Iterative Substitution

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General Idea. Repeatedly substitute the value of the recurrent part of the recurrence equation until a pattern (or a summation) is found.

Example 1. $T(n) = c + T(n-1)$ if $n \geq 1$, $T(0) = 0$

$$T(n) = c + (c + T(n-2)) = 2c + T(n-2)$$

$$T(n) = 2c + (c + T(n-3)) = 3c + T(n-3)$$

$$T(n) = 3c + (c + T(n-4)) = 4c + T(n-4)$$

It seems that for some value $k < n$:

$$T(n) = k \times c + T(n-k)$$

Since we stop when $k = n$:

$$T(n) = n \times c + T(0) = cn = \Theta(n)$$

Example 2. $T(n) = n + T(n-1)$ if $n \geq 1$, $T(0) = 0$

$$T(n) = n + (n-1) + T(n-2)$$

$$T(n) = n + (n-1) + (n-2) + T(n-3)$$

$$T(n) = n + (n-1) + (n-2) + (n-3) + T(n-4)$$

It seems that for some value $k < n$:

$$T(n) = n + (n-1) + (n-2) + (n-3) + \ldots + (n-(k-1)) + T(n-k)$$

Since we stop when $k = n$:

$$T(n) = n + (n-1) + (n-2) + (n-3) + \ldots + (n-(n-1)) + T(0)$$

$$T(n) = \sum_{i=0}^{n} i = \Theta(n^2)$$

Example 3. $T(n) = \log n + T(n-1)$ if $n \geq 1$, $T(0) = 0$

$$T(n) = \log(n) + \log(n-1) + T(n-2)$$

$$T(n) = \log(n) + \log(n-1) + \log(n-2) + T(n-3)$$

$$T(n) = \log(n) + \log(n-1) + \log(n-2) + \log(n-3) + T(n-4)$$

It seems that for some value $k < n$:

$$T(n) = \log(n) + \log(n-1) + \log(n-2) + \log(n-3) + \ldots + \log(n-(k-1)) + T(n-k)$$

Since we stop when $k = n$:

$$T(n) = \log(n) + \log(n-1) + \log(n-2) + \log(n-3) + \ldots + \log(n-(n-1)) + T(0)$$

$$T(n) = \sum_{i=1}^{n} \log(i) = \log(n!) = \Theta(n \log n)$$

Example 4. $T(n) = c + 2T(\frac{n}{2})$ if $n > 1$, $T(1) = c$

$$T(n) = c + 2(c + 2T(\tfrac{n}{4})) = c + 2c + 4T(\tfrac{n}{4})$$

$$T(n) = c + 2c + 4(c + 2T(\tfrac{n}{8})) = c + 2c + 4c + 8T(\tfrac{n}{8})$$

$$T(n) = c + 2c + 4c + 8(c + 2T(\tfrac{n}{16})) = c + 2c + 4c + 8c + 16T(\tfrac{n}{16})$$

It seems that for some value $k < \log_2 n$:

$$T(n) = 2^0 \bullet c + 2^1 \bullet c + 2^2 \bullet c + \ldots + 2^{k-1} \bullet c + 2^k \bullet T(\tfrac{n}{2^k})$$

Since we stop when $k = \log_2 n$:

$$T(n) = 2^0 \bullet c + 2^1 \bullet c + 2^2 \bullet c + \ldots + 2^{\log_2 n - 1} \bullet c + 2^{\log_2 n} \bullet T(1)$$

$$T(n) = c \times \sum_{i=0}^{\log_2 n} 2^i = c \times (2^{\log_2 n + 1} - 1) = 2cn - c = \Theta(n)$$

Example 5. $T(n) = c + T(\frac{n}{2})$ if $n > 1$, $T(1) = c$

$$T(n) = c + c + T(\tfrac{n}{4}) = 2c + T(\tfrac{n}{4})$$

$$T(n) = 2c + c + T(\tfrac{n}{8}) = 3c + T(\tfrac{n}{8})$$

$$T(n) = 3c + c + T(\tfrac{n}{16}) = 4c + T(\tfrac{n}{16})$$

It seems that for some value $k < \log_2 n$:

$$T(n) = k \bullet c + T(\tfrac{n}{2^k})$$

Since we stop when $k = \log_2 n$:

$$T(n) = c \log_2 n + T(1) = \Theta(\log n)$$

Example 6. $T(n) = n + \frac{1}{2}T(\frac{n}{2})$ if $n > 1$, $T(1) = 1$

$$T(n) = n + \tfrac{1}{2}(\tfrac{n}{2} + \tfrac{1}{2}T(\tfrac{n}{4})) = n + \tfrac{n}{4} + \tfrac{1}{4}T(\tfrac{n}{4})$$

$$T(n) = n + \tfrac{n}{4} + \tfrac{1}{4}(\tfrac{n}{4} + \tfrac{1}{2}T(\tfrac{n}{8})) = n + \tfrac{n}{4} + \tfrac{n}{16} + \tfrac{1}{8}T(\tfrac{n}{8})$$

It seems that for some value $k < \log_2 n$:

$$T(n) = \frac{n}{4^0} + \frac{n}{4^1} + \frac{n}{4^2} + \ldots + \frac{n}{4^{k-1}} + \frac{1}{2^k}T(\frac{n}{2^k})$$

Since we stop when $k = \log_2 n$:

$$T(n) = \frac{n}{4^0} + \frac{n}{4^1} + \frac{n}{4^2} + \ldots + \frac{n}{4^{\log_2 n - 1}} + \frac{1}{2^{\log_2 n}}T(1) =$$

$$T(n) = n \bullet (\sum_{i=0}^{\log_2 n - 1} (\frac{1}{4})^i) + \frac{1}{n}$$

$$T(n) \leq n \bullet (\sum_{i=0}^{\infty} (\frac{1}{4})^i) + \frac{1}{n}$$

$$T(n) \leq n \bullet (\frac{1}{1 - \frac{1}{4}}) + \frac{1}{n} \leq \frac{4n}{3} + \frac{1}{n} = O(n)$$

Since the first term in the recurrence equation is n, $T(n)$ is also $\Omega(n)$, which implies that $T(n) = \Theta(n)$.