Chapter II. Vector Spaces.

1.1 Definition A *vector space* (over ℝ) consists of a set V along with two operations '+' and '·' subject to the conditions that for all vectors $\vec{v}, \vec{w}, \vec{u} \in V$ and all *scalars* $r, s \in \mathbb{R}$:

(1) the set V is closed under vector addition, that is, $\vec{v} + \vec{w} \in V$
(2) vector addition is commutative, $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
(3) vector addition is associative, $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$
(4) there is a *zero vector* $\vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in V$
(5) each $\vec{v} \in V$ has an *additive inverse* $\vec{w} \in V$ such that $\vec{w} + \vec{v} = \vec{0}$
(6) the set V is closed under scalar multiplication, that is, $r \cdot \vec{v} \in V$
(7) scalar multiplication distributes over scalar addition, $(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$
(8) scalar multiplication distributes over vector addition, $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$
(9) ordinary multiplication of scalars associates with scalar multiplication, $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$
(10) multiplication by the scalar 1 is the identity operation, $1 \cdot \vec{v} = \vec{v}$.

1.7 Definition A one-element vector space is a *trivial* space.

1.16 Lemma In any vector space V, for any $\vec{v} \in V$ and $r \in \mathbb{R}$, we have (1) $0 \cdot \vec{v} = \vec{0}$, (2) $(-1 \cdot \vec{v}) + \vec{v} = \vec{0}$, and (3) $r \cdot \vec{0} = \vec{0}$.

2.1 Definition For any vector space, a *subspace* is a subset that is itself a vector space, under the inherited operations.

2.9 Lemma For a nonempty subset S of a vector space, under the inherited operations the following are equivalent statements.*

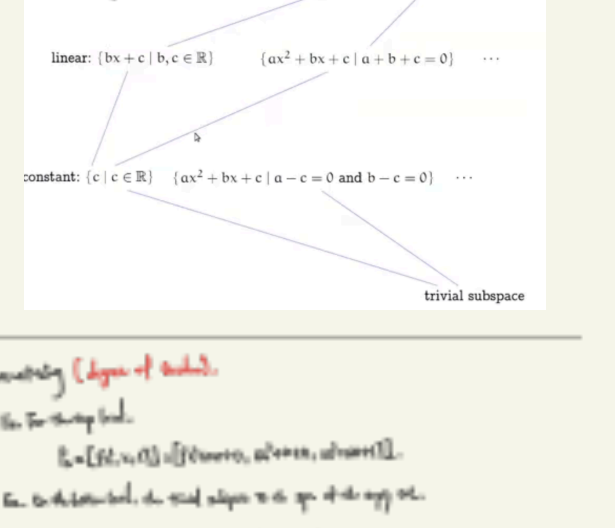
(1) S is a subspace of that vector space
(2) S is closed under linear combinations of pairs of vectors: for any vectors $\vec{s}_1, \vec{s}_2 \in S$ and scalars r_1, r_2 the vector $r_1\vec{s}_1 + r_2\vec{s}_2$ is in S
(3) S is closed under linear combinations of any number of vectors: for any vectors $\vec{s}_1, \ldots, \vec{s}_n \in S$ and scalars $r_1, \ldots, r_n$, the vector $r_1\vec{s}_1 + \cdots + r_n\vec{s}_n$ is an element of S.

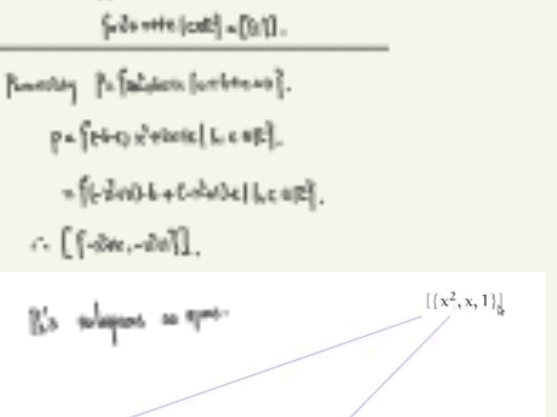
2.13 Definition The *span* (or *linear closure*) of a nonempty subset S of a vector space is the set of all linear combinations of vectors from S.

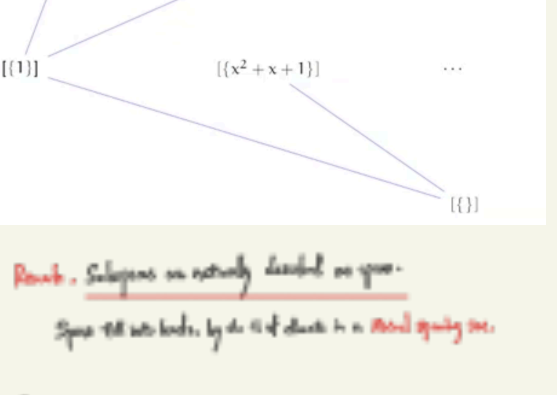
$$[S] = \{c_1\vec{s}_1 + \cdots + c_n\vec{s}_n \mid c_1, \ldots, c_n \in \mathbb{R} \text{ and } \vec{s}_1, \ldots, \vec{s}_n \in S\}$$

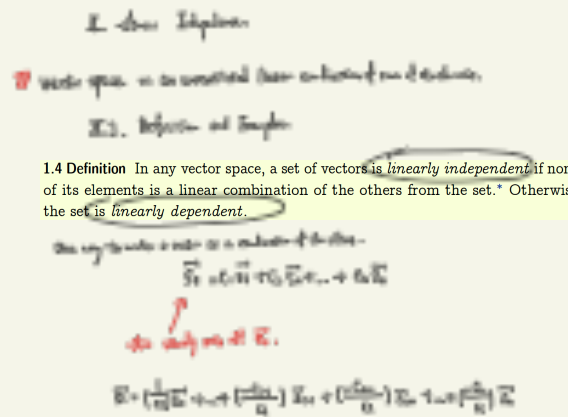
The span of the empty subset of a vector space is its trivial subspace.

2.15 Lemma In a vector space, the span of any subset is a subspace.






$\mathcal{P}_2$
linear: $\{bx + c \mid b, c \in \mathbb{R}\}$
$\{ax^2 + bx + c \mid a + b + c = 0\}$
constant: $\{c \mid c \in \mathbb{R}\}$
$\{ax^2 + bx + c \mid a - c = 0 \text{ and } b - c = 0\}$
trivial subspace



1.4 Definition In any vector space, a set of vectors is *linearly independent* if none of its elements is a linear combination of the others from the set.* Otherwise the set is *linearly dependent*.

1.5 Lemma A subset S of a vector space is linearly independent if and only if among its elements the only linear relationship $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{0}$ is the trivial one, $c_1 = 0, \ldots, c_n = 0$ (where $\vec{s}_i \neq \vec{s}_j$ when $i \neq j$).

1.2 Lemma Where V is a vector space, S is a subset of that space, and $\vec{v}$ is an element of that space, $[S \cup \{\vec{v}\}] = [S]$ if and only if $\vec{v} \in [S]$.

1.3 Corollary For $\vec{v} \in S$, omitting that vector does not shrink the span $[S] = [S - \{\vec{v}\}]$ if and only if it is dependent on other vectors in the set $\vec{v} \in [S]$.

1.14 Corollary A set S is linearly independent if and only if for any $\vec{v} \in S$, its removal shrinks the span $[S - \{\vec{v}\}] \subsetneq [S]$.

1.15 Lemma Suppose that S is linearly independent and that $\vec{v} \notin S$. Then the set $S \cup \{\vec{v}\}$ is linearly independent if and only if $\vec{v} \notin [S]$.

1.17 Corollary In a vector space, any finite set has a linearly independent subset with the same span.

1.19 Corollary A subset $S = \{\vec{s}_1, \ldots, \vec{s}_n\}$ of a vector space is linearly dependent if and only if some $\vec{s}_i$ is a linear combination of the vectors $\vec{s}_1, \ldots, \vec{s}_{i-1}$ listed before it.

1.20 Lemma Any subset of a linearly independent set is also linearly independent. Any superset of a linearly dependent set is also linearly dependent.

	$\hat{S} \subset S$	$\hat{S} \supset S$
S independent	$\hat{S}$ must be independent	$\hat{S}$ may be either
S dependent	$\hat{S}$ may be either	$\hat{S}$ must be dependent

1.1 Definition A *basis* for a vector space is a sequence of vectors that is linearly independent and that spans the space.

1.5 Definition For any $\mathbb{R}^n$

$$\mathcal{E}_n = \left\langle \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\rangle$$

is the *standard* (or *natural*) basis. We denote these vectors $\vec{e}_1, \ldots, \vec{e}_n$.

1.12 Theorem In any vector space, a subset is a basis if and only if each vector in the space can be expressed as a linear combination of elements of the subset in one and only one way.

1.13 Definition In a vector space with basis B the *representation of* $\vec{v}$ *with respect to* B is the column vector of the coefficients used to express $\vec{v}$ as a linear combination of the basis vectors:

$$\operatorname{Rep}_B(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_B$$

where $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$. The c's are the *coordinates of* $\vec{v}$ *with respect to* B.

1.18 Lemma Where B is a basis with n elements, for any set of vectors $a_1\vec{v}_1 + \cdots + a_k\vec{v}_k = \vec{0}_V$ if and only if $a_1\operatorname{Rep}_B(\vec{v}_1) + \cdots + a_k\operatorname{Rep}_B(\vec{v}_k) = \vec{0}_{\mathbb{R}^n}$.

2.1 Definition A vector space is *finite-dimensional* if it has a basis with only finitely many vectors.

2.3 Lemma (Exchange Lemma) Assume that $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ is a basis for a vector space, and that for the vector $\vec{v}$ the relationship $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$ has $c_i \neq 0$. Then exchanging $\vec{\beta}_i$ for $\vec{v}$ yields another basis for the space.

PROOF Call the outcome of the exchange $\hat{B} = \langle \vec{\beta}_1, \ldots, \vec{\beta}_{i-1}, \vec{v}, \vec{\beta}_{i+1}, \ldots, \vec{\beta}_n \rangle$.

We first show that $\hat{B}$ is linearly independent. Any relationship $d_1\vec{\beta}_1 + \cdots + d_i\vec{v} + \cdots + d_n\vec{\beta}_n = \vec{0}$ among the members of $\hat{B}$, after substitution for $\vec{v}$,

$$d_1\vec{\beta}_1 + \cdots + d_i \cdot (c_1\vec{\beta}_1 + \cdots + c_i\vec{\beta}_i + \cdots + c_n\vec{\beta}_n) + \cdots + d_n\vec{\beta}_n = \vec{0} \qquad (*)$$

gives a linear relationship among the members of B. The basis B is linearly independent so the coefficient d_ic_i of $\vec{\beta}_i$ is zero. Because we assumed that c_i is nonzero, $d_i = 0$. Using this in equation (∗) gives that all of the other d's are also zero. Therefore $\hat{B}$ is linearly independent.

We finish by showing that $\hat{B}$ has the same span as B. Half of this argument, that $[\hat{B}] \subseteq [B]$, is easy; we can write any member $d_1\vec{\beta}_1 + \cdots + d_i\vec{v} + \cdots + d_n\vec{\beta}_n$ of $[\hat{B}]$ as $d_1\vec{\beta}_1 + \cdots + d_i \cdot (c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) + \cdots + d_n\vec{\beta}_n$, which is a linear combination of linear combinations of members of B, and hence is in [B]. For the $[B] \subseteq [\hat{B}]$ half of the argument, recall that if $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$ with $c_i \neq 0$ then we can rearrange the equation to $\vec{\beta}_i = (-c_1/c_i)\vec{\beta}_1 + \cdots + (1/c_i)\vec{v} + \cdots + (-c_n/c_i)\vec{\beta}_n$. Now, consider any member $d_1\vec{\beta}_1 + \cdots + d_i\vec{\beta}_i + \cdots + d_n\vec{\beta}_n$ of [B], substitute for $\vec{\beta}_i$ its expression as a linear combination of the members of $\hat{B}$, and recognize, as in the first half of this argument, that the result is a linear combination of linear combinations of members of $\hat{B}$, and hence is in $[\hat{B}]$. QED

2.4 Theorem In any finite-dimensional vector space, all bases have the same number of elements.

2.5 Definition The *dimension* of a vector space is the number of vectors in any of its bases.

2.10 Corollary No linearly independent set can have a size greater than the dimension of the enclosing space.

2.12 Corollary Any linearly independent set can be expanded to make a basis.

2.13 Corollary Any spanning set can be shrunk to a basis.

2.14 Corollary In an n-dimensional space, a set composed of n vectors is linearly independent if and only if it spans the space.

3.1 Definition The *row space* of a matrix is the span of the set of its rows. The *row rank* is the dimension of this space, the number of linearly independent rows.

3.3 Lemma If two matrices A and B are related by a row operation

$$A \xrightarrow{\rho_i \leftrightarrow \rho_j} B \quad \text{or} \quad A \xrightarrow{k\rho_i} B \quad \text{or} \quad A \xrightarrow{k\rho_i + \rho_j} B$$

(for $i \neq j$ and $k \neq 0$) then their row spaces are equal. Hence, row-equivalent matrices have the same row space and therefore the same row rank.

3.4 Lemma The nonzero rows of an echelon form matrix make up a linearly independent set.

3.6 Definition The *column space* of a matrix is the span of the set of its columns. The *column rank* is the dimension of the column space, the number of linearly independent columns.

3.8 Definition The *transpose* of a matrix is the result of interchanging its rows and columns, so that column j of the matrix A is row j of A^T and vice versa.

3.10 Lemma Row operations do not change the column rank.

3.11 Theorem For any matrix, the row rank and column rank are equal.

PROOF Bring the matrix to reduced echelon form. Then the row rank equals the number of leading entries since that equals the number of nonzero rows. Then also, the number of leading entries equals the column rank because the set of columns containing leading entries consists of some of the $\vec{e}_i$'s from a standard basis, and that set is linearly independent and spans the set of columns. Hence, in the reduced echelon form matrix, the row rank equals the column rank, because each equals the number of leading entries.

But Lemma 3.3 and Lemma 3.10 show that the row rank and column rank are not changed by using row operations to get to reduced echelon form. Thus the row rank and the column rank of the original matrix are also equal. QED

3.12 Definition The *rank* of a matrix is its row rank or column rank.

3.13 Theorem For linear systems with n unknowns and with matrix of coefficients A, the statements

(1) the rank of A is r
(2) the vector space of solutions of the associated homogeneous system has dimension $n - r$

are equivalent.

3.14 Corollary Where the matrix A is n×n, these statements

(1) the rank of A is n
(2) A is nonsingular
(3) the rows of A form a linearly independent set
(4) the columns of A form a linearly independent set
(5) any linear system whose matrix of coefficients is A has one and only one solution

are equivalent.

3.15 Remark [Munkres] Sometimes the results of this subsection are mistakenly remembered to say that the general solution of an m equations, n unknowns system uses $n - m$ parameters. The number of equations is not the relevant number; rather, what matters is the number of independent equations, the number of equations in a maximal independent set. Where there are r independent equations, the general solution involves $n - r$ parameters.