A phase-field model of dynamic fracture

CES Seminar Work WS17/18, Karsten Paul
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Agenda

- Introduction to fracture
  - Brittle and ductile fracture
- Phase-field formulation
  - Griffith‘s theory of brittle fracture
  - Phase-field theory
  - Energy approximation
  - Strong form
- Numerical formulation
  - Weak form
  - Spatial and temporal discretization
- Numerical result
- Summary

Cracks: Brittle and ductile fracture

- Engineering designs: reliable prediction of fracture and material failure
  - Experiments: high expense and enormous costs
  - FEM: approximate crack nucleation and propagation

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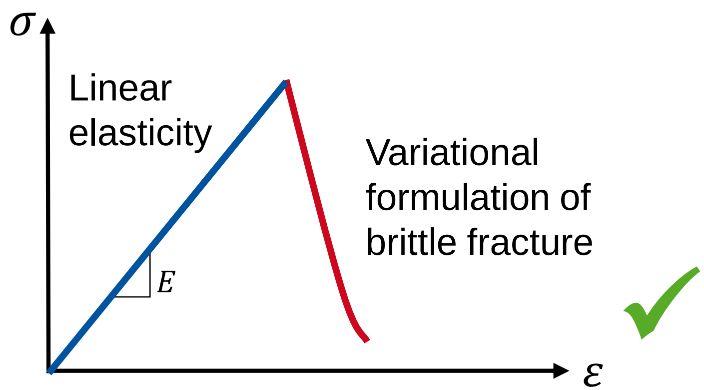
Brittle fracture



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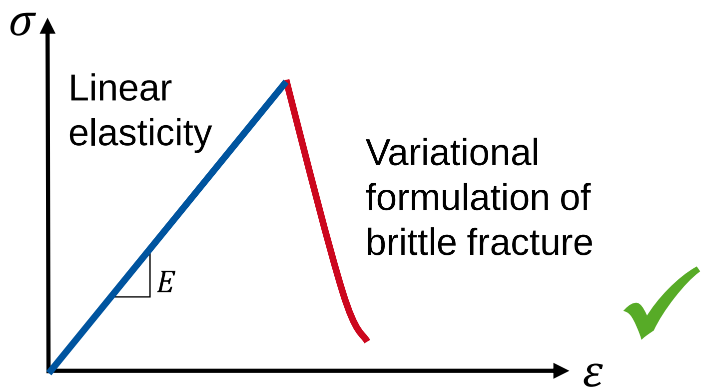
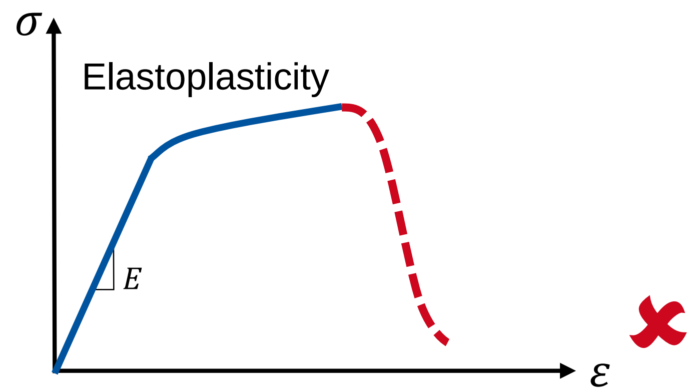


Ductile fracture



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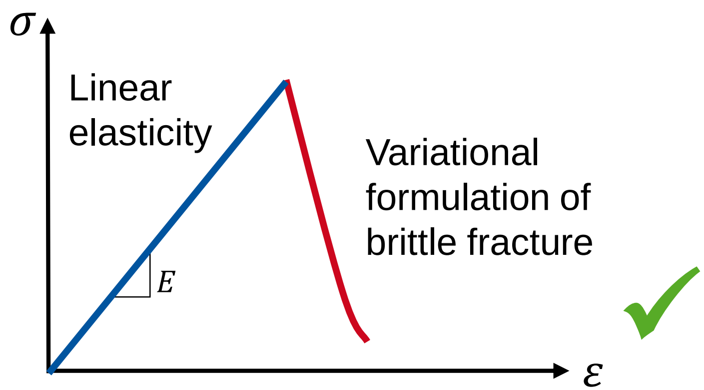

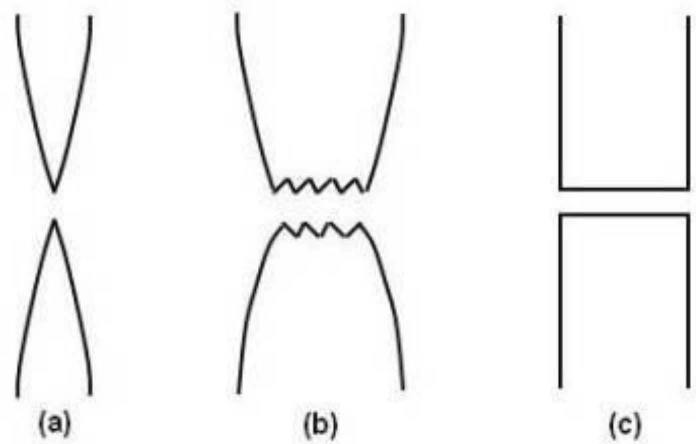

Macroscopic View of Fractures

K P Shah, The Handbook on Mechanical Maintenance

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How to describe the splitting of the material?

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Sharp interface models

- Insert **discontinuities** into the body
  - Enrich displacement field

    Cohesive segments method

    $$\boldsymbol{u}(\boldsymbol{x}, t) \approx \sum_A (N_A(\boldsymbol{x}) \boldsymbol{u}_A(t) + \ \dots)$$

  - Mesh handling

    Virtual crack closure technique

- Great drawback: 3D

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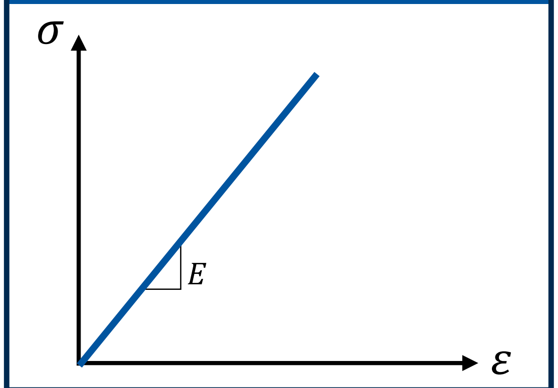
Diffuse interface models

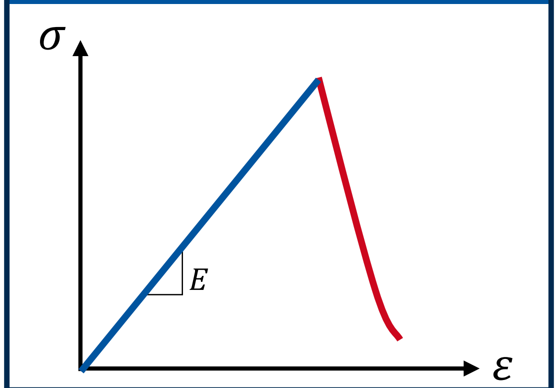
- Phase-field methods
  - Model interfaces between different phases
  - Smooth transition
  - No discontinuities

Phase-field formulation

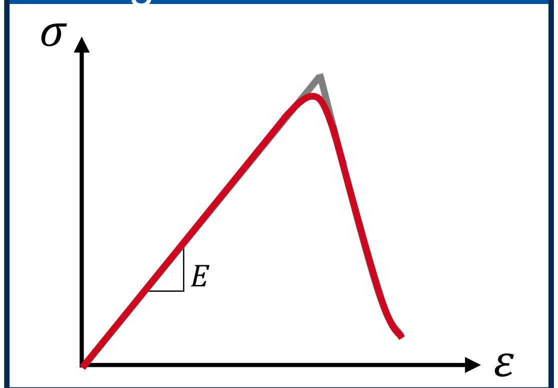
Linear elasticity



Variational formulation of brittle fracture



Phase-field formulation: Regularized version



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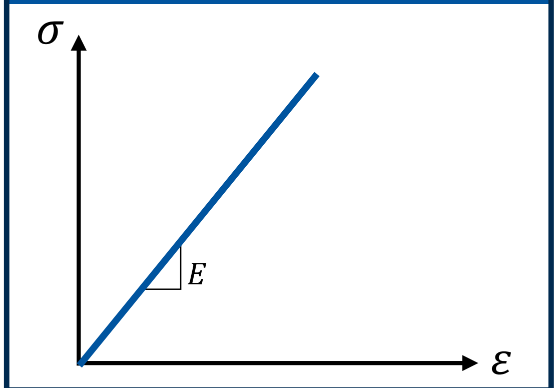

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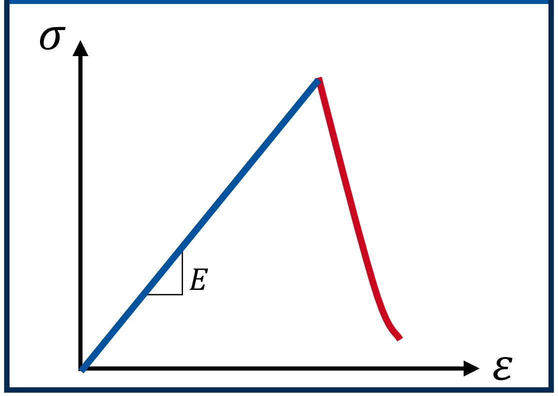

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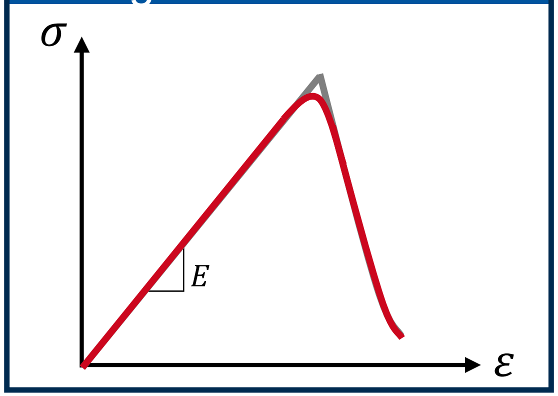
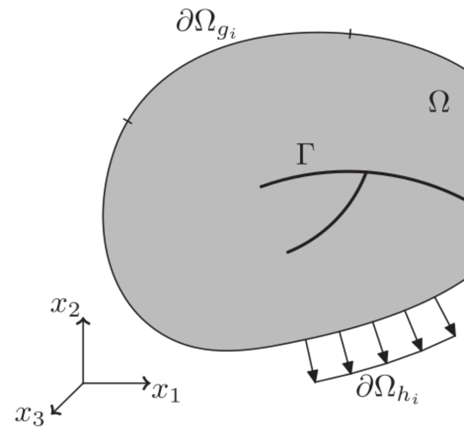


Notation and assumptions

- Small deformation and deformation gradients
- Isotropic linear elasticity

- Body Ω with boundary $\partial\Omega$
- *Fracture surface* Γ
- Dirichlet BC $\boldsymbol{u} = \boldsymbol{g}$ on $\partial\Omega_{\boldsymbol{g}_i}$
- Neumann BC $\boldsymbol{t} = \boldsymbol{h}$ on $\partial\Omega_{\boldsymbol{h}_i}$



Phase-field formulation

Griffith's theory and phase-field theory

- Elastic energy density: $\Psi_e(\boldsymbol{\varepsilon}) = \frac{1}{2}\lambda\, tr(\boldsymbol{\varepsilon})^2 + \mu\, \boldsymbol{\varepsilon}{:}\,\boldsymbol{\varepsilon}$
- Critical fracture energy density: G_c *Energy necessary to create a unit area of fracture surface*

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Recall: Hamilton's principle

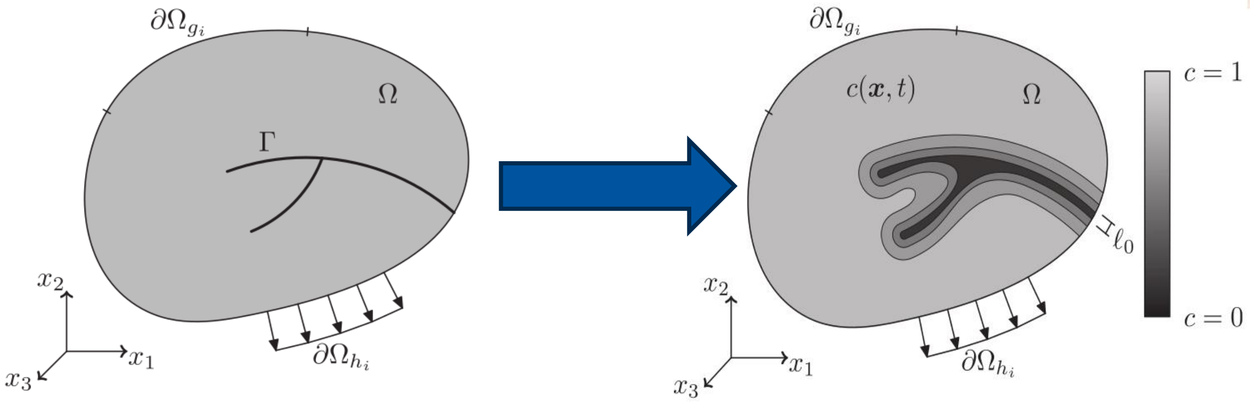
Motion of mechanical system: $J(q,\dot{q}) = \int_{t_0}^{t_1} L(q,\dot{q},t)dt \overset{!}{\rightarrow} stationary\ point$

Lagrangian (here): $L(\boldsymbol{u},\dot{\boldsymbol{u}},t) = \int_\Omega \left(\frac{1}{2}\rho\dot{\boldsymbol{u}}\dot{\boldsymbol{u}} - \Psi_e(\boldsymbol{\varepsilon})\right) d\Omega - \int_\Gamma G_c d\Gamma$

⇒ Euler-Lagrange equation to find minimizer

Griffith's theory and phase-field theory

Approximation of the fracture surface



Phase-field: $c(\boldsymbol{x}, t) \in \mathbb{R}^d$

$c = 1$: undamaged

$c = 0$: totally broken

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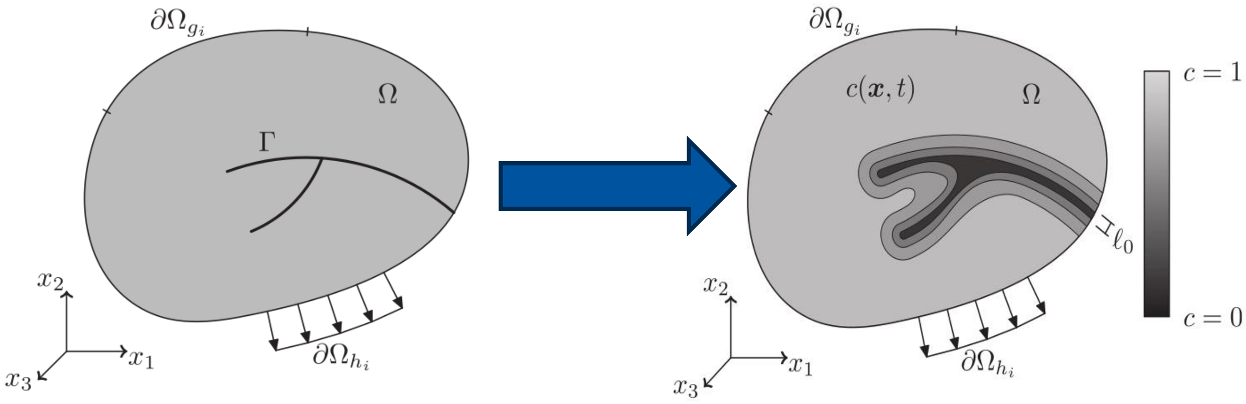

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  - Approximation: $\int_\Gamma G_c d\Gamma \approx \int_\Omega G_c \Gamma_{c,n} d\Omega$
  - Crack density functional: $\Gamma_{c,2} = \frac{1}{4l_0}\left[(c-1)^2 + 4l_0^2|\nabla c|^2\right]$

$$\Gamma_{c,4} = \frac{1}{4l_0}\left[(c-1)^2 + 2l_0^2|\nabla c|^2 + l_0^4(\Delta c)^2\right]$$

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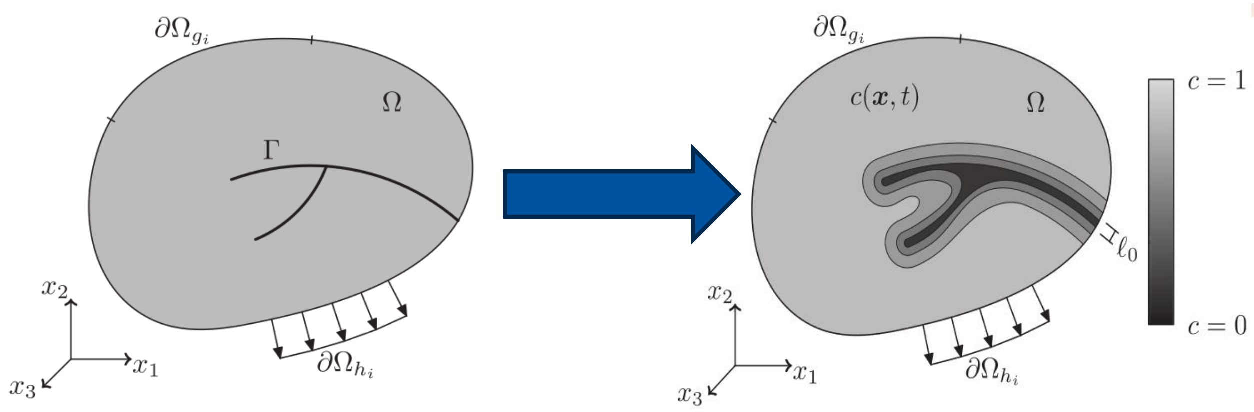

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Energy approximation and strong form

Coupling phase-field and kinematics

$$\Psi_e(\boldsymbol{\varepsilon}, c) = \underbrace{g(c)\Psi_e^+(\boldsymbol{\varepsilon})}_{\textit{Tensile}} + \underbrace{\Psi_e^-(\boldsymbol{\varepsilon})}_{\textit{Compressive}}$$

$g(c) = m(c^3 - c^2) + 3c^2 - 2c^3$ $\triangleq$ Crack growth driven by development of elastic strains

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Strong form: Governing equations (abbreviated)

- Local form of the linear momentum balance: $\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \rho \ddot{\boldsymbol{u}}$
- + BC + IC

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Strong form: Governing equations (abbreviated)

- Local form of the linear momentum balance: $\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \rho \ddot{\boldsymbol{u}}$
- Phase-field evolution, 2^{nd} order theory: $\frac{2l_0 H}{G_c} g'(c) + c - 4l_0^2 \Delta c = 1$
- Phase-field evolution, 4^{th} order theory: $\frac{2l_0 H}{G_c} g'(c) + c - 2l_0^2 \Delta c + l_0^4 \Delta(\Delta c) = 1$
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Numerical formulation

Weak form (abbreviated)

Weak form: Multiply with variation and integrate by parts

Find $\boldsymbol{u}(t)$ *and* $c(t)$ *s.t.* $\forall$ *variations* $\boldsymbol{w}$ *and* q:

$$(\rho\ddot{\boldsymbol{u}}, \boldsymbol{w})_\Omega + (\boldsymbol{\sigma}, \nabla \boldsymbol{w})_\Omega = (\boldsymbol{h}, \boldsymbol{w})_{\partial\Omega_h} + (b, \boldsymbol{w})_{\partial\Omega_g}$$

$$\left(\frac{2l_0 H}{G_c} g'(c) + c, q\right)_\Omega + (4l_0^2 \nabla c, q)_\Omega = (1, q)_\Omega \quad , \text{for } n = 2$$

$$\left(\frac{2l_0 H}{G_c} g'(c) + c, q\right)_\Omega + (2l_0^2 \nabla c, q)_\Omega + (l_0^4 \Delta c, q)_\Omega = (1, q)_\Omega \quad , \text{for } n = 4$$

+ BC + IC, $(\cdot,\cdot)_\Omega$ *:* L^2 *inner product*

- FE approximation: $\boldsymbol{u}(\boldsymbol{x}, t) \approx \boldsymbol{u}^h(\boldsymbol{x}, t) = \sum_{A=1}^{n} N_A(\boldsymbol{x})\, \boldsymbol{u}_A(t)$
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Spatial and temporal discretization

Recall

- Crack density functional: $\Gamma_{\mathrm{c},4} = \frac{1}{4l_0}\left[(c-1)^2 + 2l_0^2|\nabla c|^2 + l_0^4(\Delta c)^2\right]$
- Term in the weak form: $(l_0^4 \Delta c, q)_\Omega$

⇒ 2^{nd}-order continuous derivatives of c required

Spatial and temporal discretization

Isogeometric Analyis (IGA)

- Splines (e.g. NURBS) used for geometric domain and analysis
- Exact representation of geometries
- Smooth basis (FEA: continuous but not smooth basis)
- More accurate stress representation

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Numerical formulation

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Generalized-α scheme

- Implicit
- Newton-iteration required
- 2^{nd}-order accuracy for optimal parameters

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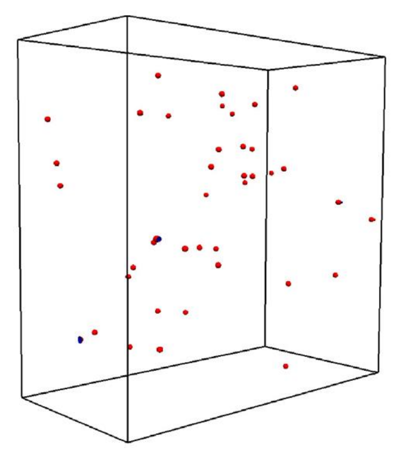
Three-dimensional crack propagation over time



Initial state

- Nucleation sites randomly seeded
- Displacement imposed on top and bottom surface

Borden, M.J., Hughes, T.J.R., Landis, C.M., Verhoosel, C.V: A higher-order phase-field model for brittle fracture: Formulation and analyis within the isogeometric analysis framework, Computer Methods in Applied Mechanics and Engineering 273 (2014), pages 100-118

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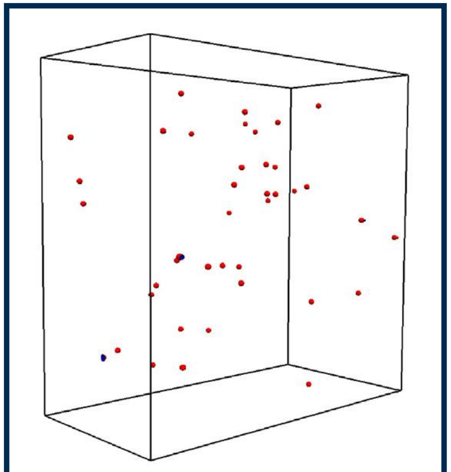

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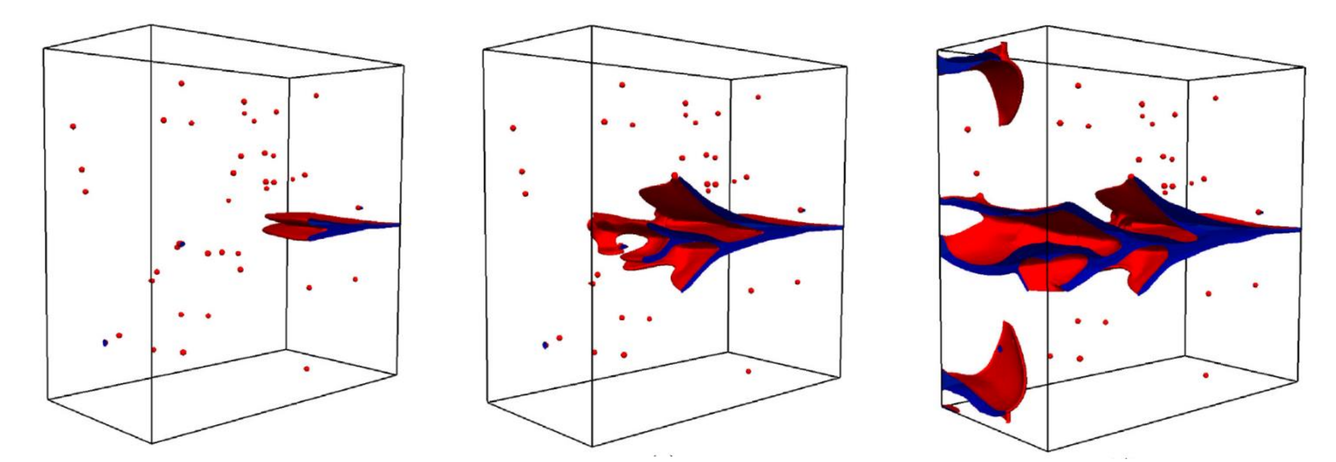

Crack progression
- Branching
- Turning
- Merging

Red: *Isosurface at* $c = 0.1$
Blue: *Area interior to this isosurface*

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Phase-field method: Models and benefits

- General differences:

Quasi-static	Dynamic
2^{nd}-order-	4^{th}-order phase-field theory
Brittle fracture	Ductile fracture
Small deformations	Large deformations
Elasticity	Elastoplasticity

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- Benefits: - 2D to 3D
  - Crack not a function of the geometry
  - No numerical tracking of crack

Thank you for your attention