Big Oh

2-7. *[3]* True or False?

(a) Is $2^{n+1} = O(2^n)$?

(b) Is $2^{2n} = O(2^n)$?

2-8. *[3]* For each of the following pairs of functions, either $f(n)$ is in $O(g(n))$, $f(n)$ is in $\Omega(g(n))$, or $f(n) = \Theta(g(n))$. Determine which relationship is correct and briefly explain why.

(a) $f(n) = \log n^2$; $g(n) = \log n + 5$

(b) $f(n) = \sqrt{n}$; $g(n) = \log n^2$

(c) $f(n) = \log^2 n$; $g(n) = \log n$

(d) $f(n) = n$; $g(n) = \log^2 n$

(e) $f(n) = n \log n + n$; $g(n) = \log n$

(f) $f(n) = 10$; $g(n) = \log 10$

(g) $f(n) = 2^n$; $g(n) = 10n^2$

(h) $f(n) = 2^n$; $g(n) = 3^n$

2-9. *[3]* For each of the following pairs of functions $f(n)$ and $g(n)$, determine whether $f(n) = O(g(n))$, $g(n) = O(f(n))$, or both.

(a) $f(n) = (n^2 - n)/2$, $g(n) = 6n$

(b) $f(n) = n + 2\sqrt{n}$, $g(n) = n^2$

(c) $f(n) = n \log n$, $g(n) = n\sqrt{n}/2$

(d) $f(n) = n + \log n$, $g(n) = \sqrt{n}$

(e) $f(n) = 2(\log n)^2$, $g(n) = \log n + 1$

(f) $f(n) = 4n \log n + n$, $g(n) = (n^2 - n)/2$

2-10. *[3]* Prove that $n^3 - 3n^2 - n + 1 = \Theta(n^3)$.

2-11. *[3]* Prove that $n^2 = O(2^n)$.

2-12. *[3]* For each of the following pairs of functions $f(n)$ and $g(n)$, give an appropriate positive constant c such that $f(n) \leq c \cdot g(n)$ for all $n > 1$.

(a) $f(n) = n^2 + n + 1$, $g(n) = 2n^3$

(b) $f(n) = n\sqrt{n} + n^2$, $g(n) = n^2$

(c) $f(n) = n^2 - n + 1$, $g(n) = n^2/2$

2-13. *[3]* Prove that if $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, then $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$.

2-14. *[3]* Prove that if $f_1(N) = \Omega(g_1(n))$ and $f_2(n) = \Omega(g_2(n))$, then $f_1(n) + f_2(n) = \Omega(g_1(n) + g_2(n))$.

2-15. *[3]* Prove that if $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, then $f_1(n) \cdot f_2(n) = O(g_1(n) \cdot g_2(n))$

2-16. *[5]* Prove for all $k \geq 1$ and all sets of constants $\{a_k, a_{k-1}, \ldots, a_1, a_0\} \in R$,

$$a_k n^k + a_{k-1} n^{k-1} + ... + a_1 n + a_0 = O(n^k)$$

2-17. *[5]* Show that for any real constants a and b, $b > 0$

$$(n + a)^b = \Theta(n^b)$$

2-18. *[5]* List the functions below from the lowest to the highest order. If any two or more are of the same order, indicate which.

n	2^n	$n \lg n$	$\ln n$
$n - n^3 + 7n^5$	$\lg n$	$\sqrt{n}$	e^n
$n^2 + \lg n$	n^2	2^{n-1}	$\lg \lg n$
n^3	$(\lg n)^2$	$n!$	$n^{1+\varepsilon}$ where $0 < \varepsilon < 1$

2-19. *[5]* List the functions below from the lowest to the highest order. If any two or more are of the same order, indicate which.

$\sqrt{n}$	n	2^n
$n \log n$	$n - n^3 + 7n^5$	$n^2 + \log n$
n^2	n^3	$\log n$
$n^{\frac{1}{3}} + \log n$	$(\log n)^2$	$n!$
$\ln n$	$\frac{n}{\log n}$	$\log \log n$
$(1/3)^n$	$(3/2)^n$	6

2-20. *[5]* Find two functions $f(n)$ and $g(n)$ that satisfy the following relationship. If no such f and g exist, write "None."

(a) $f(n) = o(g(n))$ and $f(n) \neq \Theta(g(n))$

(b) $f(n) = \Theta(g(n))$ and $f(n) = o(g(n))$

(c) $f(n) = \Theta(g(n))$ and $f(n) \neq O(g(n))$

(d) $f(n) = \Omega(g(n))$ and $f(n) \neq O(g(n))$

2-21. *[5]* True or False?

(a) $2n^2 + 1 = O(n^2)$

(b) $\sqrt{n} = O(\log n)$

(c) $\log n = O(\sqrt{n})$

(d) $n^2(1 + \sqrt{n}) = O(n^2 \log n)$

(e) $3n^2 + \sqrt{n} = O(n^2)$

(f) $\sqrt{n} \log n = O(n)$

(g) $\log n = O(n^{-1/2})$

2-22. *[5]* For each of the following pairs of functions $f(n)$ and $g(n)$, state whether $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$, $f(n) = \Theta(g(n))$, or none of the above.

(a) $f(n) = n^2 + 3n + 4$, $g(n) = 6n + 7$

(b) $f(n) = n\sqrt{n}$, $g(n) = n^2 - n$

(c) $f(n) = 2^n - n^2$, $g(n) = n^4 + n^2$

2-23. *[3]* For each of these questions, briefly explain your answer.

(a) If I prove that an algorithm takes $O(n^2)$ worst-case time, is it possible that it takes $O(n)$ on some inputs?

(b) If I prove that an algorithm takes $O(n^2)$ worst-case time, is it possible that it takes $O(n)$ on all inputs?

(c) If I prove that an algorithm takes $\Theta(n^2)$ worst-case time, is it possible that it takes $O(n)$ on some inputs?

(d) If I prove that an algorithm takes $\Theta(n^2)$ worst-case time, is it possible that it takes $O(n)$ on all inputs?

(e) Is the function $f(n) = \Theta(n^2)$, where $f(n) = 100n^2$ for even n and $f(n) = 20n^2 - n\log_2 n$ for odd n?

2-24. *[3]* For each of the following, answer *yes*, *no*, or *can't tell*. Explain your reasoning.

(a) Is $3^n = O(2^n)$?

(b) Is $\log 3^n = O(\log 2^n)$?

(c) Is $3^n = \Omega(2^n)$?

(d) Is $\log 3^n = \Omega(\log 2^n)$?

2-25. *[5]* For each of the following expressions $f(n)$ find a simple $g(n)$ such that $f(n) = \Theta(g(n))$.

(a) $f(n) = \sum_{i=1}^{n} \frac{1}{i}$.

(b) $f(n) = \sum_{i=1}^{n} \lceil \frac{1}{i} \rceil$.

(c) $f(n) = \sum_{i=1}^{n} \log i$.

(d) $f(n) = \log(n!)$.

2-26. *[5]* Place the following functions into increasing asymptotic order.
$f_1(n) = n^2\log_2 n$, $f_2(n) = n(\log_2 n)^2$, $f_3(n) = \sum_{i=0}^{n} 2^i$, $f_4(n) = \log_2(\sum_{i=0}^{n} 2^i)$.

2-27. *[5]* Place the following functions into increasing asymptotic order. If two or more of the functions are of the same asymptotic order then indicate this.
$f_1(n) = \sum_{i=1}^{n} \sqrt{i}$, $f_2(n) = (\sqrt{n})\log n$, $f_3(n) = n\sqrt{\log n}$, $f_4(n) = 12n^{\frac{3}{2}} + 4n$,

2-28. *[5]* For each of the following expressions $f(n)$ find a simple $g(n)$ such that $f(n) = \Theta(g(n))$. (You should be able to prove your result by exhibiting the relevant parameters, but this is not required for the homework.)

(a) $f(n) = \sum_{i=1}^{n} 3i^4 + 2i^3 - 19i + 20$.

(b) $f(n) = \sum_{i=1}^{n} 3(4^i) + 2(3^i) - i^{19} + 20$.

(c) $f(n) = \sum_{i=1}^{n} 5^i + 3^{2i}$.

2-29. *[5]* Which of the following are true?

(a) $\sum_{i=1}^{n} 3^i = \Theta(3^{n-1})$.

(b) $\sum_{i=1}^{n} 3^i = \Theta(3^n)$.

(c) $\sum_{i=1}^{n} 3^i = \Theta(3^{n+1})$.

2-30. *[5]* For each of the following functions f find a simple function g such that $f(n) = \Theta(g(n))$.

(a) $f_1(n) = (1000)2^n + 4^n$.

(b) $f_2(n) = n + n\log n + \sqrt{n}$.

(c) $f_3(n) = \log(n^{20}) + (\log n)^{10}$.

(d) $f_4(n) = (0.99)^n + n^{100}$.

2-31. *[5]* For each pair of expressions (A, B) below, indicate whether A is O, o, Ω, ω, or Θ of B. Note that zero, one or more of these relations may hold for a given pair; list all correct ones.

	A	B
(a)	n^{100}	2^n
(b)	$(\lg n)^{12}$	$\sqrt{n}$
(c)	$\sqrt{n}$	$n^{\cos(\pi n/8)}$
(d)	10^n	100^n
(e)	$n^{\lg n}$	$(\lg n)^n$
(f)	$\lg(n!)$	$n \lg n$

These exercises are extracted from:
Steven S. Skiena. 2008. The Algorithm Design Manual (2nd. ed.). Springer Publishing Company, Incorporated.
These exercises are also posted in the following link with an endorsement from the author:
https://www.algorist.com/algowiki/index.php/Chapter_2