

## §1 IMO

### Problem 1.1 (IMO 2019 P1)

Find all  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  such that,

$$f(2x) + 2f(y) = f(f(x + y))$$

*Proof.* Let  $P(x, y)$  the condition generated by  $x, y$ . Then,

$$\begin{cases} P(x, 0) : f(2x) + 2f(0) = f(f(x)) \\ P(0, x) : f(0) + 2f(x) = f(f(x)) \end{cases}$$

Combining these two we obtain that,

$$f(2x) = 2f(x) - f(0)$$

Now let us consider the following,

$$P(x, 1) : f(2x) + 2f(1) = f(f(x + 1))$$

However from the previous observations we can say that,

$$f(f(x + 1)) = 2f(x + 1) + f(0)$$

Thus,

$$2f(x) + 2f(1) - f(0) = 2f(x + 1) + f(0)$$

$$f(x + 1) = f(x) + f(1) - f(0)$$

Thus, we see that  $f(x) = f(0) + x \cdot C$  where  $C$  is some constant (specifically  $f(1) - f(0)$ ) for all  $x \in \mathbb{Z}$ . Now let us consider,

$$P(x, -x) : f(2x) + 2f(-x) = f(f(0))$$

We can rewrite this as,

$$f(0) + 2xC + 2f(0) - 2xC = f(0) + f(0) \cdot C$$

$$\implies 2f(0) = f(0) \cdot C$$

This implies that either  $f(0) = 0$  or  $C = 2$ . Let us consider both cases,

- If  $f(0) = 0$ , then,  $f(x) = cx$  and it is simple to verify that then  $f(x)$  must equal  $2x$  or  $0$ .
- If  $C = 2$ , then,

$$f(0) + 4x + 2(f(0) + 2y) = f(f(0) + 2(x + y)) = f(0) + 2f(0) + 4(x + y)$$

$$4x + 2f(0) + 4y = 2f(0) + 4x + 4y$$

Thus, we see the only solutions to this functional equation are  $f(x) = 2x + c$  and  $f(x) = 0$ .  $\square$

**Problem 1.2** (IMO 2022 P2)

Find all  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that for each  $x \in \mathbb{R}^+$  there exists exactly one  $y \in \mathbb{R}^+$  such that,

$$xf(y) + yf(x) \leq 2$$

*Proof.* Obviously  $f(x) = \frac{1}{x}$  works (from AM-GM).

Let us say that  $x \sim y$  if and only if the condition from the problem statement is true. Then obviously this is commutative, meaning that if  $x \sim y$  then  $y \sim x$ .

**Lemma 1.1** If  $f(x) \leq \frac{1}{x}$ , then,  $x \sim x$ .

This is simple to see due to,

$$2xf(x) \leq 2x \cdot \frac{1}{x} = 2$$

However, notice that if  $x \sim y$ , then,

$$xf(y) + yf(x) \leq 2$$

and in it must be that  $f(x) > \frac{1}{x}$  and  $f(y) > \frac{1}{y}$  which would imply that by AM-GM,

$$xf(y) + yf(x) = \frac{x}{y} + \frac{y}{x} \geq 2$$

Thus we obtain that for all  $x \in \mathbb{R}^+$  it must be that  $x \sim x$ . Thus that,

$$f(x) \leq \frac{1}{x}$$

for all  $x \in \mathbb{R}^+$ .

Notice that if  $f(a) < \frac{1}{a}$ , then, if,

$$f(a) = \frac{1}{a} - \epsilon$$

then,

$$xf(a) + af(x) = x \left( \frac{1}{a} - \epsilon \right) + af(x) \leq x \left( \frac{1}{a} - \epsilon \right) + \frac{a}{x} \stackrel{?}{\leq} 2$$

However the last inequality for  $\epsilon > 0$  has multiple solutions for  $x$ . Because,

$$x^2 \left( \frac{1}{a} - \epsilon \right) + a - 2x \leq 0$$

The only way for this inequality to have a single solution for  $x \in \mathbb{R}^+$  is if the equality has only one solution, thus the discriminant is zero,

$$\Delta = 4 - 4 \cdot a \cdot \left( \frac{1}{a} - \epsilon \right)$$

which is obviously never 0 for  $\epsilon > 0$ . Thus because for each  $x$  there is only one such  $y$  that  $x \sim y$  it must mean that,

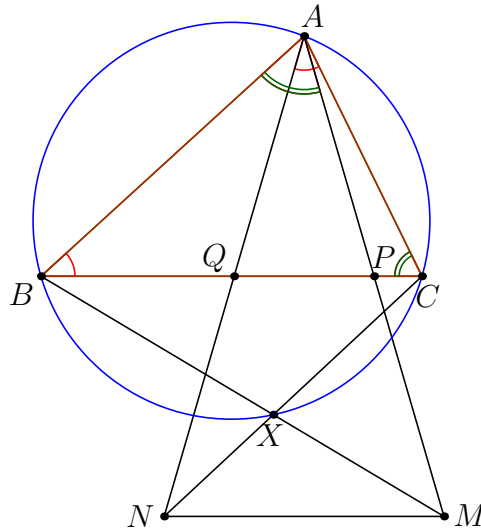
$$f(x) = \frac{1}{x}$$

for all  $x \in \mathbb{R}^+$ . □

**Problem 1.3** (IMO 2014 P4)

Let  $P$  and  $Q$  be on segment  $BC$  of an acute triangle  $\triangle ABC$  such that  $\angle PAB = \angle BCA$  and  $\angle CAQ = \angle ABC$ . Let  $M$  and  $N$  be the points on  $AP$  and  $AQ$ , respectively, such that  $P$  is the midpoint of  $AM$  and  $Q$  is the midpoint of  $AN$ . Prove that the intersection of  $BM$  and  $CN$  is on the circumference of triangle  $\triangle ABC$ .

*Proof.* Let  $\triangle ABC$  be the reference triangle in barycentric coordinates.



Then we can calculate the point  $P$ , because  $\triangle ABP \sim \triangle CBA$  and  $\triangle ACQ \sim \triangle BCA$ . Thus,

$$\frac{BA}{BC} = \frac{BP}{BA}$$

$$\implies BP = \frac{BA^2}{BC} = \frac{c^2}{a}$$

consequently,  $P = \left(0 : a - \frac{c^2}{a} : \frac{c^2}{a}\right)$ , analogously we get,  $Q = \left(0 : \frac{b^2}{a} : a - \frac{b^2}{a}\right)$ .  
Now we can calculate points  $M$  and  $N$ ,

$$M = (-a^2 : 2a^2 - 2c^2 : 2c^2)$$

$$N = (-a^2 : 2b^2 : 2a^2 - 2b^2)$$

Now we simply need to intersect  $BM$  and  $CN$  which is not hard to do considering that both are cevians in the triangle. It is not hard to see that the following point satisfies these conditions,

$$X = (-a^2 : 2b^2 : 2c^2)$$

Now we just need to check that this point lies on the circumcircle of  $\triangle ABC$  which is given by,

$$a^2yz + b^2xz + c^2xy = 0$$

Thus,

$$a^2yz + b^2xz + c^2xy = 4a^2b^2c^2 - 2a^2b^2c^2 - 2a^2b^2c^2 = 0$$

Consequently, it must be that  $X \in (\triangle ABC)$ . □