§1 IMO

Problem 1.1 (IMO 2019 P1)

Find all $f : \mathbb{Z} \to \mathbb{Z}$ such that,

$$f(2x) + 2f(y) = f(f(x+y))$$

Proof. Let $P(x, y)$ the condition generated by x, y. Then,

$$\begin{cases} P(x,0) : f(2x) + 2f(0) = f(f(x)) \\ P(0,x) : f(0) + 2f(x) = f(f(x)) \end{cases}$$

Combining these two we obtain that,

$$f(2x) = 2f(x) - f(0)$$

Now let us consider the following,

$$P(x, 1) : f(2x) + 2f(1) = f(f(x+1))$$

However from the previous observations we can say that,

$$f(f(x+1)) = 2f(x+1) + f(0)$$

Thus,

$$2f(x) + 2f(1) - f(0) = 2f(x+1) + f(0)$$
$$f(x+1) = f(x) + f(1) - f(0)$$

Thus, we see that $f(x) = f(0) + x \cdot C$ where C is some constant (specifically $f(1) - f(0)$) for all $x \in \mathbb{Z}$. Now let us consider,

$$P(x, -x) : f(2x) + 2f(-x) = f(f(0))$$

We can rewrite this as,

$$f(0) + 2xC + 2f(0) - 2xC = f(0) + f(0) \cdot C$$

$$\implies 2f(0) = f(0) \cdot C$$

This implies that either $f(0) = 0$ or $C = 2$. Let us consider both cases,

- If $f(0) = 0$, then, $f(x) = cx$ and it is simple to verify that then $f(x)$ must equal $2x$ or 0.
- If $C = 2$, then,

$$f(0) + 4x + 2(f(0) + 2y) = f(f(0) + 2(x+y)) = f(0) + 2f(0) + 4(x+y)$$

$$4x + 2f(0) + 4y = 2f(0) + 4x + 4y$$

Thus, we see the only solutions to this functional equation are $f(x) = 2x + c$ and $f(x) = 0$. $\square$

Problem 1.2 (IMO 2022 P2)

Find all $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$ there exists exactly one $y \in \mathbb{R}^+$ such that,

$$xf(y) + yf(x) \leq 2$$

Proof. Obviously $f(x) = \frac{1}{x}$ works (from AM-GM).

Let us say that $x \sim y$ if and only if the condition from the problem statement is true. Then obviously this is commutative, meaning that if $x \sim y$ then $y \sim x$.

Lemma 1.1 If $f(x) \leq \frac{1}{x}$, then, $x \sim x$.

This is simple to see due to,

$$2xf(x) \leq 2x \cdot \frac{1}{x} = 2$$

However, notice that if $x \sim y$, then,

$$xf(y) + yf(x) \leq 2$$

and in it must be that $f(x) > \frac{1}{x}$ and $f(y) > \frac{1}{y}$ which would imply that by AM-GM,

$$xf(y) + yf(x) = \frac{x}{y} + \frac{y}{x} \geq 2$$

Thus we obtain that for all $x \in \mathbb{R}^+$ it must be that $x \sim x$. Thus that,

$$f(x) \leq \frac{1}{x}$$

for all $x \in \mathbb{R}^+$.

Notice that if $f(a) < \frac{1}{a}$, then, if,

$$f(a) = \frac{1}{a} - \epsilon$$

then,

$$xf(a) + af(x) = x\left(\frac{1}{a} - \epsilon\right) + af(x) \leq x\left(\frac{1}{a} - \epsilon\right) + \frac{a}{x} \overset{?}{\leq} 2$$

However the last inequality for $\epsilon > 0$ has multiple solutions for x. Because,

$$x^2\left(\frac{1}{a} - \epsilon\right) + a - 2x \leq 0$$

The only way for this inequality to have a single solution for $x \in \mathbb{R}^+$ is if the equality has only one solution, thus the discriminant is zero,

$$\Delta = 4 - 4 \cdot a \cdot \left(\frac{1}{a} - \epsilon\right)$$

which is obviously never 0 for $\epsilon > 0$. Thus because for each x there is only one such y that $x \sim y$ it must mean that,

$$f(x) = \frac{1}{x}$$

for all $x \in \mathbb{R}^+$. $\square$