

# Secure Classification as a Service

Levelled Homomorphic, Post-Quantum Secure Machine Learning Inference  
based on the CKKS Encryption Scheme

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## What do we want?

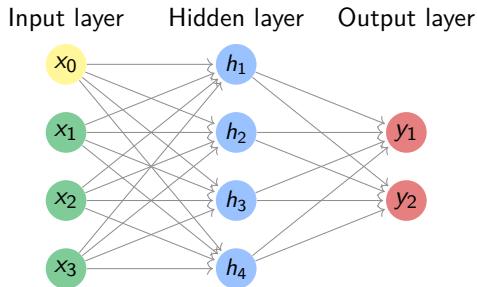
- Step 1: Encrypted Machine Learning (ML) as a service
- Step 2: Using Homomorphic Encryption
- Step 3: That is Post-Quantum secure
- Step 4: Which is somewhat fast
- Step 5: And accurate enough

And of course, an actual implementation.

## Privacy-Preserving Machine Learning (PPML)

- Machine Learning allows us to solve otherwise difficult problems.
- Development of new applications and solutions 'of numerical nature' in different fields
  - Example: Health Care with highly sensitive medical data.
  - Even more volatile results: disease indicators.
- ⇒ Demand for privacy-preserving solutions in ML applications.

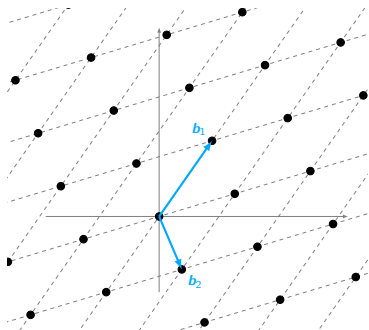
# Feedforward Neural Networks



**Figure:** The simple neural network used in our demonstrator with  $\mathbf{h} = \text{relu}(\mathbf{M}_1\mathbf{x} + \mathbf{b}_1)$  and the output  $\mathbf{y} = \text{softmax}(\mathbf{M}_2\mathbf{h} + \mathbf{b}_2)$ .

⇒ Need: Addition, Multiplication, Packing, Rotations. Trained in plain.

# Lattices



## Definition (Lattice)

A lattice  $(\mathcal{L}, +, \cdot)$  is a vector field over the integers  $(\mathbb{Z}, +, \cdot)$ , given  $n$  basis vectors  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n \in \mathbb{R}^n$ , with

$$\mathcal{L} := \left\{ \sum_{i=1}^n c_i \mathbf{b}_i \mid c \in \mathbb{Z} \right\} \subseteq \mathbb{R}^n.$$

## The Learning With Errors (LWE) Problem

### Definition (LWE-Distribution $A_{\mathbf{s}, \chi_{\text{error}}}$ )

Given a prime  $q \in \mathbb{N}$  and  $n \in \mathbb{N}$ , choose a secret  $\mathbf{s} \in (\mathbb{Z}/q\mathbb{Z})^n$ . Sampling from  $A_{\mathbf{s}, \chi_{\text{error}}}$ :

- Sample a uniformly random vector  $\mathbf{a} \in (\mathbb{Z}/q\mathbb{Z})^n$ .
- Sample a scalar 'error term'  $\mu \in \mathbb{Z}/q\mathbb{Z}$  from  $\chi_{\text{error}}$ .
- Compute a noisy inner product  $b = \mathbf{s} \cdot \mathbf{a} + \mu$ .
- Output the pair  $(\mathbf{a}, b) \in (\mathbb{Z}/q\mathbb{Z})^n \times (\mathbb{Z}/q\mathbb{Z})$ .

Search-LWE-Problem: Given  $m$  independent samples  $(\mathbf{a}_i, b_i)_{0 \leq i \leq m}$  from  $A_{\mathbf{s}, \chi_{\text{error}}}$ , find  $\mathbf{s}$ .

Published by REGEV in 2005 [6]. Lead to the FHE scheme by GENTRY in 2009 [2].

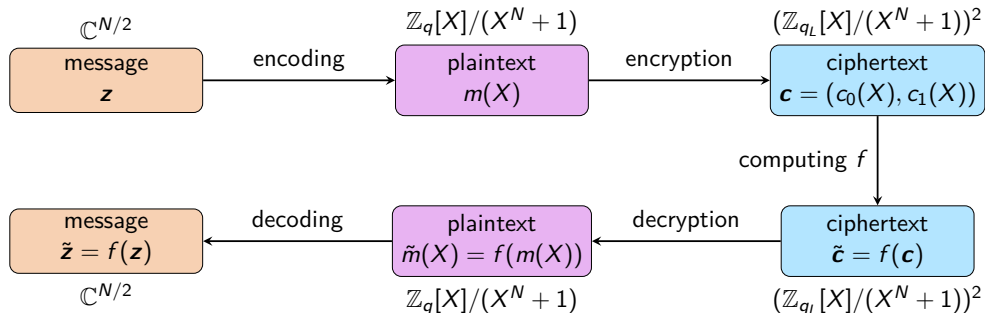
## What is CKKS?

- Levelled Homomorphic Encryption Scheme [1].

$$\forall m_1, m_2 : \mathcal{E}(m_1) + \mathcal{E}(m_2) = \mathcal{E}(m_1 + m_2) \text{ and } \mathcal{E}(m_1) \cdot \mathcal{E}(m_2) = \mathcal{E}(m_1 \cdot m_2)$$

- Enables Public-Key (Asymmetric) Cryptography.
- Approximative Floating-Point Arithmetic.
- Security based on Learning With Errors.
- Single Instruction Multiple Data (SIMD) Encoding.

# Overview of Cheon-Kim-Kim-Song (CKKS)





## Encryption and Decryption

Public key  $\mathbf{p} = (b, a)$  with  $b = -(as + \tilde{\mu})$ , secret key  $s$ , probability distributions  $\chi_{enc}$ ,  $\chi_{error}$ , plaintext (=message)  $m \in R/qR$ , ciphertext  $\mathbf{c}$ .

CKKS.

**Encrypt**( $\mathbf{p}, m$ ) Let  $(b, a) = \mathbf{p}$ ,  $u \leftarrow \chi_{enc}$ ,  $\mu_1, \mu_2 \leftarrow \chi_{error}$ , then the ciphertext is  

$$\mathbf{c} = u \cdot \mathbf{p} + (m + \mu_1, \mu_2) = (m + bu + \mu_1, au + \mu_2) \rightarrow \mathbf{c}$$

**Decrypt**( $s, \mathbf{c}$ ) Decrypt the ciphertext  $\mathbf{c} = (c_0, c_1)$  as  $m = [c_0 + c_1 s]_{q_L} \rightarrow m$

Leaves the attacker with the Learning With Errors on Rings (RLWE) problem.

## Homomorphic Addition

$$\text{CKKS.Add}(\mathbf{c}, \mathbf{c}') \quad \text{Output } \bar{\mathbf{c}} = \mathbf{c} + \mathbf{c}' = \begin{pmatrix} \delta(m + m') + b(u + u') + (\mu_1 + \mu'_1) \\ a(u + u') + (\mu_2 + \mu'_2) \end{pmatrix}^T$$

Indeed, the ciphertext  $\bar{\mathbf{c}}$  correctly decrypts back to  $\bar{m} := m + m'$ :

$$\begin{aligned} \text{CKKS.Decrypt}(s, \bar{\mathbf{c}}) &= \lfloor \delta^{-1}[\bar{c}_0 + \bar{c}_1 s]_t \rfloor \\ &= \lfloor \delta^{-1}[\delta \bar{m} + b\bar{u} + \bar{\mu}_1 + (a\bar{u} + \bar{\mu}_2)s]_t \rfloor \\ &= \lfloor [(\delta^{-1}\delta)\bar{m} + \delta^{-1}b\bar{u} + \delta^{-1}\bar{\mu}_1 + \delta^{-1}a\bar{u} + \delta^{-1}\bar{\mu}_2 s]_t \rfloor \\ &= \lfloor [\bar{m} - \cancel{\delta^{-1}a\bar{u}} - \delta^{-1}\tilde{\mu}\bar{u} + \delta^{-1}\bar{\mu}_1 + \cancel{\delta^{-1}a\bar{u}} + \delta^{-1}\bar{\mu}_2 s]_t \rfloor \\ &= \lfloor [\bar{m} + \underbrace{\delta^{-1}(\bar{\mu}_1 + \bar{\mu}_2 s - \tilde{\mu}\bar{u})}_{:=\epsilon, \|\epsilon\| \ll 1}]_t \rfloor \approx \lfloor [\bar{m}]_t \rfloor = \lfloor \bar{m} \rfloor \approx \bar{m} \end{aligned}$$

## What about Long-Term Security?

Quantum Computers affect Cryprography today:

- Problems believed to be NP-hard on classical computers can be computed in polynomial time using a quantum computer.
- No hardness proofs of the RSA or integer factorisation problems exist.
- SHOR's, GROVER's and other algorithms can 'break' many cryptographic schemes used today.
- Existence of a powerful quantum computer endangers the security of TLS, etc.

Our Webservice is (from the point of todays knowledge) still secure in the presence of a quantum computer.

## Let's Check: Secure Handwritten Digit Classification as a Service - Demo

FHE Classifier

### Classify your Secret Data

Using state-of-the-art Fully Homomorphic Encryption, directly from within the browser, based on Web Assembly.



CLEAR

CLASSIFY

Each grid cell represents one pixel in the 28x28 image.

By clicking on one of the following test images, you can load it to the canvas directly:



The 28x28 downscaled version will be classified using the

PlainCommunicator



SEALCommunicator

This will take up browser resources for a few seconds.

Prediction: 6

Probabilities

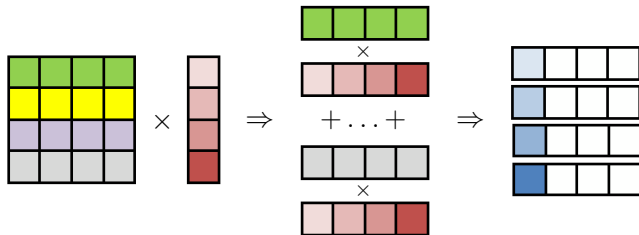
0 1 2 3 4 5 6 7 8 9

Scan the QR-Code:



Figure: <https://secure-classification.peter.waldert.at/>.

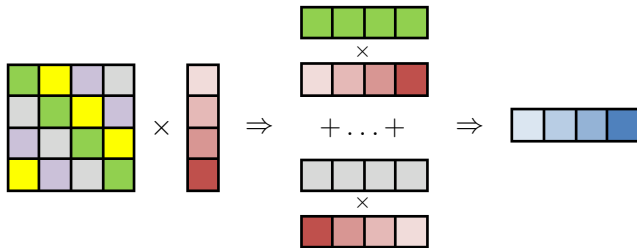
## Matrix Multiplication: The Naïve Method



$$\{M\mathbf{x}\}_i = \sum_{j=1}^t M_{ij}x_j.$$

Image adapted from [3].

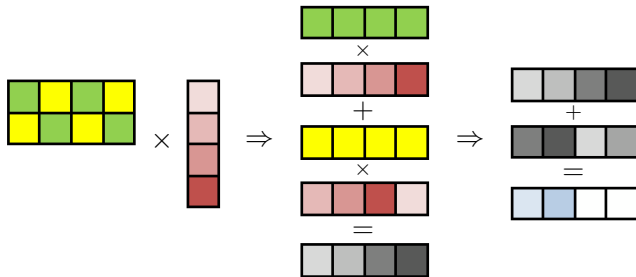
## Matrix Multiplication: The Diagonal Method



$$M\mathbf{x} = \sum_{j=0}^{t-1} \text{diag}_j(M) \cdot \text{rot}_j(\mathbf{x}).$$

Image adapted from [3].

## Matrix Multiplication: The Hybrid Method



$$M\mathbf{x} = (y_i)_{i \in \mathbb{Z}/s\mathbb{Z}} \text{ with } \mathbf{y} = \sum_{k=1}^{t/s} \text{rot}_{ks} \left( \sum_{j=1}^s \text{diag}_j(M) \cdot \text{rot}_j(\mathbf{x}) \right).$$

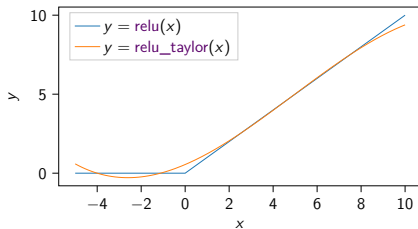
Image adapted from [3].

## Polynomial Evaluation

- In between the dense layers, we need to evaluate the  $\text{relu}(x) := \max(x, 0)$  function.  
 $\Rightarrow$  Approximate it by a series expansion...

$$\text{relu\_taylor}(x) = -0.006137x^3 + 0.090189x^2 + 0.59579x + 0.54738.$$

- The  $\text{softmax}$  activation at the end can be done by the client.





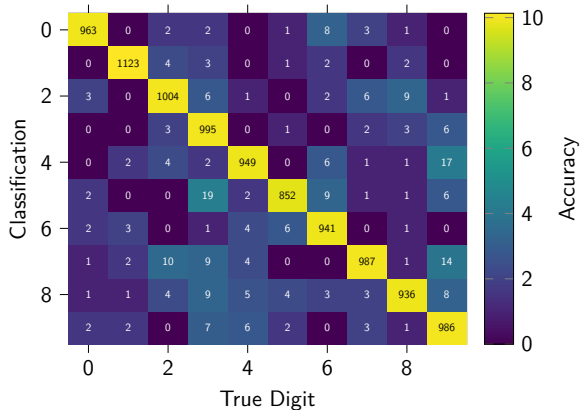
## Runtime Benchmarks & Communication Overhead

**Table:** Performance benchmarks and communication overhead of the classification procedure on an Intel® i7-5600U CPU, including the encoding and decoding steps.

Mode	SecLevel	$B_1$	$B_2$	$N$	MatMul	$T$ / s	$M$ / MiB	$\Delta$ / 1
Release	tc128	34	25	8192	Diagonal	8.39	132.72	0.0364
					Hybrid	1.35	132.72	0.0362
					BSGS	1.66	132.72	0.1433
	tc128	60	40	16384	Diagonal	17.24	286.51	0.0363
					Hybrid	3.05	286.51	0.0364
					BSGS	3.66	286.51	0.1399
	tc256	60	40	32768	Diagonal	35.24	615.16	0.0363
					Hybrid	5.99	615.16	0.0364
					BSGS	7.34	615.16	0.1399

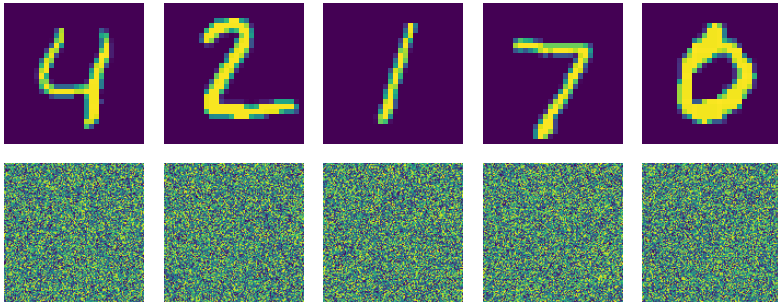
In Plain: 784 byte requests, taking 50  $\mu$ s; Encrypted: 132 MiB requests, taking 1.4 s.

## Chaos everywhere: The Confusion Matrix



Plain Accuracy: 97.6 %, Encrypted Accuracy: 97.3 %.

## Ciphertext Visualisations



## What did we do?

- Designed a neural network that solves our problem.
  - Homomorphic encryption enables private inference.
  - CKKS is inherently post-quantum secure.
- Implemented a server in C++ with a frontend interface.
- Optimized using different matrix multiplication techniques and proved that they work.
- Compared results.
- Future Work: More Complicated Networks (CNNs, etc.).

Questions?

## Glossary I

BSGS	Babystep-Giantstep	34
CKKS	Cheon-Kim-Kim-Song	8
FHE	Fully Homomorphic Encryption	6
LWE	Learning With Errors	6
ML	Machine Learning	2
MNIST	Modified National Institute of Standards and Technology	33
NP	Non-deterministic Polynomial time	11
PPML	Privacy-Preserving Machine Learning	3
RLWE	Learning With Errors on Rings	9
RSA	Rivest-Shamir-Adleman	37
SIMD	Single Instruction Multiple Data	7
TLS	Transport Layer Security	11

## Bibliography I

- [1] Jung Hee Cheon, Andrey Kim, Miran Kim and Yongsoo Song. **Homomorphic Encryption for Arithmetic of Approximate Numbers**. ASIACRYPT. 2017.
- [2] Craig Gentry. **Fully homomorphic encryption using ideal lattices**. STOC '09. 2009.
- [3] Daniel Huynh. **Cryptotree: fast and accurate predictions on encrypted structured data**. (2020). DOI: [10.48550/ARXIV.2006.08299](https://doi.org/10.48550/ARXIV.2006.08299). URL: <https://arxiv.org/abs/2006.08299>.
- [4] Yann LeCun and Corinna Cortes. **The MNIST database of handwritten digits**. 1998. URL: <http://yann.lecun.com/exdb/mnist/>.
- [5] Vadim Lyubashevsky, Chris Peikert and Oded Regev. **A Toolkit for Ring-LWE Cryptography**. IACR Cryptol. ePrint Arch. 2013.
- [6] Oded Regev. **On lattices, learning with errors, random linear codes, and cryptography**. STOC '05. 2005.
- [7] Ronald L Rivest, Adi Shamir and Leonard M Adleman. **Cryptographic communications system and method**. US Patent 4,405,829. Sept. 1983.

## Bibliography II

- [8] Peter W. Shor. **Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer.** *SIAM Journal on Computing* 26.5 (Oct. 1997), pp. 1484–1509. DOI: [10.1137/s0097539795293172](https://doi.org/10.1137/s0097539795293172).



## Some more Details...

Additional Material omitted in main talk.

- Learning With Errors on Rings
- Encoding and Decoding transformations
- The BabyStep-Giantstep method
- Proof of Diagonal, Hybrid method
- Shor's Algorithm

# Polynomial Rings

## Definition (Cyclotomic Polynomial)

Given the  $n^{\text{th}}$  roots of unity  $\{\xi_k\}$ , define  $\Phi_n \in \mathbb{Z}[X]$  as

$$\Phi_n(x) := \prod_{\substack{k=1 \\ \xi_k \text{ primitive}}}^n (x - \xi_k).$$

It is unique for each given  $n \in \mathbb{N}$ .

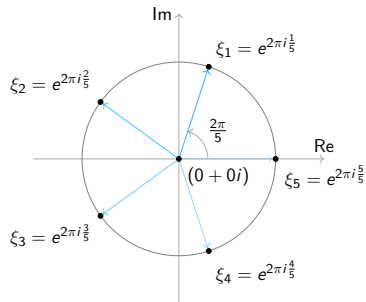


Figure: The 5<sup>th</sup> roots of unity

## Some Notation

- $\mathbb{Z}[X] := \{p : \mathbb{C} \mapsto \mathbb{C}, p(x) = \sum_{k=0}^{\infty} a_k x^k, a_k \in \mathbb{Z} \forall k \geq 0\}$ 
  - Complex-valued Polynomials with integer coefficients.
- $\mathbb{Z}_q[X] := (\mathbb{Z}/q\mathbb{Z})[X] = \mathbb{Z}[X]/q\mathbb{Z}[X]$
- $\mathbb{Z}_q[X]/\Phi_M(X) = \underbrace{\mathbb{Z}_q[X]/(X^N + 1)}_{\text{for } M=2N, \text{ powers of } 2}$  using the  $M^{\text{th}}$  cyclotomic polynomial  $\Phi_M(X)$ .
  - Its elements are polynomials of degree  $(N - 1)^1$  with integer coefficients mod  $q$ .

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<sup>1</sup>For general  $M$ , degree  $\varphi(M) - 1$

## The RLWE Problem

### Definition (RLWE-Distribution $B_{s, \chi_{error}}$ )

Given a quotient ring  $(R/qR, +, \cdot)$ , choose a secret  $s \in R/qR$ . Sampling from the RLWE distribution  $B_{s, \chi_{error}}$ :

- Uniformly randomly draw an element  $a \in R/qR$
- Sample 'noise'  $\mu \in R/qR$  from  $\chi_{error}$ .
- Set  $b = s \cdot a + \mu$ , with  $\cdot$  denoting the ring multiplication operation.
- Output the pair  $(a, b) \in R/qR \times R/qR$ .

Proven equivalent to LWE.

Use Search-RLWE to construct a cryptosystem... Idea: Attacker needs to solve LWE given the public key to recover the secret  $s$ .

## Encoding and Decoding

### CKKS.

**Encode**( $\mathbf{z}$ ) For a given input vector  $\mathbf{z}$ , output  

$$m = (\underline{\sigma}^{-1} \circ \underline{\rho}_{\delta}^{-1} \circ \underline{\pi}^{-1})(\mathbf{z}) = \underline{\sigma}^{-1}(\lfloor \delta \cdot \underline{\pi}^{-1}(\mathbf{z}) \rfloor_{\underline{\sigma}(R)}) \rightarrow m$$

**Decode**( $m$ ) Decode plaintext  $m$  as  $\mathbf{z} = (\underline{\pi} \circ \underline{\rho}_{\delta} \circ \underline{\sigma})(m) = (\underline{\pi} \circ \underline{\sigma})(\delta^{-1}m) \rightarrow \mathbf{z}$

- Three transformations:  $\underline{\sigma}^{-1}$ ,  $\underline{\rho}_{\delta}^{-1}$  and  $\underline{\pi}^{-1}$ .
- Key idea: Homomorphic property, they preserve additivity and multiplicativity.
- Allows for homomorphic SIMD operations.

## Definition (Canonical Embedding $\underline{\sigma}$ )

For a real-valued polynomial  $p \in \mathcal{S}$ , define the canonical embedding of  $\mathcal{S}$  in  $\mathbb{C}^N$  as a mapping  $\underline{\sigma} : \mathcal{S} \mapsto \mathbb{C}^N$  with

$$\underline{\sigma}(p) := (p(e^{-2\pi ij/N}))_{j \in \mathbb{Z}_d^*}$$

with  $\mathbb{Z}_d^* := \{x \in \mathbb{Z}/d\mathbb{Z} \mid \gcd(x, d) = 1\}$  the set of all integers smaller than  $d$  that do not share a factor  $> 1$  with  $d$ . The image of  $\underline{\sigma}$  given a set of inputs  $R$  shall be denoted as  $\underline{\sigma}(R) \subseteq \mathbb{C}^N$ . Let the inverse of  $\underline{\sigma}$  be denoted by  $\underline{\sigma}^{-1} : \mathbb{C}^N \mapsto \mathcal{S}$ .

## Definition (Discretisation to an element of $\underline{\sigma}(R)$ )

Using one of several round-off algorithms (cf. [5]), given an element of  $\mathbb{H}$ , define a rounding operation  $\underline{\rho}^{-1} : \mathbb{H} \mapsto \underline{\sigma}(R)$  that maps an  $\mathbf{h} \in \mathbb{H}$  to its closest element in  $\underline{\sigma}(R) \subset \mathbb{H}$ , also denoted as

$$\underline{\rho}^{-1}(\mathbf{h}) := \lfloor \mathbf{h} \rfloor_{\underline{\sigma}(R)}.$$

Further let  $\underline{\rho}_{\delta}^{-1}(\mathbf{h}) = \lfloor \delta \cdot \mathbf{h} \rfloor_{\underline{\sigma}(R)}$  denote the same rounding operation but with prior scaling by a scalar factor  $\delta$ . Note that  $\underline{\rho}$  is given directly as the identity operation because all elements of its domain are already elements of its image. Similarly,  $\underline{\rho}_{\delta}(\mathbf{y}) = \delta^{-1} \cdot \mathbf{y}$ .

## Definition (Natural Projection $\underline{\pi}$ )

Let  $T$  be a multiplicative subgroup of  $\mathbb{Z}_d^*$  with  $\mathbb{Z}_d^*/T = \{\pm 1\} = \{1T, -1T\}$ , then the natural projection  $\underline{\pi} : \mathbb{H} \mapsto \mathbb{C}^{N/2}$  is defined as

$$\underline{\pi}((z_j)_{j \in \mathbb{Z}_M^*}) := (z_j)_{j \in T}$$

Let its inverse be denoted by  $\underline{\pi}^{-1} : \mathbb{C}^{N/2} \mapsto \mathbb{H}$  and consequently defined as

$$\underline{\pi}^{-1}((z_j)_{j \in T}) := (\nu(z_j))_{j \in \mathbb{Z}_M^*} \text{ with } \nu(z_j) = \begin{cases} z_j & \text{if } j \in T \\ \bar{z}_j & \text{otherwise} \end{cases}$$



## Goal: Classify MNIST Images of Handwritten Digits

- Two major types of ML: Supervised and Unsupervised Learning
- Popular dataset: Modified National Institute of Standards and Technology (MNIST). Encode as vector of 784 entries.



**Figure:** Sample images of the MNIST database of handwritten digits [4]. The dataset contains 70,000 images of  $28 \times 28$  greyscale pixels valued from 0 to 255 as well as associated labels (as required for supervised learning).

## Theorem (Babystep-Giantstep Method)

Given a matrix  $M \in \mathbb{R}^{t \times t}$  and a vector  $\mathbf{x} \in \mathbb{R}^t$ , with  $t = t_1 \cdot t_2$  split into two Babystep-Giantstep (BSGS) parameters  $t_1, t_2 \in \mathbb{N}$  and

$$\text{diag}'_p(M) = \text{rot}_{-\lfloor p/t_1 \rfloor \cdot t_1}(\text{diag}_p(M)),$$

one can express a matrix-vector multiplication as follows:

$$M\mathbf{x} = \sum_{k=0}^{t_2-1} \text{rot}_{(kt_1)} \left( \sum_{j=0}^{t_1-1} \text{diag}'_{(kt_1+j)}(M) \cdot \text{rot}_j(\mathbf{x}) \right)$$

where  $\cdot$  denotes an element-wise multiplication of two vectors.

## Proof (Diagonal Method).

For all indices  $i \in \mathbb{Z}/t\mathbb{Z}$ ,

$$\left\{ \sum_{j=0}^{t-1} \text{diag}_j(M) \cdot \text{rot}_j(\mathbf{x}) \right\}_i = \sum_{j=0}^{t-1} M_{i,(i+j)} x_{i+j} \stackrel{[k=i+j]}{=} \sum_{k=i}^{t+i-1} M_{ik} x_k = \sum_{k=0}^{t-1} M_{ik} x_k = \{M\mathbf{x}\}_i.$$



## Proof (Hybrid Method).

For all indices  $i \in \mathbb{Z}/s\mathbb{Z}$ ,

$$\{\mathbf{y}\}_i = \left\{ \sum_{k=1}^{t/s} \text{rot}_{ks} \left( \sum_{j=1}^s \text{diag}_j(M) \cdot \text{rot}_j(\mathbf{x}) \right) \right\}_i = \sum_{k=1}^{t/s} \sum_{j=1}^s M_{i,(i+j)+ks} x_{(i+j)+ks},$$

substituting  $l = i + j + ks$  and condensing the nested sums into one single summation expression since  $\sum_{k=1}^{t/s} \sum_{j=1}^s f(j + ks) = \sum_{l=1}^t f(l)$ , we obtain

$$y_i = \sum_{l=1+i}^{t+i} M_{il} x_l = \sum_{l=1}^t M_{il} x_l = \{M\mathbf{x}\}_i.$$



## The Rivest-Shamir-Adleman (RSA) Scheme

From the integers  $\mathbb{Z}$ , define the quotient ring  $(\mathbb{Z}/q\mathbb{Z}, +, \cdot)$  for some modulus  $q \in \mathbb{N}$ .

With unpadded RSA [7],  $\mathcal{E} : \mathbb{Z}/q\mathbb{Z} \mapsto \mathbb{Z}/q\mathbb{Z}$

$$\mathcal{E}(m) := m^r \mod q \quad r, q \in \mathbb{N}$$

applied to the messages  $m_1, m_2 \in \mathbb{Z}/q\mathbb{Z}$  respectively, the following holds:

$$\begin{aligned} \mathcal{E}(m_1) \cdot \mathcal{E}(m_2) &\equiv (m_1)^r (m_2)^r \mod q \\ &\equiv (m_1 m_2)^r \mod q \\ &\equiv \mathcal{E}(m_1 \cdot m_2) \mod q \end{aligned}$$

$\Rightarrow$  A Group Homomorphism!

## SHOR's Algorithm I

Peter SHOR's algorithm was published in 1994 [8] and will be outlined here shortly as it is a core element to security considerations of modern cryptosystems. The core structure of the algorithm is

1. guessing some  $g \in \mathbb{N}$  that we hope shares a factor with a large  $N = p \cdot q$  ( $p, q, N \in \mathbb{N}$ ),
2. improving that guess  $g$  by a quantum subroutine and
3. applying EUCLID's algorithm to find  $p$  and  $q$  the factors of  $N$ .

## SHOR's Algorithm II

The core factorisation idea is the following, not specific to quantum computation: We know that for a pair  $g, N \in \mathbb{N}$ , we can always find some  $r \in \mathbb{N}$  such that

$$g^r = mN + 1, m \in \mathbb{N},$$

we are looking for a  $g^r$  that is exactly one more than a multiple of  $N$ . Rearranging,

$$g^r - 1 = mN \iff (g^{\frac{r}{2}} + 1)(g^{\frac{r}{2}} - 1) = mN$$

we have found two factors  $g^{\frac{r}{2}} + 1$  and  $g^{\frac{r}{2}} - 1$  (for even  $r$ ) that share a common factor with  $N$  and apply Euclid's algorithm to get  $p$  and  $q$ .

## SHOR's Algorithm III

Thereby, we instruct the quantum computer to raise our guess  $g$  by all possible powers  $\in \mathbb{N}$  up to some boundary in order to obtain

$$|1, g^1\rangle + |2, g^2\rangle + |3, g^3\rangle, \dots$$

which we then take modulo  $N$ , resulting in a superposition of remainders

$$|1, [g^1]_N\rangle + |2, [g^2]_N\rangle + |3, [g^3]_N\rangle + \dots$$

Here is where SHOR's key idea came in: The remainders in the above superposition expose repetitions at a period of exactly  $r$  (which, by our definition fulfils  $g^r \equiv 1 \pmod{N}$ )

$$g^x \equiv g^{x+r} \equiv g^{x+2r} \equiv \dots \equiv g^{x+ar} \pmod{N}$$



## SHOR's Algorithm IV

the remainders are periodic with frequency  $\frac{1}{r}$ .

The above can be quickly derived from  $g^r = mN + 1$ , therefore

$$g^{x+r} = g^x g^r = (\tilde{m}N + [g^x]_N)(mN + 1) = (m\tilde{m}N + [g^x]_N m + \tilde{m})N + [g^x]_N$$

is indeed congruent to  $g^x \pmod{N}$ .

From the output of

$$\text{QFT}(|1, [g^1]_N\rangle + |2, [g^2]_N\rangle + |3, [g^3]_N\rangle + \dots)$$

we obtain the dominant frequency  $\frac{1}{r}$  yielding us our desired improved guess [8].