Lyon 1

Mesh and Computational Geometry

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M2 ID3D
Image, Développement
et Technologie 3D
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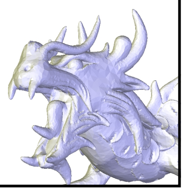

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Differential operators

- Laplace equation

$$f : U \in \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\Delta f = 0$$

- Solutions of Laplace equation :
  - Harmonic functions = kernel of the Laplacian operator
  - No local maxima or minima in U
  - Minimizing the Dirichlet energy that measures the smoothness of a function

$$E(f) = \int_U < \nabla f, \nabla f > dA$$

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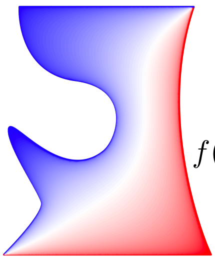
Differential operators

- Solutions of Laplace equation :
  - Used for interpolation



$$\Delta f(x) = 0 \quad x \in U$$
$$f(x) = f_0(x) \quad x \in \delta U$$

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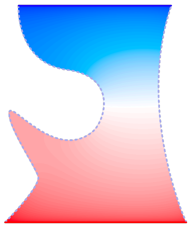
Differential operators

- Solutions of Laplace equation :
  - Used for interpolation



$$\Delta f(x) = 0 \quad x \in U$$
$$f(x) = f_0(x) \quad x \in \delta U_D \; (Dirichlet\, boundary)$$
$$\nabla f.n = g_0(x) \quad x \in \delta U_N \; (Neumann\, boundary)$$

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Differential operators

- Harmonic functions not to be misunderstood with the eigenvectors and eigen values of the Laplacian operator

$$\Delta f = \lambda f$$

When $U = \mathbb{R}$ eigenvectors are the sines and cosines functions that are used within Fourier framework

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Differential operators

- In a multivariate context, differential operators are also used in differential equations
- Generally expressed in $\mathbb{R}^k$
- Spatial notation “nabla” : Think of it as a vector of partial derivative operators!

$$\nabla = \begin{bmatrix} \delta_u \\ \delta_v \end{bmatrix}$$

  - Gradient operator of a **scalar** function f : ∇ f
  - Divergence operator of a **vector** function **V** : $\nabla \cdot$ **V**
  - Laplacian operator of a **scalar** function f : $\Delta f = \nabla . \nabla f$
  - Curl (rotational) operator of a **vector** function **V** : $\nabla \times \mathbf{V}$
- Expression of nabla depending on the coordinate system of the input domain

Cartesian coordinates u,v

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Differential operator

- A few reminders :
  - Gradient : direction (u_{max},v_{max}) of maximal slope
  - Divergence : flow traversing a unit element around (u,v)
  - Laplacian : measures the difference between the function and its mean value in a small neighborhood
    - Useful for several geometry processing tasks (interpolation by heat diffusion, spectral analysis, mean-curvature, smoothing)
  - Curl (Rotational) : does the vector field locally turn around one vector?
- Questions :
  - How could you define the “gradient” of a vector field? Jacobian matrix
  - How could you define the “laplacian” of a vector field? Vector with the Laplacian of each component

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Other use of Nabla

- Nabla transpose

  ${}_x\delta$: Derivative of the scalar field on the left

$$\nabla^t = \left[{}_x\delta,{}_y\,\delta,{}_z\,\delta\right]$$

- Jacobian matrix of a vector field **V**

$$\mathbb{J}_{\mathbf{V}} = \mathbf{V}\nabla^{\mathbf{t}} = \begin{bmatrix} \delta_x v_x, \delta_y v_x, \delta_z v_x \\ \delta_x v_y, \delta_y v_y, \delta_z v_y \\ \delta_x v_z, \delta_y v_z, \delta_z v_z \end{bmatrix}$$

- Remark : circulation of **V** around small parallelogram **A**,**B** : $curl(\mathbf{V}).(\mathbf{A}\times\mathbf{B}) = \mathbb{J}_{\mathbf{V}}\mathbf{A}.\mathbf{B} - \mathbb{J}_{\mathbf{V}}\mathbf{B}.\mathbf{A}$

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Differential operator

- Functions defining a curve

  $\mathbf{x} : [0, L] \to \mathrm{R}^3$ de classe C^3

  Tangent, normal and binormal vectors?

Vecteur tangent : $\mathbf{t}(s) = \frac{\mathbf{x}'(s)}{\|\mathbf{x}'(s)\|}$

Vecteur normal : $\mathbf{m}(s) = \frac{\mathbf{x}''(s)}{\|\mathbf{x}''(s)\|}$

Vecteur binormal : $\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{m}(s)$

Repère de Serret-Frénet : $(\mathbf{x}(s); \mathbf{t}(s), \mathbf{m}(s), \mathbf{b}(s))$

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Differential operator

- Let s be the curvilinear abscissa
  - s : length of curve between 0 and t

$$s(t) = \int_0^t \|\mathbf{X}'(t)\| dt$$

  - parameterize the curve wrt s
  - x’(s) is a unit vector

Courbure : $\kappa(s) = \mathbf{x}''(s)$

- Mesure la déviation par rapport à une droite

Torsion : $\tau(s) = \frac{det[\mathbf{x}'(s),\mathbf{x}''(s),\mathbf{x}'''(s)]}{\kappa^2(s)}$

- Mesure le défaut de planarité

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Differential operator

- Functions defining a surface

  $\mathbf{X} : \Omega \subset \mathrm{R}^2 \to \mathrm{R}^3$ de classe C^r

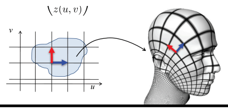
  Analogues vecteurs tangent/normal/binormal ?

On note $\mathbf{X}_u = \frac{\partial \mathbf{X}}{\partial u}$ et $\mathbf{X}_v = \frac{\partial \mathbf{X}}{\partial v}$

Plan tangent en p : $T_p\mathbf{X}$ = plan passant par p et engendré par les vecteurs $\mathbf{X}_u(p)$ et $\mathbf{X}_v(p)$

Vecteur normal : $\mathbf{n}(p) = \frac{\mathbf{X}_u(p)\times\mathbf{X}_v(p)}{\|\mathbf{X}_u(p)\times\mathbf{X}_v(p)\|}$

$(\mathbf{X}(p); \mathbf{X}_u(p), \mathbf{X}_v(p), \mathbf{n}(p))$ forme aussi un repère local



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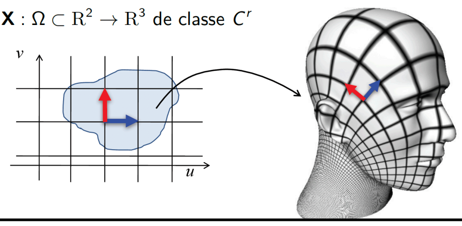
Differential operator

- Functions defining a surface
- How do unitary vectors are transformed by the parameterization?
  - Length and angles distorsion?

$\mathbf{X} : \Omega \subset \mathrm{R}^2 \to \mathrm{R}^3$ de classe C^r



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Differential operator

- Functions defining a surface?

On note de manière analogue $\mathbf{X}_{uu} = \frac{\partial \mathbf{X_u}}{\partial u} = \frac{\partial^2 \mathbf{X}}{\partial u^2}$, etc.

Première forme fondamentale :

$$\mathrm{I} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} := \begin{bmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{bmatrix}$$

Angle change

Length change

Seconde forme fondamentale :

$$\mathrm{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} := \begin{bmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{bmatrix}$$

Opérateur de forme/Application de Weingarten :

$$\mathbf{W} := \frac{1}{EG - F^2} \begin{bmatrix} eG - fF & fG - gF \\ fE - eF & gE - fF \end{bmatrix} = \begin{pmatrix} D_u n & D_v n \end{pmatrix}$$

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I = outil géométrique (**tenseur métrique**)

- Permet de mesurer aires locales, longueurs de courbes sur la surface, angles, ... $dA = \sqrt{EG - F^2}\, du dv$
- Exemple : anisotropie locale de la surface : décomposition spectrale de **I**

Propriétés différentielles ne dépendant que de **I** sont dites **intrinsèques**

- Ne dépendent pas de la paramétrisation
- Ne dépendent pas de l'espace 3D

Eigenvalues of I : maximal/minimal stretching of a tangent vector

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II = propriétés **extrinsèques** de la surface

- Dépendent du plongement dans l'espace ambiant $\mathbb{R}^3$

W détermine les directions de courbure locale de la surface

- Valeurs propres = courbures principales
- Vecteurs propres = directions principales de courbure

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Courbures principales et directions principales de courbure :

$$\mathbf{W} = \begin{bmatrix} \bar{\mathbf{t}}_1 & \bar{\mathbf{t}}_2 \end{bmatrix} \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{t}}_1 & \bar{\mathbf{t}}_2 \end{bmatrix}^{-1}$$

Courbure moyenne : $H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2} trace(\mathbf{W})$

Courbure de Gauss : $K = \kappa_1 \cdot \kappa_2 = det(\mathbf{W})$

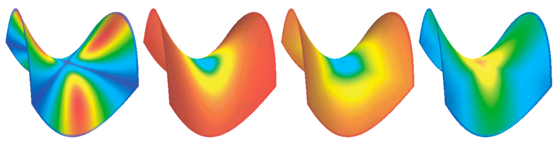

Fig. 5. *Curvature plots of a triangulated saddle using pseudo-colors: (a) Mean, (b) Gaussian, (c) Minimum, (d) Maximum.*

[Meyer et al. 2003]

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Differential operators

- Functions defined on a surface
  - Let u be a scalar function defined on a surface
  - We would like to express local variations of u
- Let consider the discrete case of a surface being approximated by a simplicial mesh
  - Function u discretized on vertices
  - Gradient of u discretized on triangles
  - Divergence of a vector (defined on triangles) discretized on vertices
- Bibliography : Keenan Crane

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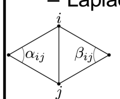
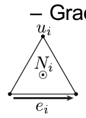
Differential operators

- Simplicial meshes
  - Laplacian Δ of u at vertex i = sum over neighbor vertices j



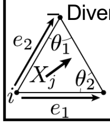
$$(Lu)_i = \frac{1}{2A_i} \sum_j (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i)$$

  - Gradient ∇ of u inside a triangle = sum over the 3 vertices I



$$\nabla u = \frac{1}{2A_f} \sum_i u_i (N \times e_i)$$

  - Divergence at vertex i of a vector X defined on faces = sum over incident faces j



$$\nabla \cdot X = \frac{1}{2} \sum_j \cot \theta_1 (e_1 \cdot X_j) + \cot \theta_2 (e_2 \cdot X_j)$$

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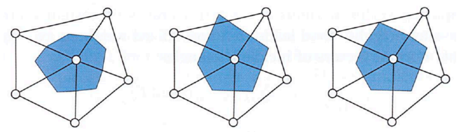
Area A_i

- Computed by duality to a vertex



Courtesy M. Botsch et al.

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Differential geometry

- How to generalize normal, curvatures?
- Consider u corresponding to each coordinate fonction in turn
  – u = ${}^t(x,y,z)$

$$\triangle_{\mathbf{x}} \mathbf{u} = -2H\mathbf{n}$$

H : mean curvature
n : normal vector

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Signals being studied in Computer Graphics

- Signals defined on $\mathbb{R}^2$
  – Scalar signals :
    - height value (terrain)
    - density of some fluid flowing in the plane
  – Vector fields
    - Surface parameterization $\Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$
    - Displacement field of some fluid flowing in the plane
- Signals defined in $\mathbb{R}^3$
    - Density of a volume material (scalar)
    - Displacement field of some fluid (vector)

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Signals being studied in Computer Graphics

- Signals defined on a surface $\mathbb{S}$
  – Scalar values :
    - Temperature, grey color …
    - Position coordinates (x, y or z)
  – Vector fields
    - Normal vector
    - Maximal/minimal curvature direction
    - Displacement field
    - …

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Problem of a triangulating a surface passing through points

- Case of points belonging to a plane
- Ideas for constructing a mesh from these data?
  - Link the projected points together avoiding crossings to produce triangular pieces of surface.

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