Trigonometry

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1 Introduction

Trigonometry is a branch of mathematics that is focused on analyzing the relations between angels and their impact on the sides of the triangle. Mostly presented on a triangle with 90 degree angle. But have you ever thought why and from where the values of trig functions come from? for example, why sin of 30 degree is $\frac{1}{2}$? Or why a cos of 1 radian $\frac{180\ degrees}{\pi}$ is approximately 0.54? Well everything comes from dangerous sounding complex numbers (also called imaginary numbers), let's dive deeper into this topic and check how and from where those formulas came from. I'll explain everything and I consider that you, as reader know what Complex numbers are.

1.1 What are sin and cosine and why are they valid?

Sin and cosine are the fundamental functions in trigonometry, they describe relationships between triangle's angle, and length of its sides. Those two functions are used to define other trigonometric functions, such as, $\tan(x)$, which is defined as $\frac{sin(x)}{cos(x)}$, or $arcsin$, which is defined as $\sin^{-1}(x)$. They are appiring also in other branches of mathematics and real life, such as complex numbers as Euler's formula; analytic geometry as polar coordinates; or in our maps as haversine formula, used to calculate distance between points on map. In my opinion understanding from where such important things come.

2 Sin, cosine and functions based on them

2.1 How we can divide them?

We can divide sin and cos functions, into 3 main groups:

1. Functions of real angle, which is our usual $\sin(x)$ or $\cos(x)$,

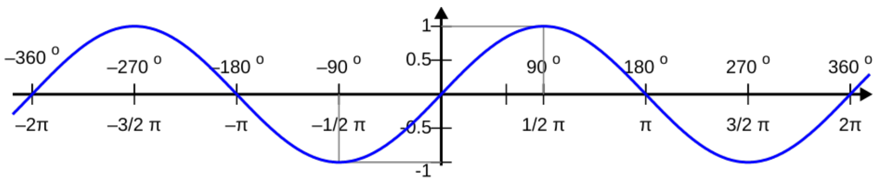

Figure 1: Plot of $\sin(x)$

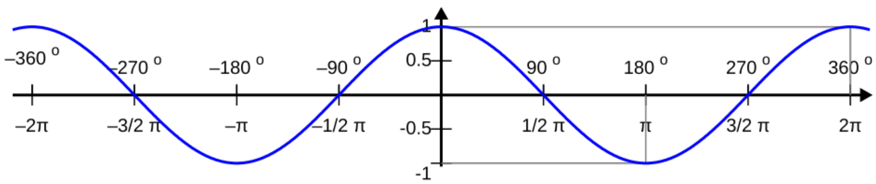

Figure 2: Plot of $\cos(x)$

2. Trigonometric functions based on them, such as: $\arcsin(x)$, $\cosh(x)$ or $\csc(x)$,
3. Functions of complex angle, usually written as $\sin(z)$ or $\cos(x + yi)$.

I sort them in such order for reason, to be exact, defining functions from the 2nd point, as name suggests, require knowledge what both $\sin(x)$ and $\cos(x)$ are. Last but not least, to define complex trigonometric functions, we need derivatives of their real forms, such as $\cosh(x)$.

2.2 Sin of real angle

Sin, function which we remember as $\frac{Opposite}{Hypotenuse}$, comes from Euler's formula. Euler's formula is way to express complex number, $e^{ix} = \cos(x) + i\sin(x)$. Let's look at the process which gives us $\sin x$.

1. $e^{ix} - e^{-ix} = [\cos(x) + i\sin(x)] - [\cos(x) - i\sin(x)]$
2. $\Rightarrow \cos(x) + i\sin(x) - \cos(x) + i\sin(x)$
3. $\Rightarrow 2i\sin(x)$
4. $\Rightarrow e^{ix} - e^{-ix} = 2i\sin(x) \Rightarrow \frac{e^{ix}-e^{-ix}}{2i} = \sin(x)$
5. $\Rightarrow \sin(x) = \dfrac{e^{ix} - e^{-ix}}{2i} \Rightarrow \sin(x) = \dfrac{z - \overline{z}}{2i}$

1. Moment where e^{ix} and e^{-ix} are rewrite as an expression with sin and cos.
2. and 3. are moments where we ar simplifying formulas, result is expression in which we can eliminate $cos(x)$ and $-\cos(x)$.
4. We are left with $2i\sin(x)$ as final expression equal to $e^{ix} - e^{-ix}$.

5. Stage in which $2i$ is eliminated by dividing both sides by itself.

How does this work?

Idea behind those operations is that, e^{-ix} is conjugation to e^{ix}, so first one can be written as $\bar{z} = x - yi$, and second as $z = x + yi$. We can see that when we subtract them, we get $x + yi - x + yi$, which is equal to $0x + 2i$, part $2i$, is same as $2i\sin(x)$, as we know that, we can divide by $2i$ and get clean $\sin(x)$.

Example

Find value of $\sin(x)$, for $x = 0.54$ (radians), using formula $\frac{e^{ix}-e^{-ix}}{2i}$.
1st step: write formula where x is replaced with value 0.54
2nd step: using modified formula, $e^{ix} = \cos(x) + i\cos(x - \frac{\pi}{2})$ we ar finding position of real and imaginary parts.
3rd step: with knowledge that $e^{-i(0.54)}$ is conjugation to $e^{i(0.54)}$, we can define that it is $0.8577 - 0.5141i$.
4th step: We subtract them, during this operation we can see that, both real parts are getting 0, and only imaginary parts can be added.
5th step: To get real number, which is between both z and $\bar{z}$, we are dividing result of subtraction by $2i$.

$$\textbf{1st: } \frac{e^{i(0.54)}-e^{-i(0.54)}}{2i}$$
$$\textbf{2nd: } e^{i(0.54)} = \cos(0.54) + i\cos(0.54 - \tfrac{\pi}{2}) = 0.8577 + 0.5141i$$
$$\textbf{3rd: } \begin{cases} e^{i(0.54)} = 0.8577 + 0.5141i \\ e^{-i(0.54)} = 0.8577 - 0.5141i \end{cases}$$
$$\textbf{4th: } (0.8577 + 0.5141i) + (-1)(0.8577 - 0.5141i)$$
$$\Rightarrow 0.8577 + 0.5141i - 0.8577 + 0.5141i$$
$$\Rightarrow 0.5141i + 0.5141i = 2(0.5141)i$$
$$\textbf{5th: } \frac{2(0.5141)i}{2i} = 0.5141$$

Answer: Value of $\sin(0.54)$ is equal to 0.5141.

2.3 Cos of real angle

Cos is usually defined as $\frac{adjacent}{hypotenuse}$ in our trigo classes, now, let's see how we can write it using Euler's formula, analogously as sin, with one exception, we will use $e^{xi} + e^{-xi}$.

1. $e^{ix} + e^{-ix} = \cos(x) + i\sin(x) + \cos(x) - i\sin(x) \Rightarrow$
2. $\Rightarrow 2\cos(x) \Rightarrow$
3. $\Rightarrow e^{ix} + e^{-ix} = 2\cos(x) \Rightarrow \frac{e^{ix}+e^{-ix}}{2} = \cos(x)$
4. $\cos(x) = \frac{e^{ix}+e^{-ix}}{2} \Rightarrow \cos(x) = \frac{z+\bar{z}}{2}$

1. As we can see, this time at first step we can eliminate $+i\sin(x)$ and $-i\sin(x)$,
2. Thank to previous operation, we ar left with $2\cos(x)$,
3. Cause $e^{ix} + e^{-ix} = 2\cos(x)$, if we divide by 2, we are left with formula equal to pure $\cos(x)$,
4. We can rewrite $\frac{e^{ix}+e^{-ix}}{2}$, as expression with conjugation: $\frac{z+\bar{z}}{2} = \cos(x)$.

How does this work?

Similarly as in sin, idea is that point between conjugations, so $z = x + yi$ and $\bar{z} = x - iy$, point between is $(x + x) + (y - y)i \Rightarrow 2x + 0i)$ is $2\cos(x)$, so if we divide result by 2 we get pure $\cos(x)$.

Example

Find value of $\cos(x)$, for x=0.99 (radians), using formula $\frac{e^{ix}+e^{-ix}}{2}$.
1st step: write formula where x is replaced with 0.99,
2nd step: using formula $e^{ix} = \sin(x + \frac{\pi}{2}) + i\sin(x)$, we can find value of both complex numbers,
3rd step: we are computing $0.5486 + i0.8360 + 0.5486 - i0.8360 = 2(0.5486)$ be cause, $\sin(x + \frac{\pi}{x}) = 0.5486$ and $\sin(x) = -0.8360$,
4th step: we divide 2(0.5486) by 2, and her 0.5486, which is value of cos(0.99).
Answer: Value of cos(0.54) is equal to 0.5486.

1st: $\frac{e^{i(0.99)}+e^{-i(0.99)}}{2}$
2nd: $e^{i0.99} = \sin(0.99 + \frac{\pi}{2}) + i\sin(x) = 0.5486 + 0.8360i$
3rd: $\begin{cases} e^{i(0.99)} = 0.5486 + 0.8360i \\ e^{-i(0.99)} = 0.5486 - 0.8360i \end{cases}$
4th: $0.5486 + i0.8360 + 0.5486 - i0.8360$
$\Rightarrow 0.5486 + 0.5486 = 2(0.5486)$
5th: $\frac{2(0.5486)}{2} = 0.5486$
Answer: Value of cos(0.99) is equal to 0.5486.

Usage - sin and cos

1. 2D matrix rotation:

$$A(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

change in cords: x, y $\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} x_{new} = x\cos(\alpha) - y\sin(\alpha) \\ y_{new} = x\sin(\alpha) + y\cos(\alpha) \end{cases}$

In those formula x, y are our start coordinates Cartesian plane. By changing angle position of point by angle α.

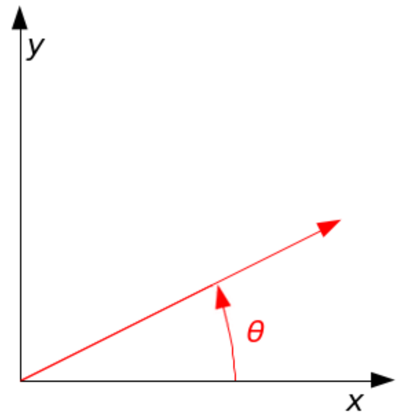

Figure 3: Change in position by angle θ. **License information at the end of document!**

2.3.1 Tangent and cotangent

Another trigonometric functions which we will look at, are tangent and cotangent (tan and cot). Those functions ar based on two previous ones. Tangent line is really important in our life, example of usage is tangent line in case of derivatives as derivative at given point. They are usually defined in two ways, one uses triangle as reference, other sin and cos:

1. Using triangle:
   (a) $\tan(x) = \frac{opposite}{adjacent}$,
   (b) $\cot(x) = \frac{adjacent}{opposite}$
2. Using $\sin(x)$ and $\cos(x)$:
   (a) $\tan(x) = \frac{\sin(x)}{\cot(x)}$,
   (b) $\cot(x) = \frac{\cos(x)}{\sin(x)}$

Basing on knowledge from previous part of document, we'll define either tan and cot.

1. $\tan(x) = -i\frac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}}$
2. $\cot(x) = i\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}}$

How does this work?

As we could see, previously formulas for both sin and cos were divided by $2i$ or 2, so why aren't we doing that now? Because both 2 are canceling:

$$\frac{2i\sin}{2\cos} = \frac{i\sin}{\cos}$$

and now as you can see we get $i\tan$, so to eliminate this i, we multiply by $-i$ (cause $i \times i = -1$ and $i \times -i = 1$). Similar rules can be aplied to *cot*.

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