AP Calculus BC Project Topic 2 4th Hour

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Part 1: Introduction

Topic: Implicit Differentiation

Sources: AoPS Calculus Book, AoPS Calculus Lecture Problems, Demidovich Mathematical Analysis Book, Khan Academy, and Myself.

Part 2: Problems

Problem 1 Find the slope of the tangent line to the circle $x^2 + y^2 = 1$ at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.
Source: Art of Problem Solving Calculus Book.

Problem 2 If $y^3 + 4y + 4 = x^2$, and both x and y are real numbers, what is the derivative of y at $x = 2$?
Source: I made it up.

Problem 3 Find $\frac{dy}{dx}$ if $x^2 + y = \ln(y^2 - 1)$.
Source: Art of Problem Solving Calculus Book.

Problem 4 Find the first derivative of $\frac{y}{x} = \ln\sqrt{x^2 + y^2}$.
Source: Demidovich Mathematical Analysis Problems.

Problem 5 The graph of $xy^4 + x^2y = 10$ has a horizontal tangent at the point (a, b). What is the value of a^2b^4?
Source: Art of Problem Solving College Math Forum

Problem 6 Prove the derivative of $y = a^x$ using implicit differentiation Hint: What if we take the ln of both sides?
Source: I made it up.

Problem 7 Find $y'(2)$ if $y = 5x^{2+3x+2xy}$, and $y(2) = -5$.
Source: Art of Problem Solving.

Problem 8 Let β be the dependent variable and let $f(\alpha, \beta) = \sin\alpha\cos\alpha + \sin\beta\cos\beta = 0$. Prove that

$$d\alpha = -d\beta.$$

Source: I made it up.

Problem 9 There are 2 lines that are tangent to the ellipse $3x^2 - 72x + y^2 + 26 = 0$, and pass through the origin. Find the equation of the line with the smaller slope.
Source: I made it up

Problem 10 2 circles, have equations

$$y^2 + x^2 = 1.$$

and

$$y^2 + (x-3)^2 = 4.$$

respectfully.
Find the equation of the line tangent to both circles.
Source: Art of Problem Solving.

Problem 11 Find $\dfrac{d^2y}{dx^2}$ if $y^2 = \dfrac{x^3}{4-x}$.
Source: Khan Academy.

Problem 12 Find $\dfrac{d^2y}{dx^2}$ at $x = 1$ and $y = 2$, if $8x^2y + 2\ln(xy) = 7$.
Source: Khan Academy.

Problem 13 Find $\dfrac{d^3y}{dx^3}$ of $x^2 + 6xy + y^2 = 4$ on the point $(1, 5)$.
Source: I made it up.

Problem 14 Let $g(x, y) = 1 + xy + (xy)^2 + (xy)^3 + ... + (xy)^{100}$. If $\dfrac{dy}{dx} = -1$ for all real x and y, find $g'\left(\frac{1}{2}, 2\right)$.
Source: I made it up.

Problem 15 Let $y = f(x)$ be a function of x such that

$$x^2y^2 + x^2 + y^2 - 1 = 0.$$

Prove that:

$$\frac{dx}{\sqrt{1-x^4}} + \frac{dy}{\sqrt{1-y^4}} = 0.$$

Source: Art of Problem Solving College Math Forum.

Part 3: Solutions

Solution for Problem 1 First let's find the expression for the slope, which is $\frac{dy}{dx}$.

$$2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x}{y}.$$

Thus, the slope at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ is $-\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -1$.

Solution for Problem 2 First, we take the implicit derivative of our given function:

$$3y^2\frac{dy}{dx} + 4\frac{dy}{dx} = 2x$$

Moving the variables, we get

$$\frac{dy}{dx}\left(3y^2 + 4\right) = 2x$$
$$\frac{dy}{dx} = \frac{2x}{3y^2 + 4}.$$

The problem asks us to find the derivative at $x = 2$, so by plugging in the x value, we can find the y value and calculate the derivative.

$$y^3 + 4y + 4 = 2^2$$
$$y\left(y^2 + 4\right) = 0$$

We can see that $y = 0$ or $y^2 + 4 = 0$, but since y is real, y must be equal to 0.
Thus, the derivative at $x = 2$ is $\frac{2 \cdot 2}{0 + 4} = \boxed{1}$.

Solution for Problem 3 Very straight-forward:

$$
\begin{aligned}
2x + \frac{dy}{dx} &= 2y\frac{dy}{dx} \cdot \frac{1}{y^2 - 1} \\
\frac{dy}{dx}\left(1 - \frac{2y}{y^2 - 1}\right) &= -2x \\
\frac{dy}{dx} &= \frac{2x}{\frac{2y - y^2 + 1}{y^2 - 1}} \\
&= \frac{2x\left(y^2 - 1\right)}{-y^2 + 2y + 1} \\
&= \frac{2x - 2xy^2}{y^2 - 2y - 1}
\end{aligned}
$$

Solution for Problem 4 Start with quotient rule:

$$
\begin{aligned}
\frac{x\frac{dy}{dx} - y}{x^2} &= \left(2x + 2y\frac{dy}{dx}\right) \cdot \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} \cdot \frac{1}{\sqrt{x^2 + y^2}} \\
\frac{x\frac{dy}{dx} - y}{x^2} &= \frac{1}{2}\left(2x + 2y\frac{dy}{dx}\right) \cdot \frac{1}{x^2 + y^2} \\
x\frac{dy}{dx} - y &= \frac{x^2 \cdot 2x}{2\left(x^2 + y^2\right)} + \frac{x^2 \cdot 2y}{2\left(x^2 + y^2\right)}\frac{dy}{dx} \\
\left(x - \frac{x^2 y}{x^2 + y^2}\right)\frac{dy}{dx} &= \frac{x^3}{x^2 + y^2} + y \\
&\boxed{\frac{dy}{dx} = \frac{x^3 + x^2 y + y^3}{x^3 + xy^2 - x^2 y}}.
\end{aligned}
$$

Solution for Problem 5 We start by finding $\frac{dy}{dx}$ because that's the expression for the slope of a tangent line to the function.

$$y^4 + 4xy^3\frac{dy}{dx} + 2xy + \frac{dy}{dx}x^2 = 0$$
$$\frac{dy}{dx}\left(4xy^3 + x^2\right) = -y^4 - 2xy$$
$$\frac{dy}{dx} = \frac{-y^4 - 2xy}{4xy^3 + x^2}$$

We need $\frac{dy}{dx} = 0$. Thus,

$$-y^4 - 2xy = 0$$
$$y^4 = 2xy$$
$$x = \frac{y^3}{2}$$

Plugging back into our original equation, we get:

$$\frac{y^3}{2} \cdot y^4 + \frac{y^6}{4} \cdot y = 10$$
$$3y^7 = 40$$
$$y = \sqrt[7]{\frac{40}{3}}$$

Now we can find the x value

$$x = \frac{1}{2}\left(\frac{40}{3}\right)^{\frac{3}{7}}$$

The problem asks for the x value squared times the y value to the power of 4.

$$a^2b^4 = \frac{1}{4}\left(\frac{40}{3}\right)^{\frac{6}{7}} \cdot \left(\frac{40}{3}\right)^{\frac{4}{7}}$$
$$= \frac{1}{4}\left(\frac{40}{3}\right)^{\frac{10}{7}}$$

Solution for Problem 6 First, we take the ln of both sides.

$$\ln y = \ln a^x$$
$$\ln y = x \ln a$$

Now we take the implicit differentiation:

$$\frac{dy}{dx}\frac{1}{y} = \ln a$$

Substitute $y = a^x$ back in recursively:

$$\frac{dy}{dx}\frac{1}{a^x} = \ln a$$
$$\frac{dy}{dx} = \ln a a^x.$$

Solution for Problem 7 Very similar to the last problem, we start by taking the ln on both sides.

$$\ln y = \ln 5x^{2+3x+2xy}$$

Applying 2 log identities, we get:

$$\ln y = \ln 5 + (2 + 3x + 2xy) \ln x$$

Now we take the implicit derivative

$$\frac{dy}{dx}\frac{1}{y} = \left(3 + 2y + 2x\frac{dy}{dx}\right) \ln x + \frac{2 + 3x + 2xy}{x}$$
$$\frac{dy}{dx}\left(\frac{1}{y} - 2x \ln x\right) = 3 \ln x + 2y \ln x + \frac{2 + 3x + 2xy}{x}$$

Since the problem gave us $y(2) = -5$, they gave us the point $(2, -5)$ and we plug that in to get $y'(2)$.

$$y'(2) = \frac{-7 \ln 2 - 6}{-\frac{1}{5} - 4 \ln 2}$$
$$= \boxed{\frac{35 \ln 2 + 30}{20 \ln 2 + 1}}.$$

Solution for Problem 8 We start by taking the implicit derivative of the given function:

$$-\cos\alpha\sin\alpha - \frac{d\beta}{d\alpha}\cos\beta\sin\beta = 0$$
$$\frac{d\beta}{d\alpha} = \frac{\cos\alpha\sin\alpha}{\cos\beta\sin\beta}.$$

If we look back to the original function, we see that $\sin\alpha\cos\alpha = -\sin\beta\cos\beta$,
Thus,

$$\frac{d\beta}{d\alpha} = -1$$
$$d\beta = -d\alpha$$
$$\boxed{d\alpha = -d\beta}.$$

Solution for Problem 9 We start by finding the expression for the slope of a tangent line to the ellipse.

$$6x - 72 + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{72-6x}{2y} = \frac{36-3x}{y}.$$

For a line to pass through the origin and have the slope $\frac{dy}{dx}$, it must be in the form:

$$y = \frac{36-3x}{y}x$$

Thus,

$$y^2 = 36x - 3x^2$$

Plugging back into the origin equation, we get:

$$3x^2 - 72x + 36x - 3x^2 + 26 = 0$$

$$-36x = -26$$
$$x = \frac{13}{18}.$$

Now we find y

$$y^2 = 36 \cdot \frac{13}{18} - 3\left(\frac{13}{18}\right)^2$$
$$y^2 = 26 - \frac{507}{324}$$
$$y = \sqrt{\frac{7917}{324}}$$

Therefore, our equation of the line is:

$$y = \frac{36 - \frac{13}{18}}{\sqrt{\frac{7917}{324}}}x$$

Solution for Problem 10 First, we find the derivative expression for both circles.
Circle 1:

$$2y\frac{dy}{dx} + 2x = 0$$
$$\frac{dy}{dx} = -\frac{x}{y}.$$

Circle 2:

$$2y\frac{dy}{dx} + 2x - 6 = 0$$
$$\frac{dy}{dx} = \frac{3-x}{y}.$$

Now we set them equal since the problem states that there is a line tangent to both circles.

$$-\frac{x}{y}\frac{3-x}{y}.$$

Cross-multiplying, we get:

$$-xy = 3y - xy$$

$$3y = 0$$

$$y = 0.$$

Now we plug y back into any of the two circles to find x.

$$0^2 + x^2 = 1$$

$$x = \pm 1.$$

Now how do we determine which one is the real x value of their intersection? Since the circle equations are simple, we know that the second circle is located to the right of the first circle, which is centered at $(0, 0)$, so x has to be 1.
Thus,

$$\frac{dy}{dx} = -\frac{1}{0} = \infty$$

This doesn't mean it doesn't exist, it means that the slope is perfectly vertical.
Thus, the equation of the tangent line is:

$$\boxed{x = 1}$$

Solution for Problem 11 Get rid of fractions:

$$y^2(4-x) = x^3$$
$$4y^2 - xy^2 = x^3$$
$$8y\frac{dy}{dx} - y^2 - 2xy\frac{dy}{dx} = 3x^2$$
$$\frac{dy}{dx} = \frac{3x^2+y^2}{2xy-8y}$$

Now we take the implicit derivative again to find the 2nd derivative:

$$8\frac{dy}{dx}\frac{dy}{dx} + 8y\frac{d^2y}{dx^2} - 2y\frac{dy}{dx} - \left(2y + 2x\frac{dy}{dx}\right)\frac{dy}{dx} - \frac{d^2y}{dx^2}2xy = 0$$
$$\frac{d^2y}{dx^2}(8y-2xy) = 6x - 8\left(\frac{dy}{dx}\right)^2 + 2y\frac{dy}{dx} + 2y\frac{dy}{dx} + 2x\left(\frac{dy}{dx}\right)^2$$
$$\frac{d^2y}{dx^2} = \frac{6x + (2x-8)\left(\frac{dy}{dx}\right)^2 + 4y\frac{dy}{dx}}{8y-2xy}$$
$$\frac{d^2y}{dx^2} = \frac{6x + (2x-8)\left(\frac{3x^2+y^2}{2xy-8y}\right)^2 + 4y\frac{3x^2+y^2}{2xy-8y}}{8y-2xy}$$

Solution for Problem 12

$$16xy + 8x^2\frac{dy}{dx} + 2\left(y + x\frac{dy}{dx}\right)\frac{1}{xy} = 0$$

$$16y + 16y\frac{dy}{dx} + 16x\frac{dy}{dx} + 8x^2\frac{d^2y}{dx^2} - 2x^{-2} - 2y^{-2}\frac{dy}{dx} + 2y^{-1}\frac{d^2y}{dx^2} = 0$$

$$\frac{dy}{dx}\left(\frac{2}{y} + 8x^2\right) = -16xy - \frac{2}{x}$$
$$\frac{dy}{dx} = \frac{-16xy - \frac{2}{x}}{8x^2 + \frac{2}{y}}$$

So our first derivative is:

$$= -\frac{8x^2y^2 + y}{4x^2y + x}$$

Now let us find the second derivative:

$$\begin{aligned}
\frac{d^2y}{dx^2}\left(8x^2+2y^{-1}\right) &= -16y-16x\frac{dy}{dx} = 16x\frac{dy}{dx}+2x^{-2}+2y^{-2}\frac{dy}{dx}\\
\frac{d^2y}{dx^2} &= \frac{2x^{-2}-16y+\left(2y^{-2}-32x\right)\frac{dy}{dx}}{8x^2+2y^{-1}}\\
&= \frac{\frac{2}{x}-16y-\frac{8x^2y^2+y}{4x^2y+x}\left(\frac{2}{y}-32x\right)}{8x^2+\frac{2}{y}}\\
&= \frac{2-32-\frac{34}{9}\cdot(-31)}{9}\\
&= \frac{-270+34\cdot 31}{81} = \boxed{\frac{784}{81}}.
\end{aligned}$$

Solution for Problem 13 Finding the first derivative: //

$$\begin{aligned}
2x+6y+6x\frac{dy}{dx}+2y\frac{dy}{dx} &= 0\\
\frac{dy}{dx} &= \frac{-2x-6y}{6x+2y}\\
\frac{dy}{dx} &= -\frac{x+3y}{3x+y}\\
\frac{dy}{dx} &= -2.
\end{aligned}$$

Now we find the second derivative:

$$2+6\frac{dy}{dx}+6\frac{dy}{dx}+6x\frac{d^2y}{dx^2}+2\frac{dy}{dx}\cdot\frac{dy}{dx}+2y\frac{d^2y}{dx^2}=0$$

We will use the above expression to find the third derivative

$$\begin{aligned}
\frac{d^2y}{dx^2}(6x+2y) &= -2\left(\frac{dy}{dx}\right)^2-12\frac{dy}{dx}-2\\
\frac{d^2y}{dx^2} &= -\frac{2\left(\frac{dy}{dx}\right)^2+12\frac{dy}{dx}+2}{6x+2y}\\
\frac{d^2y}{dx^2} &= -\frac{8-24+2}{6+10} = \frac{14}{16} = \frac{7}{8}.
\end{aligned}$$

Now let's find the third derivative:

$$6\frac{d^2y}{dx^2}+6\frac{d^2y}{dx^2}+6\frac{d^2y}{dx^2}+6x\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}\frac{dy}{dx}+2\frac{dy}{dx}\frac{d^2y}{dx^2}+2\frac{dy}{dx}\frac{d^2y}{dx^2}+2y\frac{d^3y}{dx^3}=0$$

$$\frac{d^3y}{dx^3}(6x+2y) = -18\frac{d^2y}{dx^2} - 6\frac{dy}{dx}\frac{d^2y}{dx^2}$$
$$\frac{d^3y}{dx^3} = -\frac{9\frac{d^2y}{dx^2} + 3\frac{dy}{dx}\frac{d^2y}{dx^2}}{3x+1}$$
$$\frac{d^3y}{dx^3} = -\frac{9\cdot\frac{7}{8} + 3\cdot(-2)\cdot\frac{7}{8}}{3+1}$$
$$= -\frac{\frac{21}{8}}{4} = \boxed{-\frac{21}{32}}.$$

Solution for Problem 14 This one is a really weird problem, but don't give up after just looking at it, give it a try, and find a pattern!

If we were to take the implicit derivative of the first 3 terms, we would get

$$0 + y + x\frac{dy}{dx} + 2xy\left(y + x\frac{dy}{dx}\right).$$

Since the problem gave us $\frac{dy}{dx} = -1$, plugging it in would leave us with:

$$y - x + 2xy\,(y-x)\,.$$

Notice that if we were to continue writing down terms, we would have a general structure of:

$$n\,(xy)^{n-1}\,(y-x)$$

for each term. Also, the problem wanted us to find the derivative at $x = \frac{1}{2}$ and $y = 2$, which luckily for us, makes $xy = 1$. Using this, we get:

$$y - x + y - x + y - x + ...$$
$$= 2 - \frac{1}{2} + 2 - \frac{1}{2} + 2 - \frac{1}{2} + ...$$

With 99 terms in total.

Thus, we have $2\cdot 99 - \frac{1}{2}\cdot 99 = \boxed{\frac{297}{2}}$.

Solution for Problem 15 Since nobody posted any solutions, the approach I thought of was pretty straightforward. So our goal is to prove that $\dfrac{dx}{\sqrt{1-x^4}} + \dfrac{dy}{\sqrt{1-y^4}} = 0$ is true. Notice that we have to get $1-x^4$ and $1-y^4$ somehow.

Let's start by finding $\frac{dy}{dx}$ of our given function, $f(x) = x^2y^2 + x^2 + y^2 - 1 = 0$.
Using implicit differentiation, we get the following:

$$\begin{aligned} 2xy^2 + 2y\frac{dy}{dx}x^2 + 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx}\left(2x^2y + 2y\right) &= -2xy^2 - 2x \\ \frac{dy}{dx} &= -\frac{2xy^2 + 2x}{2x^2y + 2y}. \end{aligned}$$

Factoring out $2x$ and $2y$ on the numerator and denominator respectfully, we get:

$$\frac{dy}{dx} = -\frac{2x\left(y^2+1\right)}{2y\left(x^2+1\right)}$$

$$\frac{dy}{dx} = -\frac{x\left(y^2+1\right)}{y\left(x^2+1\right)}.$$

We see that the numerator and denominator are nearly symmetrical with the x swapped with y. So how do we get $1 - x^4$? If we find out how to get that, then we could do the same thing to get $1 - y^4$ because of symmetry.

This is where the hardest part of the problem comes into play.
First, from our original equation, notice that

$$x^2y^2 + x^2 + y^2 - 1 = 0$$

can be turned into

$$x^2\left(y^2+1\right) = 1 - y^2.$$

by factoring the x^2 and moving the rest to the right-hand side.

Thus, we can square both sides to get

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^2\left(y^2+1\right)^2}{y^2\left(x^2+1\right)^2}.$$

Thus using symmetry (we could also factor y^2 out), our equation becomes:

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^2\left(y^2+1\right)^2}{y^2\left(x^2+1\right)^2} = \frac{x^2\left(y^2+1\right)\left(y^2+1\right)}{y^2\left(x^2+1\right)\left(x^2+1\right)} = \frac{\left(1-y^2\right)\left(y^2+1\right)}{\left(1-x^2\right)\left(x^2+1\right)}$$

Because $\left(a^2+b^2\right)\left(a^2-b^2\right)=a^4-b^4$ we get

$$\left(\frac{dy}{dx}\right)^2=\frac{1-y^4}{1-x^4}.$$

Now we could take the square root of both sides.

$$\frac{dy}{dx}=-\frac{\sqrt{1-y^4}}{\sqrt{1-x^4}}.$$

Note that there has to be a negative there because when we squared both sides, there was also a negative there.

Cross-multiplying, we get

$$dy\sqrt{1-x^4}=-dx\sqrt{1-y^4}$$

$$dx\sqrt{1-y^4}+dy\sqrt{1-x^4}=0$$

Finally, we divide both sides by $\sqrt{1-x^4}\sqrt{1-y^4}$ and get

$$\boxed{\frac{dx}{\sqrt{1-x^4}}+\frac{dy}{\sqrt{1-y^4}}=0}.$$

And we are done with our proof.