Fixed-Income Securities

Time Value of Money

- Interest Rate
  - Law of 72
- Discount Factor
- NPV
- IRR

Bond

- Yield & Price
- Price Sensitivity & Duration
- Convexity & Dispersion
- Yield Curve Estimation
- Defaultable Bond & Credit Spread

Fixed-Income Securities

1. Time Value of Money

- Notations
  - $M(t)$: value of investment at t
  - r: annual interest rate
  - m: #times to receive interest per annum
  - t: #years

- **Interest Rate**

$$M(t) = \begin{cases} (1+rt)M(0) & \text{simple interest rate} \\ \left(1+\frac{r}{m}\right)^{mt} M(0) & \text{discrete compounding} \\ e^{rt}M(0) & \text{continuous compounding} \end{cases}$$

  - **Law of 72**: how many yrs it takes to double the money

$$t \approx \frac{72}{r \times 100}$$

$$(\text{Derivataion: } (1+r)^t = 2 \ \longrightarrow \ t = \log_{1+r} 2)$$

  - Continuous compounding derivation:

$$\begin{aligned} \lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{mt} &= \lim_{a\to 0} e^{\frac{rt}{a}\ln(1+a)} \\ &= e^{rt\lim_{a\to 0}\frac{\ln(1+a)}{a}} \\ &= e^{rt} \end{aligned}$$

- **Discount Factor**

$$PV = B(t) \cdot FV$$

$$FV = \frac{PV}{B(t)}$$

$$B(t) = \begin{cases} \dfrac{1}{\left(1+\dfrac{r}{m}\right)^{mt}} & \text{discrete compounding} \\ e^{-rt} & \text{continuous compounding} \end{cases}$$

- **Net Present Value**
  - General Form

$$NPV = \sum_{n=0}^{N} B(t_n) C(t_n)$$

    - ♦ $t_0 = 0$
    - ♦ $B(t_0) = 1, C(t_0) =$ initial investment

  - Assume equal intervals between payments
    - ♦ Assumption:

$$t_n = \frac{n}{N} T = n\delta$$

where $\delta = \frac{T}{N}$ represents the length of an interval

    - ♦ New discount factor:

$$B(t_n) = x^n$$

$$x = \begin{cases} \dfrac{1}{\left(1+\dfrac{r}{m}\right)^{m\delta}} & \text{discrete compounding} \\ e^{-r\delta} & \text{continuous compounding} \end{cases}$$

    - ♦ Final form – Polynomial

$$NPV = \sum_{n=0}^{N} C_n x^n$$

- **Internal Rate of Return**
  - = r^* that makes NPV become 0
  - Step 1: solve x^* from the polynomial:

$$0 = \sum_{n=0}^{N} C_n x^n$$

    - ♦ Necessary conditions:
      - $C_0 < 0$
      - $\forall n$: $C_n \geq 0$. $\exists n$: $C_n > 0$.
      - $\sum_{n=0}^{N} C_n > 0$
  - Step 2: calculate r^* based on x^*:

$$r^* = \begin{cases} m\left[\left(\frac{1}{x^*}\right)^{\frac{1}{m\delta}} - 1\right] & \text{discrete compounding} \\ -\frac{\ln x^*}{\delta} & \text{continuous compounding} \end{cases}$$

  - Special cases
    - ♦ $\delta = 1, m = 1, N = 1$

$$0 = C_0 + C_1 x^* \longrightarrow r^* = \frac{C_1 + C_0}{-C_0}$$

    - ♦ $\delta = 1, m = 1, N \to \infty, \forall n$: $C_n = C$

$$0 = C_0 + \sum_{n=1}^{\infty} C_n {x^*}^n \longrightarrow r^* = \frac{C}{-C_0}$$

2. Bond

- The concepts
    - **Principal / Face value**: #cash paid to the bond holder on the predetermined future date
    - **Maturity** (date): the predetermined future date
    - **Time to maturity**: #time until maturity date
    - **Coupon-bearing bond**: the predetermined #cash paid periodically to the bond holder
    - **Coupon**: the predetermined #cash
    - **Zero-coupon bond**: no coupon

    - **Sovereign bond**: issued by national govs
    - **Municipal bond**: issued by local govs
    - **Corporate bond**: issued by corporations

- Bonds vs Bank loans
    - Bonds are often issued at auctions (i.e. primary market)
    - Bonds can be transferred from one holder to another
    - Bonds are tradable in financial markets (i.e. secondary market) until maturity
    - Bonds require **credit ratings** to assess the **default** risk of bond issuers

- Cash Flows of Bonds
    - Zero-coupon bond

t	0	1	…	$T-1$	T
$C(t)$	$-V$	0	…	0	F

- Coupon-bearing bond

t	0	1	...	$T-1$	T
$C(t)$	$-P$	C	...	C	$C+F$

- **Yield** to Maturity
  - Zero-coupon bond yield

$$y(t) = \begin{cases} \left(\frac{F}{V(t)}\right)^{\frac{1}{t}} - 1 & \text{1yr discrete compounding} \\ \frac{1}{t}\ln\frac{F}{V(t)} & \text{continuous compounding} \end{cases}$$

  - Coupon-bearing bond yield
    - Find the real root of the polynomial using **Newton's method**:

$$P(0) = C\sum_{t=1}^{T-1} B(t,y) + (C+F)B(T,y)$$

where

$$B(t,y) = \begin{cases} \frac{1}{(1+y)^t} & \text{1yr discrete compounding} \\ e^{-yt} & \text{continuous compounding} \end{cases}$$

- **Bond price**

$$V(y) = B(t,y)F$$

$$P(y) = C\sum_{t=1}^{T-1} B(t,y) + (C+F)B(T,y)$$

  - $V\ \&\ P \downarrow \ \leftrightarrow \ y \uparrow$

- **Price sensitivity** to a parallel shift in yield curve
  - Intuition of price sensitivity
    - The yield curve does not stay fixed. It frequently shifts due to business cycles, central bank interventions, market sentiments, …
  - Parallel shift:

$$\forall t: y(t) + \lambda$$

  - Sensitivity:

$$\frac{\lim_{\lambda \to 0} \dfrac{P(y+\lambda) - P(y)}{\lambda}}{P(y)} = \frac{\nabla_y P(y)}{P(y)}$$
$$= \frac{C \sum_{t=1}^{T} \nabla_y B(t,y) + F \nabla_y B(T,y)}{P(y)}$$

$$\nabla_y P(y) = \begin{cases} -C \displaystyle\sum_{t=1}^{T} \frac{tB(t,y)}{1+y} - F \frac{TB(T,y)}{1+y} & \text{1yr discrete} \\ -C \displaystyle\sum_{t=1}^{T} tB(t,y) - FTB(T,y) & \text{continuous} \end{cases}$$

  - Finalize:

$$\frac{\nabla_y P(y)}{P(y)} = \begin{cases} -\dfrac{D(y)}{1+y} & \text{1yr discrete} \\ -D(y) & \text{continuous} \end{cases}$$

where

$$D(y) = \frac{C \sum_{t=1}^{T} tB(t,y) + FTB(T,y)}{P(y)} = \textbf{duration}$$

    - $D(y) > 0 \ \rightarrow \ \frac{\nabla_y P(y)}{P(y)} < 0 \ \rightarrow P \begin{cases} \uparrow & \text{if } \lambda < 0 \\ \downarrow & \text{if } \lambda > 0 \end{cases}$
    - $D(y) = T$ for zero-coupon bond with maturity time T

- **Convexity**

$$C(y) = \frac{\nabla_y^2 P(y)}{P(y)} = \begin{cases} \dfrac{S(y) + \big(1 + D(y)\big)D(y)}{(1+y)^2} & \text{1yr discrete} \\ S(y) + \big(1 + D(y)\big)D(y) & \text{continuous} \end{cases}$$

  - **Dispersion**

$$S(y) = \frac{C\sum_{t=1}^{T}\big(t - D(y)\big)^2 B(t,y) + F\big(T - D(y)\big)^2 B(T,y)}{P(y)}$$

  - $C(y) > 0 \rightarrow$ price sensitivity = convex function
  - If $D(y)$ is the same for 2 bonds. The bond with larger $C(y)$ suffers less from upward shift ($\lambda > 0$) & gain more from downward shift ($\lambda < 0$).

- **Yield Curve Estimation**
  - Step 1: understand that the bond price of bond n is:

$$P_n = \sum_{t=1}^{T} B(t)C_n(t)$$

    where T = the longest maturity in the market.

  - Step 2: solve the system of equations:

$$\begin{bmatrix} P_1 \\ \vdots \\ P_N \end{bmatrix} = \begin{bmatrix} C_1(1) & \cdots & C_1(T) \\ \vdots & \ddots & \vdots \\ C_N(1) & \cdots & C_N(T) \end{bmatrix} \begin{bmatrix} B(1) \\ \vdots \\ B(T) \end{bmatrix}$$

  - Step 3: solve the yield curve:

$$y(t) = \begin{cases} \left(\dfrac{1}{B(t)}\right)^{\frac{1}{t}} - 1 & \text{1yr discrete} \\ \dfrac{1}{t}\ln\dfrac{1}{B(t)} & \text{continuous} \end{cases}$$

- **Defaultable Bond & Credit Spread**
  - Payment of defaultable zero-coupon bond

$$\begin{cases} \rho F & \text{Issuer defaults with } \pi \\ F & \text{Issuer doesn't default with } 1-\pi \end{cases}$$

    - π: probability of default
    - ρ: recovery rate

  - **Expected Payment**

$$(1-\pi)F + \pi\rho F = (1-(1-\rho)\pi)F$$

    - Fair price of defaultable zero-coupon bond

$$\tilde{V} = \begin{cases} \dfrac{(1-(1-\rho)\pi)F}{(1+\tilde{y})^T} & \text{1yr discrete} \\ e^{-\tilde{y}T}(1-(1-\rho)\pi)F & \text{continuous} \end{cases}$$

    - Yield on defaultable zero-coupon bond

$$\tilde{y} = \begin{cases} \dfrac{1+y}{(1-(1-\rho)\pi)^{\frac{1}{T}}} - 1 & \text{1yr discrete} \\ y - \dfrac{1}{T}\ln(1-(1-\rho)\pi) & \text{continuous} \end{cases}$$

    where y = yield on riskless bond with same maturity

  - **Credit Spread**
    - $= \tilde{y} - y$
    - $\because (1-(1-\rho)\pi) < 1 \ \ \therefore \tilde{y} > y$
    - $\pi \uparrow \ \rightarrow \ (\tilde{y} - y) \uparrow$
    - $\rho \downarrow \ \rightarrow \ (\tilde{y} - y) \uparrow$