The Triangle Equation

Sal Hasan

28th April 2023

1 Introduction to Cutter Functions

A cutter function is an implicit function $F(x, y) = 0$ that restricts the domain of definition from $\mathbb{R}$ to a subset of it via a defined relationship. We can apply a cutter function to another function $y = g(x)$ or equation $G(x, y) = 0$ by multiplying with it. Take the following cutting function as an example:

$$C_f(x) = \sqrt{x} - \sqrt{x} + 1$$

$C_f(x)$ is only defined for $x \geq 0$. This is same as defining $C_f(x) = 1$ where $x \geq 0$. If we apply it to a parabola $f(x) = x^2$ then $f(x) = x^2 C_f(x)$ would be same as defining it as $f(x) = x^2$ where $x \geq 0$.

We can extend the function above and make it general by only allowing values to be defined along in a specific direction via a unit direction vector $\hat{V}$. We define the general equation below:

$$C_{P,\theta}(x) = C_f((y - y_P)\sin(\theta) + (x - x_P)\cos(\theta))$$

where P (a starting point) and θ (an angle) are the components of a unit direction vector of our choice.

Example:
We can restrict a parabola $f(x) = x^2$ such that $y \geq -x$ by finding a unit vector $\hat{V}$ normal to $y = -x$, this is done by finding any point $P \in y = -x$ and θ the angle of inclination of the line perpendicular to $y = -x$. We can trivially find $\theta = \frac{\pi}{4}$ and choosing P to be $(0, 0)$ we get that $f(x) = x^2 C_f(y\sin(\frac{\pi}{4}) + x\cos(\frac{\pi}{4}))$ is equivalent to $f(x) = x^2$ where $y \geq -x$.

2 The Triangle Equation

Let $A(x_A, y_A)$, $B(x_B, y_B)$, $C(x_C, y_C)$ be 3 non-collinear points. Therefore points A, B, and C determine a triangle. Let $O(x_O, y_O)$ be the incenter of $\triangle ABC$ determined by the following equations:

$$O = (\frac{x_A||BC|| + x_B||AC|| + x_C||AB||}{||BC|| + ||AC|| + ||AB||}, \frac{y_A||BC|| + y_B||AC|| + y_C||AB||}{||BC|| + ||AC|| + ||AB||})$$

where AB, BC, and AC are the sides of $\triangle ABC$ and $||S||$ to be the length of side S.

We define a function $\phi(V)$ the angle of inclination between incenter O and a vertex V as the following:

$$\phi(V) = \frac{y_O - y_V}{|y_O - y_V|} \arccos(\frac{x_O - x_V}{||OV||})$$

This is done to not use piecewise functions such as **atan2** and **sign**.
$\phi(V)$ is defined for $O \neq V$ which happens **iff** A, B, and C are collinear. From our definition $\phi(V)$ is always defined.

We lastly define a function $L_{U,V}(x, y) = 0$ to be the line equation between two points U and V as the following:

$$L_{U,V}(x, y) = (x_U - x_V)y + (y_V - y_U)x + ((x_V - x_U)y_U - (y_V - y_U)x_U)$$

We can use the fact that multiplying two implicit function $F(x, y) = 0, G(x, y) = 0$ with each other $F(x, y)G(x, y) = 0$ unions the set of points (x, y) of both functions. This allows us to superpose/combine our functions defined above.

We define $T(x, y) = 0$ to be the equation of $\triangle ABC$ as the following:

$$T(x, y) = L_{A,B}(x, y)L_{B,C}(x, y)L_{A,C}(x, y)C_{A,\phi(A)}(x)C_{B,\phi(B)}(x)C_{B,\phi(B)}(x)$$

This works by creating 3 lines between the points A, B, C and then restricting the domain to be from all vertex V towards incenter. This is beautiful because it means that we are using a larger invisible triangle to make the points outside the triangle undefined. (Note: The invisible triangle is the one formed by the 3 tangents to the circumscribed circle of $\triangle ABC$ at the each respective vertex.)

3 Afterword

This paper was made as a response to Matt Parker's YouTube Video about there being no triangle equation. I am not a mathematics major student so I don't know how to write a proper mathematical paper or how to be rigorous in my definitions and explanations. I am just a math nerd who didn't like that Matt used **sign** function in the equation.