§1 IMO

Problem 1.1 (IMO 2019 P1)

Find all $f : \mathbb{Z} \to \mathbb{Z}$ such that,

$$f(2x) + 2f(y) = f(f(x+y))$$

Proof. Let $P(x, y)$ the condition generated by x, y. Then,

$$\begin{cases} P(x,0) : f(2x) + 2f(0) = f(f(x)) \\ P(0,x) : f(0) + 2f(x) = f(f(x)) \end{cases}$$

Combining these two we obtain that,

$$f(2x) = 2f(x) - f(0)$$

Now let us consider the following,

$$P(x,1) : f(2x) + 2f(1) = f(f(x+1))$$

However from the previous observations we can say that,

$$f(f(x+1)) = 2f(x+1) + f(0)$$

Thus,

$$2f(x) + 2f(1) - f(0) = 2f(x+1) + f(0)$$
$$f(x+1) = f(x) + f(1) - f(0)$$

Thus, we see that $f(x) = f(0) + x \cdot C$ where C is some constant (specifically $f(1) - f(0)$) for all $x \in \mathbb{Z}$. Now let us consider,

$$P(x, -x) : f(2x) + 2f(-x) = f(f(0))$$

We can rewrite this as,

$$f(0) + 2xC + 2f(0) - 2xC = f(0) + f(0) \cdot C$$
$$\implies 2f(0) = f(0) \cdot C$$

This implies that either $f(0) = 0$ or $C = 2$. Let us consider both cases,

- If $f(0) = 0$, then, $f(x) = cx$ and it is simple to verify that then $f(x)$ must equal $2x$ or 0.
- If $C = 2$, then,

$$f(0) + 4x + 2(f(0) + 2y) = f(f(0) + 2(x+y)) = f(0) + 2f(0) + 4(x+y)$$
$$4x + 2f(0) + 4y = 2f(0) + 4x + 4y$$

Thus, we see the only solutions to this functional equation are $f(x) = 2x + c$ and $f(x) = 0$. □

Problem 1.2 (IMO 2022 P2)

Find all $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$ there exists exactly one $y \in \mathbb{R}^+$ such that,

$$xf(y) + yf(x) \leq 2$$

Proof. Obviously $f(x) = \frac{1}{x}$ works (from AM-GM).

Let us say that $x \sim y$ if and only if the condition from the problem statement is true. Then obviously this is commutative, meaning that if $x \sim y$ then $y \sim x$.

Lemma 1.1 If $f(x) \leq \frac{1}{x}$, then, $x \sim x$.

This is simple to see due to,

$$2xf(x) \leq 2x \cdot \frac{1}{x} = 2$$

However, notice that if $x \sim y$, then,

$$xf(y) + yf(x) \leq 2$$

and in it must be that $f(x) > \frac{1}{x}$ and $f(y) > \frac{1}{y}$ which would imply that by AM-GM,

$$xf(y) + yf(x) = \frac{x}{y} + \frac{y}{x} \geq 2$$

Thus we obtain that for all $x \in \mathbb{R}^+$ it must be that $x \sim x$. Thus that,

$$f(x) \leq \frac{1}{x}$$

for all $x \in \mathbb{R}^+$.

Notice that if $f(a) < \frac{1}{a}$, then, if,

$$f(a) = \frac{1}{a} - \epsilon$$

then,

$$xf(a) + af(x) = x\left(\frac{1}{a} - \epsilon\right) + af(x) \leq x\left(\frac{1}{a} - \epsilon\right) + \frac{a}{x} \overset{?}{\leq} 2$$

However the last inequality for $\epsilon > 0$ has multiple solutions for x. Because,

$$x^2\left(\frac{1}{a} - \epsilon\right) + a - 2x \leq 0$$

The only way for this inequality to have a single solution for $x \in \mathbb{R}^+$ is if the equality has only one solution, thus the discriminant is zero,

$$\Delta = 4 - 4 \cdot a \cdot \left(\frac{1}{a} - \epsilon\right)$$

which is obviously never 0 for $\epsilon > 0$. Thus because for each x there is only one such y that $x \sim y$ it must mean that,

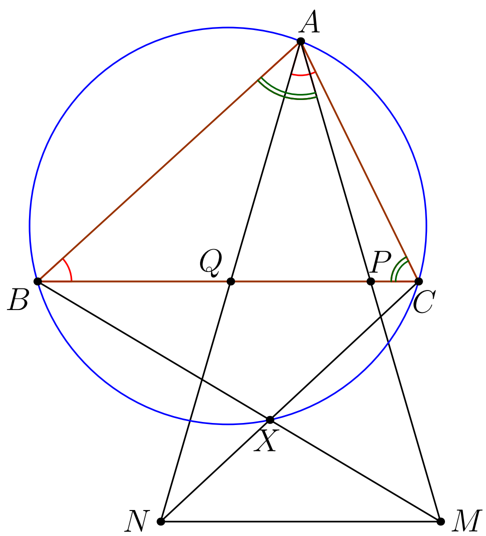
$$f(x) = \frac{1}{x}$$

for all $x \in \mathbb{R}^+$. $\square$

Problem 1.3 (IMO 2014 P4)

Let P and Q be on segment BC of an acute triangle $\triangle ABC$ such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the intersection of BM and CN is on the circumference of triangle $\triangle ABC$.

Proof. Let $\triangle ABC$ be the reference triangle in barycentric coordinates.



Then we can calculate the point P, because $\triangle ABP \sim \triangle CBA$ and $\triangle ACQ \sim \triangle BCA$. Thus,

$$\frac{BA}{BC} = \frac{BP}{BA}$$

$$\implies BP = \frac{BA^2}{BC} = \frac{c^2}{a}$$

consequently, $P = \left(0 : a - \frac{c^2}{a} : \frac{c^2}{a}\right)$, analogously we get, $Q = \left(0 : \frac{b^2}{a} : a - \frac{b^2}{a}\right)$. Now we can calculate points M and N,

$$M = \left(-a^2 : 2a^2 - 2c^2 : 2c^2\right)$$

$$N = \left(-a^2 : 2b^2 : 2a^2 - 2b^2\right)$$

Now we simply need to intersect BM and CN which is not hard to do considering that both are cevians in the triangle. It is not hard to see that the following point satisfies these conditions,

$$X = \left(-a^2 : 2b^2 : 2c^2\right)$$

Now we just need to check that this point lies on the circumcircle of $\triangle ABC$ which is given by,

$$a^2yz + b^2xz + c^2xy = 0$$

Thus,

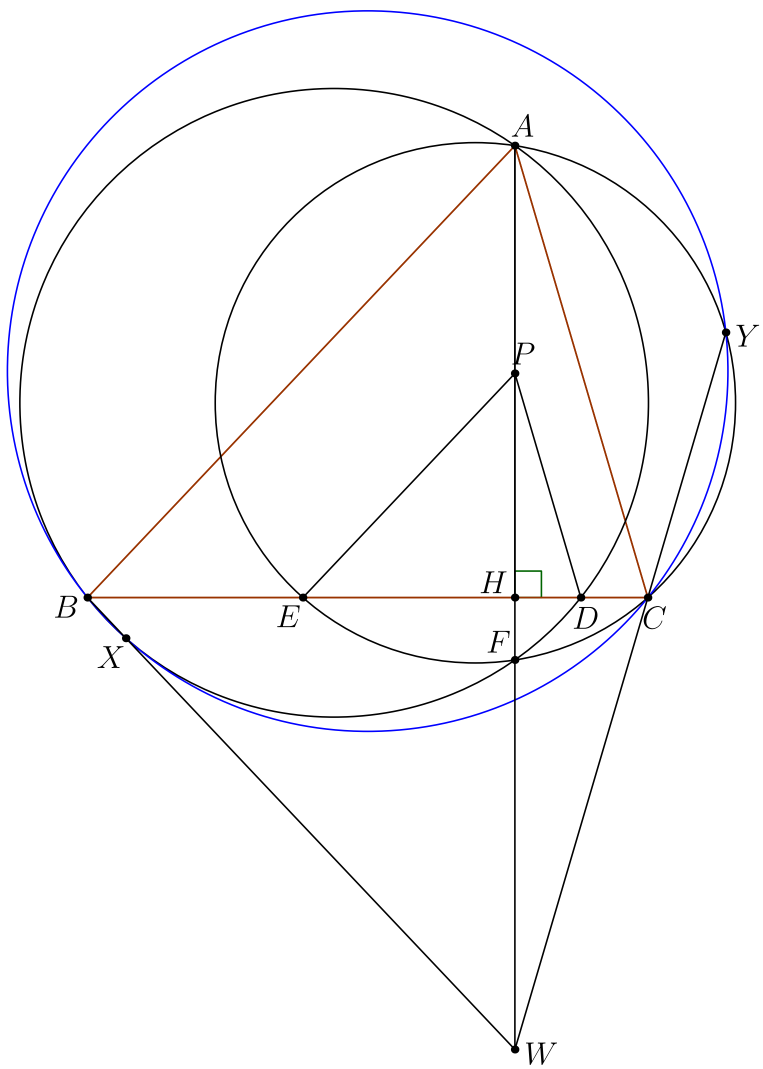
$$a^2yz + b^2xz + c^2xy = 4a^2b^2c^2 - 2a^2b^2c^2 - 2a^2b^2c^2 = 0$$

Consequently, it must be that $X \in (\triangle ABC)$. $\square$

Problem 1.4 (IMO 2022 Shortlist, G2)

In the acute-angled triangle $\triangle ABC$, the point F is the foot of the altitude from A, and P is a point on the segment AF. The lines through P parallel to AC and AB meet BC at D and E, respectively. Points $X \neq A$ and $Y \neq A$ lie on the circles ABD and ACE, respectively, such that $DA = DX$ and $EA = EY$. Prove that B, C, X and Y are concyclic.

Proof. There are a lot of observations that can be done in this problem, however this is the solution I have.



Lemma 1.2 The second intersection of (ABD) and (AEC) lies on AH.

This is true due to basic angle chase, assume that F lies on AH and (ABD), let us prove that then $AECF$ is cyclic. By Power of the Point,

$$HA \cdot HF = HD \cdot HD = HE \cdot HC$$

the last is obviously true because $PE \parallel AB$ and $PD \parallel AC$. Now let us intersect BX with AF at a point W, then the condition that $BXCY$ is cyclic by the radical center theorem is equivelent to showing that W, C and Y are colinear.

Lemma 1.3 BC is the angle bisector of $\angle ABW$

This is because,

$$\angle CBW = \angle XAD = \angle AXD = \angle ABC$$

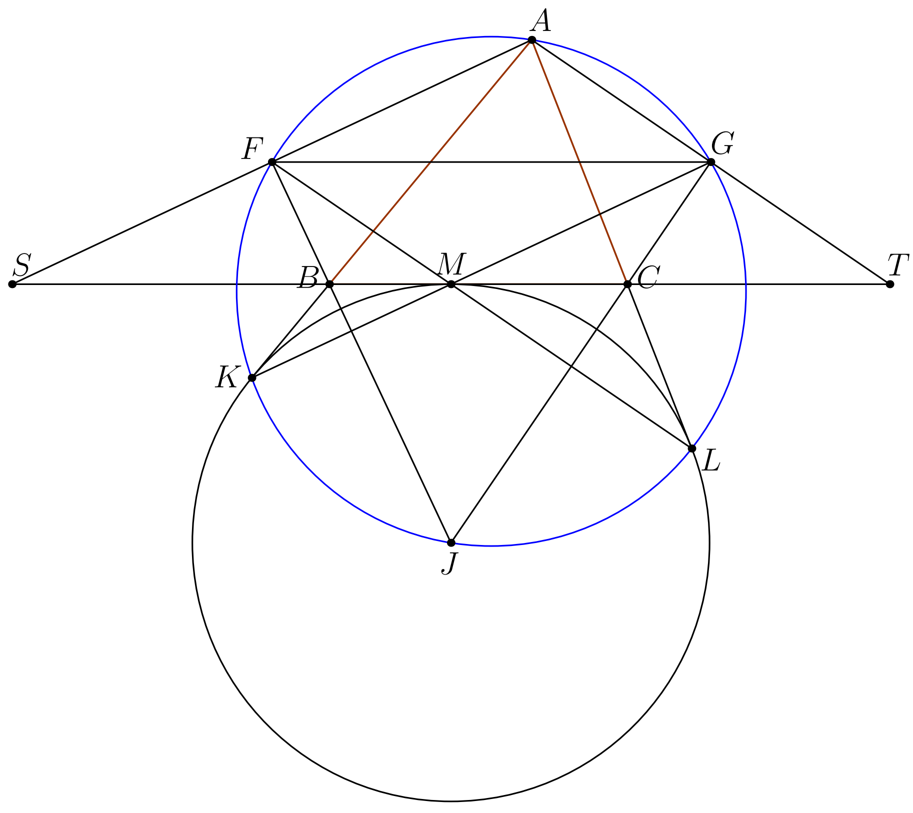
because F is the midpoint of the arc between A and X per definition. Thus, W is the reflection of A over H. The exact same argument for C shows that BC is the angle bisector of $\angle YCA$ and thus, CY must intersect AF at W.

$\square$

Problem 1.5 (IMO 2012 P1)

Given triangle ABC the point J is the center of the excircle opposite the vertex A. This excircle is tangent to side BC at M, and to lines AB and AC at K and L, respectively. Lines LM and BJ meet at F, and lines KM and CJ meet at G. Let S be the point of intersection of lines AF and BC, and let T be the point of intersection of lines AG and BC. Prove that M is the midpoint of ST.

Proof. Pascal's theorem hints us towards checking whether $FAGKJL$ is cyclic.



Notice, that because $KM \perp FJ$ and $JG \perp ML$, it must be that M is the orthocenter of FGJ. Thus, $FG \parallel BC$, consequently,

$$\angle GFJ = \angle BCJ = \angle MKJ$$

Thus, $FKJG$ is cyclic, analogously we conclude that $FKJLG$ is cyclic and by Pascal's theorem it must be that A lies on this circle as well, thus $FAGLJK$ is cyclic.

Now notice,

$$\angle FGA = \angle ALF = \angle CML = \angle GFL$$

the last step comes from $FG \parallel BC$. Thus, $AT \parallel FM$, the same for $AG \parallel FM$. Consequently, $AGMF$ is a parallelogram. Because of that it must be that FG is the midline of $\triangle AST$ and M is the midpoint of the side ST. $\square$

Problem 1.6 (IMO 2022 Shortlist C1)

A ± 1-sequence is a sequence of 2022 numbers $a_1, \ldots, a_{2022}$, each equal to either $+1$ or -1. Determine the largest C such that, for any ± 1-sequence, there exists an integer k and indices $1 \leq t_1 \leq \ldots \leq t_k \leq 2022$ so that $t_{i+1} - t_i \leq 2$ for all i, and

$$\left| \sum_{i=1}^{k} a_{t_i} \right| \geq C$$

Proof. WLOG, let there be more 1 than -1 in the ± 1-sequence. Then, let us consider the following algorithm,

Algorithm 1.4 If the next element is -1, then skip to the next one and include it. If the next element is 1, then include it. Continue this process until you have went through the entire sequence.

This algorithm will garantee that we skip at least $\lceil \frac{X}{2} \rceil$, where X is the number of -1 in the sequence. Thus,

$$\left| \sum_{i=1}^{k} a_{t_i} \right| \geq (2022 - X) - \left\lfloor \frac{X}{2} \right\rfloor \geq (2022 - 1011) - \left\lfloor \frac{1011}{2} \right\rfloor = 506$$

Consequently, it must be that $C \geq 506$. Now consider the following sequence,

$$1, 1, -1, -1, 1, 1, \ldots, -1, -1, 1, -1$$

It alternates between $1, 1$ and $-1, -1$ until the very end, where it is 1 and -1.

Notice that in each consequtive pair of identical numbers at least one of the numbers is present in our sequence. Thus, no matter our choice of indices, the total sum is bounded by,

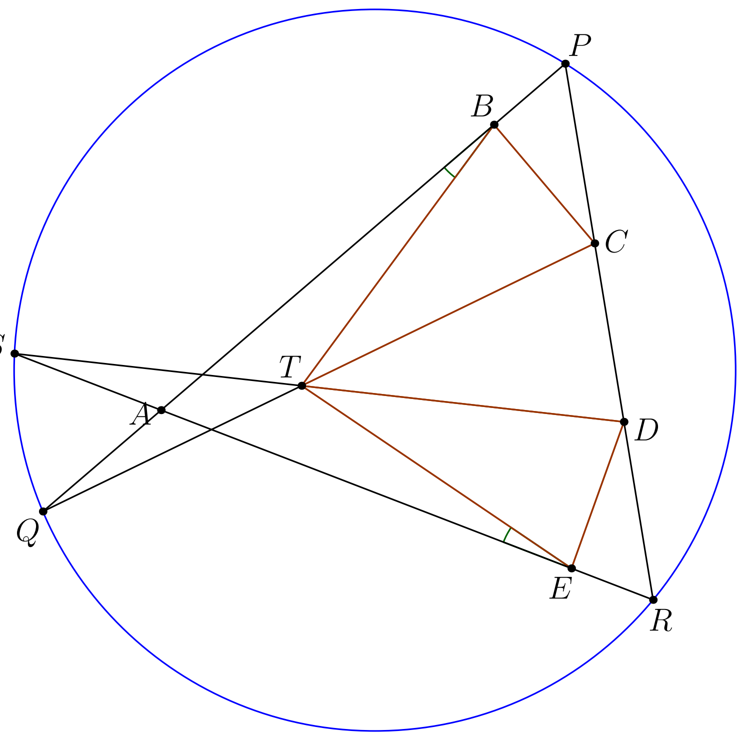
$$\left| \sum_{i=1}^{k} a_{t_i} \right| \leq 2 \cdot \frac{1010}{2} - \frac{1010}{2} + 1 = 2 \cdot 505 - 505 + 1 = 506$$

Thus, it must be that $C = 506$. $\square$

Problem 1.7 (IMO 2022 P4)

Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point T inside $ABCDE$ with $TB = TD$, $TC = TE$ and $\angle ABT = \angle TEA$. Let line AB intersect lines CD and CT at points P and Q, respectively. Assume that the points P, B, A, Q occur on their line in that order. Let line AE intersect lines CD and DT at points R and S, respectively. Assume that the points R, E, A, S occur on their line in that order. Prove that the points P, S, Q, R lie on a circle.

Proof. The first thing which one can notice is that $\triangle TDE \sim \triangle TBC$.



Now, notice that,

Lemma 1.5 $\triangle STE \sim \triangle QTB$ and $SQCD$ is cyclic.

This is because,

$$\angle STE = 180 - \angle DTE = 180 - \angle BTC = \angle QTB$$

Thus,

$$\frac{TS}{TC} = \frac{TS}{TE} = \frac{TQ}{TB} = \frac{TQ}{TD}$$
$$\implies TS \cdot TD = TC \cdot TQ$$

Consequently $SQCD$ is cyclic. Now,

$$\angle QSR = \angle QSD - \angle RSD = \angle QCD - \angle PQC = \angle QPR$$

which proves that $SQPR$ is cyclic. □

§2 EGMO

Problem 2.1 (EGMO 2025 P1)

For a positive integer N, let $c_1 < c_2 < \cdots < c_m$ be all positive integers smaller than N that are coprime to N. Find all $N \geqslant 3$ such that

$$\gcd(N, c_i + c_{i+1}) \neq 1$$

for all $1 \leqslant i \leqslant m-1$

Here $\gcd(a, b)$ is the largest positive integer that divides both a and b. Integers a and b are coprime if $\gcd(a, b) = 1$.

Proof. Notice, that if N is even, then all c_i are odd, however that implies that $c_i + c_{i+1}$ will always be even, thus never relatively prime with N. Consequently, every even N satisfies the conditions of the problem statement. If N is odd, then it is by definition relatively prime to 2, thus,

$$\gcd(N, 1+2) \neq 1 \implies 3 \mid N$$

Thus, $N = 3^\alpha x$ (where x is odd), however,

Case 1) If $x \equiv 1 \pmod 3$, then let us consider two numbers, $x+1$ and $x+3$, then,

$$\gcd(3^\alpha x, x+1) = 1$$
$$\gcd(3^\alpha x, x-2) = 1$$

But, then,

$$(3^\alpha x, 2x-1) = (3^\alpha, 2x-1) = 1$$

contradiction!

Case 2) If $x \equiv 2 \pmod 3$, then consider $x+2$ and $x-1$, then,

$$\begin{cases} \gcd(3^\alpha x, x+2) = 1 \\ \gcd(3^\alpha x, x-1) = 1 \end{cases}$$

then,

$$\gcd(3^\alpha x, 2x+1) = \gcd(3^\alpha, 2x+1) = 1$$

contradiction!

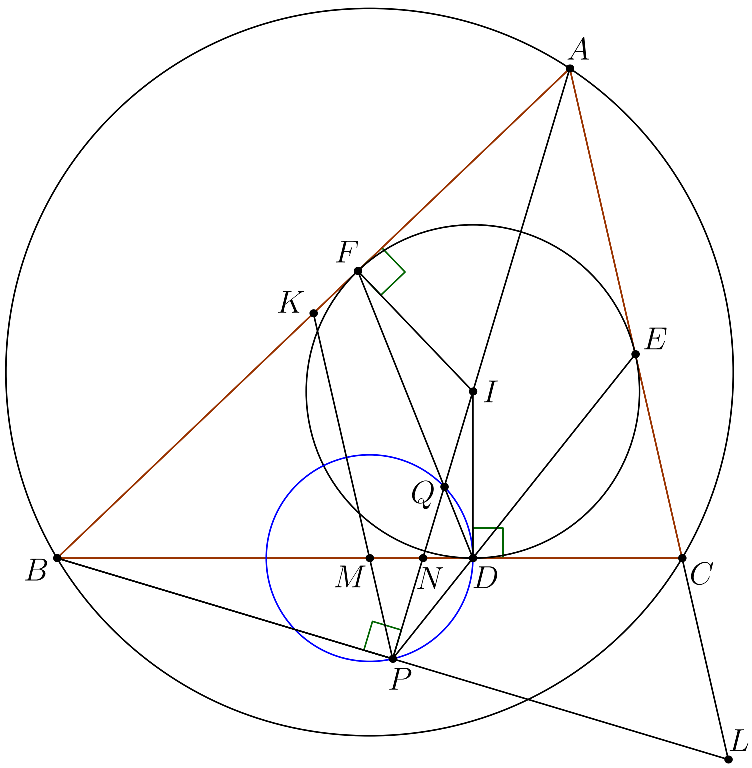
Thus $x = 1$. Consequently, the only solutions are all even numbers and all powers of three. □

§3 Korean National Olympiad

Problem 3.1 (Korea 2025 P4)

Triangle ABC satisfies $\overline{CA} > \overline{AB}$. Let the incenter of triangle ABC be ω, which touches BC, CA, AB at D, E, F, respectively. Let M be the midpoint of BC. Let the circle centered at M passing through D intersect DE, DF at $P(\neq D), Q(\neq D)$, respecively. Let line AP meet BC at N, line BP meet CA at L. Prove that the three lines EQ, FP, NL are concurrent.

Proof. Let us start by proving that A, P, Q are colinear. Let us define P as the intersection of AI and DE, then we must prove that MPD is isoseles.



This can be easily shown by angle chase, let us introduce K, the midpoint of AB it is well known that then K, M, P are colinear and $\angle BPA = 90$. Thus,

$$\begin{aligned}\angle MPD = \angle APD + \angle KPA = \angle IBC + \angle PAC = \frac{\angle A + \angle B}{2} \\ = \frac{180 - \angle C}{2} = \angle EDC = \angle MDP \quad (1)\end{aligned}$$

Thus, it must be that P lies on AI, the exact same logic shows that $Q \in AI$, thus, A, I, Q and P are colinear.

Now, notice that under reflection over AI the line FQ goes to EQ and the line LN goes to BN (since $\angle APB = 90$ it must be that B goes to L under reflection) and PF goes to PE. Thus, EQ, LN and PF are concurrent, and the intersection point is the reflection of D over AI. $\square$