# §1 IMO

**Problem 1.1** (IMO 2019 P1)

Find all $f : \mathbb{Z} \to \mathbb{Z}$ such that,

$$f(2x) + 2f(y) = f(f(x+y))$$

*Proof.* Let $P(x, y)$ the condition generated by $x, y$. Then,

$$\begin{cases} P(x,0) : f(2x) + 2f(0) = f(f(x)) \\ P(0,x) : f(0) + 2f(x) = f(f(x)) \end{cases}$$

Combining these two we obtain that,

$$f(2x) = 2f(x) - f(0)$$

Now let us consider the following,

$$P(x,1) : f(2x) + 2f(1) = f(f(x+1))$$

However from the previous observations we can say that,

$$f(f(x+1)) = 2f(x+1) + f(0)$$

Thus,

$$2f(x) + 2f(1) - f(0) = 2f(x+1) + f(0)$$
$$f(x+1) = f(x) + f(1) - f(0)$$

Thus, we see that $f(x) = f(0) + x \cdot C$ where $C$ is some constant (specifically $f(1) - f(0)$) for all $x \in \mathbb{Z}$. Now let us consider,

$$P(x, -x) : f(2x) + 2f(-x) = f(f(0))$$

We can rewrite this as,

$$f(0) + 2xC + 2f(0) - 2xC = f(0) + f(0) \cdot C$$
$$\implies 2f(0) = f(0) \cdot C$$

This implies that either $f(0) = 0$ or $C = 2$. Let us consider both cases,

- If $f(0) = 0$, then, $f(x) = cx$ and it is simple to verify that then $f(x)$ must equal $2x$ or 0.
- If $C = 2$, then,

$$f(0) + 4x + 2(f(0) + 2y) = f(f(0) + 2(x+y)) = f(0) + 2f(0) + 4(x+y)$$
$$4x + 2f(0) + 4y = 2f(0) + 4x + 4y$$

Thus, we see the only solutions to this functional equation are $f(x) = 2x + c$ and $f(x) = 0$. $\square$

**Problem 1.2** (IMO 2022 P2)

Find all $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$ there exists exactly one $y \in \mathbb{R}^+$ such that,

$$xf(y) + yf(x) \le 2$$

*Proof.* Obviously $f(x) = \frac{1}{x}$ works (from AM-GM).

Let us say that $x \sim y$ if and only if the condition from the problem statement is true. Then obviously this is commutative, meaning that if $x \sim y$ then $y \sim x$.

**Lemma 1.1** If $f(x) \le \frac{1}{x}$, then, $x \sim x$.

This is simple to see due to,

$$2xf(x) \le 2x \cdot \frac{1}{x} = 2$$

However, notice that if $x \sim y$, then,

$$xf(y) + yf(x) \le 2$$

and in it must be that $f(x) > \frac{1}{x}$ and $f(y) > \frac{1}{y}$ which would imply that by AM-GM,

$$xf(y) + yf(x) = \frac{x}{y} + \frac{y}{x} \ge 2$$

Thus we obtain that for all $x \in \mathbb{R}^+$ it must be that $x \sim x$. Thus that,

$$f(x) \le \frac{1}{x}$$

for all $x \in \mathbb{R}^+$.

Notice that if $f(a) < \frac{1}{a}$, then, if,

$$f(a) = \frac{1}{a} - \epsilon$$

then,

$$xf(a) + af(x) = x\left(\frac{1}{a} - \epsilon\right) + af(x) \le x\left(\frac{1}{a} - \epsilon\right) + \frac{a}{x} \overset{?}{\le} 2$$

However the last inequality for $\epsilon > 0$ has multiple solutions for $x$. Because,

$$x^2\left(\frac{1}{a} - \epsilon\right) + a - 2x \le 0$$

The only way for this inequality to have a single solution for $x \in \mathbb{R}^+$ is if the equality has only one solution, thus the discriminant is zero,

$$\Delta = 4 - 4 \cdot a \cdot \left(\frac{1}{a} - \epsilon\right)$$

which is obviously never 0 for $\epsilon > 0$. Thus because for each $x$ there is only one such $y$ that $x \sim y$ it must mean that,

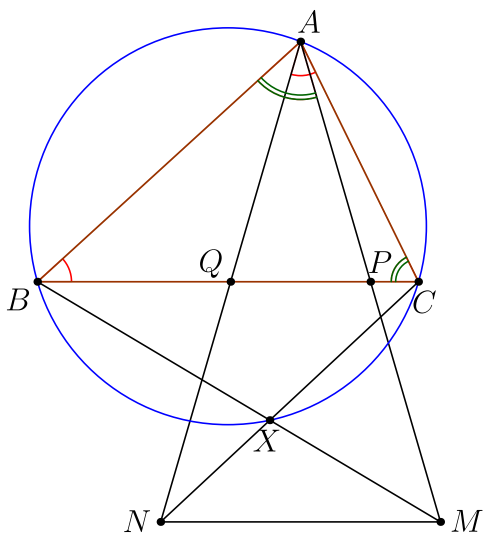
$$f(x) = \frac{1}{x}$$

for all $x \in \mathbb{R}^+$. $\square$

**Problem 1.3** (IMO 2014 P4)

Let $P$ and $Q$ be on segment $BC$ of an acute triangle $\triangle ABC$ such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let $M$ and $N$ be the points on $AP$ and $AQ$, respectively, such that $P$ is the midpoint of $AM$ and $Q$ is the midpoint of $AN$. Prove that the intersection of $BM$ and $CN$ is on the circumference of triangle $\triangle ABC$.

*Proof.* Let $\triangle ABC$ be the reference triangle in barycentric coordinates.




Then we can calculate the point $P$, because $\triangle ABP \sim \triangle CBA$ and $\triangle ACQ \sim \triangle BCA$. Thus,

$$\frac{BA}{BC} = \frac{BP}{BA}$$

$$\implies BP = \frac{BA^2}{BC} = \frac{c^2}{a}$$

consequently, $P = \left(0 : a - \frac{c^2}{a} : \frac{c^2}{a}\right)$, analogously we get, $Q = \left(0 : \frac{b^2}{a} : a - \frac{b^2}{a}\right)$. Now we can calculate points $M$ and $N$,

$$M = \left(-a^2 : 2a^2 - 2c^2 : 2c^2\right)$$

$$N = \left(-a^2 : 2b^2 : 2a^2 - 2b^2\right)$$

Now we simply need to intersect $BM$ and $CN$ which is not hard to do considering that both are cevians in the triangle. It is not hard to see that the following point satisfies these conditions,

$$X = \left(-a^2 : 2b^2 : 2c^2\right)$$

Now we just need to check that this point lies on the circumcircle of $\triangle ABC$ which is given by,

$$a^2yz + b^2xz + c^2xy = 0$$

Thus,

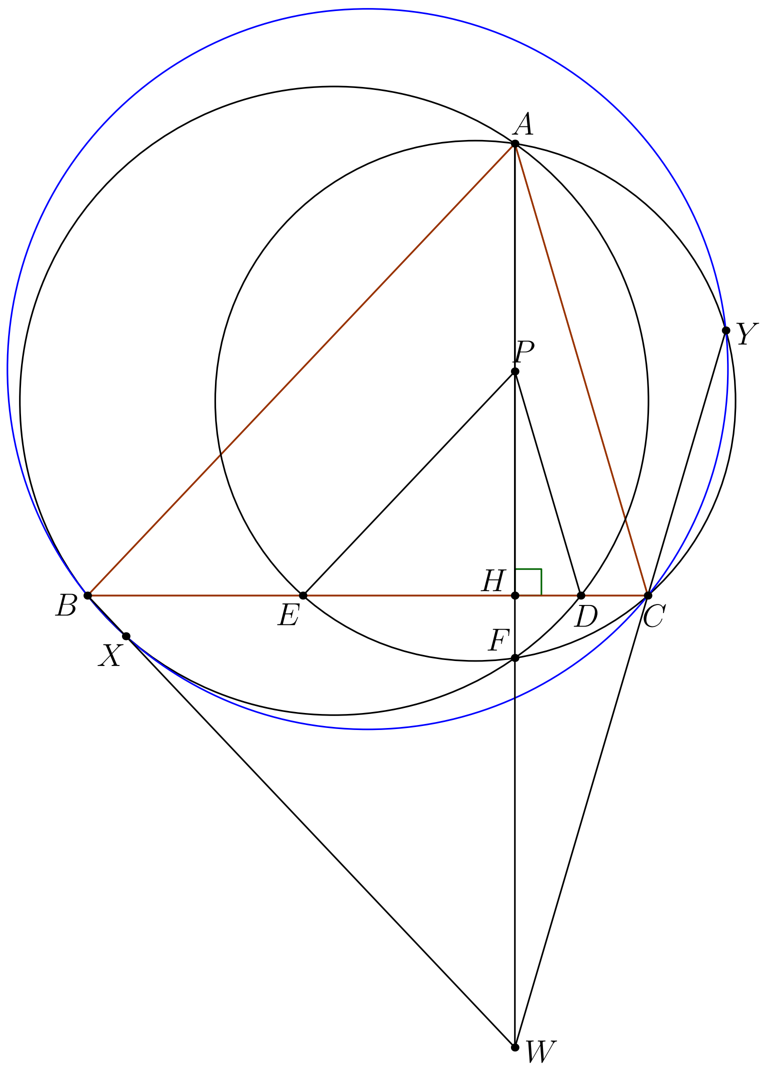
$$a^2yz + b^2xz + c^2xy = 4a^2b^2c^2 - 2a^2b^2c^2 - 2a^2b^2c^2 = 0$$

Consequently, it must be that $X \in (\triangle ABC)$. $\square$

**Problem 1.4** (IMO 2022 Shortlist, G2)

In the acute-angled triangle $\triangle ABC$, the point $F$ is the foot of the altitude from $A$, and $P$ is a point on the segment $AF$. The lines through $P$ parallel to $AC$ and $AB$ meet $BC$ at $D$ and $E$, respectively. Points $X \neq A$ and $Y \neq A$ lie on the circles $ABD$ and $ACE$, respectively, such that $DA = DX$ and $EA = EY$. Prove that $B, C, X$ and $Y$ are concyclic.

*Proof.* There are a lot of observations that can be done in this problem, however this is the solution I have.



**Lemma 1.2** The second intersection of $(ABD)$ and $(AEC)$ lies on $AH$.

This is true due to basic angle chase, assume that $F$ lies on $AH$ and $(ABD)$, let us prove that then $AECF$ is cyclic. By Power of the Point,

$$HA \cdot HF = HD \cdot HD = HE \cdot HC$$

the last is obviously true because $PE \parallel AB$ and $PD \parallel AC$. Now let us intersect $BX$ with $AF$ at a point $W$, then the condition that $BXCY$ is cyclic by the radical center theorem is equivelent to showing that $W, C$ and $Y$ are colinear.

**Lemma 1.3** $BC$ is the angle bisector of $\angle ABW$

This is because,

$$\angle CBW = \angle XAD = \angle AXD = \angle ABC$$

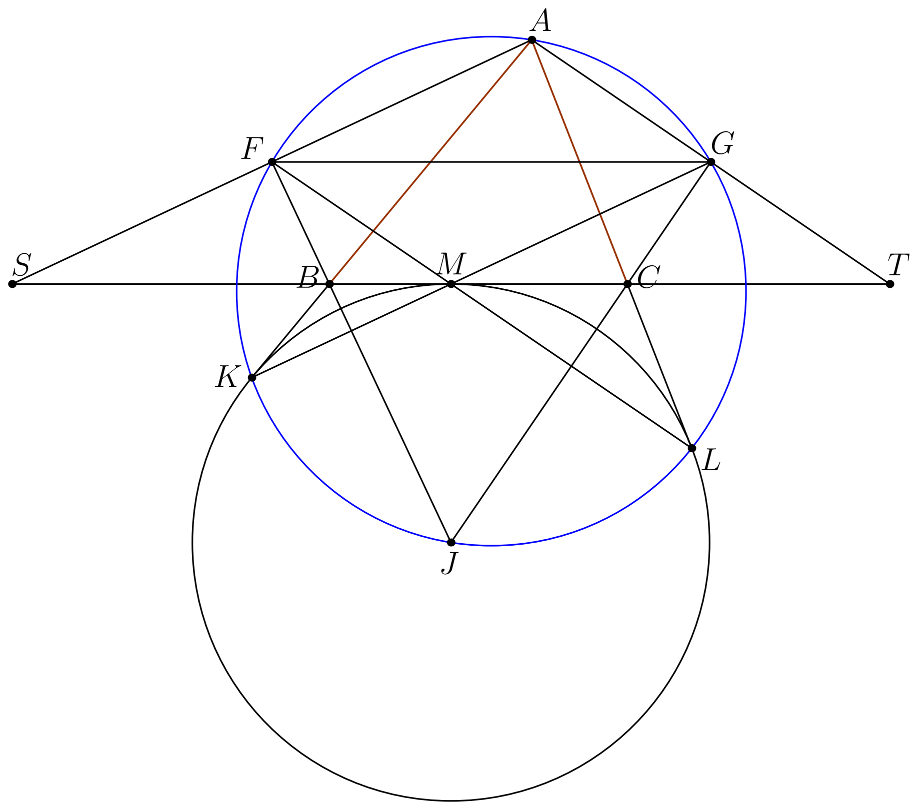
because $F$ is the midpoint of the arc between $A$ and $X$ per definition. Thus, $W$ is the reflection of $A$ over $H$. The exact same argument for $C$ shows that $BC$ is the angle bisector of $\angle YCA$ and thus, $CY$ must intersect $AF$ at $W$.

□

**Problem 1.5** (IMO 2012 P1)

Given triangle $ABC$ the point $J$ is the center of the excircle opposite the vertex $A$. This excircle is tangent to side $BC$ at $M$, and to lines $AB$ and $AC$ at $K$ and $L$, respectively. Lines $LM$ and $BJ$ meet at $F$, and lines $KM$ and $CJ$ meet at $G$. Let $S$ be the point of intersection of lines $AF$ and $BC$, and let $T$ be the point of intersection of lines $AG$ and $BC$. Prove that $M$ is the midpoint of $ST$.

*Proof.* Pascal's theorem hints us towards checking whether $FAGKJL$ is cyclic.




Notice, that because $KM \perp FJ$ and $JG \perp ML$, it must be that $M$ is the orthocenter of $FGJ$. Thus, $FG \parallel BC$, consequently,

$$\angle GFJ = \angle BCJ = \angle MKJ$$

Thus, $FKJG$ is cyclic, analogously we conclude that $FKJLG$ is cyclic and by Pascal's theorem it must be that $A$ lies on this circle as well, thus $FAGLJK$ is cyclic.

Now notice,

$$\angle FGA = \angle ALF = \angle CML = \angle GFL$$

the last step comes from $FG \parallel BC$. Thus, $AT \parallel FM$, the same for $AG \parallel FM$. Consequently, $AGMF$ is a parallelogram. Because of that it must be that $FG$ is the midline of $\triangle AST$ and $M$ is the midpoint of the side $ST$. $\square$

**Problem 1.6** (IMO 2022 Shortlist C1)

A $\pm 1$-sequence is a sequence of 2022 numbers $a_1, \ldots, a_{2022}$, each equal to either $+1$ or $-1$. Determine the largest $C$ such that, for any $\pm 1$-sequence, there exists an integer $k$ and indices $1 \leq t_1 \leq \ldots \leq t_k \leq 2022$ so that $t_{i+1} - t_i \leq 2$ for all $i$, and

$$\left| \sum_{i=1}^{k} a_{t_i} \right| \geq C$$

*Proof.* WLOG, let there be more 1 than $-1$ in the $\pm 1$-sequence. Then, let us consider the following algorithm,

**Algorithm 1.4** If the next element is $-1$, then skip to the next one and include it. If the next element is 1, then include it. Continue this process until you have went through the entire sequence.

This algorithm will garantee that we skip at least $\lceil \frac{X}{2} \rceil$, where $X$ is the number of $-1$ in the sequence. Thus,

$$\left| \sum_{i=1}^{k} a_{t_i} \right| \geq (2022 - X) - \left\lfloor \frac{X}{2} \right\rfloor \geq (2022 - 1011) - \left\lfloor \frac{1011}{2} \right\rfloor = 506$$

Consequently, it must be that $C \geq 506$. Now consider the following sequence,

$$1, 1, -1, -1, 1, 1, \ldots, -1, -1, 1, -1$$

It alternates between $1, 1$ and $-1, -1$ until the very end, where it is 1 and $-1$.

Notice that in each consequtive pair of identical numbers at least one of the numbers is present in our sequence. Thus, no matter our choice of indices, the total sum is bounded by,

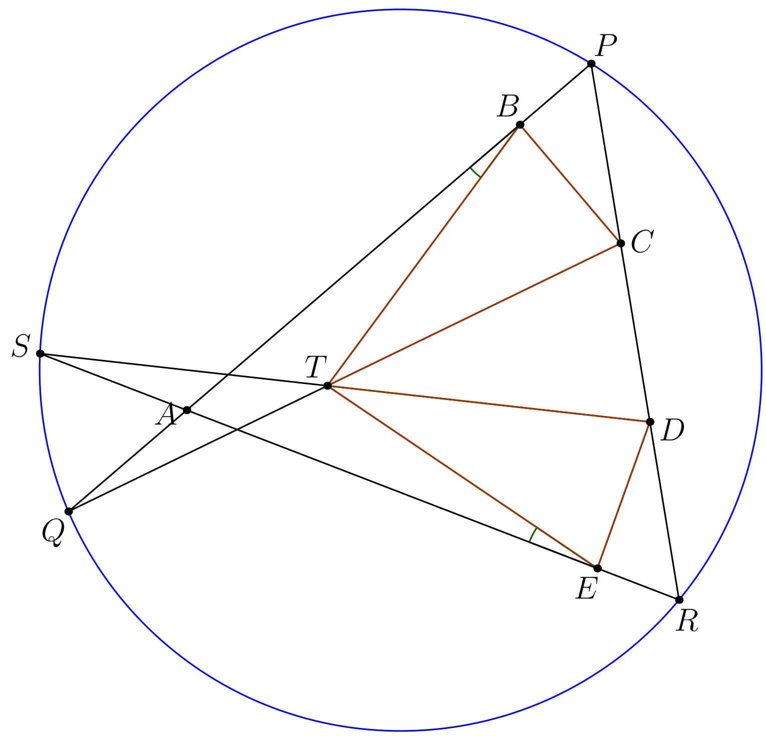
$$\left| \sum_{i=1}^{k} a_{t_i} \right| \leq 2 \cdot \frac{1010}{2} - \frac{1010}{2} + 1 = 2 \cdot 505 - 505 + 1 = 506$$

Thus, it must be that $C = 506$. $\square$

**Problem 1.7** (IMO 2022 P4)

Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point $T$ inside $ABCDE$ with $TB = TD$, $TC = TE$ and $\angle ABT = \angle TEA$. Let line $AB$ intersect lines $CD$ and $CT$ at points $P$ and $Q$, respectively. Assume that the points $P, B, A, Q$ occur on their line in that order. Let line $AE$ intersect lines $CD$ and $DT$ at points $R$ and $S$, respectively. Assume that the points $R, E, A, S$ occur on their line in that order. Prove that the points $P, S, Q, R$ lie on a circle.

*Proof.* The first thing which one can notice is that $\triangle TDE \sim \triangle TBC$.




Now, notice that,

**Lemma 1.5** $\triangle STE \sim \triangle QTB$ and $SQCD$ is cyclic.

This is because,

$$\angle STE = 180 - \angle DTE = 180 - \angle BTC = \angle QTB$$

Thus,

$$\frac{TS}{TC} = \frac{TS}{TE} = \frac{TQ}{TB} = \frac{TQ}{TD}$$
$$\implies TS \cdot TD = TC \cdot TQ$$

Consequently $SQCD$ is cyclic. Now,

$$\angle QSR = \angle QSD - \angle RSD = \angle QCD - \angle PQC = \angle QPR$$

which proves that $SQPR$ is cyclic. □

**Problem 1.8** (IMO 2012 P4)

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers $a, b, c$ such that $a + b + c = 0$, the following equality holds,

$$f^2(a) + f^2(b) + f^2(c) = 2f(a)f(b) + 2f(b)f(c) + 2f(a)f(c)$$

*Proof.* Obviously from $a = b = c = 0$ we obtain that $f(0) = 0$.

Notice, if we set into the original functional equation $b = -a$ and $c = 0$, we will obtain,

$$\begin{aligned} f^2(a) + f^2(-a) &= 2f(a)f(-a) \\ \implies (f(a) - f(-a))^2 &= 0 \\ \implies f(a) &= f(-a) \end{aligned}$$

Now, notice that a very common transformation with this type of equation is,

$$x^2 + y^2 + z^2 = 2xy + 2yz + 2xz$$

$$\begin{aligned} (x + y - z)^2 &= x^2 + y^2 + z^2 + 2xy - 2xz - 2xy \\ \implies 4xy &= (x + y - z)^2 \\ \implies xy &= \left(\frac{x + y - z}{2}\right)^2 \end{aligned}$$

Thus,

$$f(a)f(b) = \left(\frac{f(a) + f(b) - f(c)}{2}\right)^2$$

Now is the perfect time to get rid of the third variable, let $c = -a - b$ and we will obtain, (because $f(-a - b) = f(a + b)$)

$$f(a)f(b) = \left(\frac{f(a) + f(b) - f(a + b)}{2}\right)^2$$

$$f(a + b) = f(a) + f(b) \pm 2\sqrt{f(a)f(b)}$$

The nice thing about this is first of all, because $f(a)f(b)$ is always nonnegative it must be that $f$ is either always $\geq 0$ or $\leq 0$. The second thing is that now we obtained an equation equivelent to the original functional equation and we can assume that $f$ is positive (because if $f$ is a solution, then $-f$ is a solution as well). However, we can still do some preliminary steps for example obviously we want to rewrite the equation as,

$$f(a + b) = (\sqrt{f(a)} \pm \sqrt{f(b)})^2$$

$$\implies \sqrt{f(a + b)} = \pm\sqrt{f(a)} \pm \sqrt{f(b)}$$

thus if $g(x) = \sqrt{f(x)}$, then we know that,

$$g(x+y) = \pm g(x) \pm g(y)$$

Now, if $g(1) = c$, then,

$$g(2) = \pm g(1) \pm g(1) = c \pm c = 2c \vee 0$$

Now, let us consider two cases,

Case 1) If $g(2) = 2c$, then,

$$g(3) = \pm f(1) \pm f(2) = \pm c \pm 2c = 3c \vee c$$

because $f$ is $\geq 0$. Now, we again consider two cases, right now let $g(3) = 3c$. Then,

$$\begin{cases} g(4) = \pm g(2) \pm g(2) = 4c \vee 0 \\ g(4) = \pm g(1) \pm g(3) = 4c \vee 2c \end{cases}$$

$$\implies g(4) = 4c$$

inductively it is simple to continue this logic and show that $g(n) = nc$. Thus, one of the solutions for $f$ is $f(n) = cn^2$.

Case 2) If $g(2) = 2c$ and $g(3) = c$. Then,

$$\begin{cases} g(4) = \pm g(1) \pm g(3) = 0 \vee 2c \\ g(4) = \pm g(2) \pm g(2) = \pm 2c \pm 2c = 0 \vee 4c \end{cases}$$

$$\implies g(4) = 0$$

However, if you repeat this argument for higher value of $g(n)$ you will obtain a periodic function of the following form,

$$g(n) = \begin{cases} c, \text{ if } n \equiv 1 \pmod 2 \\ 2c, \text{ if } n \equiv 2 \pmod 4 \\ 0, \text{ if } n \equiv 0 \pmod 4 \end{cases}$$

which is equivelent to,

$$f(n) = \begin{cases} c, \text{ if } n \equiv 1 \pmod 2 \\ 4c, \text{ if } n \equiv 2 \pmod 4 \\ 0, \text{ if } n \equiv 0 \pmod 4 \end{cases}$$

Case 3) If $g(2) = 0$, then,

$$\begin{cases} g(3) = \pm g(1) \pm g(2) = c \\ g(4) = \pm g(2) \pm g(2) = 0 \\ \dots \end{cases}$$

Thus,

$$g(n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 2 \\ c, & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

$$\implies f(n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 2 \\ c, & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

Thus, we have obtained all solutions (all the solutions above have there negative counterparts as well). $\square$

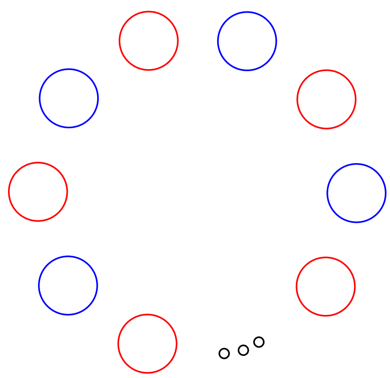
**Problem 1.9** (IMO 2013 P2)

A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied:

1. No line passes through any point of the configuration.
2. No region contains points of both colors.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

*Proof.* Let us consider the following construction (alternating the colors), I claim one needs at least 2013 lines.



To prove this simply consider all the segments connecting neighboring blue/red points, then each of those segments must be split by some line (else they wouldn't be in different regions), but each line cuts no more than 2 such segments. However, there is a total of $2 \cdot 2013$ such segments, thus a minimum of 2013 lines is required.

Now, we need to prove that 2013 lines is always enough.

**Lemma 1.6** Given $n$ points blue and $k < n$ red points, where $k$ is even, then it is possible to satisfy the conditions in the problem statement using $k$ lines.

Consider two red points, then no blue point lies on the line between them. Thus, we can consider two close enough lines parallel to the line between the two red points and divide the plane into three parts with one of the parts containing both of the red points and no other point. Thus, if we repeat this argument for the $\frac{k}{2}$ pairs of red points we will obtain a construction satisfying the conditions of the problem statement involving only $2 \cdot \frac{k}{2} = k$ lines.

**Lemma 1.7** Given $n$ points of one color and $n-1$ points of another color, a minimum of $n-1$ lines is required to satisfy the conditions in the problem statement.

If $n-1$ is even, then we are done, thus let us assume that $n-1$ is odd.

Let us construct a convex hull given all the points. If it contains two neighboring blue points, then we separate them from the rest of the points using one line and then apply the induction hypothesis on the rest of the points and obtain $1 + n - 2 = n - 1$ lines.

If at least one red point is present in the convex hull we can separate it from the rest of the points using one line. Then, we have $n-2$ red points left, which is an even number, thus we can satisfy the conditions of the problem statement using $1 + n - 2 = n - 1$ lines.

Obviously the problem follows when $n = 2014$. □

**Problem 1.10** (IMO Shortlist 2022 N2)

Find all positive integers $n > 2$ such that,

$$n! \mid \prod_{p<q\leq n} (p+q)$$

*Proof.* Let $r$ be the biggest prime $\leq n$. Then, obviously $v_r(n!) = 1$. However,

$$v_p\left(\prod_{p<q\leq n}(p+q)\right) = \sum_{p+q=r} 1$$

because $p+q < r+r = 2r$. Thus, there must be exactly one solution to $p+q = r$ for prime $p, q$. However, because $r$ is an odd prime it must be that either $p$ or $q$ is even, i.e. 2. WLOG $q = r-2$.

Notice, that it can't be that $p$, $p+2$ and $p+4$ are all prime, except for $3, 5, 7$.

Let us use the exact same logic for $q$, assuming that $q > 5$, which is true for $n \geq 11$. Notice, that,

$$x + y < p + p - 4 = 2(q-2)$$

where $x, y$ are prime (due to the above). However, notice that it also can't be that $x + y = q$ where $x, y$ are prime, also due to the argument before. Thus,

$$v_q\left(\prod_{x<y\leq n}(x+y)\right) = 0 < v_q(n!)$$

contradiction! Thus, $7 \leq n < 11$,

Case 1) If $n = 7$, (primes are $2, 3, 5, 7$)

$$\begin{aligned}\prod_{p<q\leq n}(p+q) &= (2+3)\cdot(3+5)\cdot(5+7)\cdot(2+5)\cdot(2+7)\cdot(3+7)\\ &= 5\cdot 8\cdot 12\cdot 7\cdot 9\cdot 10 \qquad (1)\end{aligned}$$

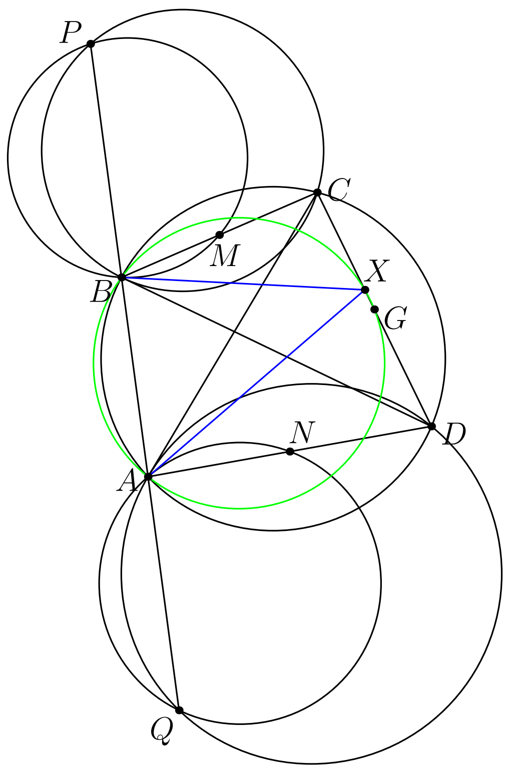
which is obviously divisible by $7!$.

Case 2) However, already for $n > 7$, there isn't enough factors in the product (because no new primes appear up until 11).

Thus, the only $n$ satisfying the problem statement is 7. □

**Problem 1.11** (IMO Shortlist 2022 G3)

Let $ABCD$ be a cyclic quadrilateral. Assume that the points $Q, A, B, P$ are collinear in this order, in such a way that the line $AC$ is tangent to the circle $ADQ$, and the line $BD$ is tangent to the circle $BCP$. Let $M$ and $N$ be the midpoints of segments $BC$ and $AD$, respectively. Prove that the following three lines are concurrent: line $CD$, the tangent of circle $ANQ$ at point $A$, and the tangent to circle $BMP$ at point $B$.




*Proof.* Notice that,

$$\measuredangle QDA = \measuredangle CAB = \measuredangle BDC$$

and,

$$\measuredangle QAD = \measuredangle BCD$$

Thus, $\triangle ADQ \sim \triangle CDB$, and $D$ is the center of the spiral similarity between the triangles.

Let $G$ be the midpoint of $CD$. Notice, under the spiral similarity $N$ goes to $G$. Thus,

$$\measuredangle GBC = \measuredangle NQA = \measuredangle DAX$$

where $X$ is the intersection of the tangent of $(ANQ)$ from $A$ with $CD$. Analogously we obtain that $\measuredangle DAG = \measuredangle X'BC$.

However, this means that $\measuredangle GBX' = \measuredangle GAX$, thus $X$ and $X'$ are the same point, the intersection of $(BAG)$ with $CD$. $\square$

**Problem 1.12** (IMO 2022 P1)

The Bank of Oslo issues two types of coin: aluminum (denoted A) and bronze (denoted B). Marianne has $n$ aluminum coins and $n$ bronze coins arranged in a row in some arbitrary initial order. A chain is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \leq 2n$, Gilberty repeatedly performs the following operation: he identifies the longest chain containing the $k^{th}$ coin from the left and moves all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be $AABBBABA \to BBBAAABA \to AAABBBBA \to BBBBAAAA \to ...$

Find all pairs $(n, k)$ with $1 \leq k \leq 2n$ such that for every initial ordering, at some point of the process there will be at most one aluminium coin adjacent to a copper coin.

*Proof.* Obviously it makes sense to split the row into consecutive blocks of coins of the same type. Then, the fact that at most one aluminium coin is adjacent to a copper coin is equivelent to there being $\leq 2$ blocks. Notice,

**Lemma 1.8** The number of blocks decreases or remains constant after every operation.

Thus, in order to not obtain a situation with $\leq 2$ blocks, it must be that the number of blocks doesn't change. However, the number of blocks doesn't change if and only if, either the pointer is on the first block, or that the pointer is on the last block. Thus, one of these two conditions must be satisfied after a certain point in time for infinitely many operations. Thus, a situation stabalizes (i.e. doesn't change the number of blocks) if and only if the size of each block is less than the distance from the pointer to the right end of the row, or that the pointer points to the first block.

Case 1) Notice, that for $n > \lceil \frac{3n}{2} \rceil$ we can consider four blocks of alternating coin types with the sizes $\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil$, then the blocks will simply rotate and there will never be 2 blocks.

Case 2) If $k < n$, then the configuration of where the first $n - 1$ coins are of one time, and then the rest is randomly distributed will never change, thus will never achieve $\leq 2$ blocks.

Case 3) If $\lceil \frac{3n}{2} \rceil \geq k \geq n$, then if there are $\geq 4$ blocks the situation won't be stable (due to the above). However, a configuration with an odd number of blocks is never stable, because at some point blocks of the same type will merge. Thus, the situation will only be stable when the number of blocks is 2, which is what is required.

Thus, $n \leq k \leq \lceil \frac{3n}{2} \rceil$. $\square$

**Problem 1.13** (IMO 2016 P4)

A set of positive integers is called fragrant if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2+n+1$. What is the least possible positive integer value of $b$ such that there exists a non-negative integer $a$ for which the set

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

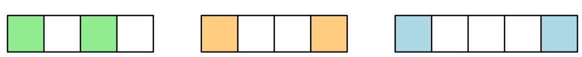
is fragrant?

*Proof.* It is well known that a degree $N$ polynomial over $\mathbb{Z}_p$ has no more than $N$ roots, thus $n^2+n+1$ is congruent to zero modulo $p$ no more than twice per $p$ consecutive values.

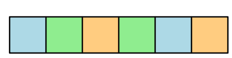
After not noticing any meaningful pattern in the problem it is reasonable to just brute force some values. Notice,

$$\begin{cases} 1+1+1=3 \\ 2^2+2+1=7 \\ 3^3+3+1=13 \\ 4^2+4+1=21=7\cdot 3 \\ 5^5+5+1=31 \\ 6^2+6+1=43 \\ 7^2+7+1=57=19\cdot 3 \\ 8^2+8+1=73 \\ 9^2+9+1=91=7\cdot 13 \\ 10^2+10+1=111=37\cdot 3 \\ 11^2+11+1=133=7\cdot 19 \end{cases}$$

Thus, it is seems apparent that the primes $3, 7$ and $19$ are interesting. They have the following patterns,



The three diagrams represent $7, 3$ and $19$, respectively, where the coloring represents distances between $P(x)$ and $P(y)$ uch that both are divisible by the appropriate prime. Using CRT we can essentially request 6 consecutive values with the following pattern,



Now, all we have to prove is that $b = 6$ is the smallest value for $b$. However, notice,

$$\begin{aligned}(n^2+n+1,(n+1)^2+n+1+1) &= (n^2+n+1,n^2+3n+3) = (n^2+n+1,2n+2) \\ &= (n^2+n+1,n+1) = (n^2,n+1) = 1 \quad (2)\end{aligned}$$

$$\begin{aligned}(n^2+n+1,(n+2)^2+n+2+1) &= (n^2+n+1,n^2+5n+7) = (n^2+n+1,4n+6) \\ &= (n^2+n+1,2n+3) \mid (2n^2+2n+2,2n^2+3n) \\ &= (-n+2,2n^2+3n) = (2-n,2n+3) = (2-n,7) \mid 7 \quad (3)\end{aligned}$$

$$\begin{aligned}(n^2+n+1,(n+3)^2+n+3+1) &= (n^2+n+1,n^2+7n+13) = (n^2+n+1,6n+12) \\ &= (n^2+n+1,3n+6) \mid (3n^2+3n+3,3n^2+6n) \\ &= (-3n+3,3n^2+6n) \mid 3(-n+1,n^2+n) = 3(1-n,n+1) \mid 3 \quad (4)\end{aligned}$$

$$\begin{aligned}(n^2+n+1,(n+4)^2+n+4+1) &= (n^2+n+1,n^2+9n+21) = (n^2+n+1,8n+20) \\ &= (n^2+n+1,2n+5) \mid (2n^2+2n+2,2n^2+5n) \\ &= (-3n+2,2n^2+5n) = (-3n+2,2n+5) = (3n-2,-n+7) \\ &= (3n-2,7-n) = (7\cdot 3-2,7-n) \mid 19 \quad (5)\end{aligned}$$

Thus, if $b \leq 5$, then we must only use the $3, 7$ and $19$ factors, however the patterns from above cannot form $b \leq 5$, contradiction! $\square$

**Problem 1.14** (IMO 2023 P1)

Determine all composite integers $n > 1$ that satisfy the following property: if $d_1$, $d_2$, …, $d_k$ are all the positive divisors of $n$ with $1 = d_1 < d_2 < \cdots < d_k = n$, then $d_i$ divides $d_{i+1} + d_{i+2}$ for every $1 \le i \le k - 2$.

*Proof.* Let us look at the largest three divisors, then they are $\{n, \frac{n}{p}, \frac{n}{q}\}$ or $\{n, \frac{n}{p}, \frac{n}{p^2}\}$. Let us consider the first case, due to the condition in the problem statement it must be that,

$$\begin{aligned} &\frac{n}{q} \mid n + \frac{n}{p} \\ \implies &\frac{1 + \frac{1}{p}}{1/q} \in \mathbb{Z} \\ \implies &\frac{p+1}{pq} \in \mathbb{Z} \end{aligned}$$

However, $p + 1$ is not divisible by $p$, thus the fraction cannot be an integer. Consequently, it must be that the three largest divisors must be $\{n, \frac{n}{p}, \frac{n}{p^2}\}$, which implies that $\{1, p, p^2\}$ are the smallest divisors. Now, let us assume that the first $k$ divisors are of the form $p^i$, then, let us prove that the $k+1$-st divisor must also be of that form. Notice, otherwise,

$$p^{k-1} \mid p^k + q$$

which is impossible! Consequently all the divisors are consecutive powers of $p$. Thus, the only $n$ for which the condition in the problem statement is satisfied are powers of a prime $p$. □

**Problem 1.15** (IMO 2019 P4)

Find all positive $(n, k)$ such that,

$$k! = (2^n - 1)(2^n - 2)\ldots(2^n - 2^{n-1})$$

*Proof.* Let us look at the 2-adic valuation of both sides,

$$v_2\Big((2^n - 1)(2^n - 2)\ldots(2^n - 2^{n-1})\Big) = v_2\Big((2^n - 2)(2^n - 4)\ldots(2^n - 2^{n-1})\Big)$$
$$= 1 + 2 + \ldots + (n-1) = \frac{n(n-1)}{2}$$

consequently,

$$v_2(k!) = \sum_{i=1}^{\infty} \left\lfloor \frac{k}{2^i} \right\rfloor = \frac{n(n-1)}{2}$$

which implies that $k > \frac{n(n-1)}{2}$. However, notice that by LTE,

$$v_3(2^n - 1) = v_3(4^{n/2} - 1) = 1 + v_3\left(\frac{n}{2}\right)$$

Thus,

$$v_3\Big((2^n-1)(2^n-2)\ldots(2^n-2^{n-1})\Big) = v_3\Big((2^n-1)\cdot 2\cdot(2^{n-1}-1)\ldots\cdot 2^{n-1}\cdot(2-1)\Big)$$
$$= v_3\Big((2^n - 1)(2^{n-1} - 1)\ldots(2^2 - 1)\cdot(2-1)\Big)$$
$$= v_3\Big((2^2 - 1)\cdot(2^4 - 1)\cdot\ldots\cdot(2^n - 1)\Big)$$
$$= v_3\left(\frac{2}{2}\right) + v_3\left(\frac{4}{2}\right) + \ldots + v_3\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + \left\lfloor\frac{n}{2}\right\rfloor$$
$$= \left[\sum_{w=1}^{\lfloor n/2\rfloor} v_3(w)\right] + \left\lfloor\frac{n}{2}\right\rfloor < \left\lfloor\frac{n}{2}\right\rfloor\cdot\left[\frac{1}{3}+\frac{1}{9}+\ldots\right] + \left\lfloor\frac{n}{2}\right\rfloor = \frac{1}{1-\frac{1}{3}}\cdot\left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor = \frac{3}{2}\left\lfloor\frac{n}{2}\right\rfloor$$

Thus, to summarize the findings we have the following,

$$\begin{cases} v_2(k!) = \frac{n(n-1)}{2} \\ v_3(k!) < \frac{3}{2}\left\lfloor\frac{n}{2}\right\rfloor \end{cases}$$

However, obviously $v_3(k!) \geq \left\lfloor\frac{k}{3}\right\rfloor$,

$$\left\lfloor\frac{k}{3}\right\rfloor < \frac{3}{2}\left\lfloor\frac{n}{2}\right\rfloor \implies \left(\frac{k}{3} - 1\right) < \frac{3}{2}\frac{n}{2}$$

$$\implies 4k + 12 < 9n \implies k < \frac{9}{4}n + 3$$

However, then,

$$\frac{n(n-1)}{2} < k < \frac{9}{4}n + 3$$

is only true for $n \leq 6$. It is simple to verify that $(1, 1)$ and $(2, 3)$ are the only solutions. $\square$

**Problem 1.16** (IMO 2022 P5)

Find all triples $(a, b, p)$ of positive integers with $p$ prime and

$$a^p = b! + p.$$

*Proof.* A relative order of variables seems to be something to look into, consequently let us consider several cases,

Case 1) If $a > b$, then we can consider the relative order between $b$ and $p$. Thus, we obtain another branching in the case logic,

- Case 1.1) Notice, that if $b \geq p$, then it must be that $a$ is divisible by $p$, since the RHS is divisible by $p$. Obviously if $b \geq 2p$, then the order of $p$ in the $RHS$ will be one, since $v_p(b! + p) = \min\{v_p(b!), p\} = \min\{\geq 2, 1\} = 1$, thus, $b < 2p$. However, since $p \mid a$ and $a > b$ it must be that $a \geq 2p$. To summarize the findings in a single inequality,

  $$a \geq 2p > b \geq p$$

  Notice, if $q \mid a$ and $b > q$, then the LHS will be divisible by $q$, however the RHS won't be divisible by $q$ contradiction! Let us assume a prime $q$ greater than $p$ divides $a$, then,

  $$a^p \geq p^{2p} \stackrel{?}{>} (2p-1)! + p \geq b! + p$$

  if the inequality with the question mark is proven, then equality is impossible, contradiction! Notice,

  $$(2p-1)! = \Big(1 \cdot (2p-1)\Big) \cdot \Big(2 \cdot (2p-2)\Big) \cdot \ldots \cdot \Big((p-1) \cdot (p+1)\Big) \cdot p \leq p^{2p-1}$$

  obviously adding $p$ won't change anything, thus the inequality is proven. If there is no other $q$, then $b \geq p^2$ and one will obtain exactly the same inequality.

- Case 1.2) If $b < p$, then,

  $$a^p - b! \geq a^p - p^p \geq (p+1)^p - p^p > p$$

  thus, equality can never be achieved!

Case 2) Now, let us consider the case, when $b \geq a$. Then, LHS is divisible by $a$ and $b!$ is divisible by $a$, consequently it must be that $a \mid p$, consequently $a = 1 \vee p$. Since, obviously $a \neq 1$ it must be that $a = p$. Thus, we are solving the following equations,

$$p^p = b! + p$$

Obviously it must be that $v_p(b!) = 1$, else the RHS would have a $p$-adic valuation of 1, which is not equal to the $p$-adic valuation of the LHS, $p$. Thus, it must be that $b < 2p$. Also, since both sides need to be divisible by $p$ it must be that $b \geq p$, consequently, $2p > b \geq p$.

$$p^p - p = p(p^{p-1} - 1) = b!$$

Notice, by LTE,

$$\begin{aligned} v_2(p^{p-1} - 1) = v_2(p-1) + v_2(p+1) + v_2(p-1) - 1 \\ = 2v_2(p-1) + v_2(p+1) - 1 \end{aligned} \tag{6}$$

Consequently, since at least one of $v_2(p-1)$ or $v_2(p+1)$ will be one for $p \geq 3$. It must be that the value above is either $2v_2(p-1)$ or $1 + v_2(p+1)$ for $p > 3$.

If $b = p$, then due to $p^p - p > p!$ for $p > 3$ we obtain a contradiction! Thus, $b \geq p + 1$, however, then, (for $p > 3$)

$$\begin{cases} v_2(b!) \geq v_2((p+1)!) > v_2(p+1) + v_2(2) = v_2(p+1) + 1 \\ v_2(b!) \geq v_2((p+1)!) > v_2(p-1) + v_2\left(\frac{p-1}{2}\right) + v_2(p+1) \geq 2v_2(p-1) \end{cases}$$

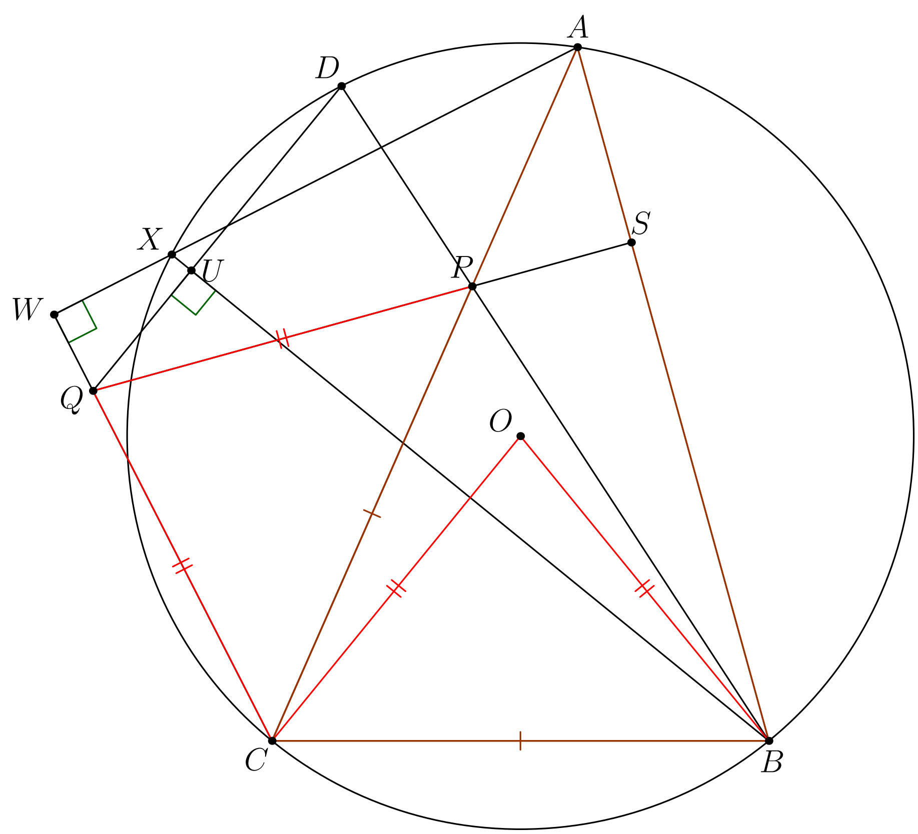
Thus, we just need to check $p = 2$ and $p = 3$, both of which obtain the only solutions $(2, 2, 2)$ and $(3, 4, 3)$.

The only exceptions are when there doesn't exist such a $q$ or when $q = 2$ and thus making the LTE application incorrect, i.e. $p \leq 4$. It is simple to verify that we obtain two solutions for $(a, b, p)$, $(2, 2, 2)$ and $(3, 3, 4)$.

$\square$

**Problem 1.17** (IMO Shortlist 2023 G2)

Let $ABC$ be a triangle with $AC > BC$, let $\omega$ be the circumcircle of $\triangle ABC$, and let $r$ be its radius. Point $P$ is chosen on $\overline{AC}$ such that $BC = CP$, and point $S$ is the foot of the perpendicular from $P$ to $\overline{AB}$. Ray $BP$ mets $\omega$ again at $D$. Point $Q$ is chosen on line $SP$ such that $PQ = r$ and $S, P, Q$ lie on a line in that order. Finally, let $E$ be a point satisfying $\overline{AE} \perp \overline{CQ}$ and $\overline{BE} \perp \overline{DQ}$. Prove that $E$ lies on $\omega$.




*Proof.* Let $U$ be the foot of the altitude from $B$ onto $QD$ and let $W$ be the foot of the altitude from $A$ onto $CQ$. Let $X = AW \cap BU$.

The condition that $PQ = r$ hints towards drawing $O$, the circumcircle of $\omega$. Notice,

**Lemma 1.9** $\triangle PQC \sim \triangle COB$

since,

$$\angle QPB = \angle SPA = 90 - \alpha = \angle OBC$$

thus, $\triangle PQC \sim \triangle COB$ by two sides and the angle between them. Now,

**Lemma 1.10** $OD \parallel CQ$

since,

$$\begin{aligned}\angle DOC = 360 - \angle COB - \angle BOD = 360 - 2\alpha - (180 - 2 \cdot \angle OBD) \\ = 180 - 2\alpha + 2(\angle OBA - \angle PBA) \\ = 180 - 2\alpha + 2(90 - \gamma - (\beta - 90 + \frac{\gamma}{2})) = 360 - 2\alpha - 2\gamma - 2\beta + 180 - \gamma \\ = 180 - \gamma\end{aligned}$$

and $\angle QCO = \angle PCB = \gamma$, consequently $OD \parallel CQ$. However, this implies that $CQDO$ is a rhombus.

**Lemma 1.11** $\angle ABX = \gamma$

since, $\angle QUX = \angle QWX = 90$, $XWQU$ is cyclic, consequently,

$$\angle ABX = \angle WQD = 180 - \angle DQC = \angle QCO = \gamma$$

This implies that $ABCX$ is cyclic, proving the problem statement. □

**Problem 1.18** (IMO Shortlist 2023 A2)

Let $\mathbb{R}$ be the set of real numbers. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(x+y)f(x-y) \geqslant f(x)^2 - f(y)^2$$

for every $x, y \in \mathbb{R}$. Assume that the inequality is strict for some $x_0, y_0 \in \mathbb{R}$.

Prove that either $f(x) \geqslant 0$ for every $x \in \mathbb{R}$ or $f(x) \leqslant 0$ for every $x \in \mathbb{R}$.

*Proof.* Notice,

**Lemma 1.12** $f(x) = \pm f(-x)$

since if we substitute $y = -x$ and the same for $x = -y$ we will obtain,

$$\begin{cases} 0 \geq f(x)^2 - f(-x)^2 \\ 0 \geq f(-x)^2 - f(x)^2 \end{cases}$$

$$\implies f(x)^2 - f(-x)^2 = 0$$

which implies that $f(x) = \pm f(-x)$.

**Lemma 1.13** There exists a $a \in \mathbb{R}$ such that $f(a) = f(-a)$.

Notice, by substituting $x := y$ and $y := x$ we obtain,

$$\begin{cases} f(x+y)f(x-y) \geq f(x)^2 - f(y)^2 \\ f(x+y)f(y-x) \geq f(y)^2 - f(x)^2 \end{cases} \implies f(x)^2 - f(y)^2 \geq -f(x+y)f(y-x)$$

Consequently,

$$f(x+y)f(x-y) \geq -f(x+y)f(y-x)$$

But, since $x+y$ and $x-y$ take any values we can rewrite this inequality as,

$$f(a)f(b) \geq -f(a)f(-b)$$

$$\implies f(-b) \geq -f(b)$$

which implies that $f(x) \geq -f(-x)$.

However, the original inequality is strict for some $x_0, y_0$ it must be that for some $b \in \mathbb{R}$ it must be that,

$$f(-b) > -f(b)$$

which due to the lemma above implies that $f(-b) = f(b)$ for some $b \in \mathbb{R}$.

Now, simply notice that, then for any $a \in \mathbb{R}$ it must be that,

$$f(a)f(b) \geq -f(a)f(b)$$

however, if $f(a)$ didn't have the same sign as $f(b)$ this inequality would not be true, consequently all the values of $f(x)$ are either positive or negative. $\square$

# §2 EGMO

**Problem 2.1** (EGMO 2025 P1)

For a positive integer $N$, let $c_1 < c_2 < \cdots < c_m$ be all positive integers smaller than $N$ that are coprime to $N$. Find all $N \geqslant 3$ such that

$$\gcd(N, c_i + c_{i+1}) \neq 1$$

for all $1 \leqslant i \leqslant m - 1$

Here $\gcd(a, b)$ is the largest positive integer that divides both $a$ and $b$. Integers $a$ and $b$ are coprime if $\gcd(a, b) = 1$.

*Proof.* Notice, that if $N$ is even, then all $c_i$ are odd, however that implies that $c_i + c_{i+1}$ will always be even, thus never relatively prime with $N$. Consequently, every even $N$ satisfies the conditions of the problem statement. If $N$ is odd, then it is by definition relatively prime to 2, thus,

$$\gcd(N, 1 + 2) \neq 1 \implies 3 \mid N$$

Thus, $N = 3^\alpha x$ (where $x$ is odd), however,

Case 1) If $x \equiv 1 \pmod 3$, then let us consider two numbers, $x + 1$ and $x + 3$, then,

$$\gcd(3^\alpha x, x + 1) = 1$$
$$\gcd(3^\alpha x, x - 2) = 1$$

But, then,

$$(3^\alpha x, 2x - 1) = (3^\alpha, 2x - 1) = 1$$

contradiction!

Case 2) If $x \equiv 2 \pmod 3$, then consider $x + 2$ and $x - 1$, then,

$$\begin{cases} \gcd(3^\alpha x, x + 2) = 1 \\ \gcd(3^\alpha x, x - 1) = 1 \end{cases}$$

then,

$$\gcd(3^\alpha x, 2x + 1) = \gcd(3^\alpha, 2x + 1) = 1$$

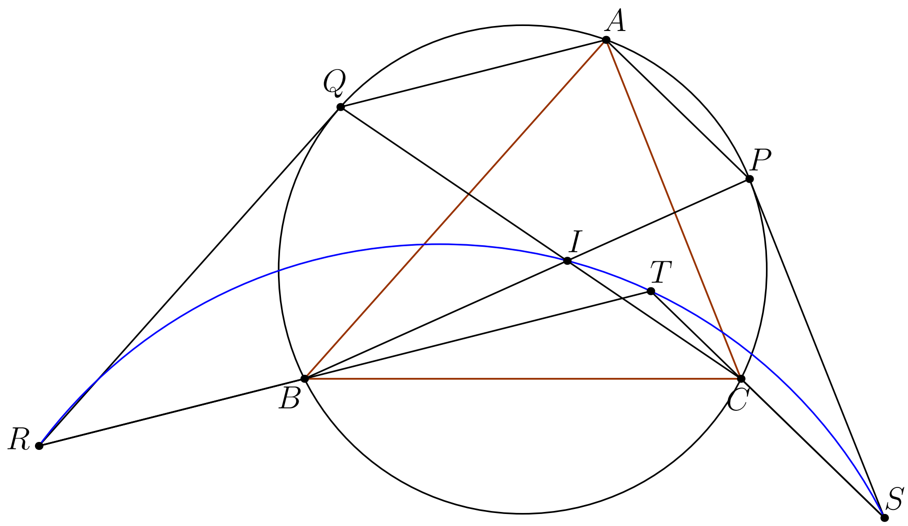
contradiction!

Thus $x = 1$. Consequently, the only solutions are all even numbers and all powers of three. □

**Problem 2.2** (EGMO 2025 P4)

Let $ABC$ be an acute triangle with incentr $I$ and $AB \neq AC$. Let lines $BI$ and $CI$ intersect the circumcircle of $ABC$ at $P \neq B$ and $Q \neq C$, respectively. Consider points $R$ and $S$ such that $AQRB$ and $ACSP$ are parallelograms (with $AQ \parallel RB, AB \parallel QR, AC \parallel SP$, and $AP \parallel CS$). Let $T$ be the point of intersection of lines $RB$ and $SC$. Prove that points $R, S, T$, and $I$ are concyclic.

*Proof.* At first it might seem unclear how to prove cyclicity of these four points.




However, after playing around with some angles it is not difficult to notice,

**Lemma 2.1** $BITC$ is cyclic.

This is because,

$$\begin{aligned}\angle BTC &= 180 - \angle TBC - \angle TCB = 180 - (\angle B - \angle ABT) - (\angle C - \angle TCA) \\ &= \angle A + \angle QAB + \angle CAP = \angle A + \frac{\angle C}{2} + \frac{\angle B}{2} = 90 + \frac{\angle A}{2} = \angle BIC \quad (7)\end{aligned}$$

Now, notice,

**Lemma 2.2** $\triangle IBR \sim \triangle ICS$

Because, ($\triangle IBQ \sim \triangle ICP$)

$$\frac{BI}{BR} = \frac{IB}{QA} = \frac{IB}{BQ} = \frac{IC}{CP} = \frac{IC}{PA} = \frac{IC}{CS}$$

and $\angle RBI = \angle ICS$ since $\angle IBT = \angle TCI$ from $ITBC$ being cyclic. But this similarity implies that $\angle BIR = \angle CIS$, thus,

$$\angle RIS = \angle BIC + \angle RIB - \angle ICS = \angle BIC = \angle RTS$$

Consequently, $RITS$ is cyclic. $\square$

**Problem 2.3** (EGMO 2025 P5)

Let $n > 1$ be an integer. In a configuration of an $n \times n$ board, each of the $n^2$ cells contains an arrow, either pointing up, down, left, or right. Given a starting configuration, Turbo the snail starts in one of the cells of the board and travels from cell to cell. In each move, Turbo moves one square unit in the direction indicated by the arrow in her cell (possibly leaving the board). After each move, the arrows in all of the cells rotate $90°$ counterclockwise. We call a cell good if, starting from that cell, Turbo visits each cell of the board exactly once, without leaving the board, and returns to her initial cell at the end. Determine, in terms of $n$, the maximum number of good cells over all possible starting configurations.

*Proof.* Notice, if $n$ is odd, then $n^2$ is odd and thus it is impossible to make a cycle (due to parity).

Let us consider the very edges of the board, the only way to go through them is using one of the two following algorithms,

| | | |
|---|---|---|
| $\cdots$ | $\rightarrow$ | $\leftarrow$ |
| | | $\nearrow$ |
| ⋰ | | $\vdots$ |

| | | |
|---|---|---|
| $\cdots$ | $\nearrow$ | $\uparrow$ |
| | | $\uparrow$ |
| ⋰ | | $\vdots$ |

where by $\nearrow$ I mean either $\rightarrow$ or $\uparrow$. Thus, due to the cycles being uniquely determined after the turn, it must be due to there being only two ways of going through the corners, only two global cycles. However, notice that the two configurations above are not equivelent under the rotation operation, thus there cannot be more than one global cycle. However, given a cycle there are only $\frac{n^2}{4}$ generators (every fourth element in the cycle).

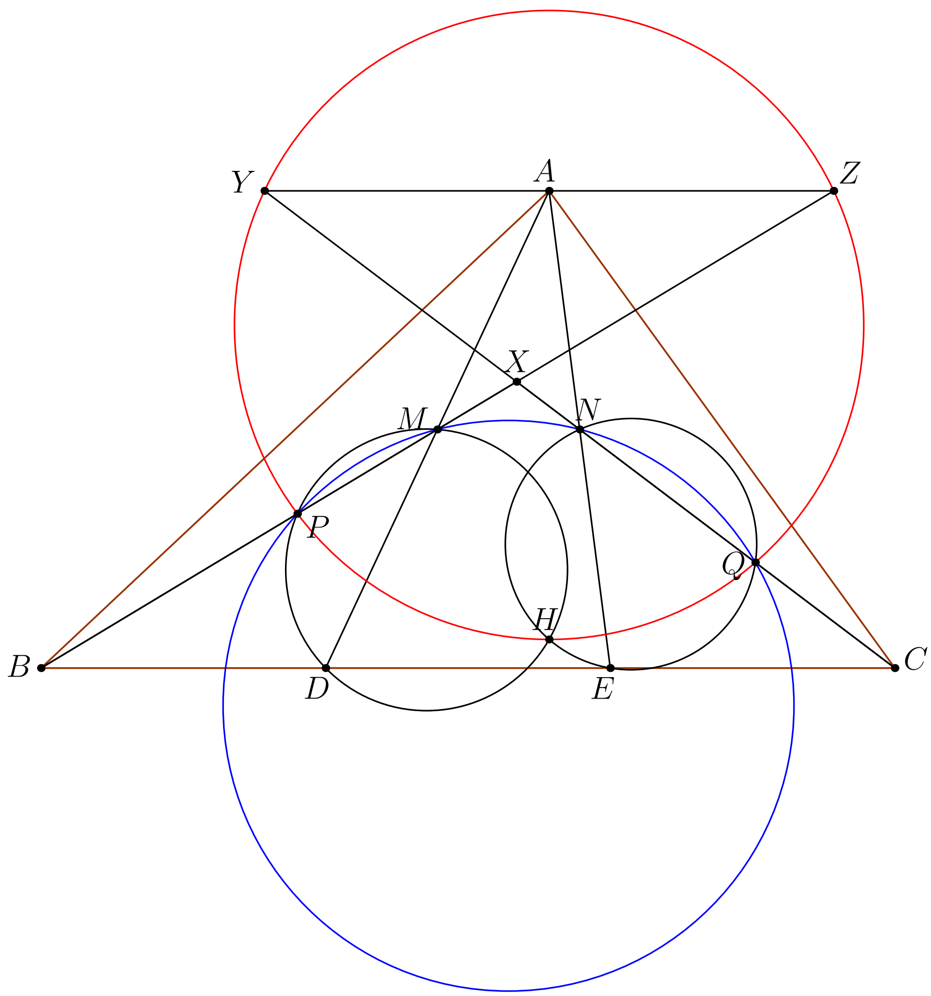
To achieve this bound, simply consider any global cycle through the entire board and appropriatly adjust the arrows so that it is a cycle from a given square. Then, due to the entire board taking the original position after four rotations, it must be that every fourth element in the cycle is a generator as well.

Thus, for all odd $n$ there are 0 good cells, however for even $n$ there are $\frac{n^2}{4}$ good cells. $\square$

**Problem 2.4** (EGMO 2025 P3)

Let $ABC$ be an acute triangle. Points $B, D, E$, and $C$ lie on a line in this order and satisfy $BD = DE = EC$. Let $M$ and $N$ be the midpoints of $AD$ and $AE$, respectively. Suppose triangle $ADE$ is acute, and let $H$ be its orthocentre. Points $P$ and $Q$ lie on lines $BM$ and $CN$, respectively, such that $D, H, M$, and $P$ are concyclic and pairwise different, and $E, H, N$, and $Q$ are concyclic and pairwise different. Prove that $P, Q, N$, and $M$ are concyclic.

*Proof.* Let us reflect $C$ and $B$ over $N$ and $M$, respectively, then we will obtain points $Y$ and $Z$. Let $X$ be the intersect of $BM$ and $CN$.



Notice,

**Lemma 2.3** $AZHE$ is cyclic.

This is because,

$$\measuredangle AHE = \measuredangle EDA = \measuredangle AZE$$

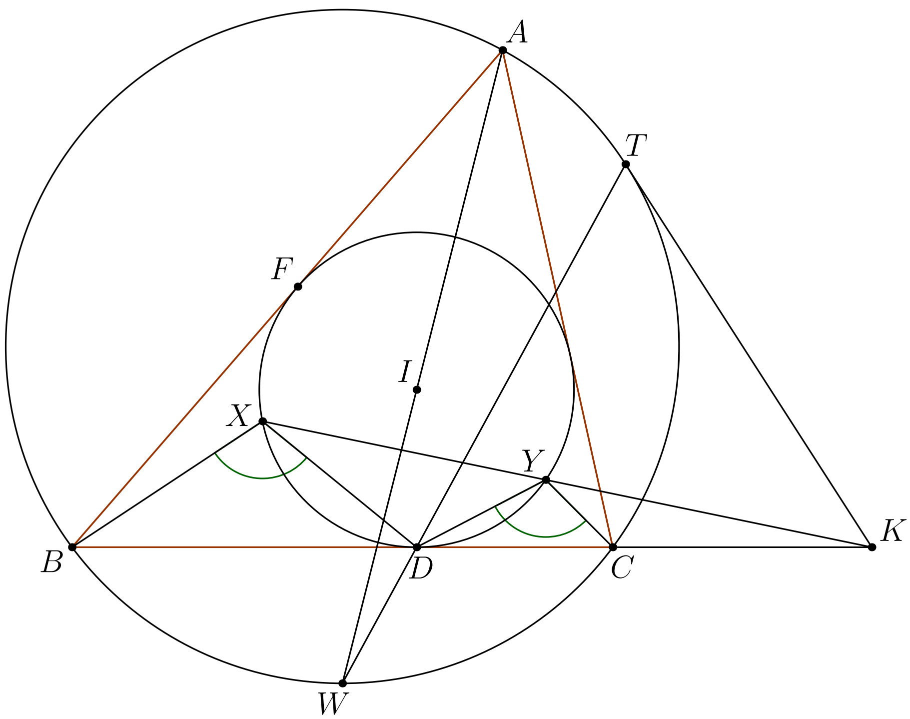
Now, notice that,

$$\measuredangle AZH = \measuredangle AEH = \measuredangle YQH$$

thus $YZHQ$ is cyclic, analogously it must be that $YZPH$ is cyclic, consequently $YZPQH$ is cyclic. However, because $MN \parallel YZ$ by Reim's lemma it must be that $MNPQ$ is cyclic. $\square$

**Problem 2.5** (EGMO 2024 P2)

Let $ABC$ be a triangle with $AC > AB$ , and denote its circumcircle by $\Omega$ and incentre by $I$. Let its incircle meet sides $BC, CA, AB$ at $D, E, F$ respectively. Let $X$ and $Y$ be two points on minor arcs $\widehat{DF}$ and $\widehat{DE}$ of the incircle, respectively, such that $\angle BXD = \angle DYC$. Let line $XY$ meet line $BC$ at $K$. Let $T$ be the point on $\Omega$ such that $KT$ is tangent to $\Omega$ and $T$ is on the same side of line $BC$ as $A$. Prove that lines $TD$ and $AI$ meet on $\Omega$.




*Proof.* Notice,

$$\measuredangle BXY = \measuredangle BXD + \measuredangle DXY = \measuredangle DYC + \measuredangle CDY = \measuredangle BCY$$

Thus, the wierd condition that $\measuredangle BXD = \measuredangle DYC$ is simply equivelent to BXYC being cyclic. However, notice that as we move $X$ on the incircle, if we define $Y$ to be the intersection of $(BXC)$ with the incircle, then $XY$ passes through a constant point. This is simply due to the power of the point, if a ray through $K$ intersects the incircle at $X'$ and $Y'$, then,

$$KC \cdot KB = KX \cdot KY = KX' \cdot KY'$$

thus, $BX'Y'C$ is cyclic. Consider a circle tangent to $KT$ at $T$ which passes through $D$, assume that it passes through another point on $BC$, let it be $D'$, then,

$$KD \cdot KD' = KT^2 = KX \cdot KY$$

however $KD^2 = KX \cdot KY$, thus $D = D'$. Thus, a circle exists which is tangent to $TK$ at $T$ and which is tangent to $BC$ at $D$, however then by the Shooting lemma it must be that $DT$ passes through the midpoint of the arc $\widehat{BC}$.

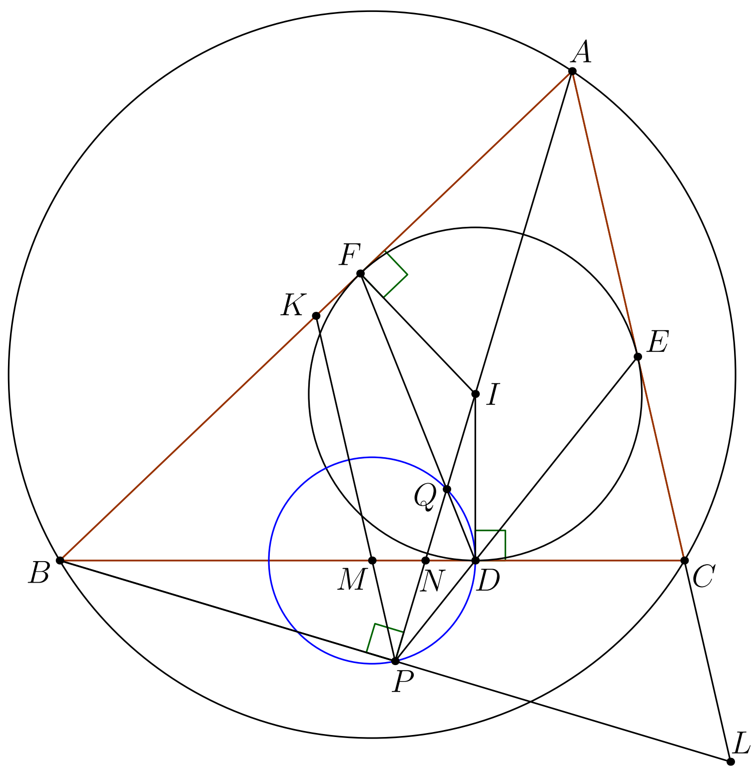
Thus, $AI$ and $TD$ intersect on $\Omega$, exactly at the midpoint of the arc $\widehat{BC}$. $\square$

# §3 Korean National Olympiad

**Problem 3.1** (Korea 2025 P4)

Triangle $ABC$ satisfies $\overline{CA} > \overline{AB}$. Let the incenter of triangle $ABC$ be $\omega$, which touches $BC, CA, AB$ at $D, E, F$, respectively. Let $M$ be the midpoint of $BC$. Let the circle centered at $M$ passing through $D$ intersect $DE, DF$ at $P(\neq D), Q(\neq D)$, respecively. Let line $AP$ meet $BC$ at $N$, line $BP$ meet $CA$ at $L$. Prove that the three lines $EQ, FP, NL$ are concurrent.

*Proof.* Let us start by proving that $A, P, Q$ are colinear. Let us define $P$ as the intersection of $AI$ and $DE$, then we must prove that $MPD$ is isoseles.




This can be easily shown by angle chase, let us introduce $K$, the midpoint of $AB$ it is well known that then $K, M, P$ are colinear and $\angle BPA = 90$. Thus,

$$\angle MPD = \angle APD + \angle KPA = \angle IBC + \angle PAC = \frac{\angle A + \angle B}{2}$$
$$= \frac{180 - \angle C}{2} = \angle EDC = \angle MDP \quad (8)$$

Thus, it must be that $P$ lies on $AI$, the exact same logic shows that $Q \in AI$, thus, $A, I, Q$ and $P$ are colinear.

Now, notice that under reflection over $AI$ the line $FQ$ goes to $EQ$ and the line $LN$ goes to $BN$ (since $\angle APB = 90$ it must be that $B$ goes to $L$ under reflection) and $PF$ goes to $PE$. Thus, $EQ$, $LN$ and $PF$ are concurrent, and the intersection point is the reflection of $D$ over $AI$. $\square$