SCIENCE
PASSION
TECHNOLOGY

Secure Classification as a Service

Levelled Homomorphic, Post-Quantum Secure Machine Learning Inference based on the CKKS Encryption Scheme

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Outline

Privacy-Preserving Machine Learning (PPML)

- Development of new applications and solutions 'of numerical nature' in different fields
  - Example: Health Care with highly sensitive medical data.
  - For instance, RNA sequences, images of skin, lab data, medical records, etc.
  - The results are even more volatile: disease indicators.
- $\Rightarrow$ Demand for privacy-preserving solutions in Machine Learning (ML) applications.
- By the way: Post-Quantum Secure Cryptosystems!

What about Long-Term Security?

Quantum Computers affect Cryprography today:

$\psi(\boldsymbol{r})$

- Problems believed to be NP-hard on classical computers can be computed in polynomial time using a quantum computer.
- No hardness proof of the integer factorisation or RSA problems exist as of today.
- SHOR's, GROVER's and other algorithms can 'break' many cryptographic schemes used today.
- The existence of a sufficiently powerful quantum computer endangers the security of TLS, etc.

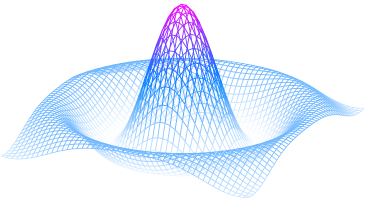


Figure: Illustration of a wave function ψ as commonly used in quantum mechanics.

Our Webservice is (from the point of todays knowledge) still secure in the presence of a quantum computer.

The Rivest-Shamir-Adleman (RSA) Scheme

From the integers $\mathbb{Z}$, define the quotient ring $(\mathbb{Z}/q\mathbb{Z}, +, \cdot)$ for some modulus $q \in \mathbb{N}$.

With unpadded RSA [8], $\mathcal{E} : \mathbb{Z}/q\mathbb{Z} \mapsto \mathbb{Z}/q\mathbb{Z}$

$$\mathcal{E}(m) := m^r \mod q \quad r, q \in \mathbb{N}$$

applied to the messages $m_1, m_2 \in \mathbb{Z}/q\mathbb{Z}$ respectively, the following holds:

$$\begin{aligned}
\mathcal{E}(m_1) \cdot \mathcal{E}(m_2) &\equiv (m_1)^r (m_2)^r \mod q \\
&\equiv (m_1 m_2)^r \mod q \\
&\equiv \mathcal{E}(m_1 \cdot m_2) \mod q
\end{aligned}$$

$\Rightarrow$ A Group Homomorphism!

Lattices

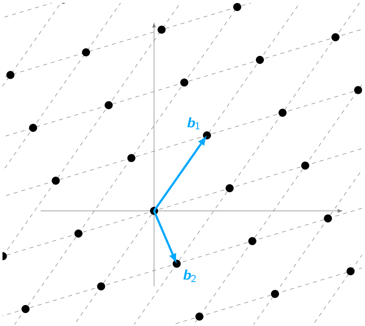


Figure: Illustration of a standard lattice $\mathcal{L}$ with two basis vectors $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$.

Definition (Lattice)

A lattice $(\mathcal{L}, +, \cdot)$ is a vector field over the integers $(\mathbb{Z}, +, \cdot)$, defined using a set of n basis vectors $\boldsymbol{b_1}, \boldsymbol{b_2}, \ldots, \boldsymbol{b_n} \in \mathbb{R}^n$, that can be introduced as a set

$$\mathcal{L} := \left\{ \sum_{i=1}^{n} c_i \boldsymbol{b}_i \,\middle|\, c \in \mathbb{Z} \right\} \subseteq \mathbb{R}^n$$

equipped with at least vector addition $+ : \mathcal{L} \times \mathcal{L} \mapsto \mathcal{L}$ and scalar multiplication $\cdot : \mathbb{Z} \times \mathcal{L} \mapsto \mathcal{L}$.

The Learning With Errors (LWE) Problem

Definition (LWE-Distribution $A_{\boldsymbol{s},\chi_{error}}$)

Given a prime $q \in \mathbb{N}$ and $n \in \mathbb{N}$, choose a secret $\boldsymbol{s} \in (\mathbb{Z}/q\mathbb{Z})^n$. Sampling from $A_{\boldsymbol{s},\chi_{error}}$:

- Sample a uniformly random vector $a \in (\mathbb{Z}/q\mathbb{Z})^n$.
- Sample a scalar 'error term' $\mu \in \mathbb{Z}/q\mathbb{Z}$ from χ_{error}, often also referred to as noise.
- Set $b = \boldsymbol{s} \cdot \boldsymbol{a} + \mu$, with $\cdot$ denoting the standard vector product.
- Output the pair $(\boldsymbol{a}, b) \in (\mathbb{Z}/q\mathbb{Z})^n \times (\mathbb{Z}/q\mathbb{Z})$.

Search-LWE-Problem: Given m independent samples $(\boldsymbol{a}_i, b_i)_{0<i\leq m}$ from $A_{\boldsymbol{s},\chi_{error}}$, find $\boldsymbol{s}$.

Published by Regev in 2005 [7]. Lead to the FHE scheme by Gentry in 2009 [2].

Some Notation

- $\mathbb{Z}[X] := \{p : \mathbb{C} \mapsto \mathbb{C}\,, p(x) = \sum_{k=0}^{\infty} a_k x^k, a_k \in \mathbb{Z}\ \forall k \geq 0\}$
  - Complex-valued Polynomials with integer coefficients.
- $\mathbb{Z}_q[X] := (\mathbb{Z}/q\mathbb{Z})[X] = \mathbb{Z}[X]/q\mathbb{Z}[X]$
- $\mathbb{Z}_q[X]/\Phi_M(X) \underbrace{= \mathbb{Z}_q[X]/(X^N + 1)}_{\text{for } M=2N\text{, powers of 2}}$ using the M^{th} cyclotomic polynomial $\Phi_M(X)$.
  - Its elements are polynomials of degree $(N-1)$[1] with integer coefficients mod q.

[1] For general M, degree $\varphi(M) - 1$

The Learning With Errors on Rings (RLWE) Problem

Corollary (RLWE-Distribution $B_{\boldsymbol{s},\chi_{error}}$)

Given a quotient ring $(R/qR, +, \cdot)$, choose a secret $s \in R/qR$. Sampling from the RLWE distribution $B_{s,\chi_{error}}$:

- *Uniformly randomly draw an element $a \in R/qR$*
- *Sample 'noise' $\mu \in R/qR$ from χ_{error}.*
- *Set $b = s \cdot a + \mu$, with $\cdot$ denoting the ring multiplication operation.*
- *Output the pair $(a, b) \in R/qR \times R/qR$.*

Proven equivalent to LWE.

Use Search-RLWE to construct a cryptosystem... Idea: Attacker needs to solve LWE given the public key to recover the secret s.

Overview of Cheon-Kim-Kim-Song (CKKS)

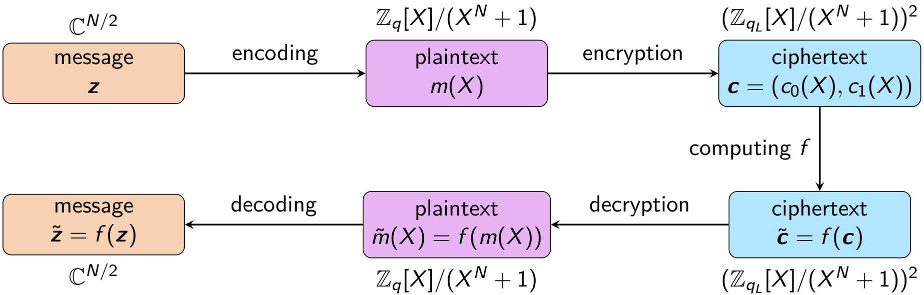


Figure: Schematic overview of CKKS [1], adapted from [3]. A plain vector $\boldsymbol{z} \in \mathbb{C}^{N/2}$ is encoded to $m = \mathrm{CKKS.Encode}(\boldsymbol{z})$, encrypted to $\boldsymbol{c} = \mathrm{CKKS.Encrypt}(\boldsymbol{p}, m)$, decrypted and decoded to a new $\tilde{\boldsymbol{z}} = \mathrm{CKKS.Decode}(\mathrm{CKKS.Decrypt}(s, \tilde{\boldsymbol{c}}))$.

Encoding and Decoding

CKKS.

Encode($\boldsymbol{z}$) For a given input vector $\boldsymbol{z}$, output
$m = (\underline{\sigma}^{-1} \circ \underline{\rho_\delta}^{-1} \circ \underline{\pi}^{-1})(\boldsymbol{z}) = \underline{\sigma}^{-1}(\lfloor \delta \cdot \underline{\pi}^{-1}(\boldsymbol{z}) \rceil_{\underline{\sigma}(R)}) \quad \rightarrow m$

Decode(m) Decode plaintext m as $\boldsymbol{z} = (\underline{\pi} \circ \underline{\rho_\delta} \circ \underline{\sigma})(m) = (\underline{\pi} \circ \underline{\sigma})(\delta^{-1} m) \quad \rightarrow \boldsymbol{z}$

- Three transformations: $\underline{\sigma}^{-1}$, $\underline{\rho_\delta}^{-1}$ and $\underline{\pi}^{-1}$.
- Key idea: Homomorphic property, they preserve additivity and multiplicativity.
- Allows for homomorphic Single Instruction Multiple Data (SIMD) operations.

Encryption and Decryption

Public key $\boldsymbol{p} = (b, a)$ with $b = -(as + \tilde{\mu})$, secret key s, probability distributions χ_{enc}, χ_{error}, plaintext (=message) $m \in R/qR$, ciphertext $\boldsymbol{c}$.

CKKS.

Encrypt$(\boldsymbol{p}, m)$ Let $(b, a) = \boldsymbol{p}$, $u \leftarrow \chi_{enc}$, $\mu_1, \mu_2 \leftarrow \chi_{error}$, then the ciphertext is $\boldsymbol{c} = u \cdot \boldsymbol{p} + (m + \mu_1, \mu_2) = (m + bu + \mu_1, au + \mu_2) \quad \rightarrow \boldsymbol{c}$

Decrypt$(s, \boldsymbol{c})$ Decrypt the ciphertext $\boldsymbol{c} = (c_0, c_1)$ as $m = [c_0 + c_1 s]_{q_L} \quad \rightarrow m$

- A public-key cryptosystem! Encrypt with $\boldsymbol{p}$, decrypt with s.
- Leaves the attacker with the RLWE problem.
- Decrypts correctly under certain conditions...

Homomorphic Addition

$$\text{CKKS.Add}(\boldsymbol{c}, \boldsymbol{c}') \quad \text{Output } \overline{\boldsymbol{c}} = \boldsymbol{c} + \boldsymbol{c}' = \begin{pmatrix} \delta(m+m') + b(u+u') + (\mu_1 + \mu_1') \\ a(u+u') + (\mu_2 + \mu_2') \end{pmatrix}^T$$

Indeed, the ciphertext $\overline{\boldsymbol{c}}$ correctly decrypts back to $\overline{m} := m + m'$:

$$\begin{aligned} \text{CKKS.Decrypt}(s, \overline{\boldsymbol{c}}) &= \lfloor \delta^{-1}[\overline{c_0} + \overline{c_1}s]_t \rceil \\ &= \lfloor \delta^{-1}[\delta\overline{m} + b\overline{u} + \overline{\mu_1} + (a\overline{u} + \overline{\mu_2})s]_t \rceil \\ &= \lfloor [(\delta^{-1}\delta)\overline{m} + \delta^{-1}b\overline{u} + \delta^{-1}\overline{\mu_1} + \delta^{-1}as\overline{u} + \delta^{-1}\overline{\mu_2}s]_t \rceil \\ &= \lfloor [\overline{m} - \cancel{\delta^{-1}as\overline{u}} - \delta^{-1}\tilde{\mu}\overline{u} + \delta^{-1}\overline{\mu_1} + \cancel{\delta^{-1}as\overline{u}} + \delta^{-1}\overline{\mu_2}s]_t \rceil \\ &= \lfloor [\overline{m} + \underbrace{\delta^{-1}(\overline{\mu_1} + \overline{\mu_2}s - \tilde{\mu}\overline{u})}_{:=\epsilon,||\epsilon||\ll 1}]_t \rceil \approx \lfloor [\overline{m}]_t \rceil = \lfloor \overline{m} \rceil \approx \overline{m} \end{aligned}$$

Goal: Classify MNIST Images of Handwritten Digits

- Two major types of ML: Supervised and Unsupervised Learning
- Popular dataset: Modified National Institute of Standards and Technology (MNIST). Encode as vector of 784 entries.

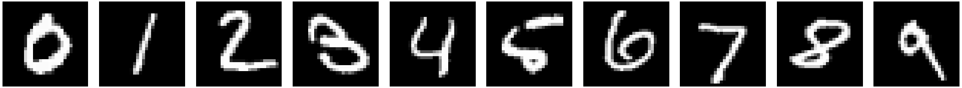


Figure: Sample images of the MNIST database of handwritten digits [5]. The dataset contains 70,000 images of 28×28 greyscale pixels valued from 0 to 255 as well as associated labels (as required for supervised learning).

Feedforward Neural Networks

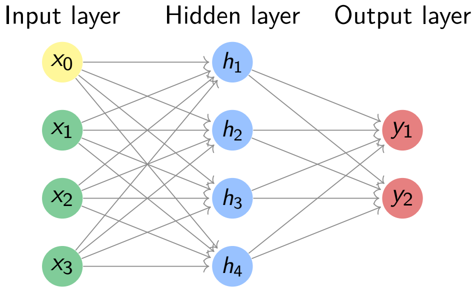


Figure: A simple neural network resembling the structure we use in our demonstrator with $\boldsymbol{h} = \text{relu}(M_1\boldsymbol{x} + \boldsymbol{b_1})$ and the output $\boldsymbol{y} = \text{softmax}(M_2\boldsymbol{h} + \boldsymbol{b_2})$.

$\Rightarrow$ Need: Addition, Multiplication, Packing, Rotations. Trained in plain.

Matrix Multiplication: The Naïve Method

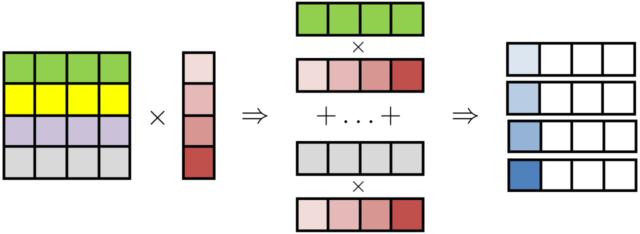


Figure: The naïve method to multiply a matrix $M \in \mathbb{R}^{s \times t}$ with a vector $\boldsymbol{x} \in \mathbb{R}^t$ (adapted from [4]).

$$\{M\boldsymbol{x}\}_i = \sum_{j=1}^{t} M_{ij} x_j \,.$$

Matrix Multiplication: The Diagonal Method

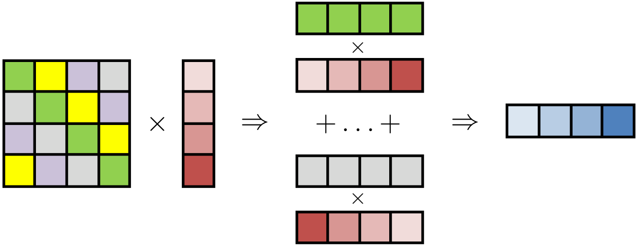


Figure: The diagonal method to multiply a square matrix with a vector (adapted from [4]).

$$Mx = \sum_{j=0}^{t-1} \mathrm{diag}_j(M) \cdot \mathrm{rot}_j(x)\,.$$

Matrix Multiplication: The Hybrid Method

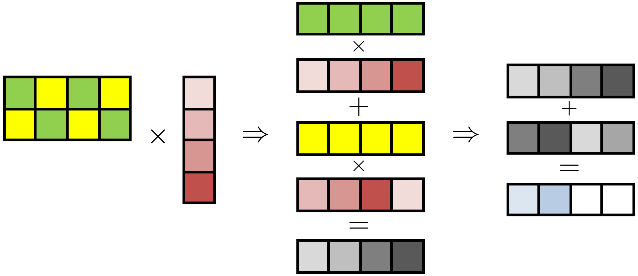


Figure: The hybrid method to multiply an arbitrarily sized matrix with a vector (adapted from [4]).

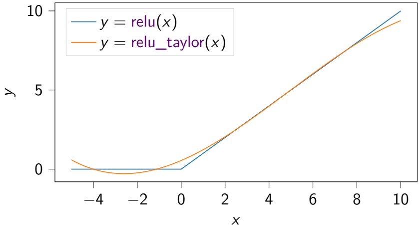
$$M\boldsymbol{x} = (y_i)_{i \in \mathbb{Z}/s\mathbb{Z}} \text{ with } \boldsymbol{y} = \sum_{k=1}^{t/s} \mathrm{rot}_{ks}\left(\sum_{j=1}^{s} \mathrm{diag}_j(M) \cdot \mathrm{rot}_j(\boldsymbol{x}) \right) .$$

Polynomial Evaluation

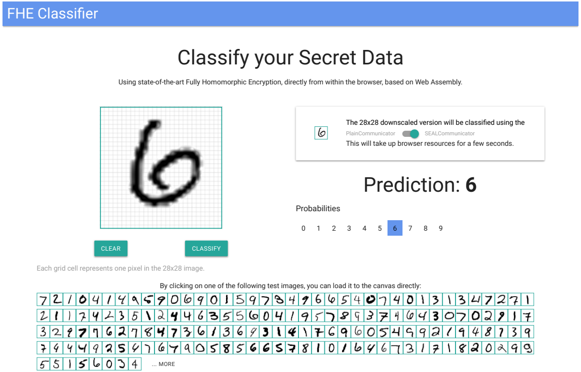
- In between the dense layers, we need to evaluate the relu$(x) := max(x, 0)$ function.
  $\Rightarrow$ Approximate it by a series expansion...

$$\text{relu\_taylor}(x) = -0.006137x^3 + 0.090189x^2 + 0.59579x + 0.54738\,.$$

- The softmax activation at the end can be done by the client.



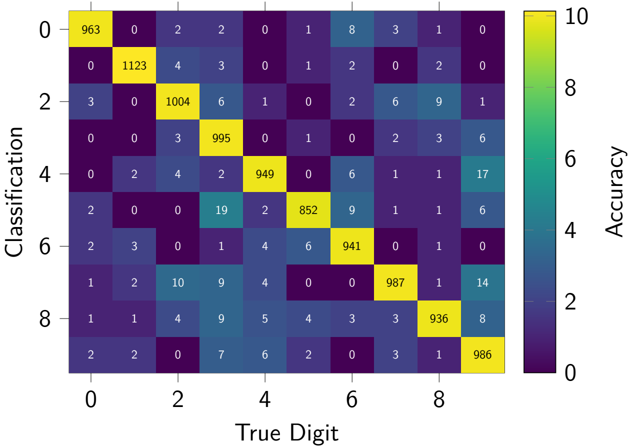
Demo: Secure Handwritten Digit Classification as a Service



Scan the QR-Code:

Figure: https://secure-classification.peter.waldert.at/.

Chaos everywhere: The Confusion Matrix



Runtime Benchmarks & Communication Overhead

Table: Performance benchmarks and communication overhead of the classification procedure on an Intel® i7-5600U CPU, including the encoding and decoding steps.

Mode	**SecLevel**	B_1	B_2	N	**MatMul**	T / s	M / MiB	Δ / 1
Release	tc128	34	25	8192	Diagonal	8.39	132.72	0.0364
					Hybrid	1.35	132.72	0.0362
					BSGS	1.66	132.72	0.1433
	tc128	60	40	16384	Diagonal	17.24	286.51	0.0363
					Hybrid	3.05	286.51	0.0364
					BSGS	3.66	286.51	0.1399
	tc256	60	40	32768	Diagonal	35.24	615.16	0.0363
					Hybrid	5.99	615.16	0.0364
					BSGS	7.34	615.16	0.1399

In Plain: 784 byte requests; Encrypted: 132 MiB requests.

Ciphertext Visualisations

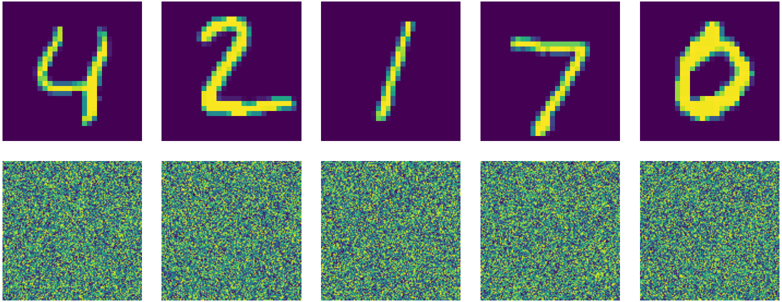


Figure: Ciphertext Visualisation: The first row corresponds to the images in plain, the second row depicts an encrypted version, namely the reconstructed polynomial coefficients $\{a_k\}$ of the ciphertext polynomial.

Conclusion

- Schemes like RSA become problematic due to Shor's Algorithm $\Rightarrow$ Lattice Crypto.
  - New Cryptosystems constructed based on Regev's LWE-problem, e.g. CKKS.
  - Encryption is homomorphic with respect to addition (and multiplication).
  - The Encoding and Decoding procedures of CKKS allow for SIMD operations needed for efficient computations.
- Image Classification of the handwritten digits can be done using a neural network.
  - The required operations can be translated to Homomorphic Encryption (HE).
  - We looked at different multiplication methods.

Questions?

Glossary I

BSGS	Babystep-Giantstep	35
CKKS	Cheon-Kim-Kim-Song	10
FHE	Fully Homomorphic Encryption	7
HE	Homomorphic Encryption	24
LWE	Learning With Errors	7
ML	Machine Learning	3
MNIST	Modified National Institute of Standards and Technology	14
NP	Non-deterministic Polynomial time	4
PPML	Privacy-Preserving Machine Learning	3
RLWE	Learning With Errors on Rings	9
RSA	Rivest-Shamir-Adleman	5

Glossary II

SIMD	Single Instruction Multiple Data	11
TLS	Transport Layer Security	4

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[2] Craig Gentry. **Fully homomorphic encryption using ideal lattices**. STOC '09. 2009.

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Some more Details...

Additional Material omitted in main talk.

- Encoding and Decoding transformations
- The BabyStep-Giantstep method
- Proof of Diagonal, Hybrid method
- Shor's Algorithm

Polynomial Rings

Definition (Cyclotomic Polynomial)

Given the n^{th} roots of unity $\{\xi_k\}$, define $\Phi_n \in \mathbb{Z}[X]$ as

$$\Phi_n(x) := \prod_{\substack{k=1 \\ \xi_k \text{primitive}}}^{n} (x - \xi_k) \,.$$

It is unique for each given $n \in \mathbb{N}$.

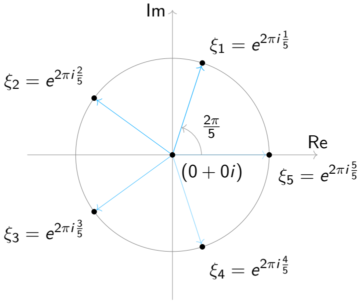

Figure: The 5^{th} roots of unity

Definition (Canonical Embedding $\underline{\sigma}$)

For a real-valued polynomial $p \in \mathcal{S}$, define the canonical embedding of $\mathcal{S}$ in $\mathbb{C}^N$ as a mapping $\underline{\sigma} : \mathcal{S} \mapsto \mathbb{C}^N$ with

$$\underline{\sigma}(p) := \left(p(e^{-2\pi ij/N})\right)_{j \in \mathbb{Z}_d^*}$$

with $\mathbb{Z}_d^* := \{x \in \mathbb{Z}/d\mathbb{Z} \mid \gcd(x, d) = 1\}$ the set of all integers smaller than d that do not share a factor > 1 with d. The image of $\underline{\sigma}$ given a set of inputs R shall be denoted as $\underline{\sigma}(R) \subseteq \mathbb{C}^N$. Let the inverse of $\underline{\sigma}$ be denoted by $\underline{\sigma}^{-1} : \mathbb{C}^N \mapsto \mathcal{S}$.

Definition (Discretisation to an element of $\underline{\sigma}(R)$)

Using one of several round-off algorithms (cf. [6]), given an element of $\mathbb{H}$, define a rounding operation $\underline{\rho}^{-1} : \mathbb{H} \mapsto \underline{\sigma}(R)$ that maps an $\boldsymbol{h} \in \mathbb{H}$ to its closest element in $\underline{\sigma}(R) \subset \mathbb{H}$, also denoted as

$$\underline{\rho}^{-1}(\boldsymbol{h}) := \lfloor \boldsymbol{h} \rceil_{\underline{\sigma}(R)} .$$

Further let $\underline{\rho_\delta}^{-1}(\boldsymbol{h}) = \lfloor \delta \cdot \boldsymbol{h} \rceil_{\underline{\sigma}(R)}$ denote the same rounding operation but with prior scaling by a scalar factor δ. Note that $\underline{\rho}$ is given directly as the identity operation because all elements of its domain are already elements of its image. Similarly, $\underline{\rho_\delta}(\boldsymbol{y}) = \delta^{-1} \cdot \boldsymbol{y}$.

Definition (Natural Projection $\underline{\pi}$)

Let T be a multiplicative subgroup of $\mathbb{Z}_d^*$ with $\mathbb{Z}_d^*/T = \{\pm 1\} = \{1T, -1T\}$, then the natural projection $\underline{\pi} : \mathbb{H} \mapsto \mathbb{C}^{N/2}$ is defined as

$$\underline{\pi}((z_j)_{j \in \mathbb{Z}_M^*}) := (z_j)_{j \in T}$$

Let its inverse be denoted by $\underline{\pi}^{-1} : \mathbb{C}^{N/2} \mapsto \mathbb{H}$ and consequently defined as

$$\underline{\pi}^{-1}((z_j)_{j \in T}) := \left(\nu(z_j)\right)_{j \in \mathbb{Z}_M^*} \text{ with } \nu(z_j) = \begin{cases} z_j & \text{if } j \in T \\ \overline{z_j} & \text{otherwise} \end{cases}$$

Theorem (Babystep-Giantstep Method)

Given a matrix $M \in \mathbb{R}^{t \times t}$ and a vector $\boldsymbol{x} \in \mathbb{R}^t$, with $t = t_1 \cdot t_2$ split into two Babystep-Giantstep (BSGS) parameters $t_1, t_2 \in \mathbb{N}$ and

$$diag'_p(M) = rot_{-\lfloor p/t_1 \rfloor \cdot t_1}(diag_p(M)) \,,$$

one can express a matrix-vector multiplication as follows:

$$M\boldsymbol{x} = \sum_{k=0}^{t_2-1} rot_{(kt_1)}\left(\sum_{j=0}^{t_1-1} diag'_{(kt_1+j)}(M) \cdot rot_j(\boldsymbol{x}) \right)$$

where $\cdot$ denotes an element-wise multiplication of two vectors.

Proof (Diagonal Method).

For all indices $i \in \mathbb{Z}/t\mathbb{Z}$,

$$\left\{\sum_{j=0}^{t-1} \mathrm{diag}_j(M) \cdot \mathrm{rot}_j(\boldsymbol{x})\right\}_i = \sum_{j=0}^{t-1} M_{i,(i+j)} x_{i+j} \overset{[k=i+j]}{=} \sum_{k=i}^{t+i-1} M_{ik} x_k = \sum_{k=0}^{t-1} M_{ik} x_k = \{M\boldsymbol{x}\}_i \,.$$

$\square$

Proof (Hybrid Method).

For all indices $i \in \mathbb{Z}/s\mathbb{Z}$,

$$\{\boldsymbol{y}\}_i = \left\{ \sum_{k=1}^{t/s} \operatorname{rot}_{ks}\left(\sum_{j=1}^{s} \operatorname{diag}_j(M) \cdot \operatorname{rot}_j(\boldsymbol{x}) \right) \right\}_i = \sum_{k=1}^{t/s} \sum_{j=1}^{s} M_{i,(i+j)+ks} x_{(i+j)+ks} \,,$$

substituting $l = i + j + ks$ and condensing the nested sums into one single summation expression since $\sum_{k=1}^{t/s} \sum_{j=1}^{s} f(j + ks) = \sum_{l=1}^{t} f(l)$, we obtain

$$y_i = \sum_{l=1+i}^{t+i} M_{il} x_l = \sum_{l=1}^{t} M_{il} x_l = \{M\boldsymbol{x}\}_i \,.$$

$\square$

SHOR's Algorithm I

Peter SHOR's algorithm was published in 1994 [9] and will be outlined here shortly as it is a core element to security considerations of modern cryptosystems. The core structure of the algorithm is

1. guessing some $g \in \mathbb{N}$ that we hope shares a factor with a large $N = p \cdot q$ ($p, q, N \in \mathbb{N}$),
2. improving that guess g by a quantum subroutine and
3. applying EUCLID's algorithm to find p and q the factors of N.

Shor's Algorithm II

The core factorisation idea is the following, not specific to quantum computation: We know that for a pair $g, N \in \mathbb{N}$, we can always find some $r \in \mathbb{N}$ such that

$$g^r = mN + 1,\ m \in \mathbb{N}\,,$$

we are looking for a g^r that is exactly one more than a multiple of N. Rearranging,

$$g^r - 1 = mN \iff (g^{\frac{r}{2}} + 1)(g^{\frac{r}{2}} - 1) = mN$$

we have found two factors $g^{\frac{r}{2}} + 1$ and $g^{\frac{r}{2}} - 1$ (for even r) that share a common factor with N and apply Euclid's algorithm to get p and q.

SHOR's Algorithm III

Thereby, we instruct the quantum computer to raise our guess g by all possible powers $\in \mathbb{N}$ up to some boundary in order to obtain

$$|1, g^1\rangle + |2, g^2\rangle + |3, g^3\rangle, \ldots$$

which we then take modulo N, resulting in a superposition of remainders

$$|1, [g^1]_N\rangle + |2, [g^2]_N\rangle + |3, [g^3]_N\rangle + \ldots .$$

Here is where SHOR's key idea came in: The remainders in the above superposition expose repetitions at a period of exactly r (which, by our definition fulfils $g^r \equiv 1 \mod N$)

$$g^x \equiv g^{x+r} \equiv g^{x+2r} \equiv \ldots \equiv g^{x+ar} \quad \mod N$$

Shor's Algorithm IV

the remainders are periodic with frequency $\frac{1}{r}$.

The above can be quickly derived from $g^r = mN + 1$, therefore

$$g^{x+r} = g^x g^r = (\tilde{m}N + [g^x]_N)(mN + 1) = (m\tilde{m}N + [g^x]_N m + \tilde{m})N + [g^x]_N$$

is indeed congruent to $g^x \mod N$.

From the output of

$$\mathrm{QFT}\left(|1, [g^1]_N\rangle + |2, [g^2]_N\rangle + |3, [g^3]_N\rangle + \ldots \right)$$

we obtain the dominant frequency $\frac{1}{r}$ yielding us our desired improved guess [9].