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MASTER THESIS

Classical Option Pricing Theory and Extensions to Deep Learning

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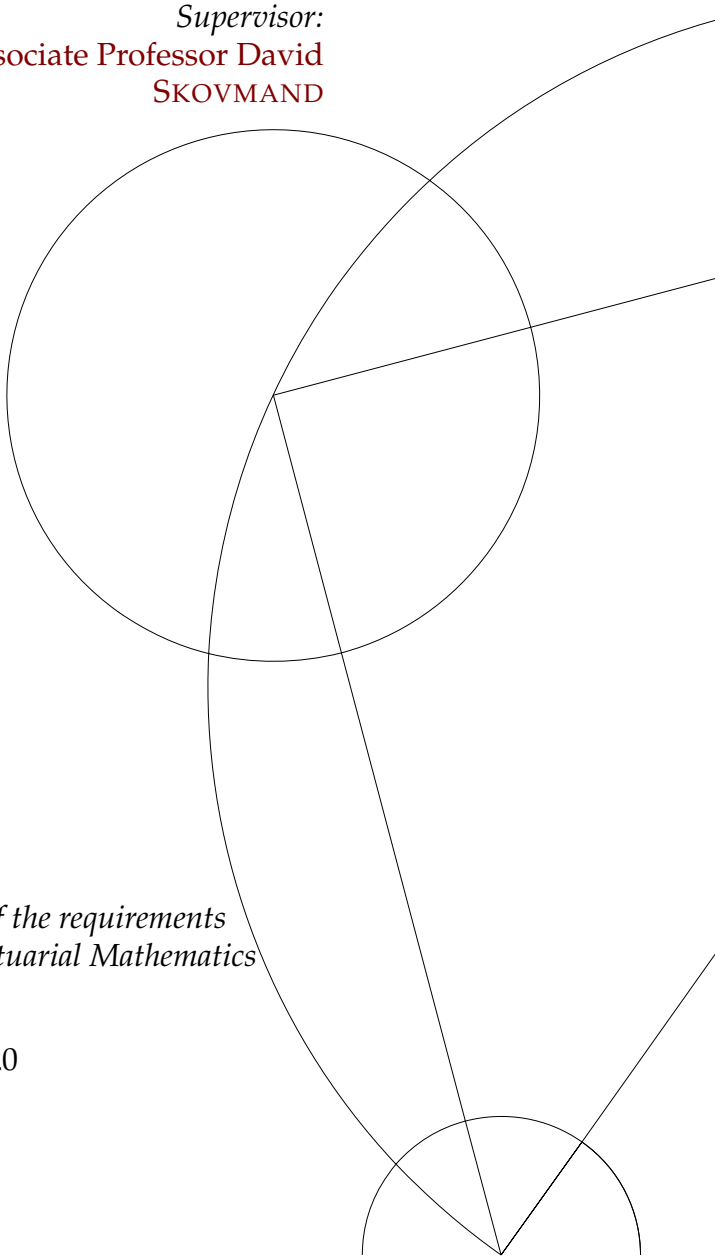

A thesis submitted in fulfillment of the requirements for the degree of Master Thesis in Actuarial Mathematics

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Declaration of Authorship

I, Peter Pommergård LIND, declare that this thesis titled, "Classical Option Pricing Theory and Extensions to Deep Learning" and the work presented in it are my own. I confirm that:

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"You were hired because you met expectations, you will be promoted if you can exceed them."

Saji Ijiyemi

UNIVERSITY OF COPENHAGEN

Abstract

Department of Mathematical Science
Science

Master Thesis in Actuarial Mathematics

Classical Option Pricing Theory and Extensions to Deep Learning

by Peter Pommergård LIND

The concepts for option pricing theory are presented and closed form solutions are provided in special cases. The options with no closed form solution are investigated through numerical methods, where both the binomial lattice model and LSM will be presented assuming the underlying Black-Scholes theory. Deep learning is then investigated to look for improvement of the existing models, where we look specifically at the MLPs regression. Our numerically study did not find any improvement in using MLPs I instead of LSM, but we believe the MLPs I can be improved with further investigation. The MLPs II was very fast, but lack the accuracy of the classical methods. Therefore the MLPs II could be beneficial in some circumstances were speed weights more than precision.

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Finally, I must express my very profound gratitude to my parents and to my family and friends for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of researching and writing this thesis. This accomplishment would not have been possible without them. Thank you.

Peter Pommergård Lind

Contents

List of Figures

List of Tables

List of Abbreviations

ATM	**A**t **T**he **M**oney
B-S	**B**lack-**S**choles
BM	**B**rownian **M**otion
FPT1	**F**undamental **P**ricing **T**heorem **I**
FPT2	**F**undamental **P**ricing **T**heorem **II**
GBM	**G**eometric **B**rownian **M**otion
ITM	**I**n **T**he **M**oney
LIBOR	**L**ondon **I**nter**b**ank **O**ffered **R**ate
LSM	**L**east **S**quare Monte Carlo **M**ethod
MLPs	**M**ulti**L**ayer **P**erceptron**s**
MRT	**M**artingale **R**epresentation **T**heorem
OTM	**O**ut **T**he **M**oney
RNVF	**R**isk **N**eutral **V**aluation **F**ormula
PDE	**P**artial **D**ifferential **E**quation
SDE	**S**tochastic **D**ifferential **E**quation
S-F	**S**elf-**F**inancing

Notations

Financial notation

c	European call option price
C	American Call option price
p	European put option price
P	American Put option price
K	Strike price
T	Maturity in years
σ	Volatility of asset
$S(0)$	Spot price
$S(T)$	Stock price at maturity
$S_i(t)$	i'th stock price at time t
r	Continuous compounding risk-free yearly interest rate
$V^h(\cdot)$	Value process of portfolio h
X	Simple Derivative
$\Phi(\cdot)$	Contract function
p_{ij}	Correlation coefficient between asset i and j
μ_i	drift of the continuous lognormal distribution of asset i
$F(t, S(t))$	pricing function of derivative depending on S(t) and time t
d	number of risky assets

Mathematical symbols

$\mathbf{A}$	matrix notation for matrix A
$\boldsymbol{a}$	vector notation for vector a
W_t	Weiner process under martingale measure Q
$\bar{W}_t$	Weiner process under probabilty measure P
$\mathbb{N}$	natural numbers: $1, 2, \ldots$
$\mathbb{R}$	real numbers
$\mathbb{R}^+$	real positive numbers including 0
$\mathbb{R}^+_*$	real positive numbers excluding 0
$\sim \mathcal{N}(\mu, \sigma^2)$	Normal distributed with mean μ and variance σ^2

Organization examples

Chapter 2	All the references are interactive marked with red lettering
Section 2.1	Under each chapter there are several sections
(Björk, 2009)	References to litteratur are in parenthesis
Equation (2.5)	Referred equations are in parenthesis
Link tensorboard 1	Links to websites are interactive marked with darkred lettering

Chapter 1

Introduction

Theory of option pricing dates back to Louis Bachelier with his PhD thesis "The Theory of Speculation" in 1900, but it was much later that option theory gained significant attention. In 1973 the world's first option exchange in Chicago opened, and in the same year Fisher Black and Myron Scholes came out with the first analytical formula for the European call option (Black and Scholes, 1973), this revolutionized market practice and option pricing theory. The idea of replication was born and the financial derivatives could now be priced by rational pricing.

The Black-Scholes model for European options is still used today, but the analytical framework cannot handle more complex products such as American put options. Since that time we have seen an increasing complexity of financial products, where big investment- and banks have increased their need for financial engineers to handle the derivative books and price derivative products. With the complexity a lot of challenges have risen in this field, where a great understanding of the products is required to handle big derivative books. Most of the existing derivatives do not have a closed form solution, so numerical methods are used to approximate the price function. To emphasize the risk of derivatives without great understanding the successful trader Warren Buffett says derivatives is "Financial weapons of mass destruction" (p. 15 (Buffett, 2002)), but on the other hand Warren Buffett has derivatives in his portfolio. A recent example of bad management of a derivative book is AIG, where AIG needed a bailout by the US government under the recent financial crisis (McDonald and Paulson, 2015). To sum up derivatives give the trader more options either to utilize arbitrage, speculate or hedge, but without care or knowledge about a derivative book the outcome can be disastrous for the owner.

What is the fair price for the right to buy or sell certain derivative for some predetermined exercise dates? And can the existing classical methods be extended by using deep learning? The thesis try to answer these questions by analytical and numerically methods in the Black-Scholes model. The pricing function for the derivatives is essential for handling a derivative book. The aim is to give an accurate and fast method for arbitrage free pricing with neural networks, where classical methods will be used for comparision. The focus in the thesis will be on financial equity contingent claims, where the prime example will be american equity put options with one or two underlying stocks.

This thesis starts presenting the basic theory on arbitrage theory to introduce the basic modeling framework for contingent claims. With this introduction the goal is to explore the applications and numerical procedures springing from the underlying theory. The classical methods in chapter 3 such as the binomial lattice approach and least square monte carlo methods (LSM) will be applied to univariate and bivariate

contingent claims, where the presentation of the classical methods are threefold. The methods give a benchmark for the neural network approaches for option pricing in later chapters. Specially the lattice approach gives strong intuition how the optimal stopping problem can be solved, where the LSM gives a framework that is easily extended to methods for pricing multivariate contingent claims. The lattice approach makes it easy to compare values with closed form solution for European options, where some special cases for exotic European option will be presented.

Deep learning theory is important for building a sound and high quality model for pricing options with neural networks, hence the basic machine learning and deep learning theory will be presented before the main chapter. Chapter 5 is the main chapter, where we try to price univariate and bivariate claims with neural networks. The aim is to look for methods which have high accuracy and fast computation time. Chapter 6 gather all the methods presented and compare them numerically. Chapter 6 contains also a discussion on the underlying Black-Scholes theory. Finally in chapter 7, we conclude on our findings and suggest what can be further investigated.

Chapter 2

Arbitrage Theory In Continuous Time Finance

Arbitrage theory in continuous time finance is a field with a lot of technical details from probability theory and stochastic calculus, where we follow the style in (Hull, 2017; Björk, 2009) to focus on intuition without going into the whelm of technicalities and proofs. The focus on this chapter will provide the basic tools and intuition for the arbitrage theory and lay the foundations for the computational finance methods. The key question is how to price derivative fairly and hedge the risk imposed by the derivative. The thesis will mainly deal with the former, where the concepts of arbitrage and replication will be important.

We start with introducing the financial markets and key concepts for building arbitrage free and complete market models (section 2.1). Then we build a framework for finding "fair" prices, i.e. finding a complete model with absence of arbitrage (section 2.2). Lastly we go into specific cases where either a closed-form solution exists or numerical methods are needed (section 2.3 and 2.4).

2.1 Financial Markets

In the financial markets there are a lot of players and different types of investments. The classical investment types are bonds and stocks, where the big players in the markets are commercial banks, investment banks, insurance companies and pension funds. Besides the classical investments types gives derivatives additional options for investment.

A derivative or a contingent claim is a financial instrument depending on an underlying asset(s), where the dependency is specified in the contract. We will focus on contingent claims with one or two underlying stocks, i.e. univariate and bivariate contingent claims, but the techniques developed can easily be extended to other types of derivatives.

To find prices of contingent claims in modeling we restrict our financial market to d risky assets $\boldsymbol{S}(t) = (S_1(t), S_2(t), \ldots, S_d(t))$ and a bank account $S_0(t)$. The probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the fundamental for modeling stochastic processes describing asset prices and trading strategies, where in the thesis the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ will be implicit assumed. Intuitively the filtration $\mathcal{F}_t$ is the information observable to time t, where the filtration $\mathbb{F}^W$ generated only by the Wiener processes $(W_t)_{0\leq t\leq T}$ will be important for having a complete

market.

The bank account is assumed to be a strictly positive adapted process $S_0 = (S_0(t))_{t\geq 0}$ and $S_0(0) = 1$, where the d risky assets are modeled by a $\mathbb{R}^d$ adapted stochastic process $\boldsymbol{S} = (\boldsymbol{S}(t))_{t\geq 0}$. The risky assets are stocks where the stocks are assumed positive $S_i(t) \geq 0$ P-a.s for all i and $t \geq 0$ by financial reasons. By using the bank account as numéraire i.e. dividing the traded asset by the bank account ($\frac{S(t)}{S_0(t)}$), this amounts to working with *zero interest*. We assume that the our financial market is frictionless.

Assumption 2.1. ***Frictionless Market:*** *We assume following institutional facts:*

- *Short positions and fractional holdings are allowed*
- *There are no bid-ask spread, i.e. selling price is equal to buying price*
- *There are no transactions costs, taxes or margin requirements of trading*
- *The market is completely liquid, i.e. it is possible to buy/sell unlimited quantities on the market. You can borrow unlimited amount from the bank by short selling*

(p. 6 (Björk, 2009))

Besides the assumptions in (Björk, 2009) we assume the market gives same uniform price for borrowing money. Stocks are fixed stochastic processes exogenously and a priori given. All the assumptions are necessary not realistic in real financial markets, but the financial market assumptions are the key to price derivative in arbitrage theory.

2.1.1 Contingent Claims

A contingent claim is a contract on a underlying asset or assets, where the price of the claim is contingent on the price behavior of the underlying asset(s). A bivariate contingent claim refers to that the option depends on two risky assets[1]. We investigate stock derivatives with different types of contracts, where we will mainly divide the derivatives into two classes.

1. Simple European derivatives
2. Exotic derivatives (e.g. American options)

Simple European options can only be exercised at maturity (time T) and they depend only on one underlying asset. Actually, we have a closed form solution for the simple European options (section 2.16). The exotic derivatives are a broad class of functions on the underlying assets, where you can e.g. have an American option where the holder can exercise from inception to maturity (section 2.4) or a contract on several underlying stocks.

Definition 2.2. European Call And Put Options: A European call option is an option where the owner of the option has the option to buy the underlying asset to price K at maturity. If the owner of the option chooses to buy the underlying asset, then the option is exercised. The contract function for the European call option is:

$$\Phi(S(T)) = \max\{S(T) - K, 0\}$$

[1]Similar refers the multivariate contingent claim to that the claim depends on two or more underlying assets

The put option is the right to sell the underlying asset to price K at maturity, hence the contract function for the European put option is:

$$\Phi(S(T)) = \max\{K - S(T), 0\}$$

Where S(T) is the price of underlying asset at maturity and K is the agreed strike price.

The American option adds the feature to the European option, that you can exercise at anytime between inception of the contract until maturity (section 2.4). For the American put option the payoff function at the stopping time is the same as for the European put at maturity (figure 2.1).

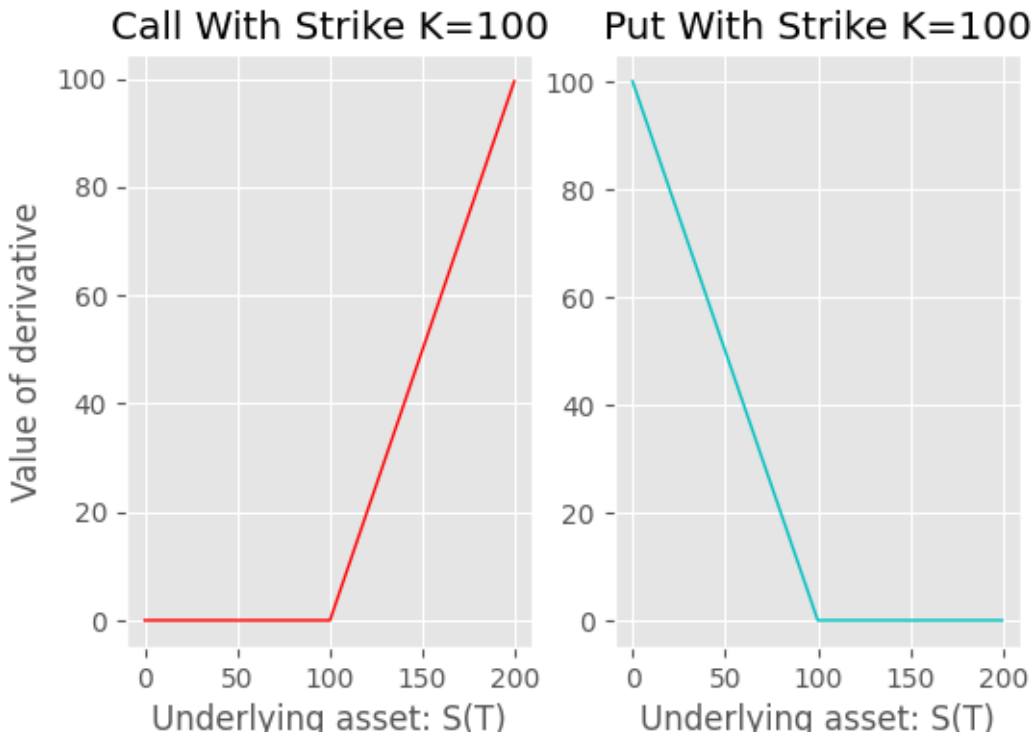

FIGURE 2.1: European options payoff at maturity with strike K=100

Figure 2.1 shows that the owner of the option has limited downside, but the graph does not take into account the initial price for the option. The profit and loss ($P\&L$) graph is also a common way of illustrating the payoff for an option, where you take the initial cost of buying the option into account. The European call and American put will be the central derivatives considered together with an American bivariate contingent claim. The task ahead is to find the initial price or the fair price for these contracts, where the concepts of completeness and arbitrage will be central.

2.1.2 Self-financing Portfolio (Without Consumption)

Before being able to use the concepts of arbitrage and completeness, the construction of the portfolio from the financial market model will be important. The portfolio is the number of each assets from the market the owner of the portfolio holds. The

value of the portfolio for a market model with the bank account and d stocks is:

$$V^h(t) = \sum_{i=0}^{d} h_i(t)S_i(t)$$

V^h is called the value process and $h_i(t)$ is number of shares of type i during the period "$[t, t + dt)$". For the definition of arbitrage (Definition 2.5) we need to restrict ourselves to self-financing (S-F) portfolios. A self-financing portfolio h, is a portfolio h which doesn't get any external injection of money.

Definition 2.3. Self-financing portfolio: A portfolio consisting of d+1 asset(s): h(t)=$(h_0(t), h_1(t), \ldots, h_d)$ is self-financing if:

$$dV^h(t) = \sum_{i=0}^{d} h_i(t)dS_i(t)$$

Where S_i is the $i'th$ asset in our portfolio, d+1 is the total number of assets in our market model and $V^h(t) = \sum_{i=0}^{d} h_i(t)S_i(t)$.

(p. 87 (Björk, 2009))

When dealing with discrete time finance the S-F portfolio is actually a budget restriction, this is important intuition for the continuous time version, because the continuous time version can be thought as the limit of the discrete version by letting step sizes in time tending to zero. To avoid pathological effects on the portfolio one often introduce the concept of an admissible portfolio:

Definition 2.4. a-admissible portfolio: For some $a \geq 0$, a portfolio h is called a-admissible if its value process $V^h(t)$ is uniform bounded from below by -a.

(p. 139 (Björk, 2009))

The definition of a-admissible portfolio is to avoid situations as the doubling strategy known from gambling and imposes a limit to the debt arrangement. The important takeaway is that the S-F portfolio is a portfolio where you only reallocate your assets through time within the portfolio.

2.1.3 Arbitrage

Arbitrage is the financial term for a "free lunch". An arbitrage opportunity produces something out of nothing without risk. For a efficient and well function market the "money pumps" cannot exist for long, because the "free lunch" would quickly be eroded by exploitation. In order to avoid making a "money machine" in our market, we want to price derivatives by not introducing arbitrage to the market.

Definition 2.5. Arbitrage: An arbitrage possibility on a financial market is an admissible self-financed portfolio h such that

$$\begin{aligned} V^h(0) &= 0 \\ P(V^h(T) \geq 0) &= 1 \\ P(V^h(T) > 0) &> 1 \end{aligned}$$

The financial market $\boldsymbol{S}$ is called arbitrage-free if there exist no arbitrage opportunities.

(p. 96 (Björk, 2009))

From the above definition we see that arbitrage is a natural financial requirement for a financial market model, because the investor in a arbitrage portfolio starts with 0 dollars, and without injecting any money, the investor is certain of not losing any money. In addition he has a positive probability by ending up with more than 0 at maturity this cannot be a well function market for both buyers and sellers. To price the derivatives fair in the model, the derivative should not introduce arbitrage to the market. A arbitrage free market is not the only desirable property for the market, we would also like to have a unique price for the derivatives. We can obtain a unique price by replicatation of the derivative cash flow with the other assets in the market model. If every derivative can be replicated the market is complete.

2.1.4 Complete Market And Replication

The replication argument in Black-Scholes paper (Black and Scholes, 1973) was groundbreaking in the sense that the attitude to risk was irrelevant for pricing, because by continuous trading in the underlying asset(s) the contingent claim cash flow could be replicated. The replication argument shows that the price is unique under the assumption investors prefer more to less. Replication is also important for risk management of the derivative books, because it tells you have to risk neutralize your exposure. A hedge[2] is simply a risk neutralization action in order to minimize the overall risk. In the definition below, we define a replication for an simple T-claim[3].

Definition 2.6. Replication and completeness for T-claim: A T-claim X can be replicated, if there exist a self-financing portfolio h such that:

- $V^h(T) = X$ P-a.s.

I.e. h is an replication portfolio for X if it is guaranteed to pay in all circumstances an amount identical to the payout of X.
The market is complete, if every derivative in the market can be replicated.
(p. 192 (Björk, 2009))

By introducing the basic concepts for how to price fair and protect ourselves against financial risk, we will in next section focus on building the financial market model.

2.2 Multidimensional Models

There is two main method for deriving arbitrage free and complete markets. The classical approach is the delta hedging approach (Black and Scholes, 1973) and (Cox and Stephen Ross, 1979)). The more advanced mathematical approach is the martingale approach (Björk, 2009). In this section we focus on the martingale approach and show that delta hedging approach coincides with the more general martingale theory. For the martingale approach the First and Second Fundamental Theorems of Mathematical Finance will be the key for obtaining a fair market. Besides the financial market assumptions in section 2.1 will we assume specific model assumptions.

[2]Note there is a subtle difference between to hegde or replicate a cash flow. The hedge gives minus the cash flow from replication

[3]Options only exercisable at maturity

2.2.1 Model Assumptions

Let us consider a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t^{\bar{W}})_{t\in[0,T]})$. Note the assumption that filtration is generated from the Wiener process and we consider a finite horizon. We assume $\bar{W}_i$ is k-dimensional and $\bar{W}$ is the only random source. A priori we assume a market $(B(t), S_1(t), S_2(t), \ldots, S_d(t))$, where $S_i(t)_{i=1,2,\ldots,d}$ are d risky assets and B(t) is the risk free asset[4]. By assumptions their dynamics are given by:

$$d\boldsymbol{S}(t) = D[\boldsymbol{S}(t)]\boldsymbol{\alpha}(t)dt + D[\boldsymbol{S}(t)]\boldsymbol{\sigma}(t)d\bar{\boldsymbol{W}}(t) \qquad S_i(0) \in \mathbb{R}^+ \qquad (2.1)$$

$$dB(t) = r(t)B(t)dt \qquad B(0) = 1 \qquad (2.2)$$

We assume $\alpha_i(t)$, $\sigma_{ij}(t)$ and the short rate $r(t)$ are adapted processes, this condition are necessary for the stochastic integrals to be well-defined. The evolution of the stocks are described by the geometric brownian motion (GBM) which has a solution to the SDE. The randomness comes from the Wiener process[5] in the GBM, which has wildly trajectories. The function $t \mapsto W_t(\omega)$ from $[0, \infty)$ to $\mathbb{R}$ is continuous, but nowhere differentiable. Furthermore the BM has nonzero quadratic variation and infinite variation, which is the reason to stochastic calculus pioneered by Itô. The BM has also well-behaved property e.g. it is a Lévy process, i.e. $W(0) = 0$ *a.s*, independent and stationary increments $W(t) - W(s)$ which is normally distributed with mean zero and variance t-s. The Brownian motion in the GBM formalizes "random shocks" dW to the stock return with volatility $\boldsymbol{\sigma}(t)$ and drift $\boldsymbol{\alpha}(t)$. Figure 2.2 illustrates three approximations to sample paths of the stocks with GBM assumption with initial value $S_i(0) = 36$.

The tool for handling BM is stochastic calculus in continuous time, because the standard calculus will not work for the wildly behaved BM. In the representation of the GBM, we used vector and matrix notation for the GBM process. The stock vector is d dimensional and the Wiener process vector is k dimensional. The volatility matrix is given by $\boldsymbol{\sigma}(t) = \{\sigma_{ij}(t)\}_{i=1,\ldots,d,j=1,\ldots,k}$ and the drift is $\alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_d(t))^T$. D(x) denotes a diagonal matrix with vector x as its diagonal and the Wiener processes instantaneous correlation matrix Σ is given by $Cov(dW_i(t), dW_j(t)) = \rho_{ij}dt$.

2.2.2 Arbitrage Free Model

The first problem we are faced with in arbitrage theory is to create a model with no arbitrage opportunities. The First Fundamental Theorem tells us how to not introduce arbitrage to our market model.

Theorem 2.7. ***First Fundamental Pricing Theorem of Mathematical Finance(FFT1):*** *The market model is free of arbitrage if and only if there exist a equivalent martingale measure, i.e. a measure $Q \sim P$ such that the processes:*

$$\frac{S_0(t)}{S_0(t)}, \frac{S_1(t)}{S_0(t)}, \cdots, \frac{S_d(t)}{S_0(t)}$$

are (local)martingales under Q.

(p. 154 (Björk, 2009))

[4] B(t) is often also referred to as the bank account

[5] A Wiener process is also called a Brownian motion (BM)

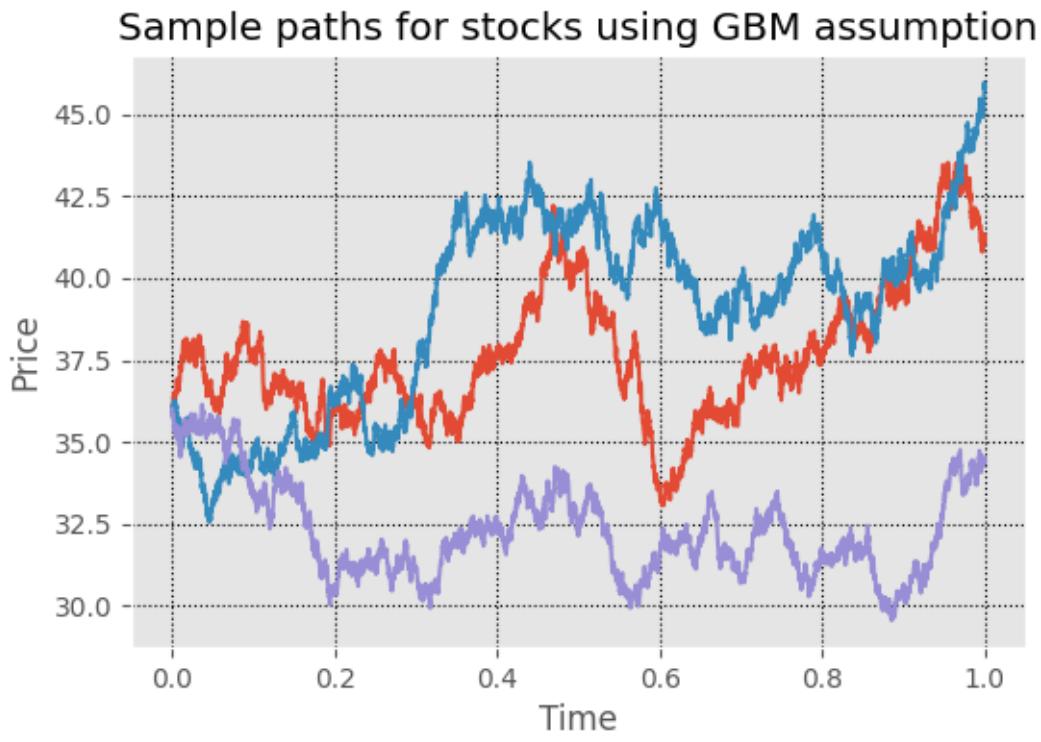

FIGURE 2.2: Three sample paths for stocks under GBM assumptions, where the spot is \$36, σ=0.2 and α=0.06

The processes are martingales is a mathematical formulation that the expected value of the discounted value coincides with the known spot value today[6]. In gambling a martingale resembles a "fair" game. From the FFT1 using the bank account $B(t)$ as numéraire it follows that:

Proposition 2.8. *We assume that $B(t) = S_0(t)$ is our numéraire and all the processes randomness comes from the Weiner process, then a equivalent measure $Q \sim P$ is martingale measure if and only if all assets $(B(t), S_1(t), \ldots, S_d(t))$ have the short rate as their local rates of return, i.e.*

$$dS_i(t) = S_i(t)r(t)dt + S_i(t)\sigma_i(t)dW^Q(t)$$

(p. 154 (Björk, 2009))

So to not introduce arbitrage to the model for the financial market, we need to ensure the Q-dynamics of S is:

$$dS(t) = D[S(t)]r(t)dt + D[S(t)]\sigma(t)dW(t) \tag{2.3}$$

The tool to obtain the dynamics in equation (2.3) is Girsanov theorem (theorem A.3). Girsanov theorem is a continuous measure transformation, where in our model we want to transform the dynamics given with the objective probability measure P to an equivalent martingale measure Q. By suitable chooses of the likelihood process L and setting $dQ = L(T)dP$, then with Girsanov theorem the transformed process is

[6] Assuming $B(0) = 1$

still a Brownian motion:

$$d\bar{W}(t) = \phi(t)dt + dW(t)$$

When applying to eq. (2.1):

$$dS(t) = D[S(t)](\alpha(t) + \sigma(t)\phi(t))dt + D[S(t)]\sigma(t)dW(t)$$

Going back to the FFT1 and the proposition hereof, we know that Q is martingale measure if and only if:

$$\boldsymbol{\alpha}(t) + \boldsymbol{\sigma}(t)\boldsymbol{\phi}(t) = \mathbf{r}(t) \quad \textit{holds with probability } 1 \textit{ for each } t \tag{2.4}$$

We disregard pathological models when doing so the term generically arbitrage free will be used.

Definition 2.9. Generically arbitrage free: The model in this section is said to be generically arbitrage free if it is arbitrage free for every (sufficiently integrable) choice of $\boldsymbol{\alpha}(t)$.

(p. 198 (Björk, 2009))

Furthermore we assume enough integrability and we have the following useful result:

Proposition 2.10. *Disregarding integrability problems the model is generically arbitrage free if and only if, for each $t \leq T$ and P-a.s. the mapping: $\boldsymbol{\sigma}(t) : \mathbb{R}^k \to \mathbb{R}^d$ is surjective, i.e. if and only if the volatility matrix $\boldsymbol{\sigma}(t)$ has rank d.*

(p. 198 (Björk, 2009))

We note that in order not to have arbitrage in our model, we need $k \geq d$, i.e. have at least as many random sources as number of risky assets.

2.2.3 Complete model

Second Fundamental Pricing Theorem is key to obtain a complete market model, i.e. a market model with unique prices and every claim can be hedged.

Theorem 2.11. ***Second Fundamental Pricing Theorem of Mathematical Finance(FFT2):*** *Assuming absence of arbitrage, the market model is complete if and only if the martingale measure Q is unique.*

(p. 155 (Björk, 2009))

In our Wiener world we have a unique martingale measure if equation 2.4 has a unique solution. The proof of proposition 2.12 will shortly reveal why the Wiener world assumption is required. If we had more random sources e.g. a Poisson process[7], than there is no guarantee that the equivalent measure transformation is of the Girsanov type above.

Proposition 2.12. *Assume that the model is generically arbitrage free and that the filtration is defined by:*

$$\mathcal{F}_t = \mathcal{F}_t^{\bar{W}} \quad t \in [0, T]$$

Then disregarding integrability problems, the model is complete if and only if k=d and the volatility matrix $\sigma(t)$ is invertible P-a.s. for each $t \leq T$

[7]The Merton's Mixed Jump-Diffusion Model is an example

Proof. The proof is based on martingale representation theorem A.2 (MRT) and converse of Girsanov theorem A.4 which uses MRT, hence the assumption about the only randomness comes from the Wiener process. By the two theorems we know every equivalent measure transformation is obtained by Girsanov theorem of the above type. Hence the martingale measure is unique if and only if the solution to (2.4) is unique. ■

(p. 200 (Björk, 2009))

Intuitively we need one independent traded assets excluding the bank account for every source of randomness (Meta-theorem 8.3.1 (Björk, 2009)).

2.2.4 Pricing and Connection to Classical Approach

The pricing formula for arbitrage free market model is the risk neutral valuation formula(RVNF):

Proposition 2.13. ***Risk Neutral Valuation Formula*** *To avoid arbitrage, $\mathcal{X}$ must be priced according to the formula:*

$$\Pi(t;\mathcal{X}) = S_0(t)E^Q[\frac{\mathcal{X}}{S_0(T)}|\mathcal{F}_t] \tag{2.5}$$

Note if we choose our numéraire $S_0(t) = B(t)$ then

$$\Pi(t;\mathcal{X}) = E^Q[\exp(-\int_t^T r(s)ds)\mathcal{X}|\mathcal{F}_t] \tag{2.6}$$

(p. 155 (Björk, 2009))

Proposition 2.13 will raise the question if there is more than one fair price for the derivative. The answer is found in FTT2, the market is complete if and only if the measure Q is unique. Intuitively it means that if you can replicate the derivative with a S-F portfolio, then the risk free position should only earn risk free interest rate. The conditional expectation in the RNVF is a natural choice for pricing, because intuitively the conditional expectation is our best estimate given the available information[8].

The classical approach in (Black and Scholes, 1973) to arbitrage free and complete market models is based on a Markovian model assumption. For the model to have the Markov property, we assume $k = d$ and the probability space is $(\Omega, \mathcal{F}, P, \mathcal{F}_t^{\bar{W}_t})$. Furthermore we assume $\boldsymbol{\alpha}$ and $\boldsymbol{\sigma}$ are deterministic and constant over time. $\boldsymbol{\sigma}$ is also assumed invertible. Under these more restrictive assumptions the risk neutral valuation formula for a simple T-claim is given by the pricing function:

$$F(t,S(t)) = \exp(-r(T-t))E^Q[\mathcal{X}|S(t)] \tag{2.7}$$

The Markov property implies that the price only depend on the current state of $\boldsymbol{S}$. Applying Kolmogorov backward equation on equation (2.7) we obtain the Black-Scholes PDE for the pricing function $F(t,S(t)) = \Pi(t;\mathcal{X})$.

[8] Mathematically known as the projection property

Theorem 2.14. ***Black Scholes PDE:*** *Consider the contract $\mathcal{X} = \Phi(\boldsymbol{S}(T))$. In order not to introduce arbitrage to the market, the pricing function $F(t,s)$ must solve the boundary value problem.*

$$F_t(t,s) + \sum_{i=1}^{n} rs_i F_i(t,s) + \frac{1}{2} tr\{\sigma^* D[S] F_{ss} D[S]\sigma\} - rF(t,s) = 0$$

$$F(T,s) = \Phi(s)$$

(p. 203 (Björk, 2009))

2.3 Classical Black-Scholes Theory

We will not do the classical delta hedging approach in (Black and Scholes, 1973). Instead we use the general multidimensional martingale approach to derive the essential formulas for pricing. To derive a closed-form solution to the European call and put option, we concentrate at a special case of the multidimensional framework, where we only have the risk free asset and one risky asset in the financial market model. We further restrict ourselves to:

Assumption 2.15. ***Black-Scholes assumptions****: We assume following ideal conditions in addition to (2.1):*

- *The short-term interest rate $r \in \mathbb{R}_*^+$[9], volatility $\sigma \in \mathbb{R}^+$ and the drift $\alpha \in \mathbb{R}$ are constant.*
- *The stock pays no dividends or other distributions.*
- *The option is a simple option ("European").*
- *No arbitrage opportunity on the market.*

(p. 640 (Black and Scholes, 1973))

The interest rate is assumed strictly positive to assure the European call and American call value coincides[10]. The above assumptions gives the Markovian model described in previous section.

We assume the underlying stock and the bank account have differentials:

$$\begin{aligned} dS(t) &= S(t) \cdot \alpha dt + S(t)\sigma d\bar{W}(t) & S(0) &\in \mathbb{R}^+ \\ dB(t) &= rB(t)dt & B(0) &= 1 \end{aligned}$$

By Itô's lemma (lemma A.1) for one dimensional process the solution to the differentials above is:

$$\begin{aligned} S(t) &= S(0) \cdot \exp\left((\alpha - \frac{1}{2}\sigma^2)t + \sigma W(t) \right) \\ B(t) &= \exp(r \cdot t) \end{aligned}$$

[9] Note that restricting the interest rate to the positive reals is not part of the Black-Scholes papers assumptions.

[10] Detailed explained in section 2.4.1

The solution of the SDE of S under Q dynamics is:

$$S(t) = S(0) \cdot \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma W(t)\right) \tag{2.8}$$

By equation (2.8) we see that the Black-Scholes model assumes that the stock price evolution produces a lognormal distribution for the price at any future time.

The closed form solution for the European call can be derived solving the Black-Scholes PDE or with the RNVF given in previous section.

Proposition 2.16. ***Black-Scholes formula for call option:*** *The price of a European call option with strike K and maturity T is given by the formula* $\Pi(t) = F(t, S(t))$*, where*

$$F(t,s) = c(t,s) = s \cdot N(d_1(t,s)) - e^{-r(T-t)} \cdot K \cdot N(d_2(t,s))$$

N is the cumulative distribution function of a standard normal distribution $\mathcal{N}(0,1)$ *and*

$$d_1(t,s) = \frac{1}{\sigma \cdot \sqrt{T-t}} \cdot \left(\ln(\frac{s}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)\right)$$
$$d_2(t,s) = d_1(s,t) - \sigma\sqrt{T-t}$$

(p. 105 (Björk, 2009))

We provided only the price for the European call option, but the European put price can readily be obtained by the put-call-parity for European options.

Proposition 2.17. ***Put-call parity:*** *Assume the call and put option has same strike price and time to maturity.*

$$p(t,s) = K \cdot \exp(-r(T-t)) + c(t,s) - s$$

(p. 126 (Björk, 2009))

The aim for this thesis is to price American put options, but the European option provide a reference price in a closed form format. The put-call-parity holds only for European options, where for the American option there is a bound on the difference in price:

$$S_0 - K \leq C - P \leq S_0 - K \cdot e^{-rT}$$

The above formula for the European call option is actually the same for an American call option, but is not true for an American put option or for call options with underlying stock paying dividends. The result for the American call option was shown by Merton (Merton, 1973), that the intrinsic value is never greater than the worth of the option given by the risk-neutral valuation formula. In section 2.4 we will show a martingale approach to prove the value of a European and American call coincides when the underlying is a non-dividend paying stock.

2.4 American Options And Optimal Stopping

The American options adds additional complexity to the pricing problem, because compared to the European option the American option can be exercised at anytime from inception to maturity. The exercise value at time t is also called the intrinsic value of the option. This section is inspired by (Björk, 2009; Shiryaev and Peskir,

2006; Elliott and Kopp, 1999) where (Shiryaev and Peskir, 2006) is specialized to optimal stopping problems and the two other references give the fundamentals for option and arbitrage theory in general.

We still assume a diffusion setting that the underlying stochastic process for the stock behaves under the risk neutral measure as a GBM. The exercise feature of the American option raises the problem of rationally to find the optimal stopping time to maximize profit. The value of the option is given by exercising the option at the optimal stopping time, hence it is a optimal stopping problem. We will assume a finite horizon $T \in \mathbb{R}_*^+$ throughout the thesis, because all the derivative will be priced in a finite timeframe. Let the gain function $G : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying:

$$E_s[\sup_{0\leq t\leq T} |G(S(t))|] < \infty \tag{2.9}$$

where S is the underlying stochastic process. If the integrability condition is satisfied on a finite interval $[0, T]$ (equation (2.9)) then the optimal stopping problem for gain function G and $s \in \mathbb{R}$ is well defined. We assumed that the underlying stochastic S(t) process is time-homogeneous, but the assumption can be relaxed. If S(t) is a time-inhomegenous we can extend the underlying process S(t) by time albeit increasing the underlying process dimension. We define the optimal value process in terms of the gain process.

Definition 2.18. For fixed $(t, x) \in [0, T] \times \mathbb{R}$, and each stopping time τ with $\tau \geq t$ the optimal value function $V(t, x)$ is defined by

$$V(t, x) = \sup_{\tau\in\mathcal{T}^T} E_{t,x}[G(S(\tau))] \tag{2.10}$$

A stopping time which realizes supremum for V is called optimal and be denoted $\hat{\tau}$.
(p. 341 (Björk, 2009))

The solution to the optimal stopping problem $\hat{\tau}$ is where supremum is attained and the price is then $V(t, x)$ for $(t, x) \in [0, T] \times \mathbb{R}$. The supremum is taken over all stopping times with respect to the natural filtration $\mathcal{F}_t$ belonging in the class of stopping times:

$$\mathcal{T}_0^T = \mathcal{T}^T = \{\tau : 0 \leq \tau \leq T\}$$

The definition of a stopping time τ can be seen in definition A.5. The intuition is that the stopping time is a random time, where we know at present time weather the process is stopped or not. To solve the optimal stopping problem some trivial solutions are immediate by martingale properties:

Proposition 2.19. *The following hold:*

- *If $G(S(t))$ is a submartingale, then it is not optimal to stop at all and $\tau^* = T$*
- *If $G(S(t))$ is a martingale, then all stopping times $\tau \in [0, T]$ are optimal*
- *If $G(S(t))$ is a supermartingale, then it is optimal to stop immediately. i.e. $\tau^* = 0$*

(p. 330 (Björk, 2009))

Examples of optimal stopping problems could be the American call and put options:

$$C(t,x) = \sup_{\tau \in \mathcal{T}_t^T} E^Q[\exp(-r(\tau - t))(S(\tau) - K)^+ | S(t) = x] \quad \textit{for } t \in [0,T] \textit{ and } x \in \mathbb{R}^+$$
$$P(t,x) = \sup_{\tau \in \mathcal{T}_t^T} E^Q[\exp(-r(\tau - t))(K - S(\tau))^+ | S(t) = x] \quad \textit{for } t \in [0,T] \textit{ and } x \in \mathbb{R}^+$$

2.4.1 American Call Without Dividends

The American call options is a special case, because the optimal stopping time is always at the options maturity. With martingale machinery it means the value-process is a submartingale which implies that $\hat{\tau} = T$ (proposition 2.19). Remember the optimal stopping problem for an american call option:

$$C(t,x) = \sup_{\tau \in \mathcal{T}_0^{T-t}} E_{t,x}^Q[\exp(-r\tau)(S(t+\tau) - K)^+]$$

Looking at the gain function:

$$\exp(-rt)(S(t) - K)^+ = (\exp(-rt)S_t - \exp(-rt)K)^+$$

Recall that the discounted price process $\exp(-r \cdot t) \cdot S_t$ is a Q-martingale and $\exp(-r \cdot t) \cdot K$ is a deterministic decreasing function in t if $r > 0$. Furthermore the function $x \mapsto (x)^+$ is convex, hence the gain function is a Q-submartingale. The last result used is that a convex and increasing function on a submartingale is still a submartingale, hence the optimal stopping time is $\hat{\tau} = T$ if $r > 0$.

2.4.2 American Put

The arbitrage-free price for an American put at time t:

$$P(x,t) = \sup_{\tau \in \mathcal{T}_t^T} E^Q[\exp(-r(\tau - t))(K - S(\tau))^+ | S(t) = x] \quad \textit{for } t \in [0,T] \textit{ and } x \in \mathbb{R}^+ \tag{2.11}$$

For the American put we need computational methods, because the American and European put does not coincides like for the call option.

Proposition 2.20. *Consider European and American put options with same maturity T and strike K. If the risk free rate $r \in \mathbb{R}_*^+$, then for any $t < T$*

$$p(x,t) < P(x,t) \tag{2.12}$$

Proof. WLOG[11] we assume that t=0. Define the stopping time

$$\tau = min\{t \geq 0 : S(t) \leq K(1 - \exp(-r \cdot (T - t)))\}$$

We consider a exercise strategy $min\{\tau, T\}$, where the strategy is not necessarily optimal. We consider two events:

1) $\tau < T$

2) $\tau \geq T$

[11] Without loss of generality

The first case is to exercise at time τ when $S(\tau) \leq K(1 - \exp(-r \cdot (T - t)))$. Here the payoff by exercising will be at least $K \exp(-r \cdot (T - \tau))$. The cash flow received is then invested into the bank account at time τ. At maturity the strategy gives the holder of the put a payoff K, where the European contract with strike K will pay less, because the stock price at maturity will be $S(T) > 0$ *a.s.* The second case is trivial, because the European and American put will give the same payoff.
Since the first case has positive probability, the American put has higher discounted expected payoff by following above strategy regardless of the spot price for the stock. ■

The above proposition shows the optimal stopping strategy is not always to hold the option to maturity, hence the theory of optimal stopping is important for pricing of American put options. The American put has consequently no closed form solution, but we can search for a lower exercise boundary b(t) such that the holder of the option should exercise when:

$$S(\tau) \leq b(\tau) \quad \tau \leq T$$

The continuation set C and stopping set $\bar{D}$ are given for an American put.

$$C = \{(t, x) \in [0, T) \times (0, \infty) : V(t, x) > G(x)\}$$
$$\bar{D} = \{(t, x) \in [0, T) \times (0, \infty) : V(t, x) = G(x)\}$$

Hence the first optimal stopping time after time t for the American put is

$$\hat{\tau} = \inf\{u \in [t, T] : P(S(u), u) = (K - S(u))^+\}$$

2.4.3 Discrete time valuation

To solve the optimal stopping problem numerical methods are required for the American put option, hence the first step is to discretize exercise dates. In chapter 3 we show two approaches to price an American put option, where both are based on calculating the expected continuation value.

Suppose the probability space $(\Omega, \mathcal{F}, P)$ is equipped with the natural discrete filtration $(\mathcal{F}_{t_n})_{n=0,1,\ldots,N}$ modeling a financial market. By discretization of time we are actually looking at a Bermudan option, but for sufficient small time steps the Bermudan option approximate the American option well. The tenor structure[12] is that the time to maturity is divided into a grid of N+1 equidistant points in time $0 = t_0 \leq t_1 \leq t_2, \cdots \leq t_N = T$, where $\Delta t_n = t_n - t_{n-1} = T/N$ for each $n = 1, \ldots, N$. A Bermudan option initialized at time t_0 has $\mathcal{T}(t_0, t_1, \ldots, t_N)$ exercise dates or decision points, where the option holder chooses to exercise or keep the option alive.

The underlying process is assumed to be Markovian with state variables $(S(t_n))_{n=0,1,\ldots,N}$ recording all necessary information about relevant financial variables adapted to the natural filtration in order to solve the optimal stopping problem with the dynamic programming principle. Furthermore is the gain process adapted to the filtration

[12]The tenor of an option is the remaining life of the option from today to maturity

and square integrable $E[\max_{0\leq n\leq N}|G(S(t_n))|^2] < \infty$, which is necessary to use regression on a finite set of functions to approximate the conditional expectation (section 3.3).

The optimal stopping problem in discrete time is to find

$$\sup_{\tau\in\mathcal{T}(0,1,\dots,T)} E^Q[G(S(\tau))] \tag{2.13}$$

For using the programmable dynamic programming principle the Snell envelope $(U(t_n))_{n=0,1,\dots,N}$ (definition A.6) of the gain function $G(S(t_n))_{n=0,\dots,N}$ is useful and defined by

$$U(t_n) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}(n,\dots,N)} E^Q[G(S(\tau_{t_n}))|S(t_n)] \quad n = 0,1,\dots,N$$

The Snell envelope $U(t_n)$ is the smallest supermartingale of the gain process $\{G(S(t_n))\}$, i.e. the smallest supermartingale dominating the gain process. Using the Snell envelope theory the optimal stopping problem can be solved with the dynamic programming principle.

$$\begin{cases} U(t_N) = G(S(t_N)) \\ U(t_n) = max\{G(S(t_n)), E^Q[U(t_{n+1})|S(t_n)])\} \quad for\ n = 0,\dots,N-1 \end{cases} \tag{2.14}$$

Where $U(t_n)$ is the discounted option value at time t_n not previous exercised and $G(S(t_n))$ is the discounted exercise value.

Equation (2.14) is known as value iteration indicates that the holder should exercise the first time $G(S(t_n)) > E^Q[U(t_{n+1})|S(t_n)])$ in order to maximize payoff from the option, hence we get the first optimal stopping time. Note that an optimal stopping time will always exist in discrete time when $T < \infty$. There is though no guarantee that the stopping time is unique. The value iteration gives the optimal value process of the gain process $G(S(t))$ (theorem A.7). So

$$U(t_n) = E^Q[G(S(\tau_{t_n})|S(t_n)]$$

with

$$\tau_{t_n} = \min\{k \geq n : U(t_k) = G(S(t_k))\}$$

and

$$E^Q[U(0)] = \sup_{\tau\in\mathcal{T}(0,\dots,T)} E^Q[G(S(\tau))] = E^Q[G(S(\hat{\tau}))]$$

The optimal stopping problem can also be solved in terms of stopping times instead of using the value process. This alternative dynamic programming principle equation is called policy iteration.

$$\begin{cases} \tau_{t_N} = t_N \\ \tau_{t_n} = t_n \cdot 1_{\{G(S(t_n))\geq E^Q[G(\tau_{t_{n+1}})|S(t_n)])\}} + \tau_{t_{n+1}} \cdot 1_{\{G(S(t_n))<E^Q[G(\tau_{t_{n+1}})|S(t_n)])\}} \quad for\ n = 0,\dots,N-1 \end{cases} \tag{2.15}$$

The first approach in chapter 3, the binomial model, uses the idea of dynamic

programming. The underlying stochastic process is approximated by a discrete binomial recombining tree, which is readily to work backward in time to calculate both European and American option prices. The American put option can be solved recursively with dynamic programming principle with value iteration:

$$\begin{cases} P(t_i) = max\{(K - S(t_i))^+, \exp(-r \cdot \Delta t) E^Q[P(t_{i+1})|S(t_i)\} & for\ i = 0, \ldots, N-1 \\ P(t_N) = (K - S(t_N))^+ \end{cases} \tag{2.16}$$

The second approach, Least Square Monte Carlo Method (LSM) model, is to combine dynamic programming and Monte Carlo simulation, but this method uses the policy iteration principle instead of the value iteration. The method uses regression to calculate the expected continuation value of simulated paths instead of discretization of the underlying stochastic process. LSM overcomes the challenge with pure simulation method by using regression, because it gives an effective way to evaluate a series of conditional expectations. Both methods will be explained in details in chapter 3.

Chapter 3

Classical Numerical Results and Benchmarks

In last section we saw that the American put was an example of an option that requires numerical procedures to be priced fair. The American put is far from the only example of a derivative without a closed-form solution. We will look at two classical valuation algorithms for pricing American options in computational finance the Cox-Ross-Rubinstein (CRR) binomial model (Cox and Stephen Ross, 1979) and the Least Square Monte Carlo (LSM (Longstaff and Schwartz, 2001)). The binomial model is an example of a strategy to approximate the option by discretization of the underlying risky asset(s) and the LSM is a method to simulate the underlying risky asset(s). Another popular choice is to solve the free boundary problem with finite difference methods, but we chose to focus on the two other numerical procedures. The chapter will also investigate valuing exotic multivariate contingent claims. We extend the binomial pricing model (Ekvall, 1996; Boyle, Evnine, and Gibbs, 1989) and LSM to multivariate contingent claims and provide some closed form solutions for exotic European options (Johnson, 1987; Ouwehand, 2006). Therefore the chapter have two purposes to gain insight into valuation for exotic options and provide some benchmarks for the neural network in the coming chapters.

3.1 Cox Ross Rubenstein Model

The classical binomial model pricing formula or the CRR model presented in this section is inspired by (Cox and Stephen Ross, 1979; Hull, 2017; Björk, 2009). The model will be used for pricing an American put option with one underlying stock and to build the foundation for pricing bivariate contingent claims with the binomial model (Boyle, Evnine, and Gibbs, 1989). The CRR model provides an intuitive and easy implementable model for valuing American and European options. The binomial model has its limitations, because for computational reasons it is not suited for valuing path dependent options or options with several underlying factors. The key difference on the binomial model and the simulation approach is that the binomial model discretize the underlying stochastic process(es) to have two possible movements per timesteps.

Assume the same market and Black-Scholes assumptions as in chapter 2, but the underlying stochastic process will be assumed to follow a multiplicative binomial process over discrete periods. We work with the financial market $(\Omega, \mathcal{F}, \mathbb{F}, P, S_0, S_1)$, where the filtration is generated by $\mathbb{F} = \sigma(\mathcal{F}_{t_n})_{n=0,1,\dots,N}$ and the sigma algebra is chosen to be $\mathcal{F} = \mathcal{F}_{t_N}$. It is well known from discrete time arbitrage theory, that the binomial market model with two assets, where $u > 1 + r > d > 0$ is a complete and

arbitrage free model. The u,d and r describes the evolution of the discrete stochastic process for the stock and the free interest rate on the bank account.

$$S_0(t_n) = S_0(t_{n-1}) \cdot \exp(\Delta t \cdot r) \quad \textit{where } S_0(0) = 1 \textit{ and } n =, \ldots, N$$
$$S_1(t_n) = S_1(0) \prod_{j=1}^{n} Y_j \quad \textit{where } Y_1, Y_2, \ldots, Y_N \textit{ are i.i.d., } S_1(0) > 0$$

Note that the interest rate is continuously compounded for computational convenience and the equidistant time step is $\Delta t = T/N$ (section 2.4.3 for notation). We assume that

$$Y_i = \begin{cases} u & \textit{with probability } p \\ d & \textit{with probability } (1-p) \end{cases}$$

Below a illustration of a discrete multiplicative binomial process with two time steps, where the binomial tree recombines[1].

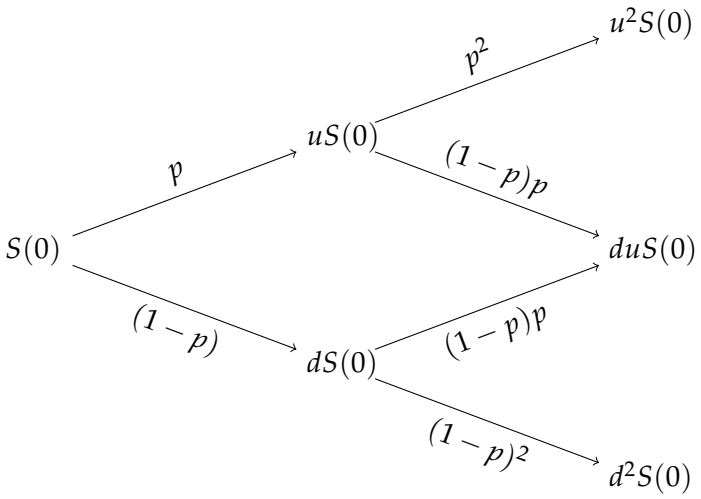

FIGURE 3.1: Price dynamics of binomial lattice model with one underlying risky asset with N=2, $S(0)$ is spot value, p objective probability measure, u and d is realizations of the stochastic variable Y_i

By constructing the discrete process for the stock it is easy to find the equivalent martingale measure.

Definition 3.1. Assume there exists a risk free asset. A probability measure Q is called a martingale measure if the following condition holds

$$s = \exp(-r\Delta t) \cdot E^{Q}[S(t + \Delta t)|S(t) = s] \tag{3.1}$$

Where Δt is a single time-step. (p. 18 (Björk, 2019))

By using definition 3.1 we find the risk neutral measure to be:

$$q = \frac{e^{r\Delta t} - d}{u - d}$$

[1]Henceforth a binomial recombining tree is referred to as a binomial lattice

The martingale measure q is unique in the binomial lattice model, because the model is complete. Remember the market $(S_0(t_n), S_1(t_n))$ for the binomial model is arbitrage free and complete, hence the pricing formula is given by the risk neutral valuation formula.

Theorem 3.2. ***Risk-neutral valuation formula (RVNF) in discrete time for a T-claim:*** *The arbitrage free price at t=0 of a T-claim X is given by*

$$\Pi(0;X) = \exp(-r\Delta t \cdot N) \cdot E^Q[X] \tag{3.2}$$

$$= \exp(-r\Delta t \cdot N) \cdot \sum_{n=0}^{N} \binom{N}{n} q^n (1-q)^{N-n} \Phi(s u^n d^{N-n}) \tag{3.3}$$

here Q denotes the martingale measure, $\Pi(0;X)$ *is the price of X to time 0 and* Δt *is a single time step.*

(p. 25 (Björk, 2019))

Theorem 3.2 gives a simple mathematical framework for pricing European options, but the model can readily be extended to American options. American put options for the binomial model will be solve with the dynamic programming approach.

$$\begin{cases} P(t_i) = max\{(K - S(t_i))^+, \exp(-r \cdot \Delta t) E^Q[P(t_{i+1})|S(t_i)])\} \quad for\ i = 0, \ldots, N-1 \\ P(t_N) = (K - S(t_N))^+ \end{cases}$$

Note that the binomial lattice model has recursive structure where the one-step transition probabilities are the same in each node. The idea is to at each decision point either exercise to gain the intrinsic value if it is greater than the expected continuation value or hold the option alive for another period. The dynamic choice to exercise or to keep the option alive gives a exercise barrier b (Figure 3.2).

I.e. we lay out all the possible path of the stock in the tree based on $S_1(0), \sigma$ and T in order to value an American put option. To construct the tree we need to specify the number of equidistant time-steps Δt ($\Delta t = \frac{T}{N}$ *where* $N = No.\ of\ steps$) for the tree, where for each step we add another possible value for the stock. We only add one more possibility for each time-step because the tree recombines[2]. Figure 3.2 is an example of a constructed tree, where the value of the option is also included by color. The decision to exercise is marked with a triangle and continuation is marked with a circle.

The u and d are chosen to be the reciprocal of each other and they are essentially functions of the volatility. The d and u is chosen such that they match the first and second moment of the Black-Scholes assumption about the lognormal returns of the stock (Detailed explanation in Appendix C.1).

$$u = \exp(\sigma\sqrt{\Delta t}) \quad d = \exp(-\sigma\sqrt{\Delta t})$$

For valuing an American put option, we value the exercise value at maturity (time T) for all possible outcomes for the price process at maturity. Then we use backward induction/dynamic programming where we compare the intrinsic value with the expected continuation value (equation (3.1)), where we choose the maximum of

[2] $(1+n)$ possible stock values after n steps

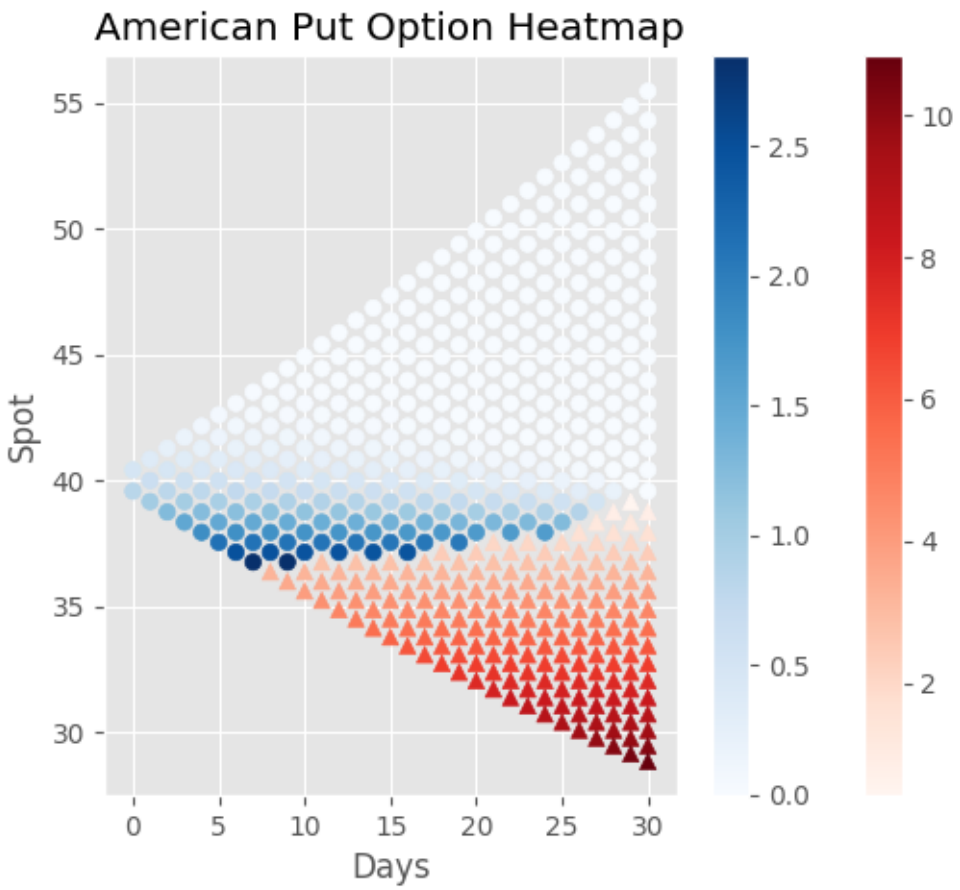

FIGURE 3.2: A valuation tree of an American put option price based on the binomial model, where the color indicate the value and the dots are marking the continuation nodes. The parameters are S(0)=40, N=30, $\Delta t = 1$ day, K=40 and u=1.0106

these two. The comparison will be applied for every node in each decision point $(t_n)_{n=0,1,\ldots,N-1}$ and all the way back in time to the initialization date. By this procedure we get present value of the American option. One design decision is to choose number of time-steps considering a trade-off between computational efficiency and accuracy. Figure 3.3 illustrates that the option value stabilizes with increasing the time-steps for an option with 1 year to maturity. The precision for the algorithm increases also with the number of steps.

We have seen central concepts arbitrage and completeness from continuous time in action for the discrete time setup. The paper (Cox and Stephen Ross, 1979) which introduced the binomial model to option pricing came after the Black-Scholes model described in section 2 (Black and Scholes, 1973). The main reason for developing a binomial model is that the discrete time approach gives a simplified model in terms of the mathematics and highlights the essential concepts in arbitrage theory. You can argue that the simpler mathematics in this model makes the binomial model more instructive and clear. Besides being easier to understand for non-mathematician it works nicely with other options than the European options like American options.

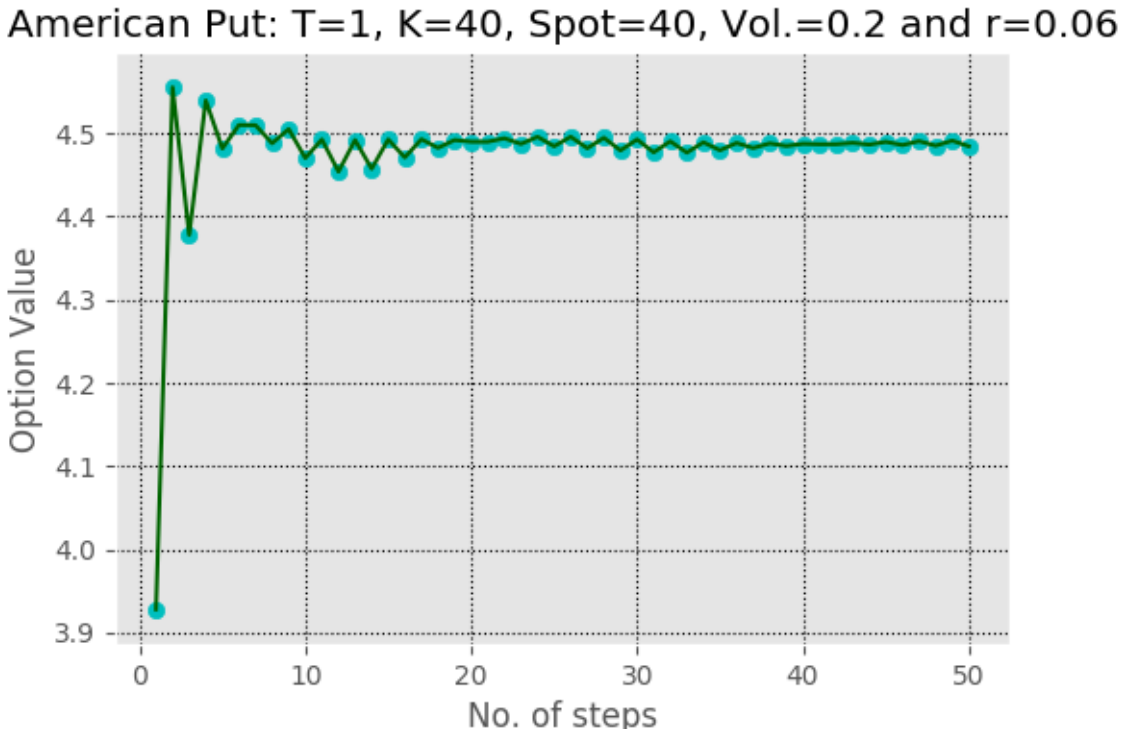

FIGURE 3.3: Price for a American put option based on the binomial model, where the independent variable is the number of time-steps. Vol. is an abbreviation for volatility.

Even though we assume the stock price moves at discrete jumps instead of the classical Black-Scholes continuous time model, it can actually be shown that the CRR binomial model will converge to the continuous time model. Hence the binomial pricing model will be equivalent with the continuous time analytical pricing model derived by Fischer Black and Myron Scholes in the limit for European options for sufficient small time steps.

Note for computational resources the path-independence payoff for the American put makes the tree recombining is important, so there is only N+1 terminal nodes at maturity. If the derivative was path-dependent e.g. an Asian option, then we have a non-recombining tree and 2^N terminal nodes where needed. This is a computational inefficient, which explains that the binomial model should not be used for path-depending derivatives. The problem with a derivative with several underlying e.g. a basket option is also the increasing number of nodes, because now you have 2^d possible one-step transitions. We will show below that the intuitive CRR binomial model can be extended to higher dimensions, but the model suffers from the curse of dimensionality.

3.2 Binomial Lattice Model For Multivariate Contingent Claims

We follow the approach in (Boyle, Evnine, and Gibbs, 1989), henceforth called the BEG method, because it is the natural extension of the Cox Ross Rubinstein model (section 3.1) for multivariate contingent claims. The idea as in the one dimensionel case is to approximate the system of underlying processes[3] with a discrete multivariate binomial lattice. The advantage is that for exotic options like the basket options the valuation of European put options is readily extended to the American put options and has high accuracy.

The BEG method has its limitation in terms of number of underlying assets and for path dependent options (see section 3.1), but it is very intuitive. The problem with increasing the number of underlying assets is that the number of one-step transition at each node is 2^d and the total number of terminal nodes after N steps is $(N+1)^d$ for path-independent derivatives. It means that the computational resources becomes an issue for high dimensional problems with the discrete approximation approach. This makes the BEG method undesirable for higher dimensions than three so we will focus on the two dimensional case. Another problem with three or more underlying assets are that some one-step transition probabilities can turnout negative with the BEG method, which makes the model nonsense in those cases. For those reasons we choose to focus on the bivariate contingent claim i.e. $d = 2$.

The model we want to approximate is the bivariate lognormal distribution, because we assume the Black-Scholes model to describe the evolution of the two risky assets (section 2.2). We restrict ourselves to the assumptions 2.15 given in the classical (Black and Scholes, 1973), hence for risk neutral pricing the SDE for the risky assets are:

$$dS_i(t) = S_i(t)r(t)dt + S_i(t)\sigma_i(t)dW^{\mathbb{Q}}(t) \quad for\ i = 1, 2, \ldots, d$$

We divide the time from inception to maturity (length T) into N equidistant intervals with length Δt, because we want the jump distribution to approximate the continuous time multivariate lognormal distribution. Each time interval has a jump size defined in terms of the volatility and the length of the interval:

$$u_i = \exp(\sigma_i\sqrt{\Delta t}) \quad and \quad u_i \cdot d_i = 1 \quad for\ i = 1, 2, \ldots, d$$

The u_i and d_i are the multiplication factor for the i'th stock, where the former is a jump up and the latter is a jump down factor for the stock. What is the probability that the stock jumps up or down? The probabilities are chosen such that the characteristics functions are equal for small time steps Δt (p. 245-246 in (Boyle, Evnine, and Gibbs, 1989) for details). The probabilities for the model with two underlying

[3] In Black-Scholes theory the stochastic process evolution for the stock is described by the GBM

risky assets are:

$$\begin{aligned} p_1 = p_{uu} &= \frac{1}{4}\left(1+\rho+\sqrt{\Delta t}(\frac{\mu_1}{\sigma_1}+\frac{\mu_2}{\sigma_2})\right) \\ p_2 = p_{ud} &= \frac{1}{4}\left(1-\rho+\sqrt{\Delta t}(\frac{\mu_1}{\sigma_1}-\frac{\mu_2}{\sigma_2})\right) \\ p_3 = p_{du} &= \frac{1}{4}\left(1-\rho+\sqrt{\Delta t}(-\frac{\mu_1}{\sigma_1}+\frac{\mu_2}{\sigma_2})\right) \\ p_4 = p_{dd} &= \frac{1}{4}\left(1+\rho+\sqrt{\Delta t}(-\frac{\mu_1}{\sigma_1}-\frac{\mu_2}{\sigma_2})\right) \end{aligned} \tag{3.4}$$

The correlation ρ between the two assets are assumed to be constant and $\mu_i = r - \frac{1}{2}\sigma_i^2$. We have illustrated below a two-dimensional lattice, where we see that the number of nodes at maturity is $(1+N)^d$.

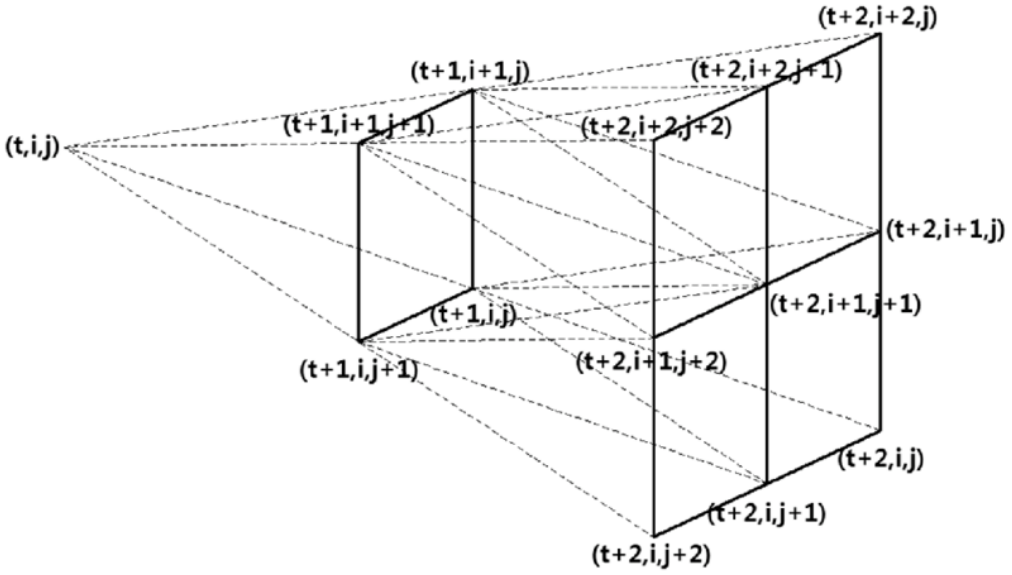

FIGURE 3.4: Evolution of binomial model with two underlying risky asset, where t is time, i is number of up movement for S_1 and j is number of up movement for S_2.

After the construction of the evolution of the underlying assets, we can like in the one dimensional CRR model recursively working backward in the multidimensional binomial lattice. For the European option the recursive formula is:

$$J_{i_1,\ldots,i_d}(t_n) = \exp(-r\Delta t)\left(p_1 J_{i_1+1,\ldots,i_d+1}(t_{n+1}) + \cdots + p_{2^d} J_{i_1,\ldots,i_d}(t_{n+1})\right)$$

Where J is the value of the option to time t_n and i is the number of times the underlying spot price at inception has been multiplied by u.
For the American option we approximate it with the Bermudan option with N decision points. For the Burmudan option the recursive formula is:

$$J_{i_1,\ldots,i_d}(t_n) = \max\{\Phi(t, S(t_n)), \exp(-r\Delta t)\left(p_1 J_{i_1+1,\ldots,i_d+1}(t_{n+1}) + \cdots + p_{2^d} J_{i_1,\ldots,i_d}(t_{n+1})\right)\}$$

With the recursive formulas we can valuate multivariate contingent claims for a variety of exotics including the American put option with several underlying risky assets.

The computational required resources increases with the number of underlying assets for the binomial lattice methods. For the BEG method we have 2^d number of one-step transition probabilities. To reduce the number of transition probability it had been tried in (Ekvall, 1996) to set all the probabilities equal to 0.5 and then calculate the jump sizes. This is the reversed process compared to the BEG method, where the jump sizes is set first and then one-step probabilities are calculated. Choosing one-step probabilities and then the jump sizes has overcome the issue with negative probabilities for higher dimensions and in (Ekvall, 1996) the algorithm seems also to have faster convergence. The reserved algorithm of BEG is called the NEK method.

The NEK and BEG approaches are both good in terms of both accuracy and computational speed for low dimensional option problems, but both still suffers from the curse of dimensionality. The simulations methods on the other hand has somewhat an advantage for higher dimensions. We choose to present the BEG approach, because it is a natural extension of the already presented CRR model, which we already build intuition upon. The key for the BEG approach is that we approximate a multivariate lognormal distribution with a discrete jump distribution.

3.3 Least Square Monte Carlo Method

We have seen that the binomial model is not suitable for path-dependent contingent claims like an Asian option and high dimensional multivariate contingent claims. Simulation methods overcome somewhat this issue. The first Monte Carlo methods was to use pure simulation techniques. These methods overcome the issue of path-dependent option, but they still suffered the curse of dimensionality. To solve the dimensionality problem the LSM was born, where the idea is to combine simulation with regression.

A pure simulation technique have three ingredients a simulation based on the assumption of the underlying asset(s) distribution to price potential future prices. From the simulation of the underlying(s) use the contract function to get the cash flow. Then discount back to present value and average over the simulated paths. The approach is suitable for the Asian option, because by simulation you have the whole path and averaging is straightforward. On the other hand the binomial model is computational fast and accurate for American options, because by discretization the underlying stock the algorithm give an effective way to get intrinsic and expected continuation values. The pure simulation method would not be ideal for American options, because a each decision point for the American option the pure simulation would need to simulate a new set of paths to estimated the expected continuation value. For reference of the pure simulation method is chapter 10 in (OVERHAUS et al., 2007).

Remember we approximate the American put option with a Bermudan put option, beacuse the computer is discrete by nature. The simulatation methods use the setup in section 2.4.3. We use again dynamic programming to calculate the price of the American put option. The problem with the pure simulation approach is the computational burden to evaluate the expected continuation value at each exercise date. The Longstaff and Schwartz LSM algorithm overcomes the exponential growing computation burden in pure Monte Carlo simulation by using regression to calculate the expected continuation value.

3.3.1 The Algorithm

To solve the optimal stopping problem numerically, we set up the mathematical framework for solving the problem inspired by (Clément, Lamberton, and Protter, 2001). The optimal problem in discrete time is:

$$\sup_{\tau \in \mathcal{T}(0,\dots,T)} E^Q[G(S(\tau))] \tag{3.5}$$

where $\mathcal{T}(0,\dots,T)$ is a class of all $(0,\dots,T)$-valued stopping times and $\Big(S(0), S(t_1), \dots, S(t_N) = S(T)\Big)$ is the underlying stochastic Markovian process describing e.g. stock price, volatility, average stock price, etc. The Markov chain $(S_{t_n})_{n=0,\dots,N}$ has state space $(E, \mathcal{E})$. Remember the discrete value process is given by

$$U(t_n) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}(t_n,\dots,t_N)} E^Q[G(S(\tau))|\mathcal{F}_n]$$

The Markov property implies that

$$E^Q[G(S(\tau_{t_{n+1}}))|\mathcal{F}_{t_n}] = E^Q[G(S(\tau_{t_{n+1}}))|S(t_n)] = f(S(t_n))$$

and we assume the initial state to be $S(0) = s$ and deterministic. We solve equation (3.5) by the theory presented in section 2.4.3, hence the dynamic programming principle on the optimal policy is

$$\begin{cases} \tau_{t_N} = t_N \\ \tau_{t_n} = t_n \cdot 1_{\{G(S(t_n)) \geq E^Q[G(S(\tau_{t_{n+1}}))|S(t_n)])\}} + \tau_{t_{n+1}} \cdot 1_{\{G(S(t_n)) < E^Q[G(S(\tau_{t_{n+1}}))|S(t_n)])\}} \end{cases} \quad for\ n = 0, \dots, N-1 \tag{3.6}$$

Equation (3.6) involves many conditions expectations, hence we need an effective algorithm for evaluating them. The pure simulation methods was expensive, because the paths required for evaluate the series of conditional expectations increases exponential with decision points (chapter 10 (OVERHAUS et al., 2007)). The solution suggested e.g. in (Longstaff and Schwartz, 2001; Tsitsiklis and Roy, 1999) is to use the cross-sectional information in simulated paths. The information is used for least square regression.

The LSM algorithm approximate the expected continuation value with respect to $S(t_n)$ by orthogonal projection on the state-space generated by finite number of basis functions of $S(t_n)$. Define the sequence $(e_j(s))_{j\geq 1}$ of real measureable functions defined on the state space $(E, \mathcal{E})$. Assume:

- The sequence $(e_j(S(t_n)))_{j\geq 1}$ is total in $L^2(\sigma(S(t_n)))$ for $n = 1, \dots, N-1$.
- if $\sum_{j=1}^m \lambda_j e_j(S(t_n)) = 0$ *a.s.* then $\lambda_j = 0$ for $n = 1, \dots, N-1$, $m \geq 1$ and $j = 1, \dots, m$

By defining the vector space generated by $(e_j(s))_{j=1,\dots,m}$ we denote the orthogonal projection from $L^2(\Omega)$ onto the vector space by $P^m_{t_n}$. We approximate the expected continuation value by the projection

$$\begin{cases} \tau^{[m]}_{t_N} = t_N \\ \tau^{[m]}_{t_n} = t_n \cdot 1_{\{G(S(t_n)) \geq P^m_{t_n}(G(S(\tau_{t_{n+1}})))\}} + \tau_{t_{n+1}} \cdot 1_{\{G(S(t_n)) < P^m_{t_n}(G(S(\tau_{t_{n+1}})))\}} \end{cases} \quad for\ n = 0, \dots, N-1$$

Where

$$P^m_{t_n}(G(S(\tau^{[m]}_{t_{n+1}}))) = \alpha^m(t_{n+1}) \cdot e^m(S(t_n)) \quad for\ n = 1, \dots, N-1$$

and the $\cdot$ on the right hand side is the usual inner product in $\mathbb{R}^m$ and $e^m = (e_1, \dots, e_m)$. By solving the above dynamic programming equation the time 0 approximate price of the Bermudan option is:

$$U^m(0) = \max\{G(S(0)), E^Q[G(S(\tau^{[m]}_{t_1}))]\}$$

Where $G(S(0))$ is deterministic by assumption, but $E^Q[G(S(\tau^{[m]}_{t_1}))]$ needs to be evaluated numerically. In the LSM algorithm the methods for evaluation the projection is Monte Carlo simulation.

From the assumption about the distribution of the underlying state space we simulate K independent paths $S^{(1)}(t_n), S^{(2)}(t_n), \dots, S^{(k)}(t_n), \dots, S^{(K)}(t_n)$ of the Markov

chain $S(t_n)$ with gain process $G(S^{(k)}(t_n))$ for $k = 1, \ldots, K$ and $n = 0, \ldots, N$. The least square estimator $\alpha^{(m,K)}(t_n) \in \mathbb{R}^m$ for the simulated paths is:

$$\alpha^{(m,K)}(t_n) = \underset{a \in \mathbb{R}^m}{\operatorname{argmin}} \sum_{k=1}^{K} \left(G(S^{(k)}(\tau_{t_{n+1}}^{k,m,K})) - a \cdot e^m(S^{(k)}(t_n)) \right)^2 \quad for\ n = 1, \ldots, N-1 \tag{3.7}$$

Where the recursively estimated stopping time is defined by:

$$\begin{cases} \hat{\tau}_{t_N}^{k,m,K} = t_N \\ \hat{\tau}_{t_n}^{k,m,K} = t_n \cdot 1_{\{G(S^{(k)}(t_n)) \geq \alpha^{m,K}(t_n) \cdot e^m(S^{(k)}(t_n))\}} + \hat{\tau}_{t_{n+1}}^{k,m,K} \cdot 1_{\{G(S^{(k)}(t_n)) < \alpha^{m,K}(t_n) \cdot e^m(S^{(k)}(t_n))\}} \end{cases} \quad for\ n = 0, \ldots, N-1 \tag{3.8}$$

An example of polynomial linear on each decision points is illustrated in figure 3.5, where the blue line is the estimated expected continuation value. From following the optimal stopping strategy by dynamic programming we derive the approximation for $U^m(0)$ from the simulated paths. The time 0 approximate price of the Bermudan option is

$$U^{m,K}(0) = \max\{G(S(0)), \frac{1}{K} \sum_{k=1}^{K} G(S^{(k)}(\hat{\tau}_{t_1}^{k,m,K}))]\} \tag{3.9}$$

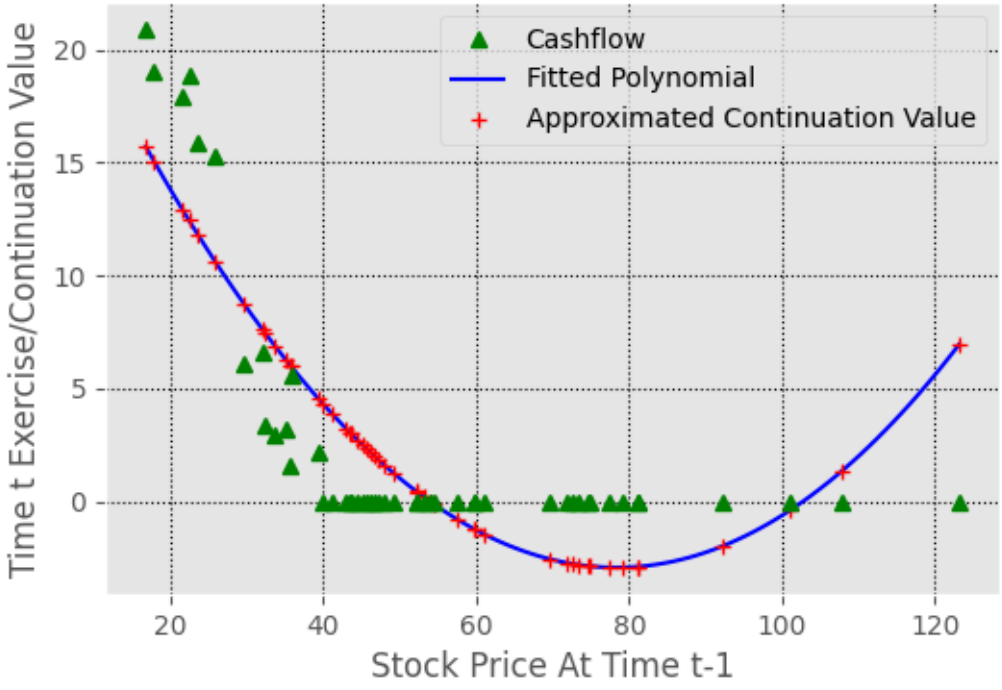

FIGURE 3.5: By zooming in on a specific point of time in the dynamic programming approach, we see how the algorithm regress the continuation value. In the illustration we used second order polynomial regression

3.3.2 American Put

The optimal stopping problem for an American put

$$P(0) = \sup_{\tau \in \mathcal{T}(0,\dots,T)} E^{Q}[e^{-r\tau} \cdot \max\{K - S(\tau), 0\}] \tag{3.10}$$

can be solved with the LSM algortihm. The stock values are modeled via Black-Scholes theory[4], hence the simulated evolution for the stock under the risk neutral valuation is given by:

$$S_i(t) = S_i(0) \cdot \exp\left(\sum_{j=1}^{d} (\sigma_{i,j} W_j(t) - \frac{1}{2}\sigma_{i,j}^2 t) + rt \right) \quad for\ i = 1, \dots, d$$

The stock paths are simulated from inception up to maturity with N decision dates. The focus in this section is on a univariate contingent claim and for convenience we assume the risk free interest rate and volatility is constant. Like in the binomial model, we first construct the paths and then work backward from maturity to inception at each exercise dates to decide the optimal stopping time.

From previous we know the dynamic programming principle on optimal policy gives the first optimal stopping time and at each decision point we regress the expected payoff by continuation of the contract and compare it to the intrinsic value. The dependent variable in the regression is the expected value of continuation and the independent variables is a set of orthogonal basis functions in $L^2(\sigma(S(t_n)))$ of the simulated paths. Typical choices for basis functions could be weighted Laguerre -, Hermit -, and Jacob polynomials. The weighted Laguerre polynomial is given by

$$\begin{aligned}
e_0(S) &= \exp(-S/2) \\
e_1(S) &= \exp(-S/2)(1 - S) \\
e_2(S) &= \exp(-S/2)(1 - 2S + S^2/2) \\
&\vdots \\
e_j(S) &= \exp(-S/2)\frac{e^S}{j!}\frac{d^j}{dS^j}(S^j e^{-S})
\end{aligned}$$

This kind of regression is a nonlinear expansion of the linear model[5]. We define regressed conditional expectation by:

$$\Psi(S(t_n); \alpha) = \sum_{j=0}^{p} \alpha_j \cdot e_j(S(t_n))$$

where α is the coefficients for the regression, e is the basis function, where the argument is the underlying Markovian process S. By using this iterative method, we arrive at the pathwise optimal stopping policy, where in figure 3.6 the optimal stopping times are shown. The figure illustrates that the option only can be exercised once, hence the gray lines after the triangles.

[4] Illustration of generated path for the GBM in figure 2.2
[5] Prediction is a linear function of the coefficients

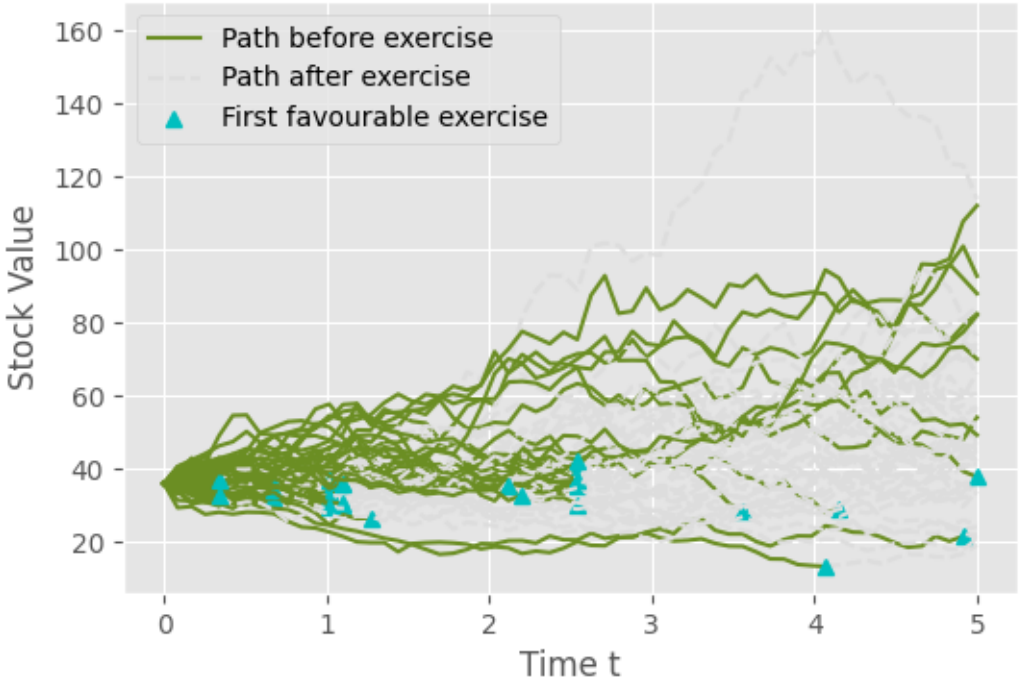

FIGURE 3.6: The optimal stopping decisions by the Least Square Monte Carlo Method

Note that for practical implementation the LSM only consider in the money path, hence equation (3.7) is changed to:

$$\alpha^{(m,K)}(t_n) = \underset{a\in\mathbb{R}^m}{\operatorname{argmin}} \sum_{k=1}^{K} 1_{\{G(S^{(k)}(t_n))>0\}} \left(G(S^{(k)}(\tau_{t_{n+1}}^{k,m,K})) - a \cdot e^m(S^{(k)}(t_n)) \right)^2 \quad for\ n = 1, \ldots, N-1$$

Hence only the in-the-money (ITM) paths are used for regression. Using only the ITM paths improves the algorithm, because the option value can never be less than zero, but including the other paths can yield negative results. The two articles (Longstaff and Schwartz, 2001; Tsitsiklis and Roy, 1999) have some differences in their approaches. The LSM consider only ITM and uses policy iteration instead of value iteration. Furthermore in (Tsitsiklis and Roy, 1999) they set $U(t_n) = \Psi(S(t_n); \alpha)$ instead of $U(t_n) = U(t_{n+1})$ in the LSM. These design choices make the LSM the most used method and it has the highest accuracy.

Note for our computational implementation we use the same paths for decision making and valuation, this gives a biased high, but the difference in practice is not noticeable. In general leads the LSM to a biased low because of the sub-optimal exercise strategy[6]. Therefore a upper bound can be computed for the option price, which is done in (Andersen and Broadie, 2004). We wrap up the LSM section by looking at how the method can be used for basket options.

[6]The LSM algorithm gives a lower bound given in appendix C.3

3.3.3 LSM Extension To Multivariate Contingent Claims

For pricing of bivariate and multivariate contingent claims, we have to account for correlation between assets. The correlation matrix Σ is given by

$$\Sigma = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} & \cdots & \rho_{1,d} \\ \rho_{2,1} & \rho_{2,2} & \cdots & \rho_{2,d} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{d,1} & \rho_{d,2} & \cdots & \rho_{d,d} \end{pmatrix} = \Sigma = \begin{pmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,d} \\ \rho_{2,1} & 1 & \cdots & \rho_{2,d} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{d,1} & \rho_{d,2} & \cdots & 1 \end{pmatrix} \tag{3.11}$$

The correlation between assets are given by $(\rho_{i,j})_{i \neq j}$ and each asset has correlation 1 with itself. We assume that $(\rho)_{i \neq j} \in (-\frac{1}{d-1}, 1]$, because then the correlation matrix is a real, symmetric and positive definite, hence Cholesky factorization can be utilized $\Sigma = LL^T$ where L is a lower triangular matrix. From the decomposition it is easy to simulate correlated assets in d-dimensional Black-Scholes model:

$$dS_i(t) = S_i(t)rdt + S_i(t)\sigma_i L_{i,.} d\boldsymbol{W}(t) \quad i \in \{1, 2, \ldots, d\} \tag{3.12}$$

Where $\boldsymbol{W}$ has values in $\mathbb{R}^d$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_d)$ is vector of volatilities.

By simulating the paths according to the Black-Scholes dynamic, it is straightforward to apply the LSM, because it follows the same algorithm just with a different contract function.

3.4 Closed Form Solutions For European Exotic Options

Most exotic options require numerical methods, but in some special cases there exist a closed form solution. We will look at some of them in this section, where the purpose is to provide benchmarks for the numerical methods. Furthermore we explore the boundaries of closed form solutions and show applications of martingale theory. Throughout the financial model and assumptions given in section 2.2 will be assumed. We derive closed form solutions to European call and put options depending on several variables, for simplicity we will focus on pricing options with 2 or 3 underlying stocks. The section is inspired by the article (Ouwehand, 2006). The exotic contingent claims we will consider are the geometric mean -, maximum - and minimum call option.

3.4.1 Geometric Basket Call Option

For a geometric basket call option the contract function is given by:

$$\Phi(S(T)) = \max\{(\prod_{i=1}^{n} S_i(T))^{\frac{1}{n}} - K, 0\}$$

The key to derive a closed form solution is the known result that the sum of normal random variables are multivariate normal distributed. This implies that the product of lognormal random variables are multivariate lognormal distributed, since:

$$\exp(x+y) = \exp(x) \cdot \exp(y)$$
$$\text{and}$$
$$X \sim \mathcal{N}(\mu, \sigma^2) \Rightarrow Y = \exp(X) \sim LN(\mu, \sigma^2)$$

Remember the assumption in section 2.2 that the stock price process follows a GBM, hence:

$$(\prod_{i=1}^{d} S_i(T))^{\frac{1}{d}} = (\prod_{i=1}^{d} S_i(0))^{\frac{1}{d}} \exp\left((r - \frac{1}{2d}\sum_{i=1}^{d}\sigma_i^2)T + \frac{1}{d}\sum_{i=1}^{d}\sigma_i W_i(T)\right) \tag{3.13}$$

By defining

$$\tilde{\sigma} = \frac{1}{d}\sqrt{\sum_{i=1}^{d}\sigma_i^2 + 2\sum_{i\neq j}\rho_{i,j}\sigma_i\sigma_j}$$
$$F = (\prod_{i=1}^{d} S_i(0))^{\frac{1}{d}} \exp\left((r - \frac{1}{2d}\sum_{i=1}^{d}\sigma_i^2)T + \frac{1}{2}\tilde{\sigma}^2 \cdot T\right)$$
$$\epsilon = \frac{\frac{1}{d}\sum_{i=1}^{d}\sigma_i W_i(T)}{\tilde{\sigma}\sqrt{T}} \sim \mathcal{N}(0,1)$$

and rewrite equation (3.13) by above definitions

$$(\prod_{i=1}^{d} S_i(T))^{\frac{1}{d}} = F \cdot \exp\left(-\frac{1}{2}\tilde{\sigma}^2 \cdot T + \tilde{\sigma}\sqrt{T}\epsilon\right)$$

This expression is one dimensional and is the standard GBM solution with zero drift and spot F. This has a known solution with Black-Scholes theory (section 2.3) and

the geometric mean call option has price

$$\Pi(t,\mathcal{X}) = \exp(-r\cdot(T-t))\Big(FN(d_1) - KN(d_2)\Big)$$

where $d_1 = \frac{\ln(\frac{F}{K})+\frac{1}{2}\tilde{\sigma}^2 T}{\tilde{\sigma}\cdot\sqrt{T}}$ and $d_2 = d_1 - \tilde{\sigma}\sqrt{T}$

The fact that the sums of normal random variables is multivariate normal distributed makes the geometric mean option easy to price in the Black-Scholes model, because the high dimensional problems can be treated as 1-dimensional problem.

3.4.2 Options On The Maximum Or The Minimum Of Several Assets

Here we restrict ourselves to consider the case with three underlying stocks like in (Ouwehand, 2006), but the formula can be generalized to higher dimensions. The contract functions we consider are:

- Best of assets or cash: $\Phi(S(T)) = \max\{S_1, S_2, \ldots, S_d, K\}$
- Call on maximum: $\Phi(S(T)) = \max\{\max(S_1, S_2, \ldots, S_d) - K, 0\}$
- Call on mininum: $\Phi(S(T)) = \max\{\min(S_1, S_2, \ldots, S_d) - K, 0\}$

Assume WLOG d=4 to avoid cumbersome calculations and notation. The section will heavily use the martingale framework developed in section 2.2 and stochastic calculus (Appendix B) to value these exotic options. The key is to choose the numéraire to a risky assets instead of the bank account. By results from section 2.2 the processes are still Q-martingales given the numéraire is strictly positive. Under the assumption of a arbitrage free and complete market it follows from risk neutral valuation formula.

$$S_0(t)E_t^{Q_0}[\frac{X}{S_0(T)}] = S_1(t)E_t^{Q_1}[\frac{X}{S_1(T)}]$$

This show that changing the numéraire does not change the price of the derivative.

3.4.2.1 Best Of Assets Or Cash

The best of assets derivative will provide the foundation for pricing best of assets or cash and call on maximum or minimum of several assets. The best of assets derivative pay out the price of the most valuable asset. The idea is then to set up 4 derivatives, where each derivative only gives a payout if the underlying stock is the most valuable out of the four stocks. This can be written with a indicator, where the payoff for the i'th asset is:

$$S_i(T)\cdot 1_{\{S_i(T)>S_j(T):i\neq j\}} \quad i = \{1,2,3,4\} \tag{3.14}$$

From equation (3.14) we see that the i'th assets is only different from zero, when it is the most valuable asset. The fair price for best of assets option $\Pi_{max}(t,\mathcal{X})$ is then given as the sum of the 4 derivatives. Hence the best of asset derivative can be found by appling RNVF (proposition 2.13) for each of them.

Let us first consider i=1 and we set S_1 to be the numéraire asset with martingale measure $\mathbb{Q}_1$. Then by RNVF:

$$\begin{aligned}\Pi_1(t,\mathcal{X}) &= S_1(t)E_t^{Q_1}[1_{\{S_1(T)>S_2(T),S_1(T)>S_3(T),S_1(T)>S_4(T)\}}] \\ &= S_1(t)Q_1[\ln(\frac{S_2(T)}{S_1(T)})<0,\ln(\frac{S_3(T)}{S_1(T)})<0,\ln(\frac{S_4(T)}{S_1(T)})<0]\end{aligned} \tag{3.15}$$

From above discussion the derivative price can be found by cycling through the 4 derivatives. Equation (3.15) can be evaluated, therefore we seek to evaluate the probability under the Q-martingale measure. By Ito's lemma (see A.1) the discounted process with stock j is

$$d(\frac{S_i(t)}{S_j(t)}) = \frac{S_i(T)}{S_j(T)}\cdot(a_i-a_j)dW^{Q_j}(t)$$

where $dW^{Q_j}(t)$ is standard d-dimensional Q_j-Wiener process and a_i is the i'th row of the covariance matrix. From the known solution of the GBM we have the distribution.

$$\ln(\frac{S_i(T)}{S_j(T)}) \sim \mathcal{N}\left(\ln(\frac{S_i(t)}{S_j(t)}) - \frac{1}{2}\sigma_{i/j}^2\cdot(T-t), \sigma_{i/j}^2(T-t)\right)$$

where $\sigma_{i/j} = (a_i - a_j)$ and $\sigma_{i/j}^2 = \sigma_i^2 + \sigma_j^2 - 2\rho_{ij}\sigma_i\sigma_j$.

Besides using the definition for d_1 and d_2 in proposition 2.16 we define:

$$d_1^{i/j} = \frac{1}{\sigma_{i/j}\cdot\sqrt{T-t}}\cdot\left(\ln(\frac{S_i(t)}{S_j(t)}) + \frac{1}{2}\sigma_{i/j}^2\cdot(T-t)\right)$$
$$d_2^{i/j} = d_1^{i/j} - \sigma_{i/j}\sqrt{T-t}$$

The correlation between $\frac{S_i(T)}{S_k(T)}$ and $\frac{S_j(T)}{S_k(T)}$ is:

$$\begin{aligned}\rho_{ij,k} &:= \frac{(a_i-a_k)\cdot(a_j-a_k)}{||a_i-a_k||\cdot||a_j-a_k||} \\ &= \frac{\rho_{ij}\sigma_i\sigma_j - \rho_{ik}\sigma_i\sigma_k - \rho_{kj}\sigma_k\sigma_j + \sigma_k^2}{\sqrt{(\sigma_i^2+\sigma_k^2-2\rho_{ik}\sigma_i\sigma_k)\cdot(\sigma_j^2+\sigma_k^2-2\rho_{jk}\sigma_j\sigma_k)}}\end{aligned}$$

Hence:

$$Q_1[\ln(\frac{S_2(T)}{S_1(T)})<0,\ln(\frac{S_3(T)}{S_1(T)})<0,\ln(\frac{S_4(T)}{S_1(T)})<0] = N_3(-d_2^{2/1},-d_2^{3/1},-d_2^{4/1},\rho_{23,1},\rho_{24,1},\rho_{34,1})$$

Cycling through each derivative, we get:

$$\begin{aligned}\Pi_{max}(t,\mathcal{X}) &= S_1(t)N_3(-d_2^{2/1},-d_2^{3/1},-d_2^{4/1},\rho_{23,1},\rho_{24,1},\rho_{34,1}) \\ &+ S_2(t)N_3(-d_2^{1/2},-d_2^{3/2},-d_2^{4/2},\rho_{13,2},\rho_{14,2},\rho_{34,2}) \\ &+ S_3(t)N_3(-d_2^{1/3},-d_2^{2/3},-d_2^{4/3},\rho_{12,3},\rho_{14,3},\rho_{24,3}) \\ &+ S_4(t)N_3(-d_2^{1/4},-d_2^{2/4},-d_2^{3/4},\rho_{12,4},\rho_{13,4},\rho_{23,4})\end{aligned} \tag{3.16}$$

We can extend the above result to best of assets and cash by letting $S_4(t) = K\exp(-r(T-$

$t)$), where K do not have any volatility and also independent of the other assets, hence (3.16) becomes:

$$\begin{aligned}\Pi_{max}(t,\mathcal{X}) &= S_1(t)N_3(-d_2^{2/1},-d_2^{3/1},d_1^1,\rho_{23,1},\rho_{24,1},\rho_{34,1})\\ &+ S_2(t)N_3(-d_2^{1/2},-d_2^{3/2},d_1^2,\rho_{13,2},\rho_{14,2},\rho_{34,2})\\ &+ S_3(t)N_3(-d_2^{1/3},-d_2^{2/3},d_1^3,\rho_{12,3},\rho_{14,3},\rho_{24,3})\\ &+ K\cdot\exp(-r(T-t))N_3(-d_2^1,-d_2^2,-d_2^3,\rho_{12},\rho_{13},\rho_{23})\end{aligned} \tag{3.17}$$

3.4.2.2 Call On Max And Call On Min

From the price of the best of asset or cash option equation (3.17) the call max option price can be derived. Note that

$$\max\{\max\{S_1,S_2,S_3\}-K,0\}=\max\{\max\{S_1,S_2,S_3\},K\}-K=\max\{S_1,S_2,S_3,K\}-K$$

The call on max option is then a best of asset or cash option subtracted the strike. Realizing that fact it is easy to price the call on max:

$$\begin{aligned}\Pi_{cmax}(t,\mathcal{X}) &= S_1(t)N_3(-d_2^{2/1},-d_2^{3/1},d_1^1,\rho_{23,1},\rho_{24,1},\rho_{34,1})\\ &+ S_2(t)N_3(-d_2^{1/2},-d_2^{3/2},d_1^2,\rho_{13,2},\rho_{14,2},\rho_{34,2})\\ &+ S_3(t)N_3(-d_2^{1/3},-d_2^{2/3},d_1^3,\rho_{12,3},\rho_{14,3},\rho_{24,3})\\ &- K\exp(-r(T-t))\cdot\Big(1-N_3(-d_2^1,-d_2^2,-d_2^3,\rho_{12},\rho_{13},\rho_{23})\Big)\end{aligned} \tag{3.18}$$

Similar ideas are needed to derive the call on min, which we will not provide[7]. The fair price for the call on min derivative is:

$$\begin{aligned}\Pi_{cmin}(t,\mathcal{X}) &= S_1(t)N_3(d_2^{2/1},d_2^{3/1},d_1^1,\rho_{23,1},-\rho_{24,1},-\rho_{34,1})\\ &+ S_2(t)N_3(d_2^{1/2},d_2^{3/2},d_1^2,\rho_{13,2},-\rho_{14,2},-\rho_{34,2})\\ &+ S_3(t)N_3(d_2^{1/3},d_2^{2/3},d_1^3,\rho_{12,3},-\rho_{14,3},-\rho_{24,3})\\ &- K\exp(-r(T-t))\cdot N_3(d_2^1,d_2^2,d_2^3,\rho_{12},\rho_{13},\rho_{23})\end{aligned}$$

To derive the put of the exotic european options we can utilize a put-call-parity (p. 6 (Ouwehand, 2006)), but it takes a different form than the one presented in chapter 2 proposition 2.17. The put-call-parity for the exotic European call options is:

$$V_c(K)+K\exp(-r\cdot(T-t))=V_p(K)+V_c(0)$$

Where $V_c(K)$ is the value of the exotic European call option.

The exotic European options serve as an benchmark for the multivariate lattice approach, because if the lattice approach is close to the European benchmark then it is reasonable to expect a close estimate to the American option. The above section is also a presentation of the martingale theory versatility.

[7]The interested reader can read more on p. 6 in (Ouwehand, 2006)

Chapter 4

Deep Learning

The chapter is inspired by (Goodfellow, Bengio, and Courville, 2016; MacKay, 2018) which the interested reader can find more information about deep learning. Deep learning experiences a renaissance, because of the technology improvements in hardware and software. The collection of data has also significant improved the field. Deep learning is a specialized field in machine learning, where you focus on a special architecture of models. Like in machine learning the basic components of a deep learning algorithm are a data set, cost function, optimization algorithm and a model. E.g. in the LSM method we assumed the linear model, data set was the simulated paths, the cost function was the mean square error and the optimization algorithm was a closed form solution of the normal equations. Deep learning is about studying neural networks which allows for greater flexibility than standard methods like linear regression.

A neural network consists of multiple layer, where the depth tells you how many layers the network has (see figure 4.2), hence the name "Deep learning". All the algorithms applied will be within supervised learning, where we try to fit the best relationship between the features and the response variable. Furthermore all our algorithms will be based on the multilayer perceptrons (MLPs) for regression, therefore most of the chapter present the theory for supervised MLPs regression. The advantage of a multilayer model is that for each layer the updated set of covariates can be more finely tuned to better explain the data making the model extremely flexible. The MLPs is also called feedforward neural networks because the information only travels forward in the neural network, through the input node(s) then through the hidden layer(s) and finally through the output node(s). First we present the basics for machine learning and then specialize the theory to deep learning.

4.1 Machine Learning Basics

The task for machine learning is to learn from data with respect to some performance measure. "A computer program is said to learn from experience E with respect to some class of tasks T and performance measure P, if its performance at tasks in T, as measured by P, improves with experience E" ((Goodfellow, Bengio, and Courville, 2016) p. 97). Classical tasks T could be classification or regression, where the two methods differ on the output type. The former has discrete outputs, where regression has continuous output. We are interested finding a price, which is naturally represented as a continuous value hence our task will be to regress the price or continuation value like in LSM. Measuring performance depends on the task, but for

regression a typical performance measure is mean squared error (MSE):

$$\frac{1}{K}\sum_{k=1}^{K}(y_k - \hat{y}_k)^2$$

There are other methods for measuring performance e.g. mean absolute error (MAE). Another measure to quantify the fit is coefficient of determination:

$$R^2 = 1 - \frac{\frac{1}{K}\sum_{k=1}^{K}(y_k - \hat{y}_k)^2}{\frac{1}{K}\sum_{k=1}^{K}(y_k - \bar{y})^2} \quad \text{where } \bar{y} \text{ is the sample mean}$$

Coefficient of determination explains how well the model explains the data compared to the empirical mean. Some care should be taken by comparing models with different capacity, because the coefficient of determination will tend to prefer the larger models. The experience comes from data, where the data can be given with or without target values $\boldsymbol{y}$. In machine learning the task is quite different depending on targets are given or not, hence the algorithms are split into two categories; supervised and unsupervised learning. The terminology supervised comes from a teacher gives you the target values $\boldsymbol{y}$ that the algorithms tries to predict from $\mathbf{X}$.

Machine learning is not about making overly complex models to fit your data perfectly, because then the model will most likely have poor performance on unseen data. This phenomenon is known as overfitting, but the models can also be too simple known as underfitting. If machine learning only cared about performance on the given data for training, then machine learning would be essentially optimization. The key difference is that machine learning wishes to obtain statistical generalization to unseen data. The practical way to evaluate the model is to measure generalization error on a test set. In machine learning the data is split into test data and training data, where the test set can only be used for evaluation after training.

The training data is used for training the model, where the training error tells how the fitted parameters fits to the training data. An unacceptable high training error could indicate the regression method could be too simple and underfitting is the issue. For training it can sometimes also be useful to have a validation data set to see how your model generalizes, because the test set cannot be used to make model choices, hence it is common to split the training set into a validation and training set. The validation set is allowed to be used in training and it tries to mimic the generalization error. A common technique for measuring validation errors is "cross-validation", where the data set is split into training and validation sets depending on the cross-validation scheme. The aim for the validation data is to approximate the model performance on unseen data, which is essentially what we want to be within an acceptable range.

After training the model we can evaluate the performance on a test set not seen in training. An unacceptable high test error when evaluating the model and very low training error could be an example of overfitting the model. Hence when training the model, we wish best of both worlds an acceptable training error and making the gap between training and test error acceptable as well. In machine learning the overfitting and underfitting concepts are expressed by the relation of bias-variance trade-off, where there is a trade-off between training error and generalization error. In case of overfitting the model has been extensively trained to get a low biases, but

the trade-off is that the model does not generalize well represented by a high variance.

A technique known as regularization is one approach to avoid overfitting. The regularization term is added to the cost function $J(\theta)$, which is essentially the function to minimize in order to train the model. There is a lot different regularization methods, where in the MLPs section 4.2 we will present some useful regularization for deep learning models. Besides the parameter estimated within the model, there is a need for model designs (hyperparameters).

Hyperparameter is the parameters not estimated within the model, but rather exogenous given values from the model designer. Some common examples in machine learning and deep learning is the learning rate, batch size, weight decay, model capacity, etc. The choice of the hyperparameters is often more important than the optimization algorithm chosen to estimate the parameters within the model, hence the need for suitable choices for hyperparameters is needed for most machine learning models. The calibration of the model to a specific task is called hyperparameter tuning, where it can be done manually or automatically. The manual choice requires expert knowledge to set the hyperparameter optimal for the given task, where the automatic procedures is often computational expensive. Examples of hyperparameters rutines are random search, grid search and Bayesian optimization.

The parameters within the model are found by minimizing the cost function $J(\theta)$. In some cases there exist either a closed form solution or the cost function is convex making the optimization problem easier to handle. The complexity of MLPs makes the cost function non convex and iterative optimization procedures are needed. The basic iterative procedure is the gradient descent where the concept is to repeatedly making small moves in parameters toward better configuration. The most popular gradient methods in deep learning are Adam, RMSProp, AdaGrad and stochastic gradient descent (SGD). The methods will be discussed in more details in section 4.2.3. To sum up a machine learning model needs a data set, a model, a cost function and an optimization algorithm. Understanding of basic machine learning will be the foundation for deep learning.

4.2 Multilayer Perceptrons

The goal of the multilayer perceptrons (MLPs) is to approximate a function $f^*(x)$, where the MLPs defines a mapping $x \in \mathbb{R}^R \mapsto f(x;\theta)$ to approximate $f^*(x)$. The task is to find the best θ such that the approximation $f(x;\theta)$ is close to the targets measured by a defined cost function $J(\theta)$. With the minimized cost function the goal is to archive statistical generalization i.e. useful results for test data not used for training.

The first step for MLPs is building the network, where we start with zooming in on a single neuron, which is one node of a directed acyclic graph (figure 4.2). Note that the MLPs is also called a feedforward network, because all the connections between the neurons are directed such that the network forms a directed acyclic graph.

4.2.1 A Single Neuron

The single neuron has a number R features of inputs x_r and one output $\hat{y}$ (see figure 4.1).

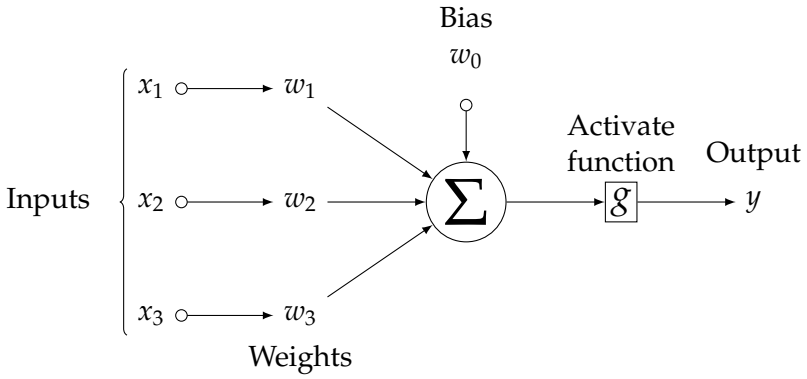

FIGURE 4.1: A single neuron

The function from inputs to output for a single neuron is:

$$\hat{y} = g(w_0 + \boldsymbol{x}^T \boldsymbol{w}) \quad where \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_R \end{pmatrix} \quad and \quad \boldsymbol{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_R \end{pmatrix}$$

The $\boldsymbol{w}$ is the weight matrix (this case a vector) and w_0 is the bias term. The term inside the function g is the activation of the neuron and it is a affine transformation denoted:

$$a = w_0 + \boldsymbol{x}^T \boldsymbol{w}$$

The function g is the activation function and it is essential for the flexibility of the MLPs. There exists numerous of activation functions only the imagination is the limit. We will list the most common and discuss them.

4.2.1.1 Activation functions

Activation functions are essential for neural network, because they allow for non-linearities and flexibility. Activation functions apply a non-linear transformation and decide whether a neuron should be activated or not. Without activation functions or the activation function being the identity function $g(a) = a$ the whole network would essentially be a linear regression model. Some popular activation functions are:

- Sigmoid function: $g(a) = \frac{1}{1+\exp(-a)}$

  This is the traditional choice, also called the logistic function. Popular in classification but can suffer from vanishing gradient in deep learning.

- Hyperbolic tangent function: $g(a) = \frac{e^a - e^{-a}}{e^a + e^{-a}}$

A scaled and shifted sigmoid function and likewise suffers from the vanishing gradient problem. The range is $(-1, 1)$ and centered at zero. Often used for hidden layers.

- ReLU function: $g(a) = \max(0, a)$.

  Rectified Linear Unit is one of the most popular choices since it does not suffer from the vanishing gradient problem. The MLPs often becomes more sparse because it sets some features to zero. It is claimed that ReLU learns multiple times faster than both the hyberbolic tangent and the sigmoid function.

- Leaky ReLU function:

$$g(a) = \begin{cases} a & if\ a \geq 0 \\ \alpha \cdot a & otherwise \end{cases}$$

  The fact that ReLU can have some hidden covariates that is zero which can be an advantage. A disadvantage of ReLU is some neurons may die out if the neurons are mapped to zero. The Leaky ReLU is designed to give such neurons a chance to get back into action, but not too easily, so $\alpha > 0$ is chosen small (typical $\alpha = 0.01$)

- ELU - exponential linear unit:

$$g(a) = \begin{cases} a & if\ a \geq 0 \\ \alpha(\exp(a) - 1) \cdot x & otherwise \end{cases}$$

  Like the Leaky ReLU the ELU is designed to avoid the dead ReLU problem.

With a understanding of a single neuron we continue to the architecture of a MLPs.

4.2.2 Architecture Of MLPs

A MLPs consists of a input layer, where all R features enter, L hidden layers, and an output layer. Each hidden layer and the output layer consists of multiple neurons, where the width of the layer is the number of neurons in that layer (m^l) (see figure 4.2). The networks inputs are called the input layer, the output layer is the output of the neural network. The layers between the input and output layer are hidden layers. This could be an explanation why the field is called Deep learning, because of a deep structure of layers. In each hidden layer a linear combination of the features from the previous layer is made and then an activation function is applied in order to create the new hidden features in that layer (see figure 4.2). The MLPs is essential a large nested function, where the input layer go through a chain of functions until reaching the output.

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{\theta}) &= \boldsymbol{f}_1 \circ \boldsymbol{f}_2 \circ \cdots \circ \boldsymbol{f}_{L+1} \\ where\ \boldsymbol{f}_i : \mathbf{R}^{m^{i-1}} &\to \mathbf{R}^{m^i} \quad i = 1, \ldots, L+1 \end{aligned}$$

Each function in the composition of functions corresponds to a layer of neurons.

$$\boldsymbol{f}_i(x) = \boldsymbol{g}(\mathbf{W}^T \boldsymbol{x} + \boldsymbol{w_0}) \quad x \in \mathbf{R}^{m^{i-1}}\ and\ \mathbf{W} \in \mathbf{R}^{\boldsymbol{m^{i-1}} \times \boldsymbol{m^i}}$$

So the function maps a vector to a vector, the hidden layers will often be denoted $\boldsymbol{h}$ where for each single neuron, we map a vector to a scalar by:

$$h_i = g(\boldsymbol{x}^T \cdot \mathbf{W}_{:,i} + (w_0)_i)$$

A layer is a vector of neurons, hence the transformation from one layer to the next can be interpreted as multiple vector to scalar transformations, where each neuron acts in parallelle. The different view motivates the initial presentation of single neuron network (section 4.2.1).

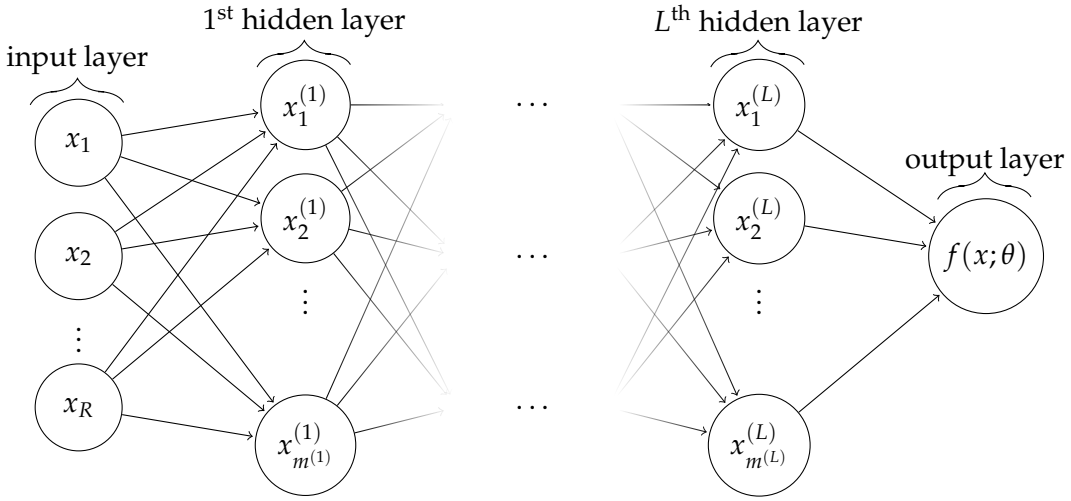

FIGURE 4.2: Multilayer perceptrons with $(L+1)$-layers with P input features and 1 output. The l^{th} hidden layer contains $m^{(l)}$ hidden neurons.

So the MLPs is not like classical linear regression in section 3.3 where a single linear transformation from input to output is applied. The unique attribute of neural network is the ability to approximate any kind of function[1], because of the flexibility with applying multiple functions to the input layer. The neural network has a lot of different design options, where e.g. hidden layers, layer width, depth, activation functions etc. are hyperparameters. In general it is recommended to use many hidden covariates (neurons) rather than too few. The danger of overfitting is than avoided by introducing some kind of penalty to avoid that the model becomes overly large, which we will discuss further in section 4.2.4.

To fit a model the model need initialization of weights, biases and activation functions. In order to measure the performance of the model, we need a function to measure the difference between the approximation $f(\boldsymbol{x};\boldsymbol{\theta})$ and the target values $f^*(\boldsymbol{x})$. The function used to quantify this approximation is the loss function, where the cost function is the average over the loss functions. The cost function tells how close the prediction $f(\boldsymbol{x};\boldsymbol{\theta})$ is to the target y. The cost function is key to improving our model in machine learning lingo training the model, hence the next section will cover model training.

[1]Universal Approximate Theorem page 194 (Goodfellow, Bengio, and Courville, 2016)

4.2.3 Training The Network

Training the network is key for building a high quality model. The performance of the model is measured by the cost function, where the cost function used in this thesis is the the empirical risk function.

$$J(\theta) = E_{(\boldsymbol{x},y)\sim\hat{p}_{data}} L(f(\boldsymbol{x};\boldsymbol{\theta}), y) = \frac{1}{k}\sum_{k=1}^{K} L(f(x_k;\theta), y_k)$$

The loss function L in empirical risk function for training is chosen to be quadratic, hence for training we penalize with mean square error (MSE) as our cost function.

From above we see that the cost function is a way to measure the approximation of $f^*(x)$ by our model $f(\boldsymbol{x};\boldsymbol{\theta})$. The training is important e.g. imagine after random initialization of parameter and construction of the MLPs we reported the output given the inputs of the model. This model would probably results in a high cost function value and bad performance because the given weights would produce a function that do not fit training data. The way out of the high cost function is to try to minimize the cost function over the weightspace hence training for MLPs is essentially a optimization of a non convex function. Remember the MLPs is a chain of functions (section 4.2.2) where the structure often makes the optimization a non convex problem hence a global minimum is seldom archived. Other pitfalls for optimization of MLPs are weight symmetry, steep cliffs, saddle points, vanishing and exploding gradient.

The actual optimization algorithms for MLPs are based on gradients, where we make small local moves. The overall goal is to find θ to reduce the test error. Within gradient methods there are batch gradient descend and mini-batch stochastic gradient descend, where the former is training on the whole data set, and the latter is only for a subset of the data set. The mini-batch methods have the advantage it can parallelized hence faster training. An epoch is the number of complete data set training cycles to update the weights. For the mini-batching techniques it is important to random sample from the whole data set in order to get unbiased gradient estimation.

The goal of the optimization algorithms is to find critical points $\nabla J(\theta) = 0$, hence to obtain the critical points the iterative methods move in the opposite direction of sign of derivative $\nabla J(\theta)$. I.e. the gradient tells us how to update the parameters with a stepsize called the learning rate η:

$$\theta_{new} = \theta_{old} - \eta\nabla J(\theta_{old})$$

$J(\theta)$ is the cost function, hence remember we use the MSE:

$$J(\theta) = \frac{1}{K}\sum_{k=1}^{K} L(f(x_k;\theta), y_k) = \frac{1}{K}\sum_{k=1}^{K}(y_k - \hat{y}_k)^2$$

Common method to estimate the parameters is gradient descent, stochastic gradient descent (SGD) and Adam, where all the optimization algorithms are iterative. Gradient descend uses the whole batch for each update, where Adam and SGD use mini-batches for each update. The Adam algorithm uses a adaptive learning rate, where the two others use constant or descending learning rate. The Adam method makes greater progress in more gently sloped directions of weightspace compared

to SGD and gradient descent.

Besides choosing a optimization procedure the initialization of the parameters are important. There is a lot different suggestions to initialize parameters, but there is no general golden rule at the moment because lack of understanding of the optimization procedure in neural nets. Often the practitioners tend to use simple and heuristic methods where it has been shown the initialization breaks weight symmetry.

The most common way of finding gradients is the backpropagation algorithm, where the basic idea is the chain rule from calculus.

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

To understand backpropagation it is often useful with a computational graph created by forward propagation, where the backpropagation compute the derivative from output layer to input layer by going backward in the computational graph. The training process is a forward-backward algorithm. Different starting values of θ will result in different parameters. The good news is that these predictors typically do not differ by very much. It is recommended to work with a set of different starting values, and then use as a final predictor the average of the individual predictors stemming from each starting value.

4.2.4 Regularization

The number of parameters and the capacity of neural networks arise often the problem of overfitting, i.e. the model does not generalize well on new data. Regularization is a technique that alter our optimization problem to discourage complex models, hence avoid the problem of overfitting. Some common methods for deep learning are parameter norm regularization, early stopping and dropout.

Early stopping is a effective and simple method for regularization. Compared to parameter norm regularization the early stopping algorithm does not harm the learning dynamics. The early stopping method is also computational efficient, hence it makes a popular regularization method for deep learning. The idea of early stopping is that the iterative training algorithm keeps improving the train error, but training too extensively leads the test error to rise. The idea is then to split your data into validation and train data sets, where you determine the best $\hat{\theta}$ and the corresponding training steps $\hat{i}$ by iterative comparing the cost function on the validation set. The algorithm stops after a predefined number of steps without improving the cost function on the validation set. The number of epochs to wait for the cost function not to improve is called the patient for the early stopping algorithm.

Like early stopping the dropout method is computationally inexpensive. The idea is to remove neuron randomly at each training loop, i.e. when updating the gradient, each node is kept with probability p, independently of each other. To perform dropout a binary mask is sampled independently for each iterations, where the probability p for 1 is another hyperparameter. The goal is to minimize $E_{\boldsymbol{\mu}} J(\boldsymbol{\theta}, \boldsymbol{\mu})$ where $\boldsymbol{\mu}$ is the mask. The result of the procedure is more robust features and a regularizing effect on most models.

There is a lot of design options for deep learning, where the choices should be specific for the given task. There is a no free lunch theorem for machine learning, which says no model is superior for all tasks (page 114 (Goodfellow, Bengio, and Courville, 2016)). The design for specific tasks are important where training error and test error can be improved by designing the model for the given task. Another aspect is the computational resources i.e. the memory space and computational time.

Pricing derivatives with deep learning methods have two clear benefits. The first is computational time, where after a model is trained, then the model is far superior to the methods presented in chapter 3. Another advantage is the non-linearities of the model making it possible to fit more complex functions. The application of deep learning in option pricing theory will be explored in next chapter.

Chapter 5

Option Pricing And Deep Learning

Deep learning can be applied to option pricing theory in different ways. We will investigate two methods using MLPs regression. The first method considered is a modified LSM algorithm, where MLPs regression is used to estimate the expected contuation value instead of the linear model. The second method use MLPs regression on a dataset $(\mathbf{X}, \boldsymbol{y})$ to infer prices. The input parameters $\mathbf{X}$ are simulated and the target variable $\boldsymbol{y}$ is found with existing pricing methods.

The two methods will be compared numerically in the next chapter. Both methods fall within supervised regression where we will use MLPs introduced in section 4.2 to approximate the mappings. The risky assets are modeled with Black-Scholes theory from earlier chapters hence the appropriate simulating methods are already presented (Chapter 2, 3). The theory in chapter 4 will be specialized for the specific task and discussed. The advantage of MLPs is that the model scale well to high dimensional data, where e.g. polynomial regression (section 3.3) is prone to overfitting and slow compared to MLPs for high dimensional tasks.

5.1 Multilayer Perceptrons Regression For Optimal Stopping

The first application of neural network is to investigate, if the LSM method can be improved by using MLPs regression to approximate continuation value instead of using the linear model in LSM. This section is influenced by (Longstaff and Schwartz, 2001; Lapeyre and Lelong, 2019; Kohler, Krzyżak, and Todorovic, 2010), where the approximate scheme with only in-the-money paths (ITM) is inspired by (Longstaff and Schwartz, 2001) and the idea of using neural network instead of the linear model comes from (Kohler, Krzyżak, and Todorovic, 2010; Lapeyre and Lelong, 2019). The algorithm as usual is based on approximating the American put option with one or several underlying assets by considering a Bermudan option.

The setup is the same as for the LSM presented in chapter 3, where the difference is the regression to estimate the expected continuation value. The main difference of the two methods lay in the capacity of the two regression models. The MLPs through its chain of functions and activation functions can capture non linearity and complex structures. Another advantage is that deep learning is known for breaking the curse of dimensionality, hence the MLPs could be better suited for multivariate contingent claims than the classical LSM.

5.1.1 Recap MLPs

We model the MLPs with a non linear function

$$s \in S \in \mathbb{R}^R \mapsto \Psi(s;\theta) \in \mathbb{R}$$

where Ψ is the function decomposition and given by

$$\Psi = a_{L+1} \circ g_L \circ a_L \circ \cdots \circ g_1 \circ a_1 \quad \textit{where } L \in \mathbb{N}$$

We refer to chapter 4 for notation, where $a_l(\boldsymbol{x}) = \mathbf{W}_l^T \boldsymbol{x} + \boldsymbol{w}_{0,l}$. We collect all the parameters of the different layers into a high dimensional parameter

$$\theta = (\mathbf{W}_l, \boldsymbol{w}_{0,l})_{l=1,\dots,L+1} \in \mathbb{R}^{N_m} \text{ with } N_m = \sum_{l=1}^{L+1} m^{(l)}(1 + m^{(l-1)})$$

We will restrict our parameter space by a increasing sequence $(\gamma_p)_{p\in\mathbb{N}}$ such that $\lim_{p\to\infty} \gamma_p = \infty$ and $p \in \mathbb{N}$ is the maximum number of neuron in each hidden layer. We define the set:

$$\Theta_p = \{\theta \in \mathbb{R} \times \mathbb{R}^p \times (\mathbb{R}^p \times \mathbb{R}^{p\times p})^{L-1} \times \mathbb{R}^p \times \mathbb{R}^{p\times p} : |\theta| \leq \gamma_p\}$$

By the definition of the set we restrict our MLPs to the set:

$$\mathcal{NN}_p = \{\Psi(\cdot;\theta) : \theta \in \Theta_p\}$$

Unfortunately the $\mathcal{NN}_p$ is not a vector space, hence the regression cannot be interpreted as an orthogonal projection as for the LSM. The MLPs method is justified by the "Universal Approximate Theorem" (theorem 5.1)

Theorem 5.1. ***Universal Approximation Theorem*** *Assume that the activation function g is non constant and bounded. Let μ denote the probability measure on $\mathbb{R}^r$, then for any $L \geq 1$, then $\mathcal{NN}_\infty$ is dense in $L^2(\mathbb{R}^r, \mu)$*
where $\mathcal{NN}_\infty = \cup_{p\in\mathbb{N}} \mathcal{NN}_\infty$

(p. 4 (Lapeyre and Lelong, 2019))

Note that the above theorem can be rephrase in terms of approximating random variables.

Remark 5.2. *Let Y be a real valued random variable such that $E[Y^2] < \infty$. Let X be a random variable taking values in R^r and $\mathcal{G}$ be the smallest σ-algebra such that X is $\mathcal{G}$ measurable. Then, there exists a sequence $(\theta_p)_{p\geq 2} \in \prod_{p=2}^{\infty} \Theta_p$ such that $E[|Y - \Psi_p(X;\theta_p)|^2] = 0$. Therefore, if for every $p \geq 2$, $\alpha_p \in \Theta_p$ solves*

$$\inf_{\theta\in\Theta_p} E[|\Psi_p(X;\theta) - Y|^2]$$

Then the sequence $((\Psi_p(X;\alpha_p))_{p\geq 2})$ converge to $E[Y|X]$ in $L^2(\Omega)$ when $p \to \infty$
(p. 5 (Lapeyre and Lelong, 2019)).

The remarks reveals that the MLPs can be used to approximate the conditional expectation instead of linear model in LSM.

5.1.2 The Algorithm

The algorithm is similar to the LSM, but the regression step is slightly different. We will use the same assumptions as in LSM section and same principles. Recall the optimal stopping problem can be solved with dynamic programming principle on the optimal policy:

$$\begin{cases} \hat{\tau}_{t_N}^{k,p,K} = t_N \\ \hat{\tau}_{t_n}^{k,p,K} = t_n \cdot 1_{\{G(S^{(k)}(t_n)) \geq \Psi_p(S^{(k)}(t_n);\hat{\theta}_{t_n}^{p,K})\}} + \hat{\tau}_{t_{n+1}}^{k,p,K} \cdot 1_{\{G(S^{(k)}(t_n)) \geq \Psi_p(S^{(k)}(t_n);\hat{\theta}_{t_n}^{p,K})\}} \end{cases} \quad for\ n = 0, \ldots, N-1$$

Where the estimator $\hat{\theta}_{t_n}^{p,M}$ is given by minimizing squared sample distance of the estimated continuation value and the realized continuation value:

$$\hat{\theta}_{t_n}^{p,M} = \underset{\theta \in \Theta_p}{\operatorname{argmin}} \sum_{k=1}^{K} 1_{\{G(S^{(k)}(t_n))>0\}} \left(G(S^{(k)}(\tau_{t_{n+1}}^{k,p,K})) - \Psi_p(S^{(k)}(t_n);\theta) \right)^2$$

Finally in analog with the LSM section the approximated time 0 price for the option is

$$U^{p,K}(0) = \max\{G(S(0)), \frac{1}{K} \sum_{k=1}^{K} G(S^{(k)}(\hat{\tau}_{t_1}^{k,p,K}))]\}$$

5.2 Multilayer Perceptrons Regression Pricing From Existing Methods

The MLPs pricing method from existing methods has a different approach than the other methods, because it is only data driven. The MLPs could easily be used to real data, which is investigated in (Gaspar, Lopes, and Sequeira, 2020). We revisit the work from (Hirsa, Karatas, and Oskoui, 2019), where we try to extend the pricing models to options with two underlying risky assets. The model will be the classical Black-Scholes model presented in earlier chapters. By choosing this model the MLPs pricing method is ready for investigation both for vanilla options and exotic options. We stress that the MLPs pricing method is not restricted to the Black-Scholes model, but can be applied to other models such as Hesten, Variance Gamma and real market data. The advantages with MLPs is the fast parameter estimation once trained. With the increased speed for pricing it can cost accuracy specially if the data is sparse, which can arise when using the method for exotic options on real market data. For practical application the accuracy is severe if the predicted price leads to arbitrage, because the aim is to price fast without introducing arbitrage. We considet the method for european call, american put and american put miminum on two assets options presented in earlier chapters.

For deep learning the hyperparameters is important for finding the right model for pricing, where different choices will be empirical presented under training. For the given task polynomial regression can also be used, but we see later why MLPs regression is preferred. The section is split into three sections "Data", "Training" and "Performance".

5.2.1 Data

The generation of labels are the computational expensive part of the MLPs method, because the method needs enough samples to approximate the function f^* well. The upside after generation of labels is that the method is computationally fast and easy to implement with basic knowledge of deep learning[1]. The labels will be generated by existing methods presented in chapter 3, where the input parameters will be uniform sampled or quasi sampled with Halton sequences.

For both European and the American univariate contingent claims the parameter in-sample will be the same, where we remember the 5 parameters for pricing an univariate contingent claims[2]. Note for simulation of labels the European call and American put options are first order homogeneous function in $(S(0), K)$, hence the valuation formulas can be modified:

$$\frac{c(S(0),K)}{K} = c(\frac{S(0)}{K}, 1) \quad \textit{and} \quad \frac{P(S(0),K)}{K} = P(\frac{S(0)}{K}, 1)$$

The alternative representation above reduces the number of parameters needed for simulation to 4, because instead of simulate both S and K, moneyness $(\frac{S(0)}{K})$ is only simulated. This property is also shared for the American bivariate contingent claim. For the European call and American put the input matrix $\mathbf{X}$ is different combinations of the 4 parameters within the parameter range (table 5.1), where it is assumed the

[1] Chapter 4
[2] E.g. the European call proposition 2.16

number of trading days are 252 in a year. The parameter ranges are risk free rate of return between 1-3 %, maturity between 1 day to 3 years, moneyness between 0.8 and 1.2 and volatility between 0.05 and 0.5.

TABLE 5.1: Parameter ranges for American put and European call options

Derivative	**r**	**T**	**Moneyness**	σ
Euro. Call	1%-3%	1d - 3y	0.8-1.2	0.05-0.5
Amer. Put	1%-3%	1d - 3y	0.8-1.2	0.05-0.5

The American put minimum option for two underlying assets will require additional parameters, because now we have two spots, two volatilities and correlation between the assets. The first order homogeneity does also hold for bivariate contingent claims.

$$\frac{P(S_1(0), S_2(0), K)}{K} = P(\frac{S_1(0)}{K}, \frac{S_2(0)}{K}, 1)$$

The parameters considered for the bivariate contingent claims are T, r, $Moneyness_1$, $Moneyness_2$, σ_1, σ_2 and ρ. So a total of 7 parameters and the given parameters ranges are given in table 5.2.

TABLE 5.2: Parameter ranges for American put minimum option on two assets

r	**T**	$Moneyness_1$	$Moneyness_2$	σ_1	σ_2	ρ
1%-3%	1d - 3y	0.8-1.2	0.8-1.2	0.05-0.5	0.05-0.5	(-0.5)-0.5

The simulation of the parameter ranges for the training data set is done by quasi-random Halton sequences sampling to obtain lower discrepancy, where the test sets are sampled with random uniform sampling. The quasi random sampling is different from uniform random sampling since the purpose is not to mimic randomness. Using Halton sequences sampling the aim is to increase accuracy by generating points that are too evenly distributed to be random. The Halton sequence and uniform sampling generates points between 0 and 1, hence we need to apply a transformation to get the parameter ranges:

$$Simulated\ point \cdot (range\ of\ parameter) + lower\ bound\ of\ parameter$$

After the simulation of the input parameters within the given ranges the labels can be generated from existing methods. For the European call option the generation of labels is done by the classical B-S formula for call options given in proposition 2.16. The B-S pricing formula is well known and has an analytical solution, hence it is relatively fast to generate labels in this model. The American options require numerical methods, hence the generation of labels are more computational expensive. The labels for the American options are generated by CRR with 100 equidistant time-steps for the american put option with one underlying stock and by BEG for two underlying assets with 50 equidistant time-steps presented in chapter 3. When the data

sets ($\mathbf{X}$, $\boldsymbol{y}$) are generated we are left with a regression task. The regression task is to predict the price $\boldsymbol{y}$ from the input parameters $\mathbf{X}$.

The data sets generated for within the parameter ranges will be referred to as the in-sample data set. The training data set is a in-sample data set, which is split up into a training and validation data set in order to approximate model performance on the test sets. The training data set is used to update the parameters internal in the model e.g. weights, biases, et cetera and fine-tune hyperparameters for choosing the optimal model design. To avoid overfitting and good generalization models the validation set is useful, because the trained model has not seen the data before evaluation on the validation set. The validation set is randomly subsampled from the training set, where the validation data set constitute 20 percent of the training set. To check the robustness of the regression we choose to sample three test sets with one in-sample and two out-of-sample data sets.

The test data sets out-of-sample is simulated by uniform sampling adjusting the parameter range for either moneyness or maturity for the options (table 5.3). The test data set has not been seen by the model in the training process, hence we get a unbiased evaluation of the model. The aim with producing different test and validation data sets is to measure the models performance at interpolation and extrapolation. The data set for the bivariate American contingent claim will be similar to the american put by adjusting either the $moneyness_1$ and $moneyness_2$ or the maturity.

TABLE 5.3: Parameter ranges for European call and American put option

Data set	Derivative	**r**	**T**	**Moneyness**	σ
In-Sample	Euro. Call	0.05-0.5	1d-3y	0.8-1.2	1%-3%
Out-Of-Money		0.05-0.5	1d-3y	0.6-0.8	1%-3%
Longer Maturity		0.05-0.5	3y-5y	0.8-1.2	1%-3%
In-Sample	Amer. Put	0.05-0.5	1d-3y	0.8-1.2	1%-3%
In-The-Money		0.05-0.5	1d-3y	0.6-0.8	1%-3%
Longer Maturity		0.05-0.5	3y-5y	0.8-1.2	1%-3%

In figure 5.1 we have visualized a simulated training data set for the American put. The marginal distributions shown is for 300.000 data samples ($\mathbf{X}, y$) generated by Halton sequences and the CRR model. The marginal distributions for the features cover the parameter range almost uniform and the simulated y lies with most values at zero and maximum at 0.387 rounded to three decimals. The marginal distributions shows that we have successfully generated parameters in the given ranges and the parameters are evenly spaced in the ranges. In the model performance section the out-of-sample and in-sample test data sets will be used to check the extrapolation and interpolation of the models. The test data sets are 60.000 generated data points with uniform sampling and the parameter ranges for the options with one underlying is given in table 5.3.

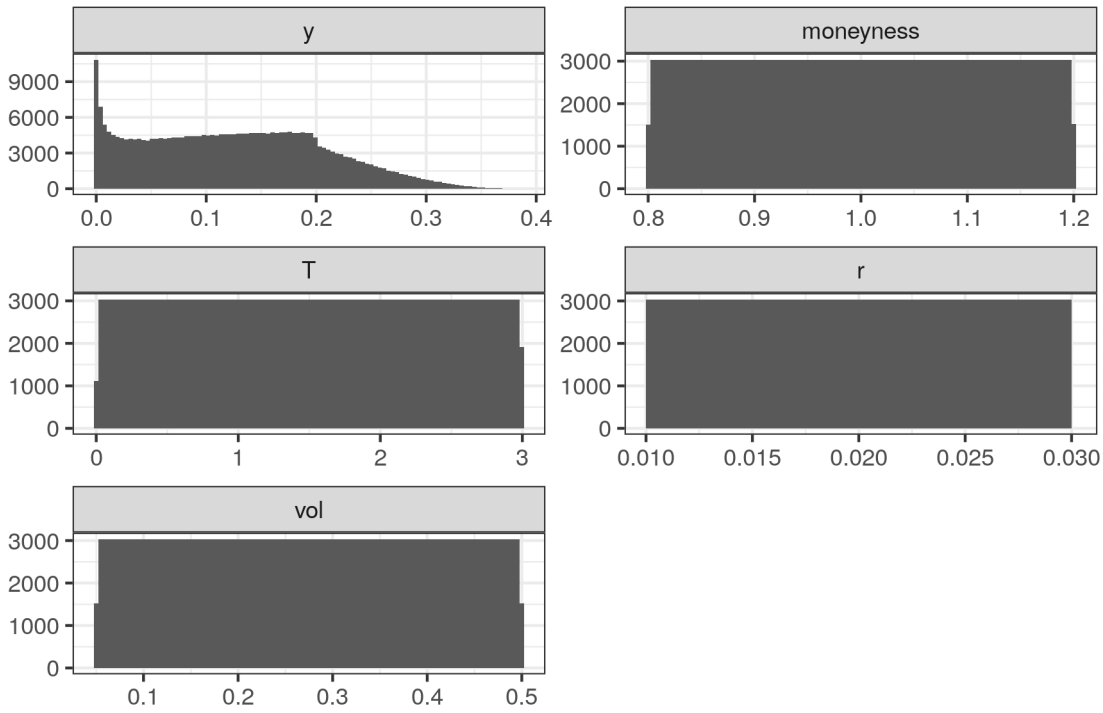

FIGURE 5.1: Quasi random simulation with Halton sequences for input variables and CRR for generation of labels for american put option.

5.2.2 Training

The aim for training is that the model will learn the pricing function f^*. Once the data set $(\mathbf{X}, \boldsymbol{y})$ is generated the model can be trained to approximate the true function f^*. We will also present a standard linear model to compare the MLPs with. We want to have a fast, robust and accurate model after training on the training set. To train the model we need a measure for the error, where the mean squared error (MSE) for regression is applied. I.e. the cost function is chosen to be the empirical risk function with a quadratic loss function:

$$J(\theta) = \frac{1}{K} \sum_{k=1}^{K} (y_k - \hat{y}_k)^2$$

The MSE penalizes outliers stronger than e.g. mean absolute error (MAE), but it penalizes small deviations less. To avoid overfitting we regularize using early stopping with a patient of 5 epochs, but with a maximum of 100 epochs for computational reasons. To update the weights the Adam optimization algorithm is chosen with learning rate η found by hyperparameter tuning.

5.2.2.1 Hyperparameter Tuning

A research area within deep learning is to fine-tune the hyperparameters to the specific task, where both manually search and automated search can be used. To choose

the best hyperparameters there is several choices, where the most basic automated task is random search and grid search. The random search is to define a predefined range to pick randomly from. This method can be effective to discover new hyperparameter values or combination, but it can be considerably more computational expensive than the grid search. The grid search is to search in a grid with each hyperparameters in each dimension. This kind of hyperparameter search has fast computational time relative to the random search, but requires some expert knowledge about the hyperparameters.

We test empirically the best hyperparameters for the MLPs, where the validation set is used. For hyperparameter tuning grid search is conducted to find the optimal set of hyperparameters where we will look at data set size, learning rate and batch size.

Firstly the European call option regression is investigated, where we vary the data set size and batch size. For the Adam optimization algorithm we choose a learning rate $\eta = 0.001$. The goal is to quantify how large a data set(d) and batch size(b) are needed for having a high quality model for the European option. The data sets for an European call option considered are in-sample data sets of size 1K, 10K, 100K, 300k and 1M data samples, where the validation set is subsampled from the training data set. The batch sizes are 64, 256 and 1024, i.e. we have the combinations expressed by the Cartesian product:

$$b \times d = \{(b,d) : b \in \{64, 256, 1024\} \; and \; d \in \{1K, 10K, 100K, 300K, 1M\}\}$$

By inspiration from (Hirsa, Karatas, and Oskoui, 2019) we train a MLPs model with 4 layers, 120 neurons in each hidden layer and 1 neuron in the output layer. In each layer we choose the activation function leaky ReLU with negative slope α=0.01, which is one of the most popular choices for activation functions. The number of data samples is relevant for real data, because for real market data there is not unlimited market data available.

Table 5.4 shows that the the model performs well for in-sample data with 10K-300K data points, hence the biggest data set with 1M seems not to be worth the computational cost. The model is only trained once on each data sets, so there is some randomness on each run, but the picture is clear. The model to interpolate prices for European call options in-sample data does not significant improve with gathering more data than 10K-300K data points, which is good news for using the method on real market data. Beware that the simulated data can underestimate the validation error, because we have a controlled setup where the parameter range is within a predetermined range. For the controlled setup with simulated data we can choose arbitrary many data points albeit making the method more computational expensive. By weighting both the computational cost and accuracy, we choose to work with 300K data points and a batch size of 64 for the European option.

The European option needs fewer parameters compared to the American put on minimum of two assets, hence the data set might need to be larger for that study. We conduct a large grid search with varying the learning rate η, the batch size b and the data set size d, where the grid of the Cartesian product of η, b and d is searched.

$$\eta \times b \times d = \{(\eta, b, d) : \eta \in \{0.0001, 0.001, 0.01\}, \; b \in \{8, 64, 256, 512, 1024\} \; and \; d \in \{1K, 100K, 300K\}\}$$

TABLE 5.4: Hyperparameter tuning of data set size and batch size for the American put bivariate contingent claim. The table shows validation loss in ascending order for different hyperparameter combinations and for the interested reader the tensorboard is online (link tensorboard 1)

Data set Size	**Batch Size**	**Training Loss**	**Validation Loss**	**Time: min:sec**
1M	256	$8.094 \cdot 10^{-7}$	$8.7764 \cdot 10^{-7}$	3:38
300K	64	$7.0562 \cdot 10^{-7}$	$1.0849 \cdot 10^{-6}$	2:49
1M	1024	$1.1596 \cdot 10^{-6}$	$1.1578 \cdot 10^{-6}$	5:58
300K	256	$9.579 \cdot 10^{-7}$	$1.3268 \cdot 10^{-6}$	2:29
300K	1024	$1.5367 \cdot 10^{-6}$	$1.4138 \cdot 10^{-6}$	9:28
1M	64	$3.4919 \cdot 10^{-7}$	$1.9197 \cdot 10^{-6}$	8:24
100K	256	$2.243 \cdot 10^{-6}$	$2.1192 \cdot 10^{-6}$	1:02
100K	64	$1.9565 \cdot 10^{-6}$	$2.5738 \cdot 10^{-6}$	1:01
100K	1024	$3.22 \cdot 10^{-6}$	$4.4754 \cdot 10^{-6}$	2:00
10K	256	$1.1179 \cdot 10^{-5}$	$1.0980 \cdot 10^{-5}$	0:37
10K	64	$1.0043 \cdot 10^{-5}$	$1.9830 \cdot 10^{-5}$	0:15
1K	64	$6.1389 \cdot 10^{-5}$	$7.8711 \cdot 10^{-5}$	0:22
10K	1024	$8.7067 \cdot 10^{-5}$	$8.1122 \cdot 10^{-5}$	0:32
1K	256	$1.2032 \cdot 10^{-4}$	$1.2504 \cdot 10^{-4}$	0:20
1K	1024	$7.5948 \cdot 10^{-3}$	$7.3595 \cdot 10^{-3}$	0:08

Besides varying the hyperparameters for η, b and d the other hyperparameters is the same values as for the European call option.

Looking at table 5.5 the best configuration for both the validation loss and training loss is for $(\eta = 0.0001, b = 8, d = 300K)$, but the computational cost is by far the most expensive by taking an hour for training with these hyperparameters. The reason lay in the lowest batch size, biggest data set and the smallest learning rate considered. The learning rate effect the computational speed, because we are training with early stopping regularization. Overall the data set with 300K performed best, which is expected, since a bigger data set gives more learning opportunities. The batch size effects heavily the computational speed, since a high batch size gives fewer iterations per epoch. Considering computational cost and the loss the 300K data sets, the learning rate $\eta = 0.001$ and a batch size of 64 or 256 are preferred.

Hyperparameter tuning is highly computational expensive and there is many opportunities to test out. We have seen with grid search the optimal choice for the learning rate, batch size and data set size for the European call option and American put on minimum for two assets. The American put option uses the same parameters as the european option, hence we choose the same hyperparameters for the univariate contingent claims. Below we will try a different model than MLPs to see if we really need deep learning for this method.

5.2.2.2 Polynomial Regression

To compare MLPs and Polynomial regression the data set and the performance metrics are the same, but the model training is obviously different. The chosen data set

TABLE 5.5: Hyperparameter tuning of data set size, learning rate and batch size for the American put bivariate contingent claim. The table shows the top 10 best performing combinations for the training loss and for the interested reader the tensorboard is online (link tensorboard 2) and the full table is given in appendix table D.2

Data set Size	**Learning Rate**	**Batch Size**	**Train Loss**	**Val. Loss**	**Time: min:sec**
300K	0.0001	8	0.0151	$3.615 \cdot 10^{-3}$	60:41
300K	0.001	64	0.0355	$6.676 \cdot 10^{-3}$	11:04
100K	0.0001	8	0.0649	0.0473	20:44
100K	0.001	8	0.0721	0.0373	21:41
300K	0.0001	64	0.0760	0.1548	13:51
300K	0.001	8	0.1046	8.8420	26:41
300K	0.01	256	0.4800	0.1400	2:37
300K	0.0001	256	1.0409	0.9533	6:40
300K	0.001	256	1.0681	0.9156	2:27
300K	0.001	512	1.0894	0.9345	3:11

size is with 300.000 samples and the performance metric is MSE. For simplicity the European call option is investigated, where we fit polynomials up to degree 6 for comparison of the model capacity and fit.

$$y_i = \beta_0 + \beta_1 \cdot x_i + \cdots + \beta_n \cdot x_i^n + \epsilon_i \quad \textit{where } n = 1, 2, \ldots, 6$$

Plotting the fit actual vs predicted target variable (figure 5.2) it is clear, that the in-sample fit improves with the increased model capacity. The linear regression is too simple for pricing European option, but the 6 order polynomial actually performs better than the MLPs for the in-sample validation set (table 5.6). Keep in mind that we do not only want good interpolation, but we also want good extrapolation for our fitted model. The out-of-sample data will reveal if the high order polynomial or MLPs have overfitted the data.

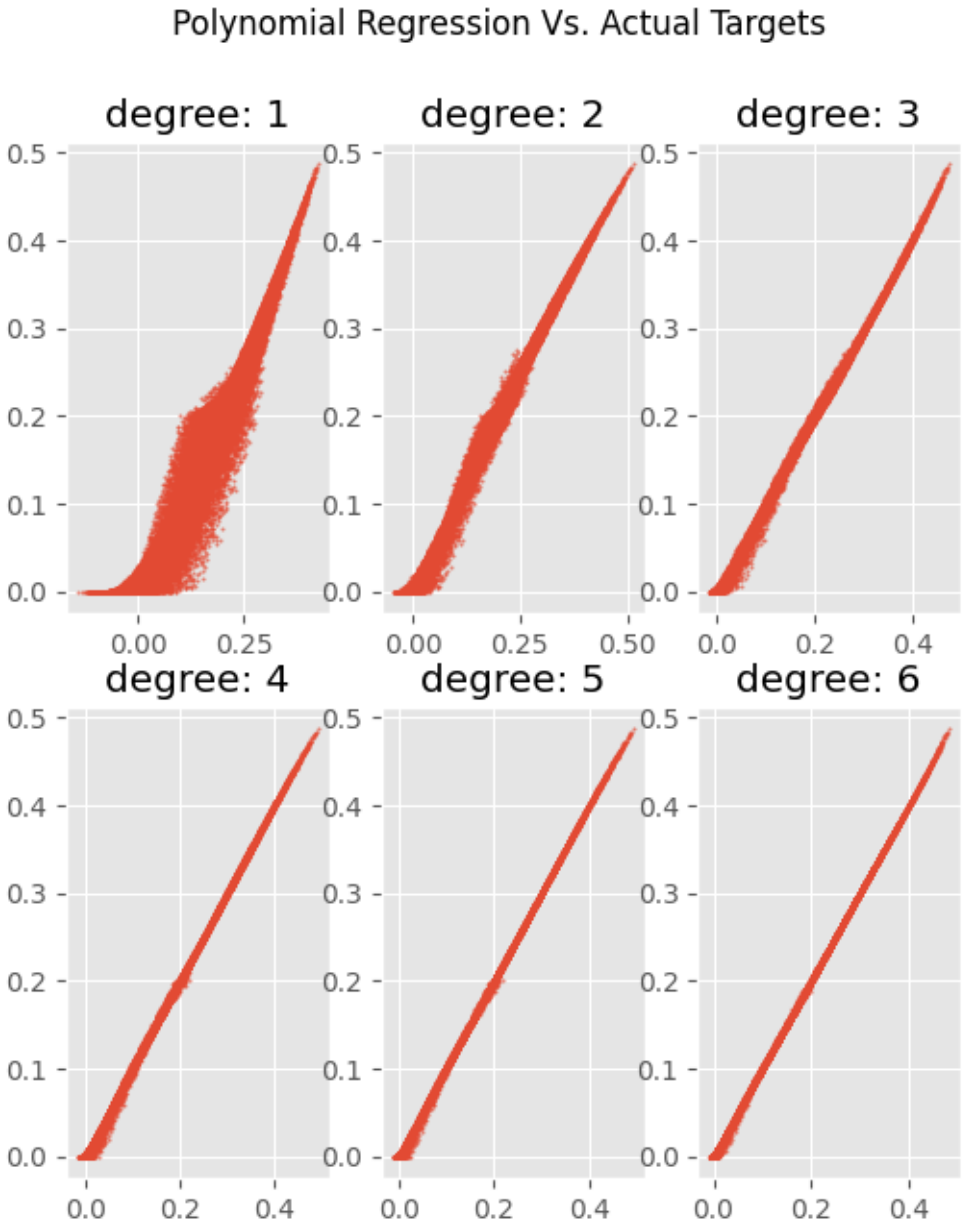

FIGURE 5.2: Predicted price based on polynomial regression of varying degree

Looking closer at table 5.6 to compare the performance of each model. The table confirms that the linear regression has a worse fit than the other models with higher capacity. The difference on the MLPs and best performing polynomial model are less than $1 \cdot 10^{-6}$. The difference is almost negligible so the fit for MLPs and polynomial regression of degree 4-6 performs all very well on the in-sample training data in terms of the validation loss.

TABLE 5.6: In-sample validation error for polynomial regression and MLPs on European call option

Model	**Valalidation Loss**
Linear Reg.	0.000631
2. degree poly.	0.000069
3. degree poly	0.000013
4. degree poly.	0.000004
5. degree poly.	0.000002
6. degree poly.	0.000001
MLPs	0.000003

5.2.3 Performance

The model performance is evaluated by MSE, RMSE, MAE and coefficient of determination, where all the measures evaluate how close the model predictions are with the actual targets. The first three measures ranges are $\mathbb{R}^+$, where the goal is to have the lowest value possible. For MSE close to 0 means that the model predictions does not differ a lot from the observed targets. The RMSE and MAE are same kind of measure, but the deviation is measured slightly different. The coefficient of determination has range $(-\infty, 1]$, where a higher value indicate a better model. Coefficient of determination provides a measure of how well observed targets are predicted by the model based on the proportion of total variation of targets explained by the model.

5.2.3.1 European Call Option

The European call option is trained with the algorithm and hyperparameters described in the training section (section 5.2.2). By the hyperparameter investigation we choose a batch size of 64, learning rate $\eta = 0.001$ and a data set of 300K samples. We compare the MLPs regression with the polynomial regression. Table 5.7 shows that the MLPs is superior at extrapolating, because the MLPs performs better on all metrics on the out-of-sample test data sets compared to fitted polynomial regressions with a degree between 1 and 6[3]. For the in-sample test data set the polynomial regression of order 6 and MLPs perform almost equally well, but the polynomial regression of 6. degree is a classical example of overfitting. Note the in-sample test data is similar to the in-sample training data, hence we expect the same magnitude of error for the test set as for the training set for in-sample testing. The performance measures show that the polynomial regression that was performing well on the in-sample data set was due to overfitting, because the high order polynomial regression does perform poorly on out-of-sample data (table 5.7). For the 6. order polynomial regression, we see a negative coefficient of determination, which means the model performs worse than the model that predicts the sample mean.

[3]The out-of-sample fits for the European call with polynomial regression are illustated in figure D.1 and figure D.2

TABLE 5.7: Performance comparison of MLPs and polynomial regression on European call option. Shown the best performing regressions in the linear model and the worst performing in terms of MSE for in-sample and out-of-sample data sets (The full table for polynomial regression is found in appendix table D.1)

Model	Data set	MSE	RMSE	MAE	Coefficent of Determination
MLPs	In-Sample	0.000000	0.000629	0.000486	0.999961
	Out-of-Money	0.000007	0.002644	0.001551	0.995911
	Long Matuiry	0.000197	0.014048	0.010061	0.986518
6. degree poly.	In-Sample	0.000001	0.000958	0.000591	0.999909
1. degree poly.		0.000636	0.025212	0.018326	0.936628
2. degree poly.	Long Maturity	0.001196	0.034577	0.026287	0.918316
6. degree poly.		0.043361	0.208233	0.111190	-1.962442
2. degree poly.	Out-Of-Money	0.000767	0.027694	0.022203	0.551246
1. degree poly.		0.005772	0.075973	0.060936	-2.377251

The MLPs has high predictive strength compared to the polynomials, because it performs well also on out-of-sample data set. The MLPs show that it predicts out-of-money call option with less than a MSE value of 10^{-5}, where the fit for the long maturity test set has a higher MSE. The European options are liquid in the markets, hence the pricing method can easily be trained on real data. The MLPs fit for in-sample data on the European call option is visualized in figure 5.3, which shows $\frac{c(S_0,K)}{K}$ predicted from the model and observed target values $\boldsymbol{y}$. The figure shows a overall close fit. For the remaining part of this section the MLPs regression will only be considered, because the polynomial regression has shown bad performance for out-of-sample data.

5.2.3.2 American Put Option

The American put option is priced with the same MLPs algorithm as for the european call. Table 5.8 shows once again a good fit, hence, we believe, we have a high quality model for the American put option[4].

TABLE 5.8: MLPs Performance on American Put Option

Data set	MSE	RMSE	MAE	R^2
In-Sample	0.000002	0.001562	0.001278	0.999634
In-The-Money	0.000012	0.003519	0.002290	0.995778
Longer Maturity	0.000193	0.013894	0.009213	0.980835

5.2.3.3 American Put On Minimum of Two Assets Option moneyness

The American put on minimum of two assets option has almost the double amount of parameters compared to the univariate contingent claims, hence the performance

[4]The out-of-sample fits for the American put with MLPs regression are illustated in figure D.3 and figure D.4

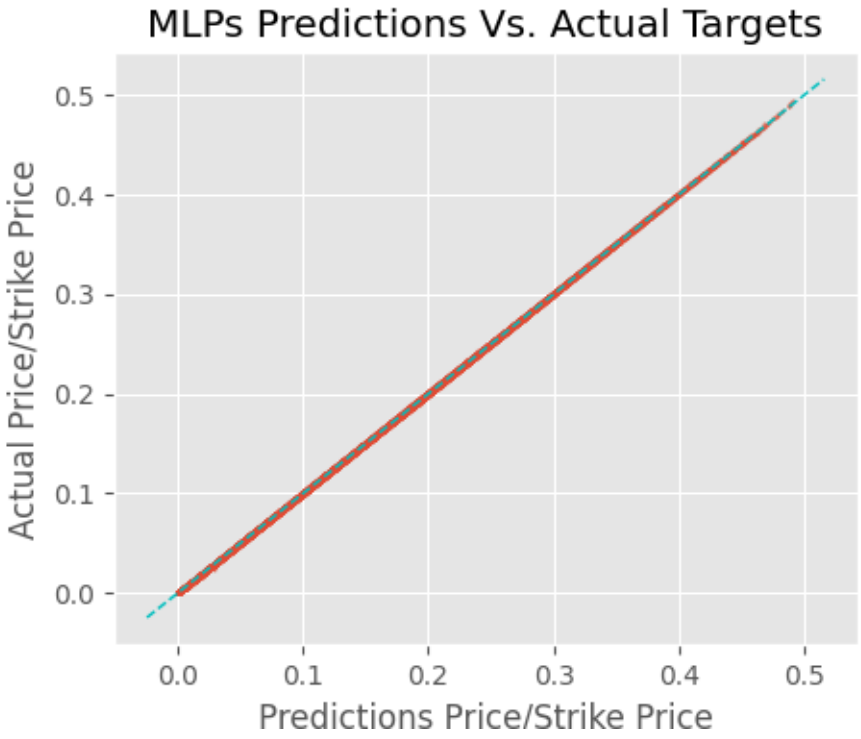

FIGURE 5.3: MLPs Performance on in-sample dataset on european call

might give slightly higher MSE. For training we used a batch size of 64 based on earlier discussions. Table 5.9 and figure 5.4 show that the MLPs also fit well the American put minimum option on two assets.

TABLE 5.9: Performance of the bivariate American put minimum contingent claim

Data set	**MSE**	**RMSE**	**MAE**	R^2
In-Sample	0.000002	0.001361	0.001008	0.999794
In-The-Money	0.000143	0.011973	0.009208	0.964370
Longer Maturity	0.000112	0.010565	0.007319	0.989863

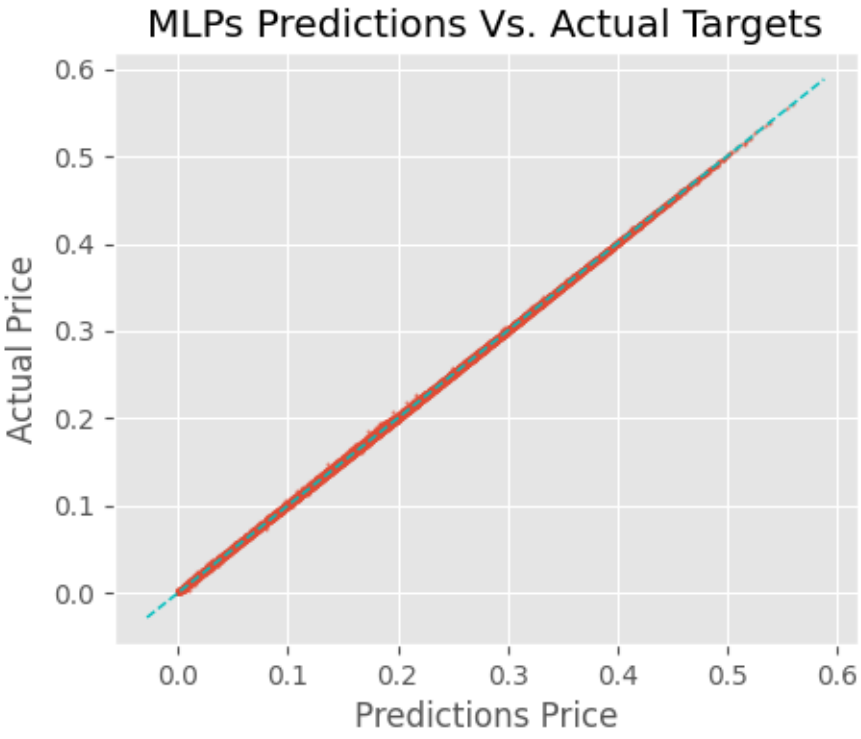

FIGURE 5.4: MLPs Performance on in-sample dataset on American bivariate contingent claim

The MLPs has shown good performance in terms of our performance measures on all the derivatives considered. Next chapter will compare this MLPs model (henceforth referred to as MLPS II) with the closed form solutions, binomial model, LSM and LSM MLPs (henceforth referred to as MLPS I). All the models in this section considered had good performance on the in-sample dataset except of the too simple models linear regression and 2. degree polynomial regression. The MLPs II was superior for out-of-sample data sets compared to the polynomial regression in terms of our performance measures.

The MLPs II is not based on a model, it is only trained on the available data. This feature makes the MLPs II pricing method versatile because the MLPs could also be used for actual market data or different models to learn patterns. The fact that the method can be extended to real market data is already shown in (Gaspar, Lopes, and Sequeira, 2020). This section showed how the MLPs can be trained to pricing derivatives with Black-Scholes theory and that the MLPs is preferred over polynomial regression.

Chapter 6

Numerical Investigation and Discussion

This chapter will compare empirically the numerical and analytical methods presented for different types of contingent claims[1]. The underlying model will be the Black-Scholes model dating back to the Black-Scholes paper (Black and Scholes, 1973)[2], because it has closed form solutions for some European options and it is thoroughly researched.

We look at the closed form solutions to the European options compared with the binomial lattice model for pricing. The closed form solution gives a measure of how the binomial lattice model approximate European options. The binomial model is readily extended to American options, hence the section also gives an indication how the binomial model approximates the American option. The American put option is then investigated, where the different numerical methods considered are compared. The last type of option we look at is the bivariate American put minimum contingent claim. After the numerical investigation the pricing methods and model assumptions are discussed.

6.1 European Options

European options are simple in the sense, that they can only be exercised at maturity. Throughout the previous chapters specially chapter 2 and 3 we have seen closed form solutions for the simple European and exotic European options. The binomial lattice models presented are numerical models that can approximate European options and American options. The closed form solutions and the numerical binomial lattice approach can be compared for European options, where a small deviation between methods for the European options indicates that the lattice approach gives reasonable prices for the American options.

The simplest case is the European call option, where we look how the CRR model and the MLPs II pricing model approximate the B-S call formula. Table 6.1 shows empirically that the CRR model price prediction converge toward the price from the Black-Scholes call formula, which is inline with the theoretical result for the CRR model that it converges to the Black-Scholes model. The MLPs II model underprice the European call option, but it performs better for this example than the CRR with 10 and 30 time-steps. The MLPs II is not practical for European call option, but

[1] The interested reader can see the implementation details in appendix D
[2] Assumption 2.15

it shows a relative close approximation, hence the increased speed for exotic options could be beneficial. Keep in mind we trained the MLPs II with 100 equidistant time-steps, hence we might have seen a better fit with a data set generated with 10^4 time-steps.

TABLE 6.1: Comparison of accuracy for the European call option, where the inputs are K=40, $S(0) = 40$, $\sigma = 0.2$, T=1 and r=0.06.

Method	**No. Steps**	**Price**
CRR	10	4.316
	30	4.369
	50	4.380
	100	4.388
	200	4.392
	500	4.394
	1000	4.395
	10000	4.396
MLPs II		4.370
Analytic form		4.396

The natural extension of the CRR model is the BEG model, where it is possible to price more exotic options. Section 3.4 showed some closed form solution for exotic European options, hence we have a benchmark for the BEG in these special cases. We choose to look at the computation time for European put minimum with two underlying assets, which has a closed form solution. Closed form solutions make it easy to look at the trade-off between accuracy and computational cost for the BEG method.

TABLE 6.2: Comparison of speed and accuracy for a European put min option, where the inputs are K=40, $S_1(0) = S_2(0) = 40$, $\sigma_1 = 0.2, \sigma_2 = 0.3$, T=1, $\rho = 0.5$ and r=0.06. Note ms is shorthand for millisecond

Method	**No. Steps**	**Price**	**Time: min:sec.ms**
BEG	10	4.248	0:00.003
	50	4.341	0:00:097
	100	4.352	0:00.591
	200	4.358	0:04.121
	500	4.361	0:59.337
	1000	4.362	9:34.164
Analytic form		4.363	

Tabel 6.2 shows that the algorithm accuracy increases with the number of equidistant time-steps, but the computational speed dramatically slows down when the time-steps increases. Therefore for exotic options the computational cost become a

factor to consider[3]. The BEG method accuracy is also tested on the European call minimum and European call maximum for 100 time-steps, where the BEG method is within 0.13 of the analytic solution (table 6.3).

TABLE 6.3: Valuation of bivariate contingent claims with K=40, $S_1(0) = S_2(0) = 40, \sigma_1 = 0.2, \sigma_2 = 0.3$, T=1, $\rho = 0.5$ and r=0.06.

Derivative type	**Method**	**No. Steps**	**Price**
European Call Minimum	BEG	100	2.475
	Analytic form		2.483
European Call Maximum	BEG	100	7.787
	Analytic form		7.800

The above tables show that the binomial model pricing model can be used for both univariate and bivariate European contingent claims. The binomial model accuracy is high for European options, hence we expect a similar good approximation for the American option. The CRR for univariate and BEG for bivariate contingent claims will be used as a benchmark for the American options, where we investigate LSM, MLPs I and MLPs II pricing methods.

Note that the BEG is not practical for pricing multivariate contingent claims with many underlying assets, because the possible states for the stochastic process increased exponentially. Therefore the bivariate contingent claim is considered instead of higher dimensional basket options in section 6.3.

6.2 American Put Option

The American put option has no analytically solution, hence numerical methods are required. We present and compare the results for the LSM, MLPs I and MLPs II pricing methods compared to the CRR model.

The LSM and MLPs I pricing method are almost identical, except the MLPs tries to utilize deep learning to regress the expected continuation value. Both pricing methods converge to the optimal value process[4], hence we except numerically that by increasing the computational burden we approach the true price. To compare the two methods we simulate 10^5 paths for the stock under the assumption that the future price of the stock is lognormal. The LSM and MLPs I pricing methods are used on the same simulated paths, but produce a different result each time due to the Monte Carlo simulations. The CRR and MLPs II[5] are not random in the sense that the output is deterministic, because both methods do not involve Monte Carlo simulation. For the LSM and MLPs I we assume 50 equidistant exercise dates for each year, where for the CRR we use 1000 equidistant time-step for the stock.

[3]Note the implementation is written in python, hence the code can be improved in terms of computational efficiency. The computations are performed on my laptop with 8GB ram and 8th Generation Intel® Core™ i5 processor

[4]See appendix C.2

[5]Note that we talk about the model after training

The MLPs I require us to set some hyperparameters, where we choose the learning rate $\eta = 0.001$, batch size of 512 and the Adam optimization algorithm. The architecture is a MLPs with three layers, where the hidden layers are with 40 neurons. The activation function is set to Leaky ReLU with 0.3 negative slope and the trained model is reused at each decision point by inspiration from (Lapeyre and Lelong, 2019). The choices are partly inspired by the work by (Lapeyre and Lelong, 2019) and empirical testing. The regression in the LSM is done with a polynomial regression of degree 10. Remember the MLPs II is trained with the same hyperparameters as for the European call option and the CRR with 100 time-steps is used to generate labels.

TABLE 6.4: Valuation of American put option with K=40 and r=0.06.

Spot	σ	**T**	**CRR**	**LSM**	**MLPs I**	**MLPs II**
36	0.2	1	4.487	4.481	4.364	4.584
36	0.2	2	4.848	4.846	4.747	4.649
36	0.4	1	7.109	7.118	6.919	7.090
36	0.4	2	8.508	8.514	8.215	8.487
38	0.2	1	3.257	3.258	3.217	3.094
38	0.2	2	3.751	3.748	3.681	3.638
38	0.4	1	6.154	6.157	6.075	6.172
38	0.4	2	7.675	7.695	7.359	7.605
40	0.2	1	2.319	2.317	2.292	2.114
40	0.2	2	2.900	2.896	2.823	2.779
40	0.4	1	5.318	5.329	5.180	5.274
40	0.4	2	6.923	6.934	6.750	6.839
42	0.2	1	1.621	1.623	1.599	1.494
42	0.2	2	2.217	2.224	2.183	2.167
42	0.4	1	4.588	4.600	4.538	4.548
42	0.4	2	6.250	6.269	6.111	6.197
44	0.2	1	1.113	1.119	1.094	1.000
44	0.2	2	1.694	1.700	1.653	1.678
44	0.4	1	3.954	3.959	3.931	3.949
44	0.4	2	5.647	5.669	5.524	5.649

Table 6.4 shows that the MLPs I always predict a lower price than the LSM, hence for our numerical study the LSM seems to be better than the MLPs I in terms of approximating the true price. The reference is the CRR model, which is a deterministic method. The MLPs II trained with the CRR model shows high variability form the CRR predicted price compared to LSM. The total deviation in absolute distance for the MLPs II is 1.56, where LSM deviation is 0.157 for the above table containing 20 prices with different unique input parameters combinations. The MLPs II is though better in total absolute deviation compared to the MLPs I which has a total absolute deviation of 2.078. This indicates that the MLPs II at this stage of development is preferred over the MLPs I in terms of speed and accuracy. The MLPs I and LSM has some uncertainty from the Monte Carlo simulation and with 10^5 paths the standard

error of the means are 0.0019 and 0.0214. The standard error[6] of the means are calculated by 100 samples[7] for the input parameters T=1, $\sigma = 0.4$, $r = 0.06$, $S(0) = 36$, $K = 40$. The empirical distribution mean is calculated by

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

and the standard error of the mean

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \quad where\ \sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})}$$

The histogram (figure 6.2) shows the variation in the estimates where the CRR price is the dashed black line. We see that the the MLPs I has higher standard error than the LSM and the center of the distribution[8] lower than the CRR price[9]. The LSM on the other hand has less variability and the center[10] is around the CRR price, hence in term of numerical stability, computational speed and accuracy the LSM is superior for the American put. The reason to the numerical instability for the MLPs I is that the optimization algorithm is random on each run compared to the linear model[11].

[6] Sometimes written in short form SE
[7] Denoted with n in the formulas
[8] Sample mean is 6.9048
[9] CRR price is 7.1094
[10] Sample mean is 7.1074
[11] E.g. polynomial regression where the solution is exact and unique

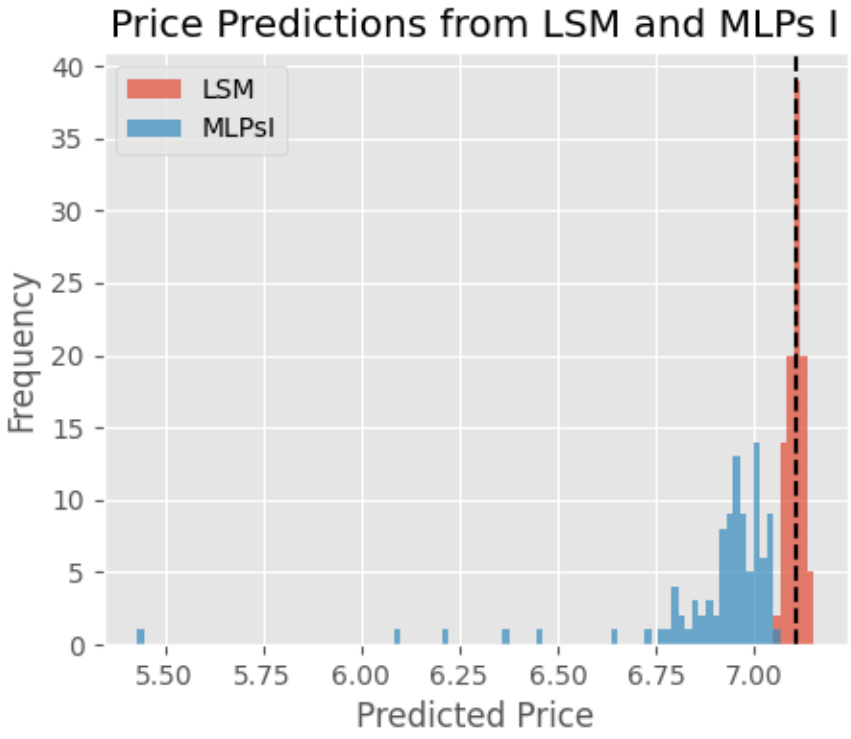

FIGURE 6.1: Predicted prices for American put option with LSM and MLPs. Parameters are T=1, $\sigma = 0.4$, $r = 0.06$, $S(0) = 36$, $K = 40$. Note CRR price is the dashed black line.

Improvements of MLPs I is needed in order for the method to challenge the existing LSM both in terms of speed and accuracy for low dimensional problem. Possible improvement of the MLPs I could be conducted by a large hyperparameter tuning study. The issue with hyperparameter tuning is that it is computational expensive. For the MLPs I the hyperparameter tuning need to be searched at every decision point. Another type of network could also be considered e.g. recurrent neural network. The inferior results from the MLPs I to the LSM show that some work still need to be done. We choose not to consider MLPs I and LSM for the next section, because of the poor results from the MLPs I pricing method in this section.

6.3 Bivariate American put minimum Option

There are many exotic American options to consider, but we choose to focus on the bivariate American put minimum option, because MLPs II is trained to predict prices on this specific option. The other advantage is that we have the deterministic method BEG to the bivariate contingent claim, hence the MLPs II can easy be compared both for accuracy and speed.

First we look at the computational cost for the BEG method by increasing the number of time-steps. By looking at table 6.5 it is clear that the computational time

is vary dependent on the number of time-steps. We know from the discussion in section 6.1 that the accuracy increases by increasing the number of time-steps. Weighting the computational cost and accuracy we choose to use the BEG method as reference price with 500 time-steps. Note that when we generated the labels for the MLPs II pricing method we used 50 equidistant time-steps. The reason to use 50 time-steps is that we simulated 300.000 data points, which is a computational heavy task. A MLPs II model trained with 500 time-steps would probably predict the price from the BEG method with 500 steps better, but it was not computational feasible. This gives an expectation that the option priced with MLPs II will have a bias price, hence we will also include the BEG with 50 time-steps in the comparison.

TABLE 6.5: Comparision of speed for the American put minimum option, where the inputs are K=40, $S_1(0) = S_2(0) = 40$, $\sigma_1 = 0.2, \sigma_2 = 0.3$, T=1, $\rho = 0.5$ and r=0.06. Note ms is shorthand for millisecond

Method	**No. Steps**	**Price**	**Time: min:sec.ms**
BEG	10	4.524	00:00.006
	50	4.594	00:00.250
	100	4.602	00:01.837
	200	4.605	00:14.025
	500	4.608	03:35.039
	1000	4.609	28:32.584

Table 6.5 shows that the price for the American put minimum is greater than the European put minimum from table 6.2. The other clear thing is that the computational time increases for the American option, because we have to compare intrinsic value and expected continuation value at each node. Both results are good sanity checks.

For visualization we plot the BEG50 and BEG500 and MLPs II for varying spots with fixed K=40, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, T=1, $\rho = 0.5$ and r=0.06. The figure shows that the BEG500[12] and BEG50 is close to each other. The total absolute deviation is 0.142 for the 21 prices, which shows that the BEG50 is suffient for training our MLPs. The total absolute deviation for the MLPs II with the BEG500 is 3.716 and with the BEG50 3.713. This is depicted in the figure, where the MLPs underprice the option for the in-sample domain with varying spot. To compare the computational speed the MLPs II took around 00:00.000349 for generate one price, which is considerable faster than the BEG with 10 time-steps. Hence we see we loose some accuracy with the MLPs II, but the pricing method is fast.

[12]BEG method with 500 time-steps

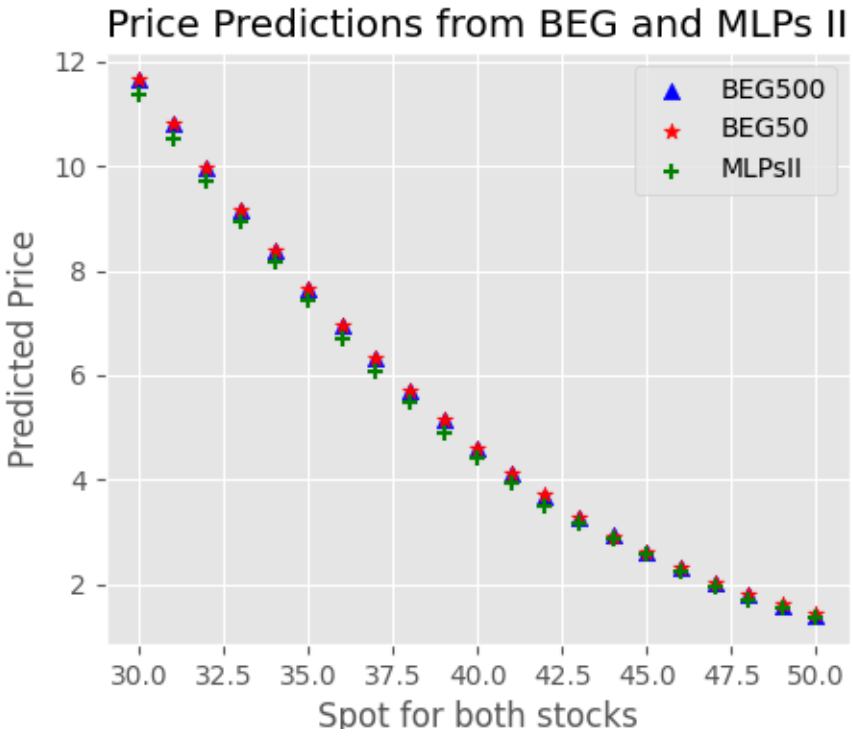

FIGURE 6.2: Scatter plot of predicted prices for American put minimum on two assets option with BEG and MLPs II. Parameters are T=1, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $r = 0.06$, $K = 40$ and ρ. Note BEG is showed for 50 time-steps and 500 time-steps.

The MLPs might give a better estimate by simulating more data samples and make the model more complex, but our hyperparameter tuning study was not conclusive on this point. Hyperparameter tuning could also be further investigated e.g. looking at depth and width of the network. The real power of pricing with MLPs II is when the computational burden becomes an issue for fast pricing.

6.4 Discussion of Pricing Methods

The idea underlying using neural networks instead of the linear model is that the neural networks scale better for high dimensional problems. We have only consider univariate and bivariate contingent claims, because in these cases we have a deterministic alternative in the binomial lattice models. This gives good indication how the model predicts. Besides if the model does not well for low dimensional problems, we would not expect the model to be any better for more complex multivariate contingent claims.

Remember the binomial lattice model has also its limitation in terms of computational resources, because the computational burden scales exponentially with the number of underlying risky assets[13]. The LSM is versatile and can be used for most derivatives, because it relies on Monte Carlo simulation and regression. For considering multivariate contingent claims the LSM is readily useful, but the linear model

[13]The same can be said about the PDE methods

for regression suffers the curse of dimensionality for basket options with many underlying risky assets. We have in mind that neural network does not suffer from the curse of dimensionality, hence the method compared to the linear model has advantages when considering multivariate contingent claims. The idea with neural network should work in theory[14], but the numerical results shown in this chapter are not satisfying for the MLPs I for the univariate case.

The MLPs II is somewhat better, but it still relies on existing option pricing methods, hence it does not solve the curse of dimensionality for basket options with many underlying assets. The MLPs II has though the advantage of the increased speed after training, which could be beneficial in some circumstances. The MLPs could also be used on real data, hence it is versatile enough to capture the volatility shew[15] for equity options. One drawback is that there should be enough data samples which is relevant if the method is used on real market data. Another drawback is that the trained model would need to be calibrated regularly, because financial scenarios can change dramatically e.g. the financial crisis in 2007-2009.

In the training face of the neural networks the derivatives of the model parameters are used for minimizing the loss function. The derivatives are calculated efficiently with backpropagation, which is important for risk management. Potentially the neural networks can speed up the risk management of the derivative books and give real time risks. Deep learning for risk management is investigated in (Savine, 2019). It should be mentioned that the above methods cannot only be used for equity options, but the LSM is e.g. widely used for fixed income[16].

The results from the MLPs I were a bit discouraging, because it was inferior to the LSM. The computational cost is higher for the MLPs I for low dimensional problems, hence a accuracy on the same level as for the LSM would make it attractive to investigate for higher dimensional problems. One explanation could be wrong hyperparameters for the model, which would require a time consuming hyperparameter search. The grid search might has its shortcomings in this aspect, because it requires some knowledge about possible well suited hyperparameters for the optimal stopping problem. A random search or Bayesian search could be beneficial in this aspect, but it would also require many computational resources.

The MLPs II lack some precision compared to LSM for the univariate contingent claim. For the bivariate the MLPs II again showed a loss of accuracy, but showed additional speed. A larger hyperparameter study could also be conducted here, a different method to generates labels or a bigger data set could have been sampled. E.g. the article (Ferguson and Green, 2018) shows good result for a European basket option with 6 underlying asset by simulating a data set of 500M samples.

6.5 Discussion of the Black-Scholes model

For pricing we utilize the Black-Scholes theory, where we assume constant volatility, interest rate is constant through time and future stock prices are lognormal distributed. It is well known that the volatility is not constant in fact it depends on the

[14] Universal approximation theorem for MLPs
[15] Explanation below in section "Discussion of the Black-Scholes model"
[16] E.g. Libor Market Model

maturity and strike. The dependency on strike and maturity can be modeled with a volatility surface. Looking only at one dependency the implied volatility for equity options is a decreasing function of the strike for real data (p. 458 (Hull, 2017)). This phenomenon is known as volatility shew for equity options.

A possible reason for the volatility shew is that investors are more concerned with falling stock prices than rising prices, hence the volatility are instantaneously negative correlated with the stock price. To overcome the issue about assuming constant volatility a model with stochastic variance can be considered. The model becomes more complex with an extra stochastic variable where for simplicity we mention the two factor Heston model. The basic model is given by the stock follow the SDE:

$$dS(t) = \alpha S(t)dt + \sqrt{V(t)}S(t)dW_S(t)$$

And the stochastic variance process $V(t)$ is the solution to the SDE:

$$dV(t) = a(\theta - V(t))dt + \epsilon\sqrt{V(t)}dW_V(t) \quad where a > 0, \theta > 0, \epsilon > 0 \text{ and } V(0) > 0$$

Where $W_S(t)$ and $W_V(t)$ have correlation ρ. The interpretation of the constants is:

- θ is long run average price variance
- a is the rate which $V(t)$ reverts to θ
- ϵ is the volatility of the volatility

The implication of ϵ is that $V(t)$ is more volatile when volatility is high. The correlation between the stock and variance process is often modeled with a negative correlation. The negative correlation between the two processes displays the real market phenomenon of volatility shew for equity options, because when the stock price drops the volatility increases. Assuming stochastic volatility the Heston model overcomes the issue with volatility shew. Another model trying to solve the non constant volatility is e.g. "Constant Elasticity of Variance model", but there exist numerous others[17].

The constant risk free interest rate through time assumption can also be discussed, because the interest rate is not constant for real market behavior. We did not investigate the American call, because it coincides with the European call for positive interest rate. Today markets interest rates are negative for the Eurozone, which was probably unheard before the last decade financial events. The decision to assume positive constant interest rate through time is set for convenience.

The Black-Scholes model assumes that the underlying risky assets evolves as a GBM, where the distribution of possible stock prices at the end of any interval is lognormal. The model is convenient, because the GBM has a analytical solution[18]. If we wanted to investigate arithmetic mean basket option the Bachelier model would be more convenient, because the future stock price is normal distributed.

$$dS_i = \sigma_i dW_i$$

[17] E.g "Merton's Mixed Jump-Diffusion Model" and "Variance-Gamma Model"
[18] SDE does not always have a analytical solution

This assumption would simplify the pricing problem of arithmetic basket options, because the sum of normals gives a multivariate normal distribution. The basket option problem in the Bachelier is then essentially one dimensional. This is similar to the the geometric mean basket option for the Black-Scholes model (section 3.4.1). A disadvantage with the Bachelier model is that it can lead to negative stock values, which is not realistic.

The Black-Scholes model has some drawbacks where you can question the assumptions, but the model is convenient to use for comparison of methods. By above discussion, we stress, that we do not necessary believe that the Black-Scholes model is the true model for real market behavior. The purpose is rather to investigate pricing methods in a convenient model.

Chapter 7

Conclusion and Further Investigation

7.1 Conclusion

We have seen different methods for pricing equity options within the Black-Scholes world. We have specifically focused on the European Call, Exotic European, American Put and American Put minimum on two stocks. The closed form solutions and the classical binomial lattice model and LSM were provided for investigating the potential of using deep learning in option pricing theory.

The numerical investigation showed that MLPs I and MLPs II pricing methods were inferior in accuracy for univariate and bivariate contingent claims, but the MLPs II had an enhanced speed compare to the classical methods after training. The results for the MLPs I were somewhat discouraging, but the theory tells us we can improve our training algorithm to perform on the same level as LSM. The CRR model was extended to the BEG method with two underlying stocks, which we saw gave a good approximation for European bivariate claims. The BEG method has its limitations in terms of pricing a basket option with many underlying stocks, which can be solved with LSM or MLPs I.

To sum up the MLPs I needs hyperparameter tuning, but the underlying theory and idea gives hope for finding a better model for high dimensional data than the classical LSM. The BEG has its limitation in high dimensions, hence LSM is preferred for higher dimensions. The MLPs II showed better performance than for polynomial regression at supervised learning. Furthermore the MLPs II can be beneficial in terms of computational speed, but has lower precision for univariate contingent claims than the LSM compared to our CRR benchmark price.

7.2 Further Investigation

The code library is written in python, which could be optimized in terms of computational speed with e.g. Julia or C++. The advantage with python is the libraries for machine learning and specially deep learning. The code could also be more generalized not only to handle specific option contracts by using the object oriented programming paradigm instead of procedural programming approach. The code was run on my CPU, but the speed could also be improved by using the GPU.

Hyperparameter tuning for the MLPs would be interesting to investigate further with e.g. the more advanced method Bayesian hyperparameter tuning. The article from (Liaw et al., 2018) looks at how hyperparameter tuning can be easy implemented, which is worth looking at. The data sets could have been bigger or the labels could have been generated with a different method. Another interesting aspect to investigate is to calculate the derivatives of the pricing function, which gives the risks for the derivative books. It looks like MLPs are superior for calculating risks in terms of speed to the classical methods, hence there is potentially an additional gain using deep learning.

Appendix A

Stochastic Calculus and Probability Theory

Theorem A.1. ***Itô's formula multidimensional*** *Let the n-dimensional process X have dynamics given by:*

$$dX(t) = \mu(t)dt + \sigma(t)dW(t) \tag{A.1}$$

Then the process $f(t, X(t))$ has stochastic differential given by:

$$df(t,X(t)) = \frac{\partial f(t,X(t))}{\partial t}dt + \sum_{i=1}^{n}\frac{\partial f(t,X(t))}{\partial x_i}dX_i(t) + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f(t,X(t))}{\partial x_i \partial x_j}dX_i(t)dX_j(t) \tag{A.2}$$

Note:

$$dW_i(t)\cdot dW_j(t) = \begin{cases} \rho_{ij}dt & \text{For correlated Weiner processes} \\ 0 & \text{For independent Weiner processes} \end{cases}$$

(p. 58-60 (Björk, 2009))

Theorem A.2. ***The Martingale Representation Theorem*** *Let W be a k-dimensional Wiener process, and assume that the filtration is defined by:*

$$\mathcal{F}_t = \mathcal{F}_t^W \quad t \in [0,T]$$

Let M be any $\mathcal{F}_t$-adapted martingale. Then there exist uniquely determined $\mathcal{F}_t$-adapted processes $h_1, \ldots, h_k$ such that M has the representation

$$M(t) = M(0) + \sum_{i=1}^{k}\int_0^t h_i(s)dW_i(s) \quad t \in [0,T]$$

if the martingale M is square integrable, then $h_1, \ldots, h_k$ are in L^2

(p. 161 (Björk, 2009)).

Theorem A.3. ***The Girsanov Theorem*** *Assume the probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t^{\bar{W}})_{t\in[0,T]})$ and let the Girsanov kernel ϕ be any d-dimensional adapted column vector process. Choose a fixed T and define the process L on $[0,T]$ by:*
$dL_t = \phi(t)^T \cdot L_t d\bar{W}_t^P$
$L_0 = 1$.
Assume that $E^P[L_T] = 1$ and define the new probability measure Q on $\mathcal{F}_T$ by:

$$L_T = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_T$$

Then

$$d\bar{W}(t) = \phi(t)dt + dW(t) \tag{A.3}$$

Where $W(t)$ is the Q-Wiener process and $\bar{W}(t)$ is the P-Wiener process.

(p. 164 (Björk, 2009))

Theorem A.4. ***The Converse of the Girsanov Theorem*** *Let $\bar{W}$ be k-dimensional standard P-Wiener process (i.e. zero drift and unit variance independent components) on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t^{\bar{W}})_{t\in[0,T]})$. Assume that there exits a probability measure Q such that $Q << P$ on $\mathcal{F}_T^{\bar{W}})$. Then there exists an adapted process ϕ such that the likelihood process L has dynamics*

$$dL(t) = L(t)\phi^T(t)d\bar{W}(t)$$
$$L(0) = 1$$

(p. 168 (Björk, 2009))

Definition A.5. Stopping time: A random variable $\tau : \Omega \to [0, \infty]$ is called a Markov time if

$$(\tau \leq t) \in \mathcal{F}_t \quad \forall t \geq 0$$

A Markov time is called a stopping time if $\tau < \infty$ P-a.s.

(p. 27 (Shiryaev and Peskir, 2006))

Definition A.6. Snell envelope Consider a fixed process Y.

- We say that a process X dominates the process Y if $X_n \geq Y_n \; P-a.s. \; \forall n$.
- Assuming that $E[Y_n] < \infty \; \forall n \leq T$, the Snell envelope S, of the process Y is defined as the smallest supermartingale dominating Y. More precisely: S is a supermartingale dominating Y, and if D is another supermartingale dominating Y, then $S_n \leq D_n \; P-a.s. \; \forall n$.

(p. 380 (Björk, 2019))

Theorem A.7. ***The Snell Envelope Theorem*** *The optimal value process V is the Snell envelope of the gain process G.*

(p. 381 (Björk, 2019))

Appendix B

Code Implementation

The code is available at my GitHub repository https://github.com/MrPPL/DeepPricing. The code is written in the python directory and the following files are used:

- European call and put option:
  ClosedForm directory and the file closedEuro.py
- Exotic European options:
  ClosedForm directory and the file rainbow2Dim.py
- CRR:
  Directory BinomialModel and file BinoHull.py
- BEG:
  Directory BinomialModel and file BEGTwoDim.py
- LSM:
  Directory DeepStopping and file AmericanPut.py[1]
- MLPs I:
  Directory DeepStopping and file AmericanPut.py [2]
- MLPs II:
  Directory DeepLearning/hirsa19/evaluation and file MLPsIIPricing.py

[1]Note that the LSM is implemented by using luphord GitHub repository https github.com/luphord/longstaff_schwartz

[2]Note that the MLPs is my own build upon the LSM method

Appendix C

Additional Material for CRR, LSM and MLPs I

C.1 Moment Matching CRR

In the CRR the stock multiplication factor for up and down movement is chosen to match the two first moments of the lognormal distribution. By matching the first moments the underlying discrete binomial stochastic process for the stock converge toward the continuous time lognormal distribution for sufficient large equidistant time-steps N. Hence the CRR model will coincide with the Black-Scholes model.

The SDE for the Black Scholes under the equivalent martingale measure Q:

$$dS(t) = rS(t)dt + \sigma S(t)dW(t)$$

Using Itô's lemma:

$$d\ln(S(t))) = (r - \frac{1}{2}))dt + \sigma dW_t \tag{C.1}$$

The solution of equation C.1 is then:

$$S(t) = S(0)\exp\left((r - \frac{1}{2}\sigma)t + \sigma W(t)\right)$$

Note that $W(t) \sim \mathcal{N}(0, t)$ implies:

$$\ln(\frac{S(t)}{S(0)}) \sim \mathcal{N}((r - \frac{1}{2}\sigma^2)t, \sigma^2 t)$$

The two first moments for the lognormal distribution is then:

$$\begin{aligned} E[\frac{S(t)}{S(0)}] &= \exp(rt) \\ E[(\frac{S(t)}{S(0)})^2] &= \exp(t \cdot (2r + \sigma^2)) \end{aligned} \tag{C.2}$$

The above derivation can be used for any time interval.

The binomial lattice model has two discrete outcomes from each state, hence the moments are:

$$\begin{aligned} E[\frac{S(t_{n+1})}{S(t_n)}] = u \cdot Q(\frac{S(t_{n+1})}{S(t_n)} = u) + d \cdot Q(\frac{S(t_{n+1})}{S(t_n)} = d) = u \cdot q + d \cdot (1-q) \\ E[(\frac{S(t_{n+1})}{S(t_n)})^2] = u^2 \cdot q + d^2 \cdot (1-q) \end{aligned} \tag{C.3}$$

We match the moments (equations (C.2) and (C.3)) and get two equations with two unknowns, where the time-step is chosen to be Δt

$$\begin{aligned} \exp(r\Delta t) &= u \cdot q + d \cdot (1-q) \quad (i) \\ \exp(\Delta t \cdot (2r + \sigma^2)) &= u^2 \cdot q + d^2 \cdot (1-q) \quad (ii) \end{aligned}$$

Multipling (i) with u+d and recognizing (ii):

$$\begin{aligned} (u+d)\exp(r\Delta t) &= u^2 \cdot q + d^2 \cdot (1-q) + ud \\ \Rightarrow (u+d)\exp(r\Delta t) &\overset{(ii)}{=} \exp(\Delta t \cdot (2r + \sigma^2)) + ud \end{aligned}$$

Remember we choose $u = \frac{1}{d}$ hence we arrive at a quadratic equation by some algebra.

$$u^2 - u\Big(\exp(-r\Delta t) + \exp(\Delta t(r + \sigma^2))\Big) + 1 = 0$$

We are interested in that the binomial model converge toward the Black-Scholes model, hence we are looking at small time increment Δt. This justify the Taylor approximation of the exponential function around zero.

$$\exp(r\Delta t + \sigma^2 \Delta t) \approx 1 + (r + \sigma^2)\Delta t + O(t^2)$$

By Taylor approximation we arrive at a simpler quadratic equation:

$$u^2 - u(2 + \sigma^2 \Delta t) + 1 = 0$$

Solving the quadratic equation above gives

$$\begin{aligned} u &= 1 + \frac{1}{2}\sigma^2 \Delta t \pm \frac{\sqrt{(2 + \sigma^2 \Delta t)^2 - 4}}{2} \\ &= 1 + \frac{1}{2}\sigma^2 \Delta t \pm \frac{\sqrt{(4 + \sigma^4 \Delta t^2 + 4\sigma^2 \Delta t - 4}}{2} \\ &= 1 + \frac{1}{2}\sigma^2 \Delta t \pm \frac{1}{2}\sigma\sqrt{\Delta t}\sqrt{\sigma^2 \Delta t + 4} \\ &\approx 1 + \frac{1}{2}\sigma^2 \Delta t \pm \sigma\sqrt{\Delta t} \\ &\approx \exp(\pm\sigma\sqrt{\Delta t}) \end{aligned}$$

Both approximations use that the time-step is small.

C.2 Convergence for LSM and MLPs I

C.2.1 LSM

In the rigorous approach in (Clément, Lamberton, and Protter, 2001) they show convergence results for the optimal value process or the Snell envelope U. We will present that $U(0)^{m,K}$ converges almost surely to $U(0)^m$ for K goes to infinity, i.e. the approximate value process by simulation and regression on a finite set of functions converge to the approximated value process with truncated orthogonal basis by letting the sample size go to infinity. Furthermore it can be shown that $U(0)^m$ converge to $U(0)$ for m goes to infinity. The second result is that the regressed value function converge to the expected continuation value by letting the number of basis function goes to infinity. The latter result is shown using the expected continuation values.

Theorem C.1. *Assume the sequence $(e_j(S(t_n)))_{j\geq 1}$ is total in $L^2(\sigma(S(t_n)))$ for $n = 1, \dots, N-1$. Then for $n = 0, \dots, N$ we have*

$$\lim_{m\to+\infty} E^Q[G(S(\tau_{t_n}^{[m]}))|\mathcal{F}_{t_n}] = E^Q[G(S(\tau_{t_n}))|\mathcal{F}_{t_n}]$$

in L^2

Proof. The proof is given by induction, where the orthogonal basis is total in L^2 is important, because $||P_{t_n}^m(E^Q[G(S(\tau_{t_{n+1}}))|\mathcal{F}_{t_n}]) - E^Q[G(S(\tau_{t_{n+1}}))|\mathcal{F}_{t_n}]||_2 \to 0 \quad for\ m \to \infty$. (more details on p. 6-7 (Clément, Lamberton, and Protter, 2001)) ■

The former result is also shown in (Clément, Lamberton, and Protter, 2001).

Theorem C.2. *Assume the sequence $(e_j(S(t_n)))_{j\geq 1}$ is total in $L^2(\sigma(S(t_n)))$ for $n = 1, \dots, N-1$ and if $\sum_{j=1}^m \lambda_j e_j(S(t_n)) = 0$ a.s. then $\lambda_j = 0$ for $n = 1, \dots, N-1$, $m \geq 1$ and $j = 1, \dots, m$. Furthermore assume that $Q(\alpha_{t_n} \cdot e(S(t_n)) = G(S(t_n))) = 0$.*
Then $U^{m,K}(0)$ converges almost surely to $U^m(0)$ as K goes to infinity.
The proof is out of scope for this thesis, see the article (Clément, Lamberton, and Protter, 2001) for a proof in details.

The two convergence results show that the convergence for the LSM algorithm, hence the LSM will approximate the optimal value process well for sufficient large sample sets and enough basis functions.

C.2.2 MLPs I

The MLPs regression method enjoys the same convergence results presented for the LSM algorithm, i.e.

Theorem C.3. *Assume that*

$$E[\max_{0\leq t_n\leq T} |G(S(t_n))|^2] < \infty$$

. Then $\lim_{p\to\infty} E^Q[G(S(\tau_{t_n}^p))|\mathcal{F}_{t_n}] = E[G(\tau_{t_n})|\mathcal{F}_{t_n}]$ in $L^2(\Omega)$ for all $n \in \{1, 2, \dots, N\}$
Proof p. 7-8 (Lapeyre and Lelong, 2019)

The above theorem states convergence of the neural network approximation, i.e. $U^p(0) \to U(0)$ which is similar to the convergence result for LSM.

Theorem C.4. ***Strong law of large numbers:*** *Assume*

A1: For every $p \in \mathbb{N}$, $p > 1$, *there exist* $q \geq 1$ *and* $\kappa_p > 0$ *such that*

$$\forall s \in \mathbb{R}^R, \forall \theta \in \Theta_p, \quad |\Psi_p(s,\theta)| \leq \kappa_p(1 + |s|^q)$$

Moreover $\forall n \in \{1, 2, \ldots, N-1\}$, *a.s. the random functions* $\theta \in \Theta_p \mapsto \Psi_p(S(t_n), \theta)$ *are continuous. Note* Θ_p *is a compact set, hence the continuity is uniform.*

A2: For q *defined in A1,* $E^Q[|S(t_n)|^{2q}] < \infty \quad \forall n \in \mathbb{N} \cup 0$

A3: $\forall p \in \mathbb{N}$, $p > 1$ *and* $\forall n \in \{1, 2, \ldots, N-1\}$,

$$P(S(t_n) = \Psi_p(S(t_n); \theta^p_{t_n}) = 0$$

A4: $\forall p \in \mathbb{N}$, $p > 1$ *and* $\forall n \in \{1, 2, \ldots, N-1\}$, *if* θ^1 *and* θ^2 *solves*

$$\underset{\theta \in \Theta_p}{\operatorname{argmin}} E^Q[|\Psi_p(S(t_n); \theta) - S(\tau^p_{t_{n+1}})|^2]$$

then $\Psi_p(s, \theta^1) = \Psi_p(s, \theta^2)$ *for almost all* x.

If A1-A4 hold, then for $\zeta \in \{1, 2\}$ *and every* $n \in \{1, 2, \ldots, N\}$

$$\lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} \left(G(S(\hat{\tau}^{k,p,K}_{t_n})) \right)^{\zeta} = E[\left(G(S(\tau^p_{t_n})) \right)^{\zeta}] \quad a.s. \tag{C.4}$$

Proof p. 13-14 (Lapeyre and Lelong, 2019)

The last result is the strong law of large numbers for the value function, i.e. $U^{p,K}(0) \to U^p(0)$ *a.s.* $\quad for\ K \to \infty$

C.3 LSM Lower Bound

The LSM approach gives a lower bound for the true price of the option given optimal stopping choice:

Proposition C.5. ***Lower Bound To True Value:*** *For any finite choice of M, K, and vector* $\theta \in \mathbb{R}^{M \times (K-1)}$ *representing the coefficients for the M basis functions at each of the K-1 early exercise dates, let* $LSM(\omega; M, K)$ *denote the discounted cash flow resulting from the following the LSM rule of exercising when the immediate exercise value is positive and greater than or equal to* $\hat{F}_M(\omega_l; t_k)$ *as defined by* θ. *Then the following inequality holds almost surely,*

$$V(X) \geq \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} LSM(\omega_i; M, K)$$

(p. 124 (Longstaff and Schwartz, 2001))

Appendix D

Additional Tables and Figures

D.1 Tables

TABLE D.1: Polynomial regression result for the European call option

Data set	**Measure**	**1. degree**	**2. degree**	**3. degree**	4. degree	**5. degree**	6. degree
In-Sample	MSE	0.000636	0.000069	0.000013	0.000004	0.000002	0.000001
	RMSE	0.025212	0.008316	0.003624	0.002052	0.001414	0.000958
	MAE	0.018326	0.006141	0.002561	0.001280	0.000867	0.000591
	R^2	0.936628	0.993105	0.998691	0.999580	0.999801	0.999909
Long Maturity	MSE	0.002662	0.001196	0.001654	0.003956	0.012255	0.043361
	RMSE	0.051593	0.034577	0.040666	0.062894	0.110702	0.208233
	MAE	0.041232	0.026287	0.028371	0.039932	0.064402	0.111190
	R^2	0.818143	0.918316	0.887014	0.729744	0.162742	-1.962442
Out-of-Money	MSE	0.005772	0.000767	0.000839	0.000944	0.001812	0.004423
	RMSE	0.075973	0.027694	0.028960	0.030724	0.042568	0.066506
	MAE	0.060936	0.022203	0.020288	0.020542	0.027125	0.041315
	R^2	-2.377251	0.551246	0.509280	0.447668	-0.060261	-1.588030

TABLE D.2: Hyperparameter tuning of dataset size and batchsize for american put minimum two assets for the interested reader see the tensorboard

Data set Size	η	**Batch Size**	**Loss**
300K	0.0001	8	0.015130897983909
300K	0.001	64	0.035523075610399
100K	0.0001	8	0.064886227250099
100K	0.001	8	0.072143875062466
300K	0.0001	64	0.075988814234734
300K	0.001	8	0.104622706770897
300K	0.01	256	0.480043411254883
300K	0.0001	256	1.04093873500824
300K	0.001	256	1.06809389591217
300K	0.001	512	1.08942472934723
100K	0.001	64	1.18230485916138
300K	0.001	1024	1.30065310001373
300K	0.01	512	1.45135951042175
100K	0.01	256	1.52623510360718
300K	0.01	64	1.75954759120941
300K	0.01	1024	1.92213356494904
100K	0.01	512	2.01193928718567
100K	0.01	64	2.02969717979431
300K	0.0001	1024	5.72439670562744
300K	0.0001	512	5.72847843170166
100K	0.0001	256	5.76278400421143
100K	0.0001	512	5.77397203445435
100K	0.0001	64	5.80239868164063
100K	0.0001	1024	5.90070772171021
100K	0.001	256	6.29003953933716
100K	0.001	512	6.52896738052368
100K	0.001	1024	6.56187915802002
1K	0.001	8	7.5845251083374
100K	0.01	8	8.07239627838135
1K	0.01	8	11.7414264678955
100K	0.01	1024	13.4287099838257
1K	0.01	64	14.1254253387451
1K	0.0001	8	24.0087566375732
1K	0.001	64	27.0112972259521
1K	0.01	512	40.0226554870605
1K	0.001	512	44.841724395752
1K	0.01	256	45.8363571166992
1K	0.01	1024	49.3933792114258
1K	0.0001	64	51.3070755004883
1K	0.001	256	52.8623466491699
1K	0.0001	512	87.4313125610352
1K	0.0001	256	89.8093795776367
1K	0.0001	1024	90.8342361450195

D.2 Figures

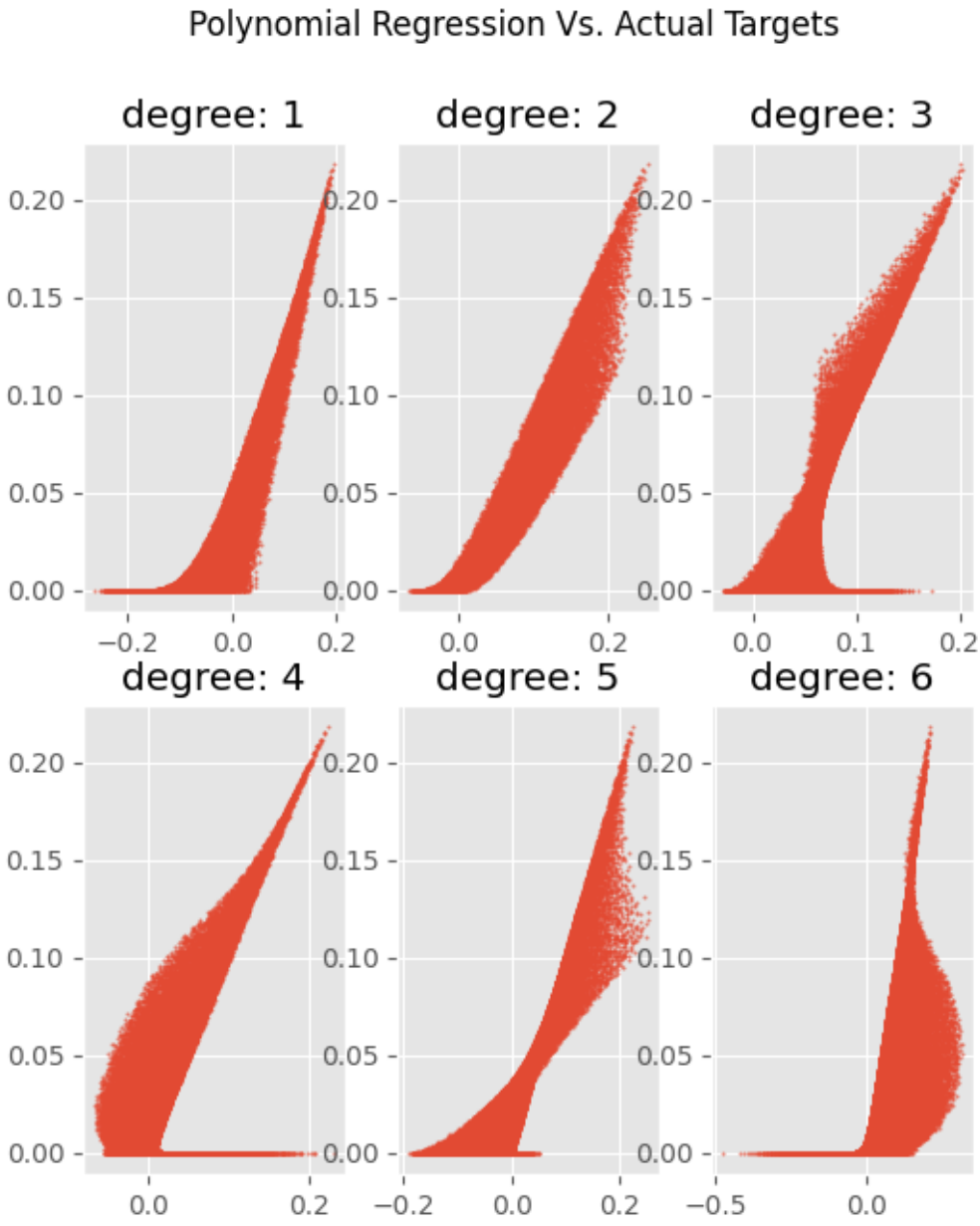

FIGURE D.1: Polynomial Regression Performance for out of money data set on the European call

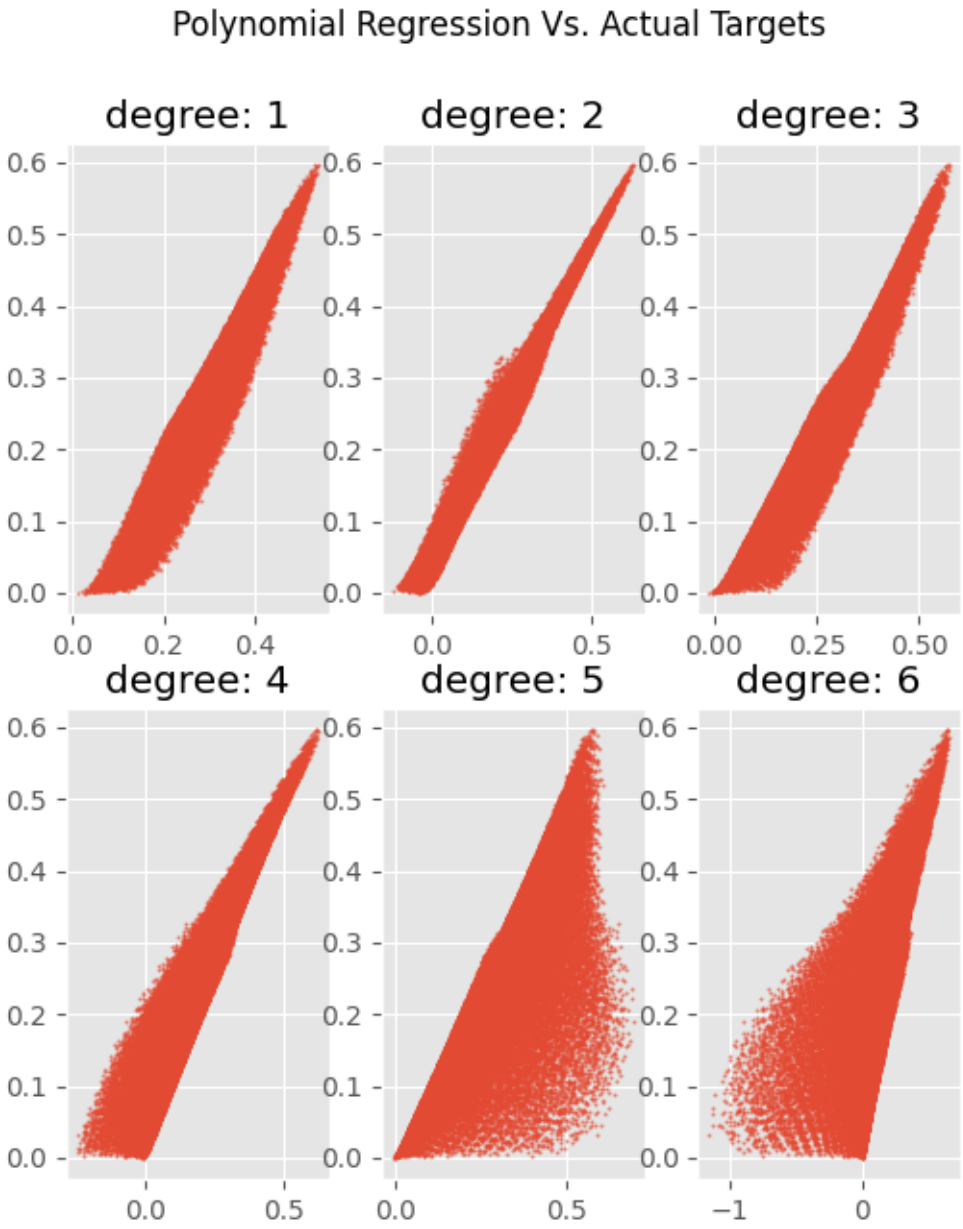

FIGURE D.2: Polynomial Regression Performance for long maturity data set on European call

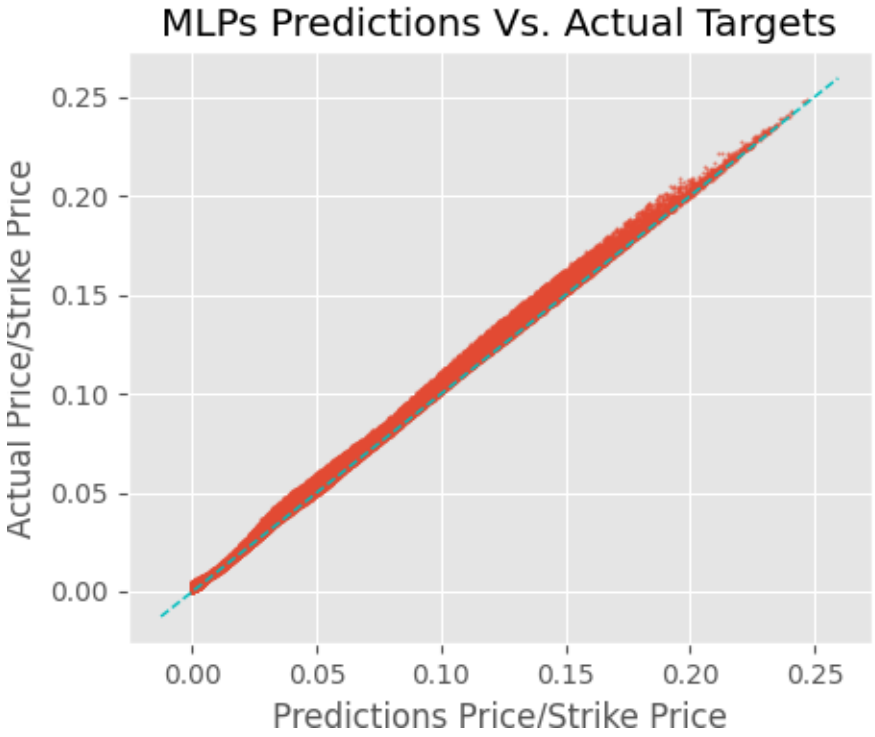

FIGURE D.3: MLPs Performance for In-the-Money data set on American Put

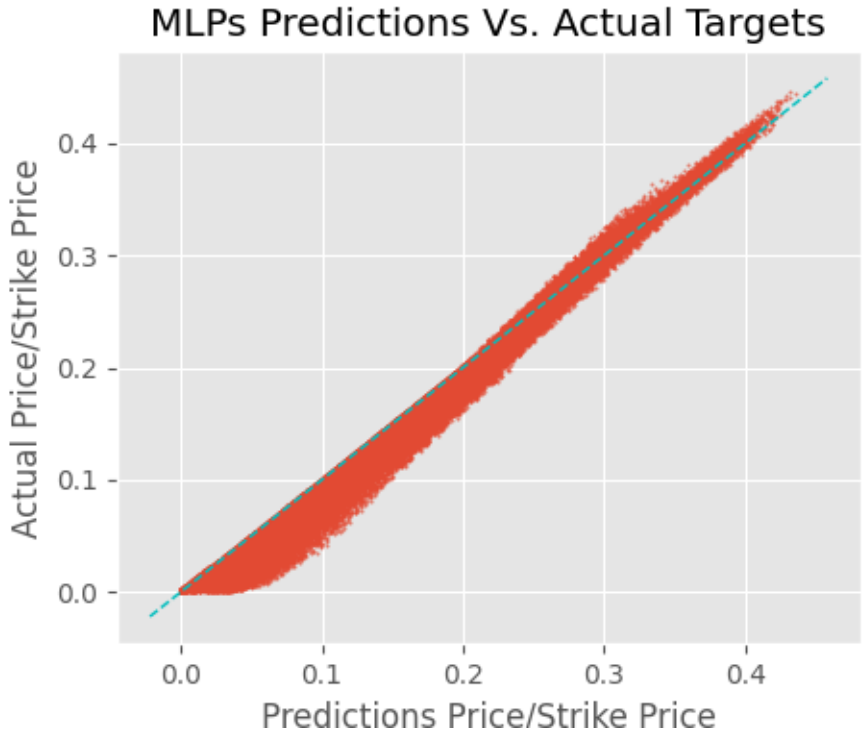

FIGURE D.4: MLPs Performance for long maturity data set on American Put

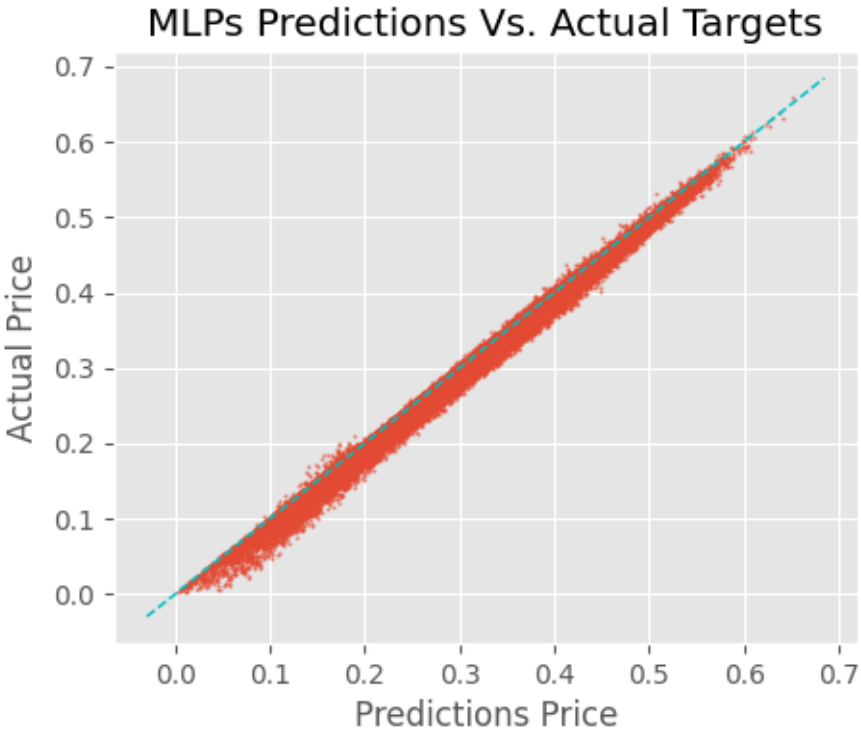

FIGURE D.5: MLPs Performance for long maturity data set on American put on minimum of two stocks

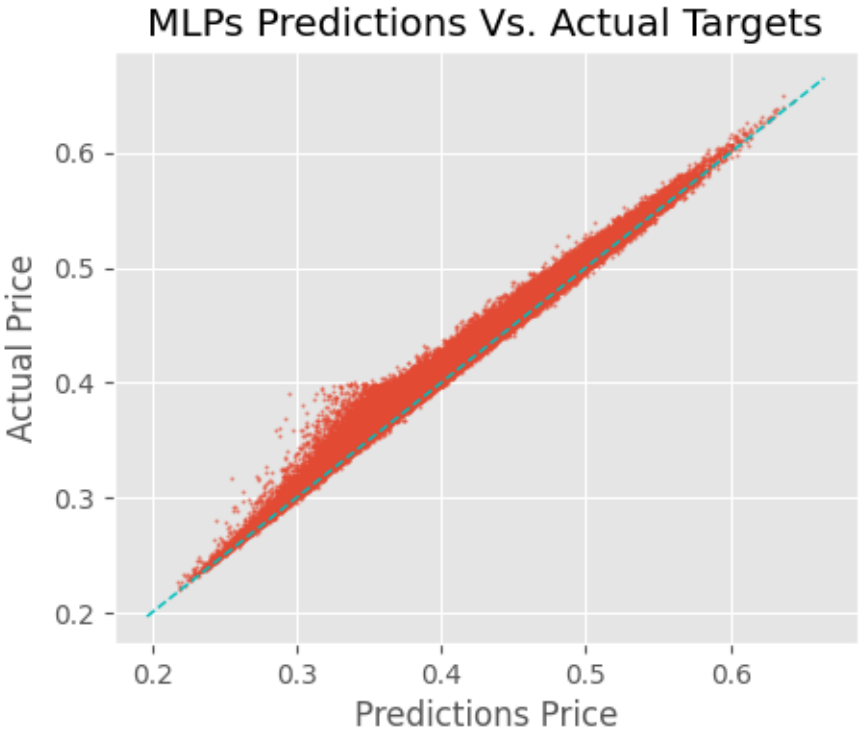

FIGURE D.6: MLPs Performance on In-the-Money data set on American put on minimum of two stocks

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