CS11313 - Spring 2023

Design & Analysis of Algorithms

Greedy Algorithms

Ibrahim Albluwi

Warmup: Change-Making Problem

Problem. Given a set of coin denominations (e.g. quarter, dime, nickel, etc.) and an amount of money, find the *minimum number of coins* that add up to this amount of money.

Example. What is the minimum number of coins needed for paying 99¢ using Jordanian currency?

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Possible Solution. Sort the denominations in decreasing order. Pick from the largest denomination as much as possible and then move to the next.

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Counter Example. Consider the denominations `{25¢, 20¢, 10¢, 1¢}`. What is the minimum number of coins needed to pay 40¢ ?

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Possible Solution. Sort the denominations in decreasing order. Pick from the largest denomination as much as possible and then move to the next.

Counter Example. Consider the denominations `{25¢, 20¢, 10¢, 1¢}`. What is the minimum number of coins needed to pay 40¢ ?

Greedy: `25¢ + 10¢ + 1¢ + 1¢ + 1¢ + 1¢ + 1¢ = 40¢` (7 coins).

Optimal: `20¢ + 20¢ = 40¢` (2 coins)

Greedy Algorithms

Greedy Algorithm.

- Makes the best choice given the available information at the moment (without looking ahead or backtracking).
- Used for solving *optimization problems*.

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Optimization problem. The problem of finding the *best* solution among all *feasible* solutions.

maximizes or minimizes an objective function

valid (*respects the constraints*)

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Example. Change Making.

Input: A set of coin denominations and an amount of money.
Objective Function: Choose the *minimum* number of coins.
Constraints: The *total coins value* must equal the given amount of money.

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We need either a proof to show that the greedy solution always works or only one counterexample to show that it doesn't!

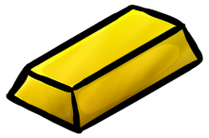
Fractional Knapsack

Input. N items, each of weight $w_i > 0$ and value $v_i > 0$. and a knapsack of weight capacity $W > 0$.

Goal. Find the *maximum* value the knapsack can hold.

Note. Taking a fraction of an item is allowed.

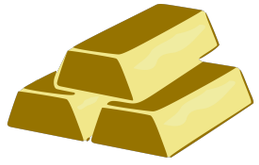
capacity = 13 kg



weight = 4 kg
value = $3000

weight = 12 kg
value = $8000

weight = 5 kg
value = $4000



weight = 6 kg
value = $6000

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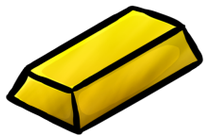

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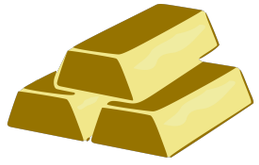

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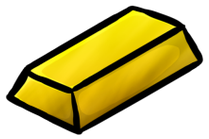
12kg of = 12/12 * \$8000 = \$8000

1kg of = 1/6 * \$6000 = \$1000

= \$9000

capacity = 13 kg



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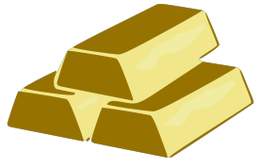

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Can we do better?

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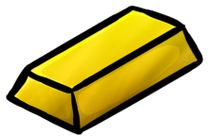

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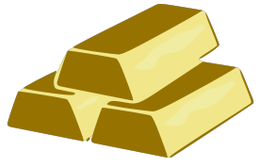

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Greedy Strategy # 2.
Take the item with the highest value per Kg first.

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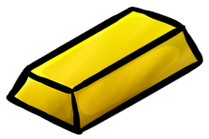

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6kg of = 6 * $1000 = $6000

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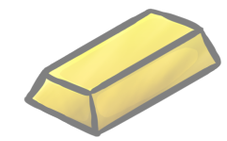

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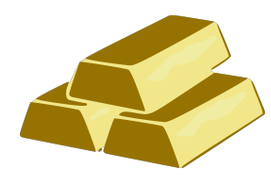

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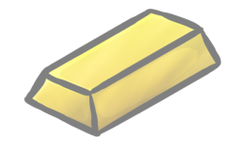

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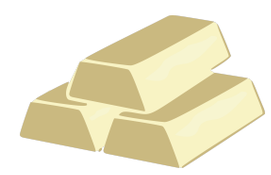

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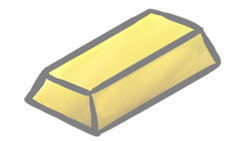

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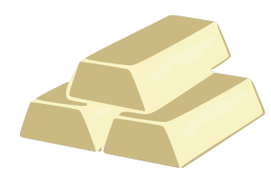

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1kg of = 1 * $750 = $750

= $11750

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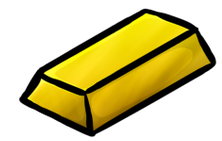

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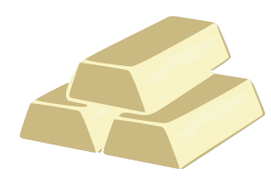

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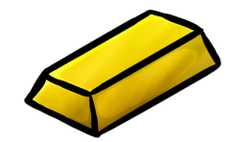

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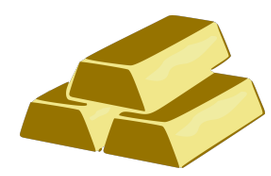

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Can we do better? **IMPOSSIBLE!**

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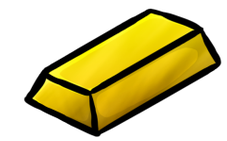

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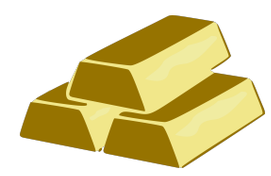

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Fractional Knapsack: Proof of Correctness

Consider a set of n items sorted by value per weight.

Item 1		Item 2		Item 3		Item 4		Item 5		· · · · · · ·		Item n
$\frac{v_1}{w_1}$	$>$	$\frac{v_2}{w_2}$	$>$	$\frac{v_3}{w_3}$	$>$	$\frac{v_4}{w_4}$	$>$	$\frac{v_5}{w_5}$	$>$		$>$	$\frac{v_n}{w_n}$

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Assume that there is an optimal solution for the fractional knapsack problem that *does not consume the item with the highest value per Kg.*

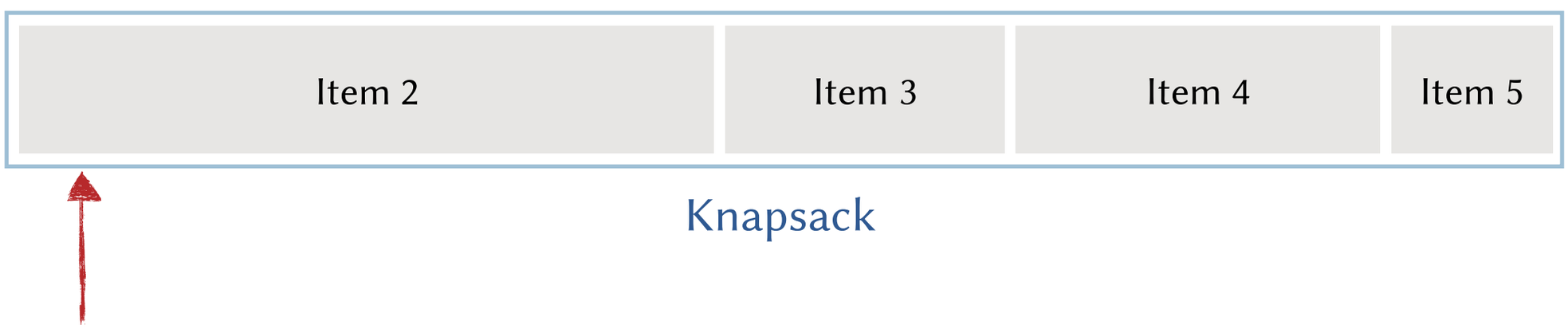
Knapsack

Fractional Knapsack: Proof of Correctness

Consider a set of n items sorted by value per weight.

Item 1		Item 2		Item 3		Item 4		Item 5		· · · · · · ·		Item n
$\frac{v_1}{w_1}$	$>$	$\frac{v_2}{w_2}$	$>$	$\frac{v_3}{w_3}$	$>$	$\frac{v_4}{w_4}$	$>$	$\frac{v_5}{w_5}$	$>$		$>$	$\frac{v_n}{w_n}$

Assume that there is an optimal solution for the fractional knapsack problem that *does not consume the item with the highest value per Kg.*



We can replace parts of this item with parts of *item 1* and get a better solution! Therefore, the solution is not optimal! (*contradiction*)

Note. This is not a proof. However, it captures the essence of the argument for why this greedy choice is optimal.

Fractional Knapsack: Proof of Correctness

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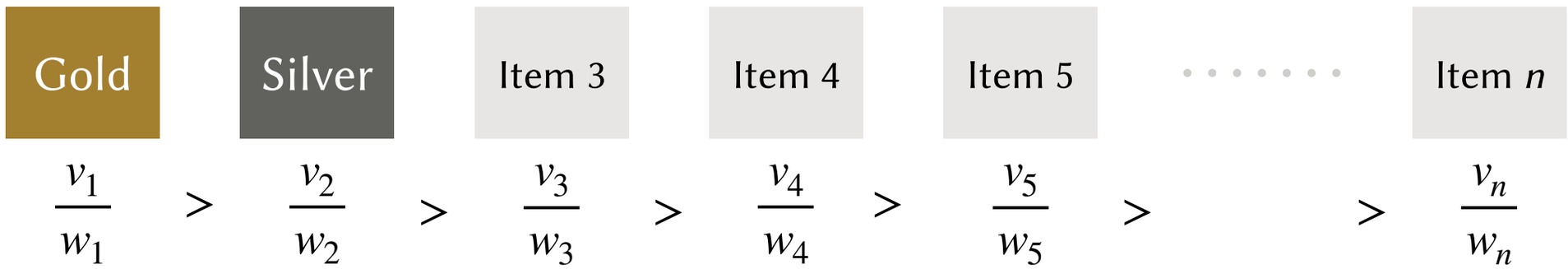
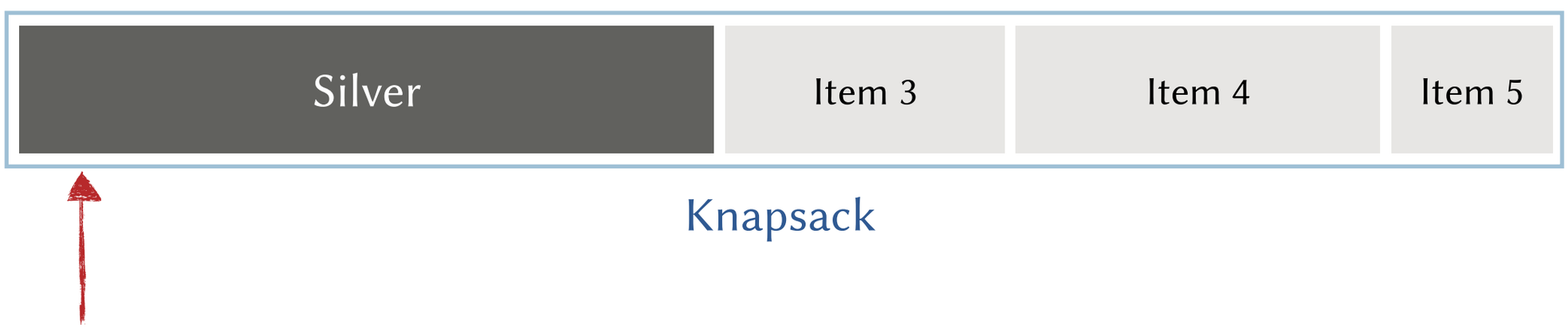


Assume that there is an optimal solution for the fractional knapsack problem that *does not consume the item with the highest value per Kg.*



We can replace parts of the silver with parts of *gold* and get a better solution! Therefore, the solution is not optimal! (*contradiction*)

Note. This is not a proof. However, it captures the essence of the argument for why this greedy choice is optimal.

Fractional Knapsack (Algorithm)

```
FRACTIONAL_KNAPSACK(w[], v[], n, W)

Sort w and v by v[i]/w[i] in decreasing order
```

Fractional Knapsack (Algorithm)

```
FRACTIONAL_KNAPSACK(w[], v[], n, W)

Sort w and v by v[i]/w[i] in decreasing order

load = 0, i = 1
while i <= n and load < W:
```

number of items

knapsack capacity

Fractional Knapsack (Algorithm)

```
FRACTIONAL_KNAPSACK(w[], v[], n, W)

Sort w and v by v[i]/w[i] in decreasing order

load = 0, i = 1
while i <= n and load < W:

    if w[i] <= W-load:

        all of the item fits inside the
        remaining capacity

    else

        only a fraction of the item fits
        inside the remaining capacity
```

Fractional Knapsack (Algorithm)

```
FRACTIONAL_KNAPSACK(w[], v[], n, W)

Sort w and v by v[i]/w[i] in decreasing order

load = 0, i = 1
while i <= n and load < W:
    if w[i] <= W-load:
        Take all of item i
        load += w[i]
        profit = profit + v[i]
    else
        only a fraction of the item fits
        inside the remaining capacity
```

Fractional Knapsack (Algorithm)

```
FRACTIONAL_KNAPSACK(w[], v[], n, W)

Sort w and v by v[i]/w[i] in decreasing order

load = 0, i = 1
while i <= n and load < W:

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        load += w[i]
        profit = profit + v[i]
    else
        Take (W - load) of item i
        profit = profit + (W - load) * v[i]/w[i]
        load = W

    i = i+1
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```

$\Theta(n \log n)$

$O(n)$

Running Time. $\Theta(n \log n)$

Lesson. Greedy algorithms are typically simple and efficient.

0 – 1 Knapsack

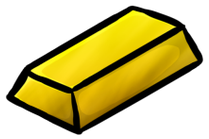
Input. N items, each of weight $w_i > 0$ and value $v_i > 0$. and a knapsack of weight capacity $W > 0$.

Goal. Find the *maximum* value the knapsack can hold.

Assumptions.

- Taking a fraction of an item is **not** allowed.
- W, w_i and v_i are integers.

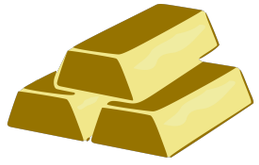
capacity = 13 kg



weight = 4 kg
value = $3000

weight = 12 kg
value = $8000

weight = 5 kg
value = $4000



weight = 6 kg
value = $6000

weight = 1 kg
value = $1000

0 – 1 Knapsack

Input. N items, each of weight $w_i > 0$ and value $v_i > 0$. and a knapsack of weight capacity $W > 0$.

Goal. Find the *maximum* value the knapsack can hold.

Assumptions.

- Taking a fraction of an item is **not** allowed.
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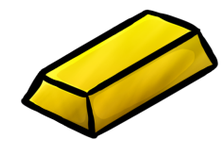
Examples.

- `$8000` (12 kg) + `$1000` (1 kg) = `$9000` (13 kg)

capacity = 13 kg



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0 – 1 Knapsack

Input. N items, each of weight $w_i > 0$ and value $v_i > 0$. and a knapsack of weight capacity $W > 0$.

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- `$8000` (12 kg) + `$1000` (1 kg) = `$9000` (13 kg)
- `$6000` (6 kg) + `$3000` (4 kg) + `$1000` (1 kg) = `$10000` (11 kg)

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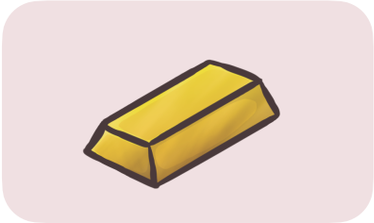

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weight = 5 kg
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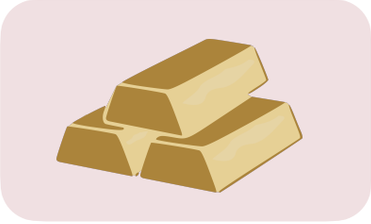

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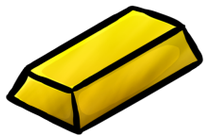

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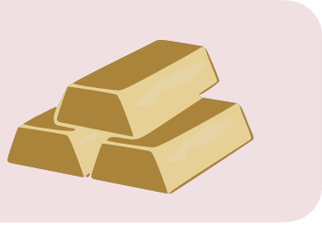

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0 – 1 **Knapsack**: Greedy Strategies

Greedy choice # 1. Take the *most valuable* item first.

Counter Example.
Greedy gives $12 but the optimal is $17.

0–1 Knapsack: Greedy Strategies

Greedy choice # 1. Take the *most valuable* item first.

Counter Example.
Greedy gives $12 but the optimal is $17.

Greedy choice # 2. Take the *lightest* item first.

0–1 Knapsack: Greedy Strategies

Greedy choice # 1. Take the *most valuable* item first.

Counter Example.
Greedy gives $12 but the optimal is $17.

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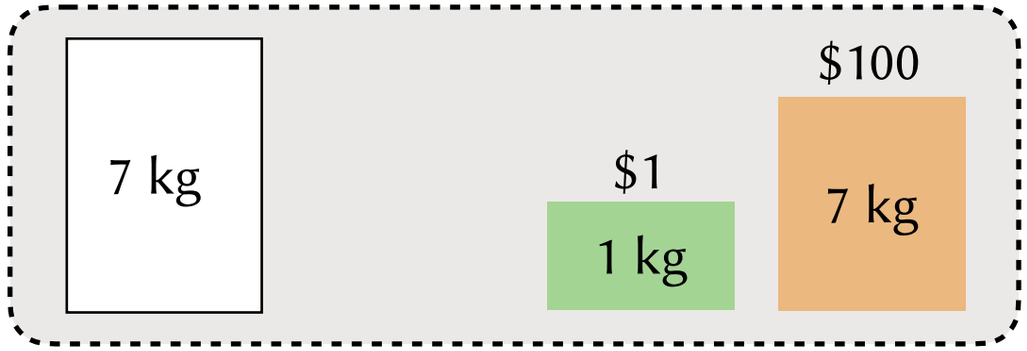

Counter Example.
Greedy gives $1 but the optimal is $100.

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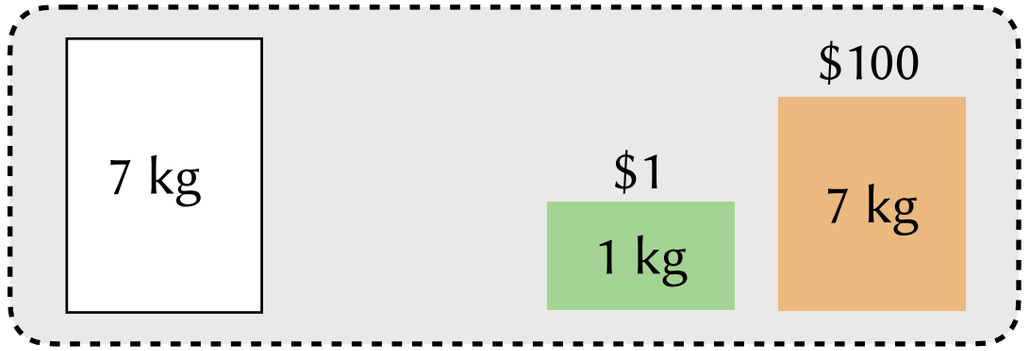

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Greedy choice # 3. Take the item with the *highest value per Kg*.

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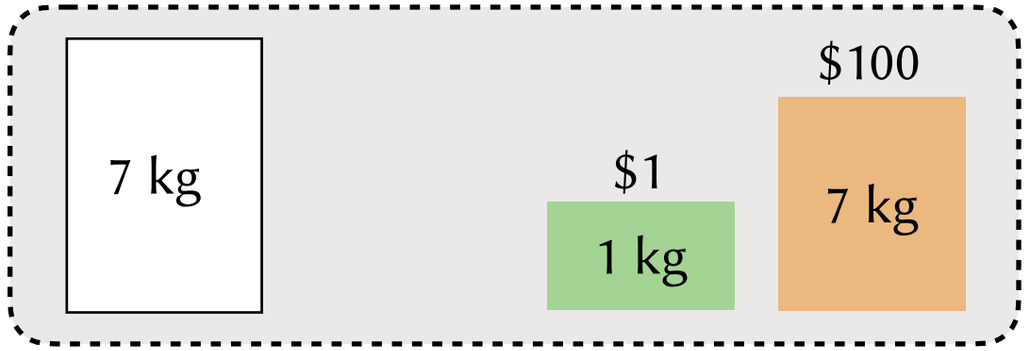

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Greedy Solutions Do not Always Work

Greedy Choice Property. A *globally* optimal solution can be reached with a sequence of l*ocally* optimal decisions.

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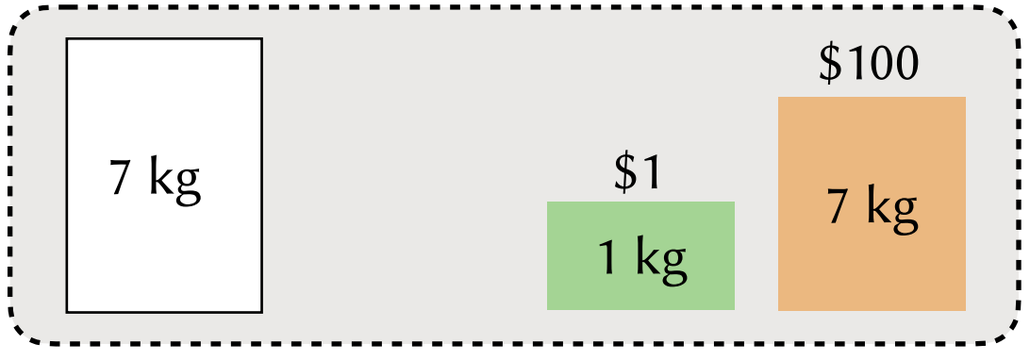

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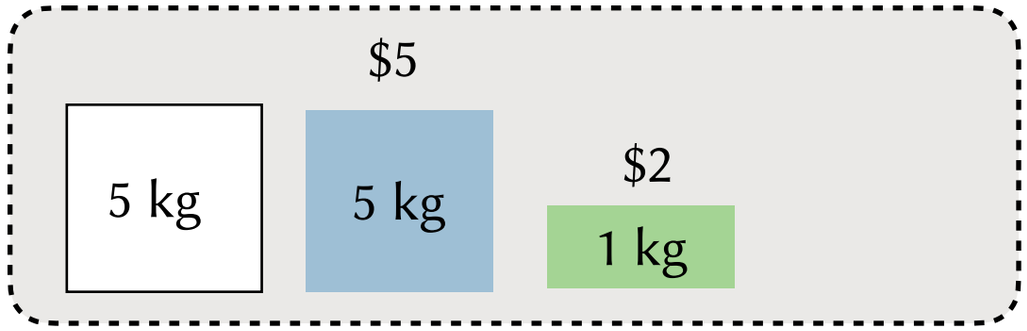

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To prove that the greedy choice property holds, we must prove that a greedy strategy produces an optimal solution for *every instance* of the problem.

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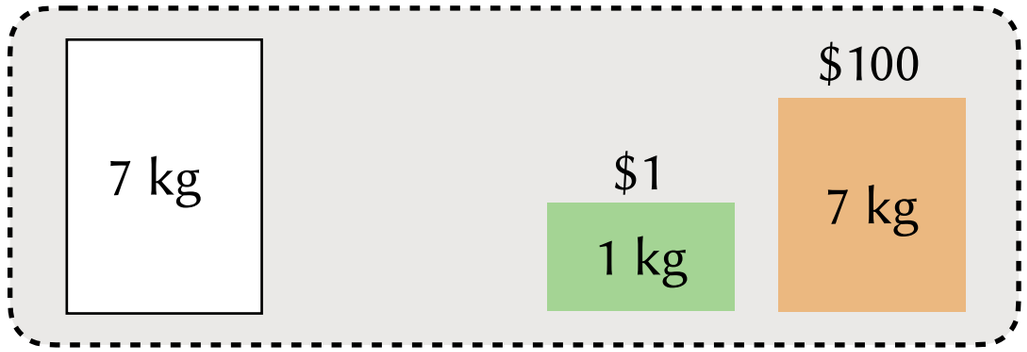

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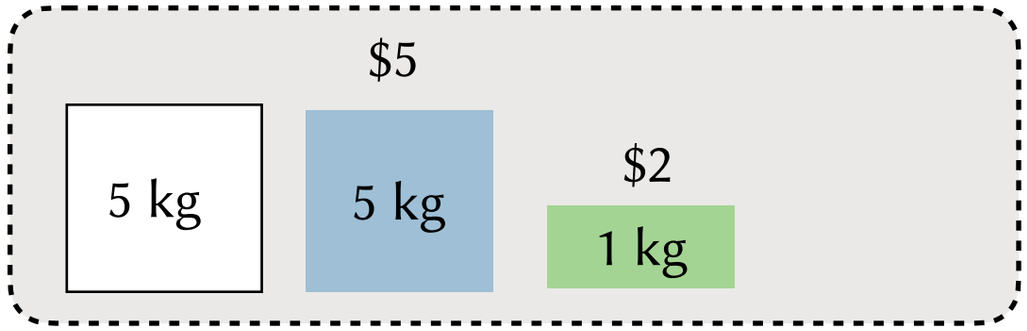

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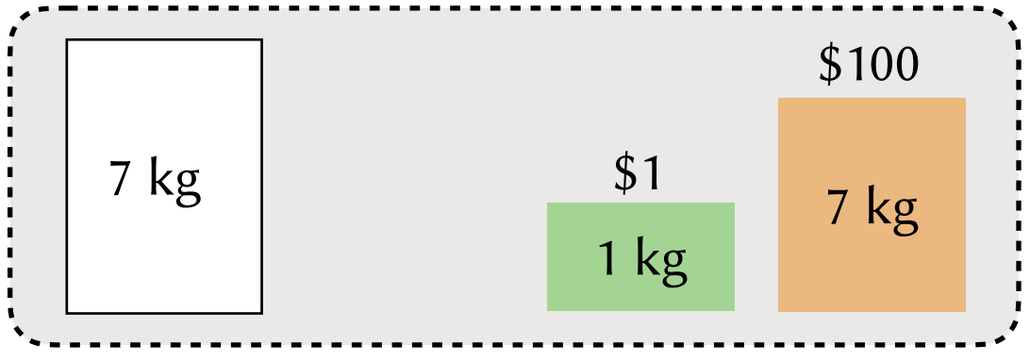

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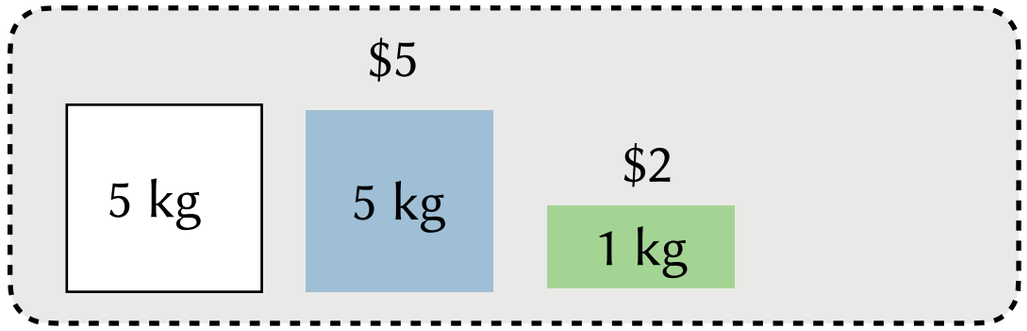

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There is no known optimal greedy strategy for the 0-1 Knapsack problem

We will learn a
dynamic programming
solution for the problem later!

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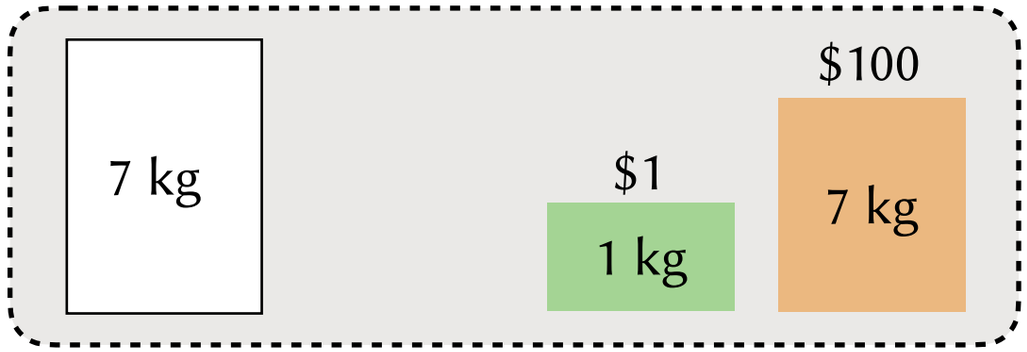

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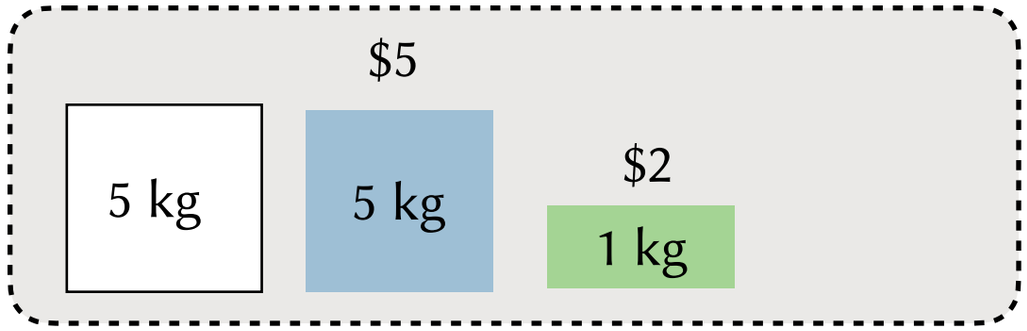

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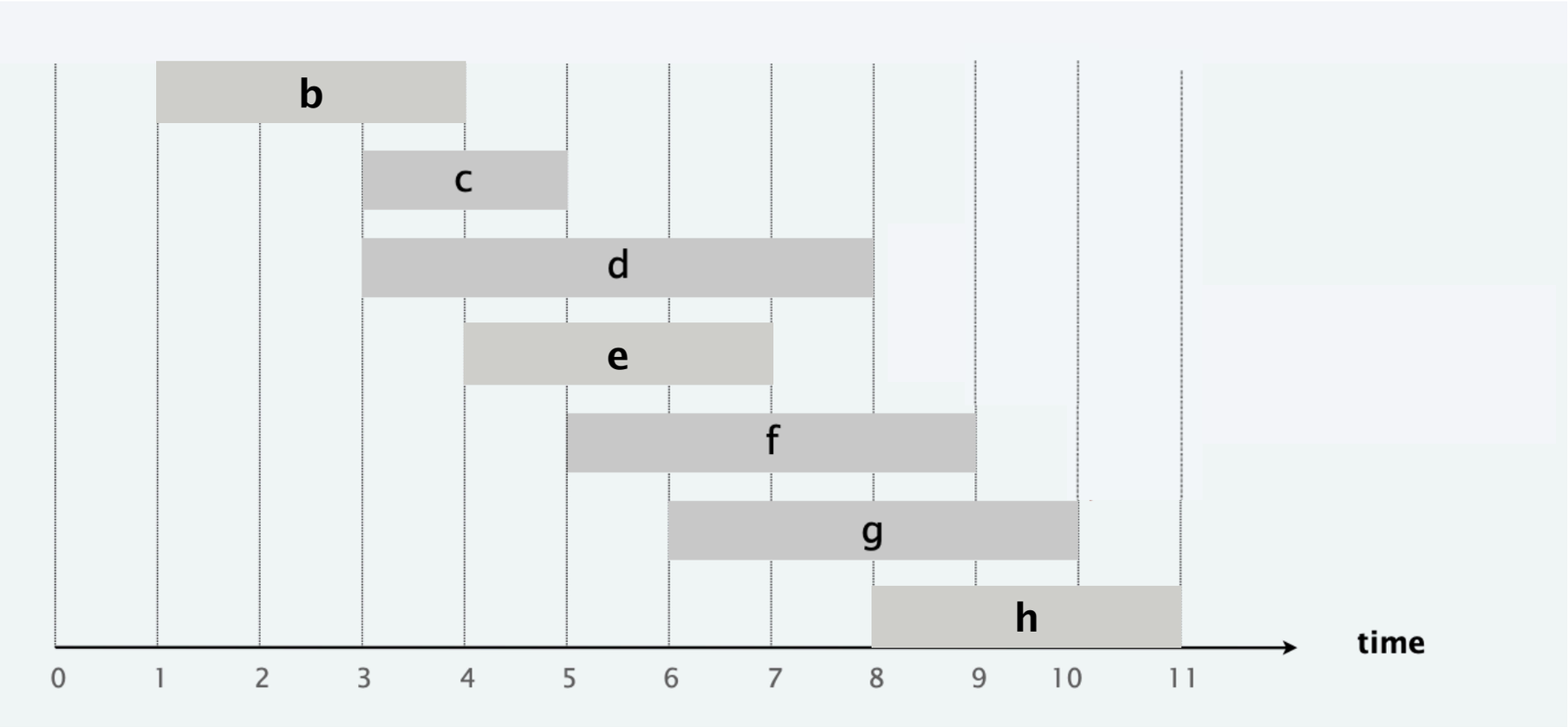
Activity Selection

Problem. Given n activities that require an exclusive use of a common resource, find the *maximum* number of non-overlapping activities.

Examples. Maximum number of jobs that can be done by a person.
Maximum number of events that can be held in a room.



adapted from an image by Kevin Wayne

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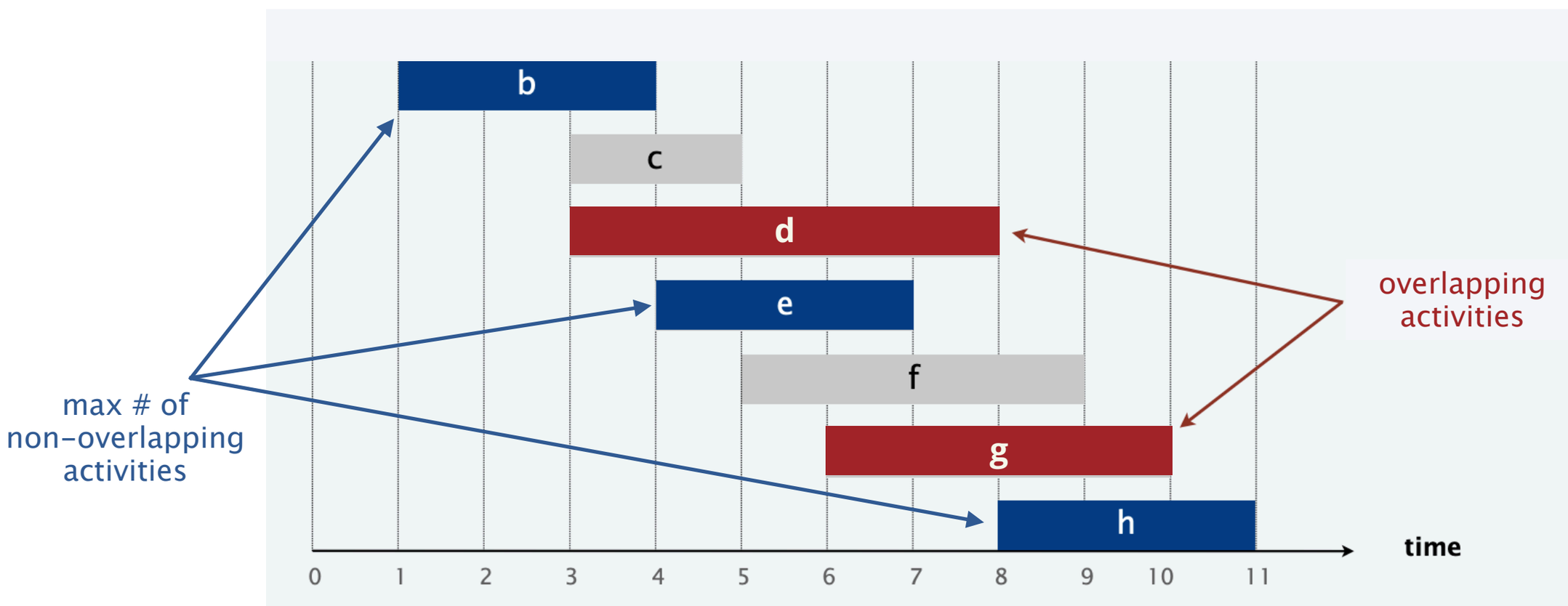

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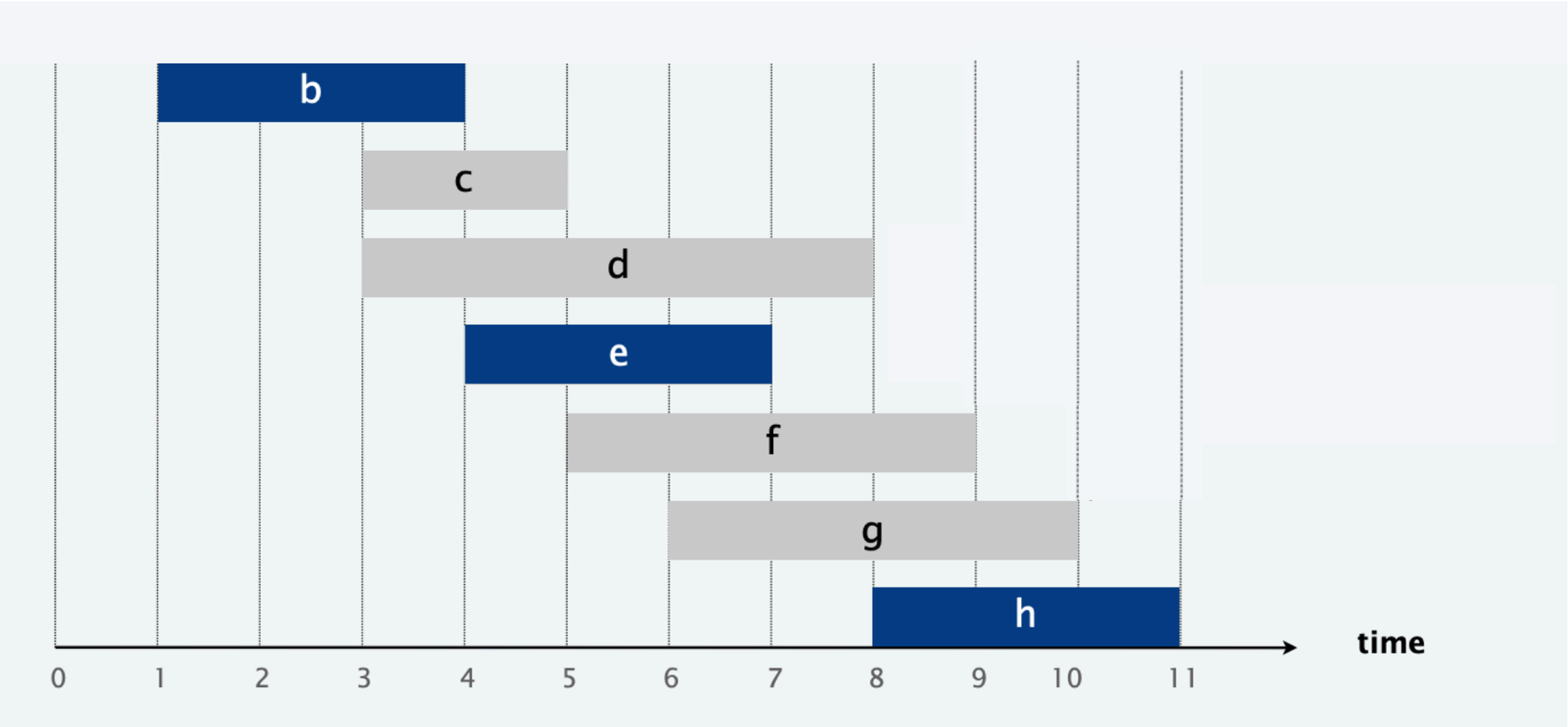
Notation. - $S = \{a_1, a_2, a_3, \ldots, a_n\}$ is the set of activities.



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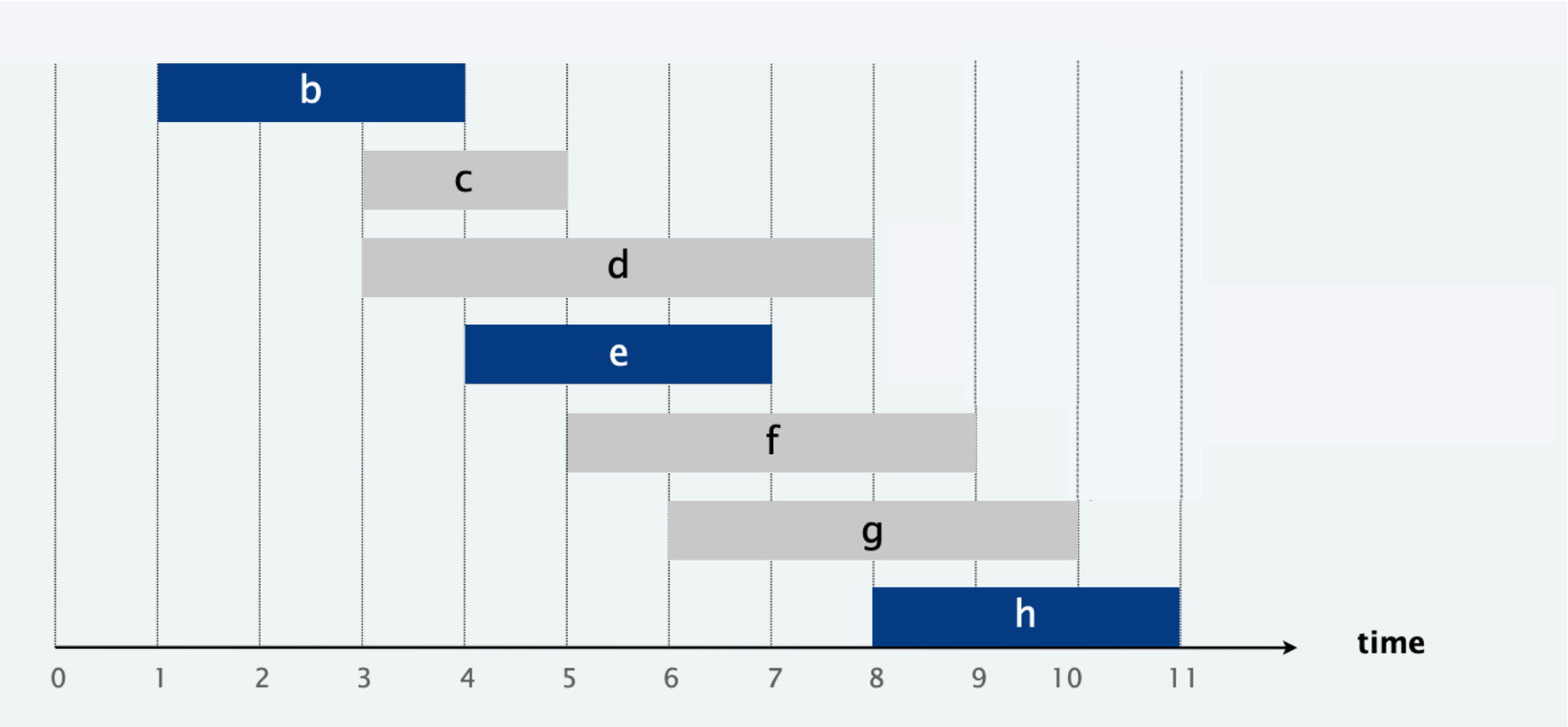

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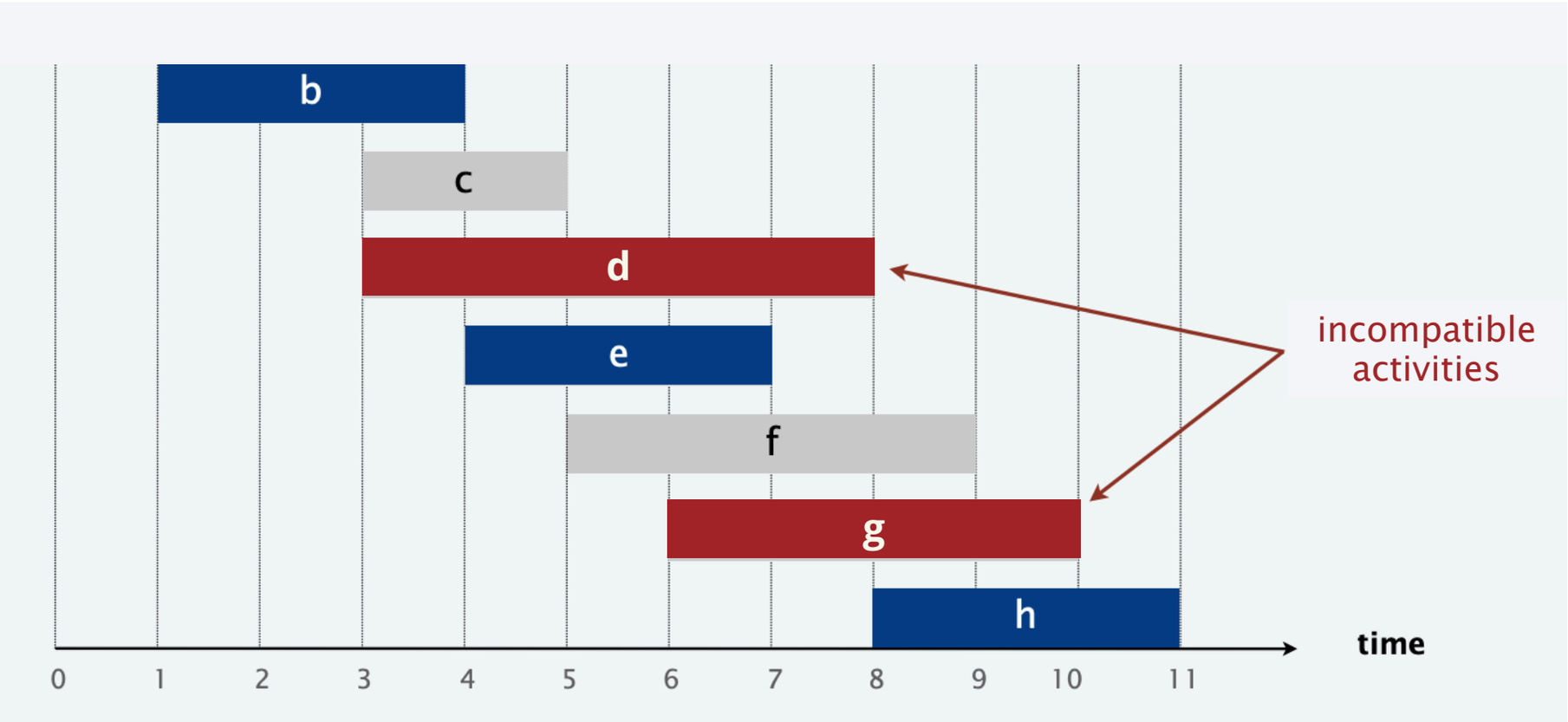
- a_i and a_j are compatible if $[\, s_i, f_i \,)$ and $[\, s_j, f_j \,)$ do not overlap.



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Activity Selection

Greedy strategy # 1. Earliest start time first.
Rationale. Start the activities as early as possible.

Activity Selection

Greedy strategy # 1. Earliest start time first.
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Counter Examples

Greedy: 1 Optimal: 2

Activity Selection

Greedy strategy # 1. Earliest start time first.
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Greedy strategy # 2. Shortest activity first.
Rationale. Shorter activities tend to have less conflicts.

Counter Examples

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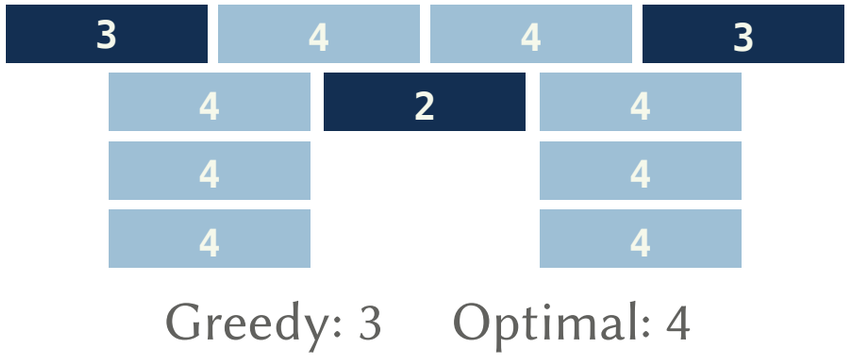

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Greedy: 3 Optimal: 4

Greedy strategy # 4. Earliest finish time first.
Rationale. Activities that finish early leave more time to be filled with other activities later.

Greedy strategy # 5. Latest start time first.
Rationale. Activities that start late leave more time to be filled with other activities earlier.

Activity Selection

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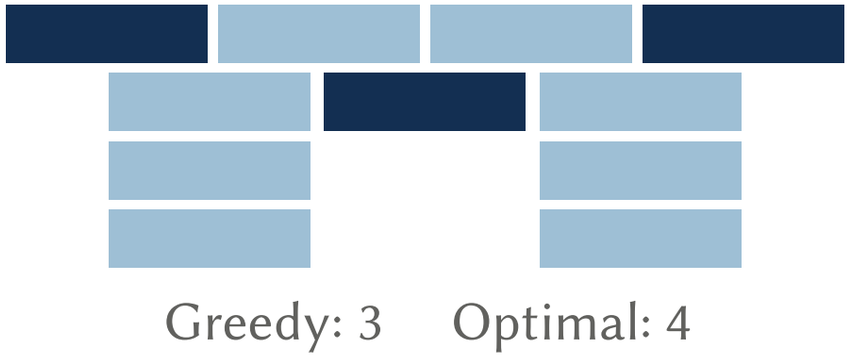

optimal

Greedy strategy # 4. Earliest finish time first.
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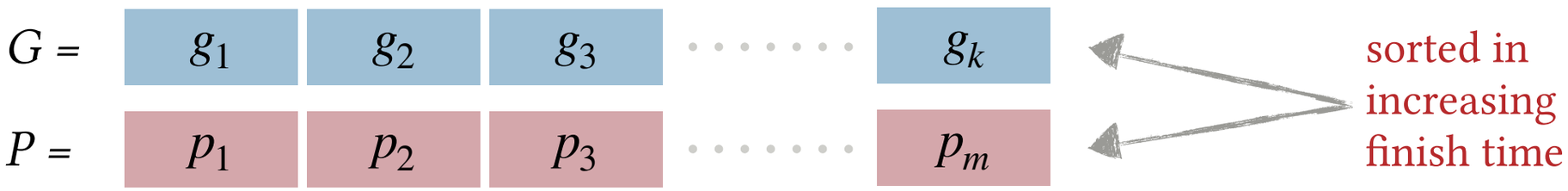
Greedy strategy # 5. Latest start time first.
Rationale. Activities that start late leave more time to be filled with other activities earlier.

Activity Selection (Optimality of Greedy Solution)

Assume P is an optimal solution with m activities and G is the greedy solution with k activities. We would like to show that $m = k$ (*i.e. G is as good as the optimal solution*).



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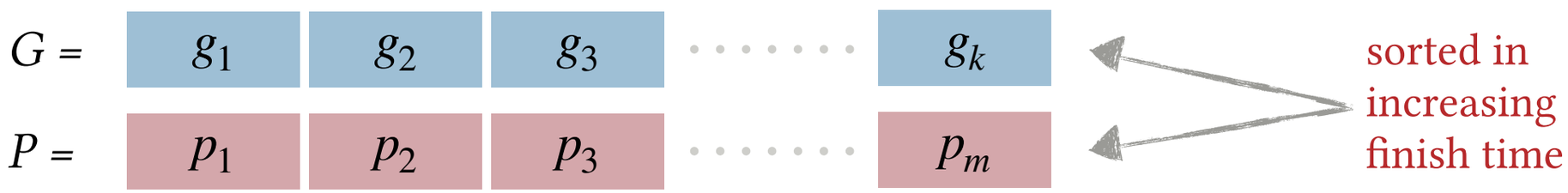

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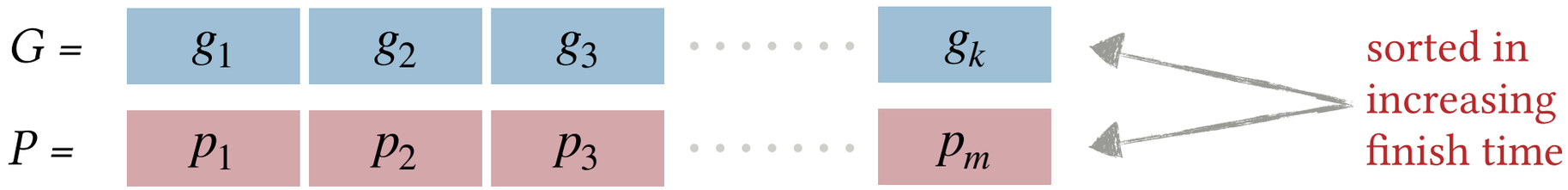
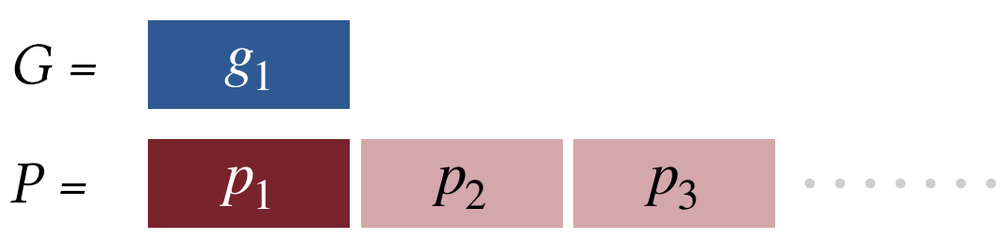


Observation 1. We can always replace p_1 in P by g_1 from G.

Proof. Since g_1 is guaranteed to finish before (or with) p_1, g_1 is compatible with $p_2 \longrightarrow p_m$. Hence, there is always an optimal solution that starts with the activity that finishes first (g_1).



If g_1 ends with p_1
it is compatible with p_2

If g_1 ends before p_1
it is compatible with p_2

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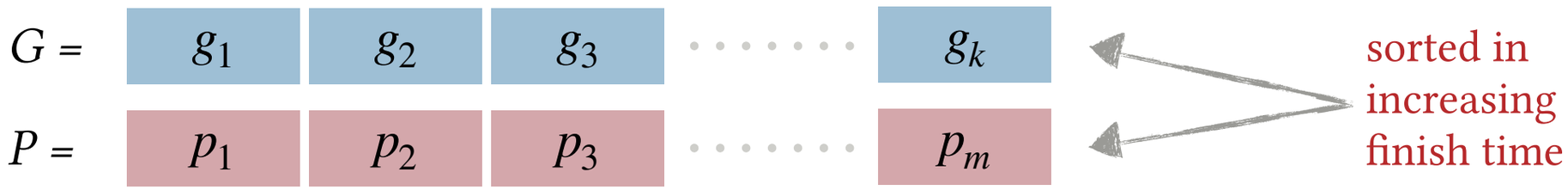

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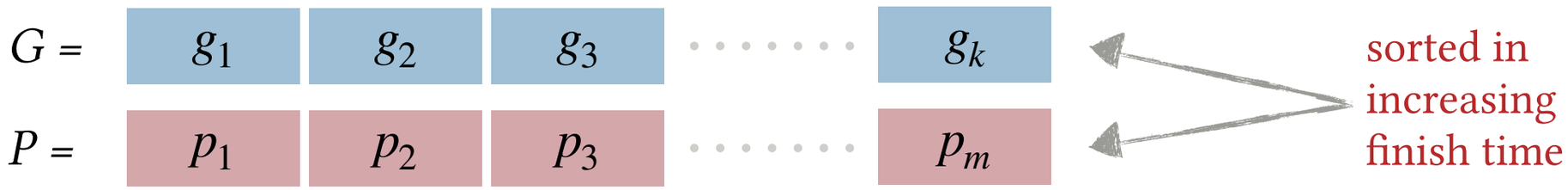

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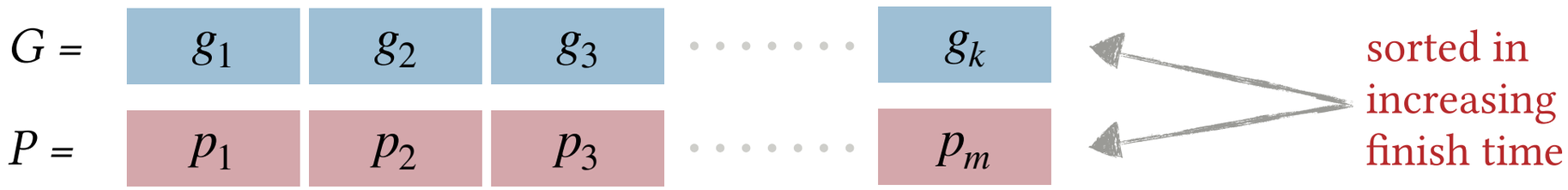

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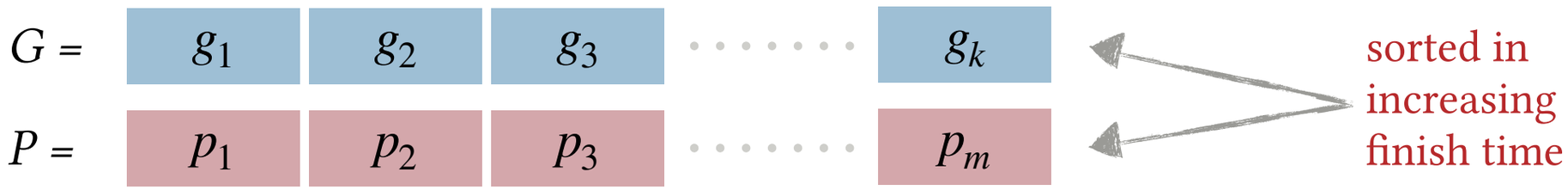

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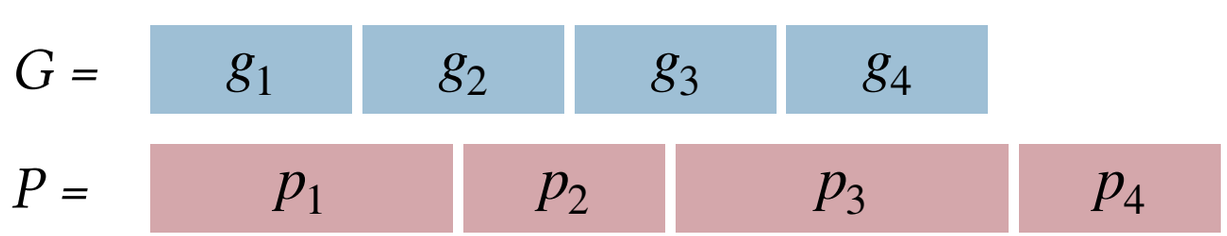

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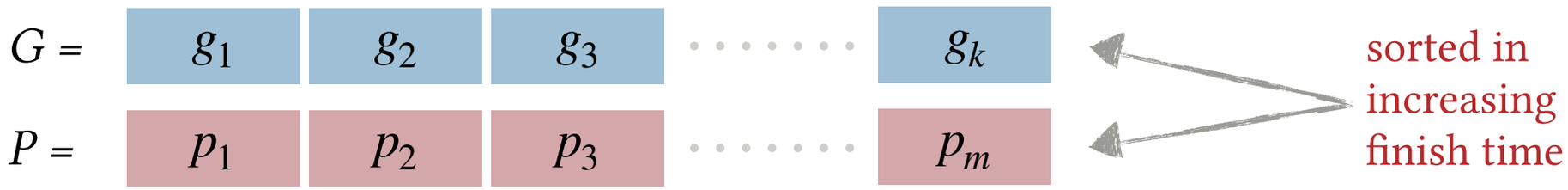

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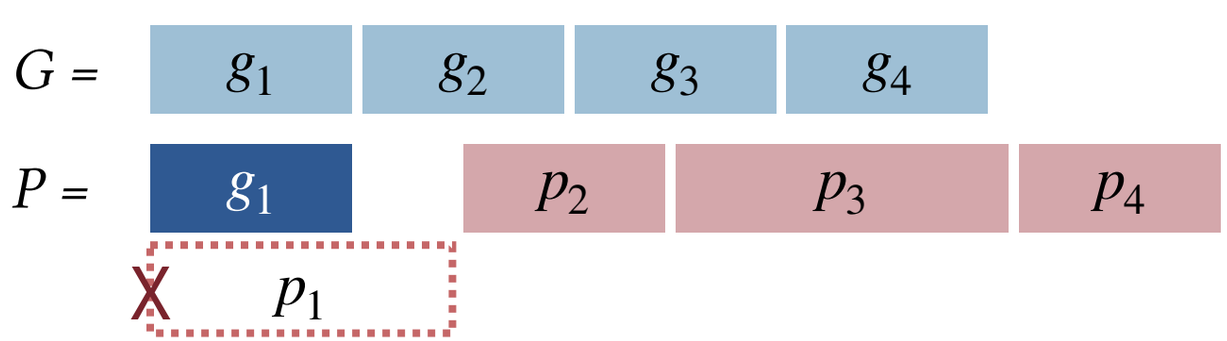

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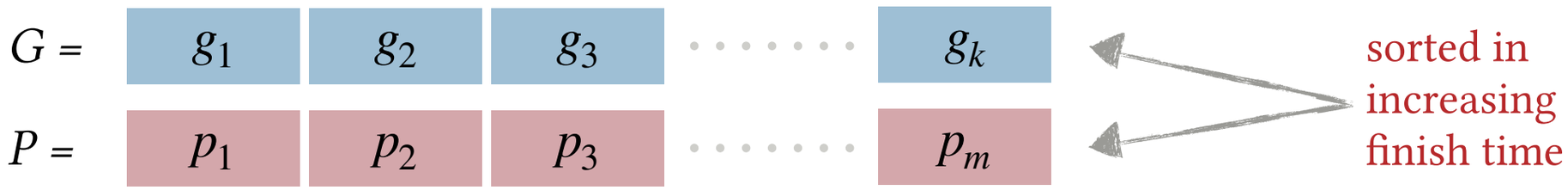

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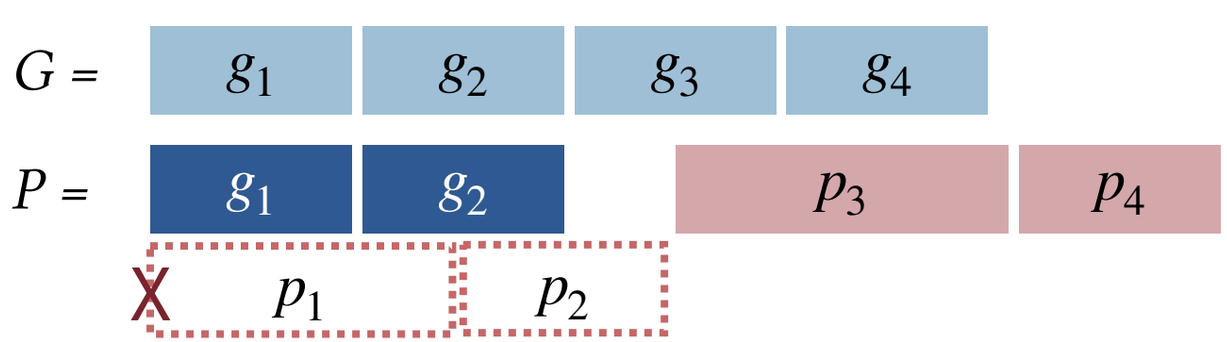

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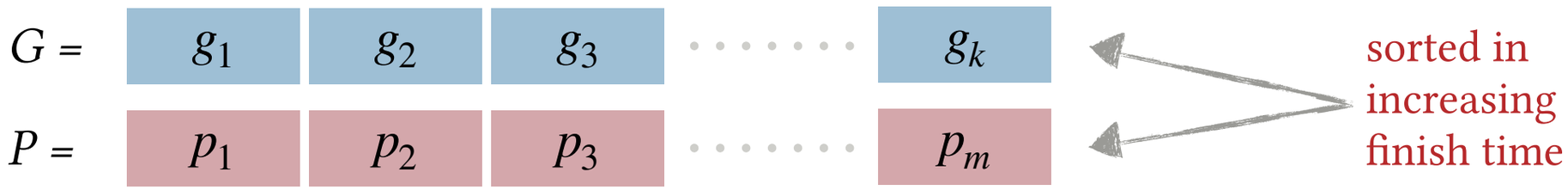

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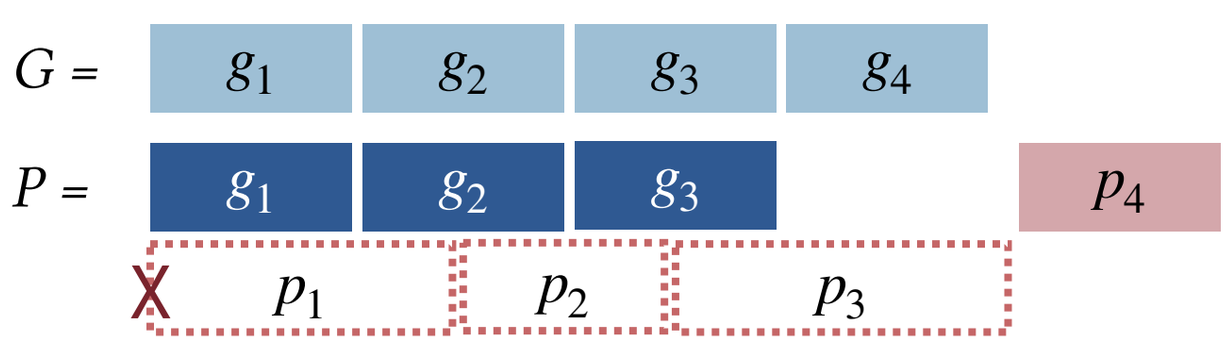

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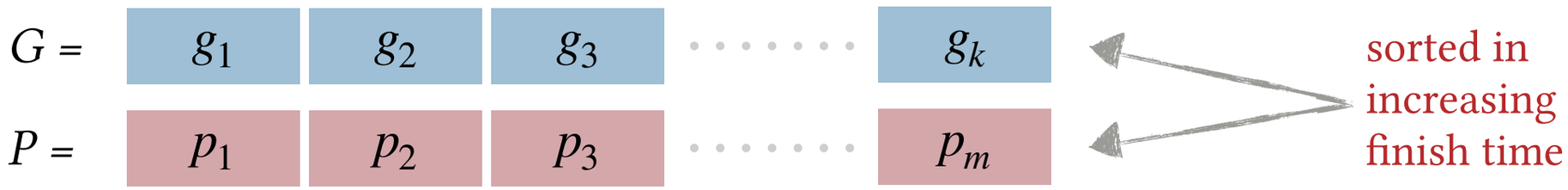

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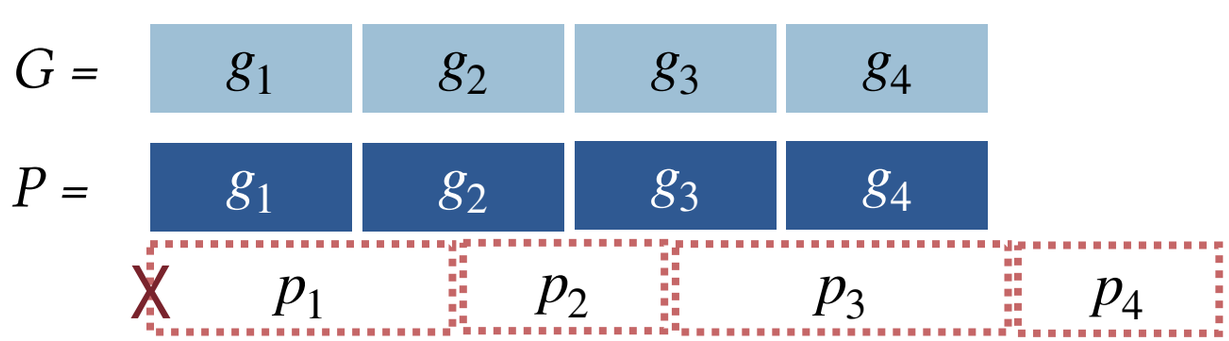

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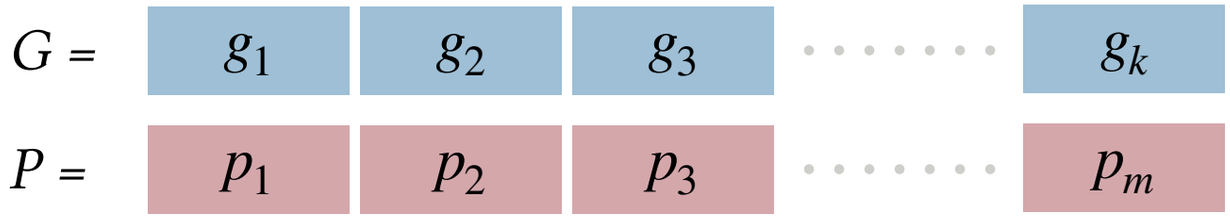

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Rolling this out, we conclude that for all $i \leq k$, every p_i can be replaced by g_i.

- If $k = m$, then G is optimal.

Activity Selection (Optimality of Greedy Solution)

Assume P is an optimal solution with m activities and G is the greedy solution with k activities. We would like to show that $m = k$ (*i.e. G is as good as the optimal solution*).

$G =$	g_1	g_2	g_3	.....	g_i	.....	g_k
$P =$	p_1	p_2	p_3	.....	p_m		

If G has more then P is not optimal!

Observation 1. We can always replace p_1 in P by g_1 from G.

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$G =$	g_1	g_2	g_3	·····	g_k	Why didn't G pick p_{i+1}?		
$P =$	p_1	p_2	p_3	·····	p_i	p_{i+1}	·····	p_m

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- If $k = m$, then G is optimal.
- If $k > m$, then P is not optimal.
- If $k < m$, then there is an activity that starts after g_k that G can still pick!

Greedy Proofs (General Pattern)

This pattern in proving the optimality of greedy solutions is commonly used:

- Assume that there is an optimal solution P and a greedy solution G.
- Show that we can always *exchange* the first choice in P with the first choice in G without making the solution worse.
- Show that the problem has an *optimal substructure* and thus the same argument applies to the solution of the subproblem after making the first choice.
- Show that P can't be better than G.

Activity Selection (Algorithm)

SELECT(s[], f[], n)

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Sort the activities by
increasing finish time

// Let k = the index of
// the last taken activity
// Let A = the indices of
// the taken activities
k = 0
Add k to A

for i=1 to n-1:
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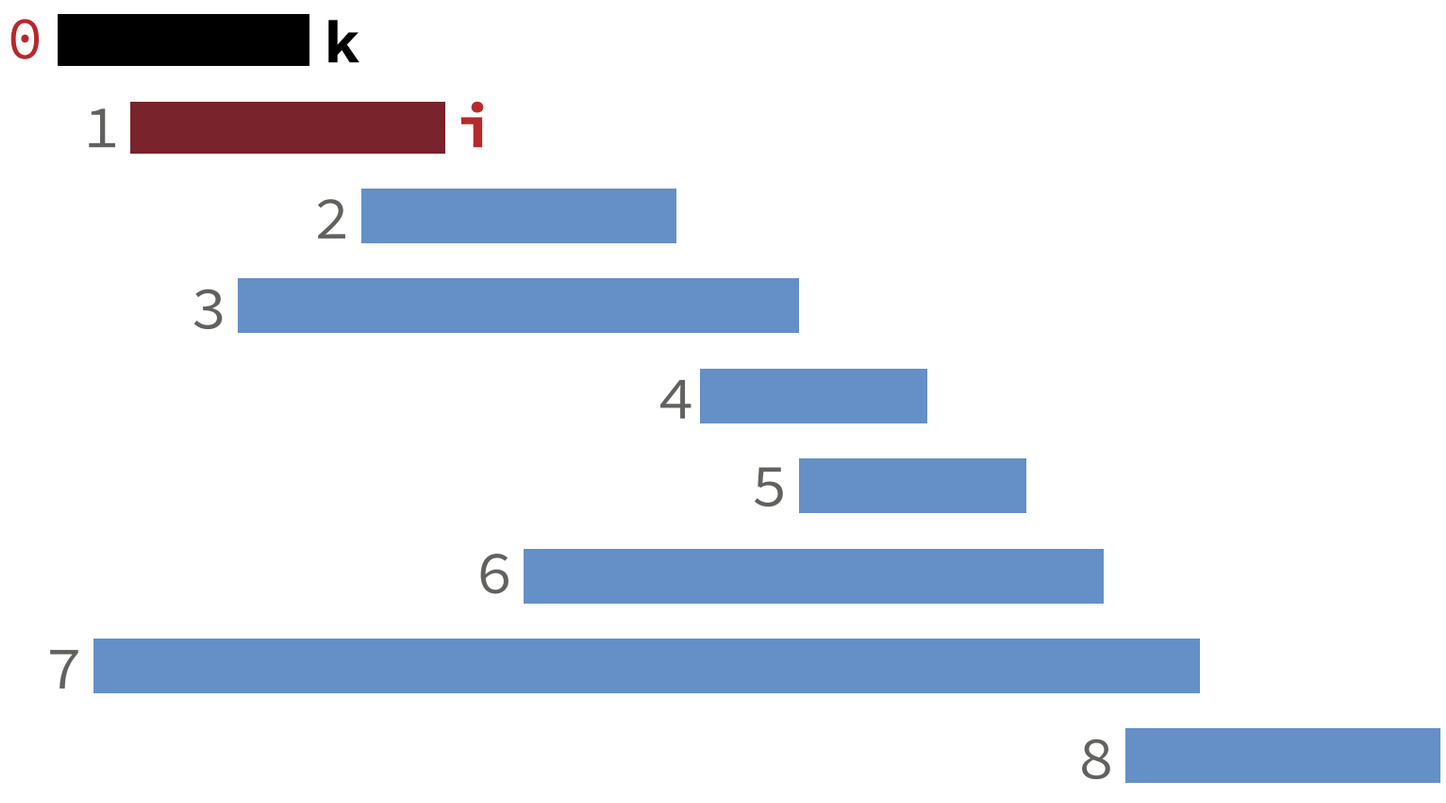

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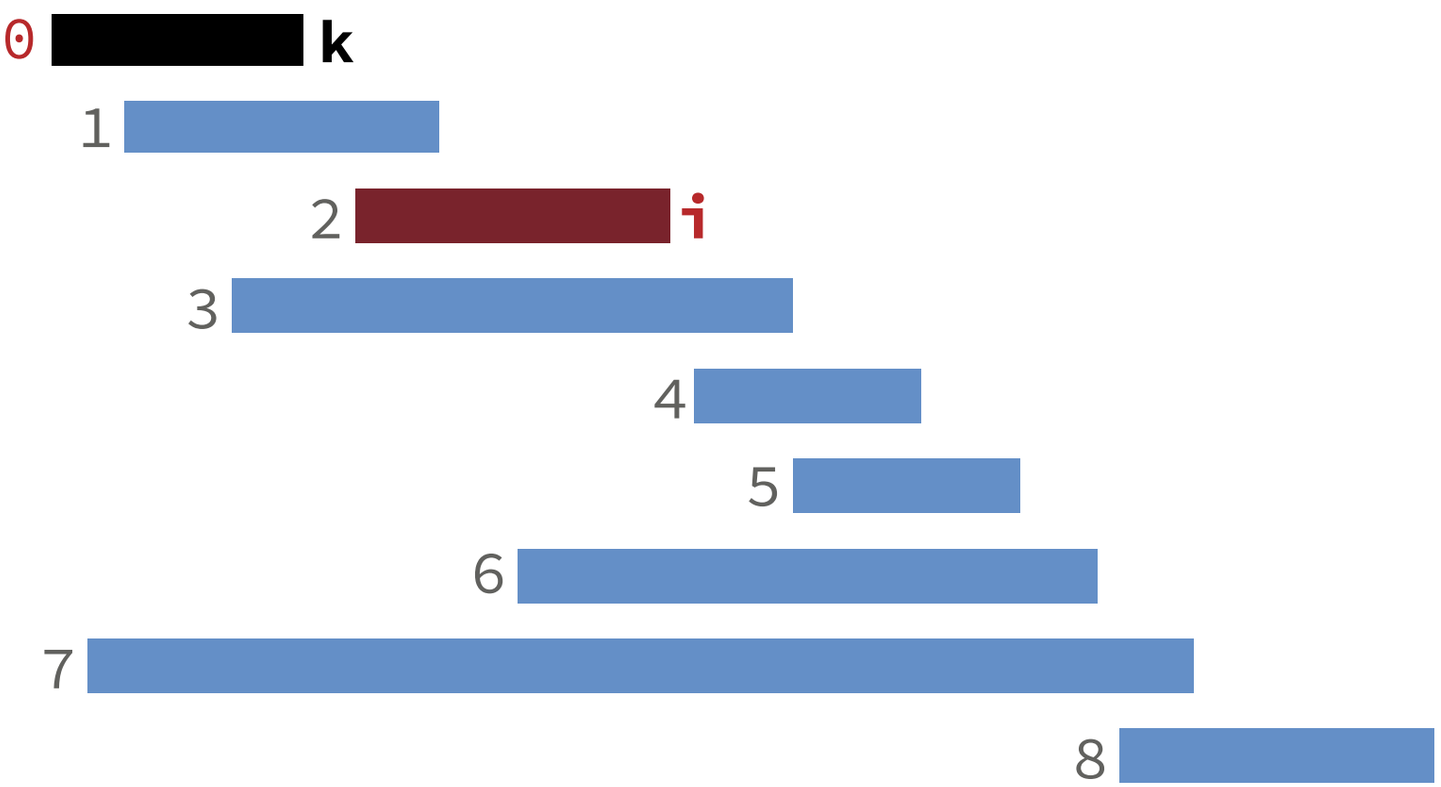

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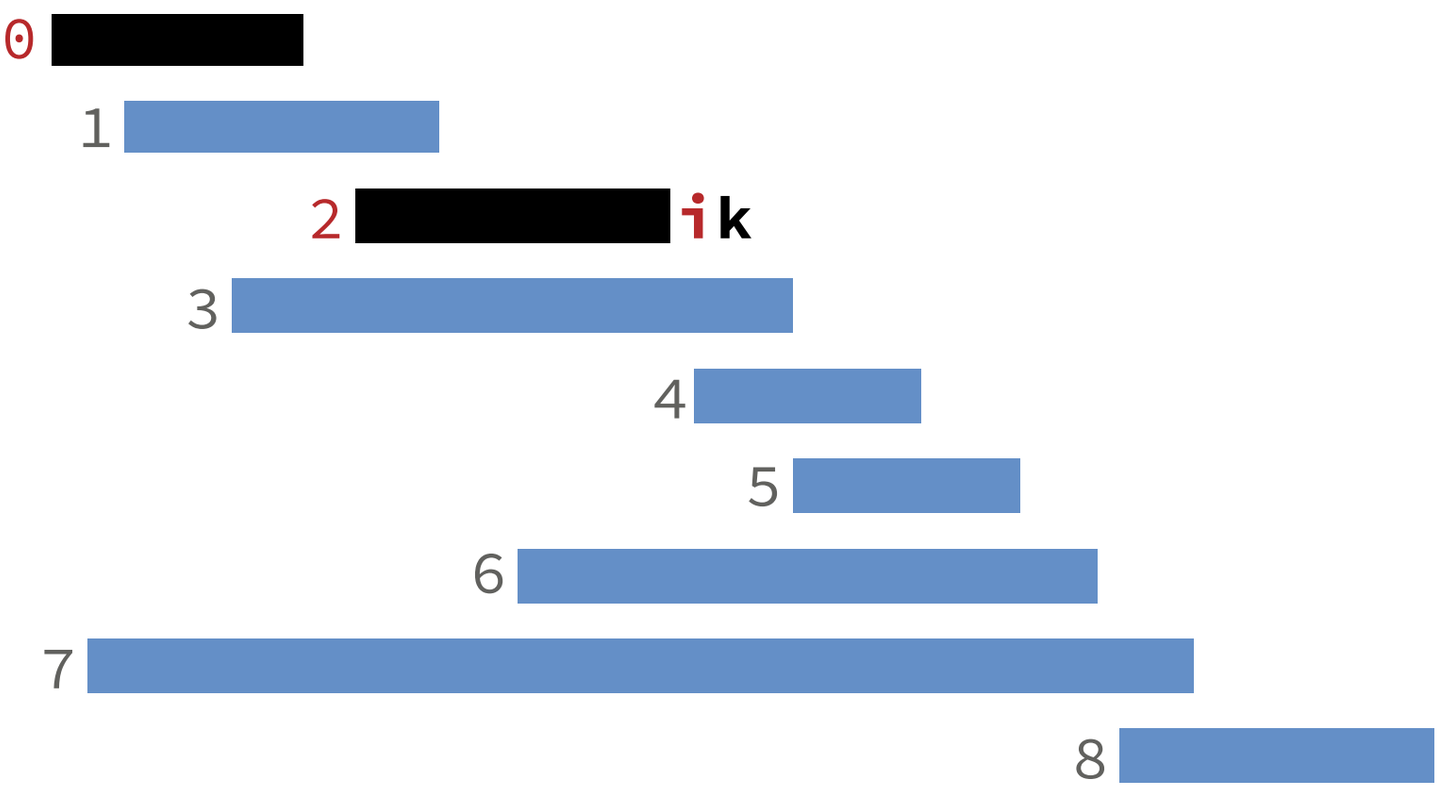

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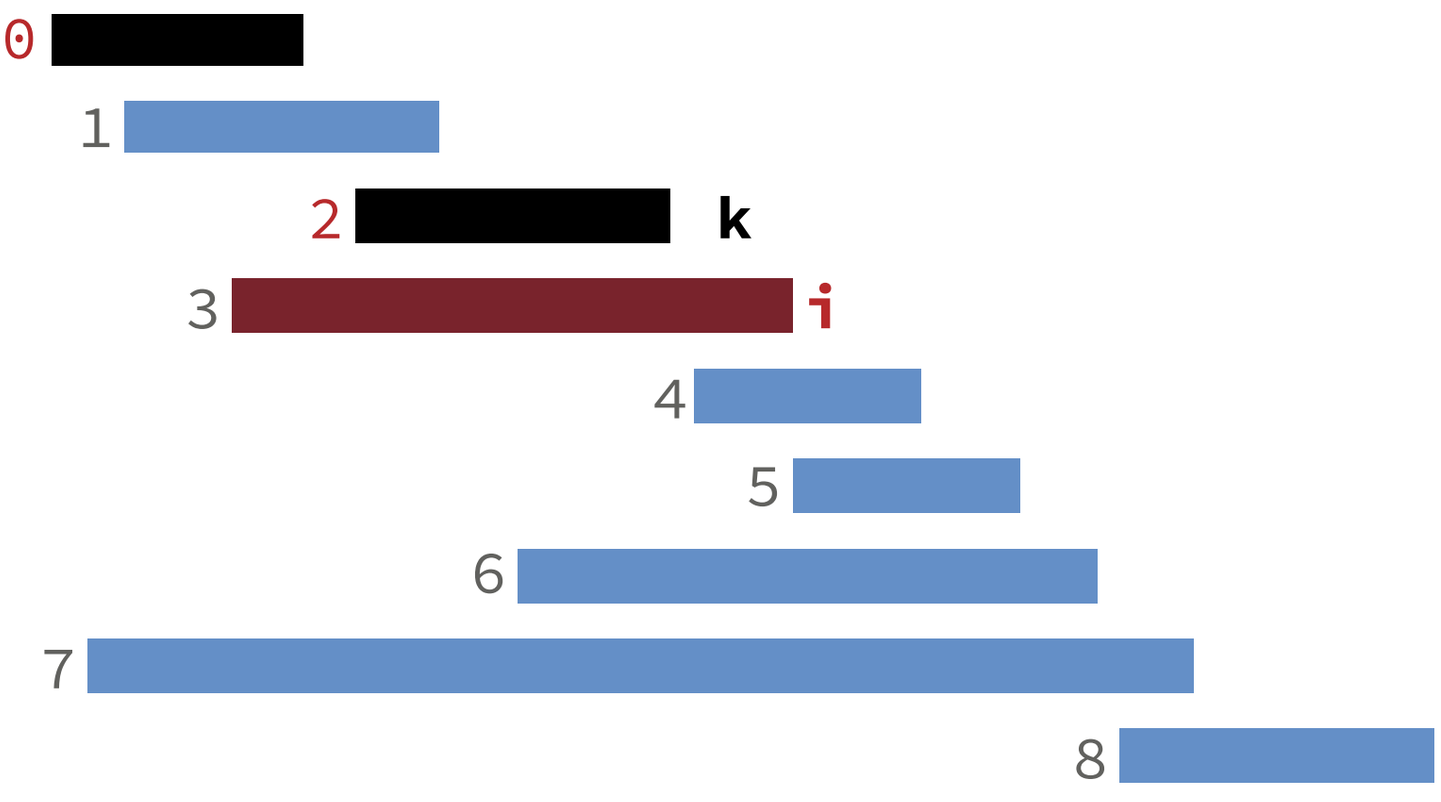

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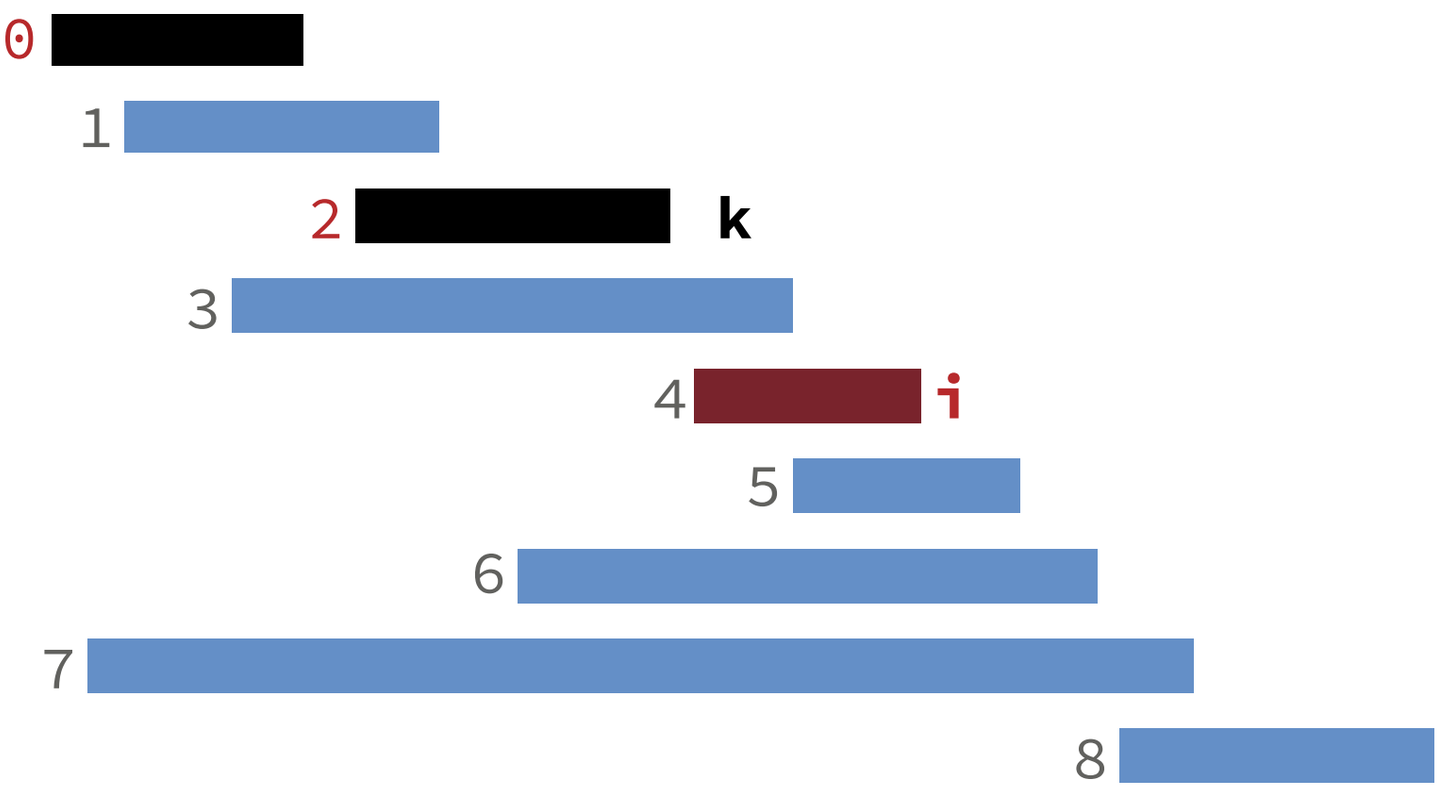

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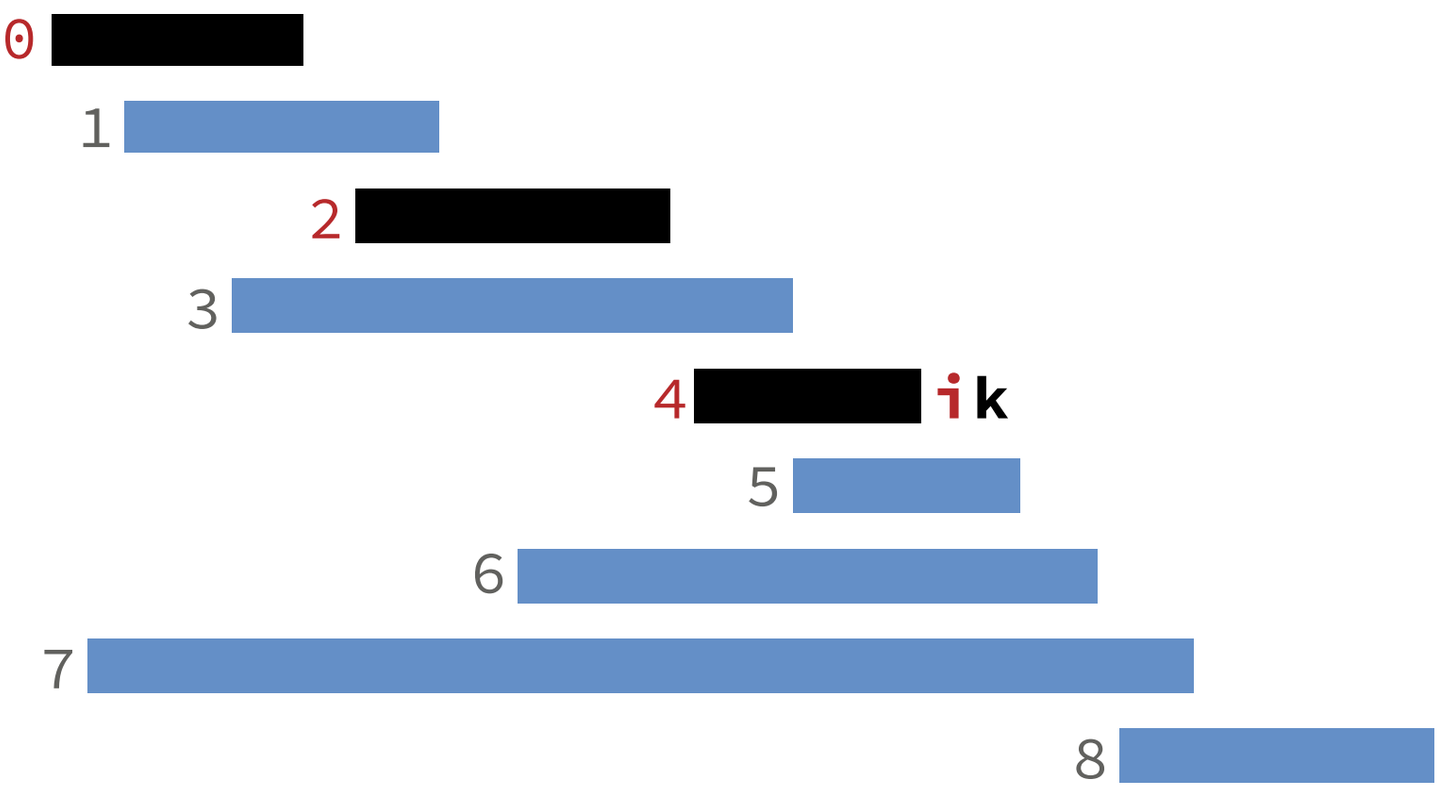

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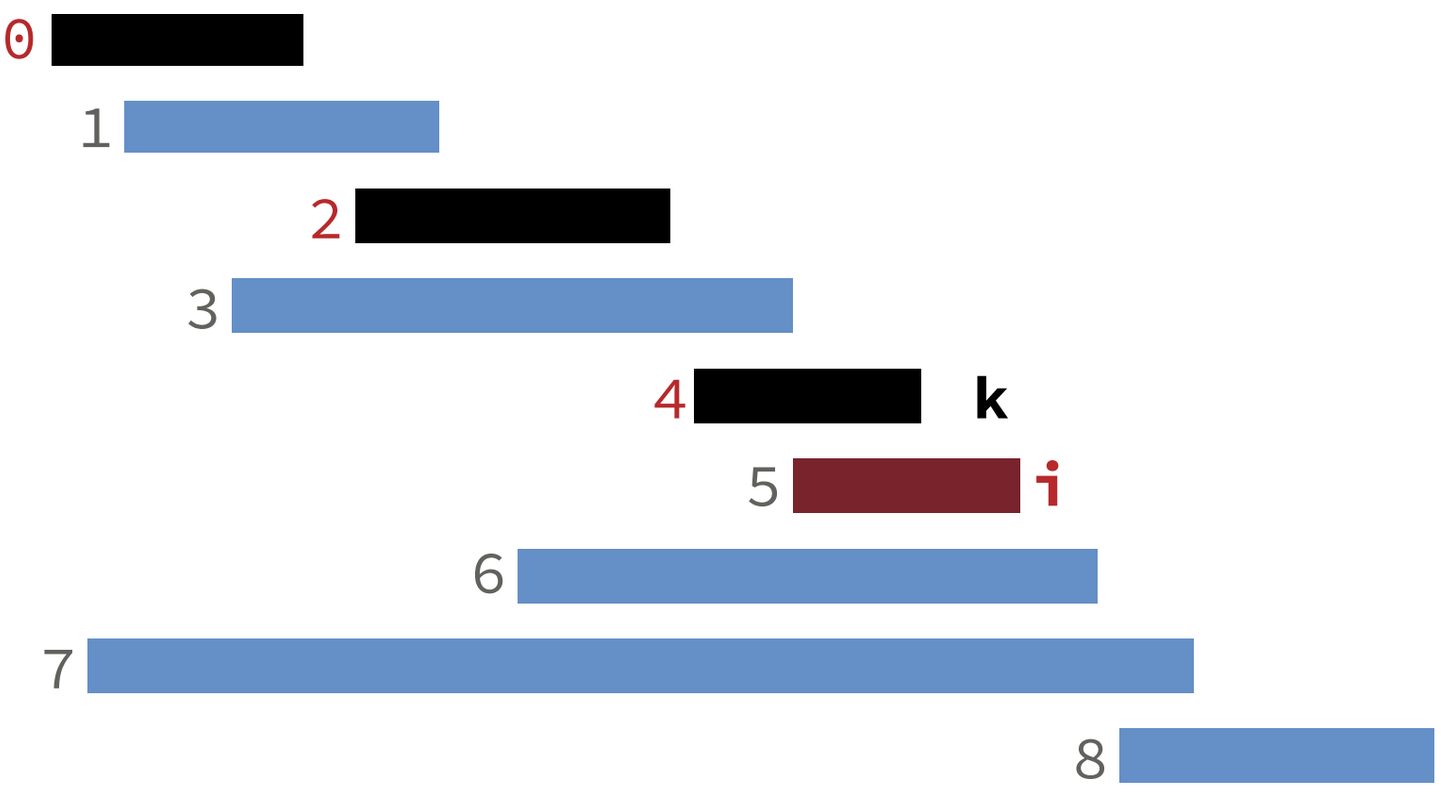

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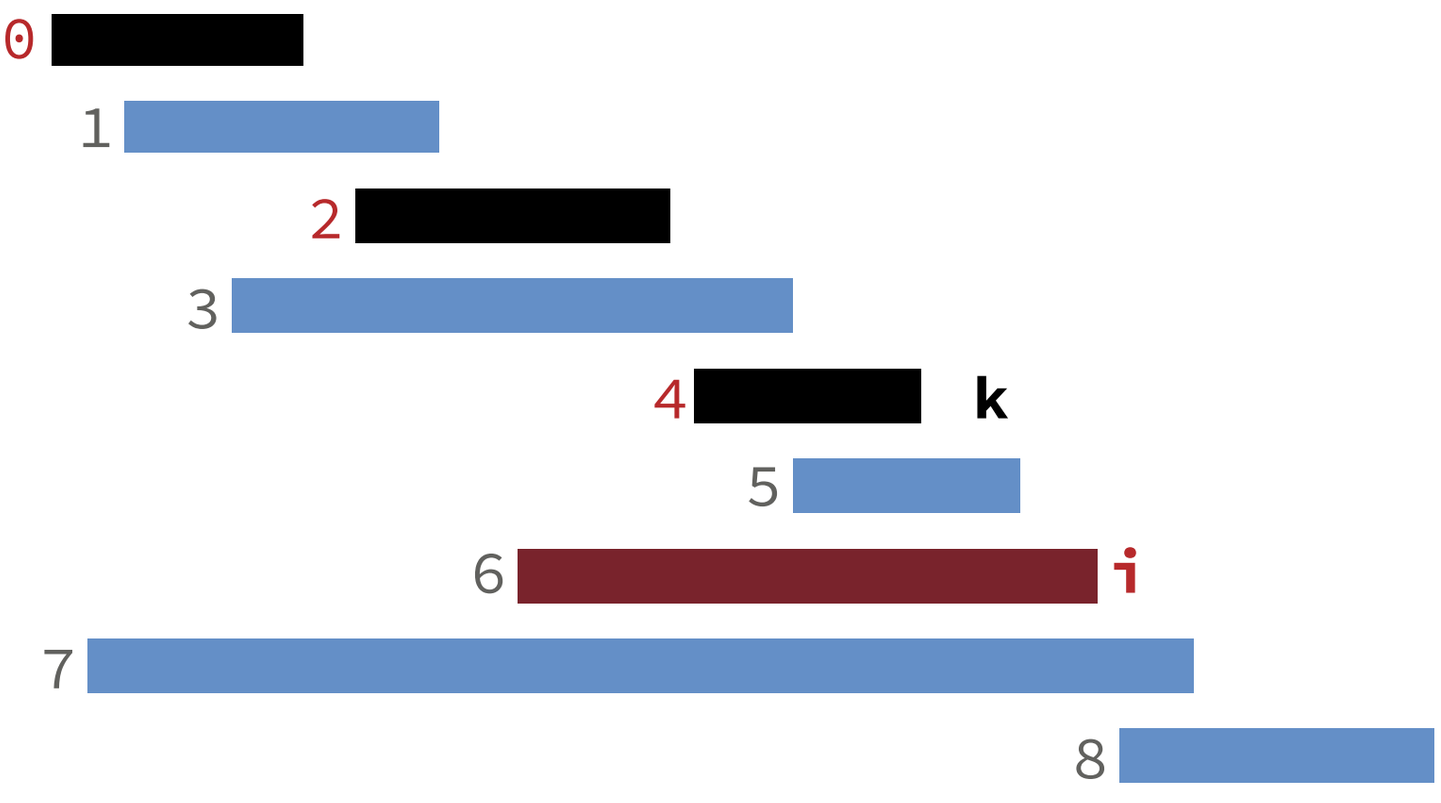

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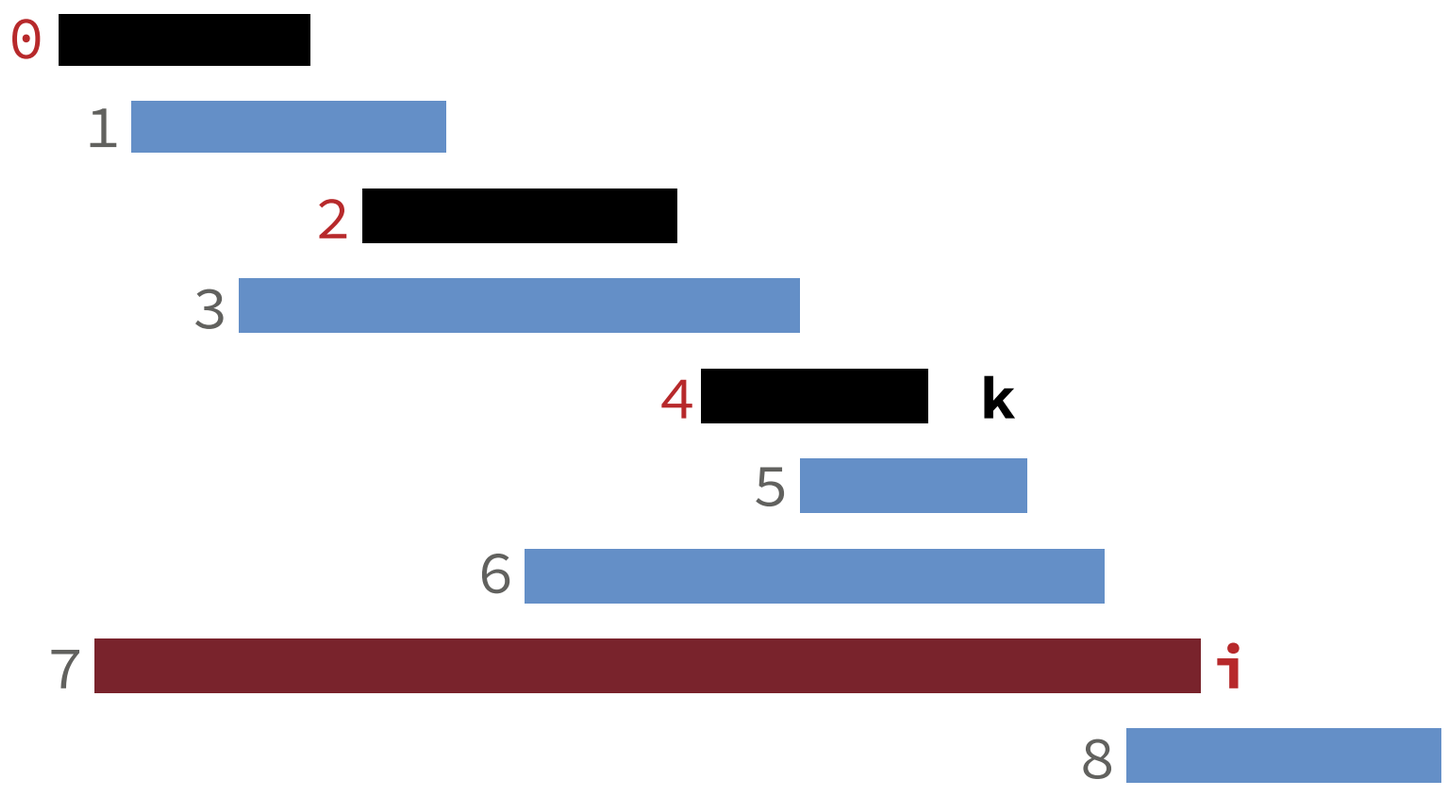

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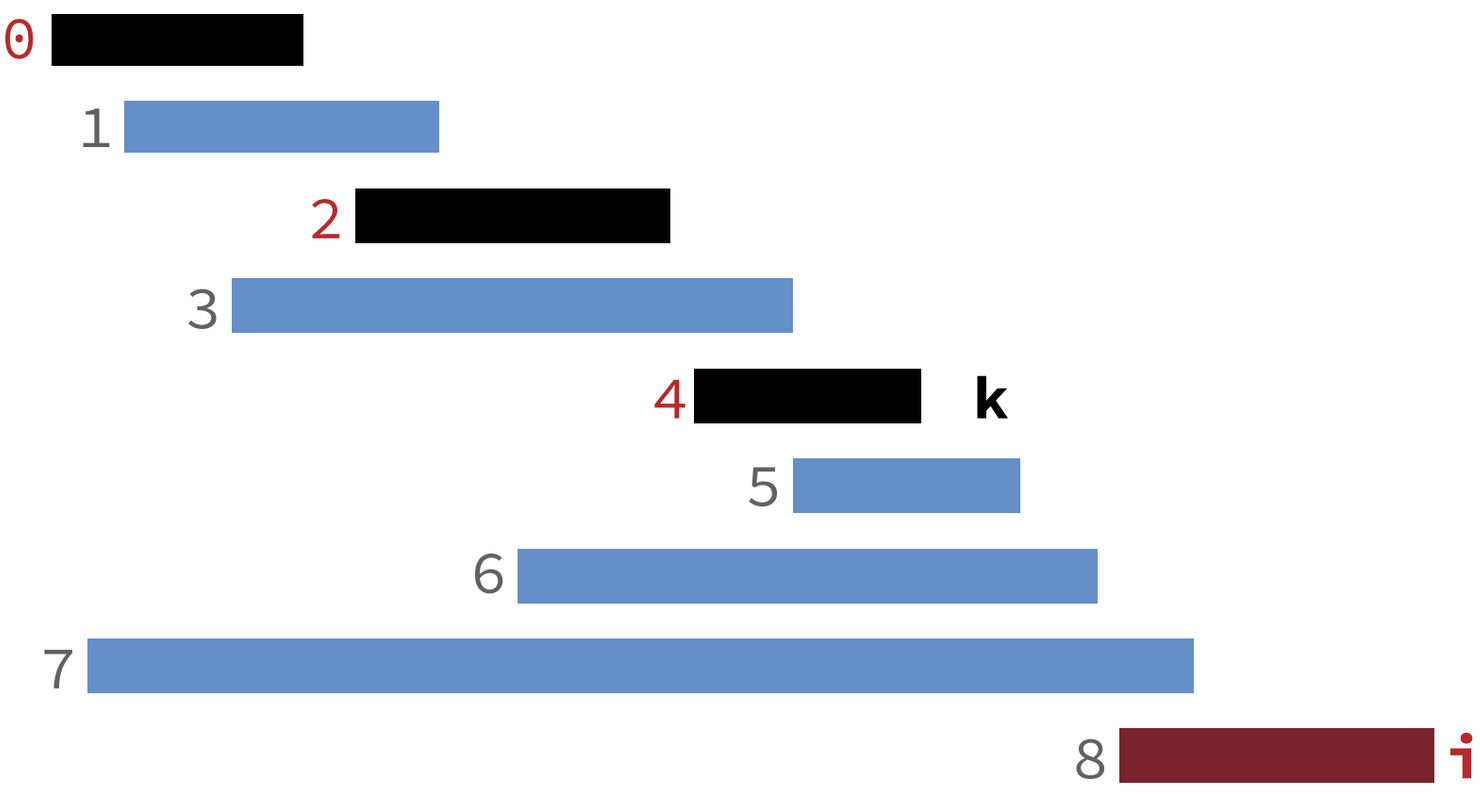

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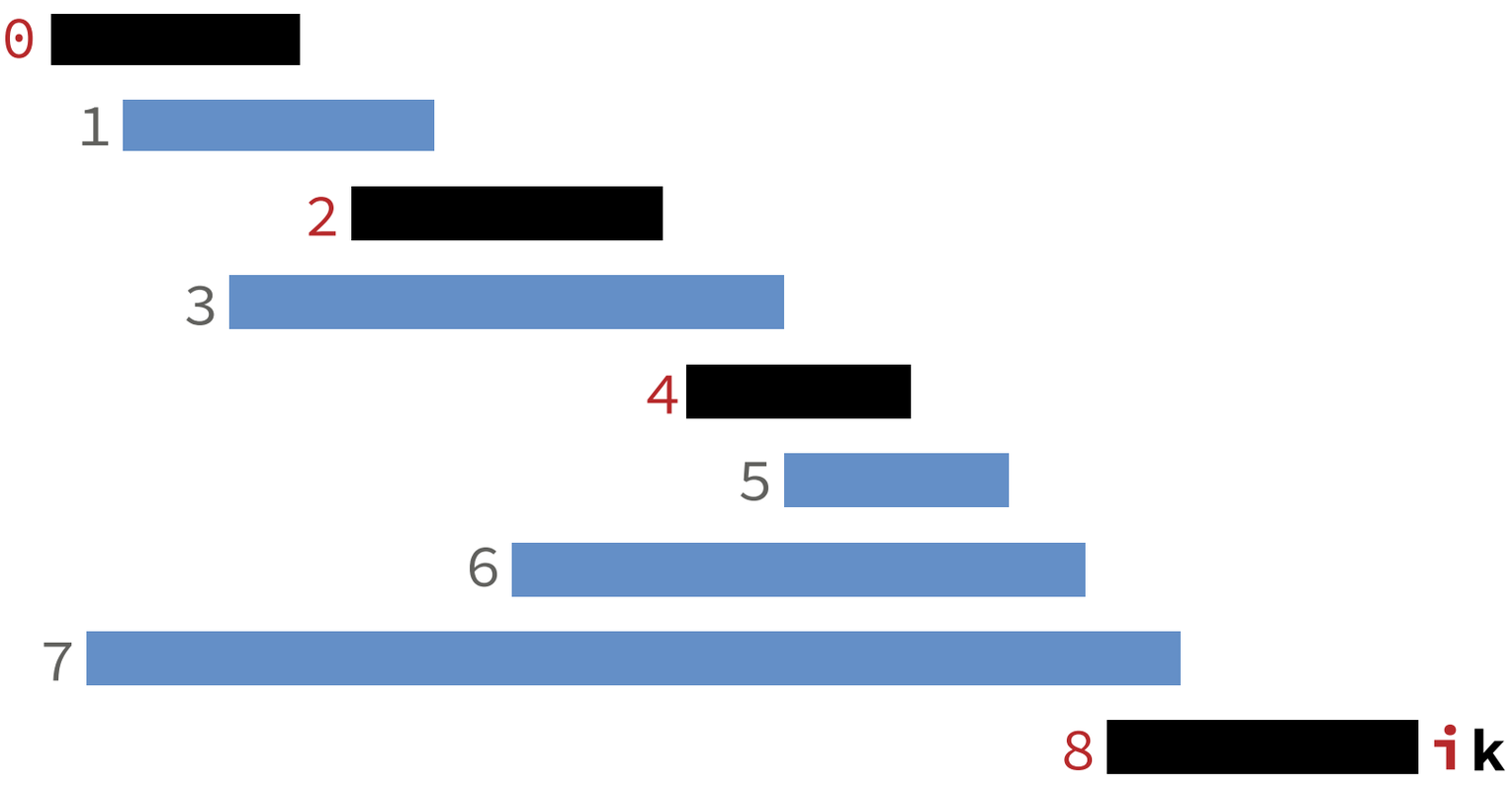

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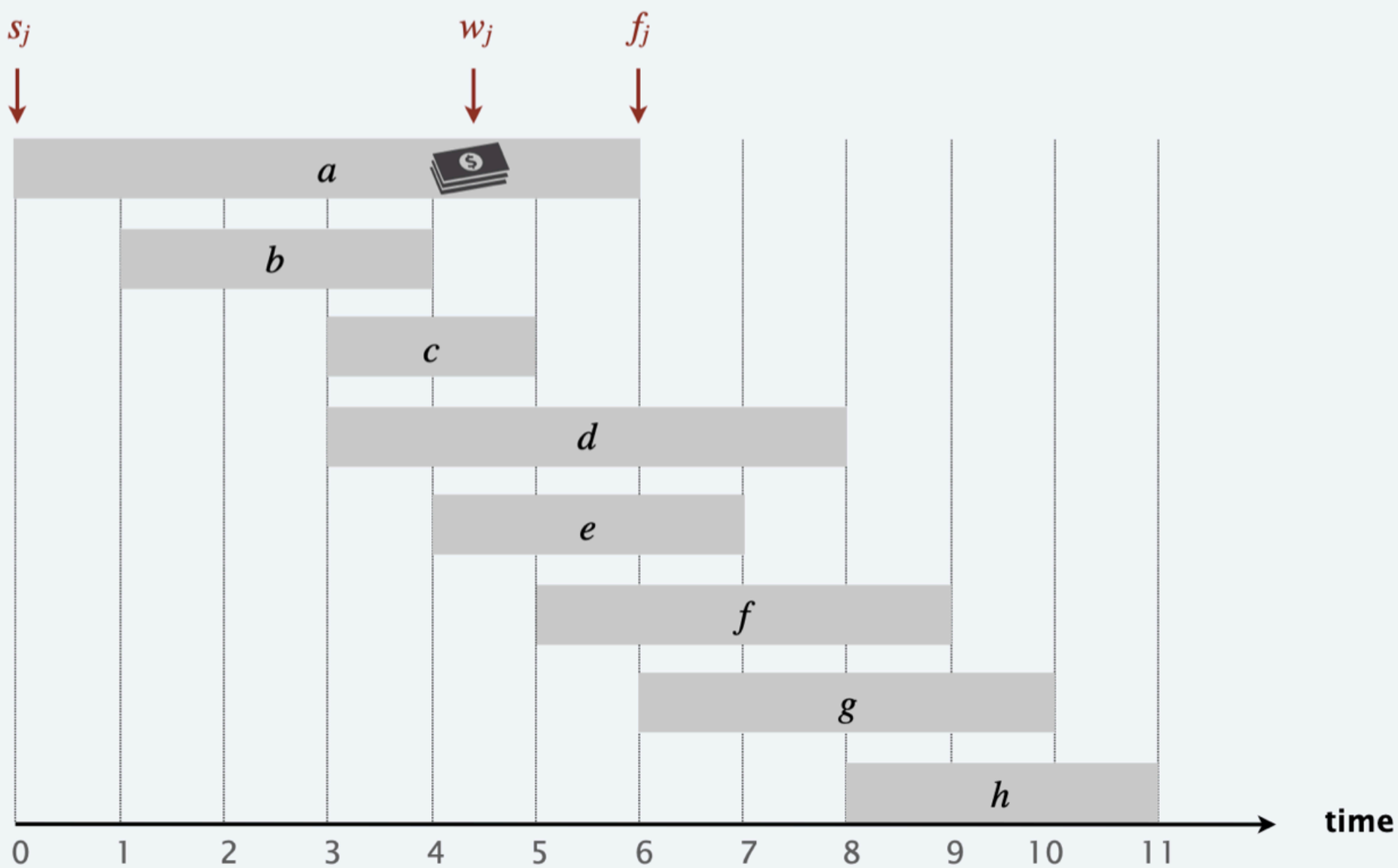
$\Theta(n \log n)$

$\Theta(n)$

Activity Selection: A Variant

Weighted Activity Selection

- Job j starts at s_j, finishes at f_j, and has weight $w_j > 0$.
- Two jobs are compatible if they don't overlap.
- Goal: find max-weight subset of mutually compatible jobs.



slide by Kevin Wayne

Exercise

Find a counterexample for each of the following greedy strategies

a. Highest value first

b. Highest value per time unit first

c. Earliest finish time first

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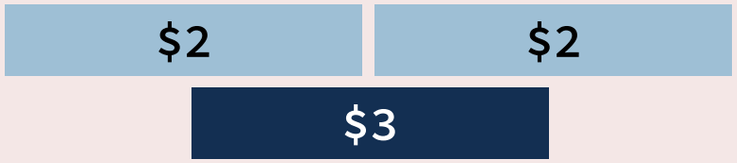
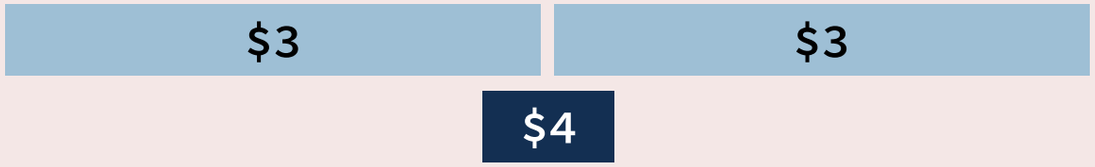


b. Highest value per time unit first



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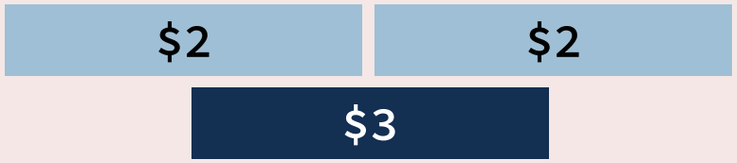

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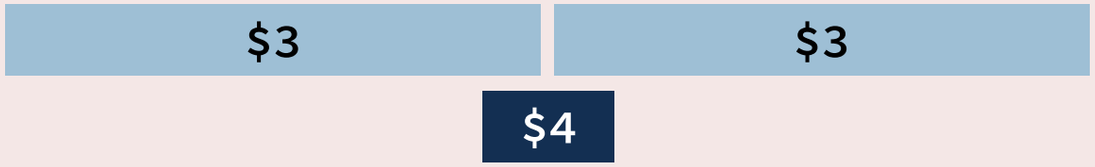


c. Earliest finish time first

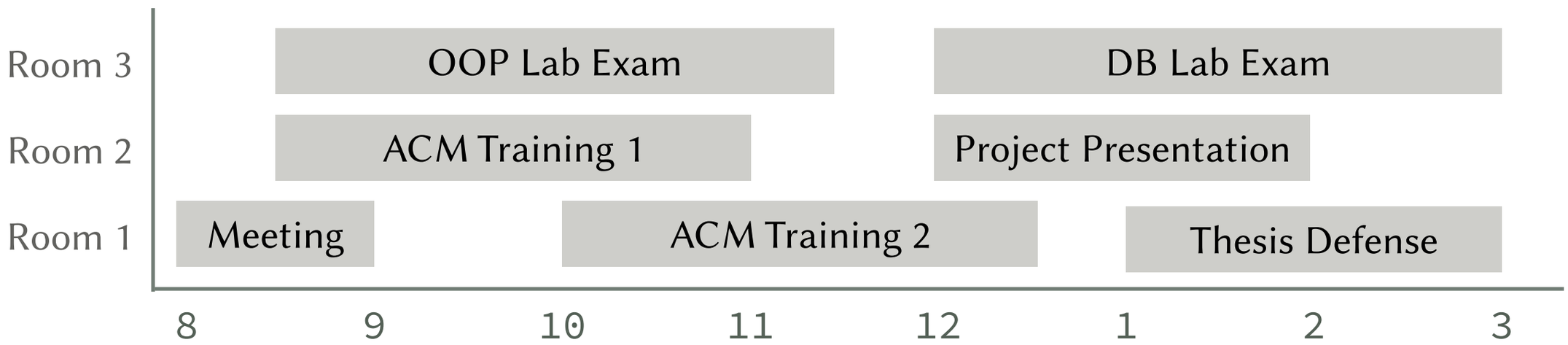
There is no known optimal greedy strategy for the weighted activity selection problem

We will learn a **dynamic programming** solution for the problem later!

Interval Partitioning

Problem. Given n activities each requiring an exclusive use of a room, find the *minimum number of rooms* such that no two activities taking place in the same room have overlapping times.

Example. At least three rooms are needed for the following activities.



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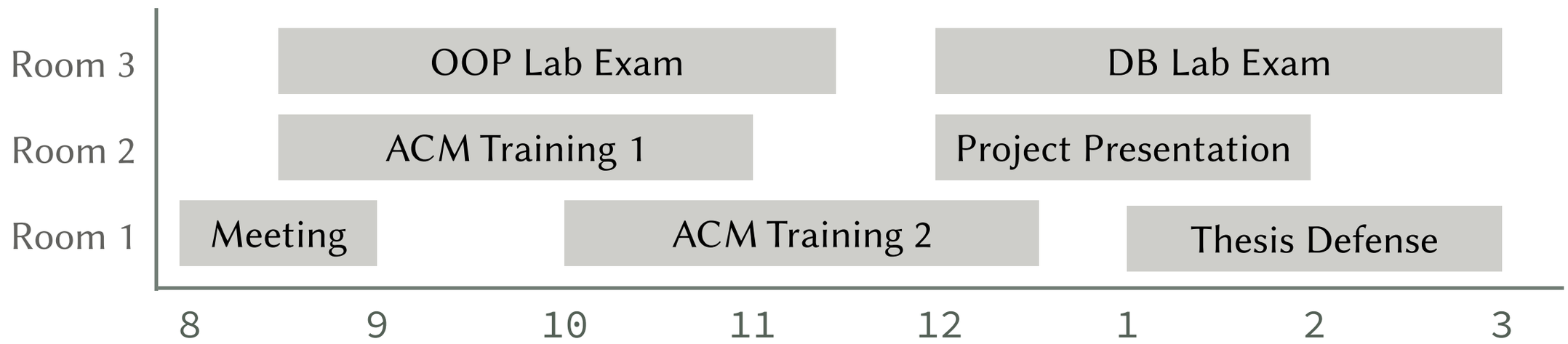
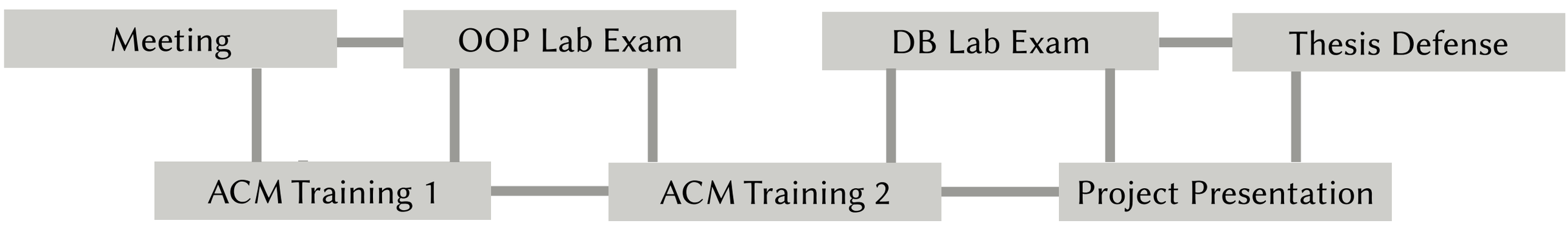


Another Formulation. Color the activities such that activities with links have different colors. What is the minimum number of colors needed?



Interval Partitioning (AKA Interval Graph Coloring)

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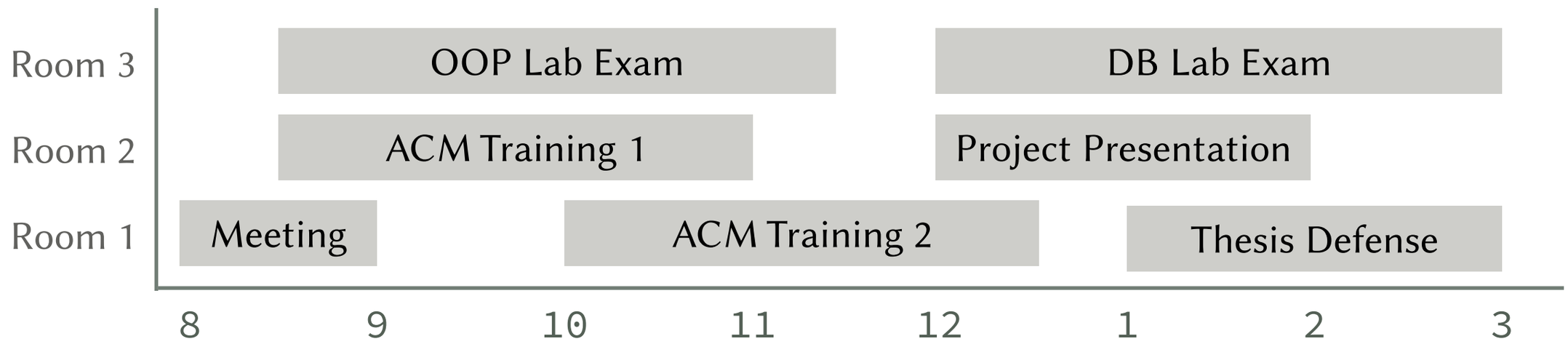

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Interval Partitioning: Greedy Algorithm

Algorithm Sketch.

- The greedy algorithm **orders** the activities based on some strategy and then processes activities one by one.

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```
ASSIGN-ROOMS(a[])

Sort the activities in a[] using some criterion
Let rooms[] be an array of lists
```

Note. This pseudocode is meant to explain the idea of the greedy algorithm only. The actual implementation of the algorithm will be different.

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   for r=0 to rooms.size-1:
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         any activity in rooms[r]:
         break
   Add a[i] to rooms[r]
```

Assuming r = a.size if the break statement does not execute

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Interval Partitioning: Unsuccessful Greedy Strategies

Example

Optimal

Greedy Strategy 1. Shortest duration first.

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Interval Partitioning: Unsuccessful Greedy Strategies

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Greedy Strategy 2. Earliest finish time first.

This could have been colored red also, but according to our algorithm, the first fitting room is picked.

Interval Partitioning: Unsuccessful Greedy Strategies

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Optimal

Greedy Strategy 1. Shortest duration first.

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Interval Partitioning: Unsuccessful Greedy Strategies

Example

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Greedy Strategy 3. Fewest conflicts first.

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Interval Partitioning: A successful Greedy Strategy

Example

Optimal

Greedy Strategy 4. Earliest start time first.

Interval Partitioning: A successful Greedy Strategy

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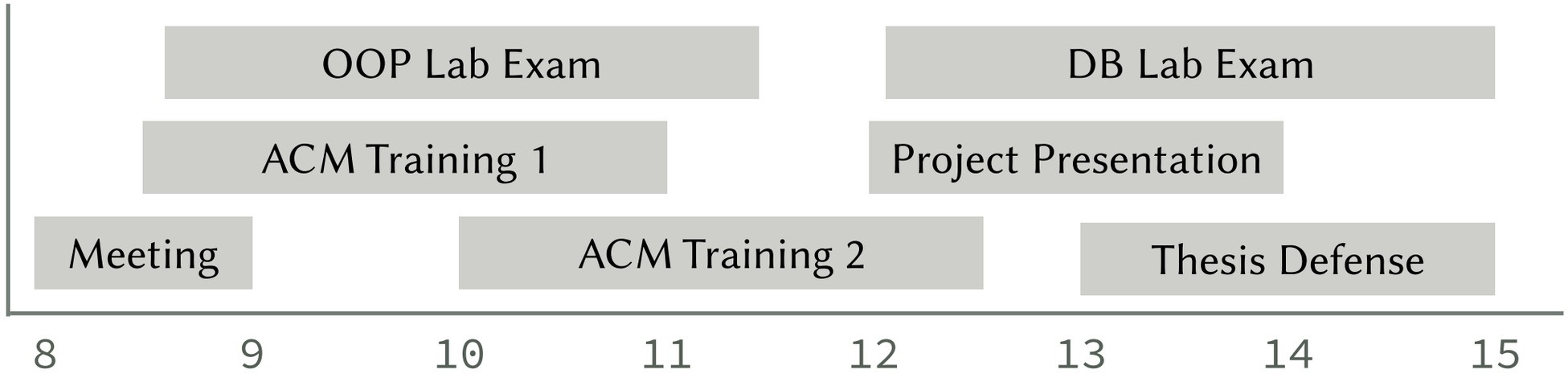

Interval Partitioning: A successful Greedy Strategy

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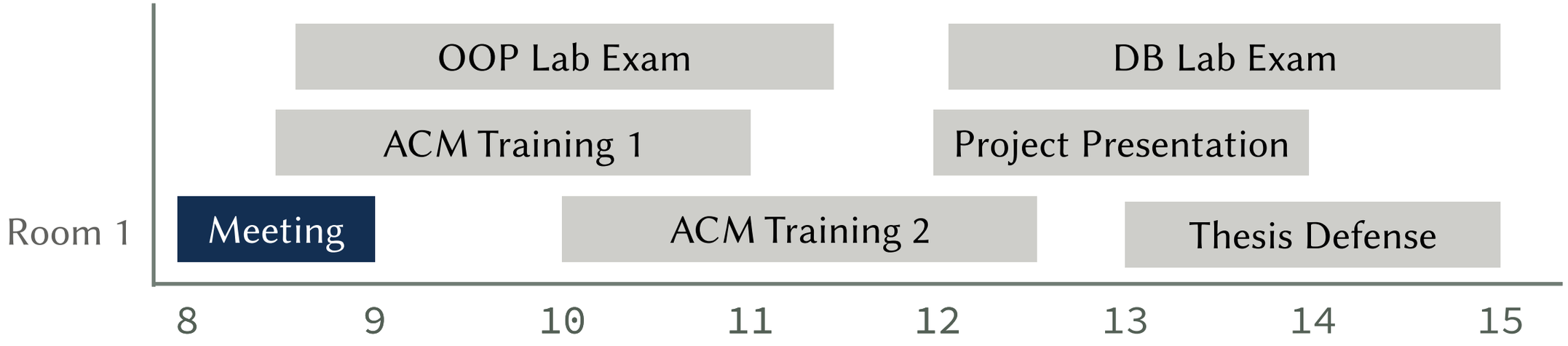

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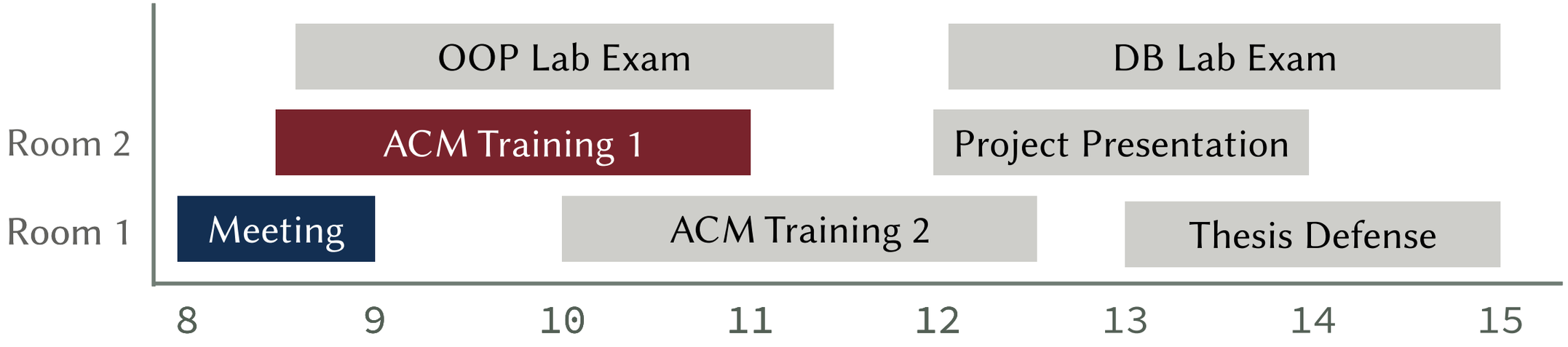

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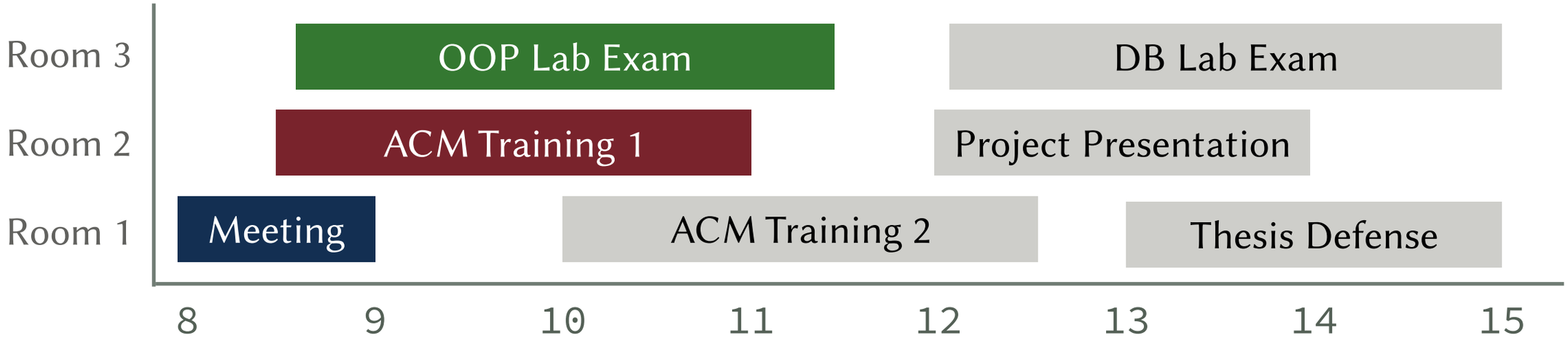

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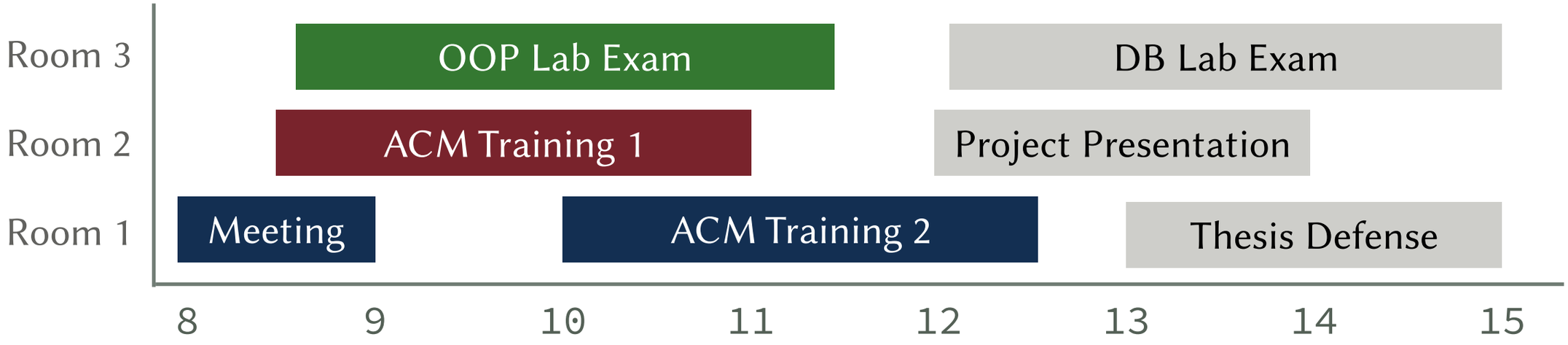

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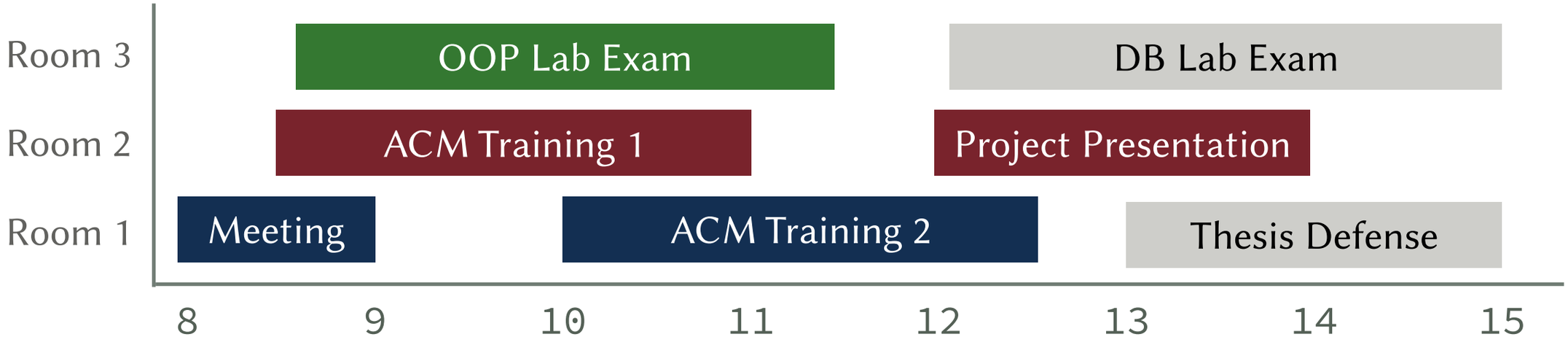

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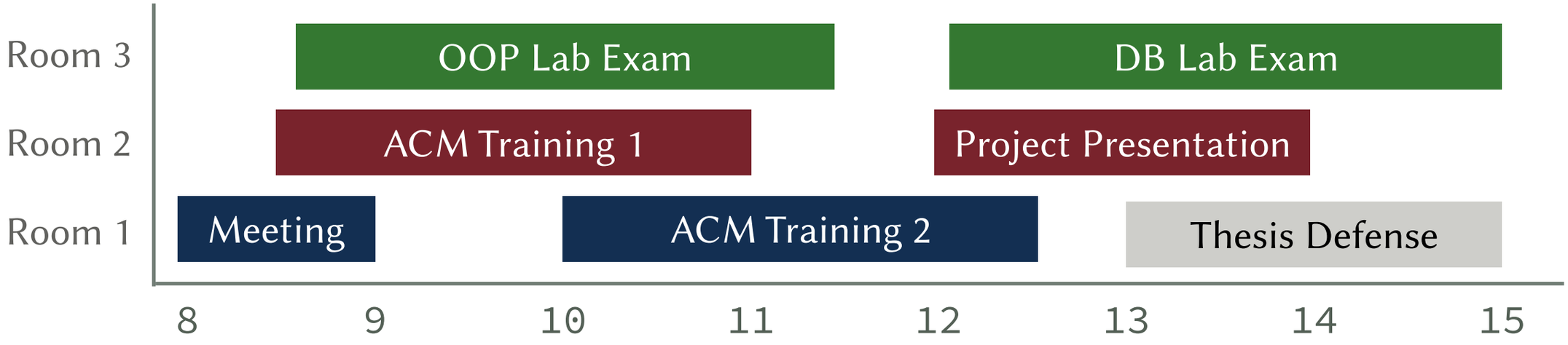

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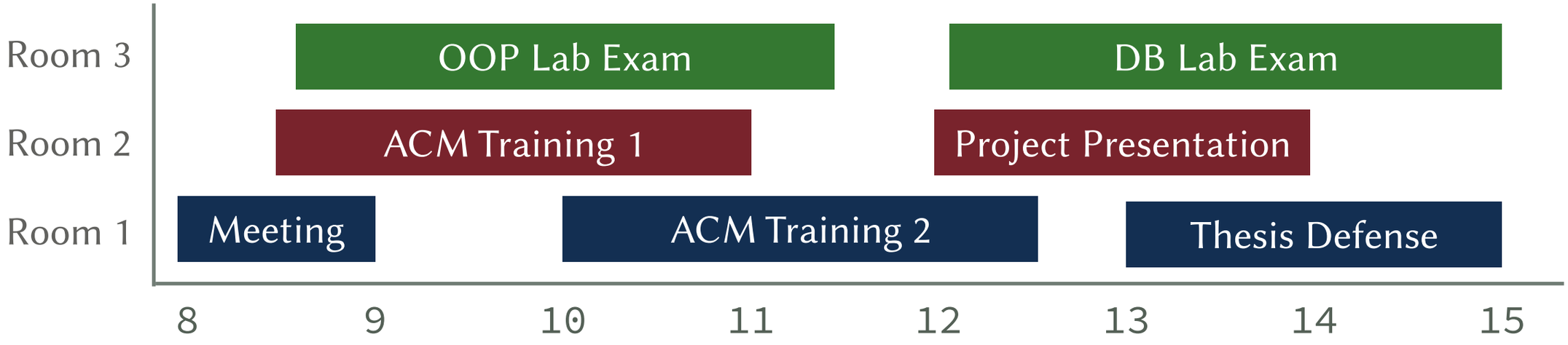

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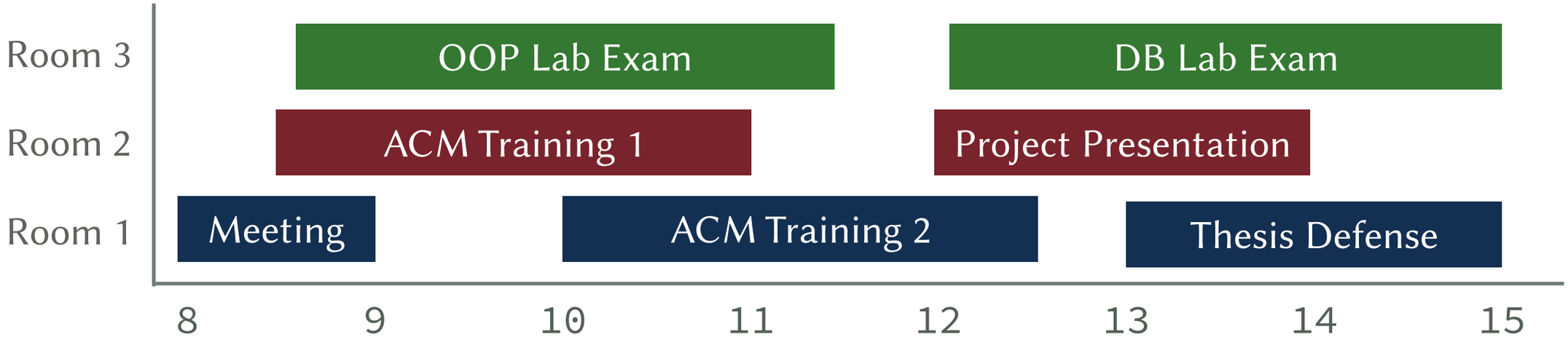
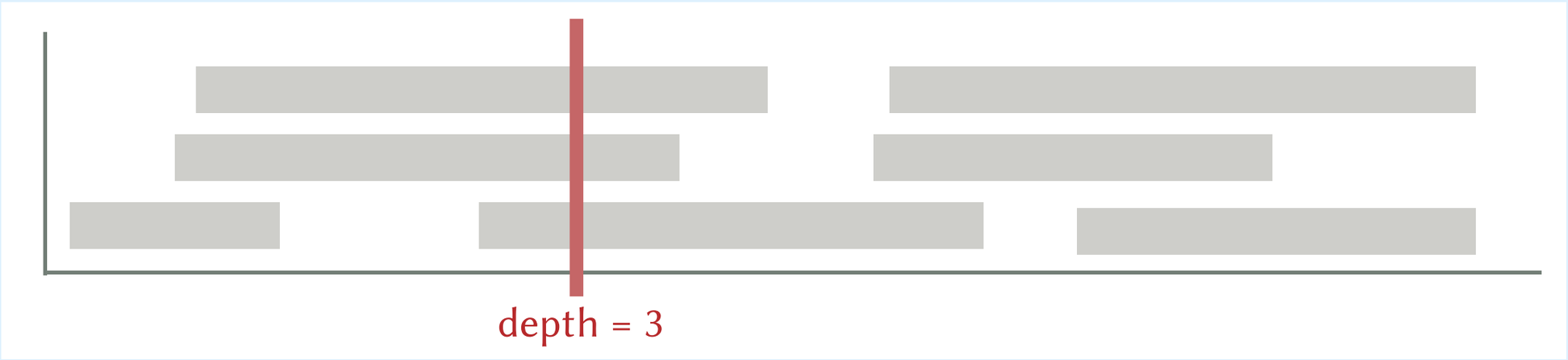


Depth. Maximum number of activities at a certain point in time.

key Observation. Number of classrooms needed ≥ depth.



Interval Partitioning: Earliest Start Time Correctness

Observation. The greedy algorithm does not schedule overlapping activities in the same room.

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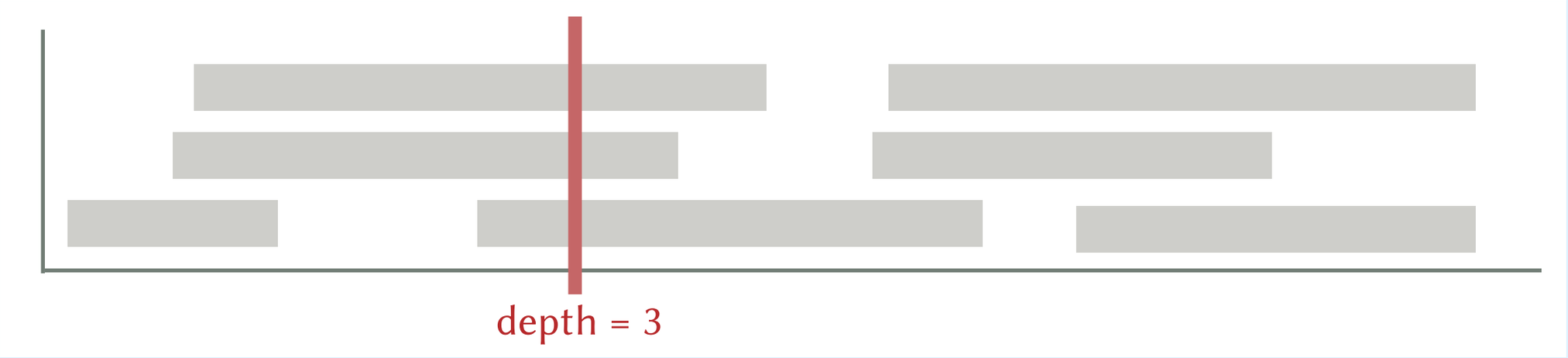

Interval Partitioning: Earliest Start Time Correctness

Observation. The greedy algorithm does not schedule overlapping activities in the same room.

Claim. The number of rooms used by the earliest-start-time strategy = **depth**.

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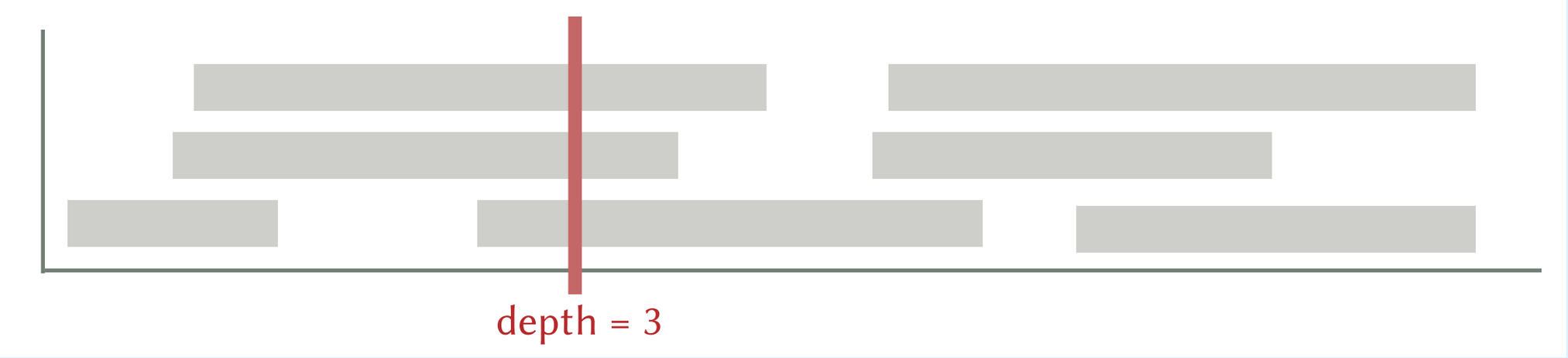

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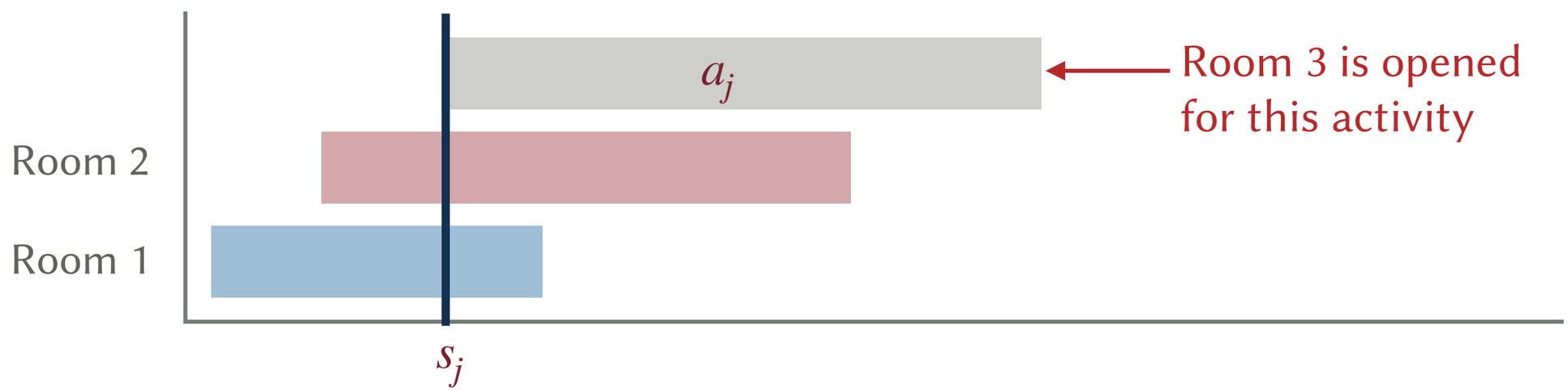

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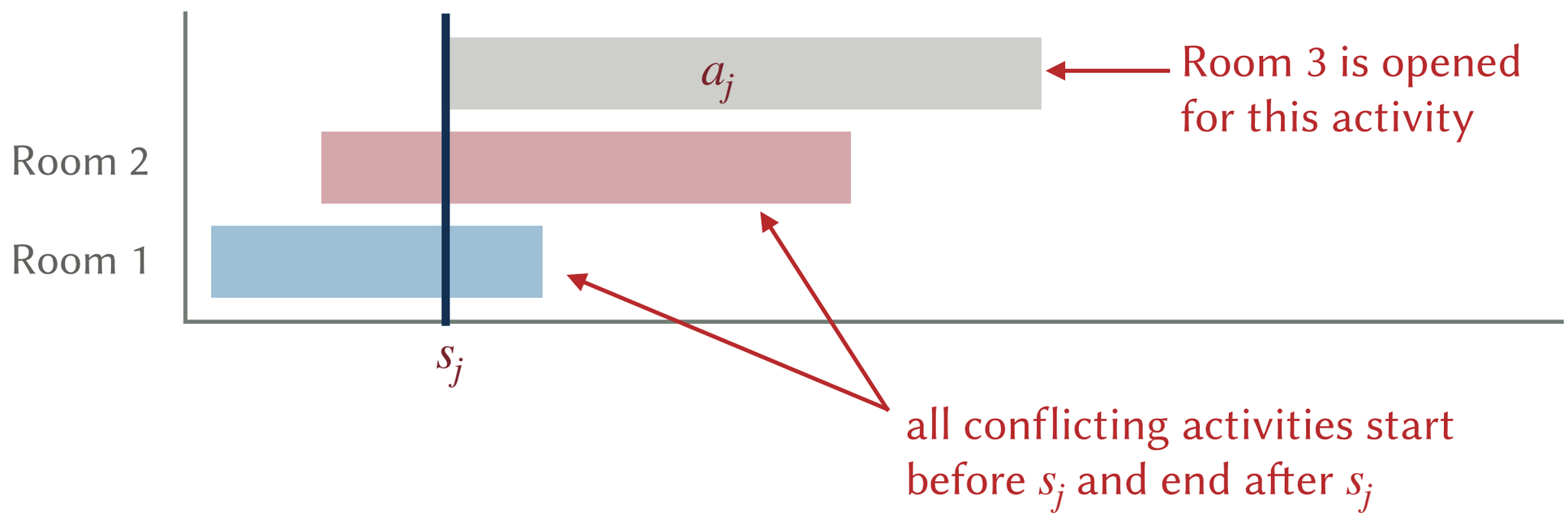

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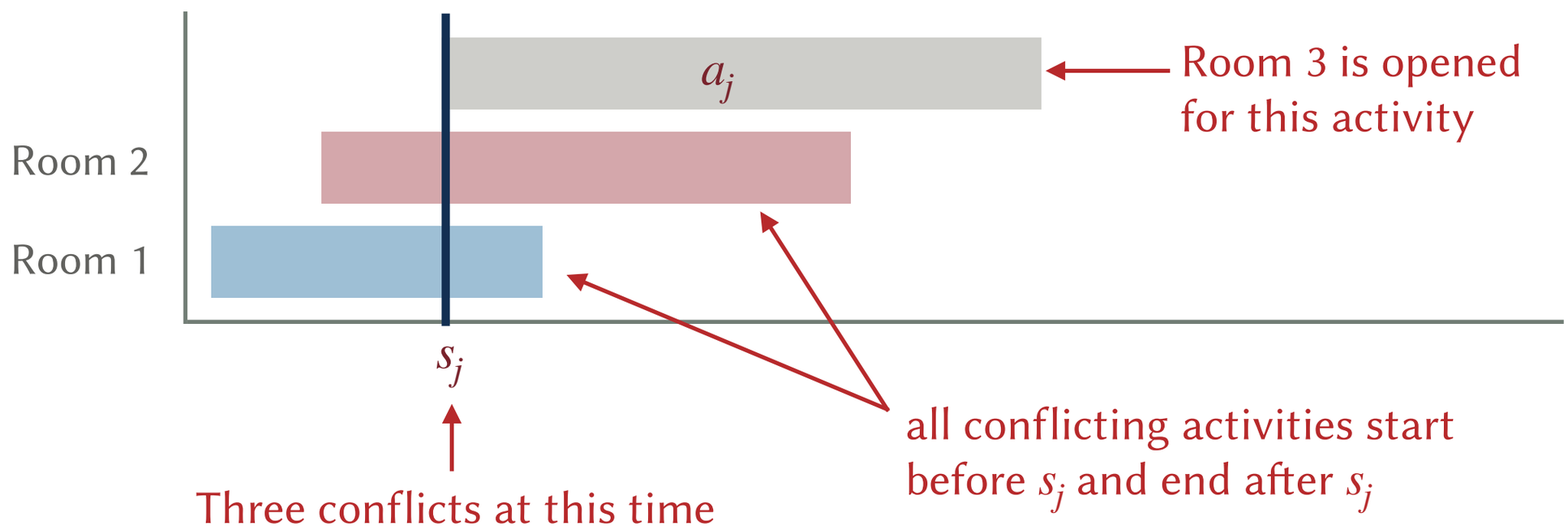

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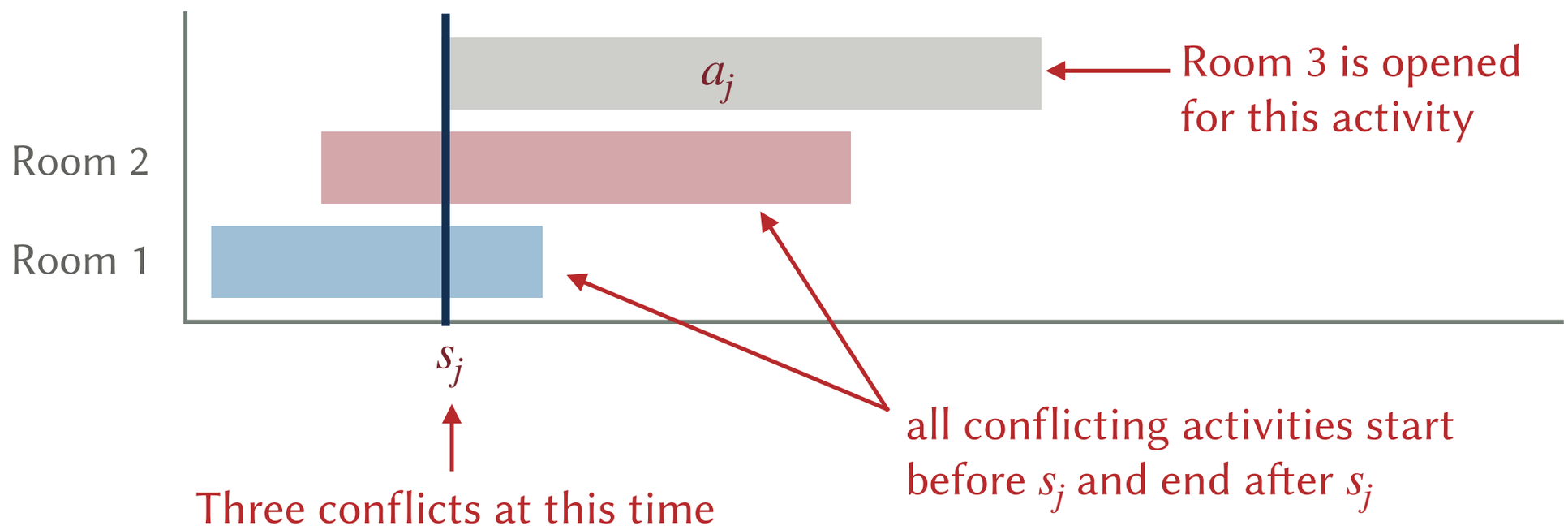
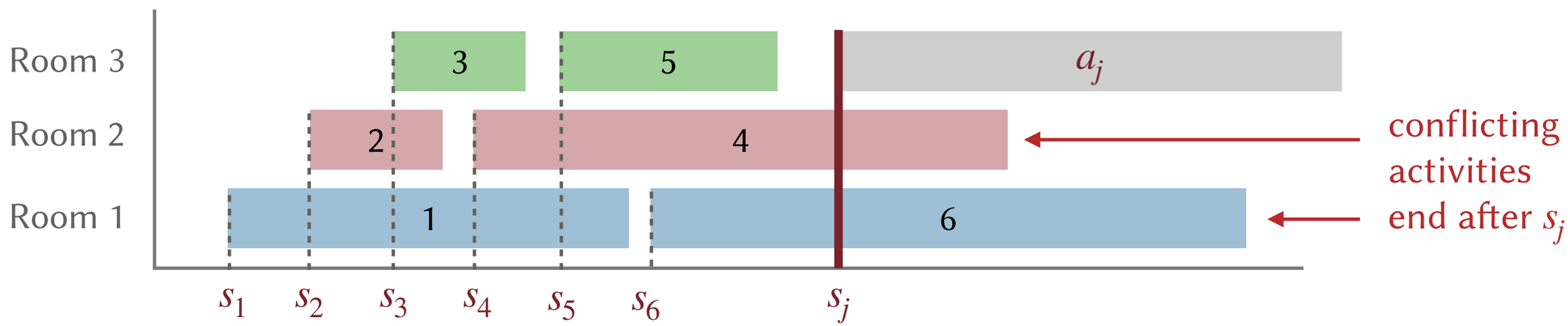


Note. Other greedy strategies do not guarantee that the k conflicts (that caused room k to open) are all at the same point in time.

Interval Partitioning: Efficient Implementation

Observation. To check if a new activity a_j conflicts with any of the activities in a room, we only need to compare s_j to the latest finish time in the room, because:

- all activities in the room start before s_j
  (because of the earliest-start-time strategy)
- a_j conflicts with an activity in the room only if a finish time is greater than s_j.
  (if $s_j >$ the latest finish time in the room, then there are no conflicts)



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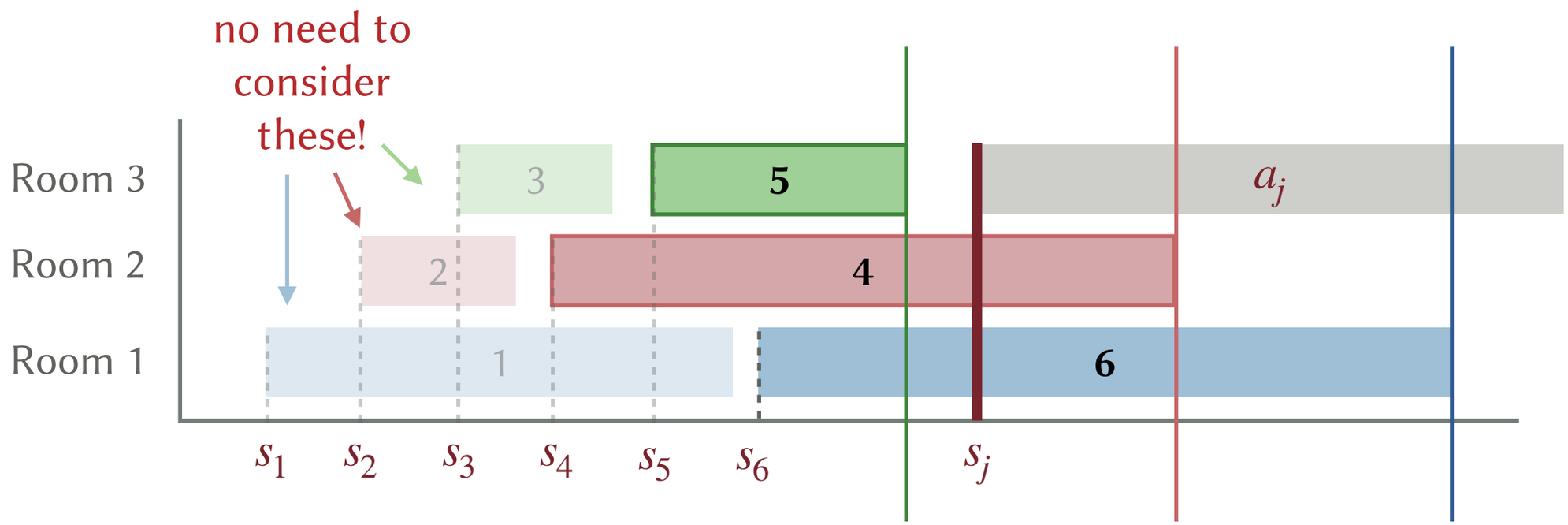

If there is a conflict with a_j in a room, then it is with the activity that has the latest finish time

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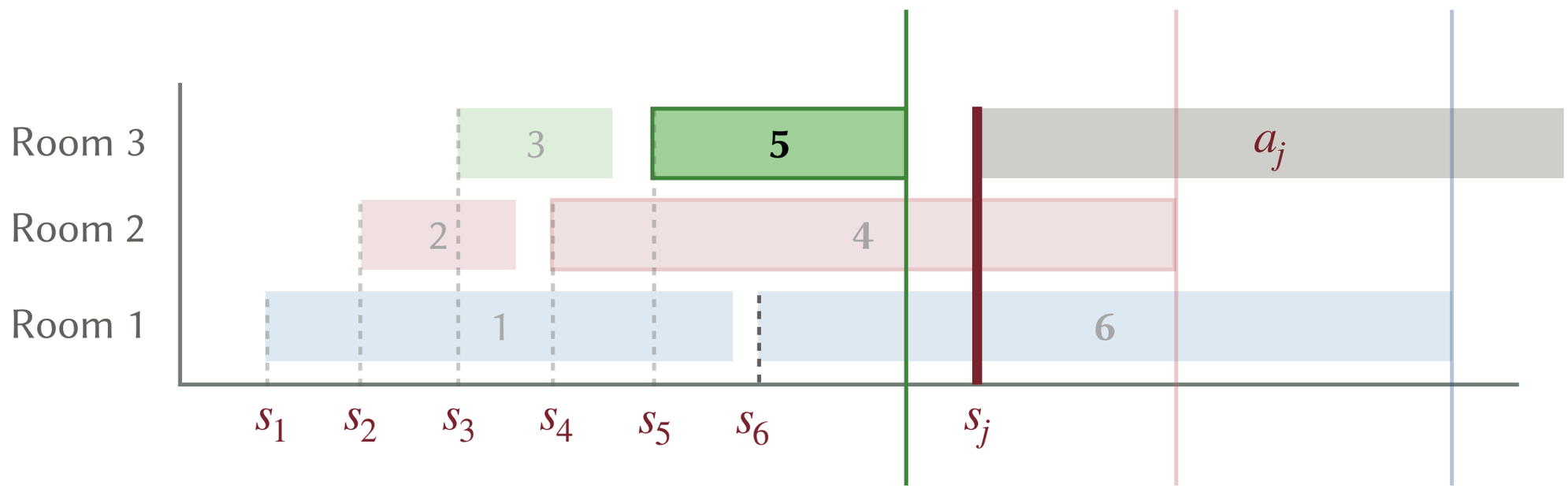

Note. We only need to check the room that **finishes first** among all the rooms!

Interval Partitioning: Efficient Implementation

COUNT-ROOMS(a[])

```
Sort the activities in a[] by start time
```

Assume each activity is an object that stores its start time and finish time

Interval Partitioning: Efficient Implementation

COUNT-ROOMS(a[])

```
Sort the activities in a[] by start time
Let rooms be a min-priority queue
    of objects of type Room
```

The key is the latest finish time in the room

Assume each room stores:
- a list of objects of type "activity"
- the latest finish time of its activities

Interval Partitioning: Efficient Implementation

COUNT-ROOMS(a[])

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Sort the activities in a[] by start time
Let rooms be a min-priority queue
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Create a new room, insert a[0] into it and
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COUNT-ROOMS(a[])

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Let rooms be a min-priority queue
    of objects of type Room

Create a new room, insert a[0] into it and
insert the new room into rooms.

for i=1 to a.size-1:
    minRoom = rooms.MIN()
    latest = minRoom.LATEST-FINISH-TIME()
```

The key is the latest finish time in the room

compare only to the earliest finishing room

Interval Partitioning: Efficient Implementation

```
COUNT-ROOMS(a[])

Sort the activities in a[] by start time
Let rooms be a min-priority queue
    of objects of type Room

Create a new room, insert a[0] into it and
insert the new room into rooms.

for i=1 to a.size-1:
    minRoom = rooms.MIN()
    latest = minRoom.LATEST-FINISH-TIME()

    if a[i].start >= latest:
        minRoom.INSERT(a[i])
        rooms.UPDATE(minRoom)
```

The key is the latest finish time in the room

compare only to the earliest finishing room

a room with no conflicts is found

the heap needs to be updated based on the new latest finish time in the room

Interval Partitioning: Efficient Implementation

```
COUNT-ROOMS(a[])

Sort the activities in a[] by start time
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    if a[i].start >= latest:
        minRoom.INSERT(a[i])
        rooms.UPDATE(minRoom)
    else
        Create a new room, insert a[i] into it
        and insert the new room into rooms.
```

The key is the latest finish time in the room

compare only to the earliest finishing room

a room with no conflicts is found

if a[i] does not fit in the earliest finishing room, then it does not fit in any room

Interval Partitioning: Efficient Implementation

```
COUNT-ROOMS(a[])

Sort the activities in a[] by start time
Let rooms be a min-priority queue
    of objects of type Room

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        rooms.UPDATE(minRoom)

    else
        Create a new room, insert a[i] into it
        and insert the new room into rooms.

return rooms.SIZE()
```

The key is the latest finish time in the room

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if a[i] does not fit in the earliest finishing room, then it does not fit in any room

Interval Partitioning: Efficient Implementation

COUNT-ROOMS(a[])

$O(n \log n)$

```
Sort the activities in a[] by start time          ← O(n log n)
Let rooms be a min-priority queue
    of objects of type Room

Create a new room, insert a[0] into it and
insert the new room into rooms.

for i=1 to a.size-1:                                ← Θ(n)
    minRoom = rooms.MIN()                           ← O(1)
    latest = minRoom.LATEST-FINISH-TIME()

    if a[i].start >= latest:
        minRoom.INSERT(a[i])                        ← O(1)
        rooms.UPDATE(minRoom)                       ← O(log n)
    else
        Create a new room, insert a[i] into it
        and insert the new room into rooms.         ← O(log n)
return rooms.SIZE()
```

in the worst case there are O(n) rooms (if d = O(n))