The Anchor Point Lemma

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1 Theory

The Anchor Point Lemma is a series of configurations surrounding circles passing through D with well-defined intersections with sides of a triangle.

1. **(Anchor Point Lemma)** Given a triangle $\triangle ABC$, let D be an arbitrary point on BC, then let DE and DF be parallel to AC and AB respectively. Let (AEF) intersect (ABC) at G, let GD intersect (ABC) at X. Prove, that if M is the midpoint of AC, then A, X and M are colinear.

2. **(Generalized Anchor Point Lemma)** Let D be an arbitrary point on BC in $\triangle ABC$. Let there be two fixed directions l_1 and l_2. Let E and F be the intersection of two lines through D parallel to l_1 and l_2 with AB and AC. Let G be the intersection of (AFE) and (ABC). Prove that DG passes through a constant point X on (ABC).

   Also prove that the circles (AFE) are all coaxial and the second constant point is the intersection of the circle with the line isogonal to AX in $\angle CAB$ with (AFE).

   > **Remark 1.1** The circle (AGD) passes through a fixed point K on BC such that AK is isogonal to AX. If $P = CE \cap BF$ then the line DP passes through a fixed point.

3. **(Advanced Anchor Point Lemma)** Let P be an arbitrary fixed point and X an abitrary fixed point on (ABC). Let D be an abritrary point on BC. Let PD intersect ACat E. Let XD intersect (ABC) at G. Let (AEG) intersect AB a second time at F. Prove that the line DF passes through a constant point Q as D moves on BC, and that (AEF) passes through a fixed point W.

   > **Remark 1.2** Similarly the to the previous lemma if $P = CE \cap BF$ then the line DP passes through a fixed point.

Note: More details about this method can be found at $https://mrmineev.com/articles/anchor-point/main.html$

2 More Advanced Theory

Let $\text{Anchor}_P^{ABC}(E)$ be the Anchor Point conjugate of point P where E is the parameter. Notice,

1. $\text{Anchor}_E^{\triangle ABC}(AC)$ is a line which passes through C and the intersection of AE with the line through B parallel to AC.

2. $\text{Anchor}_E^{\triangle ABC}(AB)$ is a line which passes through B and the intersection of AE with the line through B parallel to AC.

3. $\text{Anchor}_{(ABC)}^{\triangle ABC}(P)$ is a line parallel to AC which passes through the intersection of the line through P parallel to AB with BC.

4. Let $\mathcal{L}$ be an arbitrary line in the plane then $\text{Anchor}_E^{\triangle ABC}(\mathcal{L})$ is a line which passes through $\mathcal{L} \cap BC$.

In addition to these properties the cross-ratio is preserved.

3 Problems

1. **(The Shooting Lemma)** Consider the chord BC in the circle Ω. Let the circle ω touch BC at a point D and the circle Ω at a point E. Prove that the line DE passes through M, the middle of the larger arc $\overset{\frown}{BC}$.

2. **(Sharky-Devil Config)** Let (I) be the incircle of $\triangle ABC$, let D, E, F be the tangency points of (I) with BC, AB and AC, respectively. Let G be the second intersection of (AFE) with (ABC). Let S be the midpoint of the arc BC. Prove that S, D and G are colinear.

3. (USA TST 2008 P7) Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC. Points Q and R lie on sides AC and AB respectively, such that $PQ \parallel AB$ and $PR \parallel AC$. Prove that, as P varies along segment BC, the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$.

4. (USA TST 2012 P1) In acute triangle ABC, $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on side BC. Points D and E lie on sides AB and AC, respectively, such that $BP = PD$ and $CP = PE$. Prove that as P moves along side BC, the circumcircle of triangle ADE passes through a fixed point other than A.

5. (ELMO 2013 Shortlist G3) Given $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.

6. (APMO 2012 P4) Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side BC, by M the midpoint of BC, and by H the orthocenter of ABC. Let E be the point of intersection of the circumcircle Γ of the triangle ABC and the half line MH, and F be the point of intersection (other than E) of the line ED and the circle Γ. Prove that $\frac{BF}{CF} = \frac{AB}{AC}$ must hold.

7. **(based on Moscow Mathematical Olympiad 2015)** Let $\triangle ABC$ be an isoseles triangle. Let P and Q be points on sides AB and AC, respectively, such that $CP = AQ$. Let X be a point on BC such that $PX = PC$ and T be the second intersection of (APQ) with (ABC). Prove that $\angle ATX = 90$.

8. (IMO Shortlist 2016 G2) Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

9. (BMO Shortlist 2017 G5) Let ABC be an acute angled triangle with orthocenter H. centroid G and circumcircle ω. Let D and M respectively be the intersection of lines AH and AG with side BC. Rays MH and DG interect ω again at P and Q respectively. Prove that PD and QM intersect on ω.

10. **(APMO 2022 P2)** Let ABC be a right triangle with $\angle B = 90°$. Point D lies on the line CB such that B is between D and C. Let E be the midpoint of AD and let F be the seconf intersection point of the circumcircle of $\triangle ACD$ and the circumcircle of $\triangle BDE$. Prove that as D varies, the line EF passes through a fixed point.

    Let X be a point on the arc AB of Γ not containing C, such that $\angle AXH = \angle AFH$. Let K be the circumcenter of triangle XIA. Prove that the lines AO and KI meet on Γ.

11. (IMO Shortlist 2005 G5) Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.

12. (by @SBYT) Given a triangle $\triangle ABC$ with an incenter I, let D be a point on segment BC. Let the perpendicular line from D to CI meet BI at E and the perpendicular line from D to BI meet CI at F. Let the perpendicular line from D to BC meet EF at G. Prove that the circles (ABC) and the circle centered at G with radius GD are tangent.

4 Some of my Own

1. (AoPS) Let D be an arbitrary point on the side BC of triangle $\triangle ABC$. Let E and F be the intersections of the lines through D, parallel to AC and AB, with AB and AC, respectively. Let G be the intersection point of the circumcircle of triangle AFE with the circumcircle of triangle ABC. Let M be the midpoint of BC, and let X be the second intersection point of line AM with the circumcircle of triangle ABC. Prove that lines EF, AD, and the tangent to the circumcircle of triangle XMG at point M are concurrent.

2. (AoPS) Let D be an arbitrary point on the side BC in a triangle $\triangle ABC$. Let E and F be the intersection of the lines parallel to AC and AB through D with AB and AC. Let G be the intersection of (AFE) with (ABC). Let M be the midpoint of BC and X the intersection of AM with (ABC). Let H be the intersection of (XMG) with BC. Prove that EF is parallel to AH.

3. (AoPS) Let D be an arbitrary point on the side BC in a triangle $\triangle ABC$. Let E and F be the intersection of the lines parallel to AC and AB through D with AB and AC. Let G be the intersection of (AFE) with (ABC). Let M be the midpoint of BC and X the intersection of AM with (ABC). Let J be the intersection of (XFG) with AC. Prove that XB, AD and JM are concurrent at P.

5 A bit more theory...

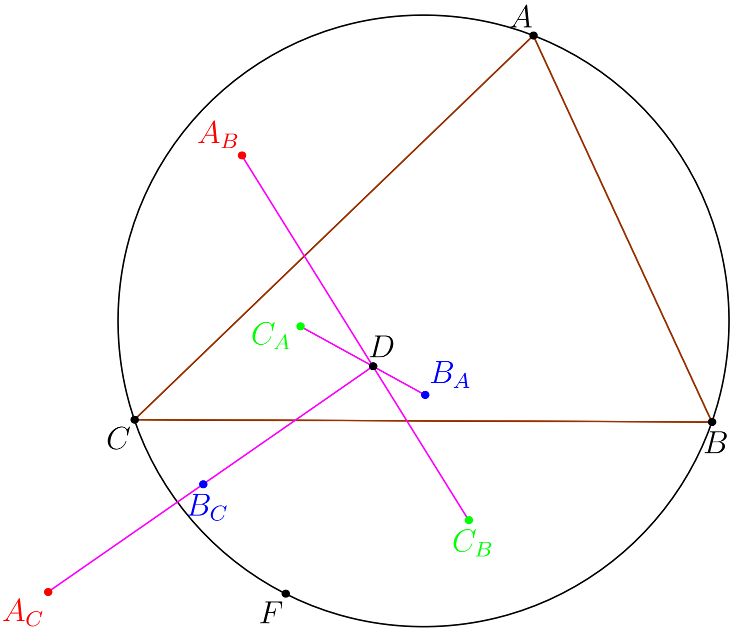
The Anchor Point Conjugate Theorem can be reformulated as,

Theorem 5.1 Fix a point $F \in (ABC)$ and an arbitrary point D in the plane. Let ω be a circle passing through A and D. Let X and Y be the second intersection points of ω with AC and AB, respectively. Let Q be the second intersection of ω with the circumcircle (ABC), and define

$$P = FQ \cap BC.$$

Then, as ω varies, the lines PX and PY each pass through a fixed point.

Let these two points be called the *Evil Anchor Point* conjugates of D in $\triangle ABC$ with respect to F, which will be denoted as $\text{Evil}_{F,W}^{\triangle ABC}(D)$ were W is either B or C depending on whether it is the point which lies on PX or PY.



1. **(Evil Characterization)** Let (ABD) meet lines AC and AF at X and Y, respectively, then $\text{Evil}_{F,B}^{\triangle ABC}(D) = BX \cap DY$.

2. (AoPS) Prove that,

$$\begin{cases} D \in \text{Evil}_{F,B}^{\triangle ABC}(D)\text{Evil}_{F,B}^{\triangle CBA}(D) \\ D \in \text{Evil}_{F,A}^{\triangle BAC}(D)\text{Evil}_{F,A}^{\triangle CAB}(D) \\ D \in \text{Evil}_{F,C}^{\triangle ABC}(D)\text{Evil}_{F,C}^{\triangle BAC}(D) \end{cases}$$

6 Extra (Unsolved)

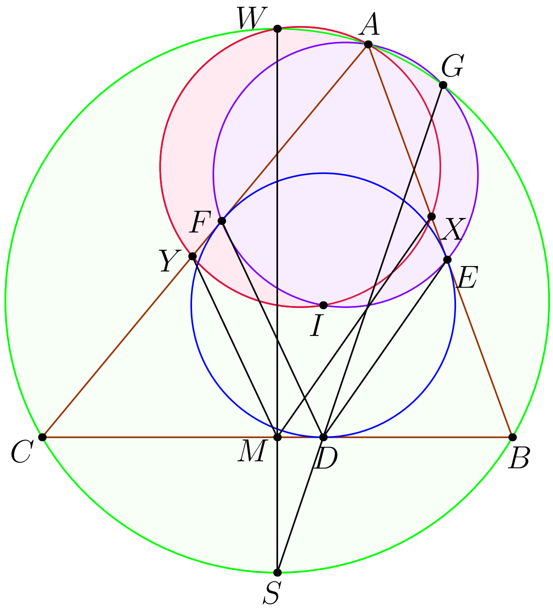
1. **(USEMO 2025 P2)** Let ABC be a fixed triangle with circumcircle ω. Consider P a variable point inside ABC. Ray BP meets side AC at Y while ray CP meets side AB at X. Let Q be the second intersection of ω and the circumcircle of triangle AXY. Let K be the second interesction of ray AP and ω.

   Prove that as P varies, the circumcircles of triangle QPK all have a common radical center.

7 Proposed Solutions to Some Problems

Problem 7.1 (Sharky-Devil Configuration)

Let (I) be the incircle of $\triangle ABC$, let D, E, F be the tangency points of (I) with BC, AB and AC, respectively. Let G be the second intersection of (AFE) with (ABC). Let S be the midpoint of the arc BC. Prove that S, D and G are colinear.



Proof. Let us introduce M the midpoint of BC and W the midpoint of the larger arc BC. Then, let X and Y be the intersections of the lines through M parallel to DE and DF with AB and AC, respectively. By the Generalized Anchor Point Lemma all that is left to prove is that $AWXY$ is cyclic.

Notice, since,

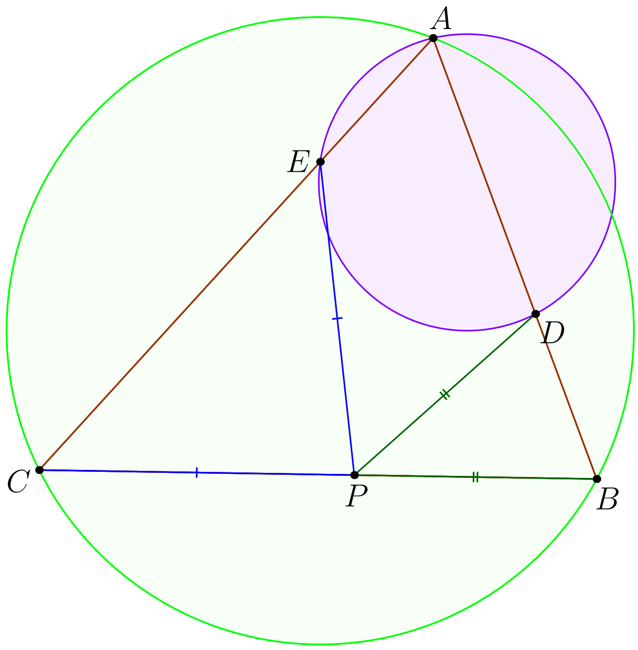
$$\measuredangle WXY = \measuredangle WAC$$

$$\measuredangle XYW = \measuredangle XAW$$

which implies that $\angle WAC = 180 - \angle BAW$, however this is only true for W being the midpoint of the larger arrc BC. Thus $WAXY$ is cyclic which proves one of the properties of the Sharky-Devil point. (Amusingly I lies on this circle as well due to $I \in (AFE)$ and I lying on the angle bisector of $\angle CAB$). ∎

Problem 7.2 (USA TST 2012 P1)

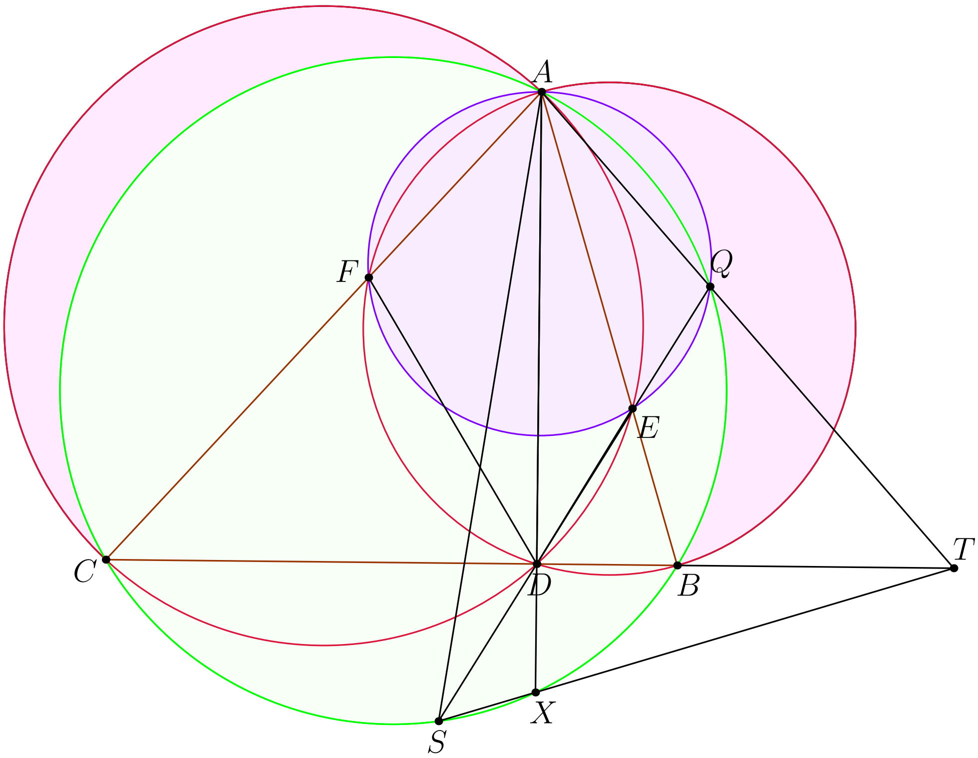
In acute triangle ABC, $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on side BC. Points D and E lie on sides AB and AC, respectively, such that $BP = PD$ and $CP = PE$. Prove that as P moves along side BC, the circumcircle of triangle ADE passes through a fixed point other than A.



Proof. Since as we move P the lines EP and PD are parallel to two fixed directions, thus by the Generalized Anchor Point Lemma it must be that (AED) passes through a fixed point. ■

Problem 7.3 (ELMO 2013 Shortlist G3)

n $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.



Proof. Notice, FD and DE point in constant directions, since $\angle CDF = \angle A = \angle EDB$. Thus, by the Generalized Anchor Point Lemma all we need to do is show for one position of D that (AFE) passes through some fixed point on the median. Let us fix D to be the foot of the altitude from A to BC. Let S be the intersection of the symmedian from A with (ABC), then, (it is well known, however the proof is outlined below)

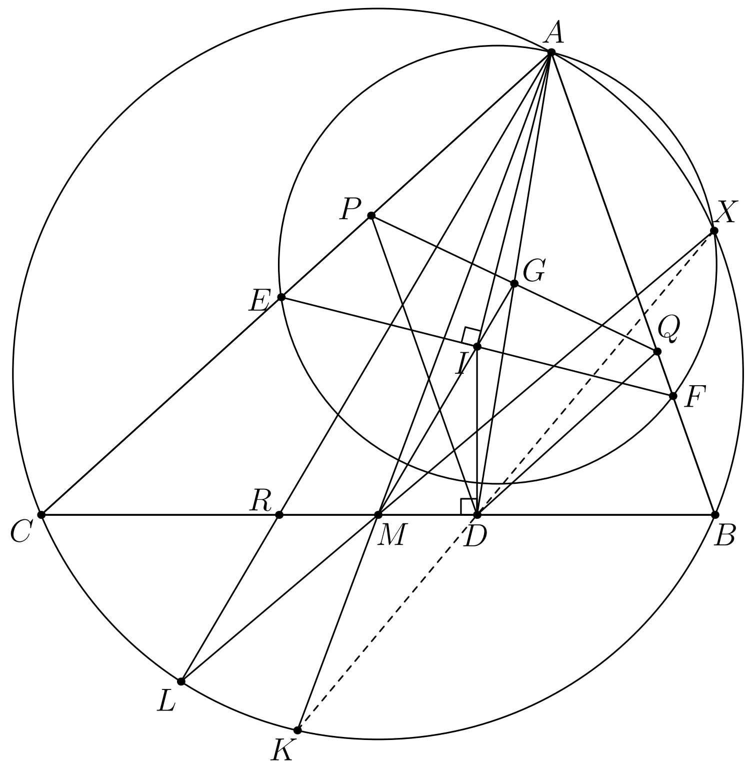
Lemma 7.1 S, D, Q are colinear.

indeed, since $ABSC$ is harmonic, by projecting from T it must be that $QBCX$ is harmonic, consequently projecting from D we obtain that Q goes to a point W on (ABC) such that $ABCW$ is harmonic, thus $W = S$, thus Q, D, S are colinear. □

Now, by the Generalized Anchor Point Lemma since S, D, Q are colinear, it must be that (AEF) passes through a fixed point lying on the isogonal line to AS in $\angle CAB$ which is the median. ■

Problem 7.4 (IMO Shortlist 2016 G2)

Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.



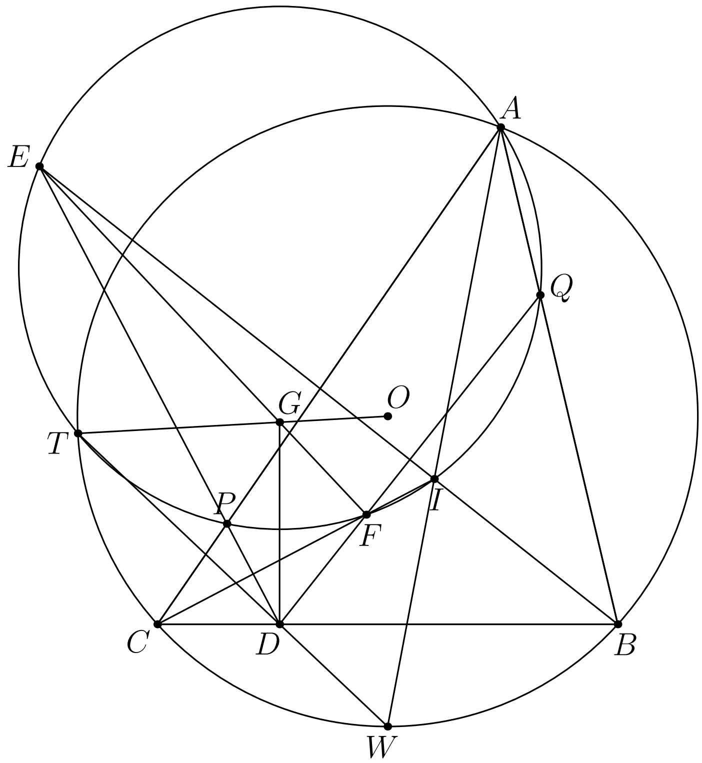
Proof. Let $K = AM \cap (ABC)$. Let P and Q be the intersections of parallel lines through D with respect to AB and AC with AC and AB, respectively.

Let $L = MX \cap (ABC)$, $R = AL \cap BC$ and $G = PQ \cap AD$. From the Anchor Point lemma we know that $GM \parallel AL$, however it is a well known lemma that M, I and G are collinear. Thus we see that $MI \parallel AR$ and the reflection of D over I lies on AR, consequntly it must be that M is the midpoint of RD.

Concluding by Butterly theorem we see that K, D and X are collinear. ■

Problem 7.5 (SBYT)

Given a triangle $\triangle ABC$ with an incenter I, let D be a point on segment BC. Let the perpendicular line from D to CI meet BI at E and the perpendicular line from D to BI meet CI at F. Let the perpendicular line from D to BC meet EF at G. Prove that the circles (ABC) and the circle centered at G with radius GD are tangent.



Proof. Let $P = DE \cap AC$, $Q = DF \cap AB$ and $T = DW \cap (ABC)$, then notice that,

Lemma 7.2 $AEQFPI$ is cyclic

Indeed, since P and Q are the reflections of D over CI and BI we know that,

$$\angle FPD = \angle FDP = \angle EQF$$

and,

$$\angle PFD = \frac{180 - 2\angle QDP}{2} = \angle A$$

consequently $APQEF$ is cyclic. By the Anchor Point Lemma we know that since as D moves PD and QD have constant directions and when D is the projection of I onto BC by Sharky-Devil we know that TD passes through W the midpoint of the arc BC. Thus TD always passes through W, but Anchor Point then tells us that the circle passes through a constant point lying on the isogonal conjugate of AW in $\angle CAB$ which implies that since I lies on the circle when D is the projection that I always lies on the circle. $\square$

Thus, since TD passes through W all we have to show to prove that (GD) is tangent to (ABC) is that T, G, O are collinear, where O is the circumcenter of $\triangle ABC$.

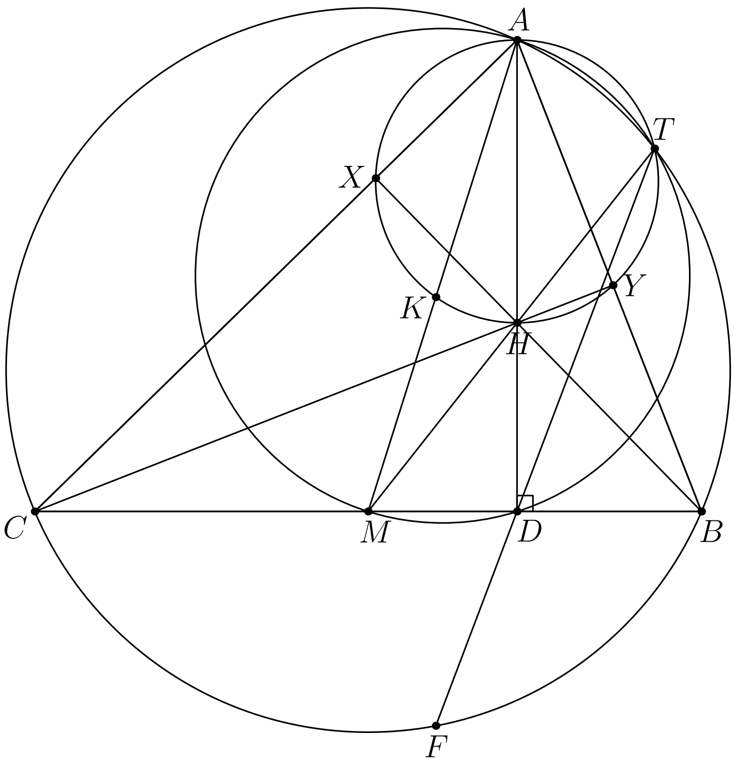
If D moves with degree 1 on BC then T moves with degree 2, similarly points F and E move with degree 1. Thus, the line EF moves with degree 2 and the line l (perpendicular from D to BC) moves with degree 1. Thus, G moves with degree 3. Thus, to show that T, G, O are collinear it sufficies to consider $2 + 3 + 1 = 6$ cases.

Consider D being equal to B, C, the midpoint of BC, the projection of I onto BC (then one needs to prove that the reflection of D over G lies on (APQ) which is trivial by angle chase), $AI \cap BC$ and finally ∞_{BC}.

Consequently since T, G, O are collinear and T, D, W are collinear it must be that (GD) is tangent to (ABC). ■

Problem 7.6 (APMO 2012 P4)

Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side BC, by M the midpoint of BC, and by H the orthocenter of ABC. Let E be the point of intersection of the circumcircle Γ of the triangle ABC and the half line MH, and F be the point of intersection (other than E) of the line ED and the circle Γ. Prove that $\frac{BF}{CF} = \frac{AB}{AC}$ must hold.

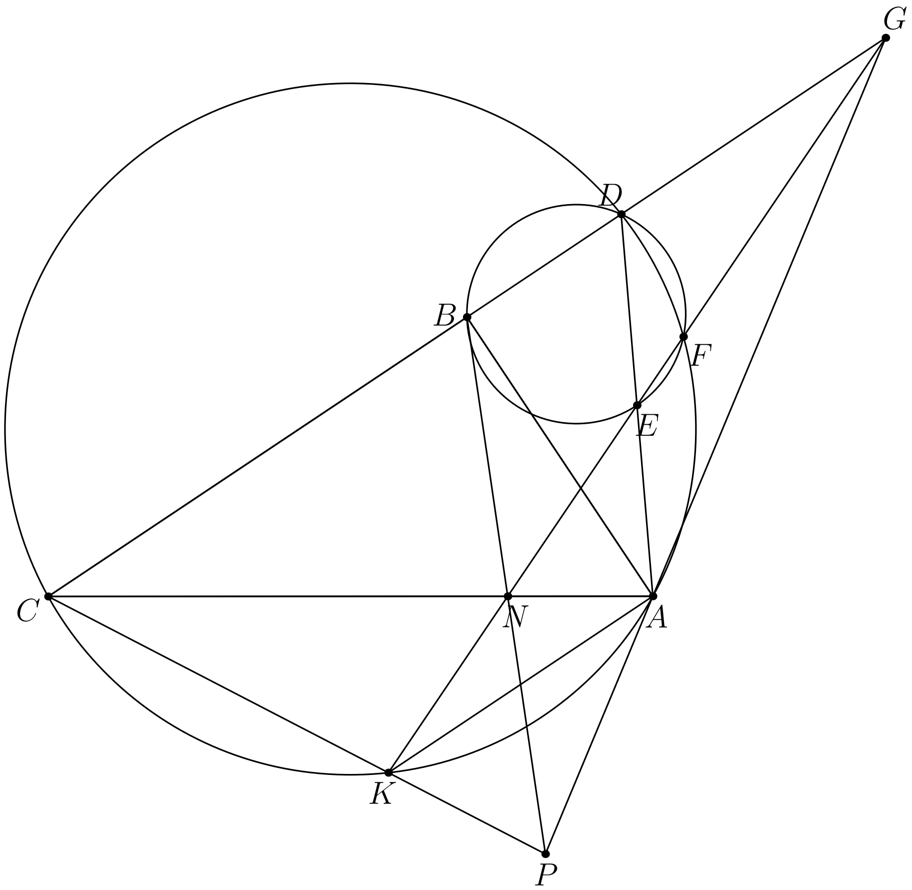


Proof. Notice, it is well know that T is the A-Queue point of $\triangle ABC$. Thus, if X and Y are the feet of the altitudes from B and C it must be that $AXHYT$ is cyclic. Notice, by **ELMO 2013 Shortlist G3** we obtain that since $AYDC$ and $AXDB$ are cyclic, consequently we know that (AXY) passes through a fixed point on the median, by Generalized Anchor Point we know that DT passes through the point F on (ABC) such that AF is the symmedian, thus $ABFC$ is harmonic and the problem statement follows. ■

Remark 7.3 The problem can also be solved by noticing that by Generalized Anchor Point since $ATMD$ is cyclic (since $\angle ATM = \angle ADM = 90$) it must be that TD passes through a point on (ABC) isogonal to AM w.r.t. $\angle CAB$.

Problem 7.7 (APMO 2022 P2)

Let ABC be a right triangle with $\angle B = 90^\circ$. Point D lies on the line CB such that B is between D and C. Let E be the midpoint of AD and let F be the seconf intersection point of the circumcircle of $\triangle ACD$ and the circumcircle of $\triangle BDE$. Prove that as D varies, the line EF passes through a fixed point.



Proof. Let K be such that $CDAK$ is a trapezoid.

Let G be the reflection of C over B. Then let $P = \text{Anchor}_K^{\triangle ABC}(G)$ then is known that the Anchor Point conjugates of points on CD will lie on CK, thus $P \in CK$. However, it is also known that the Anchor Point transformation preserves cross-ratios which in the case of lines, implies the preservation of ratios thus since,

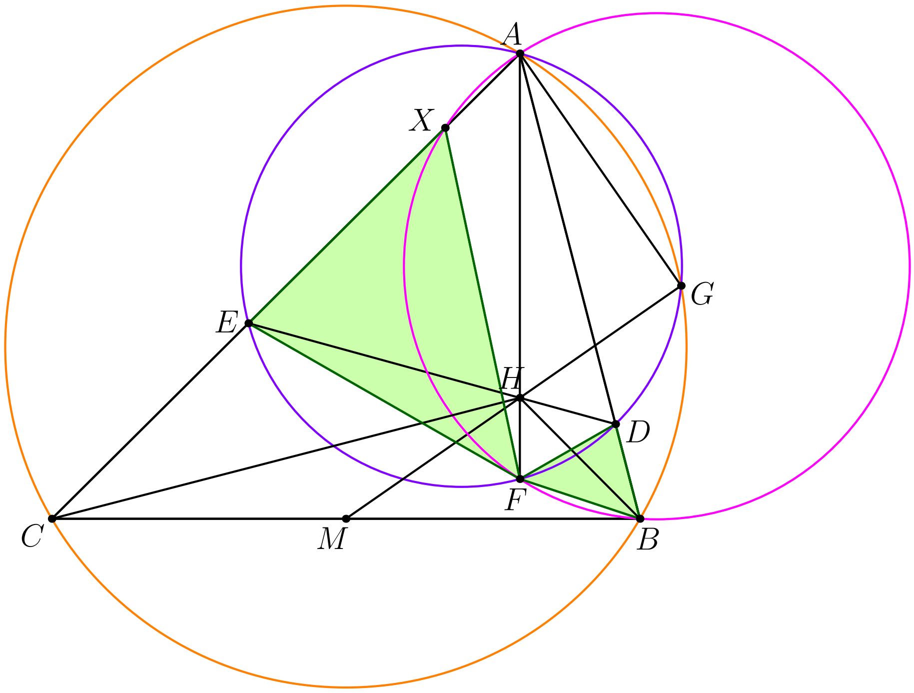
$$\begin{cases} \text{Anchor}_K^{\triangle ABC}(D) = K \\ \text{Anchor}_K^{\triangle ABC}(C) = C \end{cases}$$

thus we know that from the preservation of ratios that $DK \parallel GP$, however $AG \parallel DK$, thus G, A and P are collinear and K, E, G are collinear.

Then let $N = PB \cap AC$, trivially $N \in KG$, however now due to the definition of the Anchor Point conjugate $K \in NF$ which implies that $G \in EF$. ■

Problem 7.8 (IMO Shortlist 2005 G5)

Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.



Proof. Let $X = (AFB) \cap AC$ and $F = AH \cap (AED)$. Since AF is the isogonal line of AA' (where A' is the antipode), then we know that the Evil Anchor Point conjugates of F with respect to A' are some infinity points.

Thus, by the *Evil Anchor Point characterization* theorem we must only show that $BX \parallel EM$, which is equivalent to E is the midpoint of CX. Notice,

Lemma 7.4 $\triangle DFB \sim \triangle EFX$

Indeed since $\angle EXF = \angle ABF$ and $\angle XEF = \angle FDB$. □

Similarly,

Lemma 7.5 $\triangle CEH \sim \triangle BDH$

Indeed, since $\angle CEH = \angle HDB$ and $\angle ACH = \angle ABH$. □

Consequently we know that,

$$\begin{cases} \frac{BD}{DF} = \frac{EX}{EF} \\ \frac{CE}{EH} = \frac{DB}{DH} \\ \frac{EH}{HD} = \frac{EF}{FD} \end{cases}$$

the desired result now trivially follows,

$$EX = \frac{BD}{DF} \cdot EF = BD \cdot \frac{EH}{HD} = EH \cdot \frac{CE}{EH} = CE$$

thus $CE = EX$. ■