k - 1 + 2n_0}{2n_0} = \prod_{i=1}^{k} \frac{m_i + 1}{m_i}$$

Let us plug in one of the variables to be $2^k - 2 + 2n_0$, then,

$$\frac{2^k - 2 + 2n_0}{2n_0} = \frac{2^{k-1} - 1 + n_0}{n_0} = \prod_{i=1}^{k-1} \frac{m_i + 1}{m_i}$$

which completes the induction step.

If any one of the induction steps is not possible to perform, it must be that we are in one of the base cases mentioned above. Consequently, by induction, it must be that the desired values for m_i exist, proving the problem statement. ■

Problem 1.0.23 (IMO 2024 P1)

Determine all real numbers α such that, for every positive integer n, the integer
$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor$$
is a multiple of n. (Note that $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z. For example, $\lfloor -\pi \rfloor = -4$ and $\lfloor 2 \rfloor = \lfloor 2.9 \rfloor = 2$.)

Proof. Let $\alpha = A + \beta$, where $A = \lfloor \alpha \rfloor$. Notice,

Lemma 1.19 For the statement to be true for $n \geq N$ it must be that $\beta \leq \frac{1}{n}$ or $\beta \geq \frac{n-1}{n}$ and,
$$\sum_{k=2}^{n} \lfloor k\beta \rfloor = 0 \vee \frac{n(n-1)}{2}$$

Indeed, let us prove the statement via induction, for the base case notice that,
$$3 \mid \lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \lfloor 3\alpha \rfloor \implies 3 \mid 6A + \lfloor 2\beta \rfloor + \lfloor 3\beta \rfloor$$
Obviously $3 \mid 6A$, consequently $3 \mid \lfloor 2\beta \rfloor + \lfloor 3\beta \rfloor$. However, for this to be true it must be that either $\beta < \frac{1}{3}$, or $\beta \geq \frac{2}{3}$.

Now, to prove the step in the induction let us consider two cases for the parity of n.

Case 1) If n is odd, then,
$$n \mid \left(1 + 2 + \ldots + n\right) + \sum_{k=2}^{n} \lfloor k\beta \rfloor$$
However, for odd n we know that $n \mid 1 + 2 + \ldots + n$, consequently we can conclude that,
$$n \mid \sum_{k=2}^{n} \lfloor k\beta \rfloor$$
due to induction we may assume that $\sum_{k=2}^{n-1} \lfloor k\beta \rfloor$ is either 0 or $\frac{(n-1)(n-2)}{2}$. However, if it is zero since $\lfloor n\beta \rfloor < n$ we will not be able to obtain the neccecery divisibility unless $\beta < \frac{1}{n}$.

If $\sum_{k=2}^{n-1} \lfloor k\beta \rfloor = 1 + 2 + \ldots + (n-2) = \frac{(n-1)(n-2)}{2}$, consequently to make it divisible by n we must add at least $n-1$, which means that $\lfloor n\beta \rfloor \geq n-1$ which implies that $\beta \geq \frac{n-1}{n}$.

Case 2) If n is even, then,

$$n \mid \left(1+2+\ldots+n\right) + \sum_{k=2}^{n} \lfloor k\beta \rfloor$$

However, for an even n the smallest number required to add to $1+2+\ldots+n = \frac{n(n+1)}{2}$ is n, it must be that,

$$\sum_{k=2}^{n} \lfloor k\beta \rfloor \geq n$$

However, due to induction,

$$\sum_{k=2}^{n} \lfloor k\beta \rfloor = \left[\sum_{k=2}^{n-1} \lfloor k\beta \rfloor\right] + \lfloor n\alpha \rfloor = 0 \vee \frac{(n-2)(n-1)}{2} + \lfloor n\beta \rfloor$$

Obviously the value cannot be zero, since then the sum will be no more than $n-1$, however if the term is $\frac{(n-2)(n-1)}{2}$, then the $\lfloor n\beta \rfloor$ must be at least $n-1$ for the sum to be divisible by n, consequently $\beta \geq \frac{n-1}{n}$

Consequently the induction step is done, from this we conclude that $\beta = 0$ (since $\beta < 1$).

Lemma 1.20 The only integer solutions are even α.

Indeed, notice,

$$n \mid \sum_{k=1}^{n} k\alpha = \alpha \cdot \frac{n(n+1)}{2}$$

this divisibility holds for all odd n, however for even n we require that α is even. Trivially, all even values for α work.

Thus, the only solutions for α are even integers. ■

Problem 1.0.24 (IMO 2023 P4)

Let $x_1, x_2, \ldots, x_{2023}$ be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \ldots, 2023$. Prove that $a_{2023} \geqslant 3034$.

Proof. Trivially $a_1 = 1$. Notice,

$$A_n = (x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) = a_{n-1}^2 + \sum_{k=1}^{n-1} \frac{x_n}{x_i} + \sum_{k=1}^{n-1} \frac{x_i}{x_n}$$

Let us now consider two cases,

Case 1) If n is odd, then by AM-GM applied to both the end terms, (and since the variables are distinct the equality is not achieved in the inequality)

$$A_n \geq a_{n-1}^2 + 2(n-1) + 1$$

However, by induction $a_{n-1}^2 \geq \left(\frac{3}{2}(n-1)\right)^2$, consequently, it must be that

$$A_n \geq \left(\frac{3}{2}(n-1) + 1\right)^2$$

Case 2) If n is even, then let us apply AM-GM to the two terms in the original expression for A_n and we will obtain,

$$A_n \geq a_{n-1}^2 + 2\sqrt{(x_1 + \ldots + x_{n-1})\left(\frac{1}{x_1} + \ldots + \frac{1}{x_{n-1}}\right)}$$

Consequently, by the induction assumption,

$$A_n \geq \left(\frac{3(n-1)-1}{2}\right)^2 + 3(n-1) - 1 = \frac{9n^2 - 24n + 16}{4} + 3n - 4 = \frac{9}{4}n^2 - 3n$$

However,

$$\left(\frac{3}{2}n - 1\right)^2 = \frac{9}{4}n^2 - 3n + 1$$

consequently, unless the equality case is achieved in the AM-GM application it must be that $A_n \geq \left(\frac{3}{2}n\right)^2$, proving the desired step.

However, notice if,

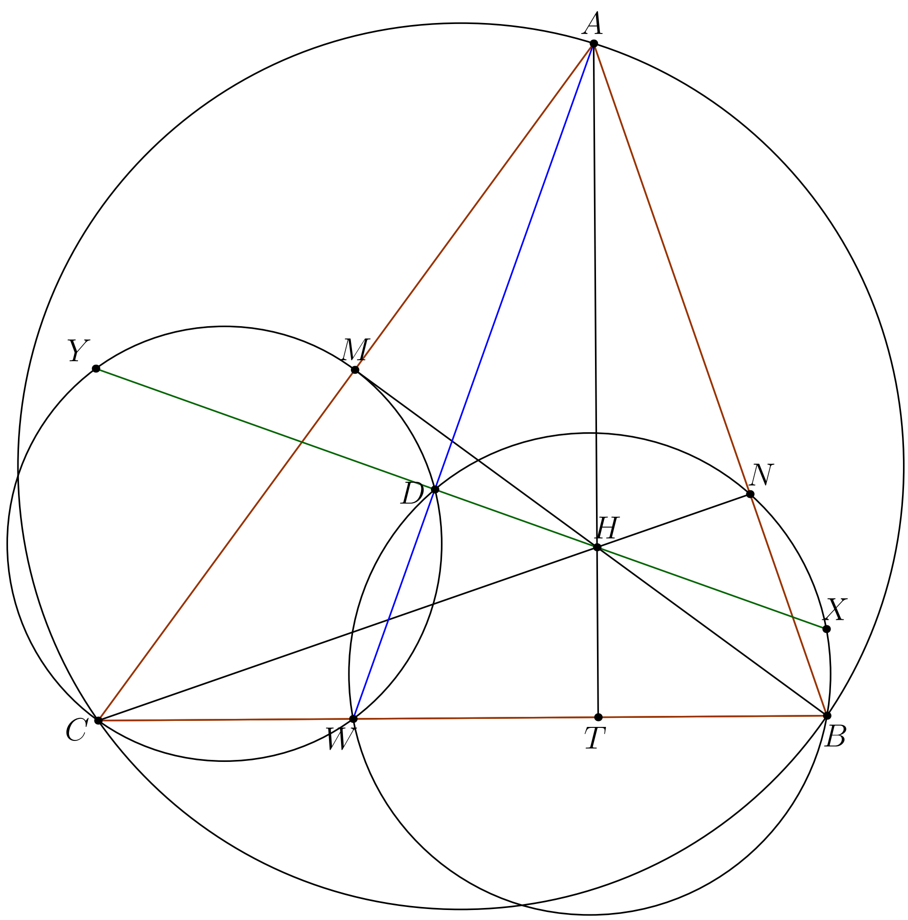
$$x_1 + \ldots + x_{n-1} = \frac{1}{x_1} + \ldots + \frac{1}{x_{n-1}}$$

then, $a_{n-2} = \sqrt{(a - x_n)\left(a - \frac{1}{x_n}\right)} = \sqrt{a^2 - a\left(x_n + \frac{1}{x_n}\right) + 1} \leq a_{n-1} - 1$, consequently by induction it must be that $x_{n-1} = 1$, however, analogously it can also be proven that $x_n = 1$, contradiction!

Consequently via induction we obtain the desired result, since $3034 = 2023 + \lfloor \frac{2023}{2} \rfloor$. ■

Problem 1.0.25 (IMO 2013 P4)

Let $\triangle ABC$ be an acute triangle with orthocenter H, and let W be a point on the side BC, between B and C. The points M and N are the feet of the altitudes drawn from B and C, respectively. Suppose ω_1 is the circumcircle of triangle $\triangle BWN$ and X is a point such that WX is a diameter of ω_1. Similarly, ω_2 is the circumcircle of triangle CWM and Y is a point such that WY is a diameter of ω_2. Show that the points X, Y, and H are collinear.



Proof. Notice, if D is the second intersection of (CMW) and (WNB), then,

Lemma 1.21 X, D, Y are colinear.

since XY is the image of $O_{(CWM)}O_{(NWB)}$ under homothepy with a coefficient 2, since D is the image under homothepy from W with coefficient 2 as well, it must be that X, D, Y lie on one line.

Now, notice,

> **Lemma 1.22** D, W and A lie on one line.

Indeed, since $CMNB$ is cyclic, then A is the radical center, consequently D, W, A are colinear.

Now, let T be the foot of the altitude from A, then since $HTNB$ is cyclic and $DWNB$ is cyclic, by Power of the Point it must be that $DHWN$ is cyclic, consequently,

$$\measuredangle WDH = \measuredangle WNH = 90$$

however, since $\measuredangle WDY = 90$, consequently X, D, H, Y are colinear, which implies that X, D and Y are colinear. ■

Problem 1.0.26 (IMO 2024 P5)

Turbo the snail plays a game on a board with 2024 rows and 2023 columns. There are hidden monsters in 2022 of the cells. Initially, Turbo does not know where any of the monsters are, but he knows that there is exactly one monster in each row except the first row and the last row, and that each column contains at most one monster.

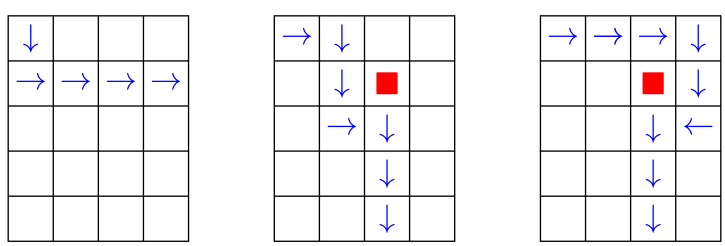
Turbo makes a series of attempts to go from the first row to the last row. On each attempt, he chooses to start on any cell in the first row, then repeatedly moves to an adjacent cell sharing a common side. (He is allowed to return to a previously visited cell.) If he reaches a cell with a monster, his attempt ends and he is transported back to the first row to start a new attempt. The monsters do not move, and Turbo remembers whether or not each cell he has visited contains a monster. If he reaches any cell in the last row, his attempt ends and the game is over.

Determine the minimum value of n for which Turbo has a strategy that guarantees reaching the last row on the n-th attempt or earlier, regardless of the locations of the monsters.

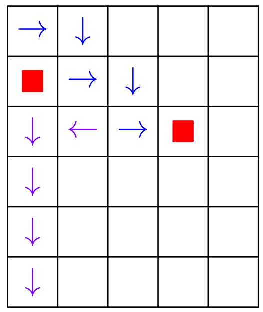
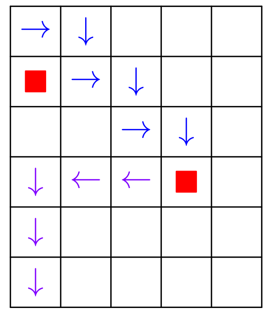
Proof. First notice observe the construction,

Lemma 1.23 It is possible to achieve $n = 3$.

Achieving this is relatively simple, with the first turn we can identify in which cell on the first row a monster is hidden (using the diagram on the left). If the first monster is not on the edges, then via two surrounding paths Turbo can make it to the last row (using the second and third diagram).



However, if the monster of the first row is on a edge we can do the ”zigzag” strategy,

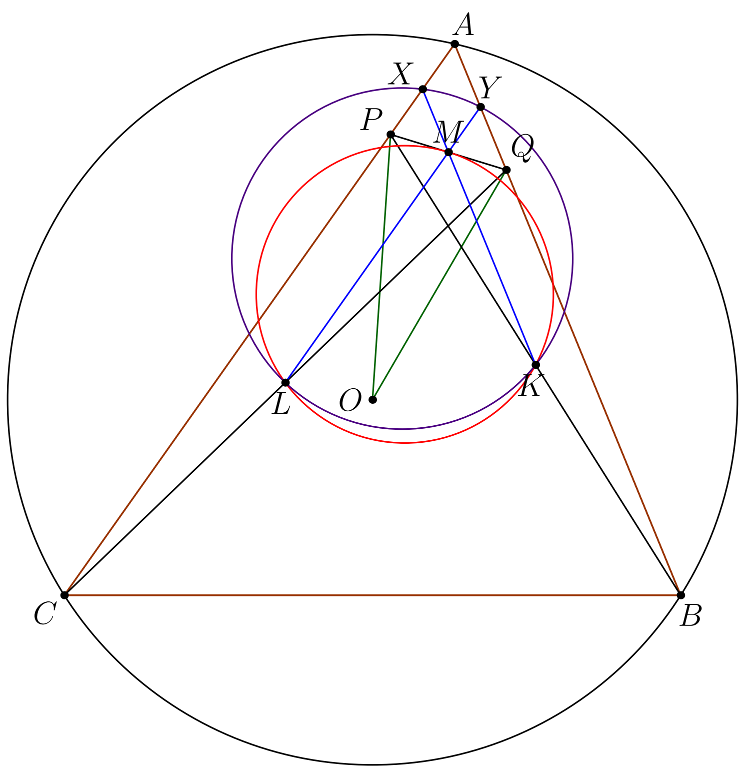
where the purple color represents the third attempt and the blue attempt represents the second attempt. (for the other position of the monster on the first row the same strategy still works).

All that is left to do, is to prove that Turbo will die at least 2 times, indeed this is simple to achieve, since Turbo can die on row 1 and then on row 2.

Thus, $n = 3$ is the smallest value for n. ■

Problem 1.0.27 (IMO 2009 P2)

Let $\triangle ABC$ be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L, M be the midpoints of BP, CQ, PQ. Suppose that PQ is tangent to the circumcircle of (KLM). Prove that $OP = OQ$.



Proof. Let X and Y be the midpoints of AP and AQ, respectively. Then, X, M and K lie on the midline in $\triangle APB$, consequently are collinear. Analogously L, M and Y are collinear. Notice,

$$\measuredangle XKL = \measuredangle PML = \measuredangle MPX = \measuredangle YXA = \measuredangle XYL$$

since $XY \parallel PQ$, $LY \parallel AC$ and $XK \parallel AB$, thus $LKXY$ is cyclic. Thus, by the Power of the Point it must be that $ML \cdot MY = MX \cdot MK$. Since all these segments are midlines in their respective triangles multiplying each segment by 2 we obtain,

$$CP \cdot PA = AQ \cdot QB$$

thus P and Q have the same power of the point with respect to (ABC). Thus it must be that $OP = OQ$. ∎

Problem 1.0.28 (ISL 2014 C2)

We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a+b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

Proof. Notice, after each step at least one of the numbers doubles, since WLOG $x \leq y$, then $x \to x + y \geq 2x$. Now, notice,

Lemma 1.24 There are no more than 2^m increases by ≤ 1 of the individual sheet.

since for a number to increase by one it must be that its counterpart in the pair is 1, however at the start there are 2^m ones, and after each such type of operation that number decreases by 1, thus no more than 2^m individual sheets increase by ≤ 1. □

Now, notice that this lemma can be generalized,

Lemma 1.25 There are no more than 2^m increases by x of the individual sheet, where $2^\alpha \leq x \leq 2^{\alpha+1}$.

The proof is essentially the same, since for an increase in such a x it must be that the counterpart in the pair is such, however after each such operation the number of such values decreases by one, thus no more than 2^m individual sheets can increase by x. □

Since the increase after each operation directly depends on the smallest value it must be that the sum of all the sheets is at least the sum obtained by doing 2^m operations on each power of two increase from the lemma. Thus, if X is the sum of sheets, then since each type of increase can be used no more than 2^m times it must be that X is at least the value obtained when performing the operations to cover the first $2 \cdot \frac{m2^{m-1}}{2^m} = 2 \cdot \frac{m}{2} = m$ layers, where the factor of two appears since each operation affects 2 sheets. Consequently, since $1 + 1 + 2 + \ldots + 2^\alpha = 2^{\alpha+1}$ (the original one is since it starts off with 1) it must be that,

$$X \geq 2^m \cdot 2^{2 \cdot \frac{m2^{m-1}}{2^m}}$$

where the 2^m appears since each layer contains 2^m sheets. Thus, since,

$$2^m \cdot 2^{2 \cdot \frac{m2^{m-1}}{2^m}} = 2^m \cdot 2^m = 4^m$$

we obtain the desired result. ■

Problem 1.0.29 (IMO Shortlist 2024 N1)

Find all positive integers n with the following property: for all positive divisors d of n, we have $d+1 \mid n$ or $d+1$ is prime.

Proof. Notice, if $n \neq 1$, then,

Lemma 1.26 $2 \mid n$

If $p \neq 2$ is the smallest divisor of n, then,

$$p+1 \mid n$$

since $p+1$ is odd and thus not prime. However, if $p+1$ divides n it must be that $p \mid p+1$ which is impossible, contradiction! $\square$

Let us assume that $n \neq 2^\alpha$ and $8 \mid n$, which implies that $9 \mid n \implies 3 \mid n$ (since 9 is not prime).

Lemma 1.27 $5 \mid n$

indeed,

$$8 \mid n \wedge 3 \mid n \implies 24 \mid n \implies 25 \mid n \implies 5 \mid n \ \square$$

Now,

Lemma 1.28 $7 \mid n$

since,

$$3 \mid n \wedge 5 \mid n \implies 15 \mid n \implies 16 \mid n$$

$$16 \mid n \wedge 3 \mid n \implies 48 \mid n \implies 49 \mid n \implies 7 \mid n \ \square$$

Now, an important lemma,

Lemma 1.29 If d is the largest odd divisor of n, then $2^{v_2(n)} > d$.

Indeed, $d+1 \mid n$ since $d+1$ is even, however $(d, d+1) = 1$, thus it must be that $d+1 = 2^\alpha$ where $\alpha \leq v_2(n)$, consequently it must be that $2^{v_2(n)} > d$. $\square$

Lemma 1.30 No required n exists. (Assuming $n \neq 2^\alpha$ and $8 \mid n$)

Notice that $5 \equiv 2 \pmod 3$ and $7 \equiv 1 \pmod 3$. Consequently either $5 \cdot 2^{v_2(n)} + 1$ or $7 \cdot 2^{v_2(n)} + 1$ is divisible by 3, thus not prime. Notice $2^{v_2(n)}$ must be larger than the largest odd divisor of n we obtain a contradiction! $\square$

Now, if $n = 2^\alpha$ it is trivial that $n = 1, 2, 4$ since $4 \mid n \implies 5 \mid n$. If $v_2(n) = 2$, then by claim 4 it must be that $n = 4 \cdot (\leq 3)$. Notice 8 doesn't work and 12 works.

Thus the only n that work are $\boxed{1, 2, 4, 12}$. $\blacksquare$

Problem 1.0.30 (ISL 2012 N1)

Call admissible a set A of integers that has the following property: If $x, y \in A$ (possibly $x = y$) then $x^2 + kxy + y^2 \in A$ for every integer k. Determine all pairs m, n of nonzero integers such that the only admissible set containing both m and n is the set of all integers

Proof. Notice,

Lemma 1.31 If $\gcd(n, m) \neq 0$, then the pair (n, m) doesn't work.

Assume that $d \mid n$ and $d \mid m$, then $x^2 + kxy + y^2$ is always divisible by d, thus if we consider the set of all integers divisible by d it would admissible. Thus, (n, m) doesn't work. □

Now,

Lemma 1.32 If $\gcd(n, m) = 1$, then the pair (n, m) works.

Notice, by substituting $x = y = n$ we obtain that kn^2 is in A for any k, analogously $km^2 \in A$ as well. Thus, by substituting $x = kn^2$ and $y = hm^2$ we obtain,

$$k^2n^4 - 2hn^2m^2 + h^2m^4 = (kn^2 - hm^2)^2 = \alpha$$

THus, since $\gcd(n^2, m^2) = 1$ it must mean that we can set $kn^2 - hm^2 = 1$, thus achieving $\alpha = 1$. However, if $1 \in A$.

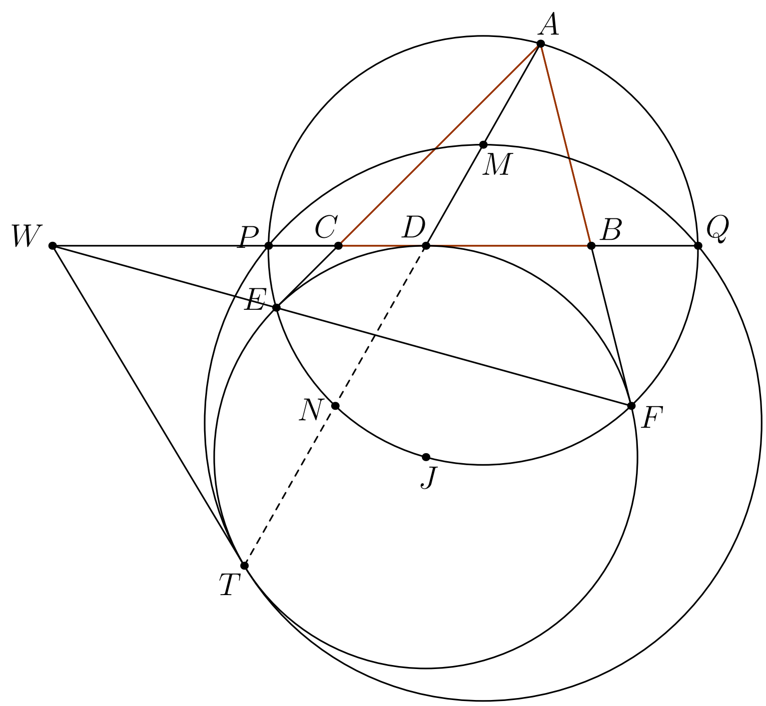
$$1 + k + 1 = 2 + k \in A$$

for any $k \in \mathbb{Z}$, thus $A = \mathbb{Z}$ □

Thus, the only pairs that satisfy the condition in the problem statement are those with $\gcd(n, m) = 1$. ■

Problem 1.0.31 (IMO Shortlist 2017 G4)

Let ABC be a triangle and let ω be the A-excircle, tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F. The circumcircle of $\triangle AEF$ intersects line BC at P and Q. Let M be the midpoint of $\overline{AD}$. Prove that the circumcircle of $\triangle MPQ$ is tangent to ω.



Proof. Let T be the intersection of AD with ω. Then, let N be the midpoint of DT,

Lemma 1.33 $N \in (AEF)$

since J is the center of $(DEFT)$ it must be that $JN \perp DT$, thus $\angle JNA = \angle JEA = \angle JFA = 90$, thus $N \in (AEF)$. $\square$

Lemma 1.34 $(PQMT)$ is cyclic

since by Power of the Point,

$$DT \cdot DM = DN \cdot DA = DP \cdot DQ$$

proving that $PQMT$ is cyclic. $\square$

Now, since $DEFT$ is harmonic since $A \in DT$ it must be that $W = BC \cap EF$ lies on the tangent to (DEF) at T. However, by the radical center theorem applied to (AEF), (MPQ) and $DEF)$ it must be that EF, BC and the radical axis of (DEF) and (PMQ) intersect at W. Consequently it must be that WT is tangent to $(MPQT)$ as well. $\blacksquare$

Problem 1.0.32 (IMO 2001 P4)

Let $n > 1$ be an odd integer and let $c_1, c_2, \ldots, c_n$ be integers. For each permutation $a = (a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$, define $S(a) = \sum_{i=1}^{n} c_i a_i$. Prove that there exist two permutations $a \neq b$ of $\{1, 2, \ldots, n\}$ such that $n!$ is a divisor of $S(a) - S(b)$.

Proof. Notice,

$$\sum_{a \in S_n} S(a) \equiv (n-1)! \cdot \frac{n(n+1)}{2} \cdot \sum_{i=1}^{n} c_i \pmod{n!} \tag{1}$$

Since if $S(a) \not\equiv S(b) \pmod{n!}$ for all $a, b \in S_n$ it must be that since $|S_n| = n!$ that all $S(a)$ are different modulo $n!$, thus,

$$\sum_{a \in S_n} S(a) \equiv \frac{n!(n!+1)}{2} \pmod{n!} \tag{2}$$

Thus,

$$(n-1)! \cdot \frac{n(n+1)}{2} \cdot \sum_{i=1}^{n} c_i \equiv \frac{n!(n!+1)}{2} \pmod{n!}$$

however the LHS is zero, since $n+1$ is divisible by two, thus LHS is divisible by $n!$, however RHS isn't divisible by $n!$, since $n!+1$ is odd, thus $n! \nmid \frac{n!(n!+1)}{2}$. Thus, LHS is not congruent to RHS, contradiction!

Thus, there exist $a, b \in S_n$ such that $n! \mid S(a) - S(b)$. ■

§2 EGMO

Problem 2.0.1 (EGMO 2025 P1)

For a positive integer N, let $c_1 < c_2 < \cdots < c_m$ be all positive integers smaller than N that are coprime to N. Find all $N \geqslant 3$ such that

$$\gcd(N, c_i + c_{i+1}) \neq 1$$

for all $1 \leqslant i \leqslant m - 1$

Here $\gcd(a, b)$ is the largest positive integer that divides both a and b. Integers a and b are coprime if $\gcd(a, b) = 1$.

Proof. Notice, that if N is even, then all c_i are odd, however that implies that $c_i + c_{i+1}$ will always be even, thus never relatively prime with N. Consequently, every even N satisfies the conditions of the problem statement. If N is odd, then it is by definition relatively prime to 2, thus,

$$\gcd(N, 1 + 2) \neq 1 \implies 3 \mid N$$

Thus, $N = 3^\alpha x$ (where x is odd), however,

Case 1) If $x \equiv 1 \pmod 3$, then let us consider two numbers, $x + 1$ and $x + 3$, then,

$$\gcd(3^\alpha x, x + 1) = 1$$

$$\gcd(3^\alpha x, x - 2) = 1$$

But, then,

$$(3^\alpha x, 2x - 1) = (3^\alpha, 2x - 1) = 1$$

contradiction!

Case 2) If $x \equiv 2 \pmod 3$, then consider $x + 2$ and $x - 1$, then,

$$\begin{cases} \gcd(3^\alpha x, x + 2) = 1 \\ \gcd(3^\alpha x, x - 1) = 1 \end{cases}$$

then,

$$\gcd(3^\alpha x, 2x + 1) = \gcd(3^\alpha, 2x + 1) = 1$$

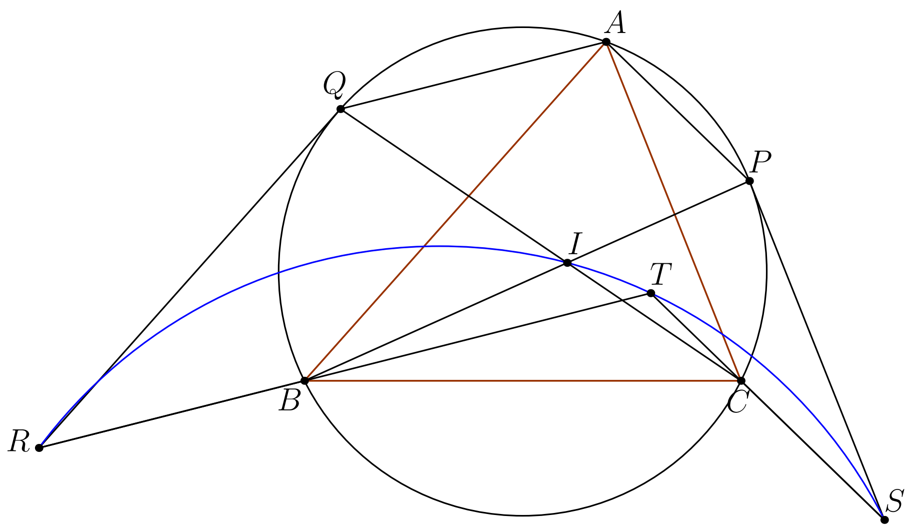
contradiction!

Thus $x = 1$. Consequently, the only solutions are all even numbers and all powers of three. ■

Problem 2.0.2 (EGMO 2025 P4)

Let ABC be an acute triangle with incentr I and $AB \neq AC$. Let lines BI and CI intersect the circumcircle of ABC at $P \neq B$ and $Q \neq C$, respectively. Consider points R and S such that $AQRB$ and $ACSP$ are parallelograms (with $AQ \parallel RB, AB \parallel QR, AC \parallel SP$, and $AP \parallel CS$). Let T be the point of intersection of lines RB and SC. Prove that points R, S, T, and I are concyclic.

Proof. At first it might seem unclear how to prove cyclicity of these four points.



However, after playing around with some angles it is not difficult to notice,

Lemma 2.1 $BITC$ is cyclic.

This is because,

$$\begin{aligned}\angle BTC &= 180 - \angle TBC - \angle TCB = 180 - (\angle B - \angle ABT) - (\angle C - \angle TCA) \\ &= \angle A + \angle QAB + \angle CAP = \angle A + \frac{\angle C}{2} + \frac{\angle B}{2} = 90 + \frac{\angle A}{2} = \angle BIC \quad (7)\end{aligned}$$

Now, notice,

Lemma 2.2 $\triangle IBR \sim \triangle ICS$

Because, ($\triangle IBQ \sim \triangle ICP$)

$$\frac{BI}{BR} = \frac{IB}{QA} = \frac{IB}{BQ} = \frac{IC}{CP} = \frac{IC}{PA} = \frac{IC}{CS}$$

and $\angle RBI = \angle ICS$ since $\angle IBT = \angle TCI$ from $ITBC$ being cyclic. But this similarity implies that $\angle BIR = \angle CIS$, thus,

$$\angle RIS = \angle BIC + \angle RIB - \angle ICS = \angle BIC = \angle RTS$$

Consequently, $RITS$ is cyclic. ■

Problem 2.0.3 (EGMO 2025 P5)

Let $n > 1$ be an integer. In a configuration of an $n \times n$ board, each of the n^2 cells contains an arrow, either pointing up, down, left, or right. Given a starting configuration, Turbo the snail starts in one of the cells of the board and travels from cell to cell. In each move, Turbo moves one square unit in the direction indicated by the arrow in her cell (possibly leaving the board). After each move, the arrows in all of the cells rotate 90° counterclockwise. We call a cell good if, starting from that cell, Turbo visits each cell of the board exactly once, without leaving the board, and returns to her initial cell at the end. Determine, in terms of n, the maximum number of good cells over all possible starting configurations.

Proof. Notice, if n is odd, then n^2 is odd and thus it is impossible to make a cycle (due to parity).

Let us consider the very edges of the board, the only way to go through them is using one of the two following algorithms,

$\cdots$	$\rightarrow$	$\leftarrow$
		$\nearrow$
$\iddots$		$\vdots$

$\cdots$	$\nearrow$	$\uparrow$
		$\uparrow$
$\iddots$		$\vdots$

where by $\nearrow$ I mean either $\rightarrow$ or $\uparrow$. Thus, due to the cycles being uniquely determined after the turn, it must be due to there being only two ways of going through the corners, only two global cycles. However, notice that the two configurations above are not equivelent under the rotation operation, thus there cannot be more than one global cycle. However, given a cycle there are only $\frac{n^2}{4}$ generators (every fourth element in the cycle).

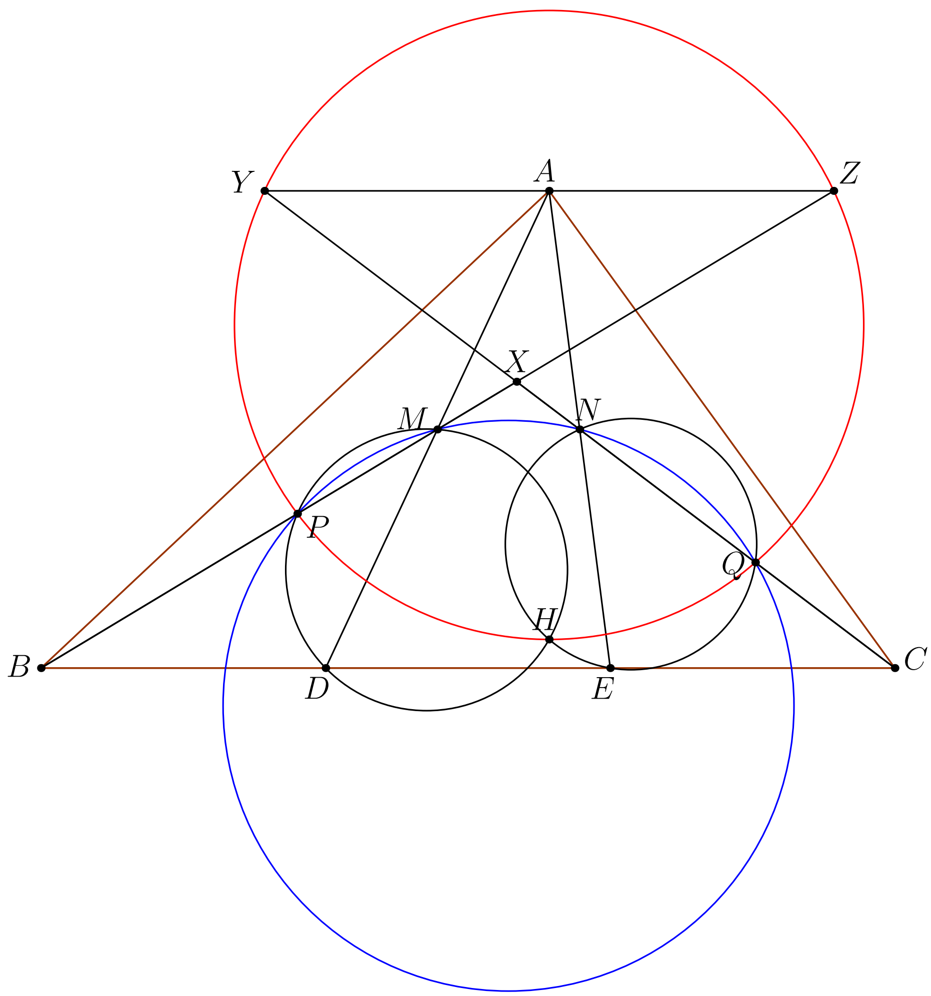
To achieve this bound, simply consider any global cycle through the entire board and appropriatly adjust the arrows so that it is a cycle from a given square. Then, due to the entire board taking the original position after four rotations, it must be that every fourth element in the cycle is a generator as well.

Thus, for all odd n there are 0 good cells, however for even n there are $\frac{n^2}{4}$ good cells. $\blacksquare$

Problem 2.0.4 (EGMO 2025 P3)

Let ABC be an acute triangle. Points B, D, E, and C lie on a line in this order and satisfy $BD = DE = EC$. Let M and N be the midpoints of AD and AE, respectively. Suppose triangle ADE is acute, and let H be its orthocentre. Points P and Q lie on lines BM and CN, respectively, such that D, H, M, and P are concyclic and pairwise different, and E, H, N, and Q are concyclic and pairwise different. Prove that P, Q, N, and M are concyclic.

Proof. Let us reflect C and B over N and M, respectively, then we will obtain points Y and Z. Let X be the intersect of BM and CN.



Notice,

> **Lemma 2.3** $AZHE$ is cyclic.

This is because,

$$\measuredangle AHE = \measuredangle EDA = \measuredangle AZE$$

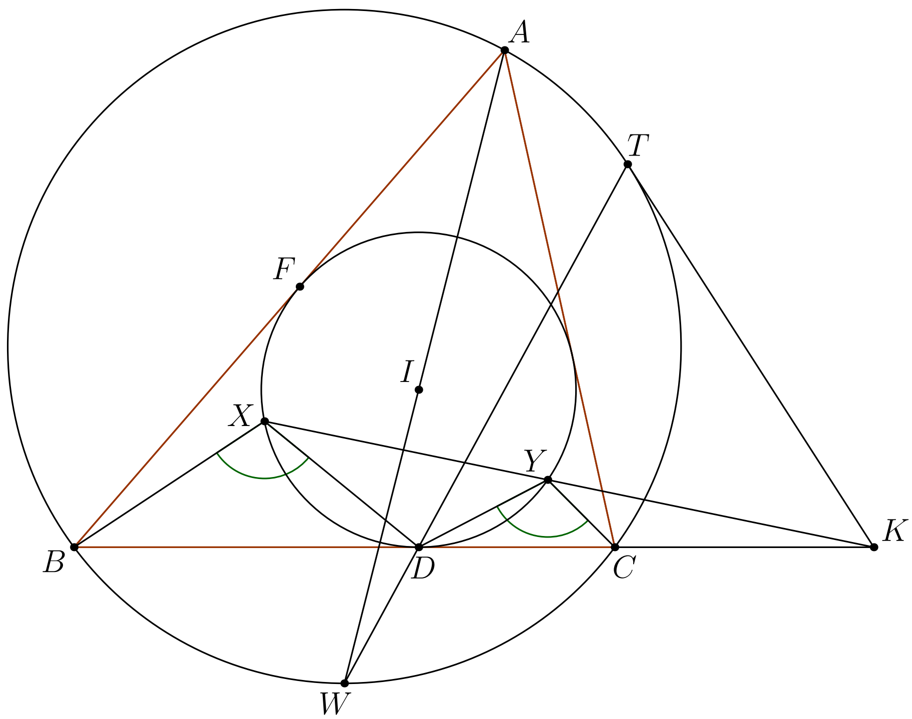
Now, notice that,

$$\measuredangle AZH = \measuredangle AEH = \measuredangle YQH$$

thus $YZHQ$ is cyclic, analogously it must be that $YZPH$ is cyclic, consequently $YZPQH$ is cyclic. However, because $MN \parallel YZ$ by Reim's lemma it must be that $MNPQ$ is cyclic. ■

Problem 2.0.5 (EGMO 2024 P2)

Let ABC be a triangle with $AC > AB$, and denote its circumcircle by Ω and incentre by I. Let its incircle meet sides BC, CA, AB at D, E, F respectively. Let X and Y be two points on minor arcs $\widehat{DF}$ and $\widehat{DE}$ of the incircle, respectively, such that $\angle BXD = \angle DYC$. Let line XY meet line BC at K. Let T be the point on Ω such that KT is tangent to Ω and T is on the same side of line BC as A. Prove that lines TD and AI meet on Ω.



Proof. Notice,

$$\measuredangle BXY = \measuredangle BXD + \measuredangle DXY = \measuredangle DYC + \measuredangle CDY = \measuredangle BCY$$

Thus, the wierd condition that $\measuredangle BXD = \measuredangle DYC$ is simply equivelent to BXYC being cyclic. However, notice that as we move X on the incircle, if we define Y to be the intersection of (BXC) with the incircle, then XY passes through a constant point. This is simply due to the power of the point, if a ray through K intersects the incircle at X' and Y', then,

$$KC \cdot KB = KX \cdot KY = KX' \cdot KY'$$

thus, $BX'Y'C$ is cyclic. Consider a circle tangent to KT at T which passes through D, assume that it passes through another point on BC, let it be D', then,

$$KD \cdot KD' = KT^2 = KX \cdot KY$$

however $KD^2 = KX \cdot KY$, thus $D = D'$. Thus, a circle exists which is tangent to TK at T and which is tangent to BC at D, however then by the Shooting lemma it must be that DT passes through the midpoint of the arc $\widehat{BC}$.

Thus, AI and TD intersect on Ω, exactly at the midpoint of the arc $\widehat{BC}$. ■

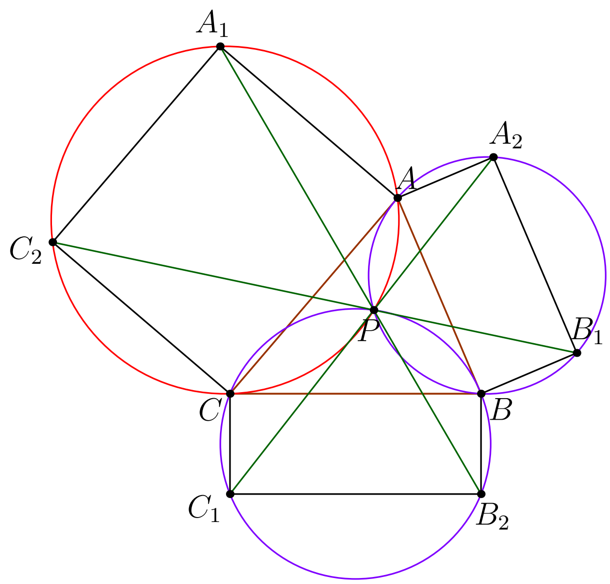
§3 USAMO

Problem 3.0.1 (USAMO 2021 P1)

Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that

$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180°.$$

Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.



Proof. Let P be the intersection of (BCC_1B_2) and (ABB_1A_2).

Lemma 3.1 C_1, P and A_2 are colinear.

Indeed, since C_1 and A_2 are the images of $O_(BCC_1B_2)$ and $O_(ABB_1A_2)$ under homothepy with coefficient 2 centered at B, consequently since PB is the radical axis it must be that C_1, P, A_2 are colinear.

Lemma 3.2 CC_2AA_1 is cyclic.

Indeed, since,

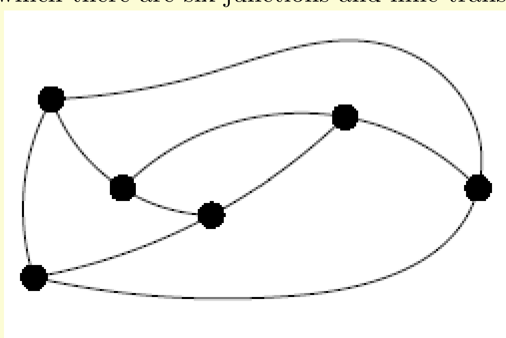
$$\angle CPA = 360 - \angle CPB - \angle APB = \angle BC_1C + \angle AB_1B = 180 - \angle CA_1A$$

where the last step is due to the condition in the problem statement.

Since CC_2AA_1P is cyclic, we can apply analogous logic from the first lemma to prove that A_1, P, B_2 and C_2, P, B_1 are colinear, proving the problem statement. ■

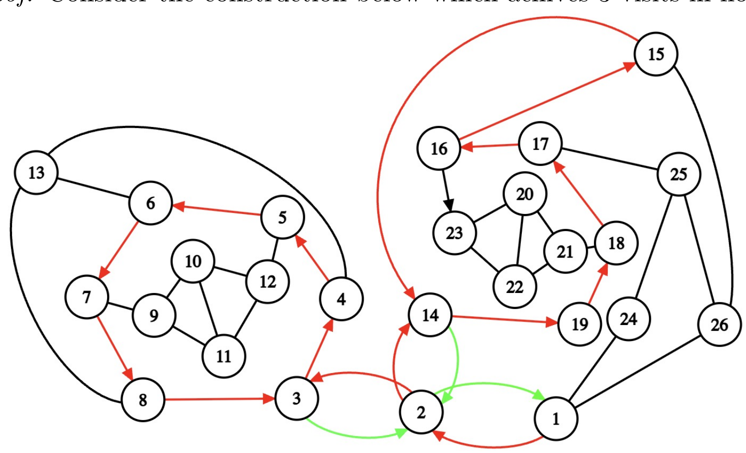
Problem 3.0.2 (USAMO 2021 P2)

The Planar National Park is a subset of the Euclidean plane consisting of several trails which meet at junctions. Every trail has its two endpoints at two different junctions whereas each junction is the endpoint of exactly three trails. Trails only intersect at junctions (in particular, trails only meet at endpoints). Finally, no trails begin and end at the same two junctions. (An example of one possible layout of the park is shown to the left below, in which there are six junctions and nine trails.)



A visitor walks through the park as follows: she begins at a junction and starts walking along a trail. At the end of that first trail, she enters a junction and turns left. On the next junction she turns right, and so on, alternating left and right turns at each junction. She does this until she gets back to the junction where she started. What is the largest possible number of times she could have entered any junction during her walk, over all possible layouts of the park?

Proof. Consider the construction below which achives 3 visits in node 2.



Now, let us prove that it is impossible to achieve more than 3 visits.

Lemma 3.3 If the visitor visits $X \to Y$, where X and Y are some edges, then the visitor cannot visit $X \to Y$ again.

indeed, from $X \to Y$ the parity can be deduced and thus the entire path from start to end as well. However, if $X \to Y$ appears a second time it must mean that there is a cycle, obviously the start node is within this cycle, consequently it must be that we visited the starting node at least twice, contradiction!

Now, notice,

Lemma 3.4 If the visitor visits $X \to Y$, where X and Y are some edges, then the visitor cannot visit $Y \to X$.

indeed, if we backtrack we will notice that the two paths follow the same path, but in reverse with the same parity, thus they can never come together and thus will never complete the path, contradiction!

Now, consider any node A, it has three edges X, Y, Z, in combination with the first and second lemma we conclude that all the visits must be of the form $\{X \leftrightarrow Y, Y \leftrightarrow Z, Z \leftrightarrow X\}$, thus no more than 3 visits can be achieved.

Thus, the answer is $\boxed{3}$. ■

§4 USA TSTST

Problem 4.0.1 (USA TSTST 2015 Day 1 P1)

Let $a_1, a_2, \ldots, a_n$ be a sequence of real numbers, and let m be a fixed positive integer less than n. We say an index k with $1 \leq k \leq n$ is good if there exists some ℓ with $1 \leq \ell \leq m$ such that $a_k + a_{k+1} + \ldots + a_{k+\ell-1} \geq 0$, where the indices are taken modulo n. Let T be the set of all good indices. Prove that $\sum\limits_{k \in T} a_k \geq 0$.

Proof. Let us say that,

Definition 4.1 An interval is *good* if the sum of elements in it is non-negative.

Let us define for each index an interval which starts at that index, which is the smallest size such that the sum of the values inside of it are non-negative. Then,

Lemma 4.2 No two intervals that are not a subset of one another intersect.

Assume that one interval is $a_l, \ldots, a_w, \ldots, a_r$ and another interval is $a_w, \ldots, a_r, \ldots, a_s$. Then, since all intervals are the smallest it must be that $\sum_{i=l}^{w-1} a_i < 0$ and that $\sum_{i=w}^{r} > 0$, however this would contradict then the minimality of the second interval since it wasn't the shortest interval starting at w with a non-negative sum.

Obviously if the second interval intersects the first due to modulo n indeces we again obtain a contradiction since any prefix in the first interval has a strictly negative sum. □

Now, notice,

Lemma 4.3 Given an interval all indices inside of it are *good*.

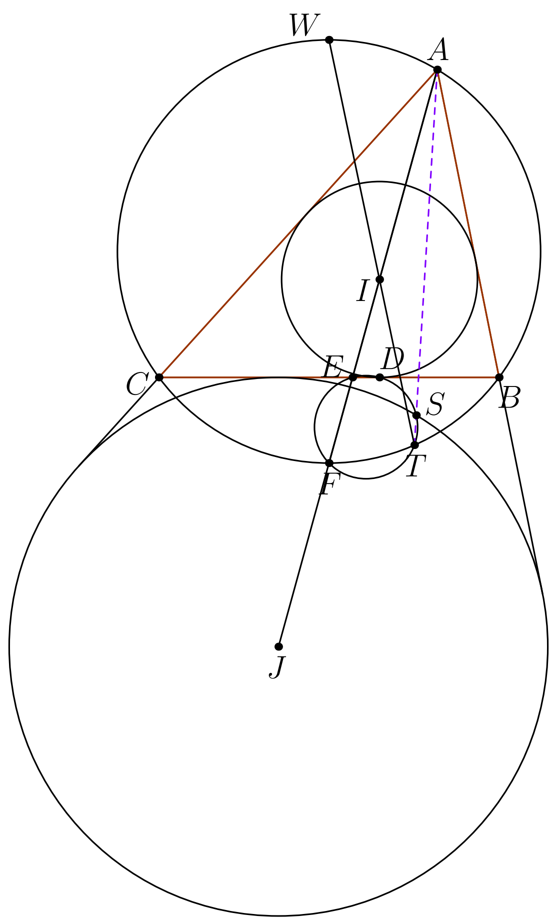
indeed, since if $a_l, \ldots, a_w, \ldots, a_r$ is a the smallest *good* interval of l it must be that $\sum_{i=l}^{w-1} a_i < 0$, thus $\sum_{i=w}^{r} a_i > 0$, thus it must be that w is *good* as well. □

But, this implies that $\sum_{k \in T} a_k$ is simply the sum of the sums of values in all independent *good* intervals. However, each *good* interval by definition has a non-negative sum, thus $\sum_{k \in T} a_k$ must be non-negative as well. ■

§5 USA TST

Problem 5.0.1 (USA TST 2016 P2)

Let ABC be a scalene triangle with circumcircle Ω, and suppose the incircle of ABC touches BC at D. The angle bisector of $\angle A$ meets BC and Ω at E and F. The circumcircle of $\triangle DEF$ intersects the A-excircle at S_1, S_2, and Ω at $T \neq F$. Prove that line AT passes through either S_1 or S_2.



Proof. Let X be the tangency point of the A-excircle with BC. Then,

Lemma 5.1 T is the A-mixtilinear tangency point.

Indeed, if T is the tangency point it is well known that $\angle BTA = \angle CTD$. Consequently,

$$\measuredangle DTF = \measuredangle BTF - \angle BTD = \measuredangle BCF - \measuredangle CTA = \measuredangle CEF$$

thus, proving that $DEFT$ is cyclic. □

Now, let S be the reflection of X over AI, trivially S lies on the A-excircle, however,

Lemma 5.2 S lies on (DEF)

indeed, since,

$$\measuredangle ASE = \measuredangle EXA = \measuredangle TFA$$

proving that $DEFS$ is cyclic. □

Thus, since T lies on the reflection of AX over AI it must be that A, X, S are collinear. ■

Problem 5.0.2 (USA TST 2016 P1)

Let $S = \{1, \ldots, n\}$. Given a bijection $f : S \to S$ an orbit of f is a set of the form $\{x, f(x), f(f(x)), \ldots\}$ for some $x \in S$. We denote by $c(f)$ the number of distinct orbits of f. For example, if $n = 3$ and $f(1) = 2$, $f(2) = 1$, $f(3) = 3$, the two orbits are $\{1, 2\}$ and $\{3\}$, hence $c(f) = 2$.

Given k bijections $f_1, \ldots, f_k$ from S to itself, prove that

$$c(f_1) + \cdots + c(f_k) \le n(k-1) + c(f)$$

where $f : S \to S$ is the composed function $f_1 \circ \cdots \circ f_k$.

Proof. Notice,

Lemma 5.3 $c(f_1) + c(f_2) \le n + c(f_1 \circ f_2)$

Assume that the statement is true for $|S| < n$, then let us prove that it is also true, when $|S| = n$. Let us interpret a bijection as a directed graph, then a bijection is simply a collection of cycles and $c(f)$ is simply the number of cycles.

If $|S| = n$, let us select an element X, then let us remove X from f_1 and f_2, i.e. if $u \to X \to v$ then one replaces it with $u \to v$. Then, notice that $f_1 \circ f_2$ is left mostly unaffected. Indeed, if $u_1 \to X \to v_1 \xrightarrow{f_2} a$ in f_1 and $b \xrightarrow{f_1} u_2 \to X \to v_2$ in f_2, then in $f_1 \circ f_2$

$$\begin{cases} u_1 \to v_2 \implies u_1 \to a \\ b \to X \implies b \to v_2 \end{cases}$$

thus we see that we create ≤ 1 new cycle and since $c(f_1^\star) + c(f_2^\star) \le (n-1) + c(f^\star)$ we see that the inequality still holds for $|S| = n$ (the cases are quite trivial). $\square$

Now, notice that by induction,

Lemma 5.4 If the statement is true for $k \le K$, then it is true for $k = K$ as well.

Indeed, since,

$$\sum_{i=1}^{K} c(f_i) = \left[\sum_{i=1}^{K-1} c(f_i)\right] + c(f_K) \le n(K-2) + c(f_1 \circ \ldots \circ f_{k-1}) + c(f_K) \le n(K-1) + c(f_1 \circ \ldots \circ f_K)$$

which proves the desired result. $\square$

Thus, by induction the statement is true for all $k \in \mathbb{N}$. ■

§6 APMO

Problem 6.0.1 (APMO 2019 P1)

Let $\mathbb{Z}^+$ be the set of positive integers. Determine all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $a^2 + f(a)f(b)$ is divisible by $f(a) + b$ for all positive integers a, b.

Proof. First,

Lemma 6.1 $f(1) = 1$

Notice, when substituting $a = b = 1$ we obtain $1 + f(1) \mid 1 + f(1)^2$, thus $1 + f(1) \mid f(1)(f(1) - 1)$, implying that $f(1) = 1$. $\square$

Now, assume that $f(x) = x$ for all $x < a$. Then,

Lemma 6.2 $f(a) = a$

since by substituting $b = a - 1$ we obtain,

$$f(a) + (a - 1) \mid a^2 + (a - 1)f(a)$$

substracting LHS from RHS $a - 1$ times we obtain,

$$f(a) + (a - 1) \mid a^2 - (a - 1)^2 = 2a - 1$$

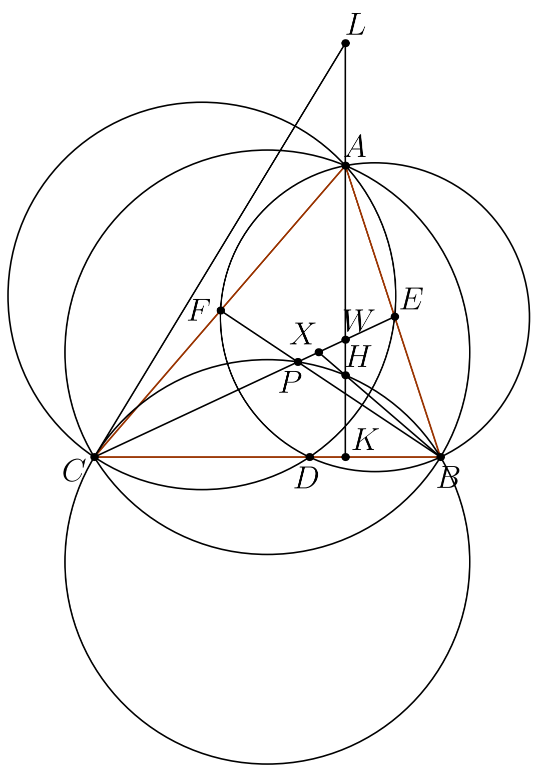
and since $f(a) + (a - 1) \geq \frac{2a-1}{3}$ ($2a - 1$ is odd) it must be that $f(a) + (a - 1) = 2a - 1$, thus $f(a) = a$. $\square$

Thus, it must be that $f(x) = x$ for all $x \in \mathbb{Z}^+$. $\blacksquare$

§7 Balkan MO

Problem 7.0.1 (Balkan MO 2025 P2)

In an acute-angled triangle ABC, H be the orthocenter of it and D be any point on the side BC. The points E, F are on the segments AB, AC, respectively, such that the points A, B, D, F and A, C, D, E are cyclic. The segments BF and CE intersect at P. L is a point on HA such that LC is tangent to the circumcircle of triangle PBC at C. BH and CP intersect at X. Prove that the points D, X, and L lie on the same line.



Proof. Notice,

Lemma 7.1 $BCPH$, $PBDE$, $CPDF$ and $AEFHP$ is cyclic

since,

$$\angle CPB = 180 - \angle ECB - \angle FBC = 180 - \angle DAE - \angle FAD = 180 - \angle A$$

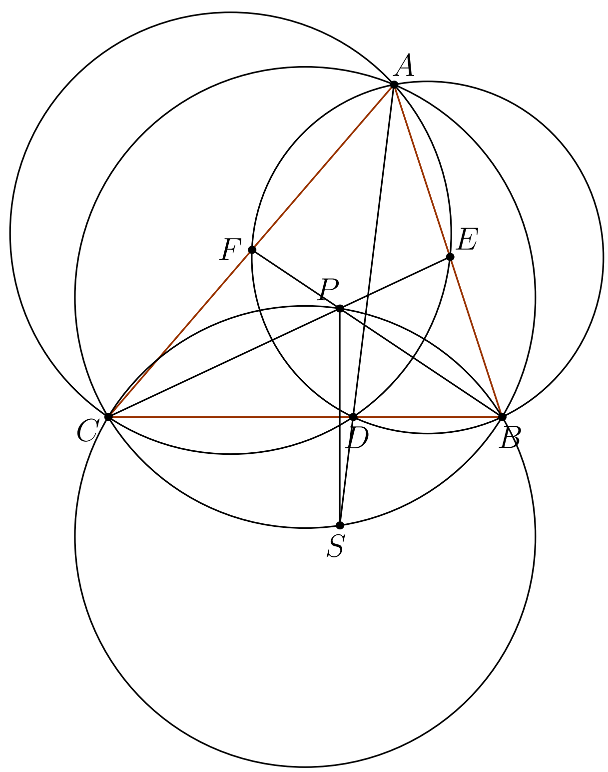
and since $\angle CHB = 180 - \angle A$ we obtain $BCPH$ is cyclic. The other cyclicities are analogous $\square$

Now, let $W = CP \cap AH$ and $K = AH \cap BC$. Then,

$$(C, P; H, B) \stackrel{C}{=} (L, W; H, K) \stackrel{X}{=} (D, C; B, K)$$

thus, to prove that L, X, D are collinear we need to prove that $(C, P; H, B) = (D, C; B, K)$.

Let S and T be the reflections of P over BC, respectively. Then, $(C, P; H, B) = (C, S; T, B)$, however then projecting through A we obtain $(C, AS \cap BC; K, B)$. Thus, to prove that L, X, D are collinear all that one has to show is that the reflection of P over BC lies on AD.



Since $EPDB$ is cyclic,

$$\angle BDS = 180 - \angle CDS = 180 - \angle CDP = 180 - \angle BFA = 180 - \angle ADB$$

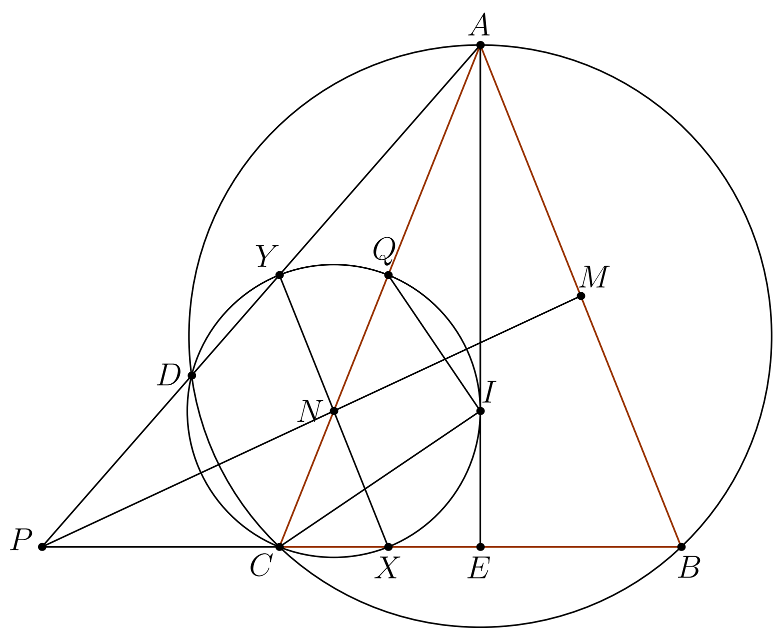
thus S, D and A are collinear.

$\blacksquare$

§8 Iran TST

Problem 8.0.1 (Iran TST 2020 Test 2 P4)

Let ABC be an isosceles triangle ($AB = AC$) with incenter I. Circle ω passes through C and I and is tangent to AI. ω intersects AC and circumcircle of ABC at Q and D, respectively. Let M be the midpoint of AB and N be the midpoint of CQ. Prove that AD, MN and BC are concurrent.



Proof. First, notice,

Lemma 8.1 $\angle CIQ = 90$

indeed, since,

$$\angle CIQ = 180 - \angle ACI - \angle CQI = 180 - \frac{\angle C}{2} - \angle CIE = 180 - \frac{\angle C}{2} - \frac{90 + \angle A}{2}$$
$$= 180 - 45 - 45 = 90$$

Consequently it must be that N is the circumcenter of (CIQ) as well. □

Let P be the intersection of NM and BC. Let X and Y be the intersections of (CIQ) with BC and AP, respectively. Then,

Lemma 8.2 N is the midpoint of XY

indeed, by **Reim's lemma** it must be that $XY \parallel AB$, and since $NX = CN$ it must be that $\triangle CNX$ is isoseles, thus X, N, Y are collinear. □

Since N is the midpoint of XY and $XY \parallel AB$ (since $\triangle CNX$ is isoseles) and P, N, M are colinear it must be that P, Y, A are collinear, however since D, Y, A are collinear it must be that P, D, A are collinear as well.

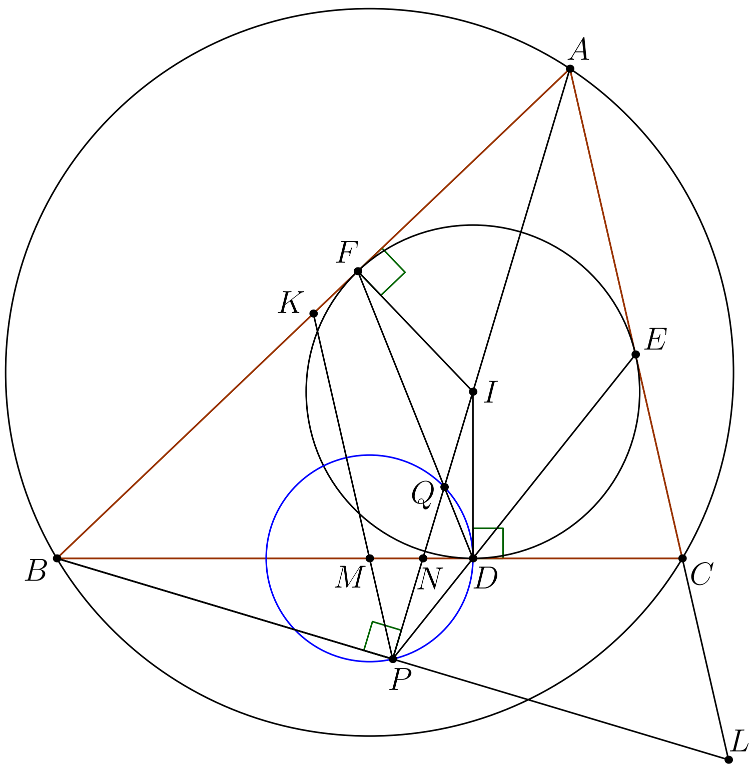
■

§9 Korean National Olympiad

Problem 9.0.1 (Korea 2025 P4)

Triangle ABC satisfies $\overline{CA} > \overline{AB}$. Let the incenter of triangle ABC be ω, which touches BC, CA, AB at D, E, F, respectively. Let M be the midpoint of BC. Let the circle centered at M passing through D intersect DE, DF at $P(\neq D), Q(\neq D)$, respecively. Let line AP meet BC at N, line BP meet CA at L. Prove that the three lines EQ, FP, NL are concurrent.

Proof. Let us start by proving that A, P, Q are colinear. Let us define P as the intersection of AI and DE, then we must prove that MPD is isoseles.



This can be easily shown by angle chase, let us introduce K, the midpoint of AB it is well known that then K, M, P are colinear and $\angle BPA = 90$. Thus,

$$\angle MPD = \angle APD + \angle KPA = \angle IBC + \angle PAC = \frac{\angle A + \angle B}{2} = \frac{180 - \angle C}{2} = \angle EDC = \angle MDP \quad (8)$$

Thus, it must be that P lies on AI, the exact same logic shows that $Q \in AI$, thus, A, I, Q and P are colinear.

Now, notice that under reflection over AI the line FQ goes to EQ and the line LN goes to BN (since $\angle APB = 90$ it must be that B goes to L under reflection) and PF goes to PE. Thus, EQ, LN and PF are concurrent, and the intersection point is the reflection of D over AI. $\blacksquare$

Problem 9.0.2 (KMO 2015 P1)

Find all functions $f : R \to R$ such that $f(x^{2015} + (f(y))^{2015}) = (f(x))^{2015} + y^{2015}$ holds for all reals x, y.

Proof. Notice,

Lemma 9.1 f is a bijection

Trivially f is surjective, since y^{2015} can be changed to an arbitrary real number. Now, assume that $f(a) = f(b)$, then let us fix x, then since f is surjective we can find the appropriate y_1 and y_2 such that,

$$\begin{cases} x^{2015} + f(y_1)^{2015} = a \\ x^{2015} + f(y_2)^{2015} = b \end{cases}$$

Then, since $f(a) = f(b)$ it must be that,

$$f(x)^{2015} + y_1^{2015} = f(x)^{2015} + y_2^{2015}$$

which implies that $y_1 = y_2$, thus that $a = b$. Thus, f is also injective. Consequently, since f is both surjective and injective it must be that f is a bijection. □

Now,

Lemma 9.2 f is an involution

since,

$$\begin{aligned} & f(x^{2015} + (f(y))^{2015}) = (f(x))^{2015} + y^{2015} \\ \implies & f(f(x^{2015} + (f(y))^{2015})) = f((f(x))^{2015} + y^{2015}) = x^{2015} + f(y)^{2015} \end{aligned}$$

where the last step is due to swapping x and y in the original functional equation. □

Thus, via substituting $y := f(y)$ we obtain,

$$f(x^{2015} + y^{2015}) = f(x)^{2015} + f(y)^{2015} \tag{$\star$}$$

Now,

Lemma 9.3 $f(0) = 0$

since f satisfies Cauchy's functional equation,

$$f(0) = 2f(0)^{2015}$$

However, when substituting $y := -f(x)$ into the original functional equation we obtain,

$$f(x^{2015} + f(-f(x))^{2015}) = f(x)^{2015} - f(x)^{2015} = 0$$

Let $x = f(0)$, then,

$$f(0 + f(0)^{2015}) = 0$$

thus, by injectivity $f(0)^{2025} = f(0) = 2f(0)^{2015}$, thus it must be that $f(0) = 0$. □

Consequently, it must be that $f(x^{2015}) = x^{2015}$ by substituting into $(\star)$ the condition $y = 0$. Consequently, we can rewrite $(\star)$ into $f(x^{2015} + y^{2015}) = f(x^{2025}) + f(y^{2025})$, thus f satisfies the Cauchy functional equation.

Since f satisfies Cauchy it must be that $f(x) = -f(x)$, since $0 = f(0) = f(x - x) = f(x) + f(-x)$. Also, notice that $f(1) = f(1)^{2015}$, thus $f(1) = \pm 1$ (cannot be zero since f is a bijection)

Lemma 9.4 (thanks @MathLuis) f is linear

Case $f(1) = 1)$ Let, $P(x) = (x + 1)^{2015} - x^{2015}$, then,

$$f(P(x)) = f((x+1)^{2015} - x^{2015}) = f(x+1)^{2015} + (-f(x))^{2015} = f(x+1)^{2015} - f(x)^{2015}$$
$$= \left(f(x)^{2015} + f(1)^{2015}\right)^{2015} - f(x)^{2015} = P(f(x))$$

However, since P has degree 2014, which is even, it must be that $P(f(x))$ is bounded from below, thus f is bounded on some continous interval, thus linear.

Case $f(1) = -1)$ Let, $P(x) = (x - 1)^{2015} - x^{2015}$, then,

$$f(P(x)) = f((x-1)^{2015} - x^{2015}) = f(x-1)^{2015} + (-f(x))^{2015} = f(x-1)^{2015} - f(x)^{2015}$$
$$= \left(f(x)^{2015} - f(1)^{2015}\right)^{2015} - f(x)^{2015} = P(f(x))$$

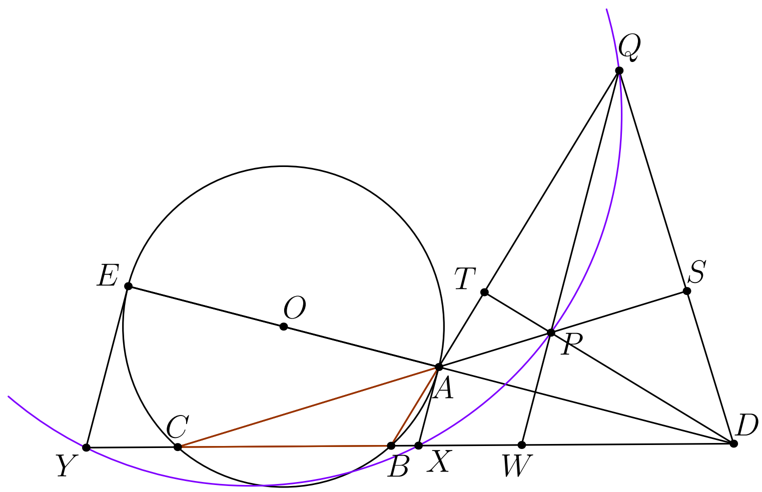
However, since P has degree 2014, which is even, it must be that $P(f(x))$ is bounded from below, thus f is bounded on some continous interval, thus linear.

□

Trivially, since $f(1) = \pm 1$, the only linear solutions are $f(x) = x$ and $f(x) = -x$. ■

Problem 9.0.3 (KMO 2023 Day 2 P2)

Let Ω and O be the circumcircle and the circumcenter of an acute triangle ABC ($\overline{AB} < \overline{AC}$). Define $D, E(\neq A)$ be the points such that ray AO intersects BC and Ω. Let the line passing through D and perpendicular to AB intersects AC at P and define Q similarly. Tangents to Ω on A, E intersects BC at X, Y. Prove that X, Y, P, Q lie on a circle.



Proof. Notice,

Lemma 9.5 $BCPQ$ is cyclic

Indeed, since P is the orthocenter of $\triangle AQD$ it must be that,

$$\measuredangle BCP = \measuredangle XAB = \measuredangle TSA = \measuredangle AQW \ \square$$

Now notice,

Lemma 9.6 $TS \parallel BC$

since,

$$\measuredangle TSC = \measuredangle TQP = \measuredangle BCP \ \square$$

Consequently,

Lemma 9.7 (PQD) is tangent to BC

since,
$$\measuredangle PQD = \measuredangle PTS = \measuredangle TDC \ \square$$

Now, by **DIT** (*Desargues' Involution Theorem*) applied to A, A, E, E it must be that there exists an involution on BC such that $B \leftrightarrow C$, $X \leftrightarrow Y$ and $D \leftrightarrow D$. However, from the tangency lemma it must be that $WD^2 = WP \cdot WQ = WB \cdot WC$, and since an involution is simply inversion and is uniquely determined by three pairs it must be that W is the center of inversion with radius WD, thus $WD^2 = WX \cdot WY$. Consequently, $WX \cdot WY = WD^2 = WB \cdot WC = WP \cdot WQ$, thus $XYPQ$ is cyclic.

■