UNIVERSITY OF COPENHAGEN

MASTER THESIS

Deep hedging

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Declaration of Authorship

I, Peter Pommergård LIND, declare that this thesis titled, "Deep hedging" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

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"You were hired because you met expectations, you will be promoted if you can exceed them."

Saji Ijiyemi

UNIVERSITY OF COPENHAGEN

Abstract

Department of Mathematical Science
Science

Master Thesis in Actuarial Mathematics

Deep hedging

by Peter Pommergård LIND

The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too…

Acknowledgements

The acknowledgments and the people to thank go here, don't forget to include your project advisor…

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List of Abbreviations

S-F	**S**elf-**F**inancing
FPT1	**F**undamental **P**ricing **T**heorem I
FPT2	**F**undamental **P**ricing **T**heorem II
B-S	**B**lack-**S**choles
BM	**B**rownian **M**otion
GBM	**G**eometric **B**rownian **M**otion

List of Symbols

c	European call option price
p	European put option price
S_0	Stock price today
K	Strike price
T	Maturity date
σ	Volatility of stock price
C	American Call option price
P	American Put option price
S_T	Stock price at option maturity
r	Continuous compounding risk-free rate
$V^h(t)$	Value process
X	Simple Derivative
Φ	Contract function
W_t	Weiner process (a synonym brownian motion).

For/Dedicated to/To my…

Chapter 1

Introduction

In recent years we have seen an increasing complexity of financial products, where big investment- and banks use a lot of money on financial engineerers in creating new innovative products. With the complexity a lot of challenges has risen in this field. Nevertheless the products can help to risk neutralize your risks. A example would be credit default swap (CDS), where you insure your risk of losing money. On the other hand the CDS was one of the main reasions that AIG needed to be safed by the US government under the recent financial crisis. In hindsight they insured to many with CDS, hence AIG was too exposed when the financial crisis in 2007 hit. A great understanding in the financial derivatives is important to understand your risks. (Zucchi, 2019)

This thesis will focus on financial derivatives, and take different approaches for pricing and hedging. We will start with the most basic derivatives European options and move toward more complex products. The European option will be the reference point for our different approaches, which will ultimately lead to pricing and hedging strategies for other derivatives. European option will be the reference point, because we have an analytic formula (The Black Scholes Formula) for the price. However when moving into other derivatives as American options the Black Scholes analytical framework breaks down, and this calls for numerical methods. In this thesis we will test deep hedging and other numerical methods.

Chapter 2

Arbitrage theory in continuous time finance

2.1 Financial markets

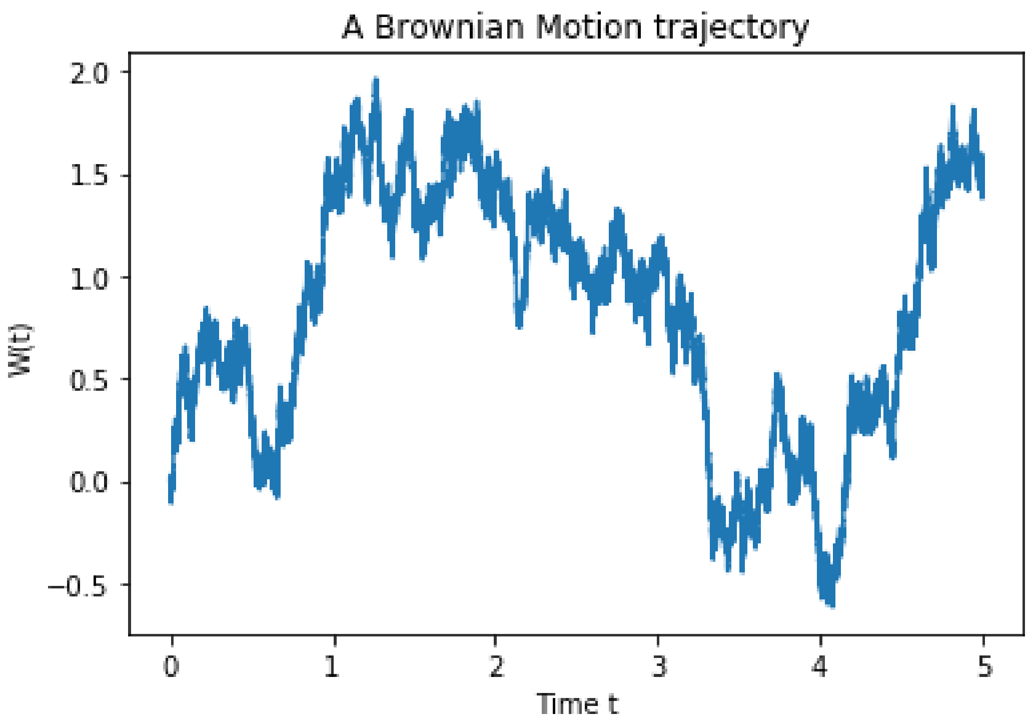

FIGURE 2.1

In the financial markets there is a lot of players and different types of investments. The classical investments are bonds and stocks, where you either lending or buying equity. The big players are banks, investment banks, insurance companies and corporations. The derivatives are depending on an underlying asset, where the dependency is specified in the contract. The options discussion in the introduction are all depending on a underlying stock. The contract can be constructed in many ways, hence it gives more options to construct your portfolio (see Appendix A for examples). When pricing financial product we use the market to price derivatives (This correspond to the equivalent martingale measure $\mathbb{Q}$ to the objective measure $\mathbb{P}$), so we do not introduce arbitrage to the market. In the classic Black Scholes formula for European options, we will assume following about the market:

Assumption 2.1.1. *We assume following institutional facts:*

- *Short positions and fractional holding are allowed*
- *There are no bid-ask spread, i.e. selling price is equal to buying price*
- *There are no transactions costs of trading.*
- *The market is completely liquid, i.e. it is possible to buy/sell unlimited quantities on the market. You can borrow unlimited amount from the bank by selling short.*

(see p. 6 (Björk, 2009))

We can discuss these assumptions at length, but in order to progress mathematically, we need to accept them for now. There is some justification for liquidity on vanilla options, because those options gets traded on large scale. Before going into the mathematics of the Black Scholes formula, we need to introduce key concepts.

2.1.1 Financial Derivatives

There a broad range of different derivatives. In this thesis, we will mainly divide derivatives into two classes.

1. Simple derivatives (T-claims)
2. Exotic derivatives

The first class is the simple derivaties or T-claims. These are simple because you can only exercise them at maturity (time T). The exotic derivatives is all kind of functions on the underlying assets, where you have more options than exercise at termination time. There are so many derivatives, hence the list will not be comprehensive at all. Some important simple derivatives will be the European calls and puts, because we can price analytically.

Definition 2.1.1. European Call Option: A Europeann call option is a option where the owner of the option has the option to exercise at maturity. The contract function for the derivative:

$$\phi(S(T)) = \max\{S(T) - K, 0\} \tag{2.1}$$

Where S(T) is the price of underlying asset at maturity and K is the agreed strike price.

For illustration of above contract see appendix A.
(Björk, 2009)

2.1.2 Self-financing portfolio (Without consumption)

A self-financing portfolio h, is a portfolio h which doesn't get any external injection of money. h is the number of each assets in our portfolio. We denote $V^h(t)$ the value of our portfolio h at time t, hence:

Definition 2.1.2. Self-financing portfolio A portfolio consisting of N+1 assets:
h(t)=$(h_0(t), h_1(t), \ldots, h_N)$ is self-financing if:

$$dV^h(t) = \sum_{i=0}^{N} h_i(t) dS_i(t) \tag{2.2}$$

Where S_i is the $i'th$ asset in our portfolio, N+1 is the total number of assets and $V^h(t) = \sum_{i=0}^{N} h_i(t)S_i(t)$

The important takeaway is that a S-F portfolio is kind of a budget restriction. You are only allowed to reallocate your assets within the portfolio but not injecting cash into the portfolio. The concept is important for the discussion of arbitrage and hedging.

2.1.3 Arbitrage

Arbitrage is the financial term for a "free lunch". An investor can profit without bearing risk, if there is arbitrage on the market. In order to avoid making a "money machine", we want to price derivatives to be arbitrage free.

Definition 2.1.3. Arbitrage: An arbitrage possibility on a financial market is a self-financed portfolio h suct that

$$\begin{aligned} V^h(0) &= 0 \\ P(V^h(T) \geq 0) &= 1 \\ P(V^h(T) > 0) &> 1 \end{aligned} \tag{2.3}$$

We say that the market is arbitrage free if there aro no arbitrage possibilities.
(see p. 96 (Björk, 2009))

From the definition a self-financing portfolio fullfilling equation (2.3) would give the possiblility for arbitrage. The investor in this portfolio starts with 0 dollars, and without injecting any money, the investor is certain of not losing any money. In addition he has a positive probability by ending up with more than 0 at maturity. Arbitrage is a way to price financial products "fair". To price "fair" and hedge againt risk will be the topics for this thesis.

2.1.4 Complete Market and Hedging

Hedging is a concepts to protect against exposure to risk. A hedge is simply a risk neutralization action in order to minimize the overall risk. In the defition below, we define a hedge for an simply T-claim (**??**).

Definition 2.1.4. Hedging and completeness for T-claim: A T-claim X can be hedged, if there exist a self-financing portfolio h s.t.:

- $V^h(T) = X$ P-a.s.

I.e. h is an hedge portfolio for X if it is guaranteed to pay in all circumstances an amount identical to the payout of X.
The market is complete, if every derivative is hedgable. (see p. 115 (Björk, 2009))

Hedging and completeness means the same for other derivatives than T-claims, but for now we will only show the concepts for the T-claim.

2.2 Black-Scholes Formula two dimensionel

In addition to our assumptions for the financial market, we also assume:

Assumption 2.2.1. *Black-Scholes assumptions We assume following ideal conditions in addition to (2.1.1):*

- *The short-term interest rate is known and is constant through time*
- *The stock price follows a Geometric Brownian Motion. The σ is constant.*
- *The stock pays no dividends or other distributions.*
- *The option is a simple option ("European" see (2.1.1)).*

(see p. 640 (Black and Scholes, 1973))

We assume the underlying stock follows a geometric brownian motion:

$$S(t) = S(0) \cdot \exp\left(\alpha - \frac{1}{2}\sigma^2)t + \sigma W(t)\right) \tag{2.4}$$

The μ and σ have clear empirical meanings.

Theorem 2.2.1. *blabla*

$$F_t(t,s) + rsF_s(t,s) + \frac{1}{2}s^2\sigma^2(t,s)F_{ss}(t,s) - rF(t,s) = 0 \tag{2.5}$$

$$F(T,s) = \Phi(s) \tag{2.6}$$

The below proposition is a consequence of the B-S equation:

Proposition 2.2.1. *Black-Scholes: The price of a European call option with strike K and maturity T is given by the formula $\Pi(t) = F(t,S(t))$, where*

$$F(t,s) = s \cdot N(d_1(t,s)) - e^{-r(T-t)} \cdot K \cdot N(d_2(t,s))$$

N is the cumulative distribution function of a standard normal distribution $\mathcal{N}(0,1)$ and

$$d_1(t,s) = \frac{1}{\sigma \cdot \sqrt{T-t}} \cdot \left(\ln(\frac{s}{K}) + (r + \frac{1}{w}\sigma^2)(T-t)\right)$$

$$d_2(t,s) = d_1(s,t) - \sigma\sqrt{T-t}$$

(see p. 105 (Björk, 2009))

The above formula for the European call option is actually the same for an American call option, but is not true for an American put option or for call options paying dividends. The result for the American call option was shown by Merton (Merton, 1973), that the intrinsic value was never greater than the worth of keeping the option given by the risk-neutral pricing formula.

(Björk, 2009)

2.3 Mathematical framework

The underlying stock is a random process, which we do not know in advance.

2.3.1 Fundamental theorems of asset pricing

Indispensable tool

2.3.2 Equivalent martingale measure

Chapter 3

Classical numerical results

3.1 Binomial Pricing model

The first numerical example will be the binomial pricing model. We will not go into mathematical details, because we already shown all the concepts and mathmatics needed for continuous time in 2. The central concepts arbitrage and completeness from continuous time also work in the discrete time setup, but the mathmatics is simpler. It can actually be shown that if you choose your parameters in this model to follow the mean and variance in the continuous time framework then the discrete time model will converge to the continuous time model. Many argue that the simpler mathematics in this model makes the binomial model more instructive and clear from an economist viewpoint. Besides being easier to understand for non-mathematician is also work nicely with other options than the European options like American options. The classical Black-Scholes model was an analytical result about European options, hence

The model gives a instructive way of thinking about arbitrage and hedging.

Firstly we choose to include the binomial pricing, because it gives a intuition and pure economic reasoning about options pricing without many technical details. Secondly it serves as a benchmark for the other algorithm's, where we can with the model both price European - and American options. This approach differs from the others, because we assume the stock price moves at discrete time instead of continuous time. This approach gives a simplified model in terms of the mathematics and highlights the essential concepts in option pricing theory arbitrage and hedging. Evenlythough our starting point is a binomial process for the development of the stock, it can be shown that in the limit the process for the stock will have a log-normal distribution. Hence the binomial pricing model will be equivalent with the continuos time analytical pricing model derived by Fischer Black and Myron Scholes in the limit (Cox and Stephen Ross, 1979) .

The classical paper (Cox and Stephen Ross, 1979) which introduce this binomial model approach to option pricing came after the Black-Scholes model described in section 2 (Black and Scholes, 1973). The

3.1.1 Arbitrage and hedging in binomial model

The important feature of the binomial model is the instructive approach to arbitrage and hedging, without repeating Chapter 2 will the focus be !!!!!!!!!!!!!!!!!!

In the binomail model the stock price can either move up or down in each discrete timepoint between !!!!!!!!!!! and maturiry. For simplicity will we consider a model with one periods after !!!initialization!!!!. The risk neutral formula is the same for the binomial model, except now the expectation is taking wrt. to the counting measure.

ss

The arbitrage price of a option will follow the martingale measure Q,

3.1.2 Numerical results

3.2 Least Square Monte Carlo Method

3.3 Comparision

Appendix A

Option contracts

This list of option contracts are far from complete, but the purpose is to illustate some payoff contracts for reference.

A.1 European Call and Put

The European options will be the most basic options, we will work with. This means not that they are not important, actually they are key for pricing options. The European call option is a contract, which pays at maturity $\Phi(S(T)) = max(S-K, 0)$.

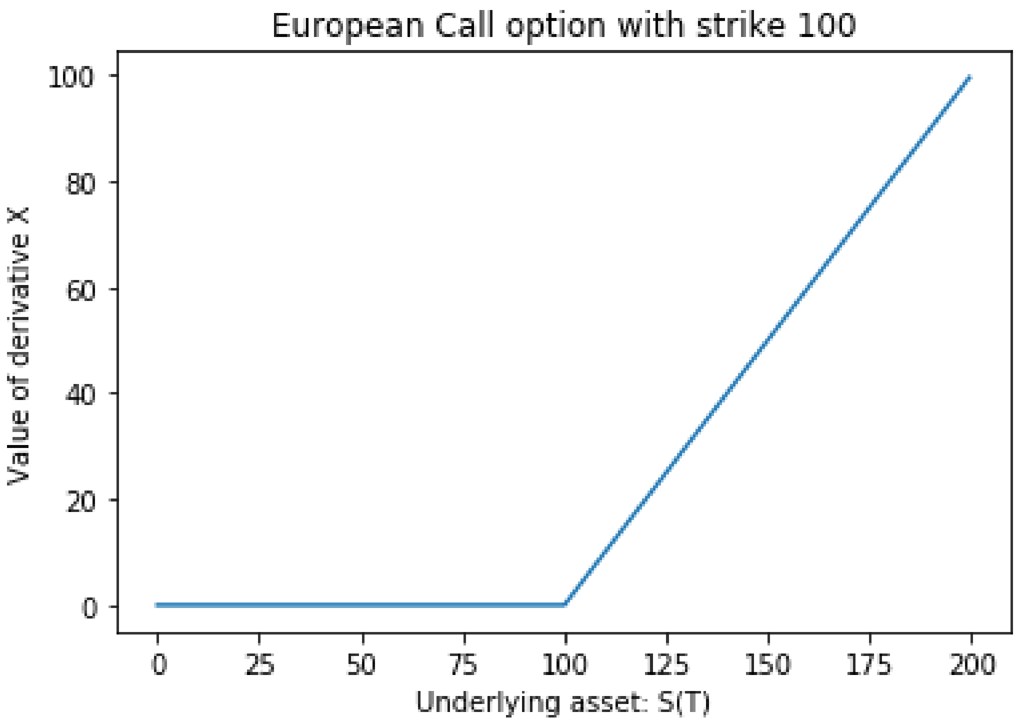

FIGURE A.1: European call with K=100

The European put is very similar to the call, except now we earn, when the stock is below the strike price K.
$\Phi(S(T)) = max(K-S, 0)$.

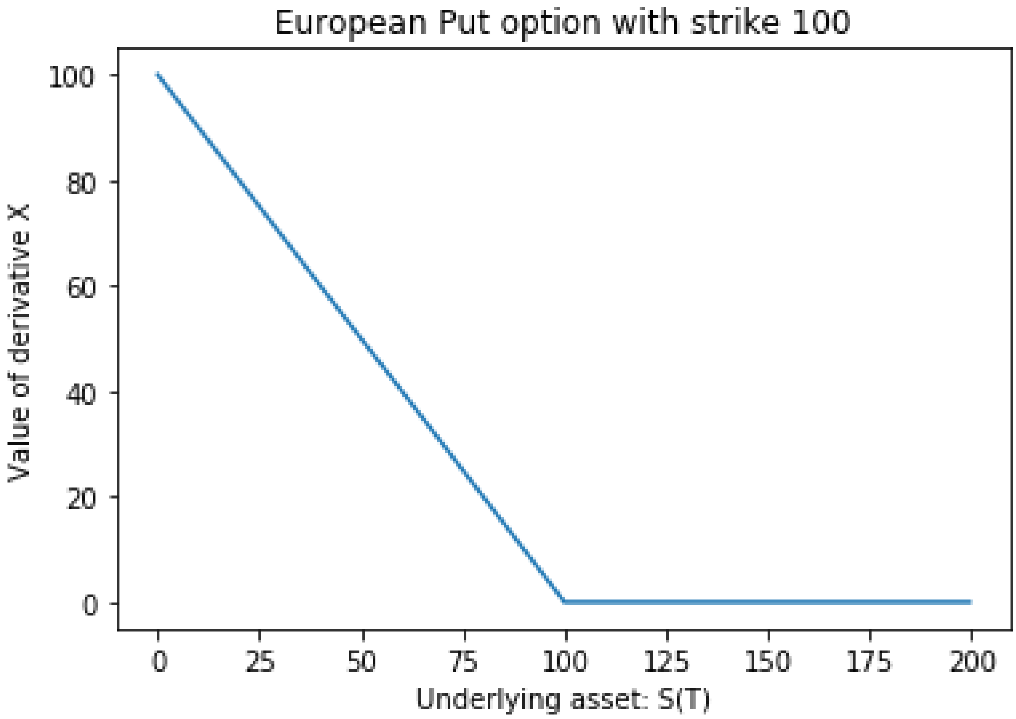

FIGURE A.2: European put with K=100

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