Design & Analysis of Algorithms

CS11313 - Fall 2023

Sorting Lower Bound

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are there sorting algorithms that
are not comparison-based?
Yes! (e.g. Radix Sort)

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proposition holds in the worst case only. E.g. Insertion sort does $\Theta(n)$ comparisons in the best case.

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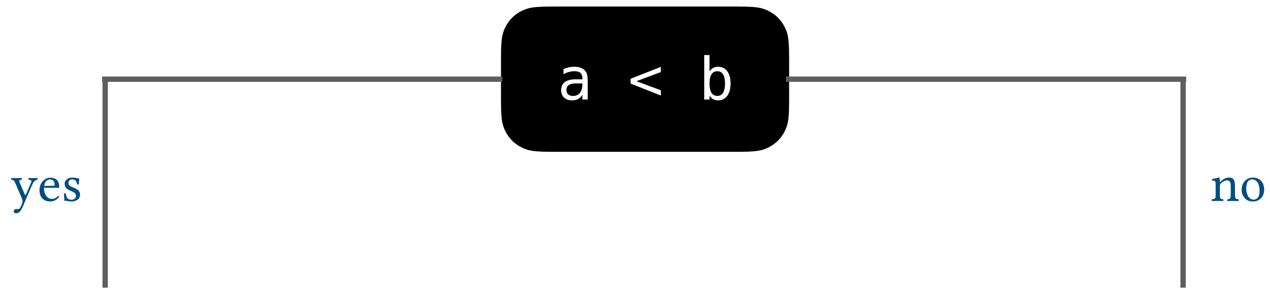
Proposition. Any comparison-based sorting algorithm performs at least $\sim n \log n$ compares in the worst case.

Put another way. For any comparison-based sorting algorithm, there must be at least one sequence of elements for which the sorting algorithm needs $\sim n \log n$ comparisons to sort.

Sorting Lower Bound

A comparison tree for three distinct keys (a, b and c)



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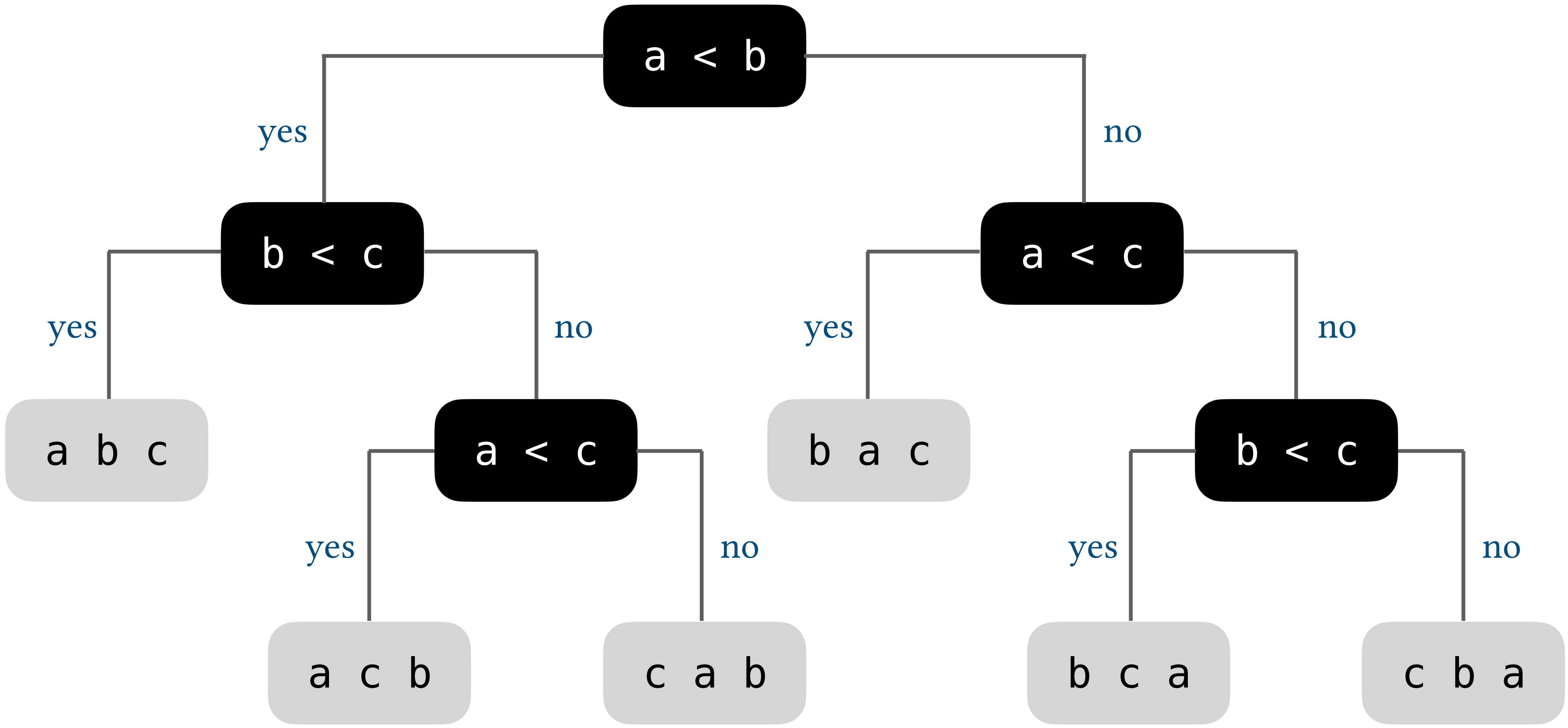

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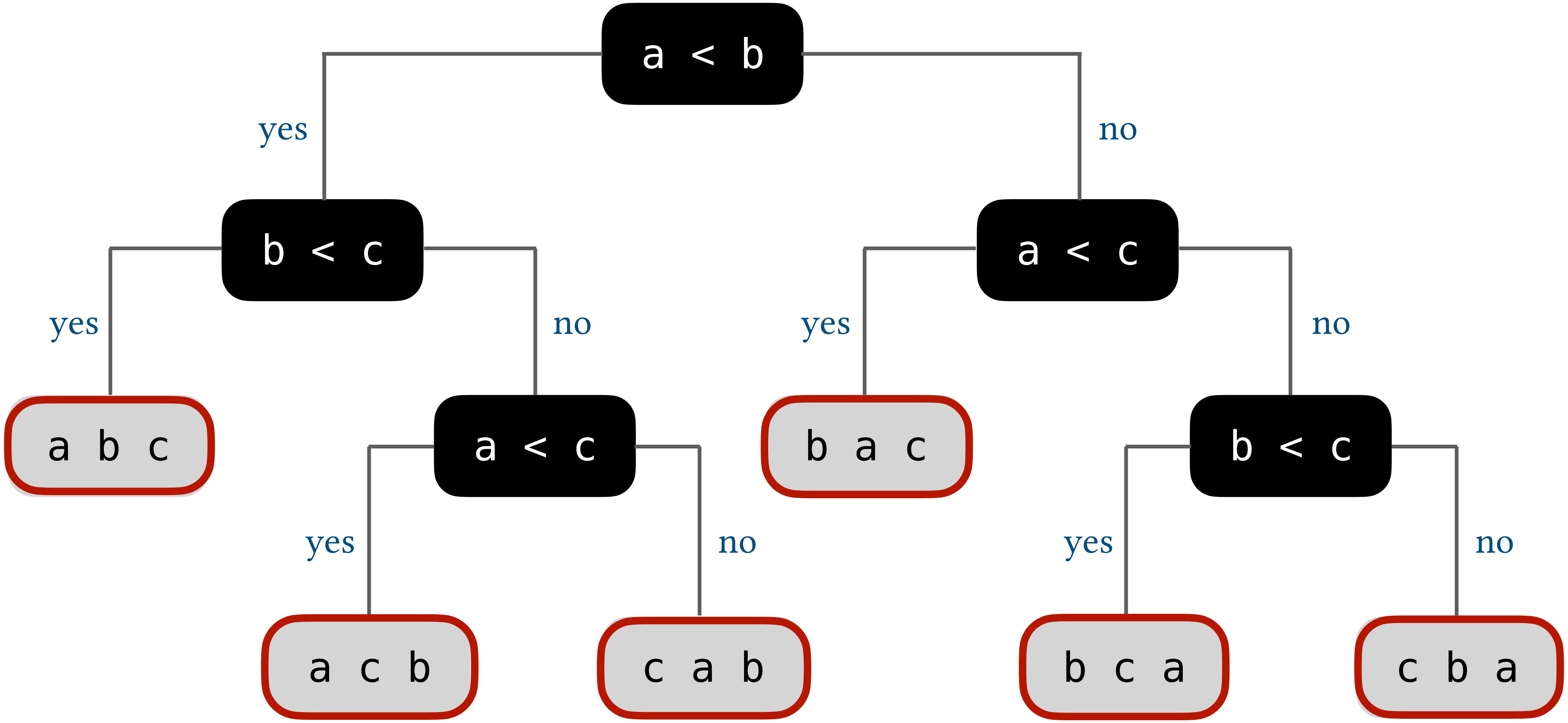

There are $n!$ unique orderings making $n!$ leaves

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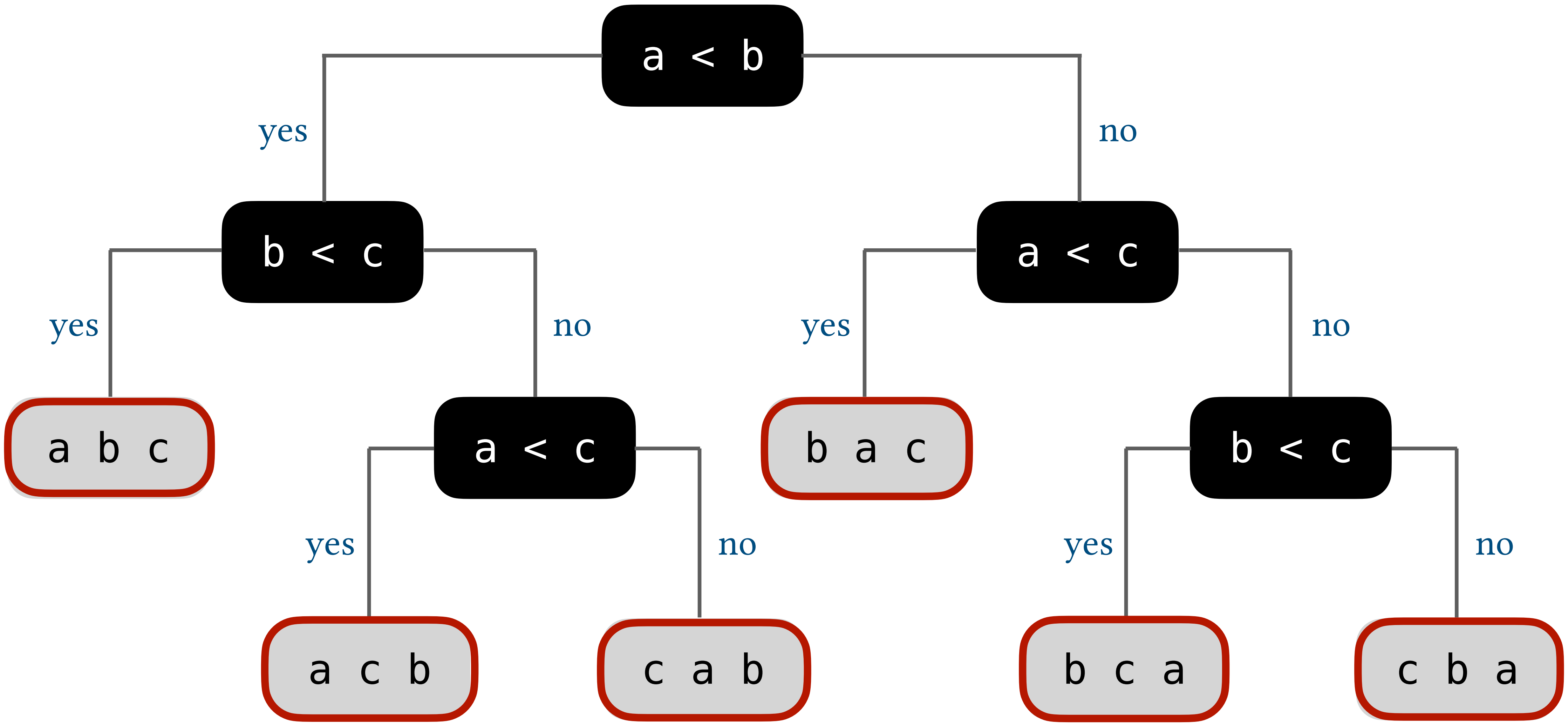

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$$\# \text{ of leaves} \quad \leq \quad 2^{\text{height}}$$

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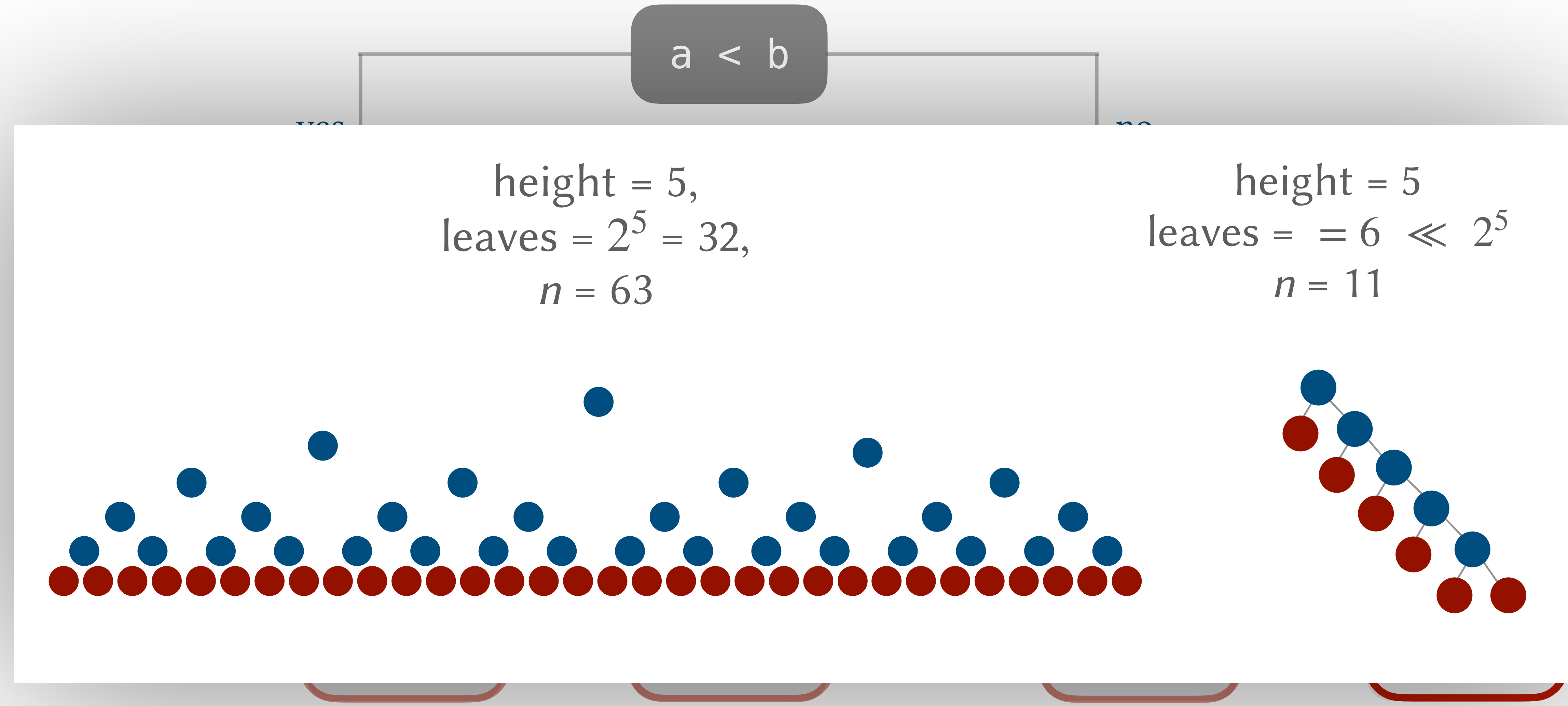

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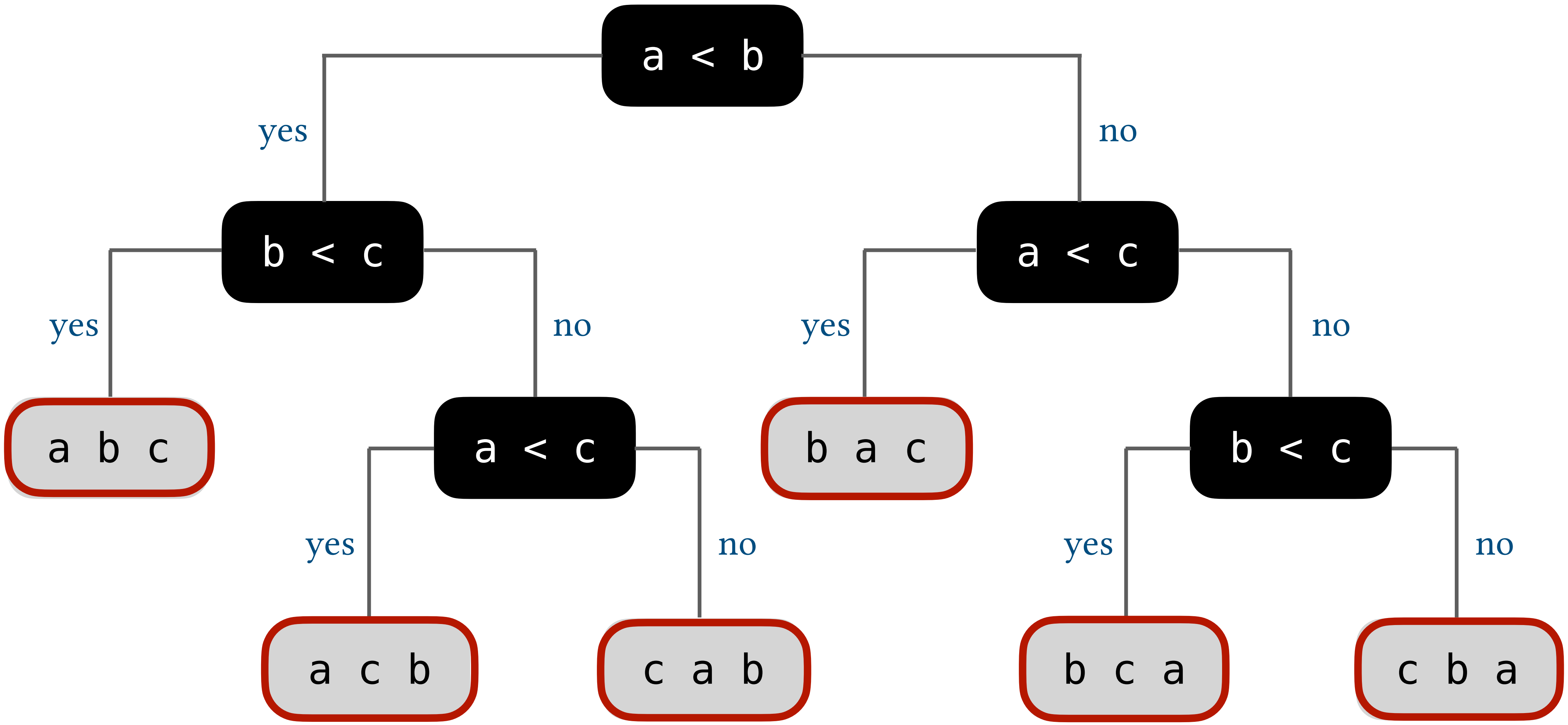

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$$\begin{aligned} \#\text{ of leaves} &\leq 2^{\text{height}} \\ n! &\leq 2^{\text{height}} \end{aligned}$$

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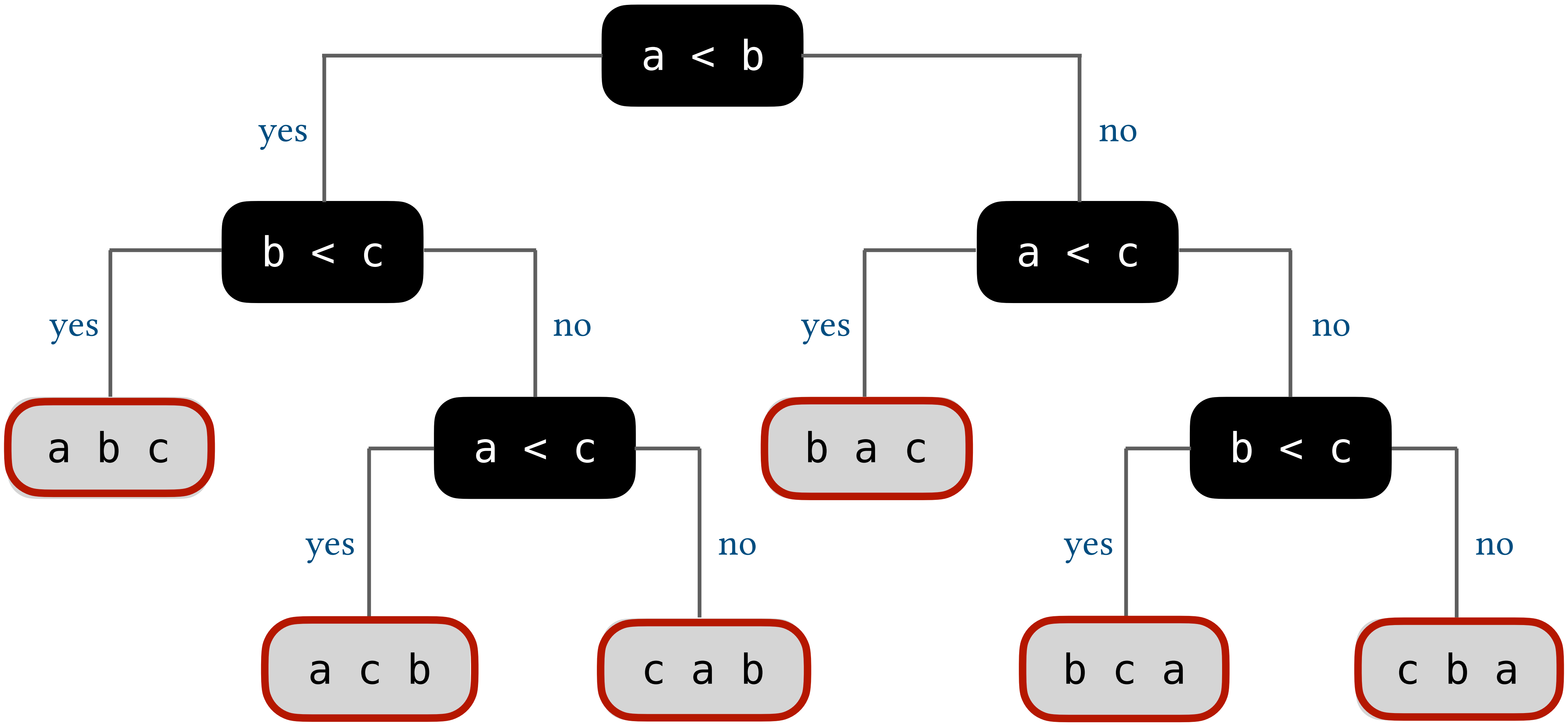

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$$
\begin{aligned}
\text{\# of leaves} &\leq 2^{\text{height}} \\
n! &\leq 2^{\text{height}} \\
\log(n!) &\leq \log(2^{\text{height}})
\end{aligned}
$$

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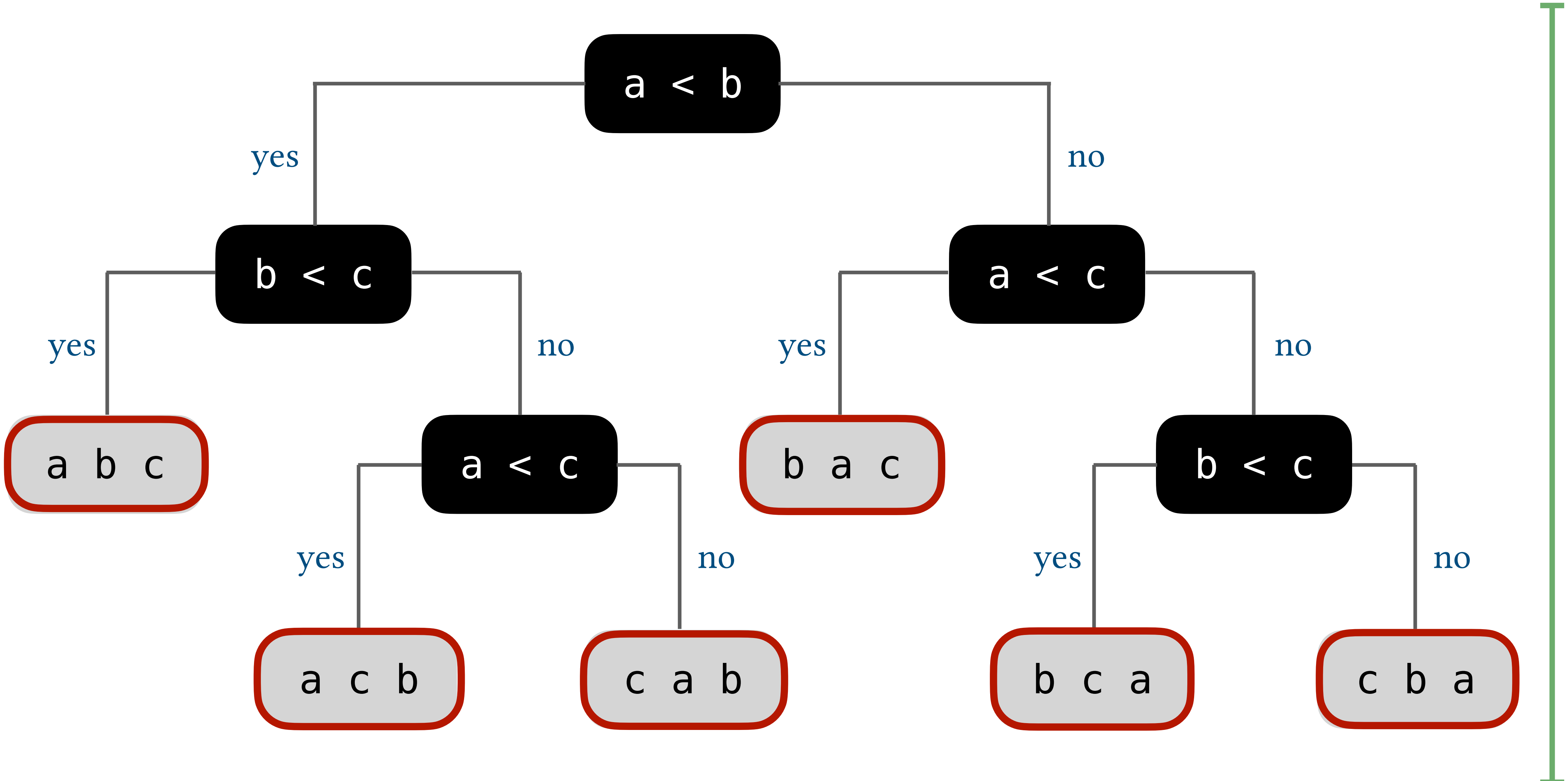

$h \geq \log_2(n!)$

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$$
\begin{aligned}
\text{\# of leaves} &\leq 2^{\text{height}} \\
n! &\leq 2^{\text{height}} \\
\log(n!) &\leq \log(2^{\text{height}}) \\
\sim n\log(n) &\leq \text{height}
\end{aligned}
$$

height represents # of comparisons

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Proof Sketch.

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  (path from the root to a leaf represents a run of a sorting algorithm).

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- The height of a binary tree with $n!$ leaves is $\geq \log_2(n!)$.
  (the height of the tree is $\log_2(n!)$ if it is a complete tree and possibly more if it is not).

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- If the longest path in the tree is $\geq \log_2(n!)$ then there must always be a sequence of input that requires $\log_2(n!)$ comparisons to be sorted.
  (the height of a binary tree is the length of the longest path from the root to a leaf).