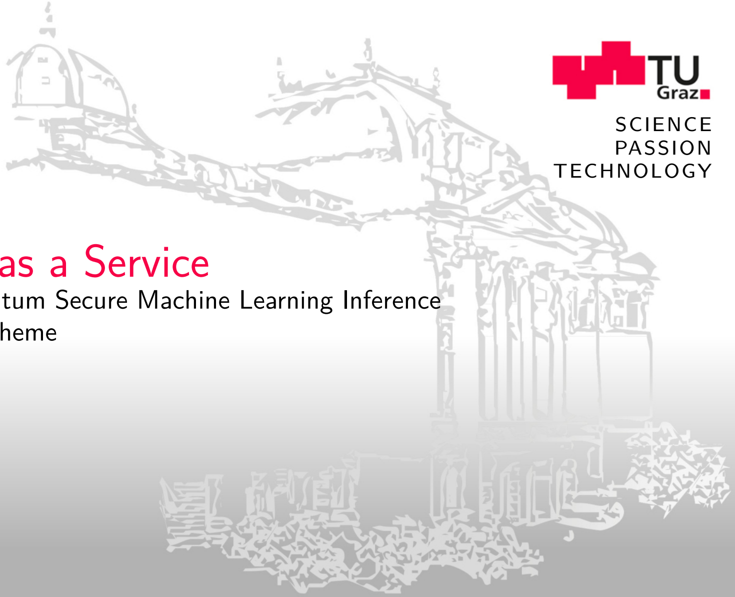
SCIENCE
PASSION
TECHNOLOGY



Secure Classification as a Service

Levelled Homomorphic, Post-Quantum Secure Machine Learning Inference based on the CKKS Encryption Scheme

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Tins and Talk, 21.07.2023

What do we want?

- Step 1: Encrypted Machine Learning (ML) as a service
- Step 2: Using Homomorphic Encryption
- Step 3: That is Post-Quantum secure
- Step 4: Which is somewhat fast
- Step 5: And accurate enough

And of course, an actual implementation.

Privacy-Preserving Machine Learning (PPML)

- Machine Learning allows us to solve otherwise difficult problems.
- Development of new applications and solutions ’of numerical nature’ in different fields
  - Example: Health Care with highly sensitive medical data.
  - Even more volatile results: disease indicators.
- ⇒ Demand for privacy-preserving solutions in ML applications.

Feedforward Neural Networks

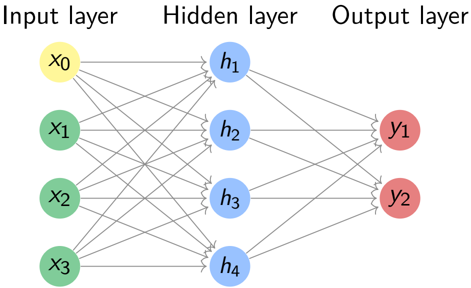


Figure: The simple neural network used in our demonstrator with $\boldsymbol{h} = \mathrm{relu}(M_1 \boldsymbol{x} + \boldsymbol{b_1})$ and the output $\boldsymbol{y} = \mathrm{softmax}(M_2 \boldsymbol{h} + \boldsymbol{b_2})$.

⇒ Need: Addition, Multiplication, Packing, Rotations. Trained in plain.

Goal: Classify MNIST Images of Handwritten Digits

- Two major types of ML: Supervised and Unsupervised Learning
- Popular dataset: Modified National Institute of Standards and Technology (MNIST). Encode as vector of 784 entries.

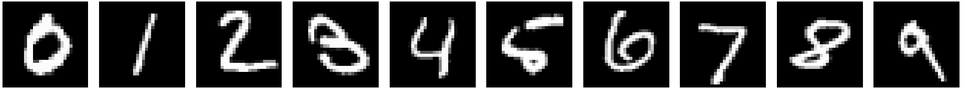
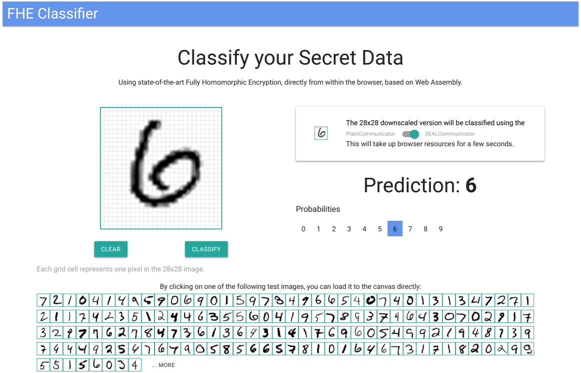


Figure: Sample images of the MNIST database of handwritten digits [4]. The dataset contains 70,000 images of 28 × 28 greyscale pixels valued from 0 to 255 as well as associated labels (as required for supervised learning).

Secure Handwritten Digit Classification as a Service - Demo



FHE Classifier

Classify your Secret Data

Using state-of-the-art Fully Homomorphic Encryption, directly from within the browser, based on Web Assembly.

The 28x28 downscaled version will be classified using the PlainCommunicator SEALCommunicator
This will take up browser resources for a few seconds.

Prediction: **6**

Probabilities

Each grid cell represents one pixel in the 28x28 image.

By clicking on one of the following test images, you can load it to the canvas directly:

Scan the QR-Code:

Figure: https://secure-classification.peter.waldert.at/.

Ring Homomorphism

Definition

Given two rings $(R, +, \cdot)$ and $(S, \oplus, \otimes)$, we call a mapping $\varphi : R \to S$ a ring homomorphism when it satisfies the following conditions:

$$\forall a, b \in R : \varphi(a + b) = \varphi(a) \oplus \varphi(b) \land \varphi(a \cdot b) = \varphi(a) \otimes \varphi(b)$$

The Rivest-Shamir-Adleman (RSA) Scheme

From the integers $\mathbb{Z}$, define the quotient ring $(\mathbb{Z}/q\mathbb{Z}, +, \cdot)$ for some modulus $q \in \mathbb{N}$.

With unpadded RSA [8], $\mathcal{E} : \mathbb{Z}/q\mathbb{Z} \mapsto \mathbb{Z}/q\mathbb{Z}$

$$\mathcal{E}(m) := m^r \mod q \quad r, q \in \mathbb{N}$$

applied to the messages $m_1, m_2 \in \mathbb{Z}/q\mathbb{Z}$ respectively, the following holds:

$$\begin{aligned} \mathcal{E}(m_1) \cdot \mathcal{E}(m_2) &\equiv (m_1)^r (m_2)^r \mod q \\ &\equiv (m_1 m_2)^r \mod q \\ &\equiv \mathcal{E}(m_1 \cdot m_2) \mod q \end{aligned}$$

$\Rightarrow$ A Group Homomorphism!

Some Notation

- $\mathbb{Z}[X] := \{p : \mathbb{C} \mapsto \mathbb{C}\,, p(x) = \sum_{k=0}^{\infty} a_k x^k, a_k \in \mathbb{Z}\ \forall k \geq 0\}$
  - Complex-valued Polynomials with integer coefficients.
- $\mathbb{Z}_q[X] := (\mathbb{Z}/q\mathbb{Z})[X] = \mathbb{Z}[X]/q\mathbb{Z}[X]$
- $\mathbb{Z}_q[X]/\Phi_M(X) \underbrace{= \mathbb{Z}_q[X]/(X^N + 1)}_{\text{for } M=2N\text{, powers of 2}}$ using the M^{th} cyclotomic polynomial $\Phi_M(X)$.
  - Its elements are polynomials of degree $(N-1)^1$ with integer coefficients mod q.

[1] For general M, degree $\varphi(M) - 1$

Polynomial Rings

Definition (Cyclotomic Polynomial)

Given the n^{th} roots of unity $\{\xi_k\}$, define $\Phi_n \in \mathbb{Z}[X]$ as

$$\Phi_n(x) := \prod_{\substack{k=1 \\ \xi_k \text{primitive}}}^{n} (x - \xi_k) \,.$$

It is unique for each given $n \in \mathbb{N}$.

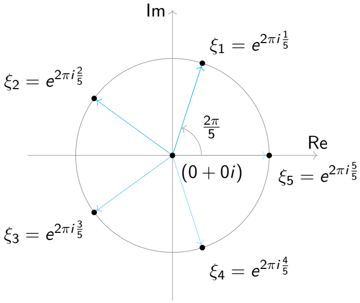
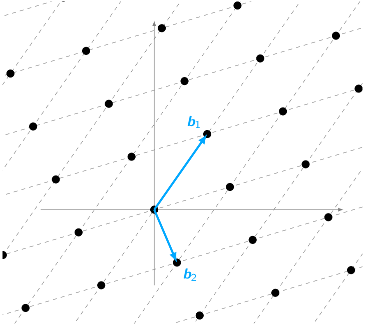

Im
Re
$\xi_1 = e^{2\pi i \frac{1}{5}}$
$\xi_2 = e^{2\pi i \frac{2}{5}}$
$\xi_3 = e^{2\pi i \frac{3}{5}}$
$\xi_4 = e^{2\pi i \frac{4}{5}}$
$\xi_5 = e^{2\pi i \frac{5}{5}}$
$\frac{2\pi}{5}$
$(0 + 0i)$

Figure: The 5th roots of unity

Lattices



Definition (Lattice)

A lattice $(\mathcal{L}, +, \cdot)$ is a vector field over the integers $(\mathbb{Z}, +, \cdot)$, given n basis vectors $\boldsymbol{b_1}, \boldsymbol{b_2}, \dots, \boldsymbol{b_n} \in \mathbb{R}^n$, with

$$\mathcal{L} := \left\{ \sum_{i=1}^{n} c_i \boldsymbol{b}_i \,\middle|\, c \in \mathbb{Z} \right\} \subseteq \mathbb{R}^n .$$

Problems I

Definition (Shortest Vector Problem (SVP))

Given a lattice $\mathcal{L}$ constructed from n basis vectors, find the shortest non-zero lattice vector $\boldsymbol{x} \in \mathcal{L} \setminus \{\mathbf{0}\}$, i.e. find $\boldsymbol{x}$ such that $||\boldsymbol{x}|| = \lambda_{min}$ [6].

Based on SVP, one can construct GapSVP, an approximative version with advantages for usage in practical problems.

Problems II

Definition (Decisional Approximate SVP (GapSVP))

Given a lattice $\mathcal{L}$ and some pre-defined function $\gamma : \mathbb{N} \mapsto \mathbb{R}$ depending on the lattice dimension n (constant for a given $\mathcal{L}$) with $\gamma(n) \geq 1$, the decisional approximate shortest vector problem is distinguishing between $\lambda_{min} \leq 1$ and $\lambda_{min} > \gamma(n)$. For other cases, it is up to the algorithm what to return.

Problems III

Definition (Short Integer Solution (SIS) Problem)

For m given vectors $(\boldsymbol{a}_i)_{0<i\leq m} \in (\mathbb{Z}/q\mathbb{Z})^n$ that comprise the columns of a matrix $A \in (\mathbb{Z}/q\mathbb{Z})^{n\times n}$ and an upper bound β, find a solution vector $\boldsymbol{z} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ such that

$$A\boldsymbol{z} = \mathbf{0} \quad \text{with} \quad ||\boldsymbol{z}|| \leq \beta\,.$$

The Learning With Errors (LWE) Problem

Definition (LWE-Distribution $A_{\boldsymbol{s},\chi_{error}}$)

Given a prime $q \in \mathbb{N}$ and $n \in \mathbb{N}$, choose a secret $\boldsymbol{s} \in (\mathbb{Z}/q\mathbb{Z})^n$. Sampling from $A_{\boldsymbol{s},\chi_{error}}$:

- Sample a uniformly random vector $a \in (\mathbb{Z}/q\mathbb{Z})^n$.
- Sample a scalar 'error term' $\mu \in \mathbb{Z}/q\mathbb{Z}$ from χ_{error}.
- Compute a noisy inner product $b = \boldsymbol{s} \cdot \boldsymbol{a} + \mu$.
- Output the pair $(\boldsymbol{a}, b) \in (\mathbb{Z}/q\mathbb{Z})^n \times (\mathbb{Z}/q\mathbb{Z})$.

Search-LWE-Problem: Given m independent samples $(\boldsymbol{a}_i, b_i)_{0<i\leq m}$ from $A_{\boldsymbol{s},\chi_{error}}$, find $\boldsymbol{s}$.

Published by Regev in 2005 [7]. Lead to the FHE scheme by Gentry in 2009 [2].

The Learning With Errors on Rings (RLWE) Problem

Definition (RLWE-Distribution $B_{\mathbf{s},\chi_{error}}$)

Given a quotient ring $(R/qR, +, \cdot)$, choose a secret $s \in R/qR$. Sampling from the RLWE distribution $B_{s,\chi_{error}}$:

- Uniformly randomly draw an element $a \in R/qR$
- Sample 'noise' $\mu \in R/qR$ from χ_{error}.
- Set $b = s \cdot a + \mu$, with $\cdot$ denoting the ring multiplication operation.
- Output the pair $(a, b) \in R/qR \times R/qR$.

Proven equivalent to LWE.

Use Search-RLWE to construct a cryptosystem... Idea: Attacker needs to solve LWE given the public key to recover the secret s.

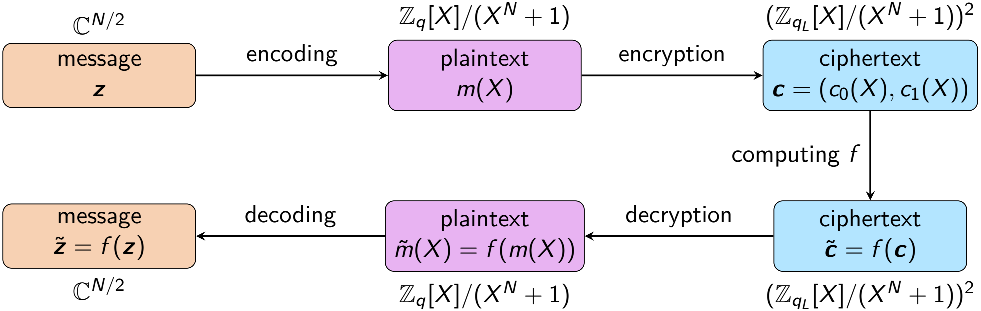
What is CKKS?

- Levelled Homomorphic Encryption Scheme [1].

$$\forall m_1, m_2 : \mathcal{E}(m_1) + \mathcal{E}(m_2) = \mathcal{E}(m_1 + m_2) \text{ and } \mathcal{E}(m_1) \cdot \mathcal{E}(m_2) = \mathcal{E}(m_1 \cdot m_2)$$

- Enables Public-Key (Asymmetric) Cryptography.
- Approximative Floating-Point Arithmetic.
- Security based on Learning With Errors.
- Single Instruction Multiple Data (SIMD) Encoding.

Overview of Cheon-Kim-Kim-Song (CKKS)



$\mathbb{C}^{N/2}$

message $\boldsymbol{z}$

encoding

$\mathbb{Z}_q[X]/(X^N+1)$

plaintext $m(X)$

encryption

$(\mathbb{Z}_{q_L}[X]/(X^N+1))^2$

ciphertext $\boldsymbol{c} = (c_0(X), c_1(X))$

computing f

ciphertext $\tilde{\boldsymbol{c}} = f(\boldsymbol{c})$

$(\mathbb{Z}_{q_L}[X]/(X^N+1))^2$

decryption

plaintext $\tilde{m}(X) = f(m(X))$

$\mathbb{Z}_q[X]/(X^N+1)$

decoding

message $\tilde{\boldsymbol{z}} = f(\boldsymbol{z})$

$\mathbb{C}^{N/2}$

Figure: Schematic overview of CKKS [1], adapted from [3]. A plain vector $\boldsymbol{z} \in \mathbb{C}^{N/2}$ is encoded to $m = \text{CKKS.Encode}(\boldsymbol{z})$, encrypted to $\boldsymbol{c} = \text{CKKS.Encrypt}(\boldsymbol{p}, m)$, decrypted and decoded to a new $\tilde{\boldsymbol{z}} = \text{CKKS.Decode}(\text{CKKS.Decrypt}(s, \tilde{\boldsymbol{c}}))$.

Encryption and Decryption

Public key $\boldsymbol{p} = (b, a)$ with $b = -(as + \tilde{\mu})$, secret key s, probability distributions χ_{enc}, χ_{error}, plaintext (=message) $m \in R/qR$, ciphertext $\boldsymbol{c}$.

CKKS.

Encrypt$(\boldsymbol{p}, m)$ Let $(b, a) = \boldsymbol{p}$, $u \leftarrow \chi_{enc}$, $\mu_1, \mu_2 \leftarrow \chi_{error}$, then the ciphertext is
$$\boldsymbol{c} = u \cdot \boldsymbol{p} + (m + \mu_1, \mu_2) = (m + bu + \mu_1, au + \mu_2) \quad \rightarrow \boldsymbol{c}$$

Decrypt$(s, \boldsymbol{c})$ Decrypt the ciphertext $\boldsymbol{c} = (c_0, c_1)$ as $m = [c_0 + c_1 s]_{q_L} \quad \rightarrow m$

Leaves the attacker with the RLWE problem.

Homomorphic Addition

$$\text{CKKS.Add}(\boldsymbol{c},\boldsymbol{c}') \quad \text{Output } \overline{\boldsymbol{c}} = \boldsymbol{c} + \boldsymbol{c}' = \begin{pmatrix} \delta(m+m') + b(u+u') + (\mu_1 + \mu_1') \\ a(u+u') + (\mu_2 + \mu_2') \end{pmatrix}^T$$

Indeed, the ciphertext $\overline{\boldsymbol{c}}$ correctly decrypts back to $\overline{m} := m + m'$:

$$\begin{aligned}
\text{CKKS.Decrypt}(s,\overline{\boldsymbol{c}}) &= \lfloor \delta^{-1}[\overline{c_0} + \overline{c_1}s]_t \rceil \\
&= \lfloor \delta^{-1}[\delta\overline{m} + b\overline{u} + \overline{\mu_1} + (a\overline{u} + \overline{\mu_2})s]_t \rceil \\
&= \lfloor [(\delta^{-1}\delta)\overline{m} + \delta^{-1}b\overline{u} + \delta^{-1}\overline{\mu_1} + \delta^{-1}as\overline{u} + \delta^{-1}\overline{\mu_2}s]_t \rceil \\
&= \lfloor [\overline{m} - \cancel{\delta^{-1}as\overline{u}} - \delta^{-1}\tilde{\mu}\overline{u} + \delta^{-1}\overline{\mu_1} + \cancel{\delta^{-1}as\overline{u}} + \delta^{-1}\overline{\mu_2}s]_t \rceil \\
&= \lfloor [\overline{m} + \underbrace{\delta^{-1}(\overline{\mu_1} + \overline{\mu_2}s - \tilde{\mu}\overline{u})}_{:=\epsilon\,,||\epsilon||\ll 1}]_t \rceil \approx \lfloor [\overline{m}]_t \rceil = \lfloor \overline{m} \rceil \approx \overline{m}
\end{aligned}$$

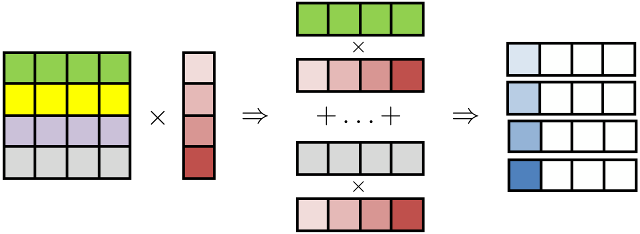
What about Long-Term Security?

Quantum Computers affect Cryprography today:

- Problems believed to be NP-hard on classical computers can be computed in polynomial time using a quantum computer.
- No hardness proofs of the RSA or integer factorisation problems exist.
- SHOR's, GROVER's and other algorithms can 'break' many cryptographic schemes used today.
- Existence of a powerful quantum computer endangers the security of TLS, etc.

Microsoft SEAL is (from the point of todays knowledge) still secure in the presence of a quantum computer.

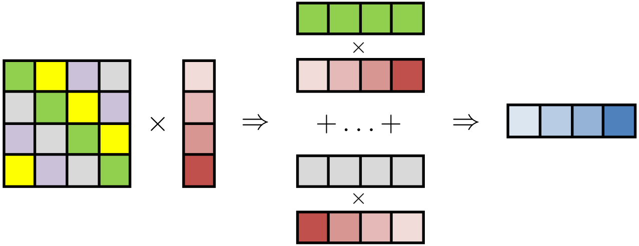
Matrix Multiplication: The Naïve Method



$$\{M\boldsymbol{x}\}_i = \sum_{j=1}^{t} M_{ij} x_j \, .$$

Image adapted from [3].

Matrix Multiplication: The Diagonal Method



$$M\boldsymbol{x} = \sum_{j=0}^{t-1} \mathrm{diag}_j(M) \cdot \mathrm{rot}_j(\boldsymbol{x}) \,.$$

Image adapted from [3].

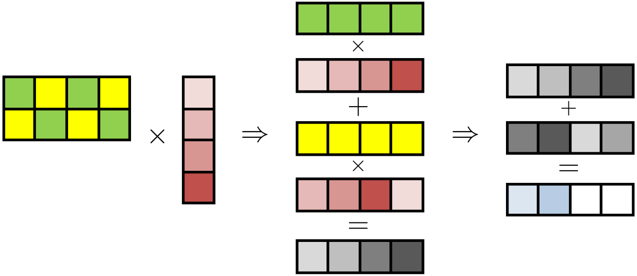
Proof (Diagonal Method).

For all indices $i \in \mathbb{Z}/t\mathbb{Z}$,

$$\left\{\sum_{j=0}^{t-1} \mathrm{diag}_j(M) \cdot \mathrm{rot}_j(\boldsymbol{x})\right\}_i = \sum_{j=0}^{t-1} M_{i,(i+j)} x_{i+j} \overset{[k=i+j]}{=} \sum_{k=i}^{t+i-1} M_{ik} x_k = \sum_{k=0}^{t-1} M_{ik} x_k = \{M\boldsymbol{x}\}_i \,.$$

□

Matrix Multiplication: The Hybrid Method



$$M\boldsymbol{x} = (y_i)_{i \in \mathbb{Z}/s\mathbb{Z}} \text{ with } \boldsymbol{y} = \sum_{k=1}^{t/s} \mathrm{rot}_{ks}\left(\sum_{j=1}^{s} \mathrm{diag}_j(M) \cdot \mathrm{rot}_j(\boldsymbol{x}) \right) .$$

Image adapted from [3].

Proof (Hybrid Method).

For all indices $i \in \mathbb{Z}/s\mathbb{Z}$,

$$\{\boldsymbol{y}\}_i = \left\{ \sum_{k=1}^{t/s} \mathrm{rot}_{ks}\left(\sum_{j=1}^{s} \mathrm{diag}_j(M) \cdot \mathrm{rot}_j(\boldsymbol{x}) \right) \right\}_i = \sum_{k=1}^{t/s} \sum_{j=1}^{s} M_{i,(i+j)+ks} x_{(i+j)+ks} \,,$$

substituting $l = i + j + ks$ and condensing the nested sums into one single summation expression since $\sum_{k=1}^{t/s} \sum_{j=1}^{s} f(j + ks) = \sum_{l=1}^{t} f(l)$, we obtain

$$y_i = \sum_{l=1+i}^{t+i} M_{il} x_l = \sum_{l=1}^{t} M_{il} x_l = \{M\boldsymbol{x}\}_i \,.$$

□

Theorem (Babystep-Giantstep Method)

Given a matrix $M \in \mathbb{R}^{t \times t}$ *and a vector* $\boldsymbol{x} \in \mathbb{R}^{t}$, *with* $t = t_1 \cdot t_2$ *split into two Babystep-Giantstep (BSGS) parameters* $t_1, t_2 \in \mathbb{N}$ *and*

$$diag'_p(M) = rot_{-\lfloor p/t_1 \rfloor \cdot t_1}(diag_p(M)),$$

one can express a matrix-vector multiplication as follows:

$$M\boldsymbol{x} = \sum_{k=0}^{t_2-1} rot_{(kt_1)}\left(\sum_{j=0}^{t_1-1} diag'_{(kt_1+j)}(M) \cdot rot_j(\boldsymbol{x}) \right)$$

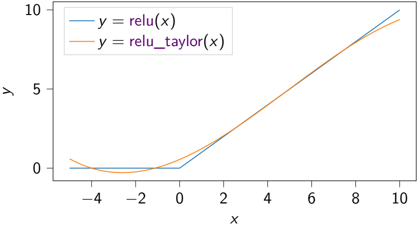
where $\cdot$ *denotes an element-wise multiplication of two vectors.*

Polynomial Evaluation

- In between the dense layers, we need to evaluate the relu$(x) := \max(x, 0)$ function.
  $\Rightarrow$ Approximate it by a series expansion...

$$\text{relu\_taylor}(x) = -0.006137x^3 + 0.090189x^2 + 0.59579x + 0.54738\,.$$

- The softmax activation at the end can be done by the client.



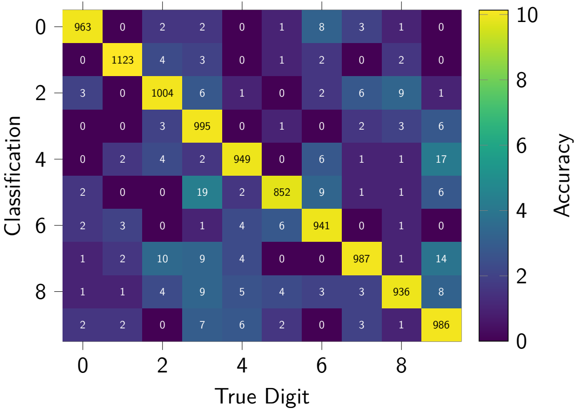
Runtime Benchmarks & Communication Overhead

Table: Performance benchmarks and communication overhead of the classification procedure on an Intel® i7-5600U CPU, including the encoding and decoding steps.

Mode	**SecLevel**	B_1	B_2	N	**MatMul**	T / s	M / MiB	Δ / 1
Release	tc128	34	25	8192	Diagonal	8.39	132.72	0.0364
					Hybrid	1.35	132.72	0.0362
					BSGS	1.66	132.72	0.1433
	tc128	60	40	16384	Diagonal	17.24	286.51	0.0363
					Hybrid	3.05	286.51	0.0364
					BSGS	3.66	286.51	0.1399
	tc256	60	40	32768	Diagonal	35.24	615.16	0.0363
					Hybrid	5.99	615.16	0.0364
					BSGS	7.34	615.16	0.1399

In Plain: 784 byte requests, taking 50 µs; Encrypted: 132 MiB requests, taking 1.4 s.

Chaos everywhere: The Confusion Matrix



963	0	2	2	0	1	8	3	1	0
0	1123	4	3	0	1	2	0	2	0
3	0	1004	6	1	0	2	6	9	1
0	0	3	995	0	1	0	2	3	6
0	2	4	2	949	0	6	1	1	17
2	0	0	19	2	852	9	1	1	6
2	3	0	1	4	6	941	0	1	0
1	2	10	9	4	0	0	987	1	14
1	1	4	9	5	4	3	3	936	8
2	2	0	7	6	2	0	3	1	986

Plain Accuracy: 97.6 %, Encrypted Accuracy: 97.3 %.

Ciphertext Visualisations

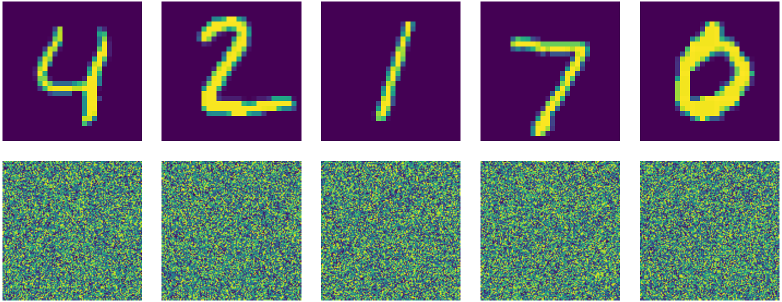


Figure: Ciphertext Visualisation: The first row corresponds to the images in plain, the second row depicts an encrypted version, namely the reconstructed polynomial coefficients $\{a_k\}$ of the ciphertext polynomial.

Questions?

Glossary I

BSGS	Babystep-Giantstep	27
CKKS	Cheon-Kim-Kim-Song	18
FHE	Fully Homomorphic Encryption	15
LWE	Learning With Errors	15
ML	Machine Learning	2
MNIST	Modified National Institute of Standards and Technology	5
NP	Non-deterministic Polynomial time	21
PPML	Privacy-Preserving Machine Learning	3
RLWE	Learning With Errors on Rings	16
RSA	Rivest-Shamir-Adleman	8
SIMD	Single Instruction Multiple Data	17
TLS	Transport Layer Security	21

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Some more Details...

Additional Material omitted in main talk.

- Learning With Errors on Rings
- Encoding and Decoding transformations
- The BabyStep-Giantstep method
- Proof of Diagonal, Hybrid method
- Shor's Algorithm

Encoding and Decoding

CKKS.

Encode($\boldsymbol{z}$) For a given input vector $\boldsymbol{z}$, output

$$m = (\underline{\sigma}^{-1} \circ \underline{\rho_\delta}^{-1} \circ \underline{\pi}^{-1})(\boldsymbol{z}) = \underline{\sigma}^{-1}(\lfloor \delta \cdot \underline{\pi}^{-1}(\boldsymbol{z}) \rceil_{\underline{\sigma}(R)}) \quad \to m$$

Decode(m) Decode plaintext m as $\boldsymbol{z} = (\underline{\pi} \circ \underline{\rho_\delta} \circ \underline{\sigma})(m) = (\underline{\pi} \circ \underline{\sigma})(\delta^{-1} m) \quad \to \boldsymbol{z}$

- Three transformations: $\underline{\sigma}^{-1}$, $\underline{\rho_\delta}^{-1}$ and $\underline{\pi}^{-1}$.
- Key idea: Homomorphic property, they preserve additivity and multiplicativity.
- Allows for homomorphic SIMD operations.

Definition (Canonical Embedding $\underline{\sigma}$)

For a real-valued polynomial $p \in \mathcal{S}$, define the canonical embedding of $\mathcal{S}$ in $\mathbb{C}^N$ as a mapping $\underline{\sigma} : \mathcal{S} \mapsto \mathbb{C}^N$ with

$$\underline{\sigma}(p) := \left(p(e^{-2\pi ij/N})\right)_{j \in \mathbb{Z}_d^*}$$

with $\mathbb{Z}_d^* := \{x \in \mathbb{Z}/d\mathbb{Z} \,|\, \mathrm{gcd}(x, d) = 1\}$ the set of all integers smaller than d that do not share a factor > 1 with d. The image of $\underline{\sigma}$ given a set of inputs R shall be denoted as $\underline{\sigma}(R) \subseteq \mathbb{C}^N$. Let the inverse of $\underline{\sigma}$ be denoted by $\underline{\sigma}^{-1} : \mathbb{C}^N \mapsto \mathcal{S}$.

Definition (Discretisation to an element of $\underline{\sigma}(R)$)

Using one of several round-off algorithms (cf. [5]), given an element of $\mathbb{H}$, define a rounding operation $\underline{\rho}^{-1} : \mathbb{H} \mapsto \underline{\sigma}(R)$ that maps an $\boldsymbol{h} \in \mathbb{H}$ to its closest element in $\underline{\sigma}(R) \subset \mathbb{H}$, also denoted as

$$\underline{\rho}^{-1}(\boldsymbol{h}) := \lfloor \boldsymbol{h} \rceil_{\underline{\sigma}(R)} \,.$$

Further let $\underline{\rho_\delta}^{-1}(\boldsymbol{h}) = \lfloor \delta \cdot \boldsymbol{h} \rceil_{\underline{\sigma}(R)}$ denote the same rounding operation but with prior scaling by a scalar factor δ. Note that $\underline{\rho}$ is given directly as the identity operation because all elements of its domain are already elements of its image. Similarly, $\underline{\rho_\delta}(\boldsymbol{y}) = \delta^{-1} \cdot \boldsymbol{y}$.

Appendix

Definition (Natural Projection $\underline{\pi}$)

Let T be a multiplicative subgroup of $\mathbb{Z}_d^*$ with $\mathbb{Z}_d^*/T = \{\pm 1\} = \{1T, -1T\}$, then the natural projection $\underline{\pi} : \mathbb{H} \mapsto \mathbb{C}^{N/2}$ is defined as

$$\underline{\pi}\left((z_j)_{j\in\mathbb{Z}_M^*}\right) := (z_j)_{j\in T}$$

Let its inverse be denoted by $\underline{\pi}^{-1} : \mathbb{C}^{N/2} \mapsto \mathbb{H}$ and consequently defined as

$$\underline{\pi}^{-1}\left((z_j)_{j\in T}\right) := \left(\nu(z_j)\right)_{j\in\mathbb{Z}_M^*} \text{ with } \nu(z_j) = \begin{cases} z_j & \text{if } j \in T \\ \overline{z_j} & \text{otherwise} \end{cases}$$

SHOR's Algorithm I

Peter SHOR's algorithm was published in 1994 [9] and will be outlined here shortly as it is a core element to security considerations of modern cryptosystems. The core structure of the algorithm is

1. guessing some $g \in \mathbb{N}$ that we hope shares a factor with a large $N = p \cdot q$ $(p, q, N \in \mathbb{N})$,
2. improving that guess g by a quantum subroutine and
3. applying EUCLID's algorithm to find p and q the factors of N.

Shor's Algorithm II

The core factorisation idea is the following, not specific to quantum computation: We know that for a pair $g, N \in \mathbb{N}$, we can always find some $r \in \mathbb{N}$ such that

$$g^r = mN + 1,\ m \in \mathbb{N}\,,$$

we are looking for a g^r that is exactly one more than a multiple of N. Rearranging,

$$g^r - 1 = mN \iff (g^{\frac{r}{2}} + 1)(g^{\frac{r}{2}} - 1) = mN$$

we have found two factors $g^{\frac{r}{2}} + 1$ and $g^{\frac{r}{2}} - 1$ (for even r) that share a common factor with N and apply Euclid's algorithm to get p and q.

Shor's Algorithm III

Thereby, we instruct the quantum computer to raise our guess g by all possible powers $\in \mathbb{N}$ up to some boundary in order to obtain

$$|1, g^1\rangle + |2, g^2\rangle + |3, g^3\rangle, \ldots$$

which we then take modulo N, resulting in a superposition of remainders

$$|1, [g^1]_N\rangle + |2, [g^2]_N\rangle + |3, [g^3]_N\rangle + \ldots .$$

Here is where Shor's key idea came in: The remainders in the above superposition expose repetitions at a period of exactly r (which, by our definition fulfils $g^r \equiv 1 \mod N$)

$$g^x \equiv g^{x+r} \equiv g^{x+2r} \equiv \ldots \equiv g^{x+ar} \mod N$$

SHOR's Algorithm IV

the remainders are periodic with frequency $\frac{1}{r}$.

The above can be quickly derived from $g^r = mN + 1$, therefore

$$g^{x+r} = g^x g^r = (\tilde{m}N + [g^x]_N)(mN + 1) = (m\tilde{m}N + [g^x]_N m + \tilde{m})N + [g^x]_N$$

is indeed congruent to $g^x \mod N$.

From the output of

$$\mathrm{QFT}\left(|1, [g^1]_N\rangle + |2, [g^2]_N\rangle + |3, [g^3]_N\rangle + \ldots \right)$$

we obtain the dominant frequency $\frac{1}{r}$ yielding us our desired improved guess [9].