# Models for dynamic fracture based on Griffith's criterion

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**Abstract** There has been much recent progress in extending Griffith's criterion for crack growth into mathematical models for quasi-static crack evolution that are well-posed, in the sense that there exist solutions that can be numerically approximated. However, mathematical progress toward dynamic fracture (crack growth consistent with Griffith's criterion, together with elastodynamics) has been more meager. We describe some recent results on a phase-field model of dynamic fracture, and introduce models for "sharp interface" dynamic fracture.

## 1 Introduction

Models for crack evolution begin with Griffith's criterion, stated for the quasi-static case. The idea is that as boundary conditions or loads slowly vary, the material can be assumed to be in elastic equilibrium at all times, subject also to the varying crack set, where the displacements are allowed to be discontinuous. Because the displacements correspond to equilibria, as a crack grows, there is a corresponding *instantaneous* change in the stored elastic energy. Griffith's criterion states that a crack grows so that the decrease in stored elastic energy (compared with the stored elastic energy corresponding to a stationary crack) balances the increase in surface energy, postulated to be proportional to the surface area of the crack (see [8]).

This principle was turned into a precise definition of quasi-static fracture evolution in [6] together with [5]. For a given varying Dirichlet condition $g$ on a time interval $[0,T_f]$, such an evolution is a family $(u(t),K(t))$ with $u(t) \in SBD_{g(t)}(\Omega)$,

$$K(t) = \bigcup_{\substack{\tau\in\mathbb{Q}\\ \tau\leq t}} S(u(\tau))$$

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where $SBD_g$ is the space of Special functions of Bounded Deformation with Dirichlet condition $g$, and $S(u(\tau))$ is the discontinuity set of $u(\tau)$ (considered to include the part of $\partial\Omega$ on which $u$ does not have the correct Dirichlet data). It further must satisfy, for all $T \in [0, T_f]$,

$$\frac{1}{2}\int_\Omega Ae(u(T)) : e(u(T))dx \leq \frac{1}{2}\int_\Omega Ae(w) : e(w)dx + \mathscr{H}^{N-1}(S(w)\setminus K(T)) \quad (1)$$
$$\forall w \in SBD_{g(T)}(\Omega)$$

$$\frac{1}{2}\int_\Omega Ae(u(T)) : e(u(T))dx + \mathscr{H}^{N-1}(K(T)) = \frac{1}{2}\int_\Omega Ae(u(0)) : e(u(0))dx \quad (2)$$
$$+\mathscr{H}^{N-1}(K(0)) + \int_0^T \int_\Omega \nabla \dot{g}(t) \nabla u(t) dx dt.$$

(1) represents global (unilateral) minimality (with given elasticity tensor $A$), and (2) represents energy balance.

[6] proposed a discrete minimization scheme for proving existence, which has been carried out under some assumptions in [7] and [4]. Note that in this model, loads cannot be considered in a reasonable way, due to the global minimality (1), but in most of the models below, they can.

## 2 Phase-field models

Computing solutions to problem (1)-(2) is quite difficult, but fortunately [1] introduced more regular functionals that $\Gamma$ converge to the above energy.

We define, for $\varepsilon > 0$, the elastic energy $\mathscr{E} : \mathrm{H}^1 \times \mathrm{H}^1 \to \mathbb{R} \cup \{+\infty\}$, and the (phase-field) surface energy $\mathscr{H} : \mathrm{H}^1 \to \mathbb{R}$, respectively, as

$$\mathscr{E}(u,v) := \frac{1}{2}\int_\Omega (v^2 + \eta_\varepsilon)|\nabla u|^2 \,\mathrm{d}x \quad \text{and} \quad \mathscr{H}(v) := \int_\Omega \left[(4\varepsilon)^{-1}(1-v)^2 + \varepsilon|\nabla v|^2\right] \mathrm{d}x,$$

where $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$. [1] showed that $\mathscr{E} + \mathscr{H}$ $\Gamma$-converges to

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathscr{H}^{N-1}(S(u))$$

defined for $u \in SBV(\Omega)$. $\mathscr{E}$ and $E$ can be easily altered for linear elasticity, with $Ae(u) : e(u)$ replacing $|\nabla u|^2$, and it is this elastic energy that we will consider. We will also consider below the kinetic energy $\mathscr{K} : \mathrm{H}^1 \to \mathbb{R}$,

$$\mathscr{K}(\dot{u}) := \frac{1}{2}\int_\Omega |\dot{u}|^2 \,\mathrm{d}x,$$

which is unaffected by the presence of a phase field in the model. The external loads at time $t$ are collected into a functional $\ell(t) \in \mathrm{H}^{-1}$,

$$\langle \ell(t), \varphi \rangle = \int_\Omega f(t) \cdot \varphi \, \mathrm{d}x + \int_{\Gamma_N} h(t) \cdot \varphi \, \mathrm{d}s \qquad \forall \varphi \in \mathrm{H}^1,$$

where $f(t) \in \mathrm{L}^2(\Omega;\mathbb{R}^3)$ and $h(t) \in \mathrm{L}^2(\Gamma_N;\mathbb{R}^3)$. We assume throughout that $\ell \in \mathrm{C}^1([0,T_f];\mathrm{H}^{-1})$. Finally, the total energy is given by

$$\mathscr{F}(t;u,\dot{u},v) := \mathscr{K}(\dot{u}) + \mathscr{E}(u,v) - \langle \ell(t), u \rangle + \mathscr{H}(v)$$

in the case of dynamics; we also write $\mathscr{F}(t;u,v)$ for the quasi-static energy, dropping the kinetic term. For simplicity below, we will take $h = 0$.

### *2.1 Quasi-statics*

Following the global minimization approach of [6], a pair $(u,v)$ is a quasi-static evolution if it satisfies global minimality and energy balance at every time: for every $T$,

$$(u(T),v(T)) \text{ minimizes } (u',v') \mapsto \mathscr{F}(T;u',v') \text{ subject to } v' \leq v(T) \tag{3}$$

$$\mathscr{F}(T;u(T),v(T)) = \mathscr{F}(0;u(0),v(0)) - \int_0^T \langle \dot{\ell}, u \rangle \, \mathrm{d}t + \int_0^T \int_\Omega \nabla \dot{g} \cdot \nabla u dx dt. \tag{4}$$

This problem was studied in [9], where it was shown that solutions converge to solutions of (1)-(2) (assuming $\ell = 0$ and the antiplane case). However, such global minimizers cannot in general be computed, which would seem to negate the advantage of the phase field approach. Yet, the natural way of trying to compute minimizers of (3)-(4) is to alternately minimize in $u$ and $v$ (see [2]), which produces pairs that are in some ways superior to solutions of (3)-(4) and (1)-(2).

First, these pairs are only stable, and are not generally global minimizers. Second, this alternate minimization can handle loads ($\ell \neq 0$), since this minimization will not constantly "see" that breaking off a piece of the material and sending it far enough away (in the direction of $f$) will always reduce the total energy (of course, this is related to producing local, not global, minimizers).

Finally, this alternate minimization approach suggests an approach to dynamics.

### *2.2 Dynamics*

A natural extension to dynamics, proposed in [3] and [10], is the following:

**Definition 1.** $(u,v)$ is a (phase field) dynamic fracture evolution if

$$\ddot{u} - \operatorname{div}((v^2 + \eta_\varepsilon) Ae(u)) = f \qquad \text{in } \Omega, \tag{5}$$

$$\mathscr{F}(T; u(T), \dot{u}(T), v(T)) = \mathscr{F}(0; u(0), \dot{u}(0), v(0)) - \int_0^T \langle \dot{\ell}, u \rangle \, \mathrm{d}t \tag{6}$$

$$v(T) \text{ minimizes } v' \mapsto \mathscr{E}(u(T), v') + \mathscr{H}(v') \text{ subject to } v' \leq v(T) \tag{7}$$

where (6) and (7) hold for every $T$ (and where for simplicity we take $h = 0$ and zero Dirichlet data) with initial conditions $u(0) = u_0$ and $\dot{u}(0) = u_1$ and as initial condition for $v$ we prescribe an arbitrary $v_0 \in \mathrm{H}^1$ which satisfies the unilateral minimality condition (7). Naturally, we look for $u(T) \in H^1_0, v(T) \in H^1$.

The idea, originating in [3], is that the principle for dynamic crack growth, (7), should be identical to that for the quasi-static setting, while $u$ should obey a wave equation corresponding to elastic stiffness $(v^2 + \eta_\varepsilon)A$. The numerical method to solve the above system, proposed in [3], is to iterate between updates in $u$ using time steps in (5), and updates in $v$ using (7). That such solutions converge, as the time step goes to zero, to a solution also obeying (5) and (7), while also satisfying (6), was proved in [10] (assuming a certain dissipation in the dynamics).

It is natural to then consider sharp interface models, i.e., models with crack sets rather than phase fields. The advantage of the above phase field approach is the fact that $v^2$ multiplies $Ae(u) : e(u)$, so that, even though there is no *instantaneous* update in $u$ due to a crack increment, there is still a decrease in stored elastic energy due to the decrease in $v$, which models a crack increment. The quasi-static setting can in this way be mimicked using a phase field. But, a new principle is needed for the sharp interface model. This can be seen from the fact that a limiting version of condition (7) is meaningless in the sharp interface case – crack increments have no effect on the stored elastic energy, so there can be no minimality condition.

We need to find other (precise) conditions for crack growth, without relying on minimality. Below we propose two, both of which extend to the quasi-static setting; we will show that there, the first principle is stronger than minimality, while the second is equivalent to it.

## 3 Sharp interface models

In observing solutions to phase field dynamic fracture [3], a striking feature is crack branching, which seems to only occur at high stress rates (and not surprisingly, is not seen in quasi-statics). This can only happen because $u$ develops large stresses at the crack "tip", in the directions of the branch, so that the minimality drives $v$ to decrease there, which "softens" the material, and results in still larger stresses (just like in quasi-statics, even though $u$ is following a wave equation and not minimality). For quasi-statics, $u$ is always in equilibrium, and so the largest stresses are straight ahead of the tip.

In seeking an equivalent (in principle) condition for crack growth, to replace minimality, it is natural to consider that a (5)-(7) solution with growing crack, and a solution for which the crack is prevented from growing, both balance energy. Furthermore, if the stresses at the crack tip are not big enough for minimality to drive the crack to grow, then a growing crack could not balance energy – the cost of the crack would exceed the reduction in elastic energy. These considerations lead to the first definition below, based only on the principle that if a crack can grow while balancing energy, then it must grow.

We first redefine the total energy $\mathscr{F}$ in the natural way, removing the phase field $v$ and the parameter $\eta_\varepsilon$ (i.e., $v^2+\eta_\varepsilon \equiv 1$), and replacing $\mathscr{H}(v)$ with $\mathscr{H}^{N-1}(K)$. We then have

**Definition 2.** $(u,K)$ is a *Maximal Dissipation* (MD) dynamic fracture evolution if:

1. $u$ is a solution of the wave equation on $\Omega \setminus K$:
$$\ddot{u} - \operatorname{div}(Ae(u)) = f \qquad \text{in } \Omega \setminus K,$$
with traction-free boundary condition on $K$ (and imposed initial conditions)
2. Energy balance: for all $T$,
$$\mathscr{F}(T;u(T),\dot{u}(T),K(T)) = \mathscr{F}(0;u(0),\dot{u}(0),K(0)) - \int_0^T \langle \dot{\ell},u\rangle \,\mathrm{d}t$$
3. Maximality: for all $T$, if a pair $(w,L)$ satisfies 1 and 2, with $K(t) \subset L(t)$ $\forall t \in [0,T]$, then $K(t) = L(t)$ for all $t \in [0,T]$

Note that 3 is just maximal dissipation with respect to set inclusion (hence the label (MD)), where subset and equality relations are taken to hold up to sets of $\mathscr{H}^{N-1}$-measure zero, and we consider solutions $u(t) \in SBD$, or $SBV$ in the anti-plane case. While the meaning of 1 is clear enough if $K$ is closed, and we conjecture that solutions will have (essentially) closed crack sets, it is worth stating a weak version (stated for zero loads):

$$\int_0^\infty \langle u,\phi_{tt}\rangle + \langle Ae(u),e(\phi)\rangle dt = 0$$

$\forall \phi \in C_0^2((0,\infty);SBD)$ with $S(\phi(t)) \subset K(t)$ $\forall t$.

The second model is motivated by the fact that the (MD) principle is very strong, and in fact strictly stronger, in the quasi-static setting, than minimality. Showing existence is expected to be quite difficult, which suggests attempting to formulate a weaker model.

The idea here is that, if the stresses at a crack tip have just become large enough for the crack to run while balancing energy, then if the crack does not run, a moment later, the stresses will be larger than necessary for the crack to run while balancing energy; in fact, it will now be possible for the crack to run and have the future energy *below* that for energy balance. To see this, notice that if the stresses are larger than necessary, then they are large enough for energy balance corresponding to a *larger* fracture toughness, i.e., a factor in front of $\mathscr{H}^{N-1}(K)$ larger than 1. But that means an energy decrease since the factor is just 1.

**Definition 3.** $(u,K)$ is a *(no) Decreasing Energy Extension* (DEE) dynamic fracture evolution if:

1. $u$ is a solution of the wave equation on $\Omega \setminus K$, just as in Definition 2
2. $(u,K)$ balances energy just as in Definition 2
3. Maximally: for all $T$, if a pair $(w,L)$ satisfies 1 with $(w(t),L(t)) = (u(t),K(t))$ for all $t \in [0,T]$, then $\frac{d}{dt}\mathscr{F}(t;w(t),\dot{w}(t),L(t))|_{t=T^+} \geq \frac{d}{dt}\mathscr{F}(t;u(t),\dot{u}(t),K(t))_{t=T^+}$

Note that condition 3 should hold for *all possible* futures of $\ell$ and $g$, so that in particular it should hold if they are all continued, from time $T$, as constants. In that case, 3 becomes:

for all $T$, if a pair $(w,L)$ satisfies 1 with $(w(t),L(t)) = (u(t),K(t))$ for all $t \in [0,T]$, then $\frac{d}{dt}\mathscr{F}(t;w(t),\dot{w}(t),L(t))|_{t=T^+} \geq 0$.

This means that there should be no extension of $(u,K)$ (corresponding to fixed future loads and boundary conditions) with decreasing energy, hence the label (no) DEE. This is the form of condition 3 that we will consider below.

## 4 General formulations: revisiting quasi-statics

**Definition 4 (Maximal Dissipation (MD)).** $(u,K)$ is a (MD) solution to an evolution problem (EP) if:

1. $(u,K)$ satisfies (EP)
2. $(u,K)$ balances total energy
3. $\forall T$, if $(w,L)$ satisfies 1) and 2), and $K(t) \subset L(t)$ $\forall t \in [0,T]$, then $K(t) = L(t)$ $\forall t \in [0,T]$.

Note that examples of (EP) include both dynamic evolutions and quasi-static evolutions. In the former case, satisfying (EP) means simply obeying elastodynamics with respect to the crack set $K$, and in the latter case, (EP) means that $u$ is in equilibrium with respect to $K$ at each time (for example, $u(t)$ is a global minimizer with respect to $K(t)$ for each $t$, in the case of globally minimizing quasi-static evoution).

**Definition 5 (Decreasing Energy Extension (DEE)).** $(u,K)$ is a (DEE) solution to an evolution problem (EP) if:

1. $(u,K)$ satisfies (EP)
2. $(u,K)$ balances total energy
3. $\forall T$, if a pair $(w,L)$ satisfies 1 with $(w(t),L(t)) = (u(t),K(t))$ for all $t \in [0,T]$, then $\frac{d}{dt}\mathscr{F}(t,w(t),L(t))|_{t=T^+} \geq 0$

where in condition 3, we assume loads and boundary conditions are continued as constants from time $T$.

We now investigate the relationship between these solutions in the context of quasi-statics, and globally minimizing quasi-static evolutions (GM). For simplicity, we will assume the anti-plane case, and $A \equiv I$. We will show that

$$(MD) \Rightarrow (DEE) \Leftrightarrow (GM)$$

**Proposition 1.** *If $(u,K)$ is a (MD) quasi-static evolution (globally minimizing), then it is a globally minimizing quasi-static evolution in the sense of* (1)-(2).

*Proof.* Suppose that $(u,K)$ is a (MD) solution but not a globally minimizing quasi-static evolution. Then there exists a time $t$ and $v \in SBV$ such that

$$\int_\Omega |\nabla v|^2 dx + \mathscr{H}^{N-1}(S(v) \setminus K(t)) < \int_\Omega |\nabla u(t)|^2 dx.$$

By continuity of the elastic plus surface energy, we can choose $\delta > 0$ such that

$$\int_\Omega |\nabla v|^2 dx + \mathscr{H}^{N-1}(S(v) \setminus K(t-\delta)) < \int_\Omega |\nabla u(t-\delta)|^2 dx.$$

We then define a competitor $(w,L)$ as follows. We first consider a smooth curve with two endpoints $C \subset \Omega$ (in 2-D; 3-D is similarly handled by choosing a smooth surface) with length greater than the total energy of $(u,K)$ for all times, and such that $\mathscr{H}^{N-1}(C \cap K(t)) = 0$ for all $t$. For $\lambda \in [0, \mathscr{H}^{N-1}(C)]$, set $C(\lambda)$ to be the curve in $C$ that, beginning at one designated endpoint of $C$, has length $\lambda$. We then choose $\lambda$ such that, with $v_\lambda$ the elastic minimizer for time $t-\delta$ with $S(v_\lambda) \subset K(t-\delta) \cup S(v) \cup C(\lambda) =: K_\lambda$, we have

$$\int_\Omega |\nabla v_\lambda|^2 dx + \mathscr{H}^{N-1}(K_\lambda \setminus K(t-\delta)) = \int_\Omega |\nabla u(t-\delta)|^2 dx$$

(we can do this since, for $\lambda = 0$, the left hand side above is less than the right, and for $\lambda = \mathscr{H}^{N-1}(C)$, we have the reverse inequality, while as $\lambda$ increases, the elastic energy is decreasing, and the surface energy is increasing and continuous).

We then set $w(t-\delta) := v_\lambda$ and $L(t-\delta) := K_\lambda$. The pair $(w,L)$ is continued for $\tau > t-\delta$ as a globally minimizing quasi-static evolution, as in [7], so that energy balance is maintained. For $\tau < t-\delta$, we set $(w,L) := (u,K)$. $(w,L)$ is then a competitor for $(u,K)$ in Definition 4, but condition 3 in that definition is violated since $\mathscr{H}^{N-1}(L(\tau) \setminus K(\tau)) > 0$ for $\tau \in [t-\delta, t]$, contradicting $(u,K)$ being a (MD) solution. This completes the proof.

**Proposition 2.** *If $(u,K)$ is a (DEE) quasi-static evolution (globally minimizing), then it is a globally minimizing quasi-static evolution in the sense of* (1)-(2).

*Proof.* If at some time $T$ $(u,K)$ is not a global minimizer, it is immediate that we can have an extension with a jump decrease in energy, so that we have

$$\frac{d}{dt}\mathscr{F}(t;w(t),L(t))|_{t=T^+} < 0,$$

contradicting $(w,L)$ being a (DEE) solution.

**Proposition 3.** *If $(u,K)$ is a a globally minimizing quasi-static evolution in the sense of* (1)-(2)*, then it is a (DEE) quasi-static evolution (globally minimizing).*

*Proof.* We give a proof of a stronger version, without the assumption that the boundary conditions are continued as constants. Suppose that $(u,K)$ is a globally minimizing quasi-static evolution, and let $t$ and an extension $(w,L)$ starting at $t$ be given. Note that $(u(t),K(t))$ is a unilateral global minimizer, and we consider a discrete globally minimizing extension: define $\tilde{u}(t+\Delta t)$ to be the minimizer of the total energy corresponding to time $t+\Delta t$, subject to the irreversibility constraint, so that the surface energy in the total energy is

$$\mathscr{H}^{N-1}(S(\tilde{u}(t+\Delta t))\setminus K(t)).$$

By usual energy estimates (see section 3.1 in [7]) we have that

$$\mathscr{F}(t+\Delta t;\tilde{u}(t+\Delta t),K(t)\cup S(\tilde{u}(t+\Delta t))) = \mathscr{F}(t;u(t),K(t)) + \int_t^{t+\Delta t}\int_\Omega \nabla\dot{g}\cdot\nabla u dx dt + o(\Delta t),$$

which also holds for $u(t+\Delta t)$ replacing $\tilde{u}(t+\Delta t)$. By the minimality of $\tilde{u}$, we have $\frac{d}{d\tau}\mathscr{F}(\tau;w(\tau),L(\tau))|_{\tau=t^+} \geq \frac{d}{d\tau}\mathscr{F}(\tau;u(\tau),K(\tau))|_{\tau=t^+}$, completing the proof.

*Remark 1.* We conclude by posing the following questions:

1. Are solutions of these dynamic models consistent with Griffith's criterion, in the sense that if $u_0$ is in equilibrium such that the initial crack set has energy release below that necessary for crack growth, and $u_1 = 0$, then they dynamic crack does not run?
2. Is there much stronger regularity of dynamic solutions than (is provable) for quasi-static evolutions?
3. Are any of these dynamic models the limit of the phase-field models? (Perhaps in principle and some situations, but probably not always true)
4. What is the quasi-static limit of the phase-field dynamic model? (Probably it is *not* a phase-field quasi-static global minimizer, except when $\mathscr{H}$ is continuous in time)
5. What is the quasi-static limit of the sharp-interface dynamic model? (Probably it is *not* a quasi-static global minimizer, as in 4.)

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