AP Calculus BC Project Topic 1 4th Hour

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May 2024

Part 1: Introduction

Topic: Indeterminate Limits

Sources: AoPS Calculus Book, AoPS Calculus Lecture Problems, Khan Academy, Mathematics Stack Exchange, and Myself.

Part 2: Problems

Problem 1 Compute $\lim_{x\to\infty} \dfrac{7x^5 - x^3 + 20}{9x^5 + 8x^4 - x + 12}$.
Source: Art of Problem Solving Calculus Book.

Problem 2 What is $\lim_{x\to 0} \dfrac{\tan x^2}{x}$.
Source: I made it up.

Problem 3 Compute $\lim_{x\to\infty} \left(x - x\cos\left(\dfrac{1}{x}\right)\right)$.
Source: Art of Problem Solving.

Problem 4 Let a be a positive constant. What is the value of $\lim_{x\to\infty} \dfrac{\log(x)}{x^a}$?
Source: Art of Problem Solving Calculus Forum.

Problem 5 Find an ordered pair (a, b) such that $\lim_{x\to 1} \dfrac{x^2 + ax + 3}{(x-1)(x-b)} = 6$.
Source: Khan Academy.

Problem 6 Compute $\lim_{x\to\infty} (nx)^{\frac{1}{mx}}$ where $n \neq 0$.
Source: I made it up.

Problem 7 What is $\lim_{x\to 0} e^{x^2} \cot x^2$?
Source: Khan Academy.

Problem 8 What is $\lim_{x\to 0} 16\left(e^x\right)^{e^x}$?
Source: Khan Academy.

Problem 9 Compute $\lim_{x\to\infty} \left(\sqrt{x^2 - x} - \sqrt{x^2 + x}\right)$.
Source: Art of Problem Solving Calculus Book.

Problem 10 What is $\lim_{x \to 0} (\cot x)^{\sin x}$?
Source: Art of Problem Solving.

Problem 11 Does the series $\sum_{n=0}^{\infty} (1+n)^{\frac{1}{n}}$ converge or diverge?
Source: Art of Problem Solving Calculus Book.

Problem 12 Find $\lim_{n \to \infty} \left(1 + \left(\frac{1}{2}\right)^x\right)^x$
Source: Mathematics Stack Exchange.

Problem 13 Calculate $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{\sum_{n=1}^{x} \frac{1}{n}}$
Source: I made it up.

Problem 14 Find $\int_1^e \frac{1}{(x-1)^2} - \frac{1}{x \ln^2(x)} dx$
Source: I made it up.

Problem 15 The graph $y = d(x)$ has a slant asymptote along the line $y = mx + b$ (with $m \neq 0$) if

$$\lim_{x \to \infty} |d(x) - (mx + b)| = 0.$$

Describe algebraically the conditions for a rational function $\frac{f(x)}{g(x)}$ to have a slant asymptote, where f and g are polynomials.
Make sure to prove that your conditions are both necessary and sufficient.
Source: Art of Problem Solving Calculus Forum.

Part 3: Solutions

Solution for Problem 1 The limit as x approaches infinity of a rational function with equal powers in the numerator and denominator is the ratio of the leading coefficients, which in this case is $\boxed{\dfrac{7}{9}}$. We can reuse that proof by dividing both numerator and denominator by x^5 which yields

$$\lim_{x\to\infty} \frac{7 - \frac{1}{x^2} + \frac{20}{x^5}}{9 + \frac{8}{x} - \frac{1}{x^4} + \frac{12}{x^5}} = \frac{7 - 0 + 0}{9 + 0 - 0 + 0} = \boxed{\frac{7}{9}}.$$

Solution for Problem 2 If we were to plug in $x = 0$, we would get $\frac{0}{0}$, so we apply L'Hopital's Rule.

$$\begin{aligned} &= \lim_{x\to 0} 2x \sec^2 x^2 \\ &= \boxed{0}. \end{aligned}$$

Solution for Problem 3 Let $y = -\frac{1}{x}$ and note that as $x \to \infty$, we have $y \to 0^-$. Then, the limit becomes

$$\lim_{y\to 0^-} \frac{-1}{y} + \frac{1}{y}\cos(-y) = \lim_{y\to 0^-} \frac{\cos(-y) - 1}{y} = \lim_{y\to 0^-} \frac{\cos(y) - 1}{y},$$

which we know equals $\boxed{0}$.

Solution for Problem 4 Since $a > 0$, we know that $\lim_{x\to\infty} x^a = \infty$. (If $a < 0$, then $\lim_{x\to\infty} x^a = 0$, and if $a = 0$, then $\lim_{x\to\infty} x^a = 1$.) Since $\lim_{x\to\infty} \log(x) = \infty$, so our limit is of the form $\dfrac{\infty}{\infty}$. Applying L'Hopital's Rule shows us that

$$\lim_{x\to\infty} \frac{\log(x)}{x^a} = \lim_{x\to\infty} \frac{1/x}{ax^{a-1}} = \lim_{x\to\infty} \frac{1}{ax^a}.$$

We recognize that, in the denominator, $x^a \to \infty$, and since $a > 0$, we know that $ax^a \to \infty$ as well. Thus, our new limit is of the form $\dfrac{1}{\infty}$, so the value of the limit is $\boxed{0}$.

Solution for Problem 5 The limit of the denominator is 0, and so the limit of the numerator must also be 0 lest the fraction approach ∞, $-\infty$, or have no limit at all. Thus, we have

$$\begin{aligned} \lim_{x\to 1}(x^2+ax+3) &= 0 \\ a+4 &= 0 \\ a &= -4. \end{aligned}$$

Since the limit is of the form $\dfrac{0}{0}$, we can compute the limit using L'Hopital's Rule to get

$$\begin{aligned} \lim_{x\to 1}\frac{x^2+ax+3}{(x-1)(x-b)} &= 6 \\ \lim_{x\to 1}\frac{2x+a}{(x-b)+(x-1)} &= 6 \\ \frac{2+a}{1-b} &= 6. \end{aligned}$$

Since $a=-4$, we have $1-b=-\dfrac{1}{3}$, and so $b=\dfrac{4}{3}$.

Our answer is $\boxed{\left(-4, \dfrac{4}{3}\right)}$.

Solution for Problem 6 We have that

$$\lim_{x\to\infty}(nx)^{\frac{1}{mx}} = \lim_{x\to\infty} e^{\left(\log\left((nx)^{\frac{1}{mx}}\right)\right)} = e^{\left(\lim_{x\to\infty}\frac{1}{mx}\log(nx)\right)}.$$

Since both the numerator and denominator in $\frac{1}{mx}\log(nx)$ approach infinity as $x\to\infty$, we can use L'Hopital's rule:

$$\lim_{x\to\infty}\frac{\log(nx)}{mx} = \lim_{x\to\infty}\frac{n/(nx)}{m} = 0.$$

Therefore, the limit of the original function is $e^0 = \boxed{1}$.

Solution for Problem 7 If we were to plug it in, we would get:

$$\lim_{x\to 0} e^{x^2} \cot x^2 = 0 \cot 0 = 0 \cdot \infty$$

which is one of our indeterminate limits.
Thus, we turn it into a fraction to apply L'Hopital's Rule.

$$\lim_{x\to 0} e^{x^2} \cot x^2 = \lim_{x\to 0} \frac{e^{x^2}}{tanx^2} = \frac{0}{0}.$$

$$= \lim_{x\to 0} \frac{2xe^{x^2}}{2x \sec^2 x^2}$$
$$= \lim_{x\to 0} \frac{e^{x^2}}{\sec^2 x^2}$$
$$= \frac{0}{1}$$
$$= \boxed{0}.$$

Solution for Problem 8 If we straight up plug $x = 0$ into the limit, we would get 0^0, which is one of our indeterminate limit forms. Thus, we would have to use the ln method.

$$y = \lim_{x\to 0} 16 \left(e^x\right)^{e^x}$$
$$\ln y = 16 \lim_{x\to 0} \ln \left(e^x\right)^{e^x}$$
$$\ln y = 16 \lim_{x\to 0} e^x \ln e^x$$
$$\ln y = 16 \cdot 1 \cdot 0$$
$$\ln y = 0.$$

So,

$$y = \boxed{1}.$$

Solution for Problem 9 We are tempted to say that both $\sqrt{x^2 - x}$ and $\sqrt{x^2 + x}$ approach x as $x \to \infty$, so the limit would be 0. However, this is incorrect. Instead, we get rid of the square roots by multiplying by the conjugate:

$$\begin{aligned}\sqrt{x^2 - x} - \sqrt{x^2 + x} &= \frac{\left(\sqrt{x^2 - x} - \sqrt{x^2 + x}\right)\left(\sqrt{x^2 - x} + \sqrt{x^2 + x}\right)}{\sqrt{x^2 - x} + \sqrt{x^2 + x}} \\ &= \frac{(x^2 - x) - (x^2 + x)}{\sqrt{x^2 - x} + \sqrt{x^2 + x}} = \frac{-2x}{\sqrt{x^2 - x} + \sqrt{x^2 + x}}.\end{aligned}$$

The limit of the last term is

$$\lim_{x\to\infty} \frac{-2x}{\sqrt{x^2 - x} + \sqrt{x^2 + x}} = \lim_{x\to\infty} \frac{-2}{\sqrt{1 - \frac{1}{x}} + \sqrt{1 + \frac{1}{x}}} = \frac{-2}{\sqrt{1 - 0} + \sqrt{1 + 0}} = -\frac{2}{2} = \boxed{-1}.$$

Solution for Problem 10 Again, if we plug in directly, we would get ∞^0. So we must take log on both sides.

$$\begin{aligned}\ln y &= \lim_{x\to 0} \ln\left(\cot x\right)^{\sin x} \\ \ln y &= \lim_{x\to 0} \sin x \ln \cot x \\ \ln y &= 0 \\ y &= \boxed{1}.\end{aligned}$$

Solution for Problem 11 We start by doing the nth term test.

$$\begin{aligned}&\lim_{n\to\infty} (1 + n)^{\frac{1}{n}} \\ &= \infty^0.\end{aligned}$$

Since that is one of our indeterminate limits, we do the ln method.

$$\begin{aligned}\ln y &= \lim_{n\to\infty} \frac{1}{n} \ln 1 + n \\ \ln y &= \frac{\infty 0}{=} \infty.\end{aligned}$$

So it diverges.

Solution for Problem 12 If we plug $n = \infty$ in, we would get ∞^{∞} which is indeterminate. Thus, we would need to use the ln trick.

$$\ln y = \lim_{x \to \infty} x \ln 1 + \left(\frac{1}{2}\right)^{x}$$

$$\ln y = x \cdot \ln 1$$

$$\ln y = 0$$

$$y = \boxed{1}.$$

Solution for Problem 13 Ok so this might look pretty overwhelming at first, so lets break it down in to two pieces. The thing on the power and the base. Lets start with the power.

$$\sum_{n=1}^{x} \frac{1}{n}$$

where $\lim_{x \to \infty}$, we could use the P-series rule to determine that since $p \leq 1$, the series diverges, it $= \infty$.

Putting this back into the original limit, we get:

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{\infty}$$

$$= 1^{\infty}$$

$$= \boxed{1}.$$

Solution for Problem 14 Lets start by simply taking the integral:

$$\int_{1}^{e} \frac{1}{(x-1)^2} - \frac{1}{x \ln^2(x)} dx$$

The first term we can use power rule. The second term we use u sub by setting $u = \ln x$. We get:

$$= -\frac{1}{x-1} + \frac{1}{\ln x}$$

Now we plug in the intervals:

$$= -\frac{1}{e-1} + \frac{1}{1} + \frac{1}{0} - \frac{1}{0} = 1 = \frac{1}{e-1} + \infty - \infty.$$

However, $\infty - \infty$ is one of our indeterminate forms. So we have to use the fraction method.

$$\lim_{x\to 1} \frac{1}{\ln x} - \frac{1}{x-1}$$
$$= \lim_{x\to 1} \frac{x - 1 - \ln x}{\ln x\,(x-1)}$$
$$= \frac{0}{0}$$

Now we apply L'Hopital's Rule:

$$\lim_{x\to 1} \frac{1-\frac{1}{x}}{\frac{1}{x} + lnx}$$
$$= \frac{1-1}{1+0} = \frac{0}{1}$$
$$= \boxed{0}.$$

Solution for Problem 15 We need $\lim_{x\to\infty} \frac{f(x) - g(x)(mx+b)}{g(x)} = 0$. This limit is 0 if the degree of the denominator is greater than the degree of the numerator. So we need

$$\deg g(x) > \deg(f(x) - g(x)(mx+b)).$$

But since $m \neq 0$, we have $\deg g(x)(mx+b) = (\deg g(x)) + 1$, so the only way that the degree of the numerator can be smaller than that of g is if the two leading terms of $g(x)(mx+b)$ cancel with $f(x)$. But remember, we get to choose the m and the b.

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots$ and let $g(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots$, with a_n and b_{n-1} nonzero. Then setting $m = a_n/b_{n-1}$ and $b = (a_{n-1} - mb_{n-2})/b_{n-1}$ will cause both the x^n and x^{n-1} terms to vanish in $f(x) - g(x)(mx+b)$.

Thus, f/g will have a slant asymptote if and only if $\deg f = \deg g + 1$.

A simpler way to compute the slant asymptote, assuming the degree of f is one more than the degree of g, is to perform the first two steps of long division. The result will be an expression

$$\frac{f(x)}{g(x)} = mx + b + \frac{r(x)}{g(x)},$$

where the quotient will be linear, and the "remainder" $\frac{r}{g}$ will have limit 0 as x approaches ∞.)