Example of Bayes predictor

TABLE DES MATIÈRES

1 BAYES ESTIMATOR AND BAYES RISK

Consider the following joint random variable (X, Y).

- $\mathcal{X} = \{0, 1, 2\}$
- $\mathcal{Y} = \{0, 1\}$.
- X follows a uniform law on $\mathcal{X}$.
- $$Y = \begin{cases} B(1/5) \text{ if } X = 0 \\ B(3/4) \text{ if } X = 1 \\ B(2/3) \text{ if } X = 2 \end{cases}$$

  With $B(p)$ a Bernoulli law with parameter p.

Compute the Bayes estimator and the bayes risk.

1.1 Solution

1.1.1 *Bayes predictor, general case*

We prove again the general result on the Bayes predictor in the case of binary classification. We have seen that the Bayes predictor is defined by

$$f^*(x) = \underset{z \in \mathcal{Y}}{\arg\min}\, E\Big[l(y, z)|X = x\Big] \tag{1}$$

Hence

$$\begin{aligned} f^*(x) &= \underset{z \in \mathcal{Y}}{\arg\min}\, E\Big[l(y, z)|X = x\Big] \\ &= \underset{z \in \mathcal{Y}}{\arg\min}\, P(Y \neq z|X = x) \\ &= \underset{z \in \mathcal{Y}}{\arg\min}\, 1 - P(Y = z|X = x) \\ &= \underset{z \in \mathcal{Y}}{\arg\max}\, P(Y = z|X = x) \end{aligned} \tag{2}$$

The optimal classifier selects the most probable output given $X = x$.

1.1.2 *Application*

In this case :

- $f^*(0) = 0$
- $f^*(1) = 1$
- $f^*(2) = 1$

1.1.3 *Bayes risk, general case*

We have also seen that using the law of total expectation, with the "0-1" loss,

$$\begin{aligned} R^* &= E\Big[l(Y, f^*(X))\Big] \\ &= E_X\Big[E_Y\Big(l(Y \neq f^*(X)|X\Big)\Big] \\ &= E_X\Big[P(Y \neq f^*(X)|X)\Big] \end{aligned} \tag{3}$$

But we have

$$P\big(Y \neq f^*(X)|X = x\big) = P\big(Y \neq f^*(x)\big) \tag{4}$$

We note $\eta(x) = P(Y = 1|X = x)$. Then,

- If $\eta(x) > \frac{1}{2}$, then $f^*(x) = 1$, and $P\big(Y \neq f^*(x)\big) = P\big(Y = 0\big) = 1 - \eta(x)$
- If $\eta(x) < \frac{1}{2}$, then $f^*(x) = 0$, and $P\big(Y \neq f^*(x)\big) = P\big(Y = 1\big) = \eta(x)$

In both cases, $P\big(Y \neq f^*(x)\big) = \min(\eta(x), 1 - \eta(x))$.
We conclude that

$$R^* = E_X\Big[\min(\eta(X), 1 - \eta(X))\Big] \tag{5}$$

1.1.4 *Application*

In this setting :

$$\begin{aligned} R^* &= \frac{1}{3}\frac{1}{5} + \frac{1}{3}\frac{1}{4} + \frac{1}{3}\frac{1}{3} \\ &= \frac{1}{3}(\frac{1}{5} + \frac{1}{4} + \frac{1}{3}) \\ &= \frac{1}{3}(\frac{12}{60} + \frac{15}{60} + \frac{20}{60}) \\ &= \frac{1}{3}(\frac{47}{60}) \\ &= \frac{47}{180} \end{aligned} \tag{6}$$