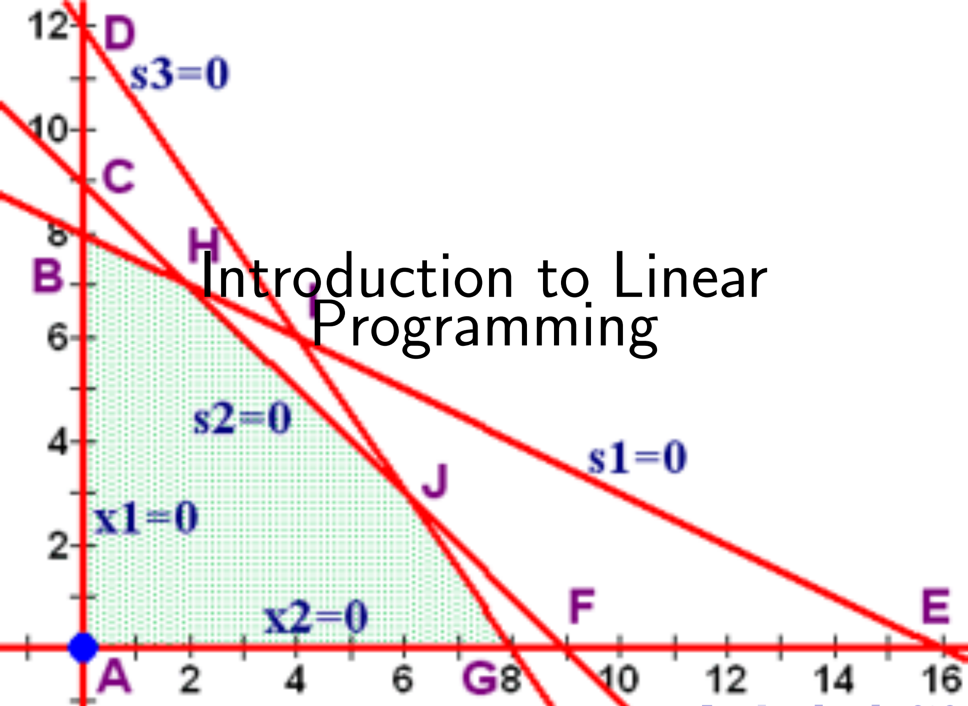
Introduction to Linear Programming



Linear Programming.

In a linear programming problem we are given a set of variables, an objective function a set of linear constrains and want to assign real values to the variables as to:

- satisfy the set of linear equations,
- maximize or minimize the objective function.

LP is of special interest because many combinatorial optimization problems can be reduced to LP: Max-Flow; Assignment problems; Matchings; Shortest paths; MinST; ...

Example.

A company produces 2 products P1, and P2, and wishes to maximize the profits.

Each day, the company *can produce* x_1 units of P1 and x_2 units of P2.
The company *makes a profit* of 1 for each unit of P1; and a profit of 6 for each unit of P2.

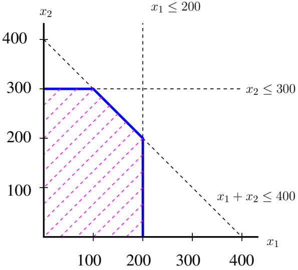
Due to supply limitations and labor constrains we have the following additional constrains: $x_1 \leq 200, x_2 \leq 300$ and $x_1 + x_2 \leq 400$.

What are the best levels of production?

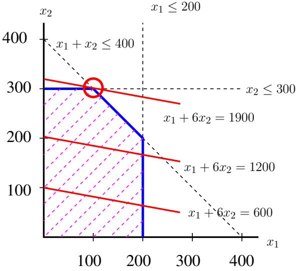
We format this problem as a linear program:

$$\begin{aligned} \text{Objective function:} \quad & \max(x_1 + 6x_2) \\ \text{subject to the constraints:} \quad & x_1 \leq 200 \\ & x_2 \leq 300 \\ & x_1 + x_2 \leq 400 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Recall a linear equation in x_1 and x_2 defines a line in $\mathbb{R}^2$. A linear inequality define a half-space.
The set of feasibles region of this LP are the (x_1, x_2) which in the convex polygon defined by the linear constrains.



x_2
$x_1 \leq 200$
400
300
200
100
$x_2 \leq 300$
$x_1 + x_2 \leq 400$
x_1
100 200 300 400



x_2
$x_1 \leq 200$
400
$x_1 + x_2 \leq 400$
300
$x_2 \leq 300$
$x_1 + 6x_2 = 1900$
200
$x_1 + 6x_2 = 1200$
100
$x_1 + 6x_2 = 600$
x_1
100 200 300 400

In a linear program *the optimum is achieved at a vertex of the feasible region*.

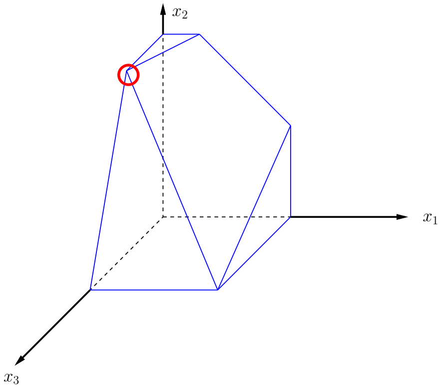
A LP is infeasible if

- The constrains are so tight that there are impossible to satisfy all of them. For ex. $x \geq 2$ and $x \leq 1$,
- The constrains are so loose that the feasible region is unbounded. For ex. $\max(x_1 + x_2)$ with $x_1, x_2 \geq 0$

Higher dimensions.

The company produces products P1, P2 and P3, where each day it produces x_1 units of P1, x_2 units of P2 and x_3 units of P3. and makes a profit of 1 for each unit of P1, a profit of 6 for each unit of P2 and a profit of 13 for each unit of P3. Due to supply limitations and labor constrains we have the following additional constrains: $x_1 \leq 200, x_2 \leq 300$, $x_1 + x_2 + x_3 \leq 400$ and $x_2 + 3x_3 \leq 600$.

$$
\begin{aligned}
\max(x_1 + 6x_2 + 13x_3)& \\
x_1 &\leq 200 \\
x_2 &\leq 300 \\
x_1 + x_2 + x_3 &\leq 400 \\
x_2 + 3x_3 &\leq 600 \\
x_1, x_2, x_3 &\geq 0.
\end{aligned}
$$


x_2
x_1
x_3

Standard form of a Linear Program.

INPUT: Given real numbers $\{c_i\}_{i=1}^n, \{a_{ji}\}_{1\leq j\leq m \& 1\leq i\leq n}\{b_i\}_{i=1}^n$
OUTPUT: real values for variables $\{x_i\}_{i=1}^n$
A linear programming problem is the problem or maximizing (minimizing) a linear function the objective function $P = \sum_{i=1}^n c_i x_j$ subject to finite set of linear constraints

$$\max \sum_{i=1}^n c_i x_j,$$
subject to:
$$\sum_{i=1}^n a_{ji} x_i = b_j \text{ for } 1 \leq j \leq m$$
$$x_i \geq 0 \text{ for } 1 \leq i \leq n$$

A LP is in standard form if the following are true:

1.- Non-negative constraints for all variables.

2.- All remaining constraints are expressed as $=$ constraints.

3.- All $b_i \geq 0$.

Equivalent formulations of LP.

A LP has many degrees of freedom:

1. It can be a maximization or a minimization problem.
2. Its constrains could be equalities or inequalities.
3. The variables are often restricted to be non-negative, but they also could be unrestricted.

Most of the *"real life"* constrains are given as inequalities.
The main reason to convert a LP into standard form is because the simplex algorithm starts with a LP in standard form.
But it could be useful the flexibility to be able to change the formulation of the original LP.

Transformations among LP forms

1.- To convert inequality $sum_{i=1}^{n} a_i x_i \leq b_i$ into equality: introduce a slack variable s_i to get $\sum_{i=1}^{n} a_i x_i + s_i = b_i$ with $s \geq 0$.
The slack variable s_i measures the amount of "non-used resource."
Ex: $x_1 + x_2 + x_3 \leq 40 \Rightarrow x_1 + x_2 + x_3 + s_1 = 40$
So that $s_1 = 40 - (x_1 + x_2 + x_3)$

2.- To convert inequality $\sum_{i=1}^{n} a_i x_i \geq -b$ into equality: introduce a surplus variable and get $\sum_{i=1}^{n} a_i x_i - s_i = b_i$ with $s \geq 0$.
The surplus variable s_i measures the extra amount of used resource.
Ex: $-x_1 + x_2 - x_3 \geq 4 \Rightarrow -x_1 + x_2 - x_3 - s_1 = 4$

Transformations among LP forms (cont.)

3.- To to deal with an unrestricted variable x (i.e. x can be positive or negative): introduce $x^+, x^- \geq 0$, and replace all occurrences of x by $x^+ - x^-$.
Ex: x unconstrained $\Rightarrow x = x^+ - x^-$ with $x^+ \geq 0$ and $x^- \geq 0$.

4- To turn max. problem into min. problem: multiply the coefficients of the objective function by -1.
Ex: $\max(10x_1 + 60x_2 + 140x_3) \Rightarrow \min(-10x_1 - 60x_2 - 140x_3)$.

Applying these transformations, we can reduce any LP into standard form, in which variables are all non-negative, the constrains are equalities, and the objective function is to be minimized.

Example:

$$
\begin{aligned}
\max(10x_1 + 60x_2 + 140x_3) \\
x_1 &\leq 20 \\
x_2 &\leq 30 \\
x_1 + x_2 + x_3 &\leq 40 \\
x_2 + 3x_3 &\leq 60 \\
x_1, x_2, x_3 &\geq 0.
\end{aligned}
$$

$$
\begin{aligned}
\min(-10x_1 - 60x_2 - 140x_3) \\
x_1 + s_1 &= 20 \\
x_2 + s_2 &= 30 \\
x_1 + x_2 + x_3 + s_3 &= 40 \\
x_2 + 3x_3 + s_5 &= 60 \\
x_1, x_2, x_3, s_1, s_2, s_3, s_4, s_5 &\geq 0.
\end{aligned}
$$

Algebraic representation of LP

Let $\vec{c} = (c_1, \dots, c_n)$ $\vec{x} = (x_1, \dots, x_n)$, $\vec{b} = (b_1, \dots, b_n)$ and A the $n \times n$ matrix of the coefficients involved in the constrains.

A L.P. can be represented using matrix and vectors:

$$
\begin{array}{rcl}
\max \sum_{i=1}^{n} c_i x_j & & \max \sum_{i=1}^{n} \vec{c}^T \vec{x} \\
\text{subject to} & \Rightarrow & \text{subject to} \\
\sum_{i=1}^{n} c_i x_j \geq b_j,\ 1 \leq j \leq m & & A\vec{x} \geq \vec{b} \\
x_i \geq 0,\ 1 \leq i \leq n & & \vec{x} \geq \vec{0}
\end{array}
$$

Given a LP

$$\begin{aligned}
&\min\ \vec{c}^T\vec{x}\\
&\text{subject to}\\
&\quad A\vec{x} \geq \vec{b}\\
&\quad \vec{x} \geq 0
\end{aligned}$$

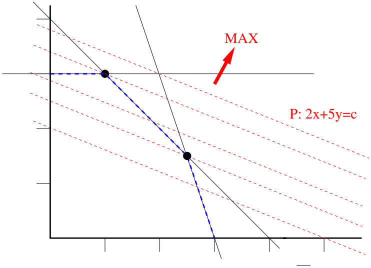
Any $\vec{x}$ that satisfies the constraints is a *feasible solution*.
A LP is *feasible* if there exists a feasible solution. Otherwise is said to be *infeasible*.
A feasible solution $\vec{x^*}$ is an optimal solution if

$$\vec{c}^T\vec{x^*} = \min\{\vec{c}^T\vec{x} \,|\, A\vec{x} \geq \vec{b}, \vec{x} \geq \vec{0}\}$$

The Geometry of LP

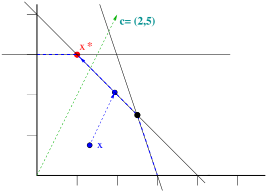
Consider P:

$$\begin{aligned} \max 2x+5y& \\ 3x+y &\leq 9 \\ y &\leq 3 \\ x+y &\leq 4 \\ x,y &\geq 0 \end{aligned}$$



Theorem: If there exists an optimal solution to P, $\vec{x}$, then there exists one that is a vertex of the polytope.

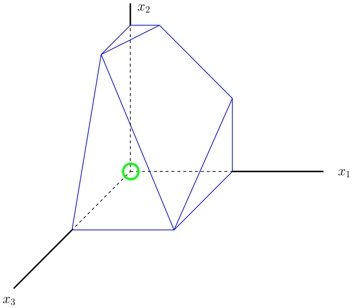
Intuition of proof If $\vec{x}$ is not a vertex, move in a non-decreasing direction until reach a boundary. Repeat, following the boundary.

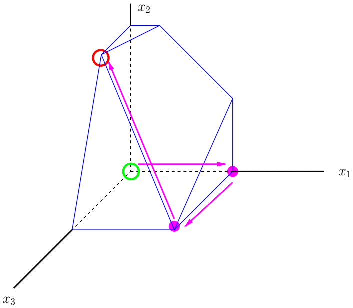


The Simplex algorithm

LP can be solved efficiently: George Dantzing (1947)

It uses hill-climbing: Start in a vertex of the feasible polytope and look for an adjacent vertex of better objective value. Until reaching a vertex that has no neighbor with better objective function.





Complexity of LP:

Input to LP: The number n of variables in the LP.

Simplex could be exponential on n: there exists specific input (the Klee-Minty cube) where the usual versions of the simplex algorithm may actually "cycle" in the path to the optimal. (see Ch.6 in Papadimitriou-Steiglitz, *Comb. Optimization: Algorithms and Complexity*)

In practice, the simplex algorithm is quite efficient and can find the global optimum (if certain precautions against cycling are taken).

It is known that simplex solves "typical" (random) problems in $O(n^3)$ steps.
Simplex is the main choice to solve LP, among engineers.

But some software packages use interior-points algorithms, which guarantee poly-time termination.

The Simplex algorithm: Searching the optimal can take long

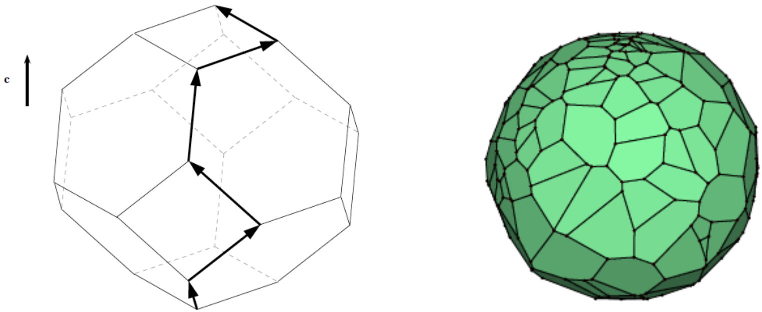


Figure: from The Nature of Computation by Moore and Mertens

LP Duality

Most important concept in LP.
Allows proofs of optimality and approximation.

LP P (PRIMAL)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

- Want to obtain upper bounds to $OPT(P)$
- The best UB?
  Take a linear combination of equations: $y^T Ax$
  $y^T Ax \leq y^T b$, because of the restrictions, for any x
  we can select y so that $y^T A = c^T$, $c^T x = y^T Ax \leq y^T b$,
  With the hope that the highest value could be close to $OPT(P)$.

The best lower bound for any x?

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y = c \\ & y \geq 0 \end{array}$$

But as we are maximizing this is equivalent to

$$\begin{array}{lll} \max & b^T y & \\ \text{s.t.} & A^T y \leq c & \quad \text{D (DUAL)} \\ & y \geq 0 & \end{array}$$

Primal-Dual

P (PRIMAL)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

D (DUAL)

$$\begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

Primal-Dual

P (PRIMAL)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

D (DUAL)

$$\begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

$$\begin{aligned} \max 2x_1 &+ 3x_2 \\ 4x_1 + 8x_2 &\leq 12 \\ 2x_1 + x_2 &\leq 3 \\ 3x_1 + 2x_2 &\leq 4 \\ x, y &\geq 0 \end{aligned}$$

$$\begin{aligned} \min 12y_1 + 3y_2 + 4y_3 & \\ 4y_1 + 2y_2 + 3y_3 &\geq 2 \\ 8y_1 + y_2 + 2y_3 &\geq 3 \\ y_1, y_2, y_3 &\geq 0, \end{aligned}$$

Duality Theorem

The following theorem is proved in any course in OR, and it is a consequence of the fact that *every feasible solution to the dual D is a lower bound on the optimum value of the primal P, and vice versa.*

Theorem (LP-duality theorem)

Let $x^* = (x_1^*, \ldots, x_n^*)$ be a finite optimal solution for the primal P, and let $y^* = (y_1^*, \ldots, y_m^*)$ be the finite optimal for the dual D, then

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_j^* .$$

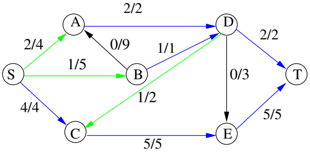
Consequences of the P-D Theorem

If P and D are the primal and the dual of a LP, then one fo the four following cases occurs:

1. Both P and D are infeasible.
2. P is unbounded and D is infeasible.
3. D is unbounded and P is infeasible.
4. Both are feasible and there exist optimal solutions x^* to P and y^* to D such that $c^T x^* = b^T y^*$.

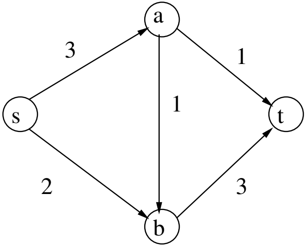
Linear programming formulation of max-flow.

$$
\begin{aligned}
\max f_{sa} + f_{sb} + f_{sc} \\
f_{sa} + f_{ba} &= f_{ad} \\
f_{sc} + f_{dc} &= f_{ce} \\
f_{sb} &= f_{bd} + f_{ba} \\
f_{ad} + f_{bd} &= f_{dc} + f_{de} + f_{dt} \\
f_{ce} + f_{de} &= f_{et} \\
f_{sa} \leq 3;\ f_{sb} \leq 5;\ f_{dt} \leq 2;\ f_{sc} &\leq 4 \\
f_{ba} \leq 9;\ f_{ad} \leq 2;\ f_{bd} \geq 1;\ f_{et} &\leq 5 \\
f_{dc} \leq 1;\ f_{ce} \leq 5;\ f_{de} &\geq 1 \\
f_{sa}, f_{sb}, f_{sc}, f_{ba}, f_{ad}, f_{bd} &\geq 0 \\
f_{dc}, f_{ce}, f_{de}, f_{dt}, f_{et} &\geq 0.
\end{aligned}
$$



Example: The Max-Flow problem

The max-flow min-cut theorem is a special case of duality.



$$
\begin{aligned}
\max \;\; f_{sa} + f_{sb} & \\
f_{sa} &\leq 3 \\
f_{sb} &\leq 2 \\
f_{ab} &\leq 1 \\
f_{at} &\leq 1 \\
f_{bt} &\leq 3 \\
f_{sa} - f_{ab} - f_{at} &= 0 \\
f_{sb} + f_{ab} - f_{bt} &= 0 \\
f_{sa}, f_{sb}, f_{ab}, f_{at}, f_{bt} &\geq 0.
\end{aligned}
$$

The dual of the previous LP:

$$
\begin{aligned}
\min\ \ 3y_{sa} + 2y_{sb} + y_{ab} + y_{at} + y_{bt} & \\
y_{sa} + u_a &\geq 1 \\
y_{sb} + u_b &\geq 1 \\
y_{ab} - u_a + u_b &\geq 0 \\
y_{at} - ua &\leq 1 \\
y_{bt} - u_b &\leq 3 \\
y_{sa}, y_{sb}, y_{ab}, y_{at}, y_{bt}, u_a, u_b &\geq 0.
\end{aligned}
$$

This D - LP defines the min-cut problem where for $x \in \{a, b\}$, $u_x = 1$ iff vertex $x \in S$, and $y_{xz} = 1$ iff $(x, z) \in$ cut (S, T).

By the LP-duality Theorem, any optimal solution to max-flow = to any optimal solution to min-cut.

Integer Linear Programming (ILP)

Consider again the Min Vertex Cover problem: Given undirected $G = (V, E)$ with $|V| = n$ and $|E| = m$, want to find $S \subseteq V$ with minimal cardinality s.t.. it covers all edges $e \in E$.
This can be expressed as a linear program of the following kind:
Let $\vec{x} \in \{0, 1\}^n$ a vector s.t. $\forall i \in V$:

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

Moreover we also ask the constrain $\forall (i, j) \in V \;\; x_i + x_j \geq 1 \quad (1)$
Define the incident matrix A of G as the $m \times n$ matrix, where

$$A_{ij} = \begin{cases} 1 & \text{if vertex } j \text{ is an endpoint of edge } i \in E \\ 0 & \text{otherwise} \end{cases}$$

Therefore we can write (1) as: $A\vec{x} \geq \vec{1}$

Integer Linear Programming

We can express the min VC problem as:

$$\begin{aligned}&\min\ \vec{1}^T\vec{x}\\&\text{subject to}\\&\quad A\vec{x} \geq \vec{1}\\&\quad \vec{x} \in (\mathbb{Z}^+ \cup \{0\})^m,\end{aligned}$$

where we have a new constrain, we require the solution to be integer.

Asking for the best possible integral solution for a LP is known as the Integer Linear Programming:

Integer Linear Programming

The ILP problem is defined:
Given $A \in \mathbb{Z}^{n \times m}$ together with $\vec{b} \in \mathbb{Z}^n$ and $\vec{c} \in \mathbb{Z}^m$, find a $\vec{x}$ that max (min) $\vec{c}^T$ subject to:

$$\min \vec{c}^T \vec{x}$$
subject to
$$A\vec{x} \geq \vec{1}$$
$$\vec{x} \in \mathbb{Z}^m,$$

Big difference between LP and ILP:
Ellipsoidal methods put LP in the class P
but ILP is in the class NP-hard.

Solvers for LP

Due to the importance of LP and ILP as models to solve optimization problem, there is a very active research going on to design new algorithms and heuristics to improve the running time for solving LP (algorithms) IPL (heuristics).

There are a myriad of solvers packages:

- GLPK: https://www.gnu.org/software/glpk/
- LP-SOLVE: http://www3.cs.stonybrook.edu/
- CPLEX: http://ampl.com/products/solvers/solvers-we-sell/cplex/
- GUROBI Optimizer: http://www.gurobi.com/products/gurobi-optimizer

LP to approximate solving ILP?

- Starting from an ILP we can relax the integrality restrictions to get a LP (the relaxad version)
- We can solve the LP in polynomial time either
  - solve the LP and find a way to round the values in the optimal solution to construct a valid solution for the ILP
  - solve the LP using primal-dual to maintain a feasible solution for the iLP and improve it the as much as possible
- in the hope that the algorithm provides a good approximation rate.

Vertex cover: LP relaxation

IP

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} x_i \\
\text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all } (i,j) \in E \\
& x_i \in \{0,1\} \quad \text{for all } i \in V
\end{aligned}
$$

LP

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} x_i \\
\text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all} (i,j) \in E \\
& x_i \geq 0 \ (x_i \in (0,1)) \quad \text{for all } i \in V
\end{aligned}
$$

Vertex cover: LP relaxation

IP

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all } (i,j) \in E \\ & x_i \in \{0,1\} \quad \text{for all } i \in V \end{aligned}$$

LP

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all} (i,j) \in E \\ & x_i \geq 0 \ (x_i \in (0,1)) \quad \text{for all } i \in V \end{aligned}$$

- Why can we drop $x_i \leq 1$?

Vertex cover: LP relaxation

IP

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all } (i,j) \in E \\ & x_i \in \{0,1\} \quad \text{for all } i \in V \end{aligned}$$

LP

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all} (i,j) \in E \\ & x_i \geq 0 \ (x_i \in (0,1)) \quad \text{for all } i \in V \end{aligned}$$

- Why can we drop $x_i \leq 1$?
- Let opt be the size of an optimal solution of the VC instance.

Vertex cover: LP relaxation

IP

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all } (i,j) \in E \\ & x_i \in \{0,1\} \quad \text{for all } i \in V \end{aligned}$$

LP

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all} (i,j) \in E \\ & x_i \geq 0 \; (x_i \in (0,1)) \quad \text{for all } i \in V \end{aligned}$$

- Why can we drop $x_i \leq 1$?
- Let opt be the size of an optimal solution of the VC instance.
- Let x^* be an optimal solution of the LP and $s^* = \sum_{i=1}^{n} x_i^*$.

Vertex cover: LP relaxation

IP

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all } (i,j) \in E \\ & x_i \in \{0,1\} \quad \text{for all } i \in V \end{aligned}$$

LP

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all} (i,j) \in E \\ & x_i \geq 0 \ (x_i \in (0,1)) \quad \text{for all } i \in V \end{aligned}$$

- Why can we drop $x_i \leq 1$?
- Let opt be the size of an optimal solution of the VC instance.
- Let x^* be an optimal solution of the LP and $s^* = \sum_{i=1}^{n} x_i^*$.
- Is there any relationship between s^* and opt?

Vertex cover: LP relaxation

IP

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all } (i,j) \in E \\ & x_i \in \{0,1\} \quad \text{for all } i \in V \end{aligned}$$

LP

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all} (i,j) \in E \\ & x_i \geq 0 \ (x_i \in (0,1)) \quad \text{for all } i \in V \end{aligned}$$

- Why can we drop $x_i \leq 1$?
- Let opt be the size of an optimal solution of the VC instance.
- Let x^* be an optimal solution of the LP and $s^* = \sum_{i=1}^{n} x_i^*$.
- Is there any relationship between s^* and opt? $s^* \leq$ opt

Vertex cover: Primal-Dual approximation

LP primal

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \text{for all} (i,j) \in E \\ & x_i \geq 0 \quad \text{for all } i \in V \end{aligned}$$

LP dual

$$\begin{aligned} \max \quad & \sum_{e \in E} z_e \\ \text{s.t.} \quad & \sum_{e=\{i,j\} \in E} z_e \leq 1 \quad \text{for all } i \in V \\ & z_e \geq 0 \quad \text{for all } e \in E \end{aligned}$$

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- Start with the integer infeasible primal solution $x = 0$, and the dual feasible solution $z = 0$.
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  - Increase all (unfrozen) variables z_e incidents with i until its dual constraint becomes tight.

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- Repeat while some constraint in primal (i,j) is unsatisfied:
  - Increase all (unfrozen) variables z_e incidents with i until its dual constraint becomes tight.
  - Set $x_i = 1$. Freeze all the increased variables z_e $(i \in e)$. If some dual constraint becomes tight set $x_j = 1$.

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- Cost of the solution computed by alg?
  At the end of the algorithm, if $z_e > 0$, z_e has been frozen. We do not know if for one (or both) endpoints the constraints got tight, but $x_i + x_j \leq 2$, for $e = (i, j)$. Therefore, we get a lower bound of 1 and an upper bound of 2. Which means that we have a 2-approximation algorithm.