Consider a 2D one-link pendulum system with no friction, attached to a ceiling and subject to the gravitational force. The Newton equations of motion balance the net force on the pendulum with the change in its momentum:

$$F + F_g = m\ddot{p},$$

where F is the constraint force of the hinge on the ceiling and F_g is the force of gravity. As usual, m is the mass of the pendulum and p is the 2D position of its centre of mass. We can rewrite these as:

$$m\ddot{p} - F = mg, \tag{1}$$

where $F_g = mg$, and g is the gravitational acceleration constant. Here, the right-hand-side corresponds to known external forces applied to the system, while the left-hand-side collects the unknowns.

Let θ represent the tilt of the pendulum where $\theta = 0$ corresponds to the stable equilibrium configuration. Then let $\omega = \dot{\theta}$ the the angular velocity of the pendulum about its centre of mass. We can now write the angular equations of motions relating the relevant rotational quantities as follows:

$$\begin{aligned} \tau &= \frac{DL}{Dt} = \dot{L} \\ r_x F_y - r_y F_x &= I\dot{\omega}, \end{aligned}$$

where I is the moment of inertia making up angular momentum $L = I\omega$. In 2D, the torque, τ, generated by the constraint is expressed in terms of the constraint force as $r_x F_y - r_y F_x$, where $r = (r_x, r_y)$ is the vector from the centre of the pendulum to the point of contact at the hinge, and $F = (F_x, F_y)$. As before, collecting unknowns on the left-hand-side gives us

$$I\dot{\omega} - r_x F_y + r_y F_x = 0. \tag{*}$$

For brevity, we denote $\tilde{r} = (r_y, -r_x)$, simplifying (*) to

$$I\dot{\omega} + \tilde{r} \cdot F = 0. \tag{2}$$

Finally, the hinge constraint ensures that the acceleration of the pendulum at the hinge, $\ddot{p}_r$, is zero:

$$0 = \ddot{p}_r = \ddot{p} - \dot{\omega}\tilde{r} - \omega^2 r.$$

Rearranging terms as before gives us:

$$-\ddot{p} + \dot{\omega}\tilde{r} = -\omega^2 r. \tag{3}$$

We model the pendulum with a rectangular block, which has inertia $I = \frac{m}{12}(w^2 + h^2)$ about its centre of mass, where w and h are its width and height respectively.

Putting equations (1), (2) and (3) together, we get the following system:

$$\begin{pmatrix} m\mathbf{I} & 0 & -\mathbf{I} \\ 0 & I & \tilde{r} \\ -\mathbf{I} & \tilde{r}^\top & 0 \end{pmatrix} \begin{pmatrix} \ddot{p}^\top \\ \dot{\omega} \\ F^\top \end{pmatrix} = \begin{pmatrix} mg^\top \\ 0 \\ -\omega^2 r^\top \end{pmatrix},$$

where $\mathbf{I}$ is the 2x2 identity matrix, and the superscript $^\top$ converts a vector into a column vector. Solving for $(\ddot{p}, \dot{\omega}, F)$ gives us the linear and angular accelerations, which we can use to update the pendulum configuration by one time step.