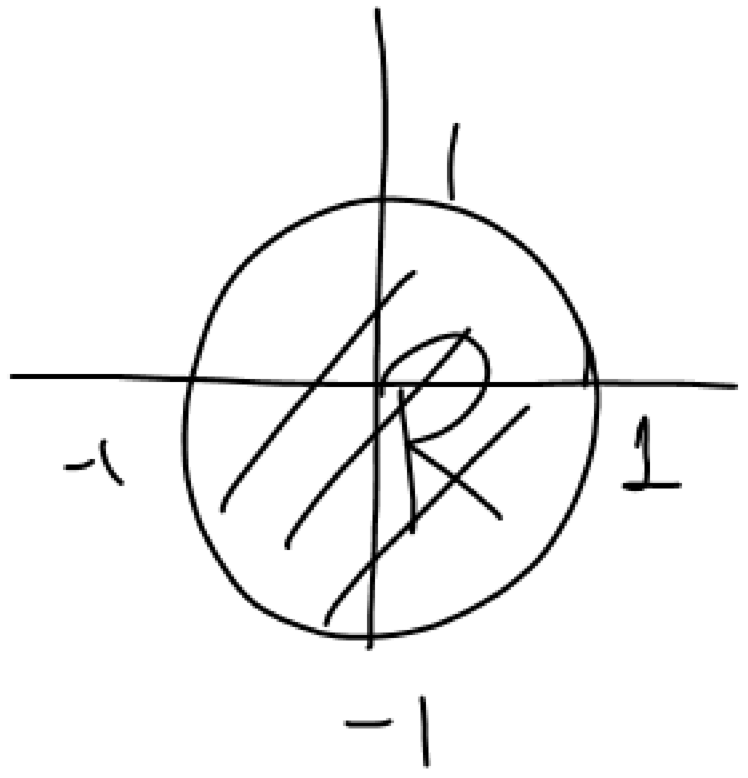
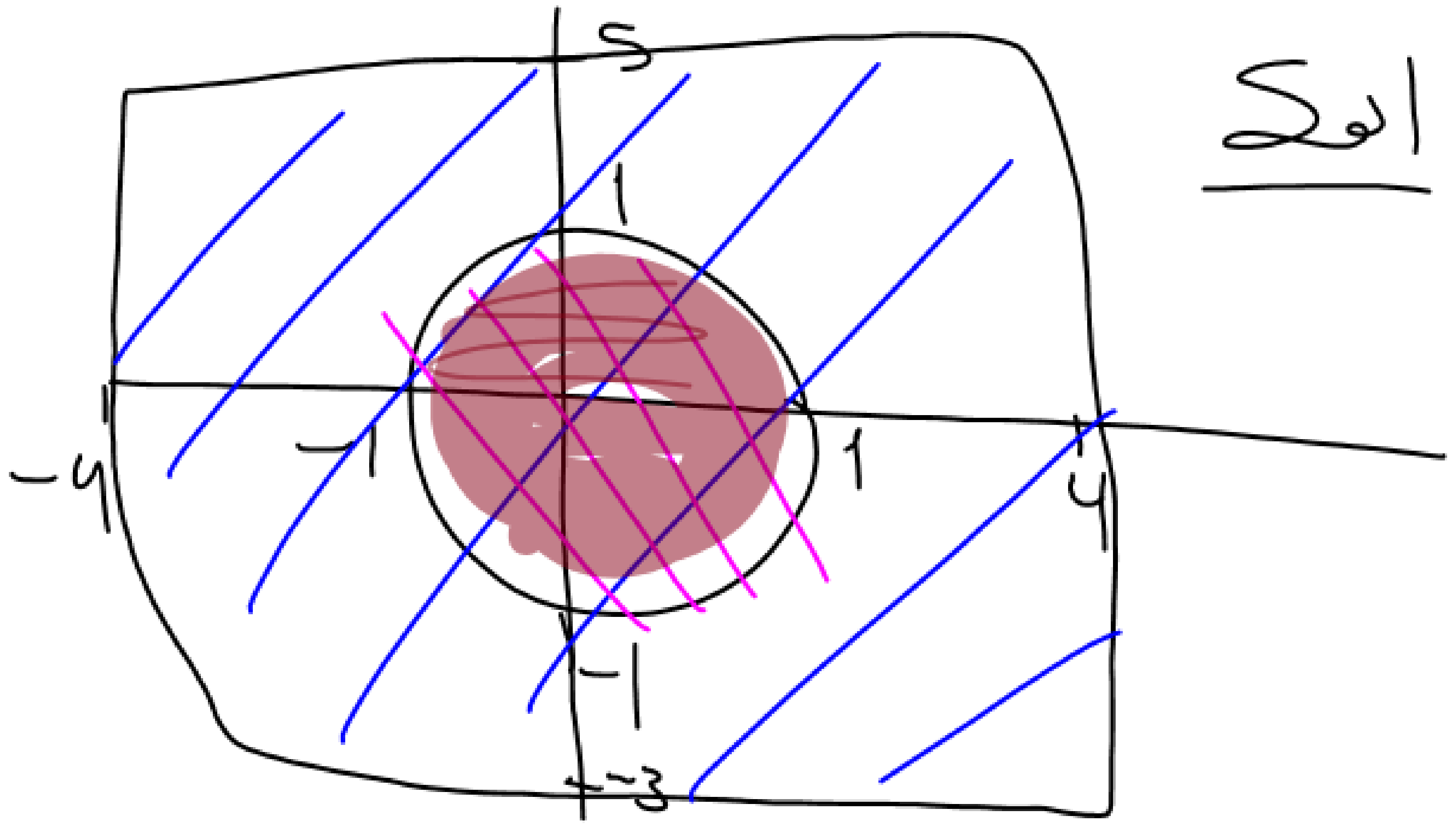
Ex $\quad f_{X,Y}(x,y) = \frac{3}{2\pi}\sqrt{x^2+y^2}$



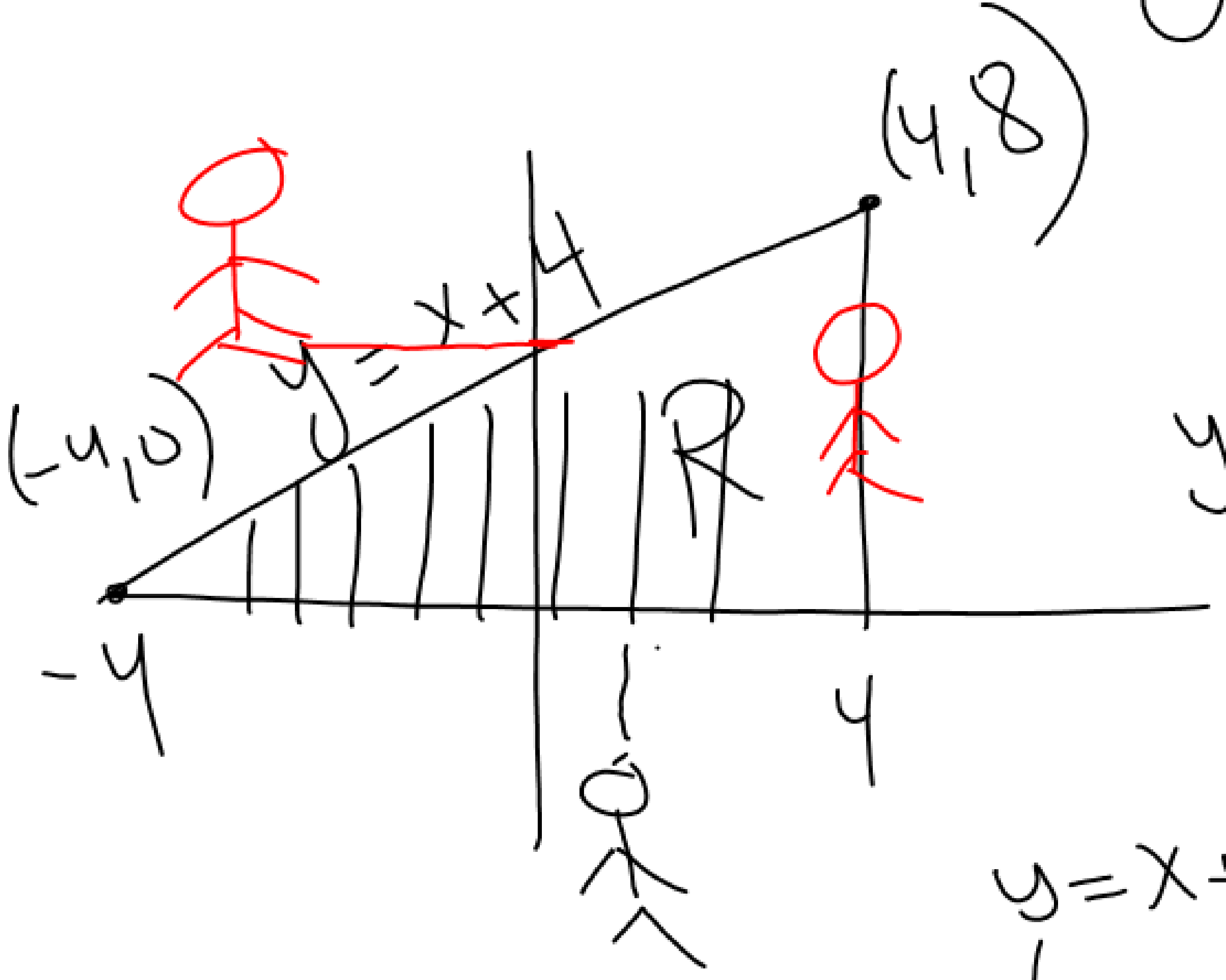
④ Find the prob. that (x,y) is inside the square $x \in [-4,4]$ and $y \in [-3,5]$



Sol $P(-4 \le X \le 4, -3 \le Y \le 5) = 1$

Ex The joint pdf of X and Y is $f_{X,Y}(x,y) = \frac{x^2+y}{256}$

$(x,y) \in R$ where R is the shaded region in the figure



$m = \frac{8-0}{4-^-4} = \frac{8}{8} = 1$

$y-0 = 1(x-^-4)$

$y = x+4$

① Find the marginal pdf of X and Y

Sol $f_X(x) = \frac{1}{256}\int_{y=0}^{y=x+4} (x^2+y)\, dy = \frac{1}{256}\left[x^2y + \frac{y^2}{2}\Big|_{y=0}^{y=x+4}\right] = \frac{1}{256}\left[x^2(x+4) + \frac{(x+4)^2}{2}\right]$

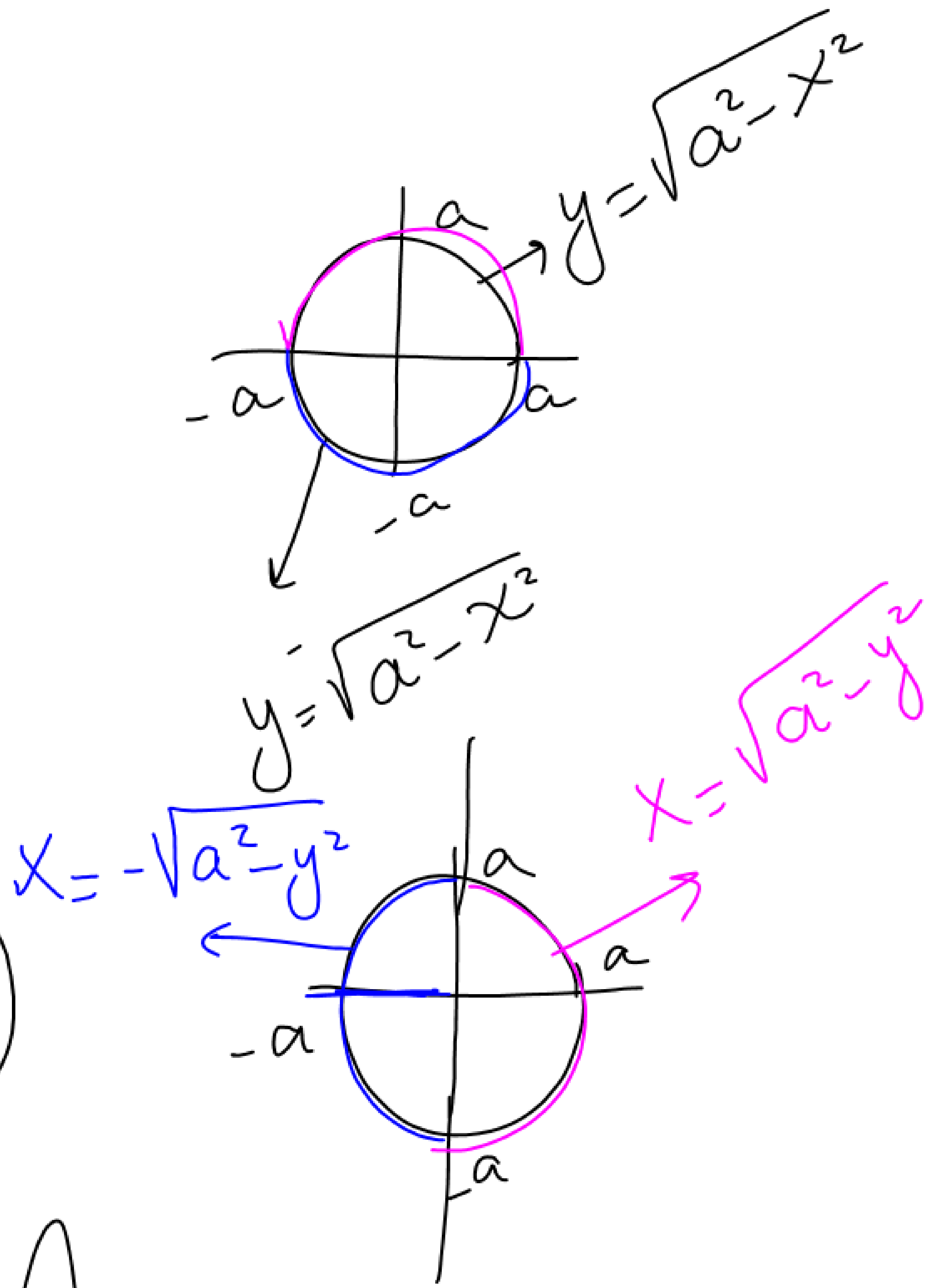
$$f_X(x) = \begin{cases} \cancel{4} & -4 \le x \le 4 \\ 0 & \text{o.w} \end{cases}$$

$$f_Y(y) = \int_{x=y-4}^{x=4} \frac{1}{256}(x^2+y)\,dx = \frac{1}{256}\left[\frac{x^3}{3} + yx \Big|_{x=y-4}^{x=4}\right]$$

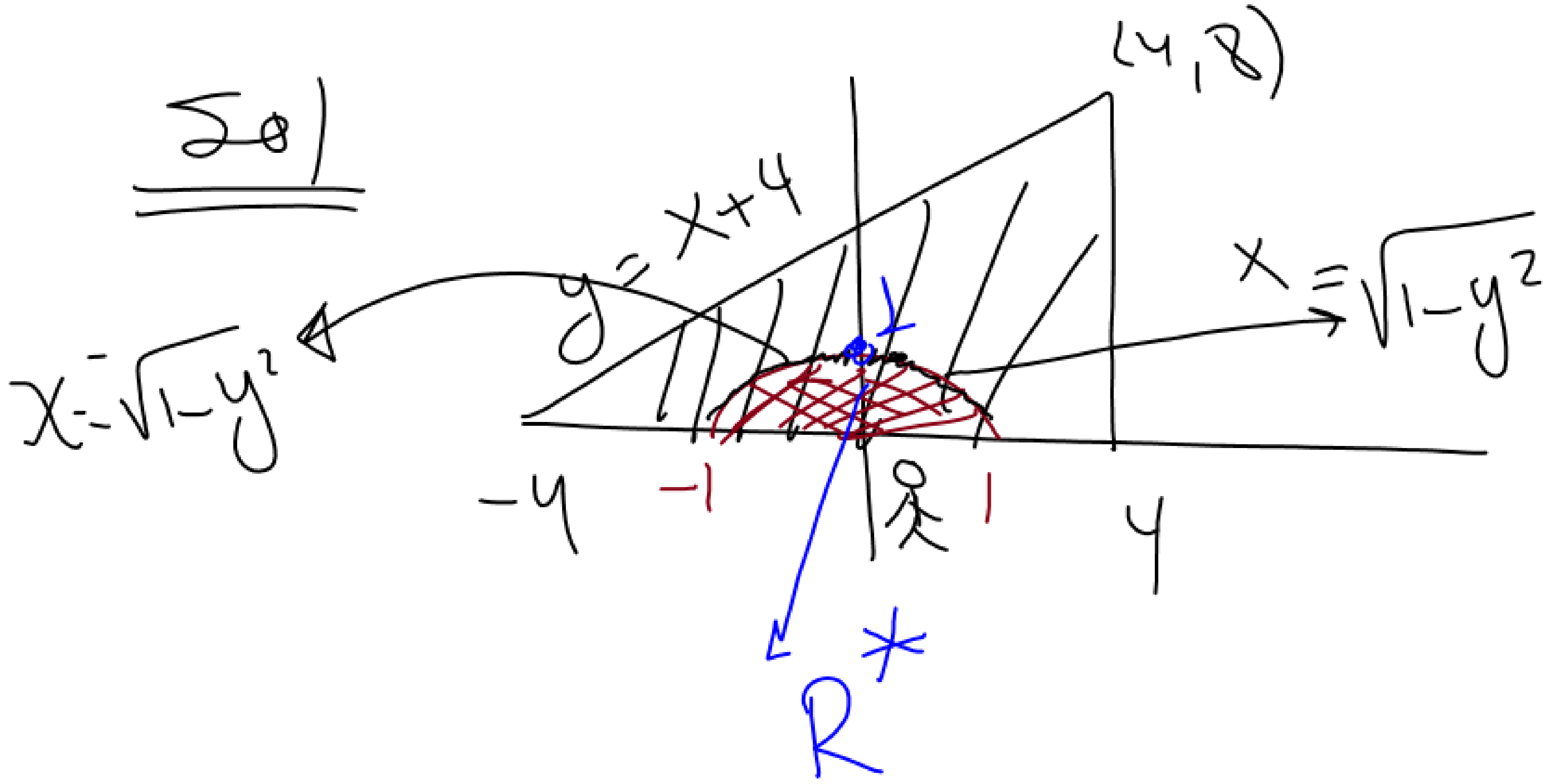
$$= \frac{1}{256}\left[\frac{(4)^3}{3} + 4y - \frac{(y-4)^3}{3} - y(y-4)\right]$$

$$f_Y(y) = \begin{cases} \quad & 0 \le y \le 8 \\ 0 & \text{o.w} \end{cases}$$

② Let $r = \sqrt{x^2+y^2}$ Find $P(r \le 1)$



Sol



$$P(\underbrace{\sqrt{x^2+y^2} \le 1})$$

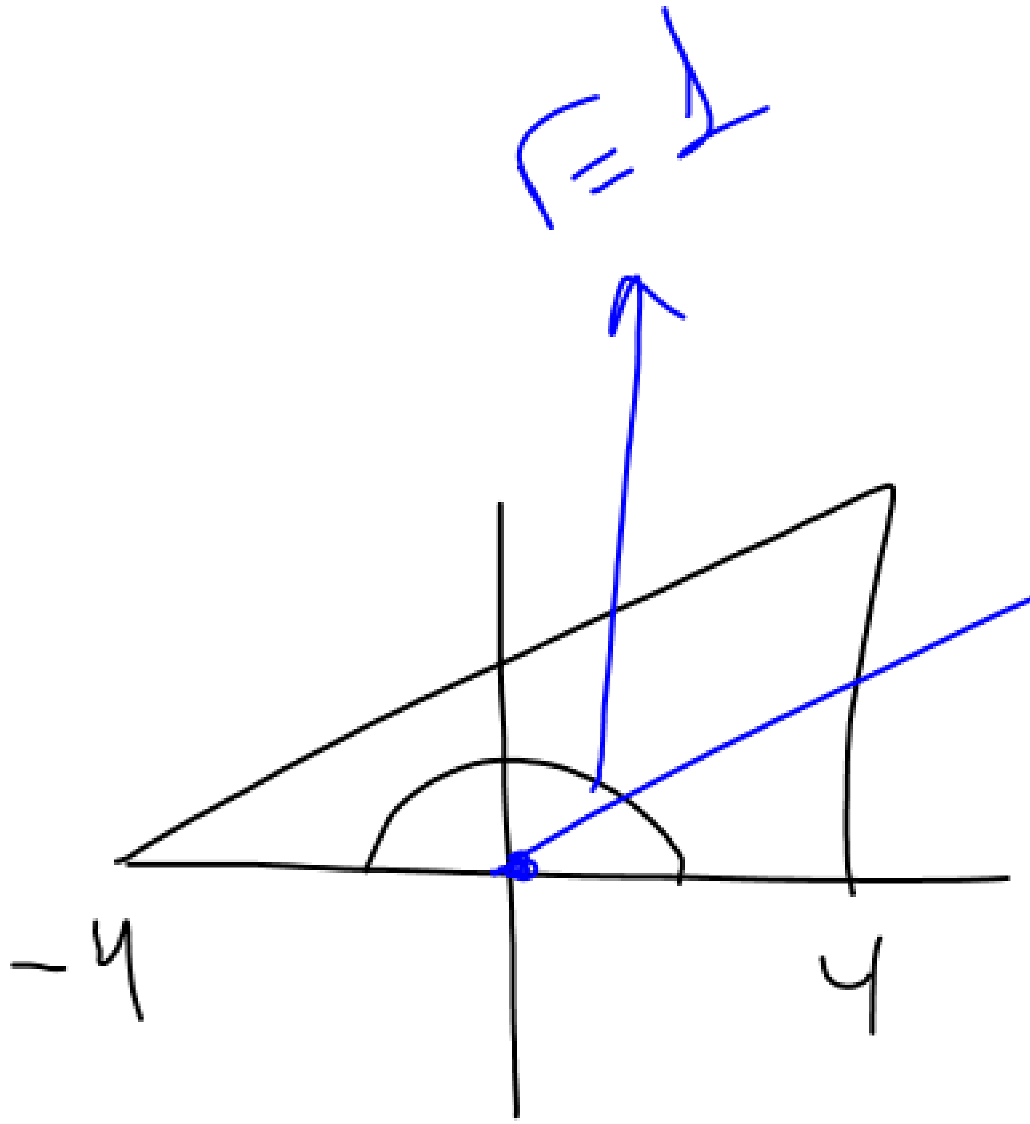
$$\iint_{R^*} \frac{1}{256}(x^2+y)\,dA$$

$$\int_{x=-1}^{x=1}\int_{y=0}^{y=\sqrt{1-x^2}} \frac{1}{256}(x^2+y)\,dy\,dx$$

OR

$$\int_{y=0}^{y=1}\int_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} \frac{1}{256}(x^2+y)\,dx\,dy$$

Not easy to evaluate



$$\frac{1}{256}\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1}(r^2\cos^2\theta + r\sin\theta)\,r\,dr\,d\theta$$

$$\boxed{x^2+y^2=a^2} \quad \boxed{r=a}$$

$$\int_{\theta=0}^{\theta=\pi}\left[\frac{r^4\cos^2\theta}{4}+\frac{r^3\sin\theta}{3}\right]_{r=0}^{r=1}d\theta = \int_{\theta=0}^{\theta=\pi}\left[\frac{\cos^2\theta}{4}+\frac{\sin\theta}{3}\right]d\theta$$

$$\int_{\theta=0}^{\theta=\pi} \left(\frac{1}{2}(1+\cos 2\theta) + \sin\theta\right) d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] - \cos\theta \Bigg|_{\theta=0}^{\theta=\pi}$$

$$= \frac{1}{2}\left[\pi + \cancel{\frac{\sin 2\pi}{2}}\right] - \underbrace{\cos\pi}_{\boxed{-1}} - \frac{1}{2}\left(0 + \cancel{\frac{\sin 0}{2}}\right) + \underbrace{\cos 0}_{1}$$

Ex Let X and Y be independent random variables such that X is an exponential with $\lambda = 5$ and Y is uniform random variable over $[0,3]$

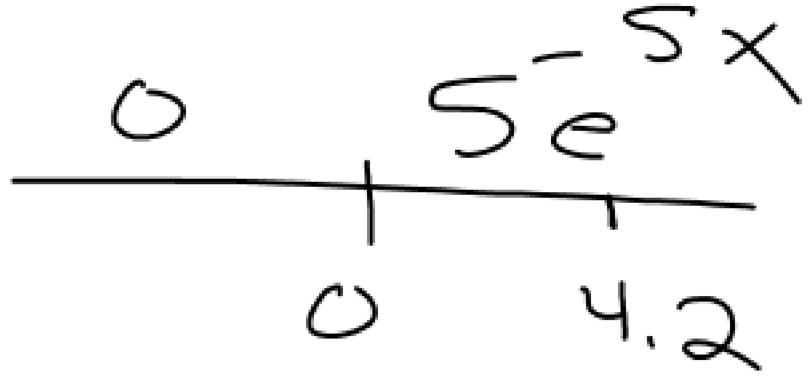
Find $P(X<4.2, Y>1.3) = P(X<4.2)P(Y>1.3)$

$= P(X<4.2)\left[1-P(Y\le 1.3)\right]$

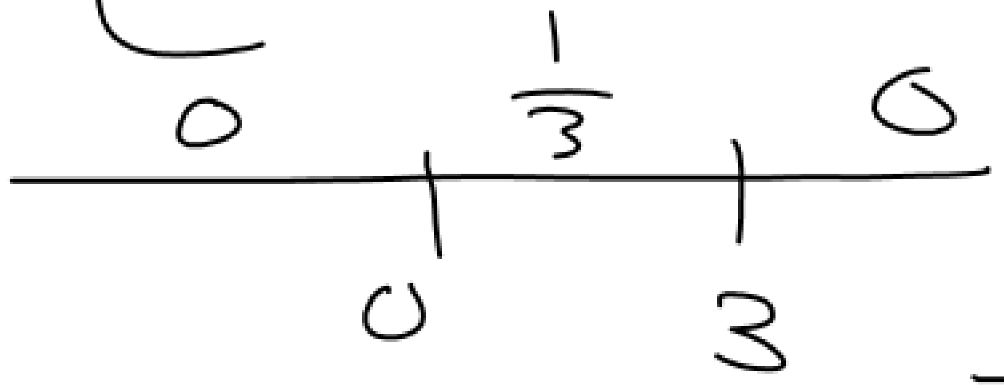
$= \left[1-e^{-5(4.2)}\right]\left[1-\frac{1.3-0}{3-0}\right]$

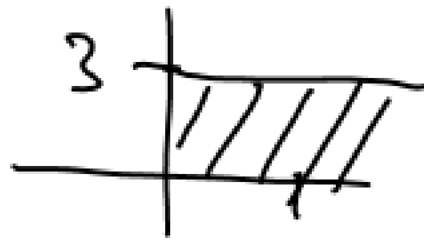
OR

$$f_X(x) = \begin{cases} 5e^{-5x} & x \ge 0 \\ 0 & \text{o.w} \end{cases}$$



$$f_Y(y) = \begin{cases} \frac{1}{3} & 0 \le y \le 3 \\ 0 & \text{o.w} \end{cases}$$





$$P(X < 4.2, Y > 1.3)$$

$$\left(\int_0^{4.2} 5e^{-5x}\,dx\right)\left(\int_{1.3}^{3} \frac{1}{3}\,dy\right)$$

OR

$$f_{XY}(x,y) = \begin{cases} \frac{5}{3} e^{-5x} & 0 \le x,\ 0 \le y \le 3 \\ 0 & \text{o.w} \end{cases}$$

(2) Find $E(3X - Y + 1)$

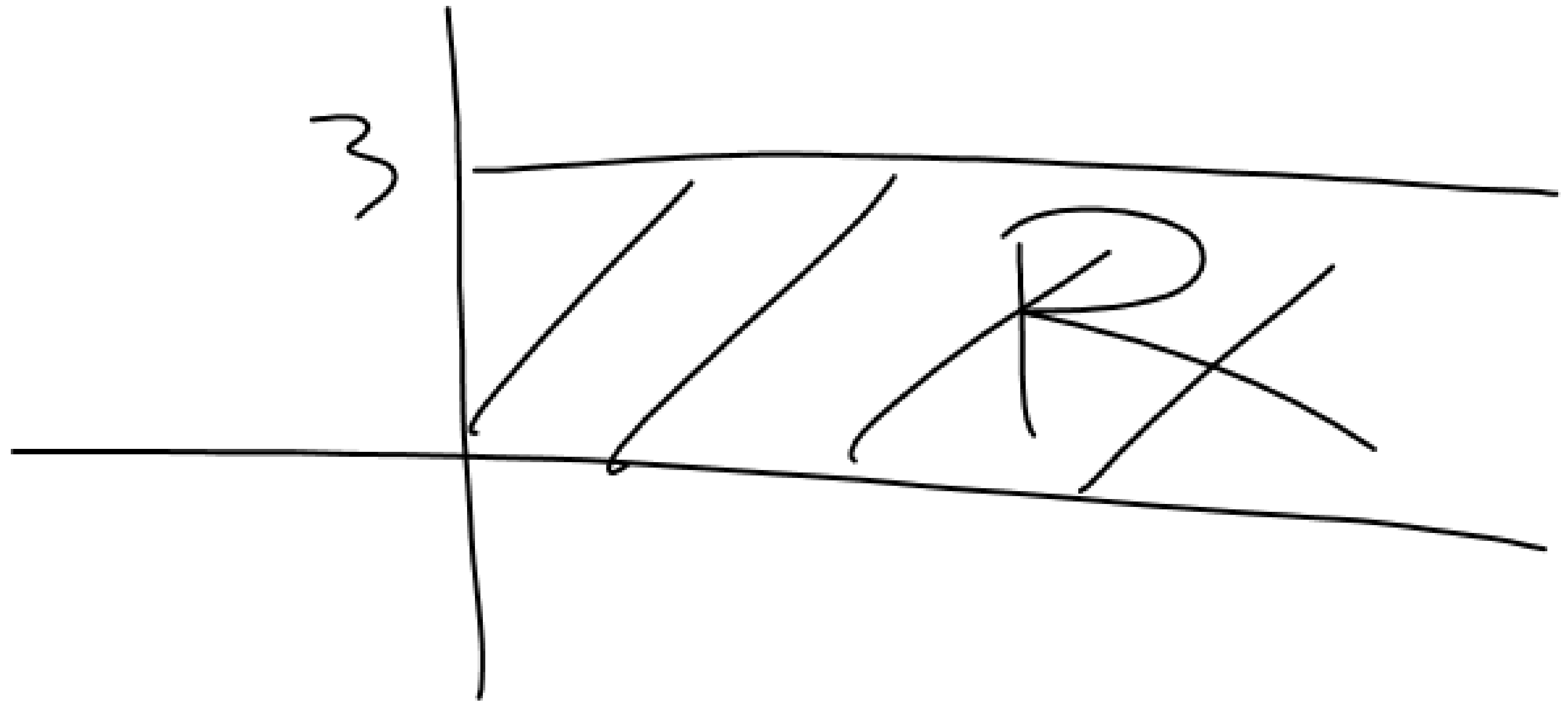
$$E(3X - Y + 1) = 3E(X) - E(Y) + 1$$

$$= \frac{3}{5} - \frac{3}{2} + 1$$

No need

OR

$$\iint_R (3x - y + 1) \frac{5e^{-5x}}{3} \, dA$$



③ Compute $Var(2X - Y + 1)$

$$= 4\,Var(X) + (-1)^2 Var(Y) \quad \text{but the } Cov(X,Y) = 0$$

indep

$$= \frac{4}{5} + \frac{9}{12}$$

3. $F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{XY}(u, v)\ dv\ du$

4. $F_Y(y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{XY}(u, v)\ du\ dv$

5. $P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{XY}(x, y)\ dy\ dx$

   The marginal PDFs are given by:

6. $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\ dy$

7. $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\ dx$

Conditional PMF for Discrete Bivariate Random Variables

1. $p_{Y|X}(y|x) = \dfrac{P[X = x, Y = y]}{P[X = x]} = \dfrac{p_{XY}(x, y)}{p_X(x)}$, provided $p_X(x) > 0$

2. $F_{X|Y}(x|y) = P[X \leq x|Y = y] = \sum_{u \leq x} p_{X|Y}(u|y)$

Conditional PDF and CDF for Continuous Bivariate Random Variables

1. $f_{Y|X}(y|X = x) = \dfrac{f_{XY}(x, y)}{f_X(x)}$, provided $f_X(x) > 0$

2. $f_{X|Y}(x|Y = y) = \dfrac{f_{XY}(x, y)}{f_Y(y)}$, provided $f_Y(y) > 0$

3. $F_X(x|Y = y) = \dfrac{\int_{-\infty}^{x} f_{XY}(u, y)du}{f_Y(y)}$

4. $F_X(x|y_1 < Y \leq y_2) = \dfrac{F_{XY}(x, y_2) - F_{XY}(x, y_1)}{F_Y(y_2) - F_Y(y_1)} = \dfrac{\int_{y_1}^{y_2} \int_{-\infty}^{x} f_{XY}(\xi, y)d\xi dy}{\int_{y_1}^{y_2} \int_{-\infty}^{\infty} f_{XY}(x, y)dxdy}$

$Var(X) = E(X - E(X))^2$

5. $f_X(x|y_1 < Y \le y_2) = \dfrac{\int_{y_1}^{y_2} f_{XY}(x,y)dx}{\int_{y_1}^{y_2}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy}$

Conditional Means and Variances for Discrete Bivariate Random Variable

1. $\mu_{Y|X} = E(Y|X) = \sum_y y p_{Y|X}(y|x)$

2.

$$\begin{aligned}\sigma^2_{Y|X} &= E[(Y - \mu_{Y|X})^2|X] = \sum_y (y - \mu_{Y|X})^2 p_{Y|X}(y|x)\\ &= E[Y^2|X = x] - (E[Y|X = x])^2\end{aligned}$$

Conditional Means and Variances for Continuous Bivariate Random Variable

1. $\mu_{Y|X} = E(Y|X = x) = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\ dy$

2.

$$\begin{aligned}\sigma^2_{Y|X} &= E[(Y - \mu_{Y|X})^2|X] = \int_{-\infty}^{\infty} (y - \mu_{Y|X})^2 f_{Y|X}(y|x)\\ &= E[Y^2|X = x] - (E[Y|X = x])^2\end{aligned}$$

<u>Ex</u> The joint pmf of X, Y is given by the table

$X \backslash Y$	0	1	2	$P(X=x)$
1	0.2	0.1	0.1	0.4
2	0.3	0.2	0.1	0.6
$P(Y=y)$	0.5	0.3	0.2	1

1) Find the marginal pmf of X and Y

2 Find the conditional pmf of X given $Y=2$

$$P(X \mid Y=2) = \frac{P(X=x, Y=2)}{P(Y=2)}$$

X	1	2
$P(X \mid Y=2)$	$\frac{P(X=1, Y=2)}{P(Y=2)} = \frac{0.1}{0.2} = \frac{1}{2}$	$\frac{P(X=2, Y=2)}{P(Y=2)} = \frac{0.1}{0.2} = \frac{1}{2}$

③ $E(X|Y=2) = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right) = \frac{1}{2} + 1 = \frac{3}{2}$

④ $Var(X|Y=2) = \left[(1)^2 \frac{1}{2} + (2)^2\left(\frac{1}{2}\right)\right] - \left(\frac{3}{2}\right)^2$