MATH 323
Inverse Theory Assignment 3

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1 Question 5.3

Based off of the provided example Matlab code, we edited our original code in the previous assignment to the following

```
rng('default')
rng(1)
N=50      %Number of z terms
no = 4    %Number of polynomial terms
z = 0:1/(N-1):1
z = z.'

mtrue=[1 .5 .25 .125].'
[]...]
H=[1,-2,0,0;1,0,-4,0;1,0,0,-8]
h=[0,0,0].'

A=[G.'*G, H.';H, zeros(length(h))]
b=[G.'*d;h]

x=A\b

mest_SP=x(1:4)

subplot(2,2,3)
plot(z,d,'ro')
hold on
plot(z,G*mtrue,'r--')
plot(z,G*mest,'g-')
plot(z,G*mest_SP,'b-')
hold off
legend('Observed data','True model','Naive model','Constrained model')
title('Model params have specific relative sizes')
```

We arrive at the following plot.

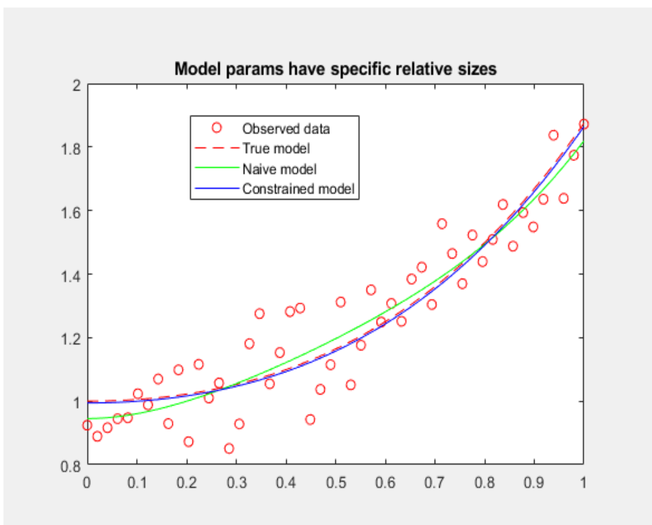


Figure 1: σ_m plot vs m

We observe that the constrained model leads to a model with a completely different parabola, pointing upwards.

Additionally, we can observe how much the model parameter values have changed by quantifying their percentage changes as follows

```
comparison=ones(1,4)
for i=1:4
    comparison(i) = -comparison(i)*(mest(i)-mest_SP(i))/mest_SP(i)*100
end
comparison

[Output]
comparison =

    4.9440  -246.0341   909.7383  -838.9130
```

In other words, compared to the unrestrained model, the model parameters have each decreased by 4.94%, -246% (increased), 909.74% and -838.91% (increased) respectively for model terms m_1,m_2,m_3 and m_4.

As for how the model parameter ratios in the unconstrained model

```
ty=1
record = ones(1,4)
for i=1:4
    record(i)= record(i)*mest(i)/ty
    ty=mest(i)
end
record(1)=0.5  %1st term is comparing with no term so it has been defaulted to 0.5

[Output]
record =

    0.5000    1.8202   -1.1700   -0.5798
```

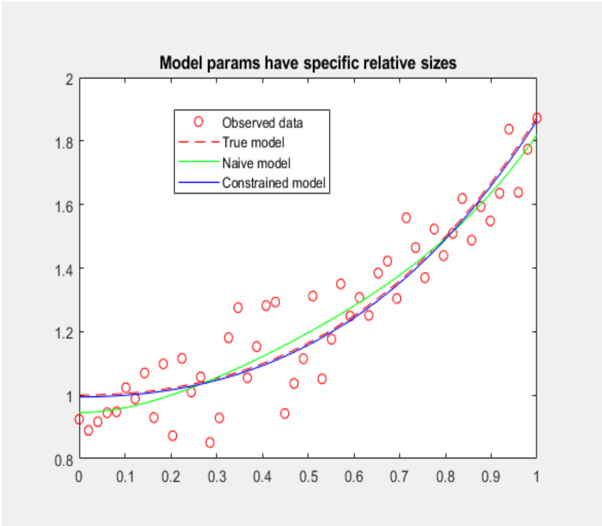
The unconstrained model has parameters that have similar ratios with respect to each other compared to the constrained model, despite them being not consistently double with respect to the succeeding model parameter.

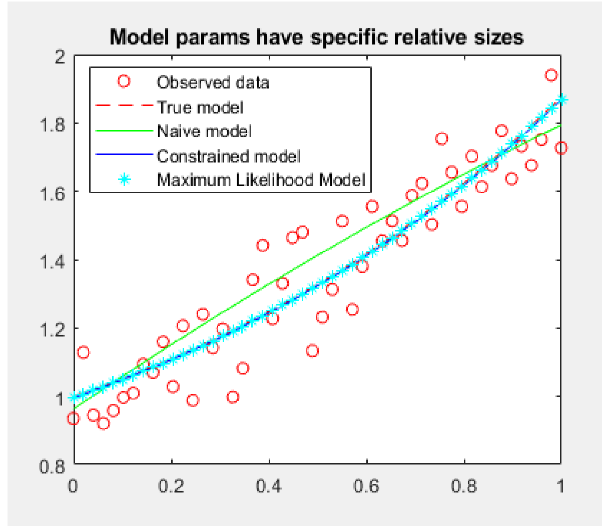
2 Question 5.4

The constrained model and unconstrained model created so far have used Lagrange multipliers. This time, we use maximum likelihood estimates to come up with the same constrained model.

```
N=50;z=[0:N-1].'/(N-1);
% Set up model
M=4
mtrue = [1, 1/2, 1/4, 1/8].'
% Set up data kernel
G = [ ones(N,1), z, z.*z, z.^3 ];
% Compute data with specified variance
sigmad=0.1
dobs=G*mtrue+random('Normal',0,sigmad,N,1);

L = 3
sigmam =0.05
r=1/sigmam
F = zeros(N+L,M)
f = zeros(N+L,1)
F(1:N,1:M)=1/sigmad*G
f(1:N)=1/sigmad*dobs
F(N+1,1)=1*r
F(N+1,2)=(-2)*r
F(N+2,1)=1*r
F(N+2,3)=(-4)*r
F(N+3,1)=1*r
F(N+3,4)=(-8)*r;


A=F.'*F
b=F.'*f
mest_ML=A\b

subplot(2,2,4)
plot(z,dobs,'ro')
hold on
plot(z,G*mtrue,'r--')
plot(z,G*mest,'g-')
plot(z,G*mest_SP,'b-')
plot(z,G*mest_ML,'c*')
hold off
legend('Observed data','True model','Naive model','Constrained model','Maximum Likelihood Model')
title('Model params have specific relative sizes')
```

The code above allows us to view the plot on the right while the one on the left is the previous plot for comparison.





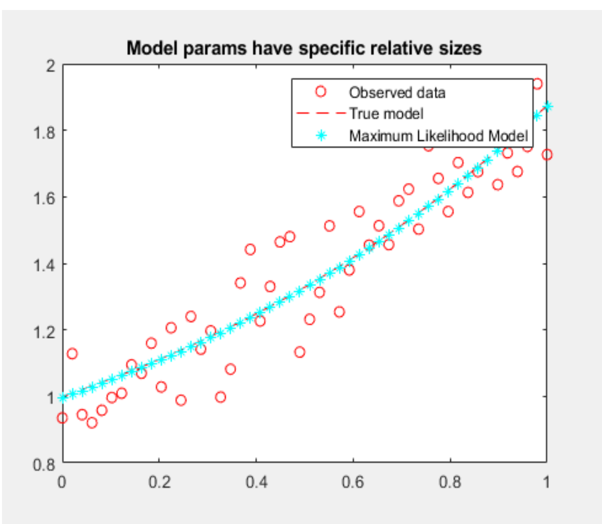
The model parameters estimated so far are as follows

Model parameters	m_{true}	Naive model m_{est}	Lagrange Multiplier m_{est}	Maximum Likelihood m_{est}
Estimates	1.0000	0.9627	0.9962	0.9958
	0.5000	0.9680	0.4981	0.5020
	0.2500	-0.1387	0.2490	0.2485
	0.1250	0.0026	0.1245	0.1241

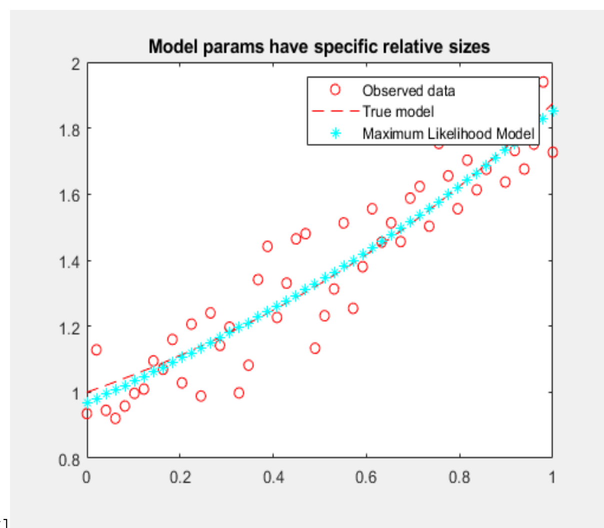
It is evident that without the constraints the fit is unable to perceive the true model parameters accurately except the first one. Interestingly, although minute, the maximum likelihood parameter estimates are slightly further from the true value compared to the Lagrange Multiplier model parameter estimates.

2.1 σ_m considerations

Considering the model parameter estimates and the plot changes with different values for σ_m, we get the following

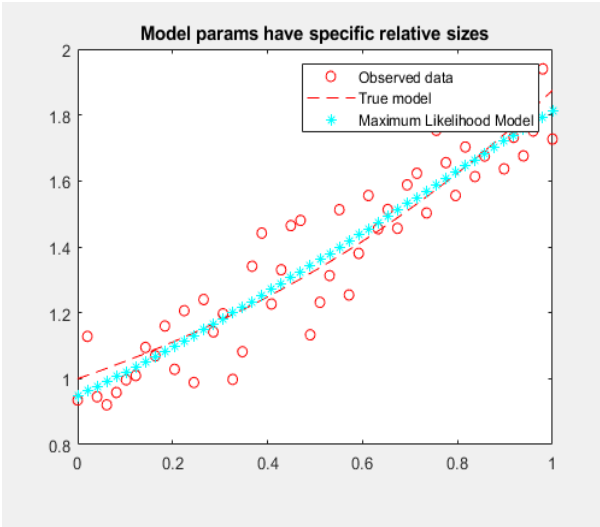


[0.5]

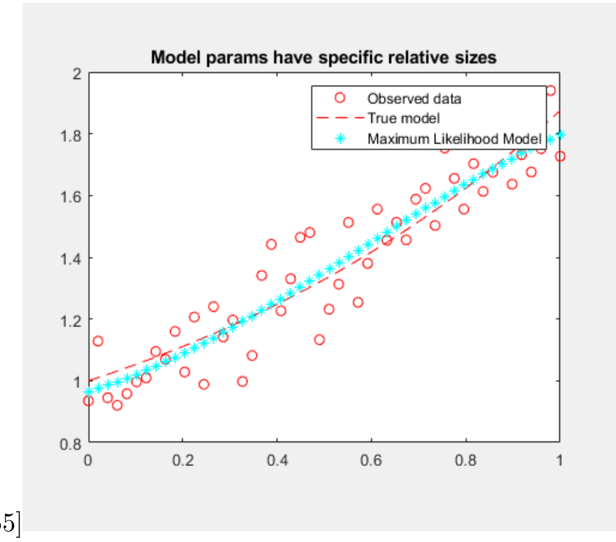


[0.55]

Figure 2: Comparison of fit of constrained model with $\sigma_m = .5$ and .55



[5.5]



[55.55]

Figure 3: Comparison of fit of constrained model with $\sigma_m = 5.5$ and 55.55

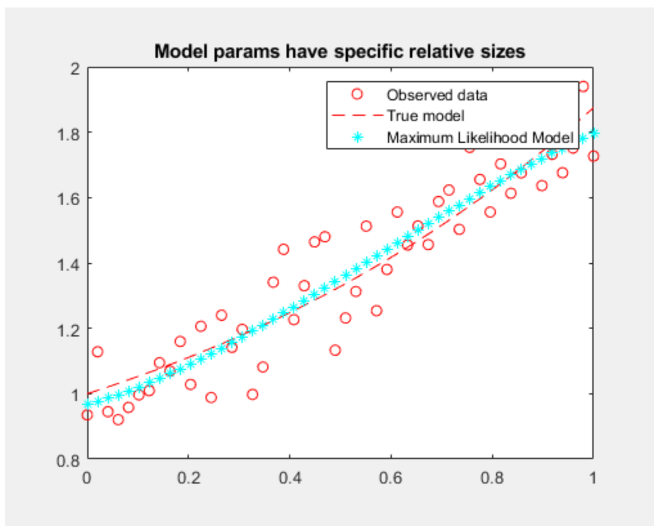


Figure 4: $\sigma_m = 555.55$

It appears to be the case that the constrained model deviates very little from the true model despite the higher model parameter variance.

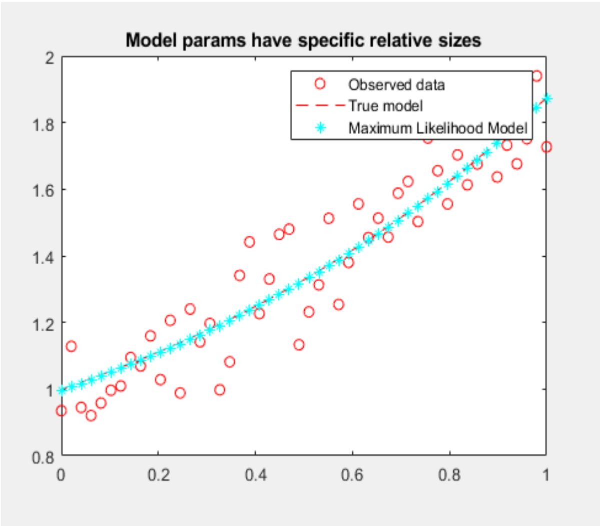
3 Question 5.5

We start by modifying Data variance $= \sigma_d^2 + \sigma_m^2$

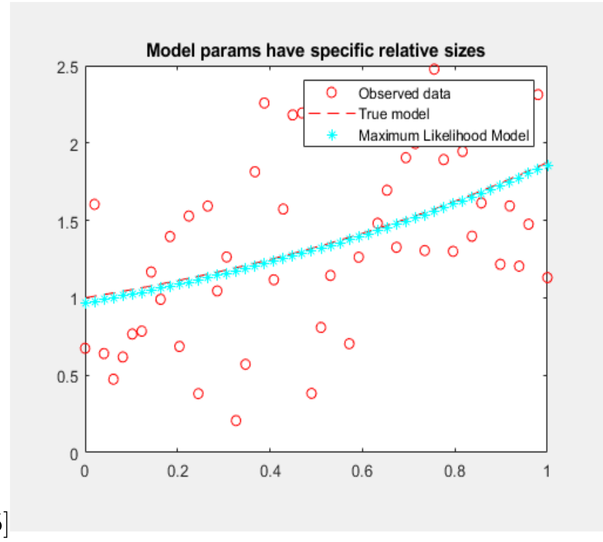
```
...
sigmam =.5  %This value will be changed later on
sigmad=(0.05^2+sigmam^2)^.5
...
```

Let us plot 3 cases whereby this dependency is evident, where we have 3 different values for σ_m to visualise the effect at different uncertainty levels for model parameters.

3.1 $\sigma_m = 0.5$



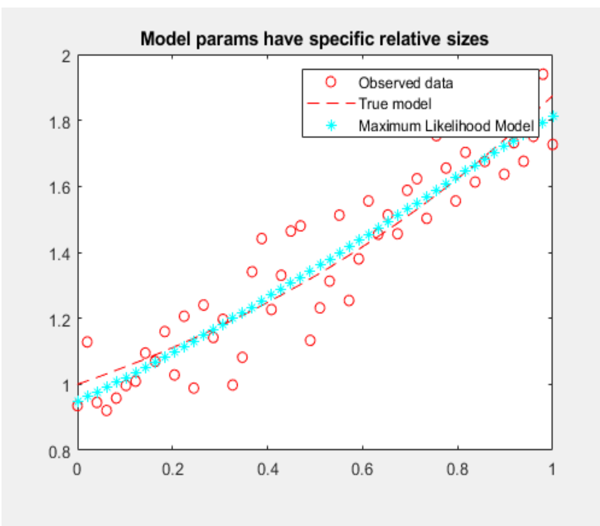
[5.5]



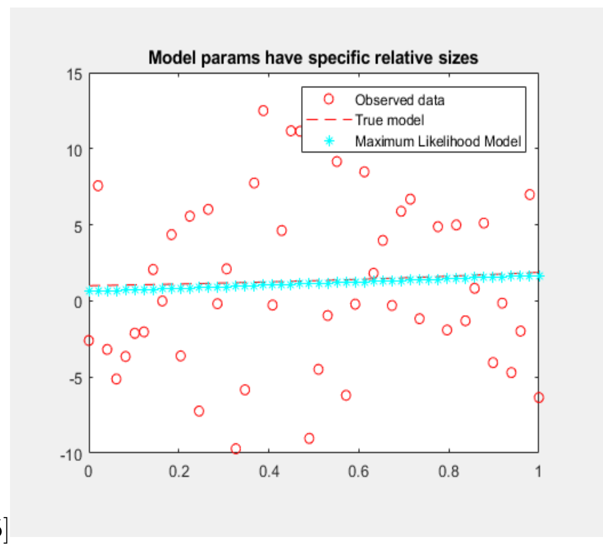
[55.55]

Figure 5: Comparison of fit of constrained model with $\sigma_m = 5.5$ and 55.55

3.2 $\sigma_m = 5.55$



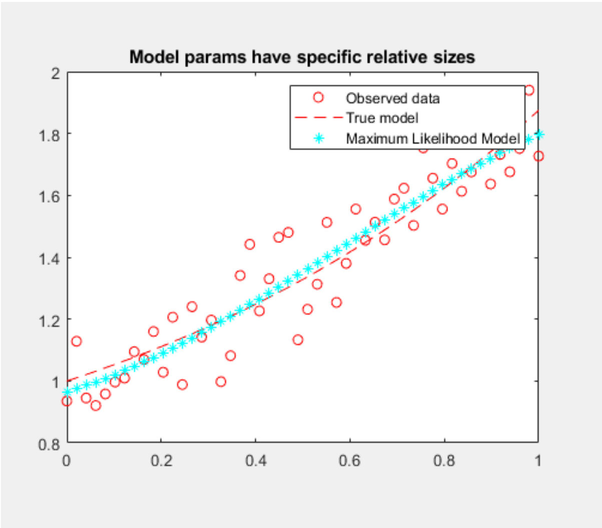
[5.5]



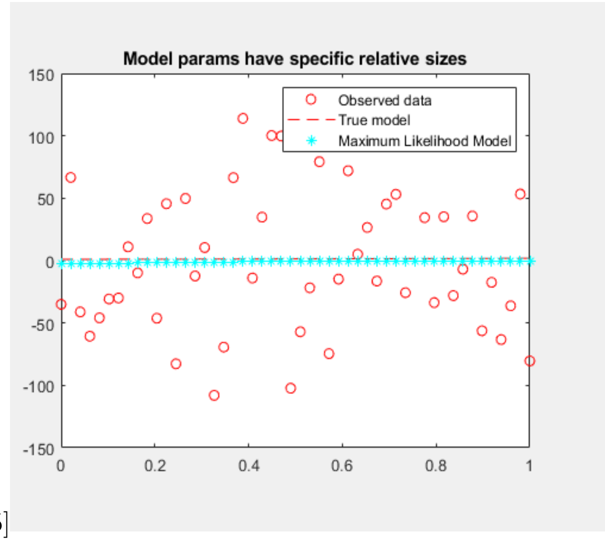
[55.55]

Figure 6: Comparison of fit of constrained model with $\sigma_m = 5.5$ and 55.55

3.3 $\sigma_m = 55.55$



[5.5]



[55.55]

Figure 7: Comparison of fit of constrained model with $\sigma_m = 5.5$ and 55.55

While the spread of the data points have changed alot, leading to the vertical scales of the plots to be increased exponentially as we increase σ_m, the constrained model and the true model appear to fit just as well to each other, although they now appear to explain the variation of the points less (obviously due to the added noise to the data).

4 Velocity structure of block model

4.1 Given measurements for t_1, t_2 and t_3

Given the schematic,

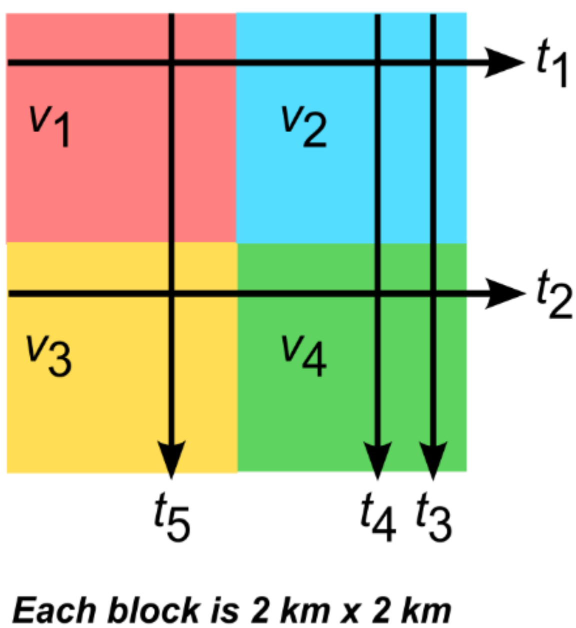


Figure 8: Velocity structure schematic with blocks of length 2x2km

We depict the problem with the following matrices

$$t_1 = \frac{2}{v_1} + \frac{2}{v_2}$$
$$t_2 = \frac{2}{v_3} + \frac{2}{v_4}$$
$$t_3 = \frac{2}{v_2} + \frac{2}{v_4}$$

Given the 3 time measurements to be 1.20, 0.87 and 0.95 seconds respectively,

$$\begin{bmatrix} 1.20 \\ 0.87 \\ 0.95 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} \mathrm{v}_1^{-1} \\ \mathrm{v}_2^{-1} \\ \mathrm{v}_3^{-1} \\ \mathrm{v}_4^{-1} \end{bmatrix} \tag{1}$$

In other words

$$G = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \end{bmatrix} \tag{2}$$

Using MATLAB to decompose this G matrix, using the singular value decomposition

```
G=[2 2 0 0;0 0 2 2;0 2 0 2]
[U,L,V]=svd(G)
r=rank(L)
Up=U(:,1:r)
Lp=L(1:r,1:r)
Vp=V(:,1:r)
d=[1.2 .87 .95]

Ginv=Vp*inv(Lp)*Up.'
m=Ginv*d.'

[Output]
m =

    0.3212
    0.2787
    0.2387
    0.1962
```

4.1.1 *a posteriori* model covariance matrix

In this case, GG^T is singular, meaning that we cannot use the least squares method to obtain the model covariance matrix. Instead , we use the minimum length solution

$$cov[\underset{\sim}{\mathbf{m}}] = \sigma^2 G^T [GG^T]^{-2} G \tag{3}$$

```
covm=0.16*G.'*(G*G.')^-2*G

[Output]
covm =

    0.0350   -0.0050    0.0250   -0.0150
   -0.0050    0.0150   -0.0150    0.0050
    0.0250   -0.0150    0.0350   -0.0050
   -0.0150    0.0050   -0.0050    0.0150
```

The covariance matrix is not with non-zero non-diagonal entries. This suggests that the fitted model to explain the values are not independent with one another, as their covariance values (non diagonal values) are non-zero.

4.2 Given measurements for t_1, t_2, t_3 and t_4

t_4 and t_3 measure the same properties of the model, such that

$$G = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \end{bmatrix} \tag{4}$$

making G have a rank of 3, with one linearly dependent row/column present. This leads to G having a determinant of 0, leading to singular matrix G^TG or GG^T.

```
G2=[2 2 0 0;0 0 2 2;0 2 0 2;0 2 0 2]
d2=[1.2 .87 .95 .88]
...
covm2=0.16*(G2.'*G2)^-1

[Output]
covm2 =

   Inf   Inf   Inf   Inf
   Inf   Inf   Inf   Inf
   Inf   Inf   Inf   Inf
   Inf   Inf   Inf   Inf
```

This seems to imply that E=0, that there is no apparent error in the model parameters.

4.3 Independent *a priori* information

```
%a priori information about velocities
%Important to take note that we need to invert the velocity values

mbar=[3 4 4 5]
mbar=mbar.^-1
mbar=mbar.'

covmbar=eye(4)*0.001

R=Ginv2*G2
mest_c=Ginv2*d2.'+(eye(4)-R)*mbar    %Equation 5.38

covm_c=Ginv2*covd_b*Ginv2.'+(eye(4)-R)*covmbar*(eye(4)-R).'  %Equation 5.41

[Output]
mest_c =

    0.3383
    0.2617
    0.2392
    0.1958

covm_c =

    0.0302   -0.0002    0.0197   -0.0097
   -0.0002    0.0103   -0.0098   -0.0003
    0.0197   -0.0098    0.0303   -0.0002
   -0.0097   -0.0003   -0.0002    0.0103
```

The *a posteriori* model covariance matrix appears to show a slight decrease in the variance of the model parameters compared to the covariance matrix for part 4.1.1 whereby only 3 measurements had been taken.

4.4 Given measurements for t_1, t_2, t_3, t_4 and t_5

```
%5 observations test

covd_c=eye(5)*0.16
G3=[2 2 0 0;0 0 2 2;0 2 0 2;0 2 0 2; 2 0 2 0]
d3=[1.2 .87 .95 .88 1.15]
[U,L,V]=svd(G3)
r=rank(L)
Up=U(:,1:r)
Lp=L(1:r,1:r)
Vp=V(:,1:r)
```

```
Ginv3=Vp*inv(Lp)*Up.'
m3=Ginv3*d3.'
covm3=0.16*G3.'*(G3*G3.')^-2*G3
covm3=0.16*(G3.'*G3)^-1

[Output]
m3 =

    0.3291
    0.2702
    0.2466
    0.1877


covm3 =

   Inf   Inf   Inf   Inf
   Inf   Inf   Inf   Inf
   Inf   Inf   Inf   Inf
   Inf   Inf   Inf   Inf
```

The covariance matrix is returning infinite values much like in the previous case in section 4.2 when we were working with 4 time measurements.

4.5 Independent *a priori* information for 5 measurements

```
%a priori information about velocities + 5 data points


R=Ginv3*G3
mest_e=Ginv3*d3.'+(eye(4)-R)*mbar    %Equation 5.38


covm_e=Ginv3*covd_c*Ginv3.'+(eye(4)-R)*covmbar*(eye(4)-R).'  %Equation 5.41

[Output]
mest_e =

    0.3374
    0.2618
    0.2383
    0.1960

covm_e =

    0.0124    0.0033    0.0019   -0.0062
    0.0033    0.0095   -0.0062   -0.0010
    0.0019   -0.0062    0.0124    0.0033
   -0.0062   -0.0010    0.0033    0.0095
```

The diagonal elements of the this final covariance matrix appear to have significantly decreased compared to the previous one in section 4.3 where we using independent *a priori* information for 4 time measurements as opposed to 5.