Portfolio Theory

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Portfolio Analysis

1. Concepts

- Return of a Portfolio

$$R_P = \sum_{n=1}^{N} w_n R_n$$

  - R_n: return on nth asset
  - w_n: the weight of nth asset in the portfolio
  - = weighted average of returns on individual assets

- **Expected Return**

$$\mu_P = \mathbb{E}\left[\sum_{n=1}^{N} w_n R_n\right] = \sum_{n=1}^{N} w_n \mu_n$$

  - = weighted average of expected returns on individual assets

- **Variance**

$$\sigma_P^2 = \mathbb{E}\left[\left(\sum_{n=1}^{N} w_n (R_n - \mu_n)\right)^2\right] = \sum_{n=1}^{N} w_n^2 \sigma_n^2 + \sum_{n=1}^{N} \sum_{m \neq n}^{N} w_n w_m \sigma_{nm}$$

  - $\sigma_{nm} = \rho_{nm} \sigma_n \sigma_m$
  - ρ_{nm}: correlation coefficient between R_n & R_m

- **Limit of Diversification**
  - = No matter how well-diversified your portfolio is, it can **never be riskless**.
  - Proof: suppose your portfolio is so diversified that $\forall n$: $w_n = \frac{1}{N}$, then

$$\begin{aligned} \sigma_P^2 &= \sum_{n=1}^{N} \frac{\sigma_n^2}{N^2} + \sum_{n=1}^{N} \sum_{m \neq n}^{N} \frac{\sigma_{nm}}{N^2} \\ &= \frac{1}{N} \bar{\sigma}_n^2 + \frac{N-1}{N} \bar{\sigma}_{nm} \end{aligned}$$

- $\bar{\sigma}_n^2 = \frac{1}{N}\sum_{n=1}^{N} \sigma_n^2$
- $\bar{\sigma}_{nm} = \frac{1}{N}\sum_{n=1}^{N} \frac{1}{N-1}\sum_{m \neq n}^{N} \sigma_{nm}$
- As $n \to \infty$: $\sigma_P^2 \to \bar{\sigma}_{nm} \neq 0 \Longrightarrow$ There is always a risk.

- The **Tradeoff** of Portfolio Selection
  - **Expected Return** → **performance measure**
  - **Variance** → **risk measure**
  - The investors want:

    ⇒ a higher return from their investments

    ⇒ a lower variation in the value of their funds.

    ⇒ HOWEVER, it is impossible to have both → **Tradeoff: risk & return**
  - **Utility maximization problem**

$$\max_{w_1,\dots,w_N} \mu_P - \gamma\sigma_P^2$$

where $\gamma \geq 0$ is the degree of risk averse.

2. Mean-Variance Approach

(i.e. **[Minimum Variance Portfolio Problem] [Markowitz Model]**)

- MVPP: maximize the utility $\mu_P - \gamma\sigma_P^2 \Leftarrow$ minimize the variance σ_P^2:

$$\min_{w_1,\dots,w_N} \sigma_P^2 = \sum_{n=1}^{N} w_n^2\sigma_n^2 + \sum_{n=1}^{N}\sum_{m\neq n}^{N} w_n w_m \sigma_{nm}$$

subject to:

$\Rightarrow\ \sum_{n=1}^{N} w_n\mu_n = \mu_P$ (def of expected return)

$\Rightarrow\ \sum_{n}^{N} w_n = 1$ (weight rule)

$\Rightarrow\ w_n \geq 0\ \forall n \in [1, \dots, N]$ (prohibition of short selling)

- Matrix form:

$$\min_{w_1,\dots,w_N} \sigma_P^2 = \boldsymbol{w}^{\boldsymbol{T}}\Sigma\boldsymbol{w}$$

subject to:

$\Rightarrow\ \boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{\mu} = \mu_P$

$\Rightarrow\ \boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{i} = \boldsymbol{1}$

$\Rightarrow\ \boldsymbol{w} \geq \boldsymbol{0}$

where:

$$\boldsymbol{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}, \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_N \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & \cdots & \sigma_N^2 \end{bmatrix}, \boldsymbol{i} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

- Notes:
    - MVPP without short selling has no closed-form solution.
    - MVPP can be solved by a quadratic programming solver.
    - Additional conditions (trading fees, taxes, upper & lower bounds of weights, etc.) can be incorporated into MVPP.

- **Risk-free**
  - = when IRR of the asset is constant in all situations
  - **Risk-free rate**: return on risk-free asset
  - **Excess return**:
    - **of asset**: $\tilde{R}_n = R_n - R_f$
    - **of market**: $\tilde{R}_{\mathcal{M}} = R_{\mathcal{M}} - R_f$
  - MVPP with risk-free asset:

$$\begin{aligned} &\min_w \sigma_P^2 = w^T \Sigma w \\ &\text{s.t. } (1 - w^T i) R_f + w^T \mu = \mu_P \end{aligned}$$

    - $1 - w^T i$: allocation weight for risk-free assets
    - w^T: allocation weight for other assets
    - $\because 1 - w^T i + w^T i = 1 \quad \therefore$ the constraint $w^T i = 1$ is unnecessary

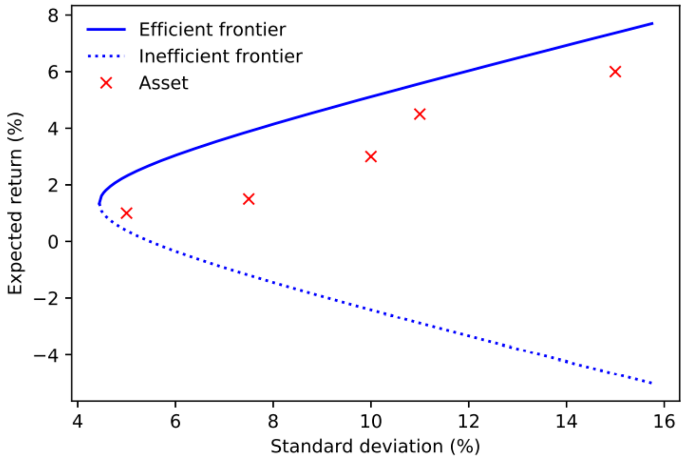
  - **Market Portfolio**

$$R_{\mathcal{M}} = \sum_{n=1}^{N} \bar{w}_n R_n \, , \bar{w}_n = \frac{M_n}{\sum_{n=1}^{N} M_n}$$

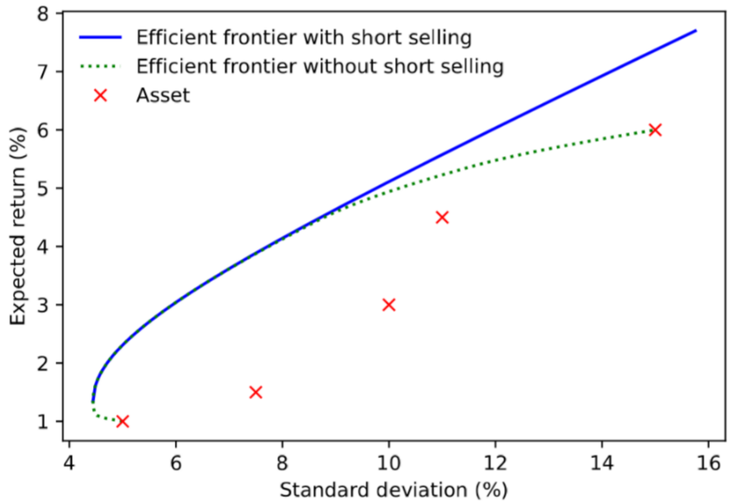
    - N: #all assets traded in the financial market
    - M_n: market capitalization of asset n

- **Optimal Portfolio Selection Procedure**
  - 1) Construct the efficient frontier.
  - 2) Set the tolerable level of risk / the target expected return.
  - 3) Pick the corresponding portfolio on the efficient frontier.

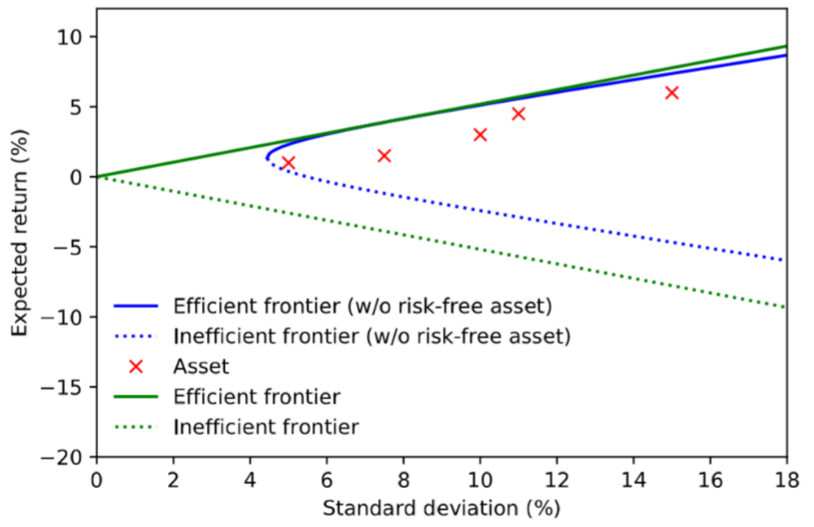
- **Minimum Variance Frontier**
  - = graph of the lowest possible variance attainable for a given expected return (i.e. (σ_P, μ_P)):



  - **Efficient frontier**: the upper half (∵ investors prefer higher return at the same risk)
  - **Two Fund Theorem**: The efficient frontier can be constructed from only two portfolios (i.e. two funds).
    - Suppose $w^{(1)}$ & $w^{(2)}$ are the two different solutions to the MVPP. Then $cw^{(1)} + (1-c)w^{(2)}\ \forall c \in \mathbb{R}$ is also a solution with the expected return at $c\mu_P^{(1)} + (1-c)\mu_P^{(2)}$.
  - Short-selling & ~~Short-selling~~



- Risk-free & ~~Risk-free~~



- **One Fund Theorem [Mutual Fund Theorem]**: Every portfolio on the efficient frontier is simply a combination of the risk-free asset and the tangent portfolio (i.e. one fund).
    - The efficient frontier (i.e. the green line) connects $(0, R_f)$ and the tangent point between the green & the blue (i.e. the **tangent portfolio**)

- Closed-form Solution
    - <u>For MVPP with short-selling</u>:

$$w = \frac{C\mu_P - A}{D}\Sigma^{-1}\mu + \frac{B - A\mu_P}{D}\Sigma^{-1}i$$

$\Rightarrow A = \mu^T\Sigma^{-1}i$

$\Rightarrow B = \mu^T\Sigma^{-1}\mu$

$\Rightarrow C = i^T\Sigma^{-1}i$

$\Rightarrow D = BC - A^2$

- Minimum Variance Frontier:

$$\sigma_P = \sqrt{\frac{C}{D}\left(\mu_P - \frac{A}{C}\right)^2 + \frac{1}{C}}$$

- For MVPP with risk-free assets:

$$w = \frac{\mu_P - R_f}{CR_f^2 - 2AR_f + B}\Sigma^{-1}\left(\mu_P - R_f i\right)$$

  - Minimum Variance Frontier:

$$\mu_P = R_f \pm \sigma_P\sqrt{CR_f^2 - 2AR_f + B}$$

- **Unknown risk & return**
  - In reality, μ & Σ are unknown → need estimation for r_{nt} [realized return on asset n at period t]

$$\bar{r}_n = \frac{1}{T}\sum_{t=1}^{T} r_{it}$$

$$s_{nm} = \frac{1}{T}\sum_{t=1}^{T}(r_{nt} - \bar{r}_n)(r_{mt} - \bar{r}_m)$$

Thus, we can replace μ & Σ with:

$$\bar{r} = \begin{bmatrix} \bar{r}_1 \\ \vdots \\ \bar{r}_N \end{bmatrix}$$

$$S = \begin{bmatrix} s_1^2 & \cdots & s_{1N} \\ \vdots & \ddots & \vdots \\ s_{N1} & \cdots & s_N^2 \end{bmatrix}$$

  - Alternative form of portfolio variance

$$\begin{aligned} w^T S w &= \sum_{n,m} w_n w_m s_{nm} \\ &= \frac{1}{T}\sum_{t=1}^{T}\sum_{n=1}^{N} w_n(r_{nt} - \bar{r}_n)\sum_{m=1}^{N} w_m(r_{mt} - \bar{r}_m) \\ &= \frac{1}{T}\sum_{t=1}^{T}\left[\sum_{n=1}^{N} w_n(r_{nt} - \bar{r}_n)\right]^2 \\ &= \frac{1}{T}\sum_{t=1}^{T}(r_{Pt} - \bar{r}_P)^2 \end{aligned}$$

- $r_{Pt} = \sum_{n=1}^{N} w_n r_{nt}$: realized return of portfolio with weights w at period t
- $\bar{r}_P = \sum_{n=1}^{N} w_n \bar{r}_n = \frac{1}{T}\sum_{t=1}^{T} r_{Pt}$

o Alternative form of MVPP

$$\min_{w,v} \widehat{\mathrm{Var}}[R_P] = \frac{1}{T} v^T v$$

subject to:

$\Rightarrow\ Dw = v$

$\Rightarrow\ w^T \bar{r} = \mu_P,\ w^T i = 1$

$\Rightarrow\ w_n \geq 0\ \ \forall n \in [1, N]$

Where

$$v = \begin{bmatrix} r_{P1} - \bar{r}_P \\ \vdots \\ r_{PT} - \bar{r}_P \end{bmatrix}, D = \begin{bmatrix} r_{11} - \bar{r}_1 & \cdots & r_{N1} - \bar{r}_N \\ \vdots & \ddots & \vdots \\ r_{1T} - \bar{r}_1 & \cdots & r_{NT} - \bar{r}_N \end{bmatrix}$$

3. Alternative Risk Measures

- **Mean Absolute Deviation**

$$\varrho^{AD}(w) = \mathbb{E}[|R_P - \mu_P|] = \frac{1}{T}\sum_{t=1}^{T}|r_{Pt} - \bar{r}_P|$$

  - Optimization problem:

$$\min_{w,v} \varrho^{AD}(w) = \frac{1}{T}\sum_{t=1}^{T}|v_t|$$

subject to:

$\Rightarrow\ Dw = v$

$\Rightarrow\ w^T\bar{r} = \mu_P,\ w^T i = 1$

$\Rightarrow\ w_n \geq 0\ \ \forall n \in [1, N]$

where $v_t = r_{P_t} - \bar{r}_P$.

- **Semivariance**

$$\varrho^{SV}(w) = \mathbb{E}\left[(R_P - \mu_P)^2 \middle| R_{P_t} \leq \mu_P\right] = \frac{1}{T}\sum_{t=1}^{T}\{[r_{Pt} - \bar{r}_P]^-\}^2$$

  - Optimization problem:

$$\min_{w,v} \varrho^{AD}(w) = \frac{1}{T}\sum_{t=1}^{T}|v_t|$$

subject to:

$\Rightarrow\ w^T\bar{r} = \mu_P,\ w^T i = 1$

$\Rightarrow\ w_n \geq 0\ \ \forall n \in [1, N]$

$\Rightarrow\ v_t \geq 0\ \ \ \forall t \in [1, T]$

$\Rightarrow\ r_{P_t} - \bar{r}_P + v_t \geq 0\ \ \ \forall t \in [1, T]$

where $v_t = \left[r_{P_t} - \bar{r}_P\right]^-$.

- **Expected Shortfall**

$$\mathrm{ES}_\alpha = \mathbb{E}[-R_P | R_P \le \mathrm{VaR}_\alpha] = \frac{1}{\alpha}\int_{-\infty}^{\infty} [R_P - \mathrm{VaR}_\alpha]^- p(R_P)\, dR_P - \mathrm{VaR}_\alpha$$

where:

- ♦ $p(R_P)$: probability density func of R_P
- ♦ VaR_α: value-at-risk where $\alpha = P\{R_P \le \mathrm{VaR}_\alpha\} \in (0,1)$ (i.e. the $(1-\alpha)$th quantile of $Y := -X$, where X is a distribution)

$$\mathrm{VaR}_\alpha(X) = -\inf\{x \in \mathbb{R}: \mathrm{cdf}_X(x) > \alpha\}$$

- o **Coherent Risk Measure**: Suppose $\mathcal{X}$ is a set of random vars, $X \in \mathcal{X}$ is the return of a portfolio, $\varrho(X)$ is the risk measure of X. ϱ is called coherent if:
  - ♦ **Monotonicity**: $\forall X, Y \in \mathcal{X}$: $P\{X \le Y\} = 1 \Rightarrow \varrho(X) \ge \varrho(Y)$

    $\Rightarrow$ A surely profitable portfolio should be less risky.
  - ♦ **Cash Invariance**: $\forall M \in \mathcal{X}$: $P\{M = R\} = 1 \Rightarrow \varrho(X + M) = \varrho(X) - R$

    $\Rightarrow$ Adding a riskless asset should reduce the risk.
  - ♦ **Sub-additivity**: $\forall X, Y \in \mathcal{X}$: $\varrho(X + Y) \le \varrho(X) + \varrho(Y)$

    $\Rightarrow$ Diversification should not make the portfolio riskier.
  - ♦ **Positive homogeneity**: $\forall \lambda \ge 0$: $\varrho(\lambda X) = \lambda \varrho(X)$

    $\Rightarrow$ The risk should be proportional to the position.

- o <u>Approximation</u>: if T is large enough, the integral can be approximated as:

$$\varrho^{ES}(w, c) = \frac{1}{\alpha T}\sum_{t=1}^{T} \left[r_{P_t} - c\right]^- - c$$

- o Optimization Problem:

$$\min_{w,v,c} \varrho^{ES}(w, c) = \frac{1}{\alpha T}\sum_{t=1}^{T} v_t - c$$

subject to:

$\Rightarrow\ w^T \bar{r} = \mu_P,\ w^T i = 1$

$\Rightarrow\ w_n \geq 0 \ \ \forall n \in [1, N]$

$\Rightarrow\ v_t \geq 0 \ \ \forall t \in [1, T]$

$\Rightarrow\ r_{P_t} - c + v_t \geq 0 \ \ \forall t \in [1, T]$

where $v_t = \left[r_{P_t} - c\right]^-$.

4. Risk Parity Approach

- Risk Parity vs Mean Variance
  - Def:
    - MV = risk & return balance
    - Risk Parity = risk allocation balance
  - Problems with MV:
    - Extremely skewed portfolios.
    - Hard to obtain a reliable estimate of expected return on any asset.
    - Useless during financial crises.

- **1/N Portfolio**: each asset takes the same risk.

$$w_n^{1/N} = \frac{1}{N}$$

- **Global Minimum Variance Portfolio**
  - OP:

$$\min_w w^T \Sigma w$$
$$\text{s.t. } w^T i = 1$$

  - Solution:

$$w^{MV} = \frac{1}{i^T \Sigma^{-1} i} \Sigma^{-1} i$$

    - If no correlation among asset returns, then

$$w_n^{MV} = \frac{\sigma_n^{-2}}{\sum_{n=1}^{N} \sigma_n^{-2}}$$

    - If $\sigma_1^2 = \cdots = \sigma_N^2$ and correlation between any two assets is fixed, then GMVP = 1/N Portfolio.

- Lagrangian for OP above:

$$\mathcal{L} = w^T \Sigma w + \lambda(1 - w^T i)$$
$$\nabla_w \mathcal{L} = \Sigma w - \lambda i = 0$$
$$\Sigma w = \lambda i$$

$$\because \sigma(w) = \sqrt{w^T \Sigma w}$$
$$\therefore \nabla \sigma(w) = \frac{1}{\sigma(w)} \Sigma w$$
$$\therefore \Sigma w = \lambda i \Longrightarrow \begin{bmatrix} \nabla_1 \sigma(w) \\ \vdots \\ \nabla_N \sigma(w) \end{bmatrix} = \begin{bmatrix} \frac{\lambda}{\sigma(w)} \\ \vdots \\ \frac{\lambda}{\sigma(w)} \end{bmatrix}$$
$$\Longrightarrow \nabla_1 \sigma(w) = \cdots = \nabla_N \sigma(w)$$

- **Risk Decomposition**

$$\begin{aligned} \sigma(w) &= \frac{1}{\sigma(w)} w^T \Sigma w \\ &= w^T \nabla \sigma(w) \\ &= \sum_{n=1}^{N} w_n \nabla_n \sigma(w) \end{aligned}$$

  - **Marginal Risk Contribution [MRC]**: $\nabla_n \sigma(w)$
  - **Total Risk Contribution [TRC]**: $w_n \nabla_n \sigma(w)$

- **Risk Parity Portfolio**
  - Condition: same TRC for all assets

$$w_1 \nabla_1 \sigma(w) = \cdots = w_N \nabla_N \sigma(w)$$

  - Derivation:

$$\sigma(w) = \sum_{n=1}^{N} w_n \nabla_n \sigma(w) \Longrightarrow \frac{w_n \nabla_n \sigma(w)}{\sigma(w)} = \frac{1}{N}$$

$$w_n = \frac{\sigma^2(w)}{(\Sigma w)_n N}$$

$$w^{RP} = \frac{\sigma^2(w^{RP})}{\Sigma w^{RP} N}$$

- OP:

$$\min_w w - \frac{\sigma^2(w)}{\Sigma w N}$$
$$\text{s.t. } w^T i = 1$$

- Solution: no closed-form solution. Can only be solved numerically.
  - If all correlation coefficients are equal:

$$w_n^{RP} = \frac{\sigma_n^{-1}}{\sum_{n=1}^{N} \sigma_n^{-1}}$$

$$w^{RP} = \frac{\sigma^{-1}}{i^T \sigma^{-1}}$$

  - If $\sigma_1^2 = \cdots = \sigma_N^2$ and correlation between any two assets is fixed, then RPP = 1/N Portfolio.

- **Maximum Diversification Portfolio**
  - OP:

$$\max_w \frac{\sigma^T w}{\sqrt{w^T \Sigma w}}$$
$$\text{s.t. } w^T i = 1$$

  - Solution:

$$w^{MD} = \frac{1}{i^T \Sigma^{-1} \sigma} \Sigma^{-1} \sigma$$

  - If all correlation coefficients are equal:

$$w_n^{MD} = \frac{\sigma_n^{-1}}{\sum_{n=1}^{N} \sigma_n^{-1}}$$

- If $\sigma_1^2 = \cdots = \sigma_N^2$ and correlation between any two assets is fixed, then MDP = 1/N Portfolio.

- Risk allocation is equalized as

$$\frac{\nabla_1 \sigma(w^{MD})}{\sigma_1} = \cdots = \frac{\nabla_N \sigma(w^{MD})}{\sigma_N}$$

5. Passive Management

- **Active Management**: balance **expected return & risk**
- **Passive Management**: minimize a discrepancy between a portfolio & the **benchmark index**
  - **Index funds**: passive management funds that mimic indices
  - ∋ stock indices, bond indices, currencies, commodities, hedge funds, etc.

- **Tracking Error**

$$\begin{aligned} e_t &= y_t - r_t^T w \\ &= y_t - [r_{1t} \quad \cdots \quad r_{Nt}] \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} \\ &= y_t - \sum_{n=1}^{N} w_n r_{nt} \end{aligned}$$

  - y_t: return on the benchmark index at time t
  - r_{nt}: return on asset n at time t
  - w_n: allocation weight for asset n

- **Tracking Error Minimization**

$$\min_w \frac{1}{T} \sum_{t=1}^{T} (y_t - r_t^T w)^2$$

$$\text{s.t.} \sum_{n=1}^{N} w_n = 1, w_n \geq 0$$

  - Matrix form:

$$\begin{aligned} &\min_w \frac{1}{T} e^T e \\ &\text{s.t.}\, e = y - Rw, w^T i = 1, w \geq 0 \end{aligned}$$