Lyon 1

# Mesh and Computational Geometry

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Master Informatique
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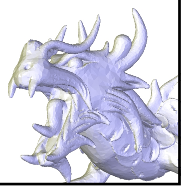

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## Triangulation of 2D points

- What if the Delaunay triangulation does not comply the boundaries of the domain or a network of constraints?
- Constraint : a segment to be inserted into the triangulation

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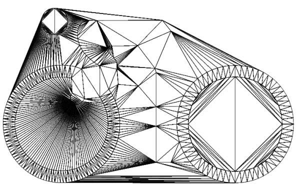
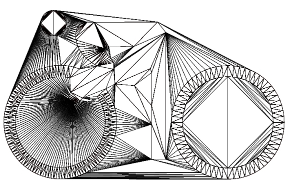
## Case of a digital 2D or 2D½ model

- Unconstrained Triangulation
- Constrained Triangulation





Images by B. Kornberger

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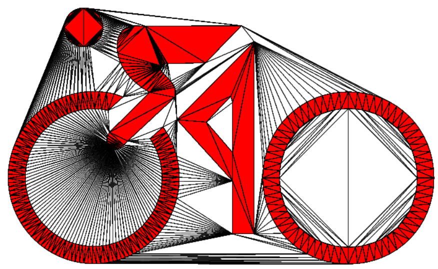
## Case of a digital 2D or 2D½ model

- Constrained triangulation



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## Constrained triangulation of 2D points

- What if the Delaunay triangulation does not comply the boundaries of the domain or a network of constraints?
- We insert the constraints into the triangulation
  - Using the *flip* operation
    - Flip of edges intersecting the constraint edges
    - Orient the constraint edges to set an order for the flip of the intersecting edges

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## Constrained triangulation of 2D points

- What about the non constrained edges?
  - Slightly modify the (locally)-Delaunay Criterion
    - A constraint is seen as a wall stopping the visibility
    - A vertex can be located inside a circumscribed circle if it is not visible from inside the triangle
  - Flip any un-constrained edge that is non locally Delaunay

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## Incremental Constrained triangulation of 2D points

- Incremental insertion of a point in a constrained 2D triangulation
  - Constrained edges cannot be flipped
  - Slightly modify the (locally)-Delaunay Criterion
    - A constraint is seen as a wall stopping the visibility
    - A vertex can be located inside a circumscribed circle if it is not visible from inside the triangle

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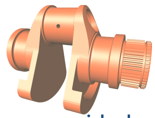
## 3D Shape Modelling

- How to obtain a mesh?
- Input data
  - Range images
  - Volumetric voxel images
  - CAD/CAM



  - Dirty meshes provided by graphic designers
  - 2D Images + stereovision

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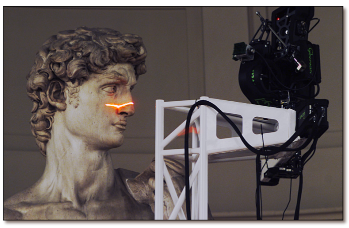
## 3D objects digitization



Laser scanner (Michelangelo Project)

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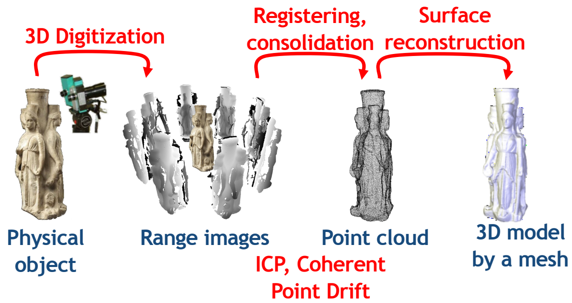
## From 3D digitization to surface reconstruction




Art3D project (ANRT)

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## Shape Reconstruction

- Problem of 2D reconstruction
  - Reconstruct a curve from 2D input point samples on that curve
- Problem of 3D reconstruction
  - Reconstruct a surface from 3D input point samples on that surface

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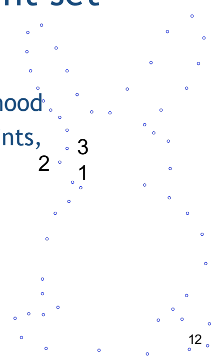
## Mesh generation from input point set

- 2D reconstruction
  - A problem from your childhood
  - Input: a finite set of 2D points, with numbers or not

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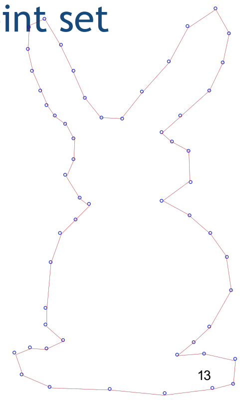
# Mesh generation from input point set

- 2D reconstruction
  - Output: A polygonal curve of the contour of the shape
  - … thanks to your pencil, your intelligence and your dexterity!



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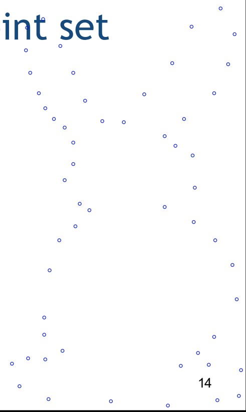
# Mesh generation from input point set

- 2D and 3D reconstruction
  - Now we remove the numbers!
- 3D reconstruction
  - How to generalize an algorithm to reconstruct a triangulated surface from input 3D points?



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# Mesh generation from input point set

- 3D Reconstruction
  - Approximation of a surface from a point set
  - Different approaches characterized by a more or less global knowledge of the surface to be produced
  - Numerous algorithms in computational geometry

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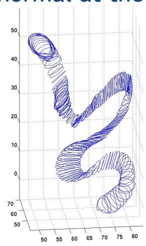
- Points 3D and additional information
  - X-ray scanner, ultrasound, MRI, PET, stereovision
  - Structured (depth images) or not
  - Possibility of an organization into slices
  - Additional information : normal at the surface



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# Mesh generation from input point set

- 2D and 3D Reconstruction
  - How to build an approximation of the shape based on the input points?
  - In what cases can we guarantee the geometric and topological validity of the surface?
  - Which additional assumptions should we provide on the sampling of the unknown surface?

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# Mesh generation from input point set

- Be careful :
  - We know how to build a 2D mesh (composed of triangles) from 2D data
  - We know how to build a 3D mesh (composed of tetrahedra) from 3D data
  - But we do not know how to build
    - A 1D mesh (composed of segments) from 2D data sampled on a curve!!
    - a 2D mesh (composed of triangles) from 3D data sampled on a surface !!!

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# Case of a digital terrain model

- Data can be parameterized as a height function with respect to a reference plane (2D ½ dimension)
- Delaunay triangulation of the points projected on the reference plane
  - Triangulation maximizing the angles of the triangle projected on the 2D plane, not the angles of the 3D triangles

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# Case of a digital terrain model

- How to better consider the height information?
  - Find the triangulation maximizing the angles of the 3D triangles
    - Remark : each pair of adjacent triangles can locally be flattened with a preservation of the angles
    - It is possible to reuse Lawson's flipping algorithm, without guaranteeing a global optimum
    - Non Delaunay edges $\Pi - (\alpha + \beta) < 0$

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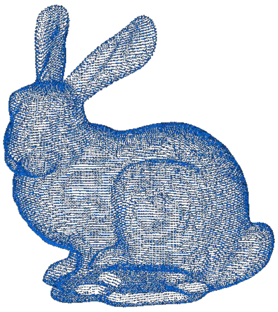
# Mesh generation from 3D point set

- Reconstruction 3D
  - Idea : link points that are close
  - What is a close point?



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# Mesh generation from 3D point set

- Reconstruction 3D
  - Idea : link points that are close
  - … This is partly what is done by 3D Delaunay, which triangulates the volume delimited by the convex hull with 3-simplexes (tetrahedra)!

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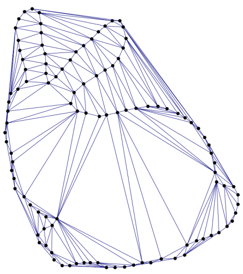
# Mesh generation from 2D or 3D point set

- 2D Reconstruction
  - Look for the curve as a sub-graph of the 2D Delaunay triangulation
  - Provided it should be present in it!
- 3D Reconstruction
  - Look for the surface mesh as a sub-graph of the 3D Delaunay triangulation



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# In the following…

- 2D reconstruction
  - Reconstruction of a 2D curve as a subset of the edges of a 2D Delaunay triangulation
- … to better understand what is happening in 3D
  - Reconstruction of a 3D surface as a subset of the triangular facets of a 3D Delaunay triangulation

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## In the following…

- We will distinguish
  - The original shape on which the input points have been sampled and which one seeks to approximate
  - The provided samples
- Care should be taken to maintain an equivalence between continuous and discrete notions

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## One of the first reconstruction algorithm

- CRUST (Amenta et al)
  - Algorithm provided with necessary and sufficient conditions on the sampling density to guarantee the result of the reconstruction
  - Minimal Sampling density characterized by using the « Local Feature Size » : distance from the surface points to the skeleton (medial axis) of the shape
  - Skeleton : set of maximal balls centers approximated by Voronoi Centers

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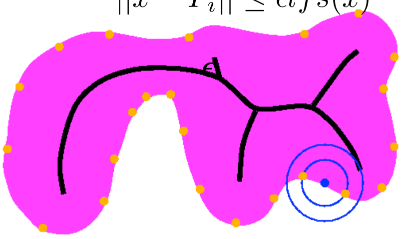
## 2D/3D Reconstruction

- Given ε, an **ε-sampling** of a shape is a set of samples $P_i$ such that for each x there is a i such that $||x - P_i|| \leq \epsilon lfs(x)$



Sampling density locally proportional to 1/lfs
Measure of the thickness and of the curvature

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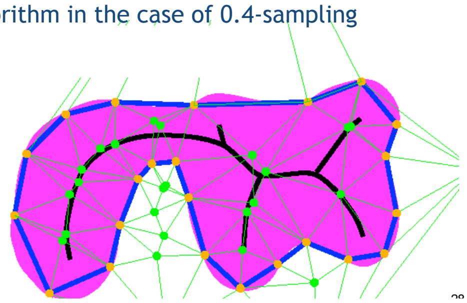
## 2D Reconstruction

- Guarantees provided by Crust
  - Adjacent points on the curve are linked by the algorithm in the case of 0.4-sampling



But additional points may also be linked.....

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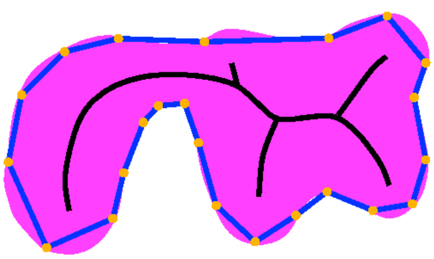
## Reconstruction

- Crust provides guarantees
  - Homoeomorphism of the curve and its approximation in the case of a 0.25-sampling



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## An other simple reconstruction algorithm

- Alpha-shape
- Ball pivoting
- Those approach require a sampling denser than a uniform density
- Easy to parallelize

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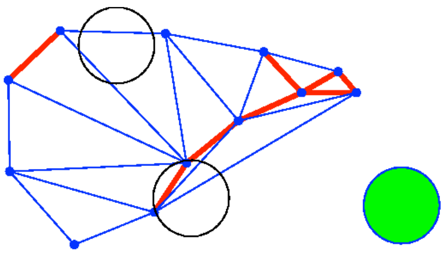
## 2D Reconstruction

- α-shape
  - pq is an edge of the α-shape if there exists an empty circle of radius 1/α that is passing through p and q



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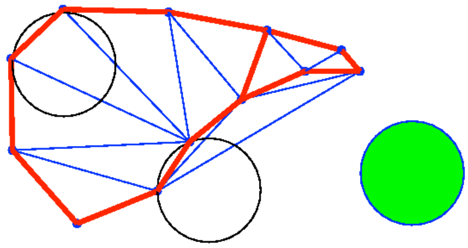
## 2D Reconstruction

- α-shape
  - pq is an edge of the α-shape if there exists an empty circle of radius 1/α that is passing through p and q



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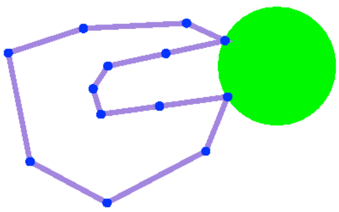
## 2D Reconstruction

- α-shape
  - If α is too small (circle too big) : pockets are closed



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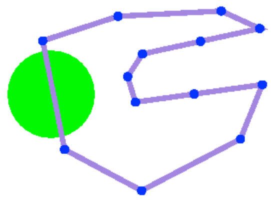
## 2D Reconstruction

- α-shape
  - If α is too big (circle too small) : the curve is opened



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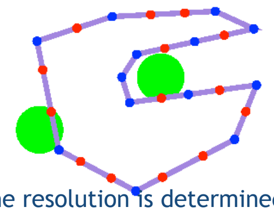
## 2D Reconstruction

- α-shape
  - Need for a sampling denser than a sufficiently fine uniform sampling



  - The resolution is determined using the radius to be used to enter in every pocket

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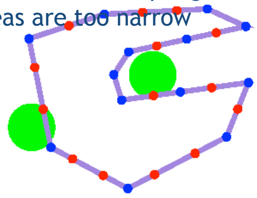
## 2D/3D Reconstruction

- α-shape
  - Need for a sampling denser than a sufficiently fine uniform sampling, but sometimes some areas are too narrow



- Algorithms also available in 3D
  - empty ball going passing through 3 points
- Ball pivoting (parallelization possibilities) 36

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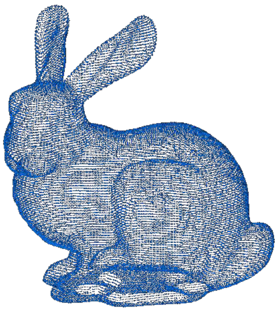
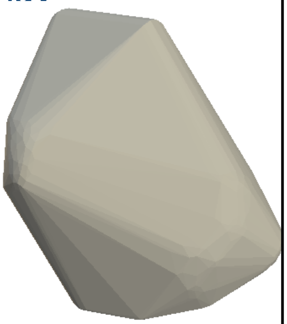
## Other algorithms based on Delaunay





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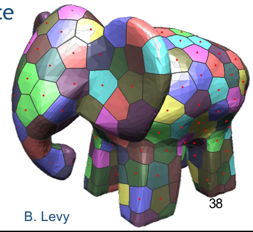
## Restricted Voronoi Diagram

- Given a set of points on a surface
  - Restricted Voronoi cell :
    - Intersection between a Voronoi cell and the surface
  - Can be used to construct triangles between points with adjacent restricted Voronoi cells



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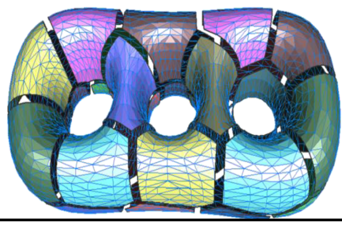
## Voronoi diagram restricted to a surface

- Restricted Voronoi diagram
  - Intersections between the 3D Voronoi cells and the surface of the original shape (if ever we have it!!)



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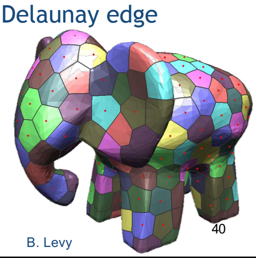
## Restricted Delaunay Triangulation

- Restricted Delaunay :
  - Defined by duality
  - Each adjacency between 2 restricted Voronoi cells results in a restricted Delaunay edge
  - Creation of triangles of restricted Delaunay by duality to a vertex of Restricted Voronoi



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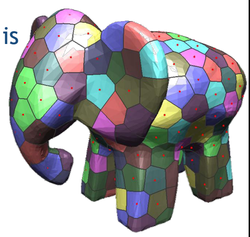
## Restricted Delaunay Triangulation

- Edelsbrunner and Shah's theorem[1997]
  - If each face of the Restricted Voronoi diagram is **homeomorph to a topological disc,** then the restricted Delaunay triangulation is homeomorphic to the unknown surface.



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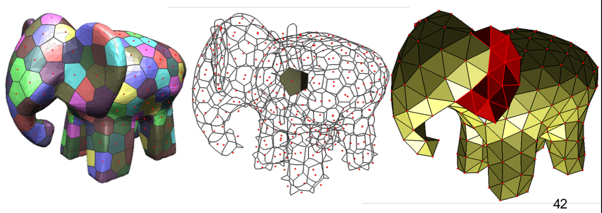
## Restricted Delaunay Triangulation

- Topological disk property not satisfied for insufficient sampling....



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## Restricted Delaunay Triangulation

- In presence of an ε-sampling with ε<0.1, the property of the topological disk is satisfied [Amenta and Bern 99]
- The problem is that we do not know the original surface, and we need it to build the restricted Voronoi cells and the associated dual triangulation...

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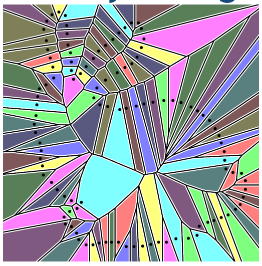
## What if the surface is unknown? Cocone : An other reconstruction approach using Voronoi diagram and Delaunay triangulation



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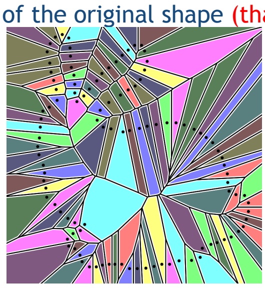
## Voronoi diagram restricted to a curve

- Restricted Voronoi diagram
  - Intersections between the 2D Voronoi cells and the curve of the original shape (that we do not know !!)



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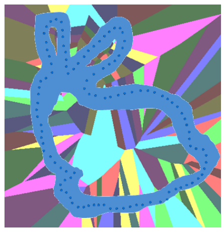
## Cocone

- Idea: find some kind of thickened version of the original surface and make a restricted Delaunay triangulation of this thickened version



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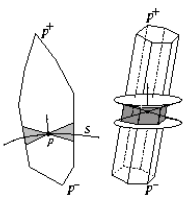
## Cocone

- Thickening of the unknown surface :
  - For this purpose, a Cocone is placed at each sampled point [Amenta et al 2000]
  - Positioning using the tangent plane
    - using unoriented normal or estimating it from Voronoi 3D cell
  - A triangle is created from every triplet of adjacent cocones
    - Residual filtering still needed (non manifoldness)

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## Other reconstruction approaches based on an approximation of the skeleton

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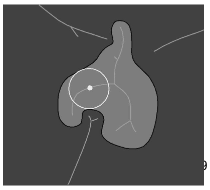
## Power Crust

- Observation :
  - any point of a compact surface is on the boundary of two maximal balls centered on the skeleton
  - an outer ball and an inner medial ball
- Use the idea that the inside of any closed surface is a union of balls



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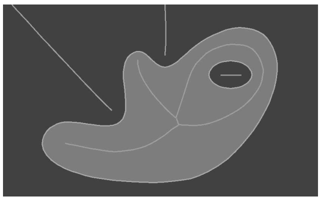
## Back to the medial axis/ skeleton

- In 2D :
  - Center of the maximal balls contained within the curve
  - All points that have more than one nearest neighbor on the curve



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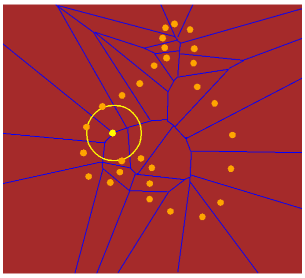
## Back to the medial axis

- 2D approximation :
  - Voronoi balls are the discreet equivalent of maximal balls



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## 2D reconstruction

- By approximating the medial axis
  - We can therefore approximate a 2D shape as a union of inner Voronoi balls
  - What we need to know is which balls are inside (resp. outside) or at least which ones have different signs

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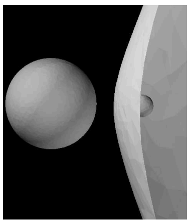
## Back to the Medial Axis

- In 3D:
  - The medial axis of a surface is a surface (possibly with pieces of curves)



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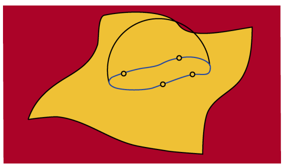
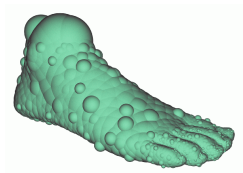
## 3D reconstruction by approximating the Medial Axis

- 3D approximation:
  - Beware of Voronoi balls that are centered on the surface (sliver)
  - Even for a good surface sampling





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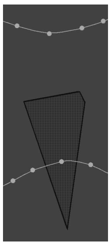
## Back to Voronoi

- In case of a dense and noise free sampling
  - Long and thin Voronoi cells,
  - Direction aligned with the normal to the surface
  - With extremities close to the medial axis



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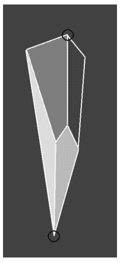
## Poles

- Poles
  - Voronoi vertices at the extremities of the elongated cells



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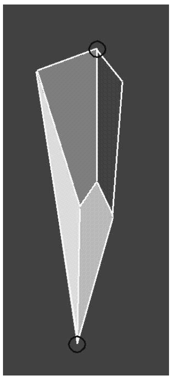
## Poles

- Approximation
  - Let $V_p$ be the Voronoi cell of a point p
  - Positive pole $p^+$ : Voronoï vertex of $V_p$ furthest from p.
  - Vector pole $pp^+$ : approximation of the normal direction at p.
  - The negative pole $p^-$ : vertex of $V_p$ furthest from $p^+$ in the direction opposite to the vector $pp^+$



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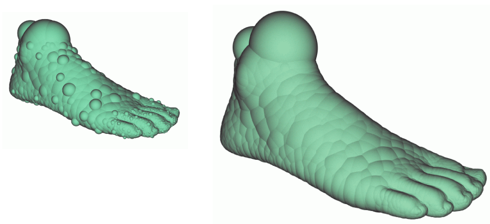
## 3D reconstruction by approximating the medial axis

- 3D approximation:
  - Retain only polar balls (centered on the poles)



Amenta and Bern 98

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## 3D reconstruction by approximating the medial axis

- 3D approximation:
  - Retain only polar balls (centered on the poles)
  - Yes, but which ones?

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## Power Crust

- It is difficult to know which polar balls are internal
- However, local criteria make it possible to distinguish polar balls of different natures
  - Two adjacent "deeply intersecting" polar balls are considered to be on the same side of the surface
  - Two "barely intersecting" polar balls are one internal and the other external

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## Power Crust

- Labelling of polar balls
- Global heuristics
  - "External" labelling of the poles incident to a large enclosing box
  - Propagation of labels:
    - For any pole p labelled "external".
      - each unlabelled neighbor q such that the polar balls associated with p and q intersect deeply is labelled "external".
      - For each point s of S considering p is the pole, the other pole is labelled internal.
    - For any pole labelled "internal".
      - Symmetrical work
  - Using a priority queue

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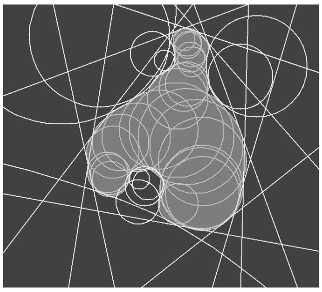
## Power Crust

- How to switch from a set of balls to a mesh?



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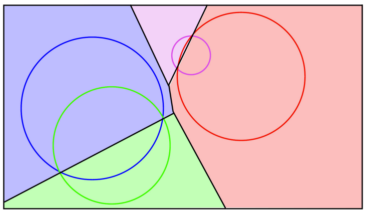
## Power Crust

- How to switch from a set of balls to a mesh?
  - Construction of a ball power diagram.



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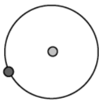
## Power diagram

- Can be seen as a Voronoï diagram of balls with an adhoc metric
  - Ball B of center c and radius r
  - Power distance of a point x with respect to B :

$$d_{pow} = d^2(c,x) - r^2$$



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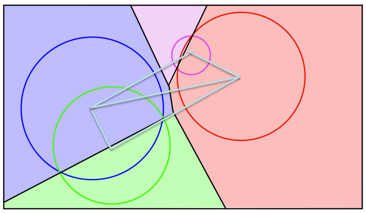
## Power Crust

- Note: The dual of a power diagram is a triangulation connecting the centers of the circles (Regular triangulation)



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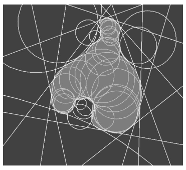
## Power Crust

- Construction of a power diagram using polar balls
- Reconstructed surface:
  - set of facets of the diagram whose dual edges link an inner pole and an outer pole.



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## Power-Crust Results

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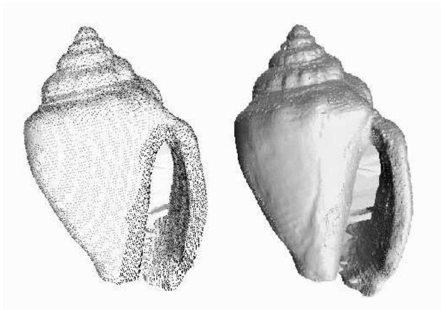
## Power Crust results



- Natural filling of the holes

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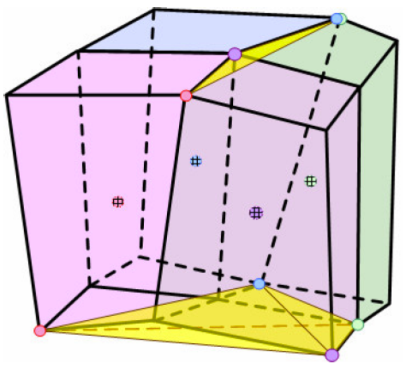
## Power-Crust Result



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## Power Crust

- Robustness:
  - The output mesh is the boundary of a solid (by construction) but does not only contain triangles
  - No surface extraction or hole filling steps
- Correctness:
  - Theoretical results that relate the geometric and topological validity of the result to the quality of the sampling

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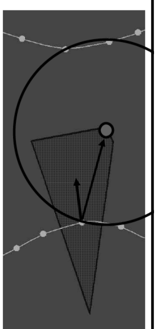
## Power Crust

- Correctness:
  - All the (wide) polar balls passing through a sample s are nearly tangential to the surface in s



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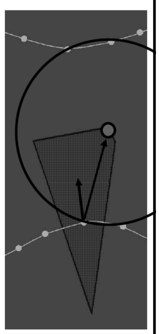
## Power Crust

- Given a ε- sampling of a surface S, the angle between the normal at S in s and the vector joining s to one of its poles is in O (ε)



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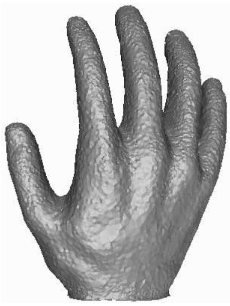
## Power Crust Results



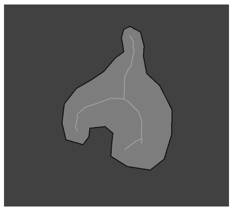
- Robustness to noise





## Power Crust

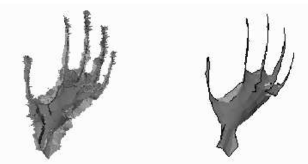
- Power crust also provides an approximation of the medial axis
- Connection of the inner poles with adjacent cells in the power diagram



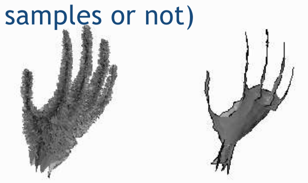




## Power Crust results



- Simplification of the skeleton (noisy samples or not)

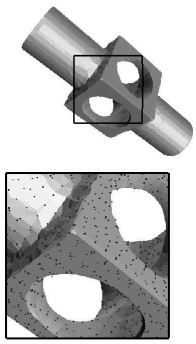






## Power Crust Results

- Sharp edges can be inferred

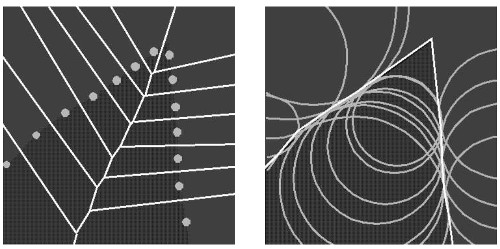






## Power Crust results

- Respect of sharp edges :
  - Ignore poles associated with malformed Voronoi cells

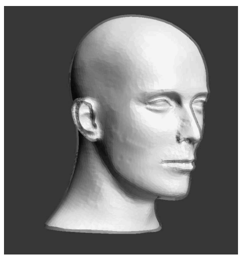






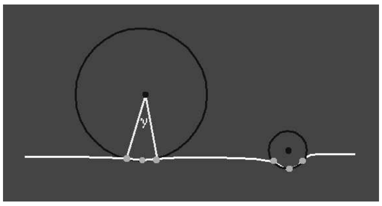
## Offset surface management

- Narrowing of the internal polar balls
- Widening of the external polar balls



## Handling noise

- By simplifying the medial axis
- Delete polar balls associated with vertices that are too close to the pole



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