Ex The joint CDF of two discrete random variables is given by

$$F_{X,Y}(x,y) = \begin{cases} \frac{1}{8} & x=1,\ y=1 \quad \to P(X\le 1, Y\le 1) \\ \frac{5}{8} & x=1\ \ y=2 \quad \to P(X\le 1, Y\le 2) \\ \frac{1}{4} & x=2,\ y=1 \quad \to P(X\le 2, Y\le 1) \\ 1 & x=2,\ y=2 \quad \to P(X\le 2, Y\le 2) \end{cases}$$

Find the pmf $P(X=x, Y=y) = p_{X,Y}(x,y)$

Sol $\underbrace{P(X\le 1, Y\le 1)}_{1/8} = P(X=1, Y=1)$

$$\tfrac{1}{8} = p_{X,Y}(1,1)$$

Y \ X	1	2	$P(Y=y)$
1	$\frac{1}{8}$	$\frac{1}{8}$	$2/8$
2	$4/8$	$2/8$	$6/8$
$P(X=x)$	$5/8$	$3/8$	1

$$P(X \le 1, Y \le 2) = p(1,1) + p(1,2)$$

$$\frac{5}{8} = \frac{1}{8} + p(1,2) \quad \therefore p(1,2) = \frac{5}{8} - \frac{1}{8} = \frac{4}{8}$$

$$P(X \le 2, Y \le 1) = p(1,1) + p(2,1)$$

$$\frac{1}{4} = \frac{1}{8} + p(2,1) \quad \therefore p(2,1) = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$

$$P(X \le 2, Y \le 2) = p(1,1) + p(1,2) + p(2,1) + p(2,2)$$

$$1 = \frac{1}{8} + \frac{4}{8} + \frac{1}{8} + p(2,2) \quad \therefore p(2,2) = 1 - \frac{1}{8} - \frac{4}{8} - \frac{1}{8} = \frac{2}{8}$$

(2) Find the marginal pmf of X, Y

Table

(Chapter 6) (Discrete) Functions of random variable

Ex The joint pmf of discrete random variables X and Y is given by the table

X \ Y	2	4	5
1	2/24	1/24	1/24
2	4/24	2/24	3/24
3	6/24	3/24	2/24

① Find the $P(X+Y=5)$

Sol Let $U = X+Y$

X \ Y	2	4	5
1	3	5	6
2	4	6	7
3	5	7	8

} $X+Y$

U	3	4	5	6	7	8	
$P(U=u)$	$\frac{2}{24}$	$\frac{4}{24}$	$\frac{7}{24}$	$\frac{3}{24}$	$\frac{6}{24}$	$\frac{2}{24}$	$\frac{24}{24}=1$

$p(1,4)+p(3,2)$

$$P(X+Y=5) = P(U=5) = 7/24$$

② Find $P(Y - X \leq 2)$

$X \backslash Y$	2	4	5
1	1	3	4
2	0	2	3
3	-1	1	2

Let $W = Y - X$

W	-1	0	1	2	3	4	
$P(W=w)$	$\frac{6}{24}$	$\frac{4}{24}$	$\frac{5}{24}$	$\frac{4}{24}$	$\frac{4}{24}$	$\frac{1}{24}$	$\frac{24}{24} = 1$

$$P(Y - X \leq 2) = P(W \leq 2) = P(W = -1) + P(W = 0) + P(W = 1) + P(W = 2)$$

$$= \frac{1}{24}\left[6 + 4 + 5 + 4\right] = 19/24$$

OR

$$P(W \leq 2) = 1 - P(W > 2) = 1 - P(W \geq 3) = 1 - \left[\frac{4}{24} + \frac{1}{24}\right] = 19/24$$

③ Let $Z = XY$ Find the pmf of Z

$X \backslash Y$	2	4	5
1	2	4	5
2	4	8	10
3	6	12	15

Z	2	4	5	6	8	10	12	15	
$P(Z=z)$	$\frac{2}{24}$	$5/24$	$\frac{1}{24}$	$\frac{6}{24}$	$2/24$	$3/24$	$3/24$	$2/24$	$\frac{24}{24}$

④ Find $P(XY>10) = P(Z>10) = P(Z \geq 12) = P(Z=12)$
$+ P(Z=15) = \frac{3}{24} + \frac{2}{24} = 5/24$

⑤ $F_Z(9) = P(Z \leq 9) = P(Z \leq 8) = P_Z(8) + P_Z(6) + P_Z(5) + P_Z(4) + P_Z(2)$

$\underline{\underline{OR}}$ $P(Z \le 9) = 1 - P(Z > 9) = 1 - P(Z \ge 10)$

$= 1 - [P(Z=10) + P(Z=12) + P(Z=15)]$

$= 1 - \frac{3}{24} - \frac{3}{24} - \frac{2}{24} = \frac{16}{24}$

6 $E(XY) = E(Z) = \sum_{\forall z} z\,P(Z=z) = \frac{1}{24}\Big[(2)(2) + 4(5) + 5(1) + 6(6)$

$+ 8(2) + 10(3) + 12(3)$

$+ 15(2)\Big] = \frac{177}{24}$

OR $E(XY) = \sum_{\forall y=2,4,5} \sum_{\forall x=1,2,3} xy\,P(X=x, Y=y)$

$= \sum_{y=2,4,5} \Big[y\,P(X=1, Y=y) + 2y\,P(X=2, Y=y) + 3y\,P(X=3, Y=y)\Big]$

sub $y=2$
then $y=4$
then $y=5$
then add them together
9 terms

Remarks

1. $E(aX+bY) = aE(X) + bE(Y)$
2. $E(aX+b) = aE(X) + b$
3. $Var(aX+b) = a^2 Var(X)$
4. $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$

where $Cov(X,Y) = E(XY) - E(X)E(Y)$

If X, Y are independent $E(XY) = E(X)E(Y)$

Hence $Cov(X,Y) = 0$

Ex The pmf of a discrete random variable X is given by

X	-2	-1	0	1	3
$P(X=x)$	$\frac{3}{14}$	$\frac{4}{14}$	$\frac{1}{14}$	$\frac{4}{14}$	$\frac{2}{14}$

1. Compute $P(|X|=1)$

$$=\frac{8}{14}$$

$\lvert X\rvert$	0	1	2	3	
$P(\lvert X\rvert=x)$	$\frac{1}{14}$	$\boxed{\frac{8}{14}}$	$\frac{3}{14}$	$\frac{2}{14}$	$\frac{14}{14}$

$P(X=-1)+P(X=1)$

② $P(X^2 \geq 4)$

X^2	0	1	4	9	
$P(X^2 = x)$	$\frac{1}{14}$	$\frac{8}{14}$	$\frac{3}{14}$	$\frac{2}{14}$	$\frac{14}{14}$

$$P(X^2 \geq 4) = P(X^2 = 4) + P(X^2 = 9)$$

$$= \frac{1}{14}\left[3 + 2\right] = 5/14$$

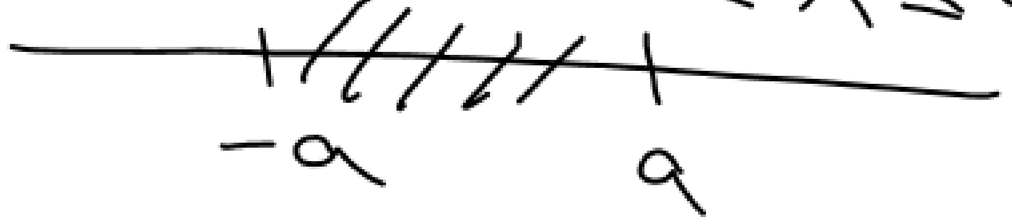
OR $P(\sqrt{X^2} \geq \sqrt{4})$

$$= P(|X| \geq 2)$$

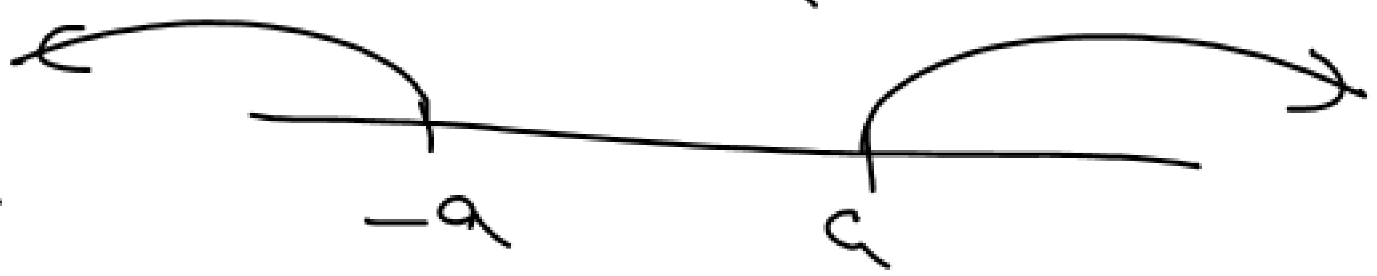
$$= P(X \leq -2) + P(X \geq 2)$$

$$= \frac{3}{14} + \frac{2}{14} = 5/14$$

$|X| \leq a$ $\quad -a \leq X \leq a$



$|X| \geq a$ $\quad X \leq -a$, $x \geq a$



③ $F_{|X|}(\sqrt{3}) = P(|X| \le \sqrt{3}) = P(|X|=0) + P(|X|=1)$

1.7

$= \frac{1}{14}[1 + 8] = 9/14$

④ $E(|X|) = \sum_{\forall x} |x| \, P(X=x)$

OR

Let $|X| = Z$

$E(Z) = \sum_{\forall z} z \, P(Z=z) = \frac{1}{14}\left[0 + 1(8) + 2(3) + 3(2)\right] = \frac{20}{14} = \frac{10}{7}$

5. $E(3X-2) = 3E(X) - 2$

$$E(X) = \sum_{\forall x} x\,P(X=x) = \frac{1}{14}\left[(-2)(3) + (-1)(4) + 0 + 1(4) + 3(2)\right] = 0$$

$$E(3X-2) = 3(0) - 2 = -2$$

OR $$E(3X-2) = \sum_{\forall x = -2, -1, 0, 1, 3} (3x-2)\,P(X=x)$$

Ex If X and Y are independent discrete random variables whose pmfs are given by

X	0	1
$P(X=x)$	$3/4$	$1/4$

Y	-1	5
$P(Y=y)$	$3/8$	$5/8$

Find the joint pmf of X, Y

Sol $P(0,-1) = P(X=0)P(Y=-1) = \frac{3}{4}\,\frac{3}{8} = \frac{9}{32}$

$P(1,-1) = P(X=1)\,P(Y=-1) = \frac{1}{4}\,\frac{3}{8} = 3/32$

$Y \backslash X$	0	1
-1	$9/32$	$3/32$
5	$15/32$	$5/32$

Ex

Assume X, Y are independent random variables whose pmfs are given by

2nd Exam

$$P(X=x)=\frac{x^2}{5} \quad x=1,2 \qquad P(Y=y)=\frac{y+1}{15} \quad y=2,4,6$$

Find the joint pmf of X and Y

Sol $$P(X=x, Y=y)=P(X=x)\,P(Y=y)=\frac{x^2}{5}\,\frac{y+1}{15} \quad \begin{array}{l} x=1,2 \\ y=2,4,6 \end{array}$$