On Job-Insertion for the Blocking-Job-Shop and its Application to the SBB Challenge

Master thesis of

Fabio Degiacomi

Supervised by

Dr. Reinhard Bürgy & Prof. Bernhard Ries

Decision Support & Operations Research Group
University of Fribourg

July 2020

Abstract

The SBB challenge is a timetable generation problem, which can be interpreted as a blocking-job-shop scheduling problem. In the blocking-job-shop, unlike the classical job-shop, the N_1 neighbour of a given solution is generally not feasible. Gröflin, Klinkert & Bürgy developed a method, conceptually based on the insertion of a single job (a SBB train) into a schedule, which recovers the feasibility of N_1 neighbours. In the present master thesis, multiple ways to improve the computational efficiency of that method are developed. The most remarkable improvement relies on the insight that a sequence of graph searches follow a 'nested' structure and can all be combined. The method is adopted to the additional constraints of the SBB challenge.

Contents

Cheatsheet

A	set of conjunctive arcs of a disjunctive graph, (V, A) acyclic
A^J	set of conjunctive arcs of the job-insertion graph G^J (inserting J)
block	'blocking' process characteristic of a job-shop problem
$C_{\max}$	makespan objective, time span 0 until all operations terminated, $C_{\max} = l^S(\sigma, \tau)$
d_{Jk}	due time of operation o_{Jk}
$\delta(.)$	function mapping an set of vertices to all arcs adjacent to any vertex of the set
E	set of disjunctive arcs of a disjunctive graph, $E = \bigcup_{\{e,\bar{e}\}\in\mathcal{E}} \{e, \bar{e}\}$
E^J	set of disjunctive arcs of the job-insertion graph G^J (inserting J)
$\mathcal{E}$	set of pairs of disjunctive arcs
$\mathcal{E}^J$	set of pairs of disjunctive arcs of the job-insertion graph G^J
e	a disjunctive arc
$\bar{e}$	mate of e, ie $\{e, \bar{e}\} \in \mathcal{E}$, $(V, A \cup \{e, \bar{e}\})$ cyclic
f	another disjunctive arc
$\bar{f}$	mate of f
G	a disjunctive graph (directed), $G = (V, A, E, \mathcal{E}, l)$
G^J	job-insertion graph (inserting J), a disjunctive graph, $G^J = (V, A^J, E^J, \mathcal{E}^J, l)$
H_G	conflict hypergraph of a disjunctive graph G
H_{G^J}	conflict graph of a job-insertion graph G^J, short-cycle property, bipartite
$h(.)$	function mapping an arc to its head vertex
$\mathcal{J}$	set of jobs of a (blocking-)job-shop problem
J	a job of a (blocking-)job-shop problem, a set of operations/vertices
J_m	job-shop problem with m machines
$l(.,.)$	function mapping two vertices to the length of an arc between them
$l(.)$	function mapping an arc to its length
$l^X(.,.)$	function mapping two vertices to the length of a longest path between them in $(V, A \cup X)$
$\mathcal{M}$	set of machines of a (blocking-)job-shop problem
m	a machine
n_J	number of operations of job J
$\mathcal{O}$	set of operations of a (blocking-)job-shop problem $\cong$ vertices of disjunctive graph
o_{Jk}	k^{th} operation of job $J \cong$ a vertex in the disjunctive graph
p_o	processing time of operation o
pm	'parallel machines,' a single operation may use multiple machines
prec	'precedence constraints' between operations of different jobs
r_{Jk}	release time of operation o_{Jk}
rcrc	'recirculation,' a jobs uses a machine for multiple operations
route	'routing options', for each job a route has to be selected
S	a selection, $S \subset E$, feasible if $(V, A \cup S)$ acyclic, complete if $\lvert S\rvert = \lvert E\rvert/2$ else partial
s_i	setup time of machine i
$succ(.)$	function mapping an operation to its successor operation
σ	dummy 'start' vertex, $t_\sigma = 0$, indegree always 0
T_{Jk}	tardiness (delay) of operation o_{Jk}
$T_{\max}$	maximum tardiness (delay)
$t(.)$	function mapping an arc to its tail vertex
t_o	entry time of operation o
τ	dummy 'end' vertex, $t_\tau = C_{\max}$, outdegree always 0
U	set of edges of a conflict graph $\cong$ set of cycles in a disjunctive graph
V	set of vertices of a disjunctive graph, $V = \mathcal{O} \cup \{\ \sigma, \tau\ \}$
w_{Jk}	weight/priority of the operation o_{Jk}
Z, Z'	cycles in a graph
$\preceq$	$a \preceq b$, if a path from a to b exists in (V, A) $\cong$ operation a precedes b of the same job

1 Background

1.1 SBB Challenge

In 2018 the Swiss Federal Railways (de: Schweizerische Bundesbahnen (SBB), fr: Chemins de fer fédéraux suisse (CFF); subsequently SBB) organized a competition, where a contestant has to solve a series of timetable generation problems [28]. Such a problem defines a set of *trains*. For each train a *route* has to be chosen. Each route consists of multiple *sections* which the train has to traverse. Each section has a *run time*, that is, the minimum time it takes the train to traverse said section. Sections may also have requirements attached: latest and/or earliest entry times. Sections have *resources* associated. The time table generation problem is then: associate each train with a route; associate each section (within a route chosen) with an entry time; such that, sections, which share a resource, are not traversed by their respective trains simultaneously. Earliest entry times have to be respected. The delay (assigning entry times later than the 'latest entry') has to be minimized. Finally, *connections*: we have to guarantee that passengers can change from some trains to others at given sections.

The core of this problem is called *blocking-job-shop problem*, which will be introduced shortly. The blocking-job-shop problem is well known to be NP-hard [21]. Given the scale of the problem that researchers at SBB want to solve, it is unlikely that we can prove the optimality of solutions within reasonable time. Hence, the hope of the SBB challenge was that someone would come up with an ingenious heuristic that could be developed further.

Academic research, while traditionally focusing on the simpler *job-shop problem*, made progress in the last 20 years on the *blocking-job-shop problem*. However, there are significant differences between the problems considered in academic literature and the SBB challenge: foremost the scale of problems studied. The problem size can be characterized by the number of *jobs* (or *trains*) n and the number of *machines* (rail segments) m. The blocking-job-shop research community has focused on a series of randomly generated problem instances, which range in size ($n \times m$) from 10×5 to 30×10. The problem instances provided by the SBB are as large as 350×450. While it is the case that the blocking-job-shop problem is very much relevant in practical applications–in contrast to the job-shop problem–it is also true, that the scale most often studied academically is too small for practical relevance.

One of the most promising approaches to solve the blocking-job-shop problem, was developed over the last years at the University of Fribourg by Heinz Gröflin, Andreas Klinkert and Reinhard Bürgy [11, 13, 23, 24]. The SBB challenge provided us with a larger example problem for which I could adopt this existing algorithm. Hence, we considered it worthwhile to work on the SBB challenge, even though the competition had formally concluded by the time it caught our attention.

1.2 Real-time Train Dispatching

The timetable generation problem, as established by the SBB challenge, is often referred to as *real-time train dispatching* in the literature. It is *not* the problem of creating an official time table (off-line), but rather, as deviations to this official timetable inevitably occur, rescheduling and rerouting trains in real time (on-line) to minimize overall delay.

In practice this problem is broken down into sub-problems, namely *conflict detection and resolution* and *speed profile generation*[12]. It is also common to solve these problems for specific areas and then integrate these partial solutions (see [12] for an overview of the employed system architectures, or [19] for a solution approach which depends on a decomposition of the problem into geographical sub units: line and station problems).

The speed profile generation problem has its own intricacies, as when we are concerned with the problem on such a practical level, we have to take heterogeneous safety and operational rules, different signaling systems, as well as conflicting objectives such as punctuality, minimization of travel time and energy efficiency into account [12]. However, as the SBB challenge's application programming interface (API) exposes essentially the real-time train dispatching, conflict detection and resolution problem, we shall focus on this aspect alone. It should be noted that completely different approaches to the conflict detection and resolution problem exist. For an overview consult [18].

In the next Sections, we will first define the blocking-job-shop scheduling problem, which is an abstraction and simplification. After the Section on disjunctive graphs–which are a common model of the blocking-job-shop–I present a short literature overview of different approaches to the blocking-job-shop problem. Most of them depend on disjunctive graphs. Only then shall we come back to real-time train dispatching and the SBB challenge, by connecting the blocking-job-shop with the SBB challenge, thereby concluding the 'background' Chapter.

1.3 (Blocking-)Job-Shop Problem

The *job-shop problem* and its variations aim to formalize *scheduling* problems. In the classical job-shop problem, a set of *jobs* $\mathcal{J} = \{J_1, \dots J_n\}$ is given. Each job consists of a sequence of operations, i.e. $J = (o_{J1}, \dots, o_{Jn_J})$, where n_J denotes the number of operations of job J. The jobs can be thought of as tasks that need to be completed, where each task decomposes into subtasks, the *operations*, which need to be done *in the given order*. Each operation occupies for its processing a *machine*, that is, there is a function $M : \mathcal{O} \to \mathcal{M}$ associating to each operation a machine. Here, $\mathcal{O}$ denotes the set of all operations and $\mathcal{M} = \{M_1, \dots, M_m\}$ is the set of all machines. Note that each operation is part of exactly one job and the operations of a job are distinct. The machines model resources that need to be acquired to complete the operations/subtasks, and let us encode constraints such as 'o_1 and o_2 cannot be processed simultaneously'. Going forward, notation will be abused slightly. Vectors use as index equivalently: (1) an operation or (2) a job and an operation index.

Each operation o has a *processing time* $p_o \in \mathbb{R}_{\geq 0}$ (or $\in \mathbb{Q}_{\geq 0}$, or $\in \mathbb{N}_0$) associated. The goal is to set the start time of each operation, that is, to define a vector $t \in \mathbb{R}_{\geq 0}^{|\mathcal{O}|}$. Commonly, we associate a job with its completion time, $C : \mathcal{J} \to \mathbb{R}_{\geq 0}, J \mapsto t_{Jn_J} + p_{Jn_J}$. The duration from 0 to $C_{\max} := \max_{J \in \mathcal{J}}\{C(J)\}$ is called *makespan*. The minimization of the makespan is an objective common to job-shop problems. A job-shop problem can easily be transformed into a linear disjunctive program:

$$
\begin{aligned}
&\min_t \max_{J \in \mathcal{J}} \{t_{Jn_J} + p_{Jn_J}\} && \\
0 &\leq t_o && \forall o \in \mathcal{O} \\
t_{Jk} + p_{Jk} &\leq t_{Jl} && \forall J \in \mathcal{J}\ \forall k\ \forall l \mid 1 \leq k < l \leq n_J \\
t_{o_1} + p_{o_1} \leq t_{o_2} &\vee t_{o_2} + p_{o_2} \leq t_{o_1} && \forall \{(o_1, o_2) \mid o_1 \in \mathcal{O}, o_2 \in \mathcal{O}, o_1 \neq o_2, M(o_1) = M(o_2)\}
\end{aligned}
$$

Variations of the job-shop problem have been classified. A specific problem is associated with a triple $\alpha \mid \beta \mid \gamma$, where α stands for the broad problem type, β is a list of process characteristics and γ defines the objective function. $J_m \mid\mid C_{max}$ is the job-shop problem introduced above. The notation introduced here is based on the textbook 'Scheduling' by Micheal L. Pinedo [21]. Elements are selected due to the relevance to the present thesis and not their importance in scheduling theory. For in-depth definitions the reader is referred to [21] page 13 et seq.

Problem types (α):

1 A scheduling problem with a single machine.

$\mathbf{J_m}$ The job-shop problem introduced above, with m machines.

Process characteristics (β):

$\mathbf{r_{Jk}}$ Release date: a job J's operation o_{Jk} cannot start execution before the time r_{Jk}.

$\mathbf{d_{Jk}}$ Due date: if job J's operation o_{Jk} completes processing after d_{Jk}, a penalty is applied.

prec Precedence constraints: an operation of a job may require operations of other jobs to be completed before being able to start.

prmp Preemptions: processing of an operation may be interrupted. The machine is freed. When the job in question resumes the operation, it has to be processed for the remaining time span.

$\mathbf{s_i}$ Machine-dependant setup times: the machine i has to be idle for the time span s_i between operations.

pm Parallel machines usage: an operation may require multiple machines.

route For each job a *route* has to be selected. Equivalently, $\mathcal{J}$ can be seen as a *set of sets* of jobs. For each $J \in \mathcal{J}$, one $j \in J$ has to be selected and the scheduling problem is solved for only those selected jobs.

block Blocking models the absence of storage: a job needs to reside on a machine at all times. As a result, a job can only take over a machine after its occupant moves to the next machine. In the blocking environment, the last family of constraints of the disjunctive program above has to be modified:

$$t_{succ(o_1)} \leq t_{o_2} \vee t_{succ(o_2)} \leq t_{o_1} \quad \forall \{(o_1, o_2) \mid o_1 \in \mathcal{O}, o_2 \in \mathcal{O}, o_1 \neq o_2, M(o_1) = M(o_2)\},$$

where $succ(o)$ denotes the successor operation of o. Two remarks are in order. Evidently the disjunctive formulation above does not work in case of the last operation. The academic literature therefore distinguishes between 'blocking' and 'ideal' (i.e. last) operations. However, as this is not a problem in practice–we can simply append a dummy operation of zero processing time, which uses no machine–I will not discuss this further. The second question, which arises, asks if we are allowed to swap two jobs J_1, J_2 if J_1 uses first machine m_1 then m_2, while J_2 uses first m_2 then m_1, after both completed processing on their respective first machine. Two variants arise: blocking *with swap* and *without swap*. But the distinction vanishes if $s_i > 0$ for all $i \in \mathcal{M}$, which is the case in the SBB challenge. I will henceforth assume 'without swap' which makes the implementation slightly easier.

rcrc Recirculation: a job may need processing by the same machine multiple times.

Objective functions (γ):

$\mathbf{C}_{\max}$ *Makespan* objective, as defined above.

$\mathbf{T}_{\max}$ We define the *tardiness* of operation o_{Jk} to be $T_{Jk} := \max\{t_{Jk} - d_{Jk}, 0\}$. The objective T_{max} is to minimize the maximum tardiness.

$(\mathbf{w_{Jk}T_{Jk}})_{\max}$ Weighted maximum tardiness. As above, with operations also having priorities.

$\sum \mathbf{w_{Jk}T_{Jk}}$ Sum of weighted tardiness.

Just as some readers doubtlessly suspect, the train scheduling problem (as defined by the SBB datasets) are of the structure $J_m \mid r_{Jk}$, d_{Jk}, prec, s_i, block, rcrc, pm, route $\mid \sum w_{Jk}T_{Jk}$. $1 \in \alpha$ and prmp $\in \beta$ are introduced here, as heuristics that we will encounter in the blocking-job-shop literature review depend on such sub-problems.

1.4 Disjunctive Graph

The aim of this Section is the introduction of a common model of the (blocking-)job-shop problem, the *disjunctive graph*. The problems we consider here are $J_m \mid\mid \gamma$ and $J_m \mid \text{block} \mid \gamma$. The disjunctive graph is a directed graph where each operation is associated with a vertex. A job J then corresponds to a subset of the vertices. Connecting the vertices, arcs are labelled with a duration ($\in \mathbb{R}_{\geq 0}$). Arcs indicate that the operation at the head of the arc cannot commence before the time when the operation at the tail started plus the duration of the arc's label. Arcs encode the idea that an operation has to precede another operation by a given time. Such graphs are also known as *precedence constraint graphs*. A given solution to a blocking-job-shop problem can be naturally represented by a precedence constraint graph. Now, we aim to represent the solution space of a (blocking-)job-shop problem by the specialized type of precedence constraint graph which is the disjunctive graph. In the solution space of the (blocking-)job-shop we encounter two types of arcs:

(i) *Conjunctive* arcs occur in every solution. These are the arcs within the same job and encode that operation o_{Jk} has to precede o_{Jk+1}.

(ii) *Disjunctive* arcs: For every two operations o_1, o_2 that use the same machine m we have a pair of disjunctive arcs. They encode: 'o_1 is scheduled on m before o_2' and 'o_2 is scheduled on m before o_1' respectively.

The disjunctive arcs occur in pairs, o_1 before o_2 *or* o_2 before o_1. A solution must contain one disjunctive arc for every pair, as we must make a choice which operation to schedule first. I offer some remarks before introducing a formal definition.

- In $J_m \mid\mid \gamma$, operations o_1, o_2 sharing m lead to the disjunctive arc pair

$$((o_1, o_2), (o_2, o_1)).$$

The labels of these arcs are the processing times of o_1 and o_2 respectively.

- In $J_m \mid \text{block} \mid \gamma$, operations o_1, o_2 sharing m lead to the disjunctive arc pair

$$((succ(o_1), o_2)), (succ(o_2), o_1)).$$

  The labels of these arcs are 0.

- Choosing between disjunctive arcs is called the *sequencing problem*. Once the sequencing problem is solved (one disjunctive arc for every pair is selected), we have to assign entry times for all operations, the *timing problem*. A side note: a *scheduling problem* is a sequencing problem followed by a timing problem. In case of the objective functions discussed here, which are *regular objectives*[23], the timing problem is simple, as an optimal solution to the timing problem is obtained by assigning earliest possible entry times for all operations. We obtain the *earliest-starting-time schedule*.

  The start time of an operation o must be set according to the start times of all operations that precede it directly, i.e. the operations o_i where an arc (o_i, o) exists. The labels of these arcs have to be taken into account. All this has to be computed according to the topological order of the operations. This is described for example in the influential work of Mc Cormick et al. (1989) [4].

- The remark above already hints at the condition, when a precedence constraint graph corresponds to feasible solutions: If the graph contains a cycle (of positive length), we would have to conclude that all operations within the cycle could never be started. No feasible solutions exist. The reverse is also true. Choosing disjunctive arcs such that the graph remains positive acyclic corresponds to feasible solutions (among them, the earliest-starting-time schedule). Subsequently, I will disregard non-earliest-starting-time schedules: this simplifies the reasoning, as we can assume a one-to-one relation between feasible schedules and acyclic graphs.

- Given a precedence constraint graph corresponding to a feasible solution, the *makespan* objective is the length of a longest path in the graph.

- The constraints of the type $0 \leq t_o \; \forall o \in \mathcal{O}$ pose no difficulty. We add an additional vertex called σ to the graph. From σ we add arcs of length 0 to the first operations of all jobs. The vertex σ is particular in the sense that its associated time t_σ is constant 0. σ has no effect on the graph being cyclic or acyclic as its indegree is always 0.

  Analogously, we could add an 'end' vertex τ, and arcs $\{(o_{Jn_J}, \tau) \mid J \in \mathcal{J}\}$. Then, $C_{\max} = t_\tau$.

Definition 1.1 (Disjunctive graph)**.** A tuple $(V, A, E, \mathcal{E}, l)$ defines a disjunctive graph, if

(i) $(V, A \cup E)$ is a directed graph.

(ii) (V, A) is an acyclic directed graph.

(iii) $\mathcal{E}$ is a set of unordered pairs of arcs; E exactly contains all arcs of $\mathcal{E}$, i.e.:

$$\forall \{e, \bar{e}\} \in \mathcal{E} \; \big(e \in E \wedge \bar{e} \in E\big).$$

  Furthermore, adding both arcs of a pair to (V, A) yields a cyclic graph. i.e.:

$$\forall \{e, \bar{e}\} \in \mathcal{E} \; \big(V, A \cup \{e, \bar{e}\}\big) \text{ is cyclic.}$$

  For $\{e, \bar{e}\} \in \mathcal{E}$, we call $\bar{e}$ the mate of e and vice versa.

(iv) $l : (A \cup E) \to \mathbb{R}_{>0}$, defines the length of an arc.

Note that it is common to allow zero-length or negative length arcs, which I do not include in this definition, as no such arcs occur in the SBB-problem. It does simplify the discussion and implementation. Under this restriction, positive-length cyclic (positive-length acyclic) is equivalent to cyclic (acyclic).

Remark 1.1. *When a disjunctive graph is used to model a (blocking-)job-shop problem: As operations correspond to vertices, I will abuse the notation and conflate operations and vertices: operation $o \in V$. As jobs are sets of operations, for a job J, $J \subset V$. Furthermore, each disjunctive arc $e \in E$ is associated with a machine m. We denote with E_m all the disjunctive arcs which model the sequencing on machine m. Then, $\bigcup_{m \in \mathcal{M}} E_m = E$.*

Directly related to the disjunctive graph is the concept of *selections*.

Definition 1.2 (Selection)**.** S is a selection in a disjunctive graph $G = (V, A, E, \mathcal{E}, l)$ if $S \subset E$ and $\forall\{e, \bar{e}\} \in \mathcal{E}\ \neg\Big(e \in S \wedge \bar{e} \in S\Big)$, that is S contains at most one element of every pair of $\mathcal{E}$. We call S complete (else partial) if S contains one element of every pair of $\mathcal{E}$, ie. $\forall\{e, \bar{e}\} \in \mathcal{E}\ \Big(e \in S \vee \bar{e} \in S\Big)$, equivalently, $|S| = |E|/2$. S is called feasible if $(V, A \cup S)$ is acyclic. A complete feasible selection corresponds to a feasible solution (earliest-entry-time schedule) of the (blocking-)job-shop problem associated with the disjunctive graph.

We conclude this Section with the definition of a critical subgraph.

Definition 1.3 (Critical subgraph)**.** For a given selection $S \subset E$, in a disjunctive graph $G = (V, A, E, \mathcal{E}, l)$ we define a subgraph of $(V, A \cup S)$. This subgraph depends on the objective function of the (blocking-)job-shop which we model with the disjunctive graph:

$\mathbf{T}_{\max}$ Let o be an operation/vertex in $(V, A \cup S)$ where maximum tardiness occurs. A longest path to o is called *critical path*. All disjunctive arcs along a critical path are called *critical arcs*. Note that, $T_{\max}$ is strictly more general than $C_{\max}$.

$\sum \mathbf{w_{Jk}T_{Jk}}$ Let $\mathcal{O}_{\text{cost}}$ be the set of all operation/vertices o where $T_o > 0$. The subgraph induced by all vertices along a longest path to any vertex in $\mathcal{O}_{\text{cost}}$ is called *critical subgraph*. A disjunctive arc within the critical subgraph is called *critical arc*.

Remark 1.2. *Note, that the critical arcs defined by a selection S exactly correspond to the choices leading to the cost of the solution corresponding to S. Hence, any change of the solution aiming to reduce the objective value, needs to modify choices represented by critical arcs.*

1.5 Literature Review: Blocking-Job-Shop

In the present review I am considering the development of different algorithms to tackle the blocking-job-shop problem that were developed over the last two decades. In effect, this complements older reviews such as [6]. To assign credit is a notoriously perilous task. In the present review I do not aim to do so, but rather, point out developments that are in *my opinion* important and hope this Section has an educational value. I proceed by introducing a construction heuristic that gave rise to multiple neighbourhood search schemes, called 'Amcc'. This is followed by the category branch & bound based attempts to solve the blocking-job-shop problem. The Section is concluded by Xie & Mati's method which takes a different approach (Adopting the Graphical Method).

Amcc Algorithm

A. Mascis & D. Pacciarelli adopted the disjunctive graph to the blocking-job-shop problem [7, 8], renaming the concept as 'alternative graph'. This, they motivate: a partial feasible selection of the (classical) job-shop problem can always be feasibly extended; a partial feasible selection of the *blocking*-job-shop problem cannot necessarily be feasibly extended [7, 8]. Here, we continue to use the term *disjunctive graph*.

Their work had a large impact on subsequent research, due to the proposition of a simple blocking-job-shop solution construction heuristic, i.e., a method, that constructs a complete feasible selection. This construction heuristic–which is outlined below–depends on another heuristic to select a pair of disjunctive arcs, of which we did not yet include either in the solution. Of the four variants proposed, *Avoid maximum current $C_{\max}$ (Amcc)* had a lasting impact on research. 'Amcc' was subsequently used for both the arc pair selection heuristic and the solution construction heuristic using said selection heuristic. For clarity I refer to them as the Amcc *rule* and Amcc *algorithm* respectively.

In the pseudo code, see Algorithm 1, the function $l^X : V^2 \to \mathbb{R}_{\geq 0}$ returns the length of the longest path in $(V, A \cup X)$ between the two input vertices, for any set of arcs X. The algorithm chooses the pairs of $\mathcal{E}$ one-by-one, each time choosing one arc, which is inserted into the partial selection S. The pair chosen, contains the 'worst' arc, the arc which leads to the highest increase in cost. First, it is attempted to insert the other arc. If this fails, the arc which leads to the highest increase in cost is added. This may fail as well, in which case the algorithm failed. Clearly, this is a weakness of the algorithm. This potential to fail is due to the fact that not every partial feasible selection can be extended (mentioned above, unlike in the classical job-shop problem).

The authors try to mitigate this with the inclusion of *static implications*: an arc (i, j) is said to statically imply (u, v), if (with very limited on-line computational effort) it can be shown that (i, j) is incompatible with the mate of (u, v), given the partial selection S. Then, when adding (i, j) to S, also all static implications of (i, j) are added to S and removed from X.

Some examples of static implication rules include [8, 10]:

- If two jobs use the same sequence of machines, scheduling one on any of the machines implies the same choice, which one goes first, on the other machines of the sequence (Only applies to the *blocking*-job-shop problem).
- A redundant arc is statically implied. (i, j) is redundant if $l(i, j) \leq l^S(i, j)$.
- If we work with an upper bound, the mate of an arc that pushes the cost above the upper bound is implied.
- The mate directly results in a cycle.

Algorithm 1: Amcc algorithm

Input : A disjunctive graph $G = (V, A, E, \mathcal{E}, l)$.
Output: A complete feasible selection S or a failure.

Initialize $S = \emptyset$
Initialize a set $X = E$
while $X \neq \emptyset$ **do**
  // the AMCC rule:
  $(h, k) = \arg\max_{(u,v)\in X}\{l^S(\sigma, u) + l(u, v) + l^S(u, \tau)\}$
  let (i, j) be the mate of (h, k)
  if $(V, A \cup S \cup \{(i, j)\}$ *is acyclic* **then**
    $S = S \cup \{(i, j)\}$
    $X = X \setminus \{(i, j), (h, k)\}$
  else if $(V, A \cup S \cup \{(h, k)\}$ *is acyclic* **then**
    $S = S \cup \{(h, k)\}$
    $X = X \setminus \{(i, j), (h, k)\}$
  else
    return *failure*
  end
end
return S

Pranzo & Pacciarelli [22] then developed a neighbourhood search based on the Amcc algorithm. A neighbour of a selection is generated by removing a subset of the selected disjunctive arcs (called *destruction phase*) followed by rerunning the Amcc algorithm to again obtain a complete selection (*construction phase*). In the destruction phase 80% of the disjunctive arcs are randomly selected for removal. This is similar to the approach of Oddi et al. [15, 16, 17] named 'iterative flattening search'; their 'relax' step corresponds to the destruction phase and their 'flatten' step corresponds to the construction phase. They however treat a slightly different problem, where machines additionally have capacities. Dabah et al. [25, 27] refine the destruction phase: a single critical arc is selected and replaced by its mate. All cycles are then identified and disjunctive arcs along those cycles are removed. It should be noted that the replacement of an arc by its mate is always feasible in the job-shop setting. The neighbourhood defined by those swaps is called N_1 neighbourhood (see [3], p394). They complemented this by clever parallelisation and significant cluster based computational power. However, the claim of results being state-of-the-art is false. The results of [23] published two years prior are superior.

Branch & Bound

A branch & bound algorithm is also commonly used to solve the job-shop problem and is a natural starting point. Branch and bound algorithms were used by D'Ariano et al. [10] and Dabah et al. [20]. A contribution by Sama, D'Ariano et al. [26] focuses on the incorporation of routing options (solving $J_m \mid \text{block, route} \mid C_{\max}$).

A branch & bound algorithm creates a tree of partial solutions. At the root is the partial solution without any decisions made. For every node in the tree, a decision variable is selected and all possible decisions are taken and result in child nodes, until, at the leaf level, we are left with solutions (no more decisions left to take). When, as is commonly done, using the disjunctive arc pairs (choose between e and $\bar{e}$) as decision variables, we get a binary tree.

The branch & bound algorithm requires a heuristic function associating with each sub-tree a valid lower cost bound. I.e., given the decisions taken so far (path from the root), the partial solution cannot be extended into a complete solution with objective value lower than LB. If we already know a valid solution, of objective value UB, and $LB \geq UB$, we can conclude that it is pointless to further investigate the mentioned sub-tree, as the already known solution cannot be improved. The sub-tree is *pruned*. The core questions here are:

- How do we choose the next decision variable?
  D'Ariano proposes the use of the Amcc rule. Here, it should be noted that complementing the Amcc algorithm with a static implication rule 'arcs that violate an upper bound are prohibited' and backtracking when failing, essentially transforms the Amcc algorithm into a branch & bound algorithm.

- What lower bound is worth computing?
  First, the current objective value of any partial solution is clearly a lower bound.

  Second, any blocking-job-shop solution is also a valid job-shop solution; the same solution has the same objective value in both problem settings. Hence, the optimal job-shop solution is a valid lower bound on the blocking-job-shop solution. As a result lower bounds developed for the job-shop are valid, though they are clearly not as effective.

  Pinedo [21] suggests solving $1 \mid\mid C_{\max}$, i.e. the single machine job-shop problem for all machines separately before taking the best/highest of these lower bounds. Carlier [5] suggests computing $1 \mid prmp \mid C_{\max}$ instead, which is a valid (though worse) lower bound which can be calculated in polynomial time ($O(|J| \log |J|)$ by the EDD rule, see [21]).

- Implications:
  The static implication rules discussed for the Amcc algorithm can be employed here. In the context of the branch & bound algorithm they are called *dynamic* implications [10].

It is clear that a branch & bound algorithm also profits from the cooperation with heuristics, as the better the upper bound, the more the search tree can be pruned. As an example, Dabah et al. [20] directly initialized their search with the best known objective value as an upper bound.

Adopting the Graphical Method

The *graphical method* was first introduced by Akers & Friedman in 1955 (for the brief outline by Akers published the subsequent year see [1]). The method solves the *two* job problem, $J_{m,n=2} \mid\mid C_{\max}$ (or the dual $J_2 \mid\mid C_{\max}$). To find a schedule containing the jobs J_1, J_2, we have to 'stretch' or 'prolong' some of the operations of J_1 and J_2. The 'stretching factors' leading to a minimal $C_{\max}$ can be found with a transformation of the problem into a shortest path problem. A 2D plot with the operations of J_1 on the x-Axis and the operations of J_2 on the y-Axis is created. The operations follow the order given by the job ($o_{J_1 1}$, $o_{J_2 1}$ start at the origin). The length of an operation is defined by the processing time. For any given point in the plot we associate the two operations (of J_1, J_2) obtained by the projections onto the axes. A point is an 'obstacle' if and only if the two associated operations share any machines. The goal is now, to find a path (called *program line*) from the origin to the top-right corner; moving only up, right and diagonally up-right. The path has to avoid all obstacle points. Intuitively, time moves forward as we move along the program line. A shortest path corresponds to optimal 'stretching factors': moving right means J_1 is running its operation while J_2 is idle (up, vice-versa); moving diagonal means both jobs are processing an operation.

Mati & Xie [14] updated this method for the blocking-job-shop, $J_{m,n=2} \mid \text{block} \mid C_{\max}$. Let r be a rectangle in the plot, defined by $o_1 \times o_2$ with $M(o_1) \cap M(o_2) \neq \emptyset$. In the classical-job-shop problem the *interior* of r is an 'obstacle'. In the blocking-job-shop the interior is well as the top and right boundary of r are 'obstacles'. The path search is adopted to account for this.

Notice that a set of scheduled jobs can be merged into a single job. At a point in time, we know which machines are currently used by all jobs of the set. Whenever this changes, we create a new operation. Each operation is associated with all machines used during its respective time window. Combining this with the graphical method, a permutation of the input jobs is required to build a solution. Mati & Xie propose the use of a tabu search to find a permutation leading to a good schedule.

1.6 The SBB Challenge as a Blocking-Job-Shop

As already stated, the academic literature mostly focuses on the $J_m \mid \text{block} \mid C_{\max}$. We hinted at the SBB problem being of the type

$$J_m \mid r_{Jk},\ d_{Jk},\ \text{prec},\ s_i,\ \text{block},\ \text{rcrc},\ \text{pm},\ \text{route},\ \mid \sum w_{Jk}T_{Jk}.$$

In the next paragraphs, we connect the SBB time table generation problem to the simple blocking-job-shop problem and we see how the different constraints of the SBB problem translate into the various (β) processing characteristics.

Each train that we need to schedule corresponds to a job. A train has to go through a sequence of track-segments, which naturally constitute the operations of the job-shop problem, as for each track segment we need to associate an entry time. A machine corresponds to an area in the track network, wherein at a point in time only a single train is allowed for safety reasons (block). Multiple track sections can lie within the same 'safety area', hence multiple operations of the same job/train use the same machine (rcrc). Further, as these safety areas overlap, a single operation needs to acquire the locks for multiple areas (pm). For the transfer of a lock (on an area) an additional safety time has to pass. This is modelled by machine-dependent setup times (s_i). The earliest/latest entry times of operations are directly mapped to the variations r_{Jk} and d_{Jk}. As we can reschedule trains to use different routes, we have the 'route' variation. Another complication of the SBB challenge is the existence of connections. A train is only allowed to depart from a station/operation after another has arrived and passengers had time to change trains. This adds the further complication 'prec', that there are dependencies between different operations of different trains.

Finally, the objective function: different trains have different priorities and some stops/operations might be more important than others, hence the weights w_{Jk}. The quality of the timetable is assessed by the sum of all weighted delays ($\sum w_{Jk}T_{Jk}$). In the Chapter 'adaptation to the SBB challenge', we return to these complications and see how they can be incorporated into the 'method' which, as presented in the next Chapter, solves the problem $J_m \mid \text{block} \mid C_{\max}$.

2 Method

2.1 Informal Introduction

The method I implemented was developed by Heinz Gröflin, Andreas Klinkert & Reinhard Bürgy [13, 23, 24] based on the underlying theory developed by Heinz Gröflin & Andreas Klinkert [11]. Before going into the more technical details, I want to give an informal overview of the idea.

The method reschedules individual jobs.

The method is based on restricting the complete problem, where each job is flexible, to the problem where the sequencing is fixed, except for a single job J which can be moved. This manifests itself as follows: from the disjunctive graph another disjunctive graph is derived; this derivation fixes all disjunctive arcs that are not adjacent to J, transforming them into conjunctive arcs. The derived disjunctive graph is called *job-insertion graph*. As we will see, the job-insertion graph has a property we can exploit to effectively manipulate the placement of J.

What purpose does this serve? First, if we want to get rid of a critical arc and replace this arc by its mate, we can reschedule one of the two jobs adjacent to the arc, including the mate instead, otherwise staying as close as possible to the original solution. In effect, this allows us to define a neighbourhood akin to the N_1 neighbourhood of the classical job-shop. Second, we can build up initial solutions by inserting jobs one-by-one. We can delete a job and reschedule it with a different routing, or place a job according to a heuristic.

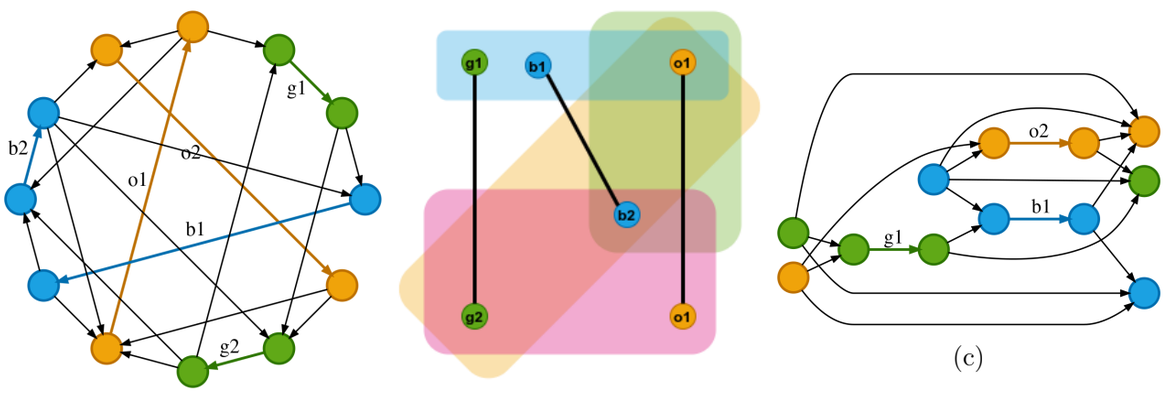
In this Chapter, I first define a conflict graph for the complete problem. This serves to illustrate why the derivation of the job-insertion graph is necessary. After the precise definition of the job-insertion graph, we will come to the property hinted at above: the *short-cycle-property*. With this, we see that the conflict graph of the job-insertion graph leads to a useful algorithm.

2.2 Conflict Hypergraph

We create a conflict graph for a complete blocking-job-shop problem. As said, this is not supposed to yield a foundation for an algorithm, but rather motivate the construction of the *job-insertion graph* that follows.

We define a conflict graph for a disjunctive graph G. The conflict graph is an undirected graph $H_G = (E, U)$. We associate vertices with elements that can be selected. If selected elements *conflict* with each other, they are connected by an edge. In the case of a blocking-job-shop problem, the elements to select are the disjunctive *arcs* of E which become the *vertices* of H_G (*arcs* $\rightarrow$ *vertices*). A selection $S \subset E$ of arcs is conflicting if $(V, A \cup S)$ contains a cycle. Cycles in the blocking-job-shop conflict graph can be made of more than two arcs, and therefore, the 'edges' in the conflict graph also connect two or more vertices. The conflict graph is a hypergraph, whose edges are sets of vertices. $\{e_1, \dots e_k\} \in U$ is an edge if $(V, A \cup \{e_1, \dots, e_k\})$ is cyclic.

Example 2.1 (Conflict hypergraph)**.** *In Figure 1, (a) we have a disjunctive graph. Here, $\mathcal{E} = \{\{b1, b2\}, \{g1, g2\}, \{o1, o2\}\}$, that is the blue arc 'b1' is the mate of the blue arc 'b2', etc. We note that, as should be the case, each of these arcs conflicts with its mate, eg. $(V, A \cup \{b1, b2\})$ contains a cycle. However there are other cycles, eg. the one including b2, g2, o1 or b1, o1, g1. Figure 1, (b) shows the conflict hypergraph. Edges between mates are represented as black lines, while other edges are represented by colored boxes around the respective vertices.*



(a) (b) (c)

Figure 1: The (a) & (b) illustrate the Example 2.1, (c) illustrates the continuation, Example 2.2.

(a) A disjunctive graph, where: V = all nodes, A = all black arcs, E = all colored arcs and $\mathcal{E} = \{\{b1, b2\}, \{g1, g2\}, \{o1, o2\}\}$

(b) The conflict hypergraph of the disjunctive graph of (a). Nodes correspond to the arcs of (a). A black line represents an edge/conflict between mates. Colored boxes represent hyperedges.

(c) Shows the graph $G = (V, A \cup S)$ with $S = \{g1, b1, o2\}$. As expected G is acyclic.

Let us link the conflict graph to the original problem: A solution to the blocking-job-shop problem is feasible if and only it corresponds to a selection which is acyclic in the associated disjunctive graph. Furthermore, a selection corresponds to a set of vertices in the conflict graph and (by construction) we know that the selection is acyclic in the disjunctive graph if and only if it is an *independent* set (*stable* set) of the conflict graph. An independent set of vertices in H_G of size $|E|/2$ corresponds to a complete feasible selection, as from all pairs in $\mathcal{E}$ exactly one arc has to be chosen.

Example 2.2 (Conflict hypergraph)**.** *$S = \{g1, b1, o2\}$ is an independent set of the conflict graph of Figure 1, (b) of size $|E|/2$. S corresponds to a complete feasible selection. Figure 1, (c) shows $(V, A \cup S)$.*

Definition 2.1 (Elementary edge)**.** An edge of H_G, $u \in U$ is called elementary if u is set-wise minimal in U, i.e. $\nexists u' \in U$ $(u \subset u')$. The conflict corresponding to an elementary edge is called elementary conflict.

Since we are only interested in the stable sets of the conflict graph, it is clear that we do not need to include *non-elementary* edges in the conflict graph. An example of a non-elementary edge in Figure 1, (b) is the 'yellow' edge, as it contains the 'green' edge. We adapt the definition of H_G: let $U' = \{\, u \mid u \in U, u \text{ elementary} \,\}$, $H_G = (E, U')$.

Based on the above observations, we could define the following algorithm to solve the blocking-job-shop problem: Enumerate independent sets of the conflict graph of size $|E|/2$, and choose the one with the best objective in the associated disjunctive graph. Return this as a solution. However, even finding a single independent set of the given size is a hard problem. This approach does not work, but I think it is important to see this and to contrast this with the method below.

2.3 Job-Insertion Graph

The job-insertion graph is also a disjunctive graph as is the case with the 'complete' disjunctive graph which we defined for a blocking-job-shop problem. The difference is that we restrict ourselves to the following sub-problem:

Given a feasible schedule, integrate an additional job.

We call this problem the *job-insertion problem*. To recall the connections between the elements introduced so far: A blocking-job-shop problem is modelled by a disjunctive graph. A solution to the blocking-job-shop problem corresponds to a selection in that disjunctive graph. The job insertion problem is a sub-problem, which we will also model with a disjunctive graph, called the job-insertion graph. A solution to the job-insertion problem corresponds to a selection in the job-insertion graph. Recall that we defined conflict (hyper-)graphs for disjunctive graphs.

Definition 2.2 (Job-insertion graph)**.** Given is a blocking-job-shop problem with jobs $\mathcal{J}$, a specific job $J \in \mathcal{J}$ and a complete feasible selection S to the problem $\mathcal{J} \setminus J$. For S i.e.:

$\forall \{e, \bar{e}\} \in \mathcal{E}$ not adjacent to J, $e \in S \vee \bar{e} \in S$.

$\forall \{e, \bar{e}\} \in \mathcal{E}$ adjacent to J, $e \notin S \wedge \bar{e} \notin S$.

Associated with the blocking-job-shop problem we have the graph $G = (V, A, E, \mathcal{E}, l)$. The job-insertion graph $G^J = (V, A^J, E^J, \mathcal{E}^J, l)$ is constructed as follows:

$A^J := A \cup S$

$\mathcal{E}^J := \mathcal{E}$ restricted to arcs adjacent to J.

$E^J := E$ restricted to arcs adjacent to J.

In effect, this construction fixes the sequencing (solution of the sequencing problem) of the jobs $\mathcal{J} \setminus J$. The disjunctive arcs of those jobs are added to the conjunctive (fixed) arcs A. The disjunctive arcs adjacent to J are not restricted–we are free to insert J at any position.

In the next Section, we will introduce the *short-cycle property*. We will show that the job-insertion graph does have this property. With this we will come back to the conflict graph. The conflict graph of the job-insertion problem will allow us to define a useful algorithm. This is to be contrasted with the attempt to derive an algorithm from the conflict graph for the complete problem.

2.4 Short-Cycle Property

Notation: For a set of vertices X, we denote all arcs adjacent to some vertex of X by $\delta(X)$.

For every cycle visiting J, there is a 'shorter' cycle visiting J once.

The short-cycle property is a property of a job-insertion (disjunctive) graph constructed for a job J, $J \subset V$. We use the fact $E \subseteq \delta(J)$, that is, all disjunctive arcs of the graph are adjacent to the job J. This follows from the construction of G^J: $E^J \subseteq \delta(J)$, indeed $E = \delta(J) \setminus \{\text{conjunctive arcs of } J\}$. Furthermore, we know that the vertices of J form a path in (V, A). This path is not reachable (in (V, A)) from any vertex in $V \setminus J$.

Definition 2.3 (Short-cycle property)**.** A disjunctive graph $G = (V, A, E, \mathcal{E}, l)$ has the short-cycle property if for any cycle Z in $(V, A \cup E)$, there exists a cycle Z' in $(V, A \cup E)$ with $Z' \cap E \subseteq Z \cap E$ and $|Z' \cap E| = 2$.

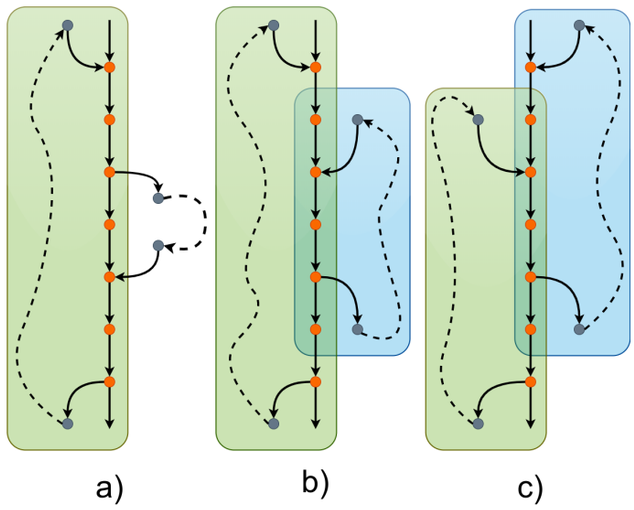


Figure 2: Illustration of the short cycle property. The Figure shows a part of the graph G^J. The orange vertices are within the job J, grey vertices are of the other jobs $\mathcal{J} \setminus J$. Dashed lines indicate that 'somewhere' in the graph a path exists. We show the 3 cases of a cycle Z visiting J two times (see Proposition 2.3). The colored areas indicate the short cycles Z' which satisfy the conditions $Z' \cap E \subseteq Z \cap E$ and $|Z' \cap E| = 2$.

Proposition 2.3. *The job-insertion graph $G^J = (V, A^J, E^J, \mathcal{E}^J, l)$ has the short cycle property.*

Proof. We proof the proposition by induction over the number of times a cycle visits the job J.

- **$\mathbf{n = 1}$ (base case)**:
  In this case we enter and leave J, hence $|Z \cap E| = 2$. ✓

- **$\mathbf{n = 2}$**:
  The treatment of this case is not necessary for the proof, but tries to illustrate the proposition. We choose arbitrarily one vertex where the cycle Z enters J and call this vertex a. Let b be the vertex where Z first leaves J, after a. Let c be the vertex where Z reenters J, after b. And, let d be the first vertex where Z leaves J, after c. Within J vertices are ordered by reachability (in (V, A)), we write $v \preceq w$ if $\{v, w\} \subseteq J$ and w is reachable from v.

  We have the following six possibilities, see Figure 2 for an illustration of the cases.

  (i) $a \preceq b \preceq c \preceq d \implies$ case a).

  (ii) $a \preceq c \preceq d \preceq b \implies$ case b).

  (iii) $a \preceq c \preceq b \preceq d \implies$ case c).

  (iv) - (vi) The above cases, with $a \leftrightarrow c, b \leftrightarrow d$.

  Which is exhaustive, given the constraints, $a \preceq b$, $c \preceq d$ (a cycle cannot leave J before it enters it). As indicated by the colored boxes in Figure 2, at least one short cycle exists in all cases. ✓

- **$\mathbf{n \rightsquigarrow n + 1}$ (inductive step)**:
  We need to prove that a cycle entering J $n + 1$ times has a short cycle. For this we can use the induction hypothesis: any cycle entering J n times has a short cycle. Let Z be a cycle which enters J $n+1$ times. We choose arbitrarily a vertex a where the cycle Z leaves J. Let b be the first vertex after a where Z reenters J. We differentiate two cases:

  (i) **$\mathbf{a \preceq b}$**:
  Define a cycle Z', equal to Z, with the path $a \to b$ replaced by the path $a \to b$ within J. Then Z' enters J n times. From the induction hypothesis it follows that a short cycle exists. ✓

  (ii) **$\mathbf{\neg(a \preceq b)}$**:
  $b \to a$ (within Z) followed by $a \to b$ (within J) is a valid short cycle. ✓

□

In the next Section we will see the importance of the short cycle property.

2.5 Conflict Graph

We recall the conflict graph $H_G = (E, U)$, defined for a disjunctive graph $G = (V, A, E, \mathcal{E}, l)$ in Section 2.2. The disjunctive arcs of G equal the vertices of H_G.

Analogously, we now create the conflict graph of a job-insertion graph G^J, $H_{G^J} = (E^J, U)$, and restrict the edges of U to elementary edges (see Definition 2.1). The short-cycle property now guarantees that each edge in U is a 'normal' edge connecting *two* vertices. If this were not the case, there would exist an edge $u \in U$ with $|u| > 2$ and $(V, A^J \cup u)$ cyclic. Then, by the short-cycle property a short cycle would exist. This short cycle would correspond to an edge $u' \subset u$, with $|u'| = 2$. Then, u would not be elementary, a contradiction to the construction. The conflict graph H_{G^J} is not a hypergraph. I adapt the definition of the edge set to this insight: $U = \{(e, f) \mid (V, A^J \cup \{e, f\}) \text{ cyclic}\}$, that is two vertices in the conflict graph are connected if adding them to (V, A^J) adds a cycle to that graph.

Proposition 2.4. *The job-insertion conflict graph is bipartite.*

Proof. Let $\{e, \bar{e}\} \in \mathcal{E}^J$. Then either e or $\bar{e}$ has its head in $J \subset V$: $\big(h(e) \in J\big) \vee \big(h(\bar{e}) \in J\big)$. And, the other arc has its tail in J: $\big(t(\bar{e}) \in J\big) \vee \big(t(e) \in J\big)$. This is the case, as the two arcs represent the decisions: an operation outside J is scheduled before the operation in J and vice-versa. We define $W := \{e \in E^J \mid h(e) \in J\}$, $\bar{W} := E^J \setminus W$. Then $W, \bar{W}$ are disjoint and $W \cup \bar{W} = E^J$. It remains to be shown that W and $\bar{W}$ are independent in H_{G^J}. We consider $(V, A^J \cup W)$. We know that the subgraphs of (V, A^J) induced by $V \setminus J$ and J are acyclic, and that all arcs in W go from $V \setminus J$ to J. Hence $(V, A^J \cup W)$ is acylic since a cycle would require an arc from J to $V \setminus J$ and no such arc exists. We conclude that W is independent in H_{G^J}, by the definition of U. An analogous argument shows that $\bar{W}$ is independent in H_{G^J}. H_{G^J} is bipartite. □

2.6 Closure Operator

The previous definitions and properties (disjunctive graph, insertion graph with corresponding bipartite conflict graph) allow us to define a 'closure operator', which forms the core of the algorithm. The closure operator takes as input a feasible solution (complete feasible selection) of a blocking-job-shop problem, as well as, a disjunctive arc that we want to 'force' into the solution and returns a feasible solution–including the forced arc–that is as 'similar' as possible to the input solution. On 'similar': The new solution preserves the sequencing decisions for all except one job. That job is sequenced only 'earlier' (or only 'later'). The change in sequencing is minimized.

How does this work: Denote the input selection with S. First a job J at the head or tail of the arc to be forced in is identified. The job-insertion graph is constructed.

$$G^J = (V, A^J, E^J, \mathcal{E}^J, l) = \left(V, A \cup \underbrace{\Big(S \setminus (\underbrace{\delta(J) \cap E}_{*}) \Big)}_{**}, \underbrace{\delta(J) \cap E}_{*}, \delta(J) \cap \mathcal{E}, l \right)$$

* $\delta(J) \cap E$ are all the disjunctive arcs adjacent to the job J.

** (*) are removed from the selection S given as input.

Next, we construct the conflict graph $H_{G^J} = (E^J, U)$. We select the forced arc e (a vertex) in H_{G^J}. If e is connected to f, we know that f cannot be part of the resulting solution. Therefore, $\bar{f}$ has to be part of the solution. Hence, $\bar{f}$ is selected. The closure operator then takes the *closure* of this selection process, yielding a subset of E^J which is stable in H_{G^J}. This subset is then completed with the input selection S to form a new feasible solution, which is the result of the closure operator.

Example 2.5 (Closure operator)**.** *In Figure 3 a tiny example with six pairs of disjunctive arcs is introduced and a closure is derived to illustrate the description of the method above.*

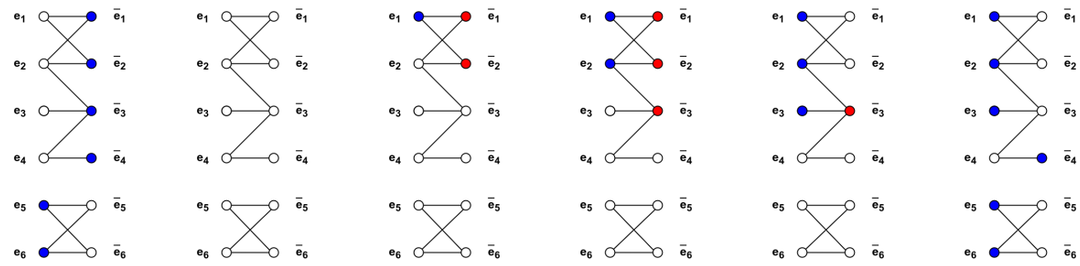


Figure 3: $\mathcal{E}^J = \{\{e_1, \bar{e_1}\}, \ldots, \{e_6, \bar{e_6}\}\}$. The input to the closure operator is the initial selection $S = \{\bar{e_1}, \bar{e_2}, \bar{e_3}, \bar{e_4}, e_5, e_6\}$, as well as the constraint, that e_1 has to be included in the selection. The conflict graph $H_{G^J} = (E^J, U)$ is given in (a), with the initial selection S marked in blue. In a first step, (b), the selection is cleared. Next, (c), the arc to be added, e_1 is selected. Since $(e_1, \bar{e_1}) \in U$ and $(e_1, \bar{e_2}) \in U$, the arcs $\bar{e_1}$ and $\bar{e_2}$ cannot be selected. Hence, (d), the mate of $\bar{e_2}$ which is e_2 is selected. Now, as $(e_2, \bar{e_3}) \in U$, $\bar{e_3}$ cannot be selected. In (e), after selecting e_3, we detect that the closure is complete. Finally, (f), the new partial feasible selection $S' = \{e_1, e_2, e_3\}$ is complemented with S resulting in a new complete feasible selection $\{e_1, e_2, e_3, \bar{e_4}, e_5, e_6\}$.

Remark 2.6. *For a given critical arc e which we aim to remove, we have the choice of fixing all jobs $\mathcal{J}$ except $J_1 \mid h(e) \in J_1$ or $J_2 \mid t(e) \in J_2$. What is the difference?*

- *Choosing J_1: We remove an arc e which enters J_1 by forcing in an arc $\bar{e}$ that leaves J_1. A short cycle through $\bar{e}$ must therefore contain an arc f which enters J_1. f must be replaced with $\bar{f}$, which leaves J_1. The same argument holds for $\bar{f}$, indeed for the complete process. More and more arcs leave J_1, while fewer and fewer arcs enter J_1. This has the effect that J_1 moves backward in time through the time table, or, in a Gantt-chart, to the left. We call this version of the closure the left-closure.*

- *The exact opposite.*
  (Choosing J_2: We remove an arc e which leaves J_2 by forcing in an arc $\bar{e}$ that enters J_2. A short cycle through $\bar{e}$ must therefore contain an arc f which leaves J_2. f must be replaced with $\bar{f}$, which enters J_2. The same argument holds for $\bar{f}$, indeed for the complete process. More and more arcs enter J_2, while fewer and fewer arcs leave J_2. This has the effect that J_2 moves forward in time through the time table, or, in a Gantt-chart, to the right. We call this version of the closure the right-closure.)

Remark 2.7. *A proof of termination and correctness is based on the conflict graph being bipartite (see Proposition 2.4).*

2.7 Local Search

With the closure-operator we can create a complete feasible selection S' from a given complete feasible selection S such that a given arc $e \in S$, $e \notin S'$. How do we use this to find a solution to the blocking-job-shop problem? Recall the Definition 1.3 where critical arcs are introduced. A critical arc corresponds to a decision that leads to some penalty. If we use the closure-operation to remove a critical arc, we find a similar, possibly better solution. In this way we define a neighbourhood, i.e., S has as neighbours all S' such that a critical arc e in S exists with $S' = \text{closure}(S, \bar{e})$. Given the neighbourhood of feasible selections, we can employ a suitable meta-heuristic. We use a tabu-search to find a solution to the blocking-job-shop problem and avoid getting permanently stuck in locally optimal solutions.

A tabu search generates a sequence of solutions $S^{(0)}, S^{(1)}, S^{(2)}, \ldots$ such that $S^{(i+1)}$ is the neighbour of $S^{(i)}$ with minimum cost, which is not *tabu*. *tabu arcs* are accumulated in a *LIFO* queue. When we move to a next solution, the critical arc e being removed to create the neighbour becomes a tabu arc. A neighbour is tabu if its objective is worse than the best objective value in $S^{(0)}, \ldots, S^{(i)}$ and contains at least one tabu arc. The tabu queue is kept at a size smaller than a specified parameter of the search (often between 6 and 9). To generate an initial solution $S^{(0)}$ multiple possibilities exist. Conceptually, we could fix an order of the jobs and schedule the jobs on all machines according to this order. This yields a (bad) valid initial solution. In practice, we insert the jobs one-by-one as is described in Section **??**, *Routing Possibilities*.

The search can be terminated at any point. $S^{(i)}$ with minimum cost is returned as a solution.

3 Implementation & Optimization

In this Chapter on the implementation we will see how the theoretical algorithm discussed in the Chapter 2 'method' is implemented. As outlined, this algorithm solves the problem $J_m \mid \text{block} \mid C_{\max}$. The discussion focuses around the fact that, neither the job-insertion graph G^J, nor the conflict graph $H_{G^J} = (E^J, U)$ are computed or stored in memory. These optimizations are part of the method as implemented by [23], but are documented in more detail. In line with those optimizations, I present the changes developed during this master thesis, namely: a faster way to update the entry times, a more efficient way to calculate the closure operation and a termination criterion which helps to further speed up the closure operation.

In a next Chapter, I will then describe how the algorithm is adopted to the additional complexity of the SBB problem: i.e., r_Jk, d_Jk, operation based earliest and latest entry; s_i, machine dependent setup time; *rcrc*, recirculation or usage of the same machine over multiple operations; *route*, routing possibilities; *pm*, parallel machine usage; *prec*, precedence constraints between jobs (connections); and the change in objective function to $\sum w_{Jk}T_{Jk}$. Theoretically, these elements can be incorporated with ease, although the proposed changes to incorporate *prec* and *route* should be improved. In any case, the implementation becomes significantly more complex.

3.1 Transitive Arcs

Definition 3.1 (Transitive arc)**.** Recall, we denote with $E_m \subset E$ the disjunctive arcs defining the job-sequence on machine m. An disjunctive arc (i, j) of a graph $(V, A \cup E_m)$ is called transitive (else non-transitive), if a path from i to j exists and $l(i, j) \leq l^{E_m}(i, j)$ (the arc is shorter than the path from i to j).

As we only ask if paths exist, and if so are only interested in the longest path in the graph $(V, A \cup S)$, transitive arcs can be ignored.

Example 3.1. *Assume we have 3 jobs a, b, c that use a machine m for their respective operations o_a, o_b, o_c. If the jobs use machine m in the order $a \to b \to c$, we have the following arcs is the disjunctive graph:*

- $succ(o_a) \to o_b$.
- $succ(o_b) \to o_c$.

The arc '$succ(o_a) \to o_c$' is transitive, we do not save it in the graph, as it is implicitly present.

For each machine we store only the non-transitive arcs. Assume machine m is used by n jobs. A complete selection now contains $n-1$ arcs defining the order in which m is used instead of $\frac{n(n-1)}{2}$ arcs.

Remark 3.2. *If arc lengths are non-negative and the length of disjunctive arcs is 0, or only dependent on the machine m (as is the case in the SBB problem, see s_i), it is guaranteed that the path $succ(o_a) \to o_b \to succ(o_b) \to o_c$ is at least as long as $succ(o_a) \to o_c$. However, if the above condition is not met, we have to ensure that longest paths are always present in the disjunctive graph.*

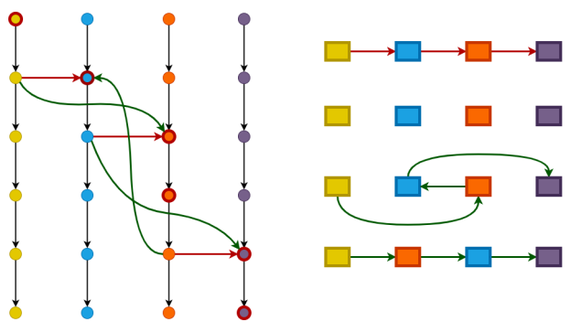


Figure 4: Illustration of the step-by-step replacement of an arc $e =$ 'blue' $\to$ 'orange' by its mate $\bar{e} =$ 'orange' $\to$ 'blue'. Only non-transitive arcs are present.

As we have seen, a common operation is the replacement of an arc e by its mate $\bar{e}$. Figure 4 illustrates how this operation is performed in the setting of storing non-transitive arcs only. On the left, some operations of four jobs are visualized. Operations circled in red share a machine. We assume that, before the operation is executed the red disjunctive arcs are present. On the right,

the replacement is shown, step-by-step. Let e be the arc $succ$('blue') $\rightarrow$ 'orange' and $\bar{e}$ be the arc $succ$('orange') $\rightarrow$ 'blue'. First, e, alongside two arcs–which become transitive–are removed. Then $\bar{e}$ is inserted, alongside two arcs which become non-transitive. Note, that this operation corresponds to swapping the order in which the jobs 'blue' and 'orange' use the machine.

During the run time of the algorithm we store for each machine the sequence of jobs that use this machine, similar to the right side of Figure 4. This allows for more efficient job-insertions and route-swaps (faster runtime, minimal memory overhead).

3.2 Conflict Graph

Why is it not necessary to form the conflict graph explicitly? We use the conflict graph to query information like 'enumerate the neighbours of e'. In the disjunctive graph, this is equivalent to 'find all disjunctive arcs $f \in E^J$ such that $(V, A^J \cup \{e, f\})$ is cyclic'.

Every edge in the conflict graph is used at most once when we transform a selection into a neighbour selection. Consider the example of Figure 3. First, (c), we find all cycles containing e_1. Then, (d), we need to find all cycles containing e_2. Finally, (e), we need to find all cycles containing e_3. All the remaining edges in H_{G^J} ($\cong$ cycles in G^J) are never used.

What are the necessary cycle searches?

- In case of the left-closure (as introduced in Remark 2.6): Recall, arcs such as e which *enter* J are replaced by arcs $\bar{e}$ that *leave* J. Whenever we insert $\bar{e}$, we have to check if a path exists between $h(\bar{e}) \rightarrow t(\bar{e}) \quad (t(\bar{e}) \in J,\ h(\bar{e}) \notin J)$. Of a potential cycle, only the disjunctive arc reentering J is relevant. Hence we search paths from $h(\bar{e})$ into the vertex interval $[o_{J1}, \ldots, t(\bar{e})]$. This is done efficiently with a forward path search.

- In case of the right-closure: Recall, arcs such as e which *leave* J are replaced by arcs $\bar{e}$ that *enter* J. Whenever we insert $\bar{e}$, we have to check if a path exists between $h(\bar{e}) \rightarrow t(\bar{e})$ $(t(\bar{e}) \notin J,\ h(\bar{e}) \in J)$. Of a potential cycle, only the disjunctive arc first leaving J is relevant. Hence we search backward paths from $t(e)$ into the vertex interval $[h(\bar{e}), \ldots, o_{Jn_J}]$. This is done efficiently with a backward path search.

3.3 Job-Insertion Graph

Why is it not necessary to form and store the job-insertion graph? We review the example of Figure 3: Forming the job-insertion graph corresponds to the step (a) $\rightarrow$ (b), that is: inserting all the edges of E^J not already present into the current solution $(V, A \cup S)$, such that when we start the cycle search for the 'forced-in' arc e_1 (e_1 substitutes $\bar{e_1}$), we do actually detect the conflicts with $\bar{e_1}$ and $\bar{e_2}$. But, if we did not complete the graph to include all E^J we would still find the conflicts with the original selection S, and those are exactly the conflicts we are looking for. As a result the step (e) $\rightarrow$ (f) is also no longer necessary since, for all $\{e, \bar{e}\}$ that were unaffected, the correct arc is already present.

However, a complication arises, if this is combined with the choice not to store transitive arcs in the graph: an initial cycle search might not reveal a cycle as the respective conflicting arc is transitive, but, later in the process becomes non-transitive.

How can this be resolved? The full set of path searches could be redone until no more cycles are found. Or, alternatively, the search can be resumed whenever an arc becomes non-transitive and another search already passed by the tail vertex (left-closure) or passed by the head vertex (right-closure).

The implementation of [23] handles the problem in a clever way: In a first step, the transitive arcs which would become non-transitive, if J were removed, are inserted. Then, during the execution of the closure operator, the arcs defining the sequences of $\mathcal{J} \setminus J$ remain unchanged. Only the current location, where to eventually insert J is tracked. With the closure completed, J is inserted, and for each machine the one arc becoming transitive with the insertion of J is removed.

3.4 Naive Closure

Having seen what we actually save and compute we will now put together the closure algorithm. For simplicity, I just present the left-closure. The right-closure, as we have seen, is analogous. For a first, simple version see Algorithm 2. As mentioned in the Section above, this only works if transitive arcs are present. The job-insertion graph still needs to be created. We shall see how this is avoided.

Algorithm 2: Naive left-closure

Input : A graph G corresponding to the current selection S: $G = (V, A \cup S)$, a disjunctive arc a to remove from the selection.

Output: A modified graph G, corresponding to a modified selection S, which is similar to the input S but does not contain the arc a.

```
1  Set<Arc> arcsToRemove = {a}
2  while arcsToRemove.Count > 0 do
3  |  Arc e = arcsToRemove.Pop()
4  |  if G.ArcExists(e) then
5  |  |  Arc ē = G.SwapInMate(e)
6  |  |  foreach Arc f ∈ G.ForwardPathSearchIntoRange(h(ē), [o_J1, t(ē)]) do
7  |  |  |  arcToRemove.Add(f)
8  |  |  end
9  |  end
10 end
```

In Algorithm 2 two functions are called:

- SwapInMate(e): Removes e from the graph and inserts $\bar{e}$. With respect to transitive arcs, this is done exactly as visualized in Figure 4. The arc $\bar{e}$ is returned.
- ForwardPathSearchIntoRange(vertex, vertexSet): Searches all paths from 'vertex', that end in 'vertexSet' and returns a list of arcs, containing the last arc of each such path. The way this function is used, it returns all disjunctive arcs to which $\bar{e}$ is connected in the conflict graph H_{G^J}.

In the next Section we shall see how this naive version is optimized and simultaneously solves the problem, that in the present version, transitive arcs need to be present.

3.5 Closure: All-at-once?

The 'all-at-once' closure optimization presented here is the core contribution made over the development of this thesis.

In this and the following Section we develop the optimization of the *left*-closure. However, all these optimizations also directly apply to the right-closure. The differences–left versus right–are briefly stated in Remark 3.5.

The key insight for the optimization presented shortly is that the series of path searches to be done, follow a 'nested' structure. Observe that, the later within J the operation $t(\bar{e})$ occurs, the larger the target vertex interval of the path search. Indeed, $t(\bar{f}) \preceq t(\bar{e})$ implies $\{o_{J1}, \ldots, t(\bar{f})\} \subseteq \{o_{J1}, \ldots, t(\bar{e})\}$. Hence, for any vertex $v \in G$ such that a path exists form $h(\bar{f}) \to v$ and $h(\bar{e}) \to v$, the forward path search (v onward) to be completed for $\bar{e}$ renders the path search for $\bar{f}$ irrelevant. Before producing the respective Algorithm 3, we go through an example.

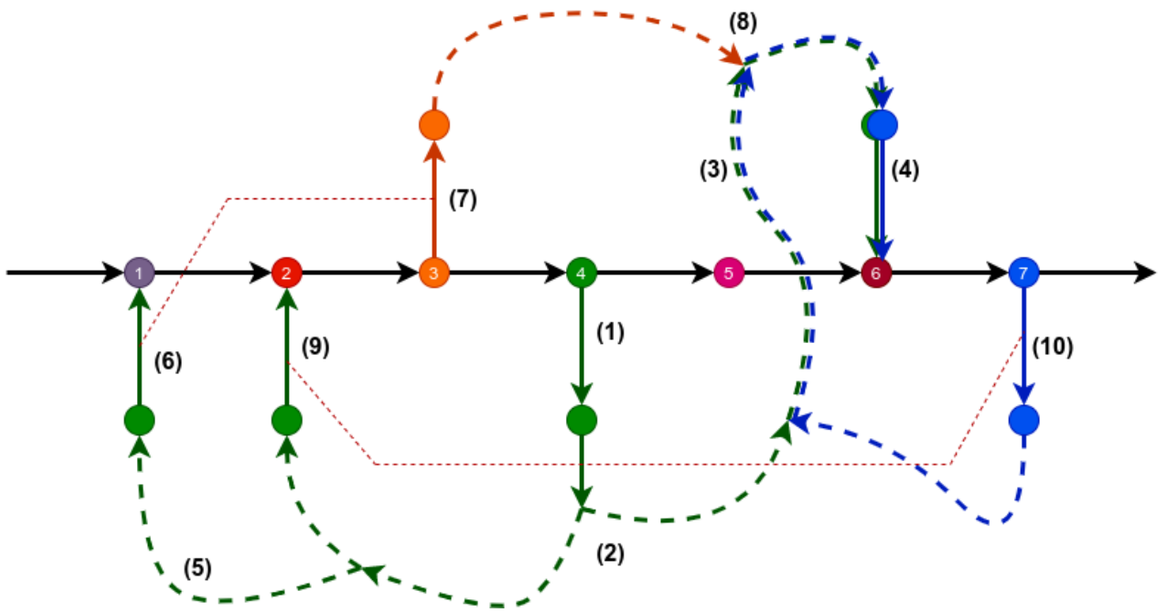


Figure 5: Illustration of Example 3.3.

Example 3.3. *Closure, all at once. Below, I outline step-by-step how the algorithm computes a left-closure as illustrated in Figure 5. Black arcs correspond to conjunctive arcs within the selected job. Numbered, colored vertices are the vertices of the selected job. Dashed red lines match disjunctive arcs with their mates ((6) ↔ (7), (9) ↔ (10)). Dashed curves indicate that we understand this part of the graph as a black-box. We only know a path exists. Numbers mark locations in the graph, which may be visited multiple times. The green/blue double line indicates that this part of the graph is traversed twice.*

(1) *We assume the green arc inserted at (1) is the one, of which we take the left closure (i.e., the mate of this arc was removed and this one inserted). We start by exploring the graph, searching for a path back into the job. While doing so, we color the visited vertices with 'green-4'.*

(2,3,4) *We found a first path back into the job. But the vertex where we enter is 'brown-6'. As* $6 > 4$*, there is no cycle.*

(5,6) *We found a second path back into the job. This time we encounter the vertex 'violet-1'. As* $1 \leq 4$ *we have a cycle. Hence, the arc at (6) has to be removed and is replaced by its mate, the arc at (7).*

(7) *After the arc insertion at (7) we continue to explore the graph from there. Now, we color the vertices with 'orange-3'.*

(8) *We encounter a vertex that is already colored. Since the color of the vertex, is 'green-4' is larger than 'orange-3' we can stop the search here. This is exactly the scenario described in the first paragraph of Section 3.5 ('key insight'):* v *is the vertex at (8) with incoming green and orange arcs.* $\bar{f}$ *corresponds to the arc at (7) and* $\bar{e}$ *corresponds to the arc at (1).*

(9) *We continue the search with 'green-4' and find a 3rd path back into the job (9). Analogous to (6) we have to replace the arc at (9) with its mate, the arc at (10).*

(10,3,8) *After inserting the mate at (10) we continue our exploration of the graph, now marking vertices with 'blue-7'. We encounter an already colored vertex. But since the vertex color is 'green-4' and* $4 < 7$*, we have to redo the search here. We continue and pass by (3) and (8).*

(4) *We revisit (4). But this time, the current color 'blue-7' is larger than 'brown-6'. Hence, we have a cycle and the vertex at (4) has to be replaced by its mate (which is not shown in the figure).*

Remark 3.4. *What did we save by combining the path searches as illustrated by the above example? When we met the green colored path of the graph, while performing the path search for the vertex 3 at (7), we could skip going forward. Note, that when coloring with a higher color such as 'blue (7)', we minimize the risk of having to partially redo the search. Therefore, we implement path search using a priority-queue to always first explore where we get the most 'definite' results. The colors correspond to priorities. Furthermore, we know the priorities which we might encounter in advance. Those correspond to the operations within a Job. A bucket-queue (also called bucket-priority-queue or bounded-height-priority-queue) provides an efficient implementation for this use case. The use of this variant of the algorithm led to a decrease in necessary execution time of at least 70%, when compared to the naive version, Algorithm 2 (on medium sized SBB instances).*

Algorithm 3: All at once left-closure

Input : A disjunctive graph G and a selection S: $G = (V, A \cup S)$, a disjunctive arc a to remove from the selection.

Output: A modified graph G, corresponding to a modified selection S, which is similar to the input S but does not contain the arc a.

```
1  Queue<Arc> Q = new BucketQueue()
2  let ā be s.t. ā ∈ G.SwapInMate(a) and t(ā) ∈ J
3  Q.Add(ā, priority = any)
4  int[] P = new int[G.Count]
5  Initialize P: P[o_jk] = { k, if j = J
                           { 0, otherwise
6  while Q.Count > 0 do
7      Arc a = Q.Pop()
8      if not G.ArcExists(a) then
9          continue
10     else if h(a) ∈ J and P[t(a)] ≥ P[h(a)] then
11         foreach Arc b ∈ G.SwapInMate(a) do
12             if P[t(b)] > 0 then
13                 Q.Add(b, priority = P[t(b)])
14             end
15         end
16     else if P[t(a)] > P[h(a)] then
17         P[h(a)] = P[t(a)]
18         foreach Arc b ∈ G.OutgoingArc(h(a)) do
19             Q.Add(b, priority = P[t(b)])
20         end
21     end
22 end
```

In Algorithm 3 these insights are included to form an improved version of the left-closure. Here, Q is an arc collection keeping track where in the graph $(V, A \cup S)$ we have to continue the path search. P is an array of priorities (colors), such that we assign to each vertex in G a priority. How this works is illustrated in Figure 5. At every step a single arc a is removed from Q and treated:

- G.ArcExists(): a disjunctive arc could have been removed, due to another detected conflict.
- The next *if* block treats the case of reaching back into the job J. If the priority we currently use is larger or equal to the priority of the operation of the job, we found a cycle. This last arc must be the disjunctive arc reaching back into J, which is removed. A key difference here is that *SwapInMate()* now has to return up to 3 arcs: the mate inserted, and the arcs that become non-transitive. Recall the issue described in Section 3.3, that is, an arc b becoming non-transitive might be part of a cycle. To ensure cycles of this type are found, we have to check if $t(b)$ was visited by a path search, and if so, continue the path search at $h(b)$. With the priority vector P this is easy: if $P[t(b)] > 0$ the path search is continued. Hence, the arc is inserted into Q.
- Finally, the 'default' case. If we did not reach the $h(a)$ yet (priority of $h(a) = 0$) or we reach $h(a)$ again but with a higher priority, then the search continues: we insert all arcs continuing forward from $h(a)$ into Q and adapt the priority of $h(a)$.

3.6 Termination Criterion

We developed a termination criterion for forward path search of the left closure. This also yields a significant speedup of the resulting algorithm when tested on the SBB problem instances.

The idea of the termination criterion is to partition the graph into two subgraphs: A and B. For any given time t, we can partition the acyclic graph into the vertex sets

$$A := \{v \mid t_v \leq t\} \text{ and } B := \{v \mid t_v > t\}.$$

Furthermore, no vertex of A is reachable from any vertex in B. Assuming that the left-closure computation affects no arcs adjacent to or in B, we can establish that A remains unreachable from B. If we then–during the forward path search of the left closure–encounter a vertex of B we can skip this branch of the path search.

We choose $t = \max_{a \in \delta(J)}\{\, t_{h(a)} \,\}$. Recall Remark 2.6: during the left-closure the job J moves to the left, or backward in time. Hence, we know that arcs in B will never be affected by the operations we perform.

Algorithm 4: All at once left-closure with termination criterion

Input : A graph G and a selection S: $G = (V, A \cup S)$, a disjunctive arc a to remove from the selection.

Output: A modified graph G, corresponding to a modified selection S, which is similar to the input S but does not contain the arc a.

```
1  Queue<Arc> Q = new BucketQueue()
2  let ā be s.t. ā ∈ G.SwapInMate(a) and t(ā) ∈ J
3  Q.Add(ā, priority = any)
4  int[] P = new int[G.Count]
5  Initialize P: P[o_jk] = { k, if j = J
                           { 0, otherwise
6  bool[] B = new bool[G.Count]
7  Initialize B: B[i] = { true,  if t_i > max_{b∈δ(J)}{ t_h(b) }
                        { false, otherwise
8  while Q.Count> 0 do
9      Arc a = Q.Pop()
10     if not G.ArcExists(a) then
11         continue
12     else if B[h(a)] then
13         continue
14     else if h(a) ∈ J and P[t(a)] ≥ P[h(a)] then
15         foreach Arc b ∈ G.SwapInMate(a) do
16             if P[t(b)] > 0 then
17                 Q.Add(b, priority = P[t(b)])
18             end
19         end
20     else if P[t(a)] > P[h(a)] then
21         P[h(a)] = P[t(a)]
22         foreach Arc b ∈ G.OutgoingArc(h(a) do
23             Q.Add(b, priority = P[t(b)])
24         end
25     end
26 end
```

Algorithm 4 shows how the termination criterion can easily be incorporated into the previous version of the left-closure (line 6, 7, 12, 13). With this we conclude the discussion of the left-closure.

Remark 3.5. *All of the optimizations discussed here also apply to the right-closure. Here, I only remark that:*

- *The direction of the path search is reversed.*
- *The priorities of operations in J are reversed.*
- *The termination criterion must be defined as a minimum of incoming arcs and the definition of A and B is reversed.*

3.7 Updating the Entry Times

Next, I present a simple way to improve the effectiveness of the recalculation of the entry times (vector t) after we modified the graph $(V, A \cup S)$. This concludes this Chapter, where we focus on improvements of the method to solve the problem $J_m \mid \text{block} \mid \gamma$. Thereafter, in the next Chapter, we will adopt the method to the SBB challenge.

We recall the standard way to compute the entry times: The entry time of the dummy vertex σ remains, $t_\sigma = 0$. The rest of the graph is traversed in topological order and the entry times are updated as follows:

$$t_v = \max\{\ t_u + l(u,v) \mid (u,v) \in (A \cup S)\ \}$$

While this is very simple to compute it has the drawback that we have to recompute entry times of the complete graph, even though most neighbours generated by the algorithm are close to the original selection and have, for a large part of the graph, identical entry times. I suggest a simple improvement: When the graph is modified we store vertices in a priority queue Q such that the following invariant holds:

Definition 3.2 (Update times, Invariant)**.**

- The entry times of all vertices topologically before all elements of Q remain correct.
- The entry time of a vertex in Q is correct, if there is no other vertex in Q that topologically precedes the first vertex.

The priority with which a vertex is inserted into Q corresponds to its (possibly wrong) entry time. The modifications necessary to the graphs methods *addArc()* and *removeArc()* are simple. Algorithm 5 shows how the entry times are updated.

Algorithm 5: Update Entry Times

Input : A graph $G = (V, A \cup S)$ and a priority queue Q. The vector t of entry times of all operations. (Precondition: Invariant, Definition 3.2)
Output: A updated vector t of entry times.

```
1  while Q.Count > 0 do
2  |  Vertex current = Q.Pop()
3  |  foreach Arc a ∈ G.OutgoingArcs(current) do
4  |  |  if t_h(a) < t_t(a) + l(a) then
5  |  |  |  t_h(a) = t_t(a) + l(a)
6  |  |  |  Q.Push(h(a))
7  |  |  else if t_h(a) > t_t(a) + l(a) then
8  |  |  |  Time newTime = max_{b∈G.IncomingArcs(h(a))}{ t_t(b) + l(b) }
9  |  |  |  if not t_h(a) = newTime then
10 |  |  |  |  t_h(a) = newTime
11 |  |  |  |  Q.Push(h(a))
12 |  |  |  end
13 |  |  end
14 |  end
15 end
```

Remark 3.6. *A proof of termination and correctness of Algorithm 5 is simple. Correctness: the invariant defined above (precondition) is also a loop-invariant of the algorithm, and termination with $Q = \emptyset$ proofs correctness. Termination: $t_{current}$ is monotone increasing.*

On larger SBB instances the time necessary to recompute the entry times in the standard way made up approximately 30% of the computation time. With the modification described, this drops to 0.5%.

4 Adaptation to the SBB challenge

In the previous Chapter we established the SBB challenge to be of the following problem type:

$$J_m \mid \text{block},\ r_{Jk},\ d_{Jk},\ \ s_i,\ \text{rcrc},\ \text{pm},\ \text{route},\ \text{prec} \mid \sum w_{Jk}T_{Jk}.$$

These additional constraints fall into one of three groups:

- Elements that can be incorporated into the method naturally and without any problems at all: $s_i, r_{Jk}, d_{Jk}, \sum w_{Jk}T_{Jk}$.
- Elements that change the data structures and implementation somewhat, but pose no theoretical challenges: rcrc, pm.
- route and prec: Here, the implementation is tricky.

I will discuss the 'complications' in the order mentioned above:

s_i, machine-dependent setup times: In the Section 1.4 we established that the length of a disjunctive arc is 0. Instead we simply use the machine-dependent setup time as arc label.

r_{Jk}, an operation o_{Jk} cannot start before r_{Jk}. We add arcs from σ to o_{Jk} of length r_{Jk}. The indegree of σ remains 0, hence these arcs cannot be part of a cycle.

d_{Jk}, due time of operation o_{Jk}, see $\sum w_{Jk}T_{Jk}$

$\sum w_{Jk}T_{Jk}$, the weighted tardiness objective. No modification is necessary, recall how we defined critical arcs (Section 1.3) and the local search (Section 2.7).

4.1 *rcrc*, Recirculation and Multi-Operation Machine Use

If a machine m is used by consecutive operations o_1, o_2 of the same job, we would have to insert the arc $(succ(o_1) \to o_2) = (o_2 \to o_2)$ of length s_m, which renders the graph immediately cyclic. Instead, we have to make the following adaptation: assume e is the disjunctive arc which hands over machine m after o_2 to another job ($t(e) = succ(o_2)$), then we have to ensure that $h(\bar{e}) = o_1$. The implementation changes: To avoid ubiquitous iteration through a job to find the start or end of a group of operations which share a machine, these groups or blocks of machine usages are made the core datatype on which we operate. In the code I call such a group a *MachineOccupation*.

4.2 *pm*, Parallel Machines

In the SBB problems, an operation's use of multiple machines is ever-present and as a result, multiple machines are often transferred to other jobs after a given operation. I opted for the one-to-one correspondence between arcs and machine hand-overs, which leads to parallel arcs (disjunctive arcs are annotated with a machine). Alternatively, we could avoid parallel arcs. The motivation for the chosen approach (include parallel arcs) is:

- Ease of implementation: no need to query / compare the machine usages of the head and tail operations. Arcs can never get lost (ie. they merge, as a job that only uses one of the two machines is scheduled differently, but later fail to split), which could lead to bugs that are very difficult to track down. The merging and splitting of arcs would necessitate keeping track of/computing setup times, which is otherwise 'free'.
- The one-to-one correspondence between arcs and the 'conceptual' nodes of the conflict graph continues to hold. If this correspondence were to break down, we would perform multiple steps in the conceptual conflict graph at once. The algorithm would become more difficult to reason about.
- Replacing one of two parallel arcs leads to a cycle which is detected immediately. Hence the other one is also directly replaced by its mate, with minimal overhead.

4.3 *route*, Routing Possibilities

To incorporate routing possibilities I used a very simple and probably insufficient approach. It is easy to delete a job from a selection. We can include a job J into a solution: insert all its machines into the machine sequences: That is, pick and remove an disjunctive arc e present in the disjunctive graph approximately where we want to insert our job. Add the two arcs necessary $(t(e) \to J,\ J \to h(e))$. Then, take the left-closure of all the left arcs or take the right-closure of all the right arcs. The improved left-closure algorithm (algorithm 4) can handle this very effectively: we initialize Q to include all the 'left arcs'.

To choose the approximate position of the job J I use the following heuristic: each machine is positioned in the machine sequence as late as possible so that the inserted job is on time. Then a left-closure is used: The left-closure schedules J earlier. This way we guarantee that J itself does not cause any delay penalty. The 'dual' of this approach would be to schedule J as early as possible and then use a right-closure. This has not been tried.

For the case when J was already present in the solution before (i.e., a route swap) I also implemented the swap such that the machines (and their disjunctive arcs) used by both routes remain unchanged.

4.4 *prec*, Precedence Constraints or Connections

The connections, or precedence constraints between operations of different jobs are the last complication posed by the SBB challenge. I tried to model the connection problem as follows: Assume o_1 of J_1 has to precede o_2 of J_2 by t_{conn}. A dummy machine is then added such that a disjunctive arc pair $\{(o_1, o_2), (\text{succ}(o_2), \text{predecessor}(o_1))\}$ is added to $\mathcal{E}$. The length of the disjunctive arcs is set to t_{conn}. The two arcs signify: 'the connection is met' and 'the connection is missed'. If the 'the connection is missed' arc is included in a solution a penalty is applied. Currently this penalty consists of a constant added to a value depending linearly on the time difference $t_{o_2} - t_{o_1}$. Finally, to improve solutions we add 'critical arcs' if a connection is missed: First the arc $(\text{succ}(o_2), \text{predecessor}(o_1))$ is considered critical. Second, disjunctive arcs along longest paths $\sigma \to o_1$ are considered critical.

It should be noted that this changes the semantics of the problem, as due times are the only soft constraints of the SBB problem, while this way, precedence constraints are also modelled as soft constraints.

5 Results & Conclusion

5.1 Validation Experiment

The algorithm as described above was run on two sets of problem instances. In addition to the SBB challenge (see below), those are the 40 blocking-job-shop problem ($J_m \mid \text{block} \mid C_{\max}$) instances proposed by Lawrence [2]. This is done to validate the implementation of the algorithm. These results can be compared to Pranzo & Pacciarelli, 2016 [22], who also provided results for the blocking no-swap problem (as is solved by the implemented algorithm), see Figure 6. I computed 40′000 iterations of the tabu search for each problem instance, while Pranzo & Pacciarelli stopped their algorithm after 60 seconds. My implementation took approximately 3 hours on a 3.2 GHz processor for all 40 instances combined. The computation power is therefore roughly comparable to Pranzo & Pacciarelli ($40 \times 10 \times 60\text{s} = 6\text{h}40\text{m}$, 3.0Ghz) [22]. I conclude that the validation experiment is a success (Figure 6): Even though the algorithm is implemented to deal with hundreds of jobs and a number of additional constraints, the results indicate that the implementation is at least competitive and compares advantageously when dealing with slightly larger ($\text{la}31 - 35$) problem instances.

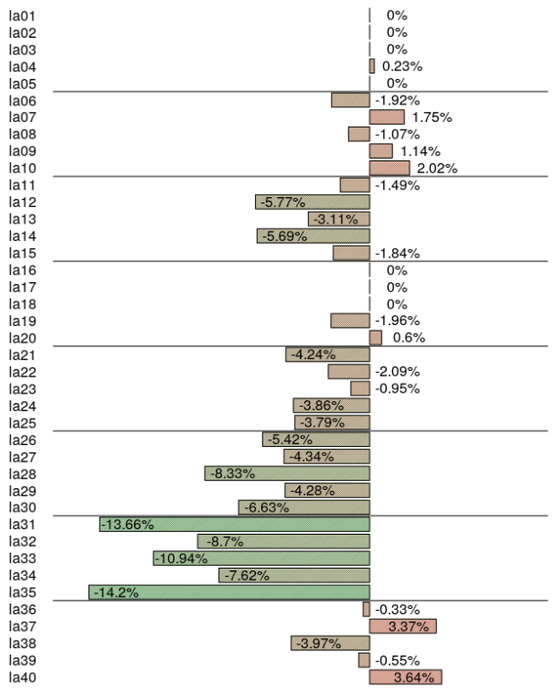


Figure 6: Results obtained by this implementation compared to Pranzo & Pacciarelli [22]: Relative difference of the objective value ($C_{\max}$) is reported. The best result of [22] corresponds to 100%. Problem sizes ($n \times m$) are: $\text{la}01 - 05$, 10×5; $\text{la}06 - 10$, 15×5; $\text{la}11 - 15$, 20×5; $\text{la}16 - 20$, 10×10; $\text{la}21 - 25$, 15×10; $\text{la}26 - 30$, 20×10; $\text{la}31 - 35$, 30×10; $\text{la}36 - 40$, 15×15.

5.2 SBB Challenge

Problem Instances & Results

For the challenge the SBB [28] provided a set of 9 problem instances of the type

$$J_m \mid \text{block},\ r_{jk},\ d_{jk},\ \ s_i,\ \text{rcrc},\ \text{pm},\ \text{route},\ \text{prec} \mid \sum w_{jk} T_{jk}.$$

These vary significantly in size and complexity (see Table 1). Largely, four groups can be differentiated.

(i) The first instances $1 - 4$ are relatively small (up to 148 jobs) and can be solved easily. It is worth noting that no competitor was able to find a solution with 0 cost for the fourth instance, but solutions corresponding to a cumulative delay of 0.08 minutes are easily found.

(ii) The 5^{th} instance 'with obstruction' is the hardest to solve and resulted being the sole criterion to separate between the top 5 entries in the SBB leaderboard.

The other instances–with the exception of 4–are solved to 0 cost by the best 5 algorithms submitted. To solve instance 5, the routing seems to be important, while precedence constraints are satisfied by all better solutions. Since I focused on this instance, I did not develop the incorporation of precedence constraints further.

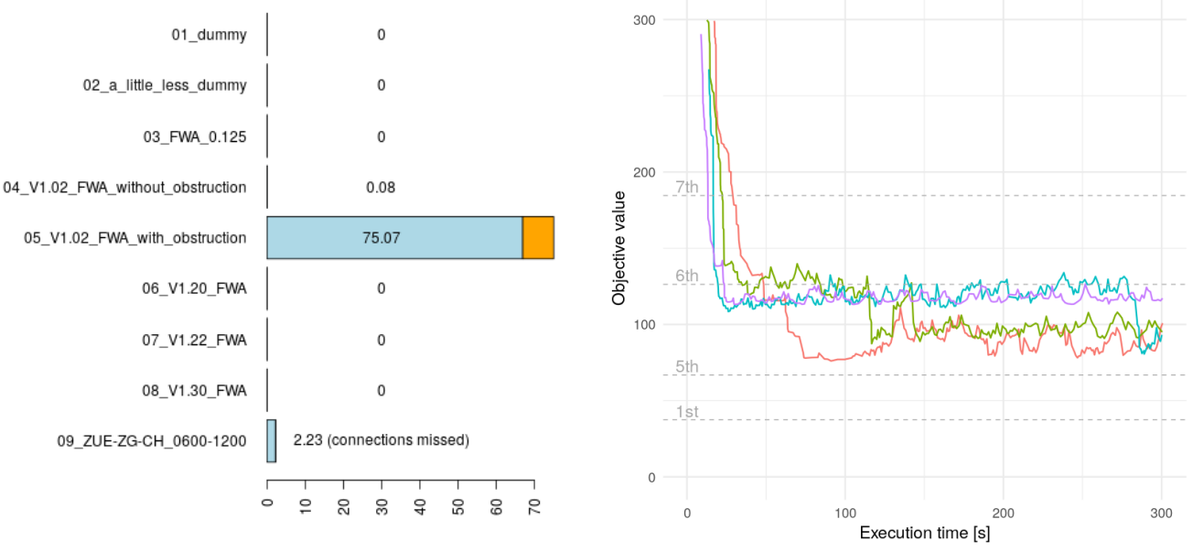
Instance	#Jobs	Average #Operations / Job	Average #Routes / Job	#Connections
1	4	73.25	2	0
2	58	75.05	1.10	2
3	143	61.62	1.10	22
4	148	75.24	1.46	31
5	149	74.91	1.46	31
6	365	93.60	8.38	25
7	467	96.50	8.53	25
8	133	101.69	53.27	0
9	287	125.65	208.40	734

Table 1: Key indicators of the problem instances provided by the SBB.

(iii) Instances $6-8$ range in size from 133 to 467 jobs. The number of possible routes is larger, but the instances are otherwise easy to solve. Even while restricting the routing to a randomly chosen route, a solution of 0 cost is often found. The precedence constraints (connections) seem to be fulfilled by all solutions of acceptable quality and do not need to be enforced.

(iv) The last instance 9 contains many more precedence constraints, some of which are hard to satisfy. The routing choices of many jobs are too numerous for our simplistic route selection approach.

Figure 7 (a) displays the best results obtained with my implementation. Note, that the solution produced for instance 9 is not feasible, as I disregarded connection constraints. Adding the respective $10'000$ points penalty the present results would be rated 9^{th} rank in the long concluded competition (including post-challenge submissions). Restricting to instance 5, the algorithm ranks 6^{th}. Though it should also be mentioned that results of quality equal to the algorithm ranked 6^{th} is generally available after $30-90$ seconds (single core, 3.2 GHz, see Figure 7 (b)).



(a) (b)

Figure 7: (a) Penalties of the results obtained: The blue bars correspond to the delay penalty. The route-choice penalty is displayed in orange. The best results submitted to the SBB accumulate a cost of 37.45 for instance 5 and a cost of 0.1 for instance 4.
(b) Shows the evolution of the objective value while the algorithm is creating a solution for problem instance 5. Each colored line corresponds to a run of the algorithm. Dashed horizontal lines indicate the objective values of solutions submitted to SBB by competitors. The algorithm uses a single core at 3.2 GHz. The plot is cropped: initial solutions have objective values ~ 4800.

Conclusion

Given the results outlined above, I conclude that our approach was partially successful. The results are good enough to warrant the further development of our approach for problems of the given type. However, the results also show that the method has shortcomings. In the next and last Section, I try to identify some shortcomings and provide three ideas to improve the method.

5.3 Future Improvements

The precedence constraints of the connections and numerous routing possibilities clearly require more sophisticated treatment. First, I focus on the core method and the associated $J_m \mid \text{block} \mid \sum w_{Jk}T_{Jk}$ problem. Two problems are identified:

(i) The search is confined to solutions relatively similar to each other. A naive remedy is to restart the algorithm several times, see e.g., [22] where the authors restart an algorithm 10 times instead of allocating more computation time to the algorithm.

(ii) Even solutions close to the current solution are not explored, since multiple steps have to be taken, where the intermediate steps make the solution worse.

To address *(i)* I propose the idea outlined in the next Paragraph *Diversification. Secondary Critical Arcs* outlines an idea to address the problem *(ii)*.

Diversification

A good approach would consist in the generation of a large number of initial solutions, which would then be refined with a local search. A key consideration here is to guarantee that the initial solutions are sufficiently diverse. I am convinced that the Kendall-τ rank distance would be a good starting point to quantify the distance between (initial) solutions. The Kendall-τ distance counts the swaps necessary to transform one permutation into another. Applying this to the sequences in which jobs use machines defines a distance, which is consistent with the algorithm. Each move to a neighbor solution boils down to multiple calls of *SwapInMate()*, each of which corresponds to a swap in job-permutation of the respective machine. A simple implementation (based on merge-sort) computes the Kendall-τ distance in $O(n \log n)$ time, where n is the length of the permutation.

Secondary Critical Arcs

The neighbours we generate swap out a single critical arc, without considering the potential gain of doing so. The potential gain, or the maximum reduction in delay of removing a critical arc, could easily be calculated. Assume job J_A is delayed significantly due to a critical arc from a job J_B. But, removing the critical arc $J_B \to J_A$ would immediately make another arc $J_C \to J_A$ critical and the decrease in delay would be marginal. Furthermore, it is likely that the sequence of removing $J_B \to J_A$, then removing $J_C \to J_A$ is not explored, as it is likely that the intermediary solution is worse (since there is not much to gain from the arc swap).

Aiming to improve an early version of the algorithm of Cesta & Oddi, Michel & Van Hentenryck [9] came to a similar conclusion: Their algorithm is based on iteratively applying 'destruction' and 'construction' phases (see Section 1.5, Paragraph Amcc Algorithm). In a destruction phase some critical arcs are removed. Michel & Van Hentenryck predicted and observed that redoing multiple smaller destruction phases would lead to an improvement of the algorithm.

I assume that an extension and more in-depth analysis of the critical subgraph is warranted. I.e., we could define: an arc a is *relevant* if the removal of critical arcs lead to a being critical. Relevant arcs are ordered in a hierarchy: arcs necessary to turn an arc into a critical arc are predecessors of said arc. We could then choose relevant arcs which correspond to a chain in the hierarchy such that the potential gain is maximized and the number of arcs selected is minimized (or, select at most 2, 3 arcs). A neighbour of a current solution is then defined as the removal of all arcs selected in such a way.

Route Selection

In this last paragraph, I identify a problem on how we deal with the 'route' processing characteristic and propose a simple augmentation to the method. A problem with the routing, as implemented, is that a large number of route choices cause difficulties, as they must be re-explored at every step in the search. So far, I tried to restrict the route changes to changes, such that operations along critical paths are affected. Another simple idea on how to reduce the number of route options is proposed below.

For the purpose of illustration I will use notation borrowed from the study of regular languages: if x and y are track segments then $x \circ y$ means x is traversed before y, and $x \mid y$ means that we can either traverse x or y. A language defined only by *concatenation* $(. \circ .)$ and *alternative* $(. \mid .)$ describes all routing alternatives.

Example 5.1. *The language L defines a possible set of routes for a given job. Every word $w \in L$ corresponds to a specific route:*

$$L = a \circ \Big(b \mid \big(c \circ (d \mid e) \circ f\big)\Big) \circ (g \mid h) \circ i.$$

Assume, the job runs currently along the route $acdfgi$. The remaining routes are

$$\{\ \mathbf{acefgi},\ \mathbf{abgi},\ acefhi,\ abhi,\ \mathbf{acdfhi}\ \}.$$

The current implementation would explore all of them.

I propose that we limit the route choice neighbours at a step in the local search to route changes which 'flip' a single alternative in the language formula. In the example above, the proposed neighbour routes correspond to the routes listed in bold. This way, we still guarantee that all routing possibilities are eventually reachable. If this is effective is apriori not clear. If our route language is of the type

$$a_1 \mid a_2 \mid \ldots \mid a_n$$

we would still need to explore $n-1$ neighbours. However, if the language is of the type

$$(a_1 \mid b_1) \circ (a_2 \mid b_2) \circ \ldots \circ (a_n \mid b_n)$$

we would need to explore n instead of $2^n - 1$ neighbours.

References

[1] Akers Jr SB (1956) Letter to the editor—A graphical approach to production scheduling problems. *Operations Research* 4(2) 244-245.

[2] Lawrence S (1984) Resource constrained project scheduling: An experimental investigation of heuristic scheduling techniques (Supplement). *Graduate School of Industrial Administration, Carnegie-Mellon University.*

[3] Nemhauser GL, Wolsey LA (1988) Integer and combinatorial optimization. *John Wiley & Sons.*

[4] McCormick ST, Pinedo LM, Shenker S, Wolf B (1989) Sequencing in an assembly line with blocking to minimize cycle time. *Operations Research*, 37(6) 925-935.

[5] Carlier J, Pinson E (1994) Adjustment of heads and tails for the job-shop problem. *European Journal of Operational Research*, 78(2) 146-161.

[6] Hall NG, Sriskandarajah C (1996) A Survey of Machine Scheduling Problems with Blocking and No-Wait in Process. *Operations Research*, 44(3) 510-525.

[7] Mascis A, Pacciarelli D (2000) Machine Scheduling via Alternative Graphs. *RT-DIA-46-2000.*

[8] Mascis A, Pacciarelli D (2002) Job-shop scheduling with blocking and no-wait constraints. *European Journal of Operational Research*, 143(3) 498-517.

[9] Michel L, Van Hentenryck P (2004) Iterative Relaxations for Iterative Flattening in Cumulative Scheduling. *ICAPS*, 4 200-208.

[10] D'Ariano A, Pacciarelli D, Pranzo M (2007) A branch and bound algorithm for scheduling trains in a railway network. *European Journal of Operational Research*, 183(2) 643-657.

[11] Gröflin H, Klinkert A (2007) Feasible insertions in job shop scheduling, short cycles and stable sets. *European Journal of Operational Research*, 177(2) 763-785.

[12] Mazzarello M, Ottaviani E (2007) A traffic management system for real-time traffic optimisation in railways. *Transportation Research Part B: Methodological*, 41(2) 246-274.

[13] Gröflin H, Klinkert A (2009) A new neighborhood and tabu search for the blocking job shop. *Discrete Applied Mathematics*, 157(17) 3643-3655.

[14] Mati Y, Xie X (2010) Multiresource shop scheduling with resource flexibility and blocking. *IEEE transactions on automation science and engineering*, 8(1) 175-189.

[15] Oddi A, Cesta A, Policella N, Smith SF (2010) Iterative flattening search for resource constrained scheduling. *Journal of Intelligent Manufacturing*, 21(1) 17-30.

[16] Oddi A, Rasconi R, Cesta A, Smith SF (2011) Iterative flattening search for the flexible job shop scheduling problem. *In Twenty-Second International Joint Conference on Artificial Intelligence.*

[17] Oddi A, Rasconi R, Cesta A, Smith SF (2012) Iterative improvement algorithms for the blocking job shop. *In Twenty-second international conference on automated planning and scheduling.*

[18] Cacchiani V, Huisman D, Kidd M, Kroon L, Toth P, Veelenturf L, Wagenaar J (2014) An overview of recovery models and algorithms for real-time railway rescheduling. *Transportation Research Part B: Methodological*, 63 15-37.

[19] Lamorgese L, Mannino C (2015) An exact decomposition approach for the real-time train dispatching problem. *Operations Research*, 63(1) 48-64.

[20] Dabah A, Bendjoudi A, AitZai A (2016) Efficient parallel B&B method for the Blocking Job Shop Scheduling Problem. *International Conference on High Performance Computing & Simulation (HPCS)*, 784-791.

[21] Pinedo ML (2016) Scheduling: Theory, algorithms, and systems. Fifth Edition, *Springer International Publishing.*

[22] Pranzo M, Pacciarelli D (2016) An iterated greedy metaheuristic for the blocking job shop scheduling problem. *Journal of Heuristics*, 22(4) 587-611.

[23] Bürgy R (2017) A neighborhood for complex job shop scheduling problems with regular objectives. *Journal of Scheduling*, 20(4) 391-422.

[24] Bürgy R, Gröflin H (2017) The no-wait job shop with regular objective: a method based on optimal job insertion. *Journal of Combinatorial Optimization*, 33(3) 977-1010.

[25] Dabah A, Bendjoudi A, AitZai A (2017) An efficient Tabu Search neighborhood based on reconstruction strategy to solve the blocking job shop scheduling problem. *Journal of Industrial & Management Optimization*, 13(4) 2015-2031.

[26] Samà M, D'Ariano A, Corman F, Pacciarelli D (2017) A variable neighbourhood search for fast train scheduling and routing during disturbed railway traffic situations. *Computers & Operations Research*, 78 480-499.

[27] Dabah A, Bendjoudi A, AitZai A (2019) Efficient parallel tabu search for the blocking job shop scheduling problem. *Soft Computing*, 23(24) 13283-13295.

[28] Train Schedule Optimisation Challenge. SBB CFF FFS. Crowd AI. `https://www.crowdai.org/challenges/train-schedule-optimisation-challenge` (Accessed June, 3rd, 2020)