Mesh and Computational Geometry

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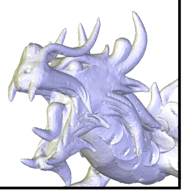
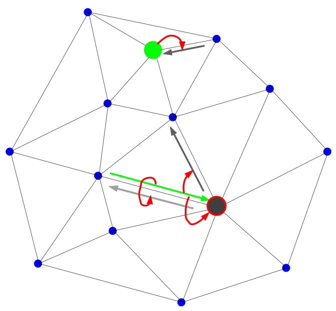


Data structures with edges

- Representation based on ½ edges and vertices
  - ½ edge :
    - Access to the coupled ½ edge
    - Access to the next ½ edge
    - Access to the target vertex
  - Vertex :
    - Access to an ½ edge oriented towards the vertex
    - Access to the underlying point

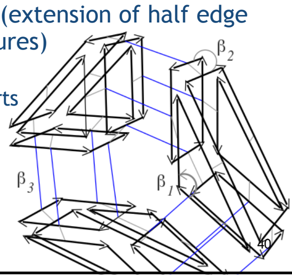
Data structures



Data structures

- Case of a 3D triangulation
  - Structure based on tetrahedrons and vertices

  OR
  - Combinatorial maps (extension of half edge or dart-based structures)
    - β_1 : next
    - β_2 et β_3 : coupled darts



CGAL

Data structures

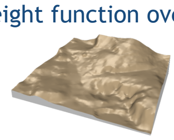
- Case of nD triangulations
  - n-simplex and vertex based structures
  - Combinatorial Maps
    - β_1 : next
    - β_2, β_3 … β_n : coupled darts

Local surface variations

- How to study and characterize surface variations?
  - When it is defined as a height function over a plane?

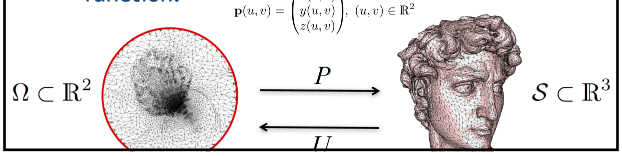
    z=f(x,y)

    Multivariate scalar function



  - When it is parameterized as a vector function?

$$\mathbf{p}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \ (u,v) \in \mathbb{R}^2$$



Interest

- Local surface variations
  - Used to compute geometric information such as normal vectors, curvature, and higher order information …
- Useful for :
  - Surface analysis
  - Surface rendering
  - Surface texturing
  - Constructing a better parameterization with less distorsion

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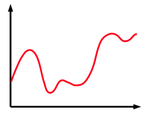
Differential operators

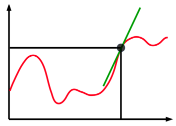
- Reminder :
  - Given a univariate function $f : \mathbb{R} \rightarrow \mathbb{R}$



  - The derivative of f is another function $\frac{\partial f}{\partial x}$ that describes the growing speed of f

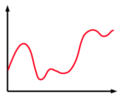
$f' : \mathbb{R} \rightarrow \mathbb{R}$

$$f'(x) = \lim_{a \to 0} \frac{f(x+a) - f(x)}{a}$$



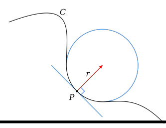
Differential operators

- Reminder :
  - The Laplacian (second derivative) of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is a measure of the difference between the value of f at any point P and the average value of f in the vicinity of P



  - It is linked to the curvature of the curve (inverse of the osculating circle radius)

$$\kappa(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$$



Differential operators

- In a multivariate context, differential operators are also used in differential equations
- Generally expressed in $\mathbb{R}^k$
- Spatial notation “nabla” : Think of it as a vector of partial derivative operators! $\nabla = \begin{bmatrix} \delta_u \\ \delta_v \end{bmatrix}$
  - Gradient operator of a scalar function f : ∇f
  - Divergence operator of a a vector function **V** : $\nabla . \mathbf{V}$
  - Laplacian operator of a scalar function f : $\Delta f = \nabla . \nabla f$
  - Curl (rotational) operator of a vector function **V** : $\nabla \times \mathbf{V}$
- Expression of nabla depending on the coordinate system of the input domain

Cartesian coordinates u,v

Differential operator

- A few reminders :
  - Gradient : direction (u_{max},v_{max}) of maximal slope
  - Divergence : flow traversing a unit element around (u,v)
  - Laplacian : measures the difference between the function and its mean value in a small neighborhood
    - Useful for several geometry processing tasks (interpolation, heat diffusion, spectral analysis, mean-curvature, smoothing)
  - Curl (Rotational) : does the vector field locally turn around one vector?
- Questions :
  - How could you define the “gradient” of a vector field? Jacobian matrix
  - How could you define the “laplacian” of a vector field? Vector with the Laplacian of each component

Other use of Nabla

- Nabla transpose

  ${}_x\delta$: Derivative of the scalar field on the left

$$\nabla^t = \left[{}_x\delta, {}_y\delta, {}_z\delta\right]$$

- Jacobian matrix of a vector field **V**

$$\mathbb{J}_{\mathbf{V}} = \mathbf{V}\nabla^{\mathbf{t}} = \begin{bmatrix} \delta_x v_x, \delta_y v_x, \delta_z v_x \\ \delta_x v_y, \delta_y v_y, \delta_z v_y \\ \delta_x v_z, \delta_y v_z, \delta_z v_z \end{bmatrix}$$

- Remark : circulation of **V** around small parallelogram **A**,**B** : $curl(\mathbf{V}).(\mathbf{A} \times \mathbf{B}) = \mathbb{J}_{\mathbf{V}}\mathbf{A}.\mathbf{B} - \mathbb{J}_{\mathbf{V}}\mathbf{B}.\mathbf{A}$

Differential operator

- Functions defining a curve

  $\mathbf{x} : [0, L] \to \mathbb{R}^3$ de classe C^3

  Tangent, normal and binormal vectors?

Vecteur tangent : $\mathbf{t}(s) = \frac{\mathbf{x}'(s)}{\|\mathbf{x}'(s)\|}$
Vecteur normal : $\mathbf{m}(s) = \frac{\mathbf{x}''(s)}{\|\mathbf{x}''(s)\|}$
Vecteur binormal : $\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{m}(s)$
Repère de Serret-Frénet : $(\mathbf{x}(s); \mathbf{t}(s), \mathbf{m}(s), \mathbf{b}(s))$

Differential operator

- Let s be the curvilinear abscissa
  - s : length of curve between 0 and t

$$s(t) = \int_0^t \|\mathbf{X}'(t)\| dt$$

  - parameterize the curve wrt s
  - x'(s) is a unit vector

Courbure : $\kappa(s) = \mathbf{x}''(s)$

- Mesure la déviation par rapport à une droite

Torsion : $\tau(s) = \frac{det[\mathbf{x}'(s), \mathbf{x}''(s), \mathbf{x}'''(s)]}{\kappa^2(s)}$

- Mesure le défaut de planarité

Differential operator

- Functions defining a surface

  $\mathbf{X} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ de classe C^r

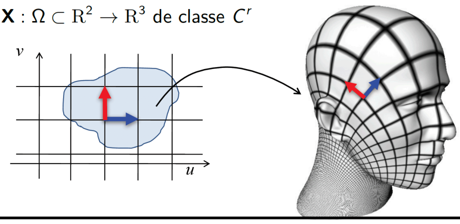
  Analogues vecteurs tangent/normal/binormal ?

On note $\mathbf{X}_u = \frac{\partial \mathbf{X}}{\partial u}$ et $\mathbf{X}_v = \frac{\partial \mathbf{X}}{\partial v}$
Plan tangent en p : $T_p\mathbf{X}$ = plan passant par p et engendré par les vecteurs $\mathbf{X}_u(p)$ et $\mathbf{X}_v(p)$
Vecteur normal : $\mathbf{n}(p) = \frac{\mathbf{X}_u(p) \times \mathbf{X}_v(p)}{\|\mathbf{X}_u(p) \times \mathbf{X}_v(p)\|}$
$(\mathbf{X}(p); \mathbf{X}_u(p), \mathbf{X}_v(p), \mathbf{n}(p))$ forme aussi un repère local

Differential operator

- Functions defining a surface
- How do unitary vectors are transformed by the parameterization?
  - Length and angles distorsion?

  $\mathbf{X} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ de classe C^r



Differential operator

- Functions defining a surface?

  On note de manière analogue $\mathbf{X}_{uu} = \frac{\partial \mathbf{X_u}}{\partial u} = \frac{\partial^2 \mathbf{X}}{\partial u^2}$, etc.

  Première forme fondamentale :

Angle change

$$\mathbf{I} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} := \begin{bmatrix} \mathbf{x}_u^T\mathbf{x}_u & \mathbf{x}_u^T\mathbf{x}_v \\ \mathbf{x}_u^T\mathbf{x}_v & \mathbf{x}_v^T\mathbf{x}_v \end{bmatrix}$$

Length change

  Seconde forme fondamentale :

$$\mathbf{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} := \begin{bmatrix} \mathbf{x}_{uu}^T\mathbf{n} & \mathbf{x}_{uv}^T\mathbf{n} \\ \mathbf{x}_{uv}^T\mathbf{n} & \mathbf{x}_{vv}^T\mathbf{n} \end{bmatrix}$$

  Opérateur de forme/Application de Weingarten :

$$\mathbf{W} \quad := \quad \frac{1}{EG - F^2} \begin{bmatrix} eG - fF & fG - gF \\ fE - eF & gE - fF \end{bmatrix}$$

$\mathbf{I}$ = outil géométrique (**tenseur métrique**)

- Permet de mesurer aires locales, longueurs de courbes sur la surface, angles, ... $dA = \sqrt{EG - F^2}\, dudv$
- Exemple : anisotropie locale de la surface : décomposition spectrale de $\mathbf{I}$

Propriétés différentielles ne dépendant que de $\mathbf{I}$ sont dites **intrinsèques**

- Ne dépendent pas de la paramétrisation
- Ne dépendent pas de l'espace 3D

Eigenvalues of I : maximal/minimal stretching of a tangent vector

II $=$ propriétés **extrinsèques** de la surface

- Dépendent du plongement dans l'espace ambiant $\mathbb{R}^3$

W détermine les directions de courbure locale de la surface

- Valeurs propres $=$ courbures principales
- Vecteurs propres $=$ directions principales de courbure

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Courbures principales et directions principales de courbure :

$$\mathbf{W} \;=\; \left[\ \bar{\mathbf{t}}_1 \ \ \bar{\mathbf{t}}_2\ \right] \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} \left[\ \bar{\mathbf{t}}_1 \ \ \bar{\mathbf{t}}_2\ \right]^{-1}$$

Courbure moyenne : $H = \frac{\kappa_1+\kappa_2}{2} = \frac{1}{2}\mathit{trace}(\mathbf{W})$

Courbure de Gauss : $K = \kappa_1 \cdot \kappa_2 = \mathit{det}(\mathbf{W})$

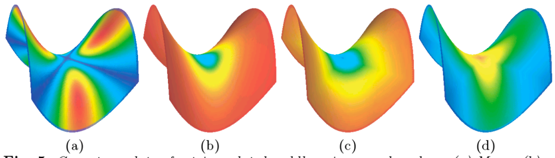

Fig. 5. *Curvature plots of a triangulated saddle using pseudo-colors: (a) Mean, (b) Gaussian, (c) Minimum, (d) Maximum.*

[Meyer et al. 2003]

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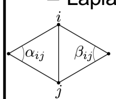
Differential operators

- Functions defined on a surface
  - Let u be a scalar function defined on a surface
  - We would like to express local variations of u
- Let consider the discrete case of a surface being approximated by a simplicial mesh
  - Function u discretized on vertices
  - Gradient of u discretized on triangles
  - Divergence of a vector (defined on triangles) discretized on vertices

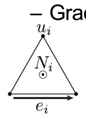
Differential operators

- Simplicial meshes
  - Laplacian $\triangle$ of u at vertex i = sum over neighbor vertices j



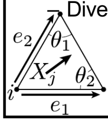
$$(Lu)_i = \frac{1}{2A_i}\sum_j(\cot\alpha_{ij} + \cot\beta_{ij})(u_j - u_i)$$

  - Gradient ∇ of u inside a triangle = sum over the 3 vertices I



$$\nabla u = \frac{1}{2A_f}\sum_i u_i(N \times e_i)$$

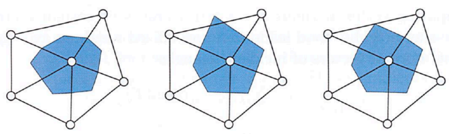
  - Divergence at vertex i of a vector X defined on faces = sum over incident faces j



$$\nabla \cdot X = \frac{1}{2}\sum_j \cot\theta_1(e_1 \cdot X_j) + \cot\theta_2(e_2 \cdot X_j)$$

Area A_i

- Computed by duality to a vertex



Barycentric cell   Voronoi cell   Mixed Voronoi cell

Courtesy M. Botsch et al.

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Differential geometry

- How to generalize normal, curvatures?
- Consider u corresponding to each coordinate fonction in turn
  - u = ^{t}(x,y,z)

$$\triangle_{\mathbf{X}}\mathbf{u} = -2H\mathbf{n}$$

H : mean curvature

n : normal vector

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Signals being studied in Computer Graphics

- Signals defined on $\mathbb{R}^2$
  - Scalar signals :
    - height value (terrain)
    - density of some fluid flowing in the plane
  - Vector fields
    - Surface parameterization $\Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$
    - Displacement field of some fluid flowing in the plane
- Signals defined in $\mathbb{R}^3$
    - Density of a volume material (scalar)
    - Displacement field of some fluid (scalar)

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Signals being studied in Computer Graphics

- Signals defined on a surface $\mathbb{S}$
  - Scalar values :
    - Temperature, grey color …
    - Position coordinates (x, y or z)
  - Vector fields
    - Normal vector
    - Maximal/minimal curvature direction
    - Displacement field
    - …

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