CS11313 - **Fall** 2023

Design & Analysis of Algorithms

Selection

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Warmup Quiz

How can we find the maximum m elements in an array of size n ?

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Answer 1. Perform m iterations of selection sort.
Running time: $\Theta(mn)$.

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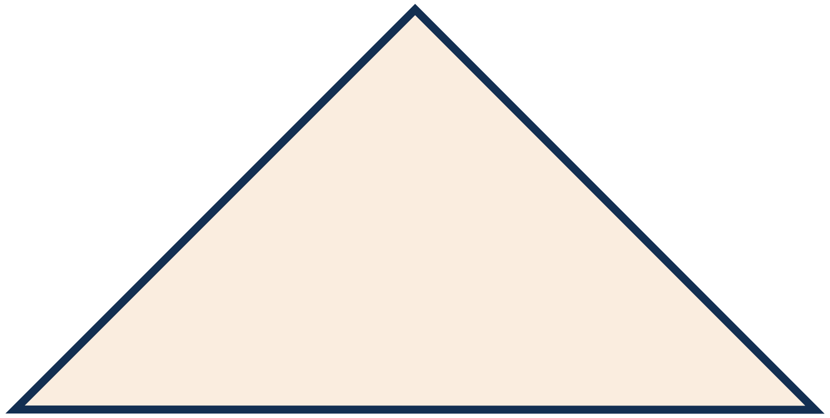
Answer 5.

Example. $m = 4$

```
a[] = 1 5 9 8 4 11 3 0 7 8 6 10 2
```



min-PQ

```
for each element k in a[]:
    minPQ.INSERT(k)
    if (minPQ.size > m)
        minPQ.DEL-MIN()
```

Answer 4. Insert all elements into a min-PQ. Remove the minimum element whenever the size of the min-PQ exceeds m.
Running time: $\Theta(n \log m)$

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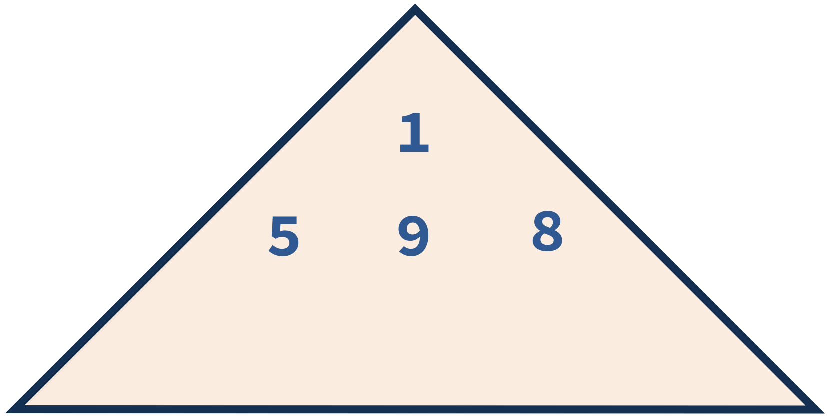

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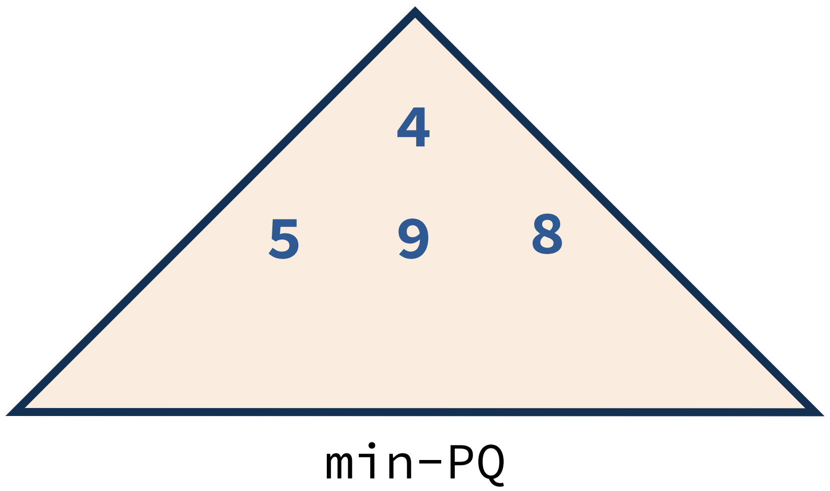


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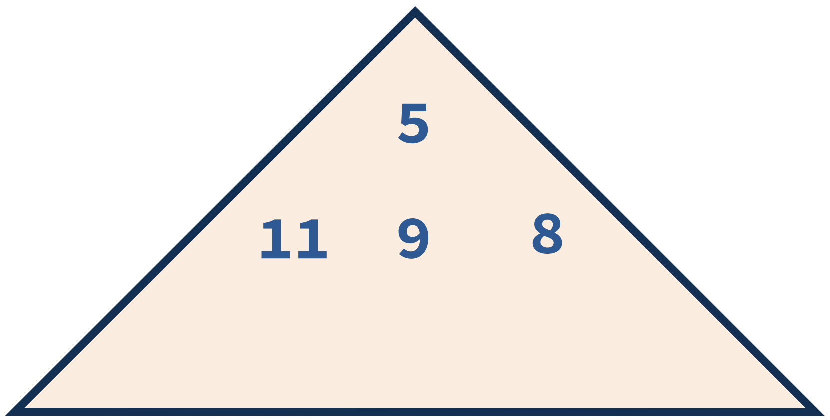
`a[] = 3 0 7 8 6 10 2`



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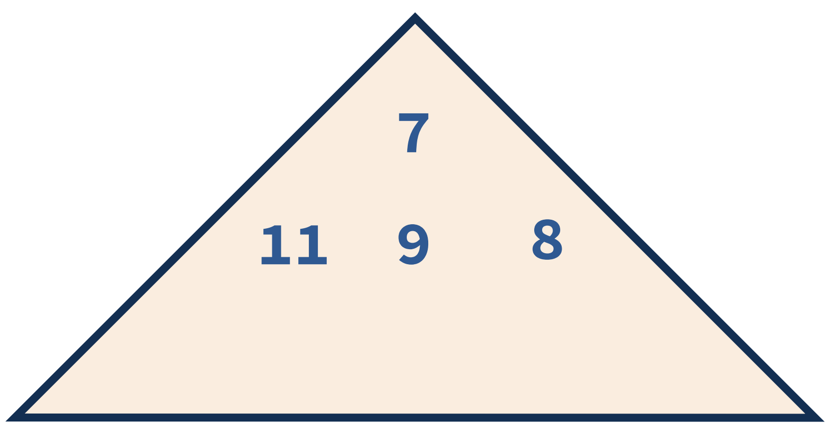

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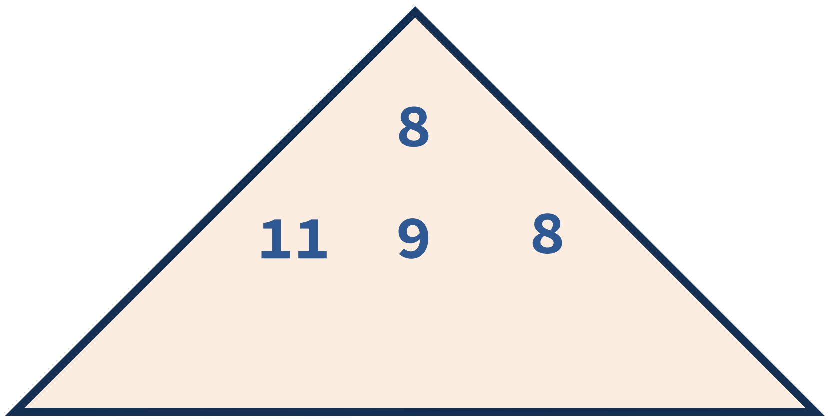

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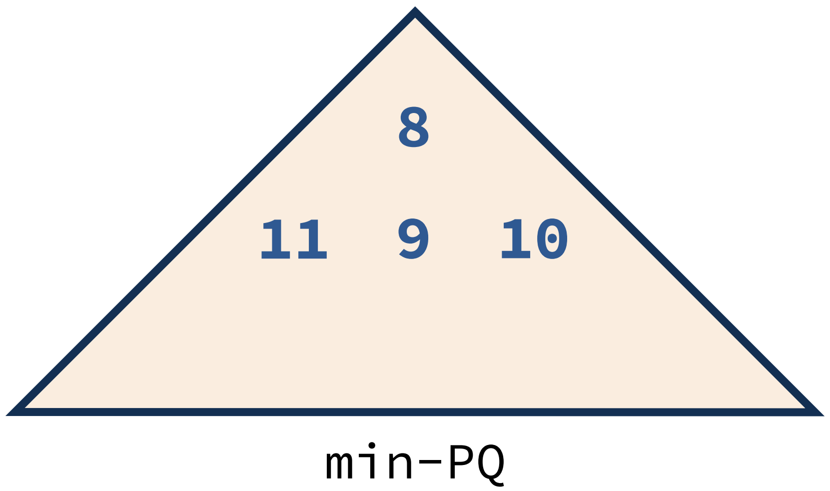
Running time: $\Theta(mn)$.

Answer 5.

Example. $m = 4$

a[] = 2

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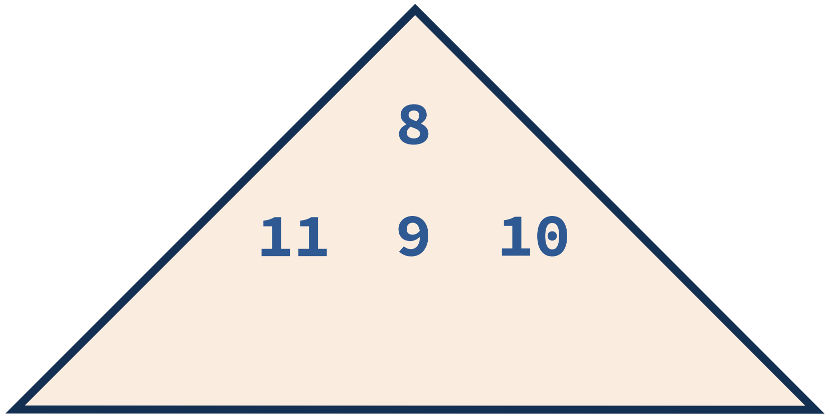

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Answer 1. Perform m iterations of selection sort.
Running time: $\Theta(mn)$.

Answer 2. Sort the array using Merge Sort and take the last m elements.
Running time: $\Theta(n \log n)$.

Answer 3. Insert all elements into a max-PQ and then remove m elements.
Running time: $\Theta(n \log n)$ to insert + $O(m \log n)$ to remove = $\Theta(n \log n)$

Answer 4. Insert all elements into a min-PQ. Remove the minimum element whenever the size of the min-PQ exceeds m.
Running time: $\Theta(n \log m)$

Selection

Problem. Find the element with rank k in an arbitrary array of size n.

Examples. $k = 0$ (minimum), $k = n - 1$ (maximum), $k = \frac{n}{2}$ (median).

Relation to Sorting.

- Repeated selection leads to sorting.
- If the array is sorted, selection is easy!

Candidate Solutions.

- Perform k iterations of selection sort. $\longleftarrow \Theta(kn)$
- Sort in ascending order and then get the element at index k. $\longleftarrow O(n \log n)$
- Insert the elements into a binary heap, keep the min k+1 elements. $\longleftarrow O(n \log k)$

Can we do better?

Is selection as hard as sorting?
(requires $\sim n \log n$ compares
in the worst case if $k = \frac{n}{2}$)

Quickselect Demo

Assume $k = \frac{n}{2}$ (5 in the example below).

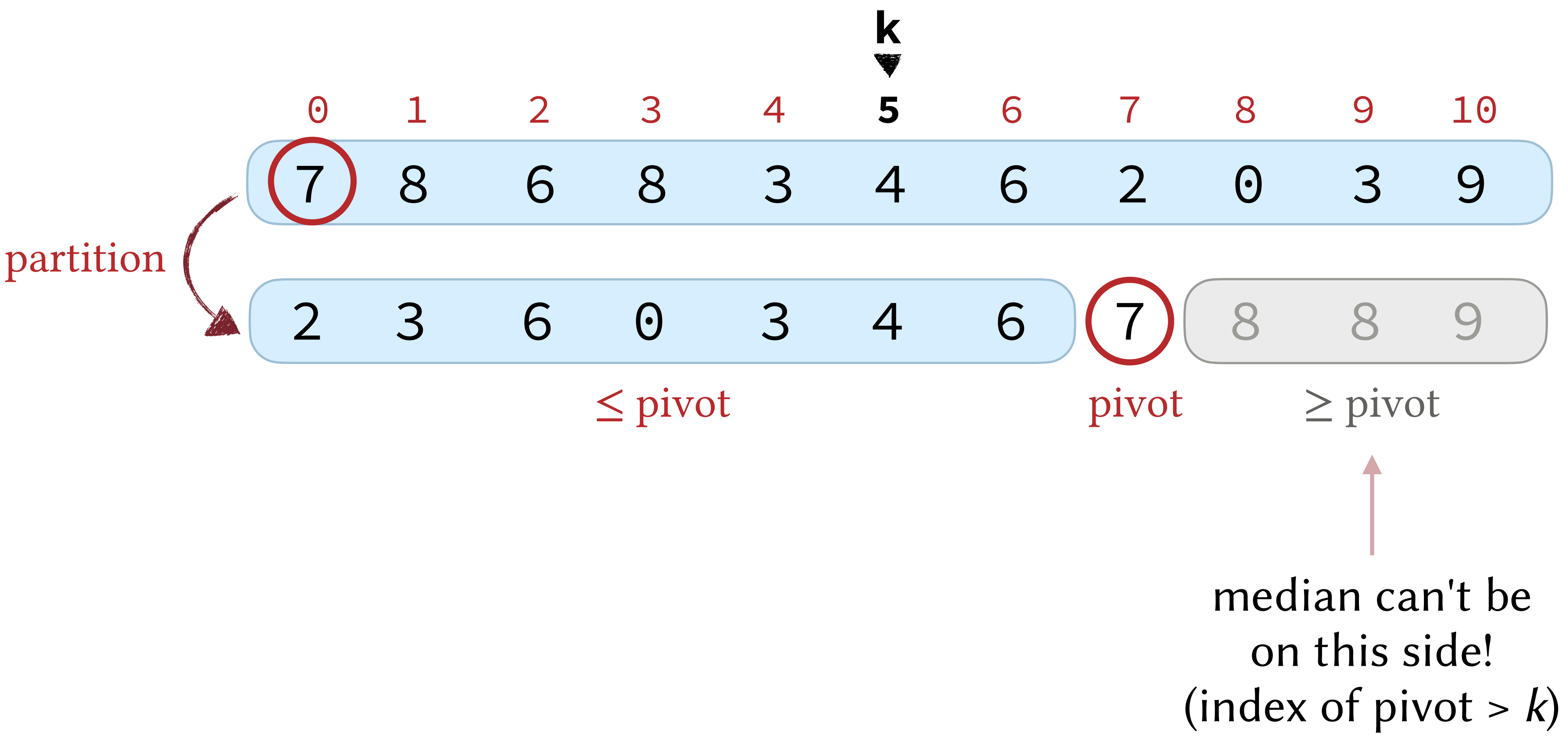
k

0	1	2	3	4	**5**	6	7	8	9	10
7	8	6	8	3	4	6	2	0	3	9

which element should be at this
index if the elements were sorted?

Quickselect Demo

Assume $k = \frac{n}{2}$ (5 in the example below).



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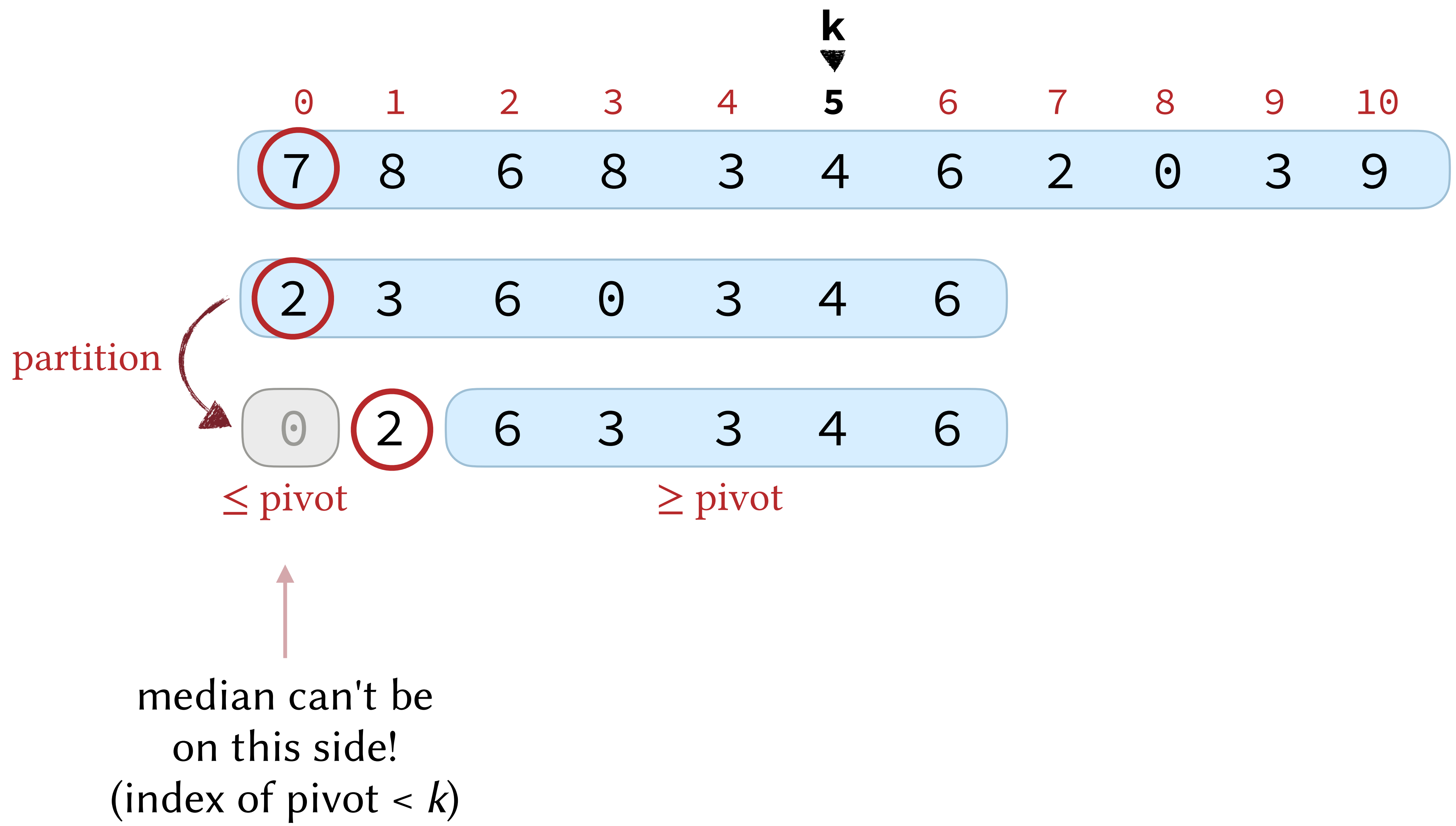

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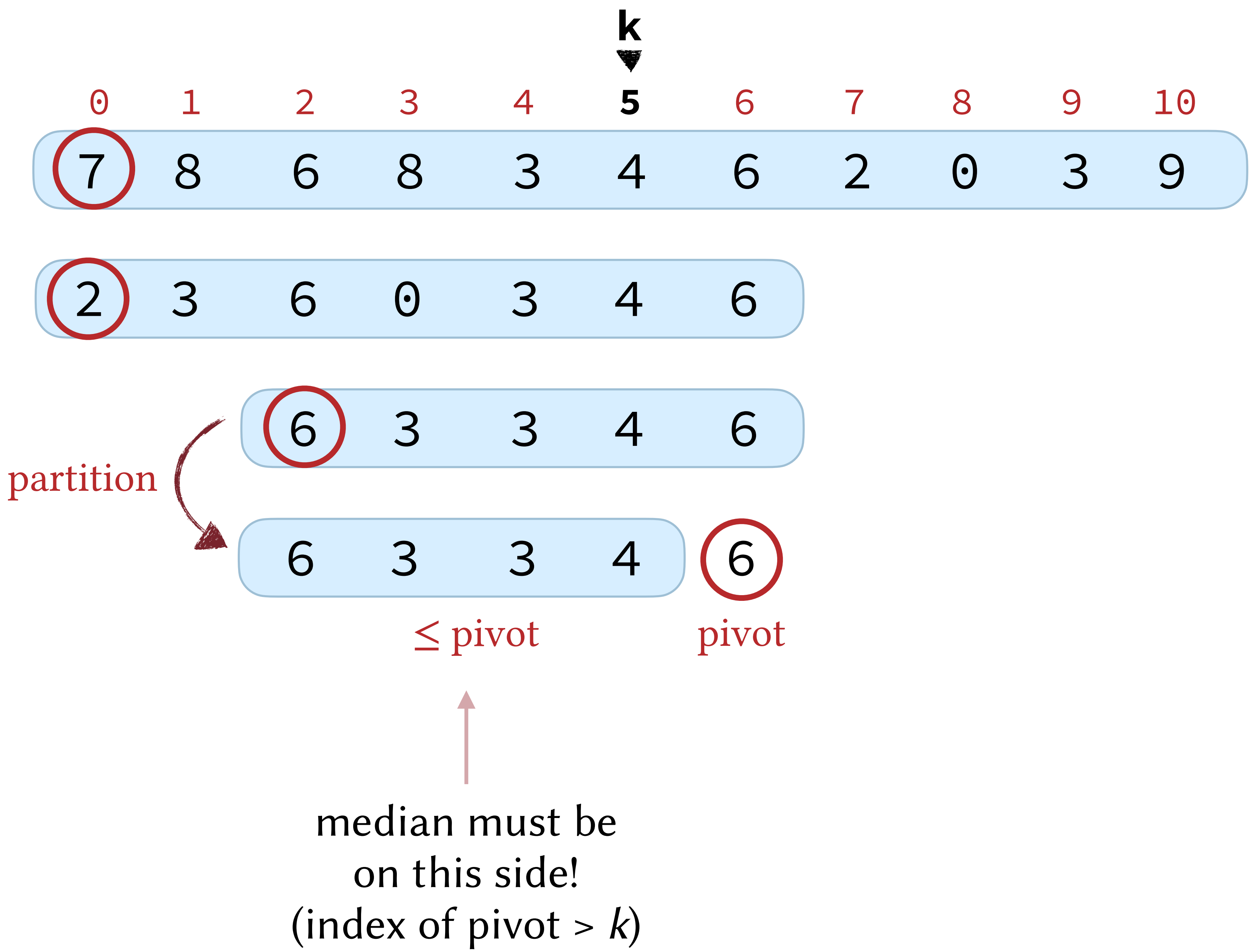

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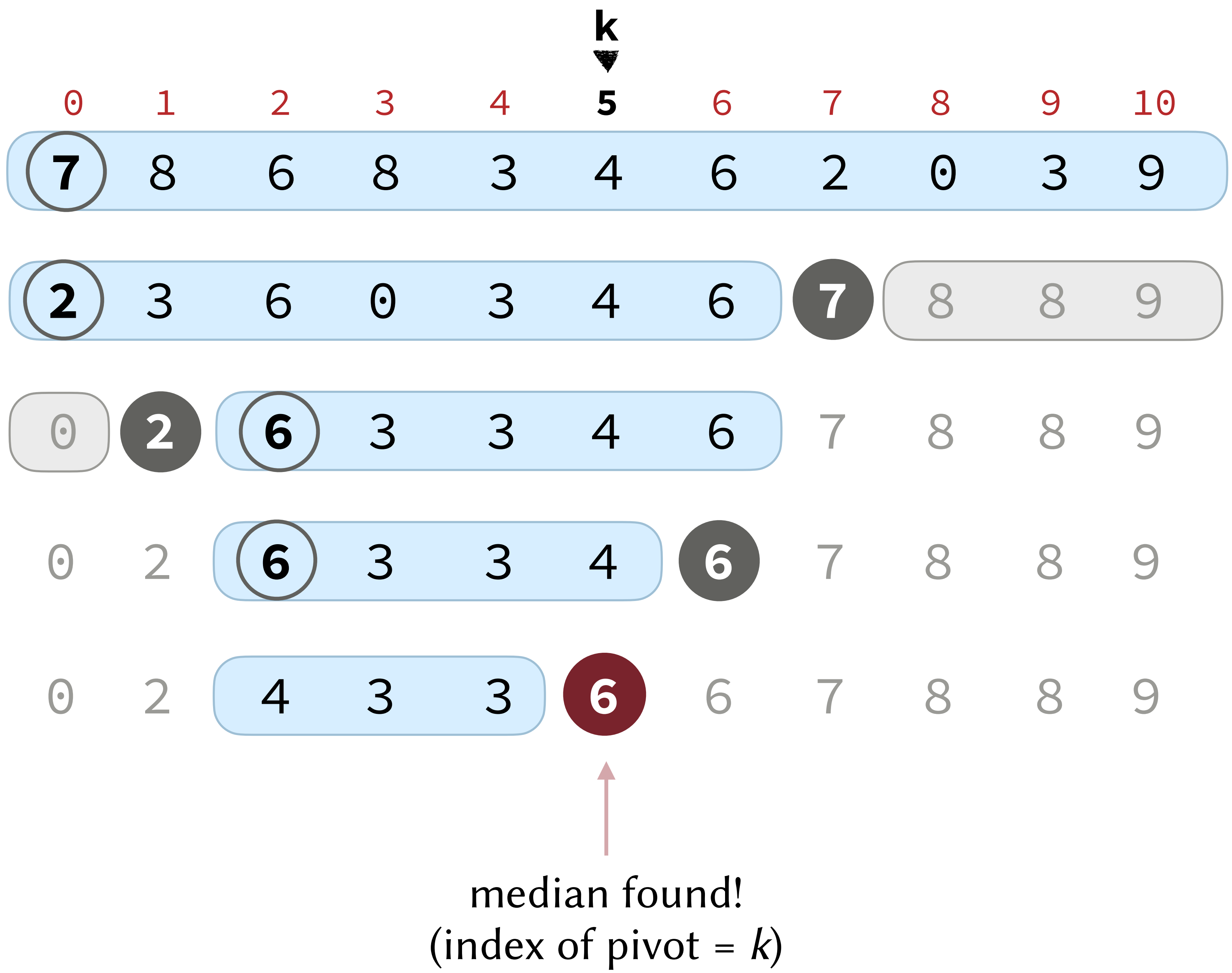


Quickselect Algorithm

```
SELECT(a[], first, last, k)
    SHUFFLE(a, first, last)
    QUICK-SELECT(a, first, last, k)
```

to guard against the worst case (or pick pivot randomly)

assuming *k* is a valid index

Quickselect Algorithm

```
SELECT(a[], first, last, k)

  SHUFFLE(a, first, last)
  QUICK-SELECT(a, first, last, k)
```

```
QUICK-SELECT(a[], first, last, k)

  if (first >= last):
     return a[k]

  p = PARTITION(a, first, last)

  if p == k:
      return a[k]
  if k > p:
      return QUICK-SELECT(a, p+1, last, k)
  else:
      return QUICK-SELECT(a, first, p-1, k)
```

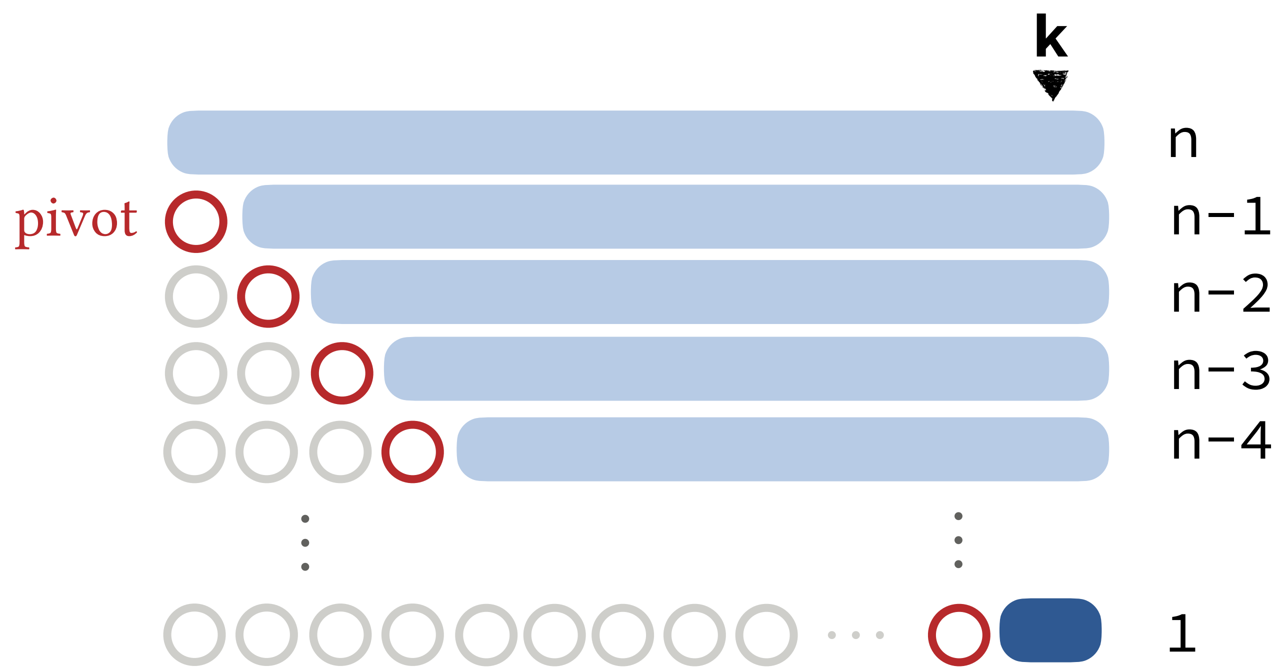
Quickselect Analysis

Best Case. Element at rank k found immediately after the first partitioning step: $\Theta(n)$.

Worst Case. Element at rank k found after $n-1$ partitioning steps:

$$n + (n-1) + (n-2) + \ \dots \ + 1 = \Theta(n^2)$$

Example 1. $k = n - 1$



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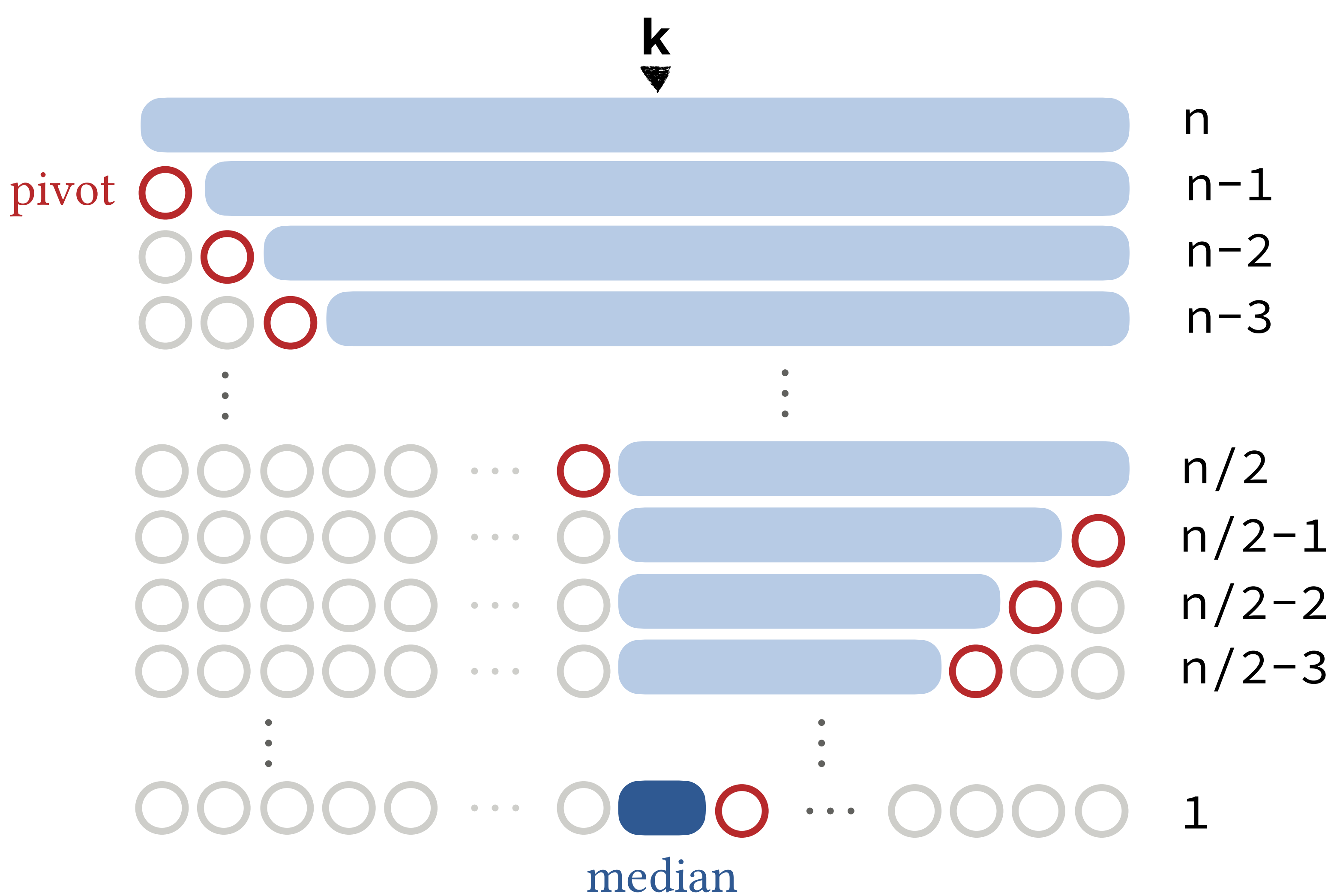
Example 2. $k = \dfrac{n}{2}$



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probabilistically almost-impossible if the array is shuffled!

Quickselect Analysis

Best Case. Element at rank k found immediately after the first partitioning step: $\Theta(n)$.

Worst Case. Element at rank k found after $n-1$ partitioning steps:

$$n + (n-1) + (n-2) + \ \dots\ + 1 = \Theta(n^2)$$

Expected Case. $\Theta(n)$

Intuition. Partitioning always gets rid of around half of the remaining elements.

$$T(n) = T(\tfrac{n}{2}) + \sim n$$

time to partition an array of size *n*

time to select from an array of size *n*

time to select from an array of size n/2

Quickselect Analysis

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n

time to partition
the whole array

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$$n \ + \ \frac{n}{2}$$

time to partition
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Expected Case. $\Theta(n)$

Intuition. Partitioning always gets rid of around half of the remaining elements.

$$T(n) = T(\tfrac{n}{2}) + \sim n$$

$$n + \frac{n}{2} + \frac{n}{4} + \dots + 1 = n(1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{n}) = \Theta(n)$$

Remember!

$$\sum_{i=0}^{\log_2 n} 2^i = \quad 2^{\log_2 n + 1} - 1 = \quad 2n - 1$$

$$1 + 2 + 4 + 8 + \ldots + n$$

$$n + \frac{n}{2} + \frac{n}{4} + \ldots + 4 + 2 + 1$$

$$n \times (1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{n})$$

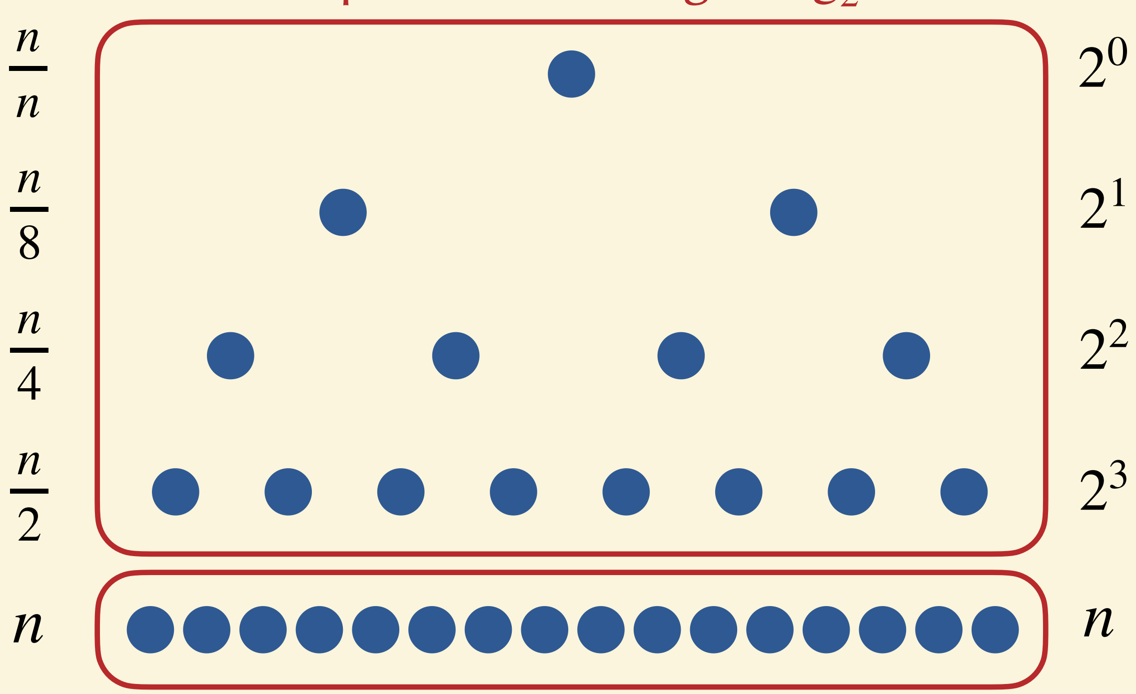
$$\leq 2$$

Remember!

$$\sum_{i=0}^{\log_2 n} 2^i = 2^{\log_2 n + 1} - 1 = 2n - 1$$

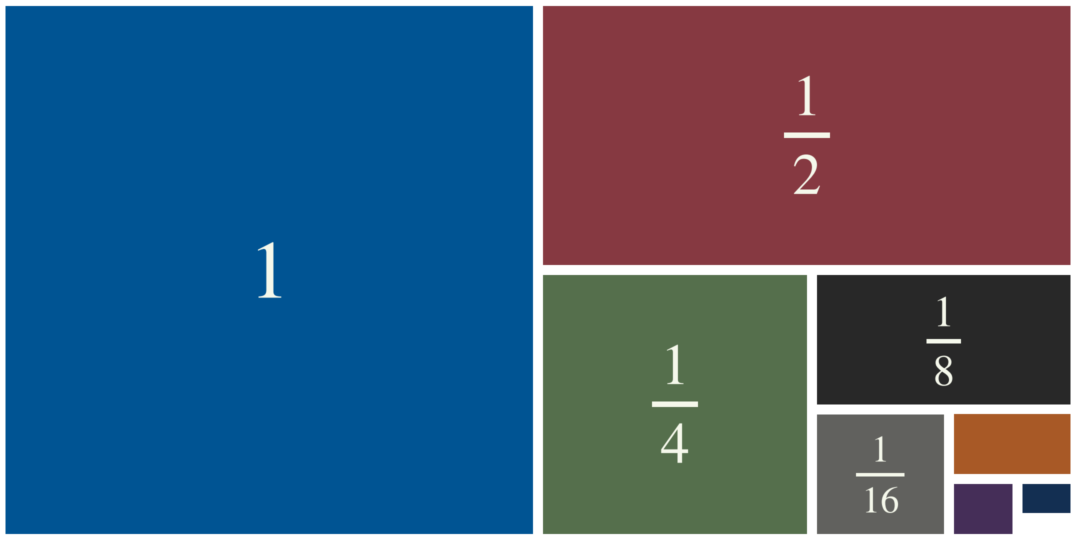
$$1 + 2 + 4 + 8 + \ldots + n$$
$$= 2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^{\log_2 n}$$

n -1 internal nodes in a complete tree of height $\log_2 n$



n leaves in a complete tree of height $\log_2 n$

$$n + \frac{n}{2} + \frac{n}{4} + \ldots + 4 + 2 + 1$$
$$n \times \left(1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{n}\right)$$



≤ 2

Analysis Notes

Remember: Code that follows the pattern below has a running time of $\Theta(n \log n)$

```
foo(n)
  if (n == 0): return
  foo(n / 2)
  foo(n / 2)
  linear(n)
```

solve *two* subproblems of half the size.

do a *linear* amount of work.

Remember: Code that follows the pattern below has a running time of $\Theta(n)$

```
foo(n)
  if (n == 0): return
  foo(n / 2)
  linear(n)
```

solve *one* subproblems of half the size.

do a *linear* amount of work.

Quickselect Analysis

Best Case. Element at rank k found immediately after the first partitioning step: $\Theta(n)$.

Worst Case. Element at rank k found after $n - 1$ partitioning steps:

$$n + (n-1) + (n-2) + \ \ldots \ + 1 = \Theta(n^2)$$

Expected Case. $\Theta(n)$

Intuition. Partitioning always gets rid of around half of the remaining elements.

$$T(n) = T(\tfrac{n}{2}) + \Theta(n)$$

$$n + \frac{n}{2} + \frac{n}{4} + \ldots + 1 = \Theta(n)$$

Can we do better?

Is selection as hard as sorting?
(requires $\sim n \log n$ compares
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$$T(n) = T(\tfrac{n}{2}) + \Theta(n)$$

$$n + \frac{n}{2} + \frac{n}{4} + \dots + 1 = \Theta(n)$$

Can we do better?

Theoretically. Selection can be done in linear time in the worst case using the Median of Medians algorithm. (Blum, Floyd, Pratt, Rivest, and Tarjan 1973).

Practically. Quickselect is faster in practice.

JOURNAL OF COMPUTER AND SYSTEM SCIENCES **7**, 448–461 (1973)

Time Bounds for Selection*

MANUEL BLUM, ROBERT W. FLOYD, VAUGHAN PRATT, RONALD L. RIVEST, AND ROBERT E. TARJAN

Department of Computer Science, Stanford University, Stanford, California 94305

Received November 14, 1972

The number of comparisons required to select the i-th smallest of n numbers is shown to be at most a linear function of n by analysis of a new selection algorithm—PICK. Specifically, no more than $5.430\dot{5}\, n$ comparisons are ever required. This bound is improved for extreme values of i, and a new lower bound on the requisite number of comparisons is also proved.

Introselect

From Wikipedia, the free encyclopedia

In computer science, **introselect** (short for "introspective selection") is a selection algorithm that is a hybrid of quickselect and median of medians which has fast average performance and optimal worst-case performance. Introselect is related to the introsort sorting algorithm: these are analogous refinements of the basic quickselect and quicksort algorithms, in that they both start with the quick algorithm, which has good average performance and low overhead, but fall back to an optimal worst-case algorithm (with higher overhead) if the quick algorithm does not progress rapidly enough. Both algorithms were introduced by David Musser in (Musser 1997), with the purpose of providing generic algorithms for the C++ Standard Library that have both fast average performance and optimal worst-case performance, thus allowing the performance requirements to be tightened.[1] However, in most C++ Standard Library implementations that use introselect, another "introselect" algorithm is used, which combines quickselect and heapselect, and has a worst-case running time of $O(n \log n)$[2].

Introselect

Class	**Selection algorithm**
Data structure	Array
Worst-case performance	$O(n)$
Best-case performance	$O(n)$

Quicksort Improvement

What is the order of growth of the running time of quicksort is if a linear time median finding algorithm is used to pick the pivot?

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What is the order of growth of the running time of quicksort is if a linear time median finding algorithm is used to pick the pivot?

Answer. If the pivot is always the median, the array is always split into almost equally-sized partitions. Therefore, the algorithm would run in $\Theta(n \log n)$.

However: the overhead for finding the pivot would be high and the algorithm would be slower in practice compared to just picking the pivot randomly.

optional

Streaming Median

Assume that you receive an arbitrary stream of numbers. How can we efficiently report the median at any point in time?

Streaming Median

Assume that you receive an arbitrary stream of numbers. How can we efficiently report the median at any point in time?

Solution 1. Maintain a max-heap:

`insert()`: $O(\log n)$

`median()`: $O(n \log n)$
Remove from the heap the first $\frac{n}{2}$ elements to reach the median and then insert them back.

Solution 2. Maintain an unordered array:

`insert()`: $\Theta(1)$
Add to the end of the list. Note that the array might resize, so the running time is amortized.

`median()`: $\Theta(n)$
Use `Quickselect` to find the median. Note that this is the expected case if the array is shuffled.

Solution 3. Maintain a sorted array:

`insert()`: $O(n)$
Search for the right position and then shift any elements that come after.

`median()`: $\Theta(1)$
The median is always at index $\frac{n}{2}$.

Streaming Median

Solution 4. Use a max-heap to store the lower half of the elements ($\le$ median) and a min-heap to store the upper half of the elements ($>$ median).

Assume that the max-heap is named **`left`** and the min-heap is named **`right`**. Ensure that:

- Any element in **`left`** is smaller than or equal to all the elements in **`right`**.
- **`left`**`.size() - `**`right`**`.size()` is `0` (equal) or `1` (**`left`** is larger by 1).

Therefore, **`left`** always has $\lceil \frac{n}{2} \rceil$ elements and the **median** is always the **maximum** element in **`left`**.

```
          left   right
```
Example: `[1 2 `**`3`**` • 4 5 6]`
```
            median
```

```
          left   right
```
Example: `[1 2 3 `**`4`**` • 5 6 7]`
```
            median
```

Streaming Median

Solution 4. Use a max-heap to store the lower half of the elements (≤ median) and a min-heap to store the upper half of the elements (> median).

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Example: `[1 2 3 • 4 5 6]` **Example**: `[1 2 3 4 • 5 6 7]`

General Idea:

- **Insert** the new element into the left heap if it is less than or equal to the current median and to the right if it is greater than the current median.

- **Rebalance** the heaps by moving an element from the larger heap to the smaller heap if the size invariant is violated.

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Example: `[1 2 3 • 4 5 6]` **Example**: `[1 2 3 4 • 5 6 7]`

```
insert(k):

    If k <= left.max(): left.insert(k)
    Else:               right.insert(k)
```

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```
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```

insert k in **`left`** if $k \leq$ median

insert k in **`right`** if $k >$ median

median

Streaming Median

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Example: `[1 2 `**`3`**` • 4 5 6]` **Example**: `[1 2 3 `**`4`**` • 5 6 7]`

```
insert(k):

    If k <= left.max(): left.insert(k)
    Else:               right.insert(k)

    If left.size()  > right.size()+1: right.insert(left.delMax()).
```

more than $\lceil \frac{n}{2} \rceil$ elements are in **`left`**

Streaming Median

Solution 4. Use a max-heap to store the lower half of the elements ($\leq$ median) and a min-heap to store the upper half of the elements ($>$ median).

Assume that the max-heap is named **`left`** and the min-heap is named **`right`**. Ensure that:

- Any element in **`left`** is smaller than or equal to all the elements in **`right`**.
- **`left`**`.size() - `**`right`**`.size()` is `0` (equal) or `1` (**`left`** is larger by 1).

Therefore, **`left`** always has $\lceil \frac{n}{2} \rceil$ elements and the **median** is always the **maximum** element in **`left`**.

Example: `[1 2` **`3`** `• 4 5 6]` **Example**: `[1 2 3` **`4`** `• 5 6 7]`

```
insert(k):

    If k <= left.max(): left.insert(k)
    Else:               right.insert(k)

    If left.size()  > right.size()+1: right.insert(left.delMax()).
```

remove the max from left
and insert it into right

Streaming Median

Solution 4. Use a max-heap to store the lower half of the elements ($\le$ median) and a min-heap to store the upper half of the elements ($>$ median).

Assume that the max-heap is named **`left`** and the min-heap is named **`right`**. Ensure that:

- Any element in **`left`** is smaller than or equal to all the elements in **`right`**.
- **`left`**`.size() - `**`right`**`.size()` is `0` (equal) or `1` (**`left`** is larger by 1).

Therefore, **`left`** always has $\lceil \frac{n}{2} \rceil$ elements and the **median** is always the **maximum** element in **`left`**.

Example: `[1 2 `**`3`**` • 4 5 6]` **Example**: `[1 2 3 `**`4`**` • 5 6 7]`

```
insert(k):

    If k <= left.max(): left.insert(k)
    Else:               right.insert(k)

    If left.size()  > right.size()+1: right.insert(left.delMax()).
    If right.size() > left.size():     left.insert(right.delMin()).
```

If **`right`** is larger than **`left`** remove the min from **`right`** and insert it into **`left`**.

Streaming Median

Solution 4. Use a max-heap to store the lower half of the elements (≤ median) and a min-heap to store the upper half of the elements (> median).

Assume that the max-heap is named **`left`** and the min-heap is named **`right`**. Ensure that:

- Any element in **`left`** is smaller than or equal to all the elements in **`right`**.
- **`left`**`.size() - `**`right`**`.size()` is `0` (equal) or `1` (**`left`** is larger by 1).

Therefore, **`left`** always has $\lceil \frac{n}{2} \rceil$ elements and the **median** is always the **maximum** element in **`left`**.

Example: `[1 2` **`3`** `• 4 5 6]` **Example**: `[1 2 3` **`4`** `• 5 6 7]`

```
insert(k):
    If k <= left.max(): left.insert(k)
    Else:               right.insert(k)

    If left.size()  > right.size()+1: right.insert(left.delMax()).
    If right.size() > left.size():    left.insert(right.delMin()).
```

insert into the correct heap

rebalance the heaps if necessary

Streaming Median

Solution 4. Use a max-heap to store the lower half of the elements ($\le$ median) and a min-heap to store the upper half of the elements ($>$ median).

Assume that the max-heap is named **`left`** and the min-heap is named **`right`**. Ensure that:

- Any element in **`left`** is smaller than or equal to all the elements in **`right`**.
- **`left`**`.size() - `**`right`**`.size()` is `0` (equal) or `1` (**`left`** is larger by 1).

Therefore, **`left`** always has $\lceil \frac{n}{2} \rceil$ elements and the **median** is always the **maximum** element in **`left`**.

Example: `[1 2 3 • 4 5 6]` **Example**: `[1 2 3 4 • 5 6 7]`

```
insert(k):
    If k <= left.max(): left.insert(k)
    Else:               right.insert(k)

    If left.size()  > right.size()+1: right.insert(left.delMax()).
    If right.size() > left.size():    left.insert(right.delMin()).
```

insert into the correct heap — $O(\log n)$

rebalance the heaps if necessary — $O(\log n)$

Streaming Median

Solution 4. Use a max-heap to store the lower half of the elements ($\leq$ median) and a min-heap to store the upper half of the elements ($>$ median).

Assume that the max-heap is named **`left`** and the min-heap is named **`right`**. Ensure that:

- Any element in **`left`** is smaller than or equal to all the elements in **`right`**.
- **`left`**`.size() - `**`right`**`.size()` is `0` (equal) or `1` (**`left`** is larger by 1).

Therefore, **`left`** always has $\lceil \frac{n}{2} \rceil$ elements and the **median** is always the **maximum** element in **`left`**.

Example: `[1 2` **`3`** `• 4 5 6]` **Example**: `[1 2 3` **`4`** `• 5 6 7]`

Running Time:

`insert()`: $O(\log n)$
Inserting into the left or the right heaps is $O(\log n)$ and rebalancing is $O(\log n)$.

`median()`: $\Theta(1)$
The median is always `left.max()`.