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EXISTENCE OF SOLUTIONS TO A REGULARIZED MODEL OF DYNAMIC FRACTURE

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Existence and convergence results are proved for a regularized model of dynamic brittle fracture based on the Ambrosio–Tortorelli approximation. We show that the sequence of solutions to the time-discrete elastodynamics, proposed by Bourdin, Larsen & Richardson as a semidiscrete numerical model for dynamic fracture, converges, as the time-step approaches zero, to a solution of the natural time-continuous elastodynamics model, and that this solution satisfies an energy balance. We emphasize that these models do not specify crack paths *a priori*, but predict them, including such complicated behavior as kinking, crack branching, and so forth, in any spatial dimension.

Keywords: Dynamic fracture mechanics; phase-field approximation; crack path; existence of solutions; convergence; energy balance.

AMS Subject Classification: 74R10, 74R15, 74H20

1. Introduction

The starting point for models predicting fracture is Griffith's criterion,[14] originally formulated in the quasi-static setting. It assumes that, as a crack grows, the displacement field is instantly in a new equilibrium (new, since the displacement may be discontinuous across the crack increment). The resulting decrease in stored elastic energy can then be balanced with the work required to create the crack increment, postulated to be proportional to the newly created area. The constant of proportionality is usually labeled *fracture toughness*. In other words, the rate of elastic energy decreases per unit area, the (quasi-static) *energy release rate*, is proportional to the fracture toughness. Griffith's criterion stipulates that the crack grows only if the energy release rate equals the fracture toughness. The crack is *stable*, if the energy release rate does not exceed the fracture toughness, and it is labeled *unstable* if it does exceed the fracture toughness.[17]

Traditionally, these ideas could be formalized only for relatively simple crack topologies, often only for a pre-defined crack path. Only recently was the theory of brittle fracture freed from this restriction.[1,12] Ambrosio & Braides[1] propose minimizing the sum of stored elastic energy and surface energy of discontinuity sets, to obtain displacements that are stable in the sense of Griffith. That is, for displacements $u \in \mathrm{SBV}(\Omega)$, the space of *special functions of bounded variation*, with Ω representing the reference configuration of a body (u taking real values, modeling anti-plane displacement), they consider energy functionals of the form

$$E(u) := \frac{\mu}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + G_c \mathcal{H}^{N-1}(S(u)). \tag{1.1}$$

We usually refer to $\frac{\mu}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x$ as the *elastic energy* and to $G_c\mathcal{H}^{N-1}(S(u))$ as the *surface energy*. Here, and throughout, μ denotes the stiffness and G_c the fracture toughness, $S(u)$ denotes the discontinuity set of u, $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure, and the minimization is performed subject to a Dirichlet condition. (For the time being, we ignore the problem of a crack forming along $\partial\Omega$, releasing u from the Dirichlet data there; we will address this issue in Sec. 2.1.) The idea is that, if u is a minimizer of E, then adding any increment to its crack set $S(u)$ cannot reduce the elastic energy by more than the cost of the increment in surface energy. Therefore, the "crack" $S(u)$ is stable in the sense of Griffith.

The first well-posed (by which we mean, throughout the paper, that existence can be shown) mathematical models of quasi-static fracture can be found in Dal Maso, Francfort & Toader,[9] Francfort & Larsen,[11] and Francfort & Marigo.[12] In these references, the Dirichlet data u_D is varying in time and an evolution u is sought such that, at each time t, $u(t)$ minimizes E subject to the Dirichlet boundary condition, and subject to an irreversibility constraint on the crack set. More precisely, it is required that

$$\begin{aligned}\frac{\mu}{2}\int_\Omega |\nabla u(t)|^2\,\mathrm{d}x \le \frac{\mu}{2}\int_\Omega |\nabla w|^2\,\mathrm{d}x + G_c\mathcal{H}^{N-1}\big(S(w)\setminus C(t)\big)&\\ \forall\, w \in \mathrm{SBV}(\Omega) \text{ s.t. } w|_{\partial\Omega} = u_D(t)|_{\partial\Omega},&\end{aligned} \tag{1.2}$$

where $C(t)$ denotes the *crack set* at time t, which is essentially the union of discontinuity sets $S(u(\tau))$, $\tau \le t$. Additionally, an energy balance formula is stipulated so that a suitably defined energy functional, including the work done by the boundary condition, is constant in time.

The strategy for proving existence of solutions to this model, proposed in the paper of Francfort & Marigo,[12] is based on a time discretization. At step $t_i^n = i/n$, the solution $u_n(t_i^n)$ is a minimizer of

$$w \mapsto \frac{\mu}{2}\int_\Omega |\nabla w|^2\,\mathrm{d}x + G_c\mathcal{H}^{N-1}\big(S(w)\setminus \cup_{j<i} S(u_n(t_j^n))\big),$$

subject to $w = u_D(t_i^n)$ on $\partial\Omega$, $i = 1,\dots,n$, $n \geq 1$. It was hoped that limits of these sequences of discrete trajectories $(u_n(t_1^n),\dots,u_n(t_n^n))$, as $n \to \infty$, would satisfy, among other things, the unilateral minimality condition (1.2) and the correct energy balance.

Proving the unilateral minimality was not straightforward (see Dal Maso & Toader[9]), but a method was introduced and shown to work in the anti-plane case in Francfort & Larsen,[11] and then generalized in Dal Maso, Francfort & Toader[8] to the case of nonlinear elasticity. We emphasize that the main achievement of Ambrosio & Braides,[1] Dal Maso, Francfort & Toader,[8] Dal Maso & Toader,[9] Francfort & Larsen,[11] and Francfort & Marigo[12] was to formulate and establish well-posedness of a model able to *predict crack paths*. In particular, crack kinking, crack branching, or indeed the far more complex three-dimensional situation, do not require additional modeling, but are naturally included in the formulation. This observation remains true for the dynamic model, which we propose in the following.

The difficulties in formulating models for dynamic fracture consistent with Griffith's criterion are readily apparent; indeed, we know of no well-posed models other than the one formulated herein. The main issue seems to be to find a precise mathematical principle corresponding to Griffith's criterion, which replaces unilateral minimality in the quasi-static setting. In our view, a dynamic model of fracture should obey the following three principles:

- *Elastodynamics*: Away from the crack set C, the governing principle is elastodynamics, for example, for anti-plane displacements,

  $$\rho\ddot{u} - \mu\Delta u = f \quad \text{in } \Omega\setminus C,$$

  with traction-free boundary conditions on either side of the crack, or

  $$\rho\ddot{u} - \Delta(\mu u + k\dot{u}) = f \quad \text{in } \Omega\setminus C,$$

  where the term $-k\Delta\dot{u}$ models elastic dissipation.
- *Energy Balance*: The evolution should satisfy an energy balance formula, akin to that found in the quasi-static setting, but now including kinetic energy.
- *Maximal Dissipation*: If the crack can propagate while balancing energy, then it should propagate.

The first principle requires no further comment. The principle of energy balance in dynamic fracture is known as *Mott's extension* of Griffith's energy concept.[17] Finally, the *maximal dissipation* principle follows a recent formulation of Larsen.[16] It further narrows down the set of admissible trajectories, which could still be very large if only energy balance is imposed (for instance, an elastodynamic solution for a stationary crack always conserves energy), and replaces unilateral minimality in the quasi-static fracture model. Indeed, in the quasi-static setting, the maximal dissipation principle implies unilateral minimality.[16]

In Bourdin, Larsen & Richardson,[4] a discrete-time candidate for such a model is proposed, based on the Ambrosio–Tortorelli approximation,

$$E_\varepsilon(u,v) := \frac{\mu}{2}\int_\Omega (v^2+\eta_\varepsilon)|\nabla u|^2\,\mathrm{d}x + G_c\int_\Omega [(4\varepsilon)^{-1}(1-v)^2 + \varepsilon|\nabla v|^2]\,\mathrm{d}x,$$

which Γ-converges, as $0 < \eta_\varepsilon \ll \varepsilon \to 0$, to the Griffith energy E; see Ambrosio & Tortorelli.[2] The point is that minimizing E_ε results in $(u_\varepsilon, v_\varepsilon)$, where u_ε approximates the minimizer u of E and v_ε provides an approximation of the crack set $S(u)$. Furthermore, the regularized elastic and surface energies converge independently to their sharp-interface versions; cf. (1.1). An analysis of this approximation in the quasi-static setting is provided by Giacomini.[13] The Ambrosio–Tortorelli approximation is particularly convenient for numerical implementation and was proposed in Bourdin, Francfort & Marigo[5] and Bourdin[3] for the simulation of the quasi-static model.

The observation which allows for an extension to dynamic fracture is that, for this approximation, there can be an instant decrease in the elastic energy when v decreases (i.e. the crack grows), even if u is held fixed. Hence, we will consider a model in which u follows elastodynamics (with stiffness $a(t) := v^2(t) + \eta_\varepsilon$) and v behaves identically as in the quasi-static setting, i.e. at every time t, $v(t)$ minimizes $v \mapsto E_\varepsilon(u(t), v)$ subject to an appropriate irreversibility constraint.

Bourdin, Larsen & Richardson[4] formulate this idea as a *numerical model*: given $T_f > 0$ and a positive integer N_f, at each discrete time $t_i = ih$, $i = 1, \ldots, N_f$, with $h = T_f/N_f$, $u(t_i)$ is computed using a time-discrete wave equation (cf. Sec. 3.1) with stiffness $(v_h(t_{i-1})^2 + \eta_\varepsilon)$, followed by the computation of $v_h(t_i)$ achieved by minimizing $v \mapsto E_\varepsilon(u_h(t_i), v)$ subject to $v \le v_h(t_{i-1})$. This approach was motivated by Bourdin[3] and Bourdin, Francfort & Marigo[5] where an alternate minimization procedure in the u and v variables was used for the simulation of the quasi-static problem. Note that, for a given discrete wave equation and time-step size h, this algorithm uniquely determines the discrete trajectory $(u_h(t), v_h(t))_{t\in[0,T_f]}$, (or, briefly, (u_h, v_h)) obtained by continuous piecewise affine interpolation of the sequence of values $(u_h(t_i), v_h(t_i))_{i=0}^{N_f}$. We also remark that several steps in our convergence proof in Sec. 3 were inspired by the convergence analysis of the alternate minimization algorithm in Burke, Ortner & Süli.[6]

If this numerical model is reasonable, then the pairs (u_h, v_h) it produces should balance energy (up to numerical errors) and converge to the solution of a corresponding time-continuous model. In this paper, we will prove that this is indeed the case: any accumulation point (u, v) of the family $\{(u_h, v_h) : h > 0\}$ of discrete trajectories is a solution to the time-continuous crack propagation problem: u solves the continuous-time wave equation, v is minimal, and the trajectory (u, v) balances energy. We were unable to prove our third postulate (maximal dissipation), and therefore believe that the formulation of our model might be underconstrained. We will return to this point in the conclusion of the paper.

We also note that, while there are other models for fracture based on crack regularization (see, e.g. Hakim & Karma[15]), they are typically based on phase-field models whose connection to the Griffith model is at best unclear. While we do not prove convergence of our model to a dynamic Griffith model, such a rigorous connection has been shown in the static and quasi-static settings by Giacomini.[13] We refer to Sec. 4 of Bourdin, Larsen & Richardson[4] for a more complete discussion.

We conclude the Introduction by noting that, in our analysis below, we add an elastic dissipation term, which helps in the analysis. Furthermore, we consider a more general, vector-valued case, instead of the anti-plane situation. Note also that, for simplicity of exposition, we take all physical constants (i.e. all constants except for ε and η_ε) to be 1.

2. Formulation of the Model

Suppose that Ω is a bounded open set in $\mathbb{R}^3$ with Lipschitz continuous boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, where Γ_D, Γ_N are disjoint measurable sets and $\mathcal{H}^2(\Gamma_D) > 0$. We use the usual notation for Lebesgue and Sobolev spaces, omitting the domain Ω whenever there is no ambiguity. For example, we shall write H^1 instead of $H^1(\Omega)$, and so forth. The space of displacements obeying the homogeneous Dirichlet boundary condition is denoted $H^1_D(\Omega;\mathbb{R}^3) := \{u \in H^1(\Omega;\mathbb{R}^3) : u|_{\Gamma_D} = 0\}$ (or simply H^1_D). Its dual is denoted $H^{-1}(\Omega;\mathbb{R}^3) = H^1_D(\Omega;\mathbb{R}^3)^*$ (or simply H^{-1}). Spaces of trajectories are denoted, as usual, by $L^p((0,T_f);X), W^{k,p}((0,T_f);X)$, $C^k([0,T_f];X)$, and so forth, where X is (a subset of) a Banach space. To simplify the notation, we shall usually write $L^p(X), W^{k,p}(X)$, $C(X)$ instead. If, e.g. $u \in L^p(H^1)$, then we will usually write $u(t) := u(\cdot,t)$. Throughout, the symbol $\|\cdot\|$ denotes the L^2-norm on Ω. We shall denote by $1 + H^1_D$ the affine variety in H^1 containing all functions $u \in H^1$ such that $u|_{\Gamma_D} = 1$.

We remark that the Arzelá–Ascoli theorem for metric spaces (see Sec. IV.6.7 in Dunford & Schwartz[10]) implies that $H^1(H^1)$ is *compactly* embedded in $C(L^2)$. That is, if a sequence $(u_j)_{j=1}^\infty \subset H^1(H^1)$ is uniformly bounded in $H^1(H^1)$, then there exists a subsequence (not relabeled) and $u \in H^1(H^1)$ such that

$$u_j \to u \quad \text{in } C(L^2). \tag{2.1}$$

Let $A \in L^\infty(\Omega;\mathbb{R}^{3^4})$ be the elastic modulus tensor, with $A^{\alpha\beta}_{ij}(x) = A^{\beta\alpha}_{ji}(x)$ for a.e. $x \in \Omega$, satisfying the following ellipticity condition: there exists $c_0 > 0$ such that $A(x)\zeta : \zeta \geq c_0|\zeta|^2$ for all $\zeta \in \mathbb{R}^{3\times3}_{\mathrm{sym}} = \{\zeta \in \mathbb{R}^{3\times3} : \zeta = \zeta^{\mathrm{T}}\}$ and for a.e. $x \in \Omega$; equivalently,

$$\sum_{i,j=1}^{3}\sum_{\alpha,\beta=1}^{3} A^{\alpha,\beta}_{ij}(x)\zeta^i_\alpha\zeta^j_\beta \geq c_0\sum_{i=1}^{3}\sum_{\alpha=1}^{3}|\zeta^i_\alpha|^2, \quad \forall\,\zeta \in \mathbb{R}^{3\times3}_{\mathrm{sym}} \quad \text{for a.e. } x \in \Omega.$$

For $\zeta \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$ and $x \in \Omega$, we define $|\zeta|^2_{A(x)} := A(x)\zeta : \zeta$. Further, for $u \in H^1(\Omega;\mathbb{R}^3)$, let

$$e(u) := \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}}) \quad \text{and} \quad \|e(u)\|^2_A := \int_\Omega |e(u)|^2_A\,\mathrm{d}x.$$

For $\eta > 0$ and $\varepsilon > 0$, we define the elastic energy $\mathcal{E} : H^1 \times H^1 \to \mathbb{R} \cup \{+\infty\}$, and the (phase-field) surface energy $\mathcal{H} : H^1 \to \mathbb{R}$, respectively, as

$$\mathcal{E}(u, v) := \frac{1}{2}\int_\Omega (v^2 + \eta)|e(u)|_A^2 \,\mathrm{d}x \quad \text{and} \quad \mathcal{H}(v) := \int_\Omega [(4\varepsilon)^{-1}(1 - v)^2 + \varepsilon|\nabla v|^2]\mathrm{d}x.$$

We define the functional $\mathcal{K} : L^2 \to \mathbb{R}$ by

$$\mathcal{K}(w) = \frac{1}{2}\int_\Omega |w|^2 \,\mathrm{d}x,$$

and note that, for $u \in H^1(L^2)$, $\mathcal{K}(\dot{u}(t))$ denotes the kinetic energy of u at time t. The external forces at time $t \in [0, T_f]$ are collected into a functional $\ell(t) \in H^{-1}$, where we recall that $H^{-1} = H^{-1}(\Omega; \mathbb{R}^3)$ denotes the dual of $H^1_D(\Omega; \mathbb{R}^3)$,

$$\langle \ell(t), \varphi\rangle := \int_\Omega f(t) \cdot \varphi \,\mathrm{d}x + \int_{\Gamma_N} g(t) \cdot \varphi \,\mathrm{d}s, \quad \forall\, \varphi \in H^1_D,$$

where $f(t) \in L^2(\Omega; \mathbb{R}^3)$ and $g(t) \in L^2(\Gamma_N; \mathbb{R}^3)$. We assume that $\ell \in C^1(H^{-1})$; a sufficient condition for this would be $f \in C^1(L^2)$ and $g \in C^1(L^2(\Gamma_N; \mathbb{R}^3))$. Finally, the total energy is given by

$$\mathcal{F}(t; u, w, v) := \mathcal{K}(w) + \mathcal{E}(u, v) - \langle \ell(t), u\rangle + \mathcal{H}(v).$$

In order to model a crack at the Dirichlet boundary, it is common to extend the domain, and to impose the "Dirichlet condition" on a set of finite measures. In order to avoid distraction from the main issues (dynamics and energy balance), we chose to impose the boundary condition $v = 1$ on Γ_D. Intuitively, with this boundary condition, the Ambrosio–Tortorelli functional should still give a good approximation to the Griffith functional, however, we stress that we do not know of a rigorous justification for this.

We seek a solution $(u,\, v)$ of the system

$$\begin{aligned} \ddot{u} - \operatorname{div}\big(a(t)\, Ae(u + \dot{u})\big) &= f(t) && \text{in } \Omega, \\ \nu^{\mathrm{T}} a(t)\, Ae(u + \dot{u}) &= g(t) && \text{on } \Gamma_N, \\ (u, v) &= (0, 1) && \text{on } \Gamma_D, \end{aligned} \tag{2.2}$$

for $t \in (0, T_f]$, where $a(t) = [v(t)]^2 + \eta$, with initial conditions $u(0) = u_0 \in H^1_D$ and $\dot{u}(0) = u_1 \in H^1_D$, and satisfying the crack stability condition

$$\mathcal{E}(u(t), v(t)) + \mathcal{H}(v(t)) = \inf_{\substack{v-1\in H^1_D \\ v\le v(t)}} \mathcal{E}(u(t), v) + \mathcal{H}(v). \tag{2.3}$$

Note that we require (2.3) to hold for *every* $t \in [0, T_f]$. As initial condition for v we prescribe an arbitrary $v_0 \in 1 + H^1_D$, $0 \le v_0 \le 1$ a.e. in Ω, that satisfies the unilateral minimality condition (2.3). Stated in this way the system may still be severely underconstrained (see the discussion in the Introduction); hence we also impose the energy balance formula

$$\begin{aligned} &\mathcal{F}(T; u(T), \dot{u}(T), v(T)) \\ &\quad = \mathcal{F}(0; u_0, \dot{u}_0, v_0) - \int_0^T \|a^{1/2} e(\dot{u})\|_A^2 \,\mathrm{d}t - \int_0^T \langle \dot{\ell}, u\rangle \mathrm{d}t \quad \forall\, T \in [0, T_f]. \end{aligned} \tag{2.4}$$

The additional term $-\int_0^T \|a^{1/2} e(\dot{u})\|_A^2 \,\mathrm{d}t$ which is not contained in the total energies describes elastic dissipation. In Sec. 2.1 we give a brief formal argument to motivate this energy balance formula.

The main result of this paper is the following theorem, which is deduced as a direct consequence of Theorem 3.1 below.

Theorem 2.1. *Under the above conditions there exists at least one trajectory* $(u, v) \in (H^2(L^2) \cap W^{1,\infty}(H^1_D)) \times W^{1,\infty}(1 + H^1_D)$ *satisfying* (2.2) *in the weak sense, i.e.*

$$(\ddot{u}, \varphi) + (aAe(u + \dot{u}), e(\varphi)) = \langle \ell(t), \varphi\rangle \quad \forall\, \varphi \in H^1_D(\Omega; \mathbb{R}^3) \quad \textit{for a.e. } t \in (0, T_f], \tag{2.5}$$

with $u(0) = u_0, \dot{u}(0) = u_1, v(0) = v_0$. *The unilateral minimality condition* (2.3) *and the energy balance condition* (2.4) *are satisfied for all times* $t \in (0, T_f]$.

Remark 2.1. (Boundary conditions) In order to avoid an overly cluttered notation, we restricted the generality of the boundary conditions in Theorem 2.1. As a matter of fact, our proof extends without major changes to the cases of (i) a time-dependent Dirichlet condition $u(t) = u_D(t)$ on Γ_D; and (ii) a pure traction problem (i.e. $\Gamma_D = \emptyset$).

To see this, note that case (i) can be reduced to our problem, provided $u_D \in C^2(H^1) \cap C^3(L^2)$. In case (ii), we face the potential difficulty that the Korn inequality

$$(Ae(w), e(w)) \geq c_0 \|\nabla w\|^2 \quad \forall\, w \in H^1(\Omega)^3$$

(where $c_0 > 0$) fails. However, the slightly weaker Gårding inequality,

$$(Ae(w), e(w)) \geq c_0 \|\nabla w\|^2 - c_1 \|w\|^2 \quad \forall\, w \in H^1(\Omega)^3,$$

still holds. Since the terms involving time-derivatives can be used to control the negative contribution, this is sufficient to extend our proofs.

Remark 2.2. (More general models) Furthermore, we note that our proofs apply verbatim to more general wave equations, including in particular the case of anti-plane strain, in-plane strain, and in-plane stress, as well as more general coefficients. For example, the wave equation

$$\rho \ddot{u} - \operatorname{div}\big(aA\big(e(u) + ke(\dot{u})\big)\big) = f,$$

where $\rho, k \in L^\infty(\Omega)$ are uniformly positive can also be treated by the same analysis. We also point out that k, the dissipation, can be taken arbitrarily small.

The dissipation term is not only crucial for our analysis, but also opens up interesting modeling questions. For example, it may allow us to investigate whether time-rescaled limits of dynamic solutions converge to a quasi-static solution.

2.1. *A formal argument for energy balance*

In this section we present a heuristic argument for the energy balance formula (2.4), which was the original motivation for pursuing the analysis in this paper.

Suppose that (u, v) is a solution to the model introduced above. Let us assume, furthermore, that $u \in C^2(H^1)$, that $v \in C^1(H^1)$, and, for simplicity, that $\ell \equiv 0$. Setting $a := v^2 + \eta$, and omitting for ease of writing the t-dependence from our notation on the right-hand side in the chain of equalities below, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t; u(t), \dot{u}(t), v(t)) &= \frac{1}{2}(\dot{a}Ae(u), e(u)) + (aAe(u), e(\dot{u})) + (\ddot{u}, \dot{u}) + \mathcal{H}'(v)[\dot{v}] \\ &= \big\{(\ddot{u}, \dot{u}) + \big(aA(e(u + \dot{u})), e(\dot{u})\big)\big\} - (aAe(\dot{u}), e(\dot{u})) \\ &\quad + \big[\big(v|e(u)|_A^2, \dot{v}\big) + (2\varepsilon)^{-1}\big((v-1), \dot{v}\big) + 2\varepsilon(\nabla v, \nabla\dot{v})\big].\end{aligned}$$

Since $\dot{u}(t) \in H^1_D$, $t \in (0, T_f)$, the group of terms enclosed in curly brackets vanishes. Suppose, furthermore that, at $t \in (0, T_f)$, $v(t)$ is a *global minimizer* of $\mathcal{E}(u(t), \cdot) + \mathcal{H}(\cdot)$ (ignoring the inequality constraint); then, the group in square brackets represents the first-order criticality condition for this minimization problem (tested with $\dot{v}(t)$), and thus vanishes as well. Hence, we would obtain the desired energy balance formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t; u(t), \dot{u}(t), v(t)) = -(aAe(\dot{u}), e(\dot{u})).$$

This formal argument is made rigorous in Sec. 3.8 below.

3. Proof of the Existence Theorem

3.1. *Time discretization*

We set $v_h^0 = v_0$, $u_h^0 = u_0$, $u_h^0 - u_h^{-1} = hu_1$ and, for $n = 1, 2, \ldots, N_f$, $N_f \geq 2$, $h = T_f/N_f$, solve

$$(\delta^2 u_h^n, \varphi) + \big(a_h^{n-1}Ae(u_h^n + \delta u_h^n), e(\varphi)\big) = \langle \ell(t_n), \varphi\rangle \quad \forall\, \varphi \in H^1_D(\Omega; \mathbb{R}^3), \tag{3.1}$$

$$v_h^n := \operatorname*{argmin}_{v-1 \in H^1_D, v \leq v_h^{n-1}} \big\{v \mapsto \mathcal{E}(u_h^n, v) + \mathcal{H}(v)\big\}, \tag{3.2}$$

where

$$\delta^2 u_h^n := \frac{\delta u_h^n - \delta u_h^{n-1}}{h}, \quad n \geq 1, \quad \delta u_h^n := \frac{u_h^n - u_h^{n-1}}{h}, \quad n \geq 0.$$

Due to the positivity of $a_h^{n-1} := [v_h^{n-1}]^2 + \eta$ and the uniform convexity of $\mathcal{E}(u, \cdot) + \mathcal{H}(\cdot)$, it is obvious that (3.1) and (3.2) are well-defined, i.e. there exists a *unique* family $(u_h^n, v_h^n)_{n=1}^{N_f}$ that solves the time-discrete problem.

In the remainder of this section we shall prove that, upon defining suitable interpolants and extracting a subsequence, the family $(u_h^n, v_h^n)_{n=0}^{N_f}$ converges to a solution of (2.5), (2.3) and (2.4), as $h \to 0$.

Theorem 3.1. *For $N_f \in \mathbb{N}$ let $(u_h^n, v_h^n)_{n=0}^{N_f}$ be the solution of the time-discretization defined by (3.1) and (3.2). Then, there exists a subsequence $h_k \searrow 0$ ($N_f^k \nearrow \infty$, with $T_f = N_f^k h_k$ fixed) and a trajectory $(u, v) \in (H^2(L^2) \cap W^{1,\infty}(H_D^1)) \times W^{1,\infty}(1 + H_D^1)$ such that*

$$(u_{h_k}, v_{h_k}) \to (u, v) \quad \textit{strongly in } H^1(H^1 \times H^1) \quad \textit{as } k \to \infty,$$

where u_{h_k}, v_{h_k} denote the piecewise affine interpolants as defined in (3.22) below. Moreover, the trajectory (u, v) is a solution of (2.3)–(2.5).

3.2. *Preliminary remarks*

We begin by stating some simple facts about the phase-field variable. First, we note that, for given $u_h^n \in H_{\mathrm{D}}^1(\Omega; \mathbb{R}^3)$, the function v_h^n is equivalently characterized as the solution in $1 + H_D^1$ of the variational inequality

$$\partial_v \mathcal{E}(u_h^n, v_h^n)[\psi - v_h^n] + \mathcal{H}'(v_h^n)[\psi - v_h^n] \geq 0 \quad \forall \psi \leq v_h^{n-1}, \;\; \psi - 1 \in H_D^1(\Omega; \mathbb{R}). \quad (3.3)$$

Lemma 3.1. *The phase-field variables satisfy the maximum principle*

$$0 \leq v_h^n \leq v_h^{n-1} \quad \textit{a.e. in } \Omega \quad \forall n = 1, \ldots, N_f. \quad (3.4)$$

Proof. The upper bound in (3.4) holds by definition, while the lower bound follows from a simple truncation argument. It can be readily seen from the definition of $\mathcal{E}$ and $\mathcal{H}$ that, if we had $v_h^n \leq \delta < 0$ on a set of positive measure, then the admissible trial function $\psi = \max(0, v_h^n)$ would strictly lower the energy. □

Our second lemma, even though it is elementary, lies at the heart of many of the calculations that we carry out below. At least formally, it can be thought of as a time-discrete counterpart of Griffith's second criterion (see also Sec. 2.1). In the same manner as Griffith's second criterion, it follows immediately from the stationarity of v_h^n, $n = 1, \ldots, N_f$.

Lemma 3.2. *For all $n = 1, \ldots, N_f$, we have that*

$$\partial_v \mathcal{E}(u_h^n, v_h^n)[\delta v_h^n] + \mathcal{H}'(v_h^n)[\delta v_h^n] = 0. \quad (3.5)$$

Equivalently,

$$(|e(u_h^n)|_A^2 v_h^n, \delta v_h^n) + (2\varepsilon)^{-1}(v_h^n - 1, \delta v_h^n) + 2\varepsilon(\nabla v_h^n, \nabla \delta v_h^n) = 0. \quad (3.6)$$

Proof. The test functions $\psi_1 = v_h^{n-1}$ and $\psi_2 = 2v_h^n - v_h^{n-1}$ are both admissible for (3.3). Testing (3.3) with ψ_1 gives

$$\partial_v \mathcal{E}(u_h^n, v_h^n)[\delta v_h^n] + \mathcal{H}'(v_h^n)[\delta v_h^n] \leq 0,$$

while testing with ψ_2 gives the opposite inequality. This establishes (3.5). The identity (3.6) is simply an explicit form of (3.5). $\square$

3.3. A priori *estimates*

In this section, we will collect a number of *a priori* estimates on the discrete family $(u_h^n, v_h^n)_{n=1}^{N_f}$, which will then allow us to extract weakly convergent subsequences. Constants appearing in this analysis may depend on $\varepsilon > 0$, $\eta > 0$, Ω, the initial and boundary data, and on the applied loads, but are independent of h.

The first, and in some sense "natural", *a priori* bounds for the discrete evolution law (3.1), (3.2) are given in the following lemma.

Lemma 3.3. *There exists a constant* C_1, *independent of* h, *such that*

$$\begin{aligned} \max_{1\le n\le N_f} \{\|\delta u_h^n\|^2 + \|e(u_h^n)\|_A^2 + \|v_h^n\|^2 + \|\nabla v_h^n\|^2\} \\ + \sum_{n=1}^{N_f} h\|e(\delta u_h^n)\|_A^2 + h\sum_{n=1}^{N_f} hD_h^n \le C_1, \end{aligned} \tag{3.7}$$

where the numerical dissipation terms D_h^n, $n = 1, \ldots, N_f$, *are non-negative and are defined as follows*:

$$\begin{aligned} D_h^n := \frac{1}{2}\|\delta^2 u_h^n\|^2 + \frac{1}{2}\|(a_h^{n-1})^{1/2}\,|e(\delta u_h^n)|_A\|^2 \\ + \frac{1}{2}\|(\delta v_h^n)\,|e(u_h^n)|_A\|^2 + (4\varepsilon)^{-1}\|\delta v_h^n\|^2 + \varepsilon\|\nabla\delta v_h^n\|^2. \end{aligned} \tag{3.8}$$

Proof. Testing (3.1) with $\varphi = h\delta u_h^n$, we obtain

$$\begin{aligned} (\delta u_h^n - \delta u_h^{n-1}, \delta u_h^n) + h\big(a_h^{n-1}Ae(\delta u_h^n), e(\delta u_h^n)\big) \\ + \big(a_h^{n-1}Ae(u_h^n), e(u_h^n) - e(u_h^{n-1})\big) = \langle \ell(t_n), u_h^n - u_h^{n-1}\rangle. \end{aligned} \tag{3.9}$$

The first term on the left-hand side is rewritten as follows:

$$\begin{aligned} (\delta u_h^n - \delta u_h^{n-1}, \delta u_h^n) &= \frac{1}{2}\|\delta u_h^n\|^2 + \frac{1}{2}\|\delta u_h^n\|^2 - (\delta u_h^n, \delta u_h^{n-1}) \\ &\quad + \frac{1}{2}\|\delta u_h^{n-1}\|^2 - \frac{1}{2}\|\delta u_h^{n-1}\|^2 \\ &= \mathcal{K}(\delta u_h^n) - \mathcal{K}(\delta u_h^{n-1}) + \frac{1}{2}h^2\|\delta^2 u_h^n\|^2. \end{aligned} \tag{3.10}$$

A similar computation yields

$$\begin{aligned} &\big(a_h^{n-1}Ae(u_h^n), e(u_h^n) - e(u_h^{n-1})\big) \\ &\quad = \mathcal{E}(u_h^n, v_h^n) - \mathcal{E}(u_h^{n-1}, v_h^{n-1}) + \frac{1}{2}h^2\|(a_h^{n-1})^{1/2}\,|e(\delta u_h^n)|_A\|^2 \\ &\qquad - \frac{1}{2}\int_\Omega (a_h^n - a_h^{n-1})|e(u_h^n)|_A^2\,\mathrm{d}x. \end{aligned} \tag{3.11}$$

The last term on the right-hand side of (3.11) is further re-expressed, first by writing

$$a_h^n - a_h^{n-1} = (v_h^n)^2 - (v_h^{n-1})^2 = h(v_h^n + v_h^{n-1})\delta v_h^n = 2h\, v_h^n\, \delta v_h^n - h^2|\delta v_h^n|^2,$$

and then employing (3.6), as follows:

$$\begin{aligned}
&-\frac{1}{2}\int_\Omega (a_h^n - a_h^{n-1})|e(u_h^n)|_A^2\,\mathrm{d}x \\
&\quad = \left\{(2\varepsilon)^{-1}(v_h^n - 1, v_h^n - v_h^{n-1}) + 2\varepsilon(\nabla v_h^n, \nabla v_h^n - \nabla v_h^{n-1})\right\} \\
&\qquad + \frac{1}{2}h^2\|(\delta v_h^n)\,|e(u_h^n)|_A\|^2.
\end{aligned}$$

Upon replacing $v_h^n - v_h^{n-1} = (v_h^n - 1) - (v_h^{n-1} - 1)$ in the first term on the right-hand side, the combined term in the curly brackets can be manipulated, with the same algebraic manipulations as in (3.10), so that we arrive at

$$\begin{aligned}
&-\frac{1}{2}\int_\Omega (a_h^n - a_h^{n-1})|e(u_h^n)|_A^2\,\mathrm{d}x \\
&\quad = \mathcal{H}(v_h^n) - \mathcal{H}(v_h^{n-1}) + h^2\big((4\varepsilon)^{-1}\|\delta v_h^n\|^2 + \varepsilon\|\nabla\delta v_h^n\|^2\big) \\
&\qquad + \frac{1}{2}h^2\|(\delta v_h^n)\,|e(u_h^n)|_A\|^2.
\end{aligned} \tag{3.12}$$

Thus, summing (3.9) over n, and using (3.10)–(3.12) to replace the left-hand side, we obtain

$$\begin{aligned}
&[\mathcal{K}(\delta u_h^N) + \mathcal{E}(u_h^N, v_h^N) + \mathcal{H}(v_h^N)] - [\mathcal{K}(\delta u_0) + \mathcal{E}(u_0, v_0) + \mathcal{H}(v_0)] \\
&\quad + \sum_{n=1}^N h\|(a_h^{n-1})^{1/2}\,|e(\delta u_h^n)|_A\|^2 + h\sum_{n=1}^N hD_h^n = \sum_{n=1}^N \langle \ell(t_n), u_h^n - u_h^{n-1}\rangle,
\end{aligned} \tag{3.13}$$

where $N \in \{1, 2, \ldots, N_f\}$, and where

$$\begin{aligned}
D_h^n := &\frac{1}{2}\|\delta^2 u_h^n\|^2 + \frac{1}{2}\|(a_h^{n-1})^{1/2}\,|e(\delta u_h^n)|_A\|^2 \\
&+ \frac{1}{2}\|(\delta v_h^n)\,|e(u_h^n)|_A\|^2 + (4\varepsilon)^{-1}\|\delta v_h^n\|^2 + \varepsilon\|\nabla\delta v_h^n\|^2.
\end{aligned}$$

We estimate the right-hand side in (3.13), using Korn's inequality and a weighted Cauchy inequality, as follows:

$$\begin{aligned}
\sum_{n=1}^N h\langle \ell(t_n), \delta u_h^n\rangle &\le \left(\sum_{n=1}^N h\|\ell(t_n)\|_{H^{-1}}^2\right)^{1/2}\left(\sum_{n=1}^N h\|\nabla\delta u_h^n\|^2\right)^{1/2} \\
&\le C\left(\sum_{n=1}^N h\|\ell(t_n)\|_{H^{-1}}^2\right)^{1/2}\left(\sum_{n=1}^N h\|e(\delta u_h^n)\|_A^2\right)^{1/2} \\
&\le \frac{C}{2\eta}\sum_{n=1}^N h\|\ell(t_n)\|_{H^{-1}}^2 + \frac{\eta}{2}\sum_{n=1}^N h\|e(\delta u_h^n)\|_A^2.
\end{aligned}$$

Inserting this estimate into (3.13) and noting that $a_h^{n-1} \geq \eta$, we obtain

$$[\mathcal{K}(\delta u_h^N) + \mathcal{E}(u_h^N, v_h^N) + \mathcal{H}(v_h^N)] - [\mathcal{K}(\delta u_0) + \mathcal{E}(u_0, v_0) + \mathcal{H}(v_0)] \\ + \frac{\eta}{2} \sum_{n=1}^{N} h \|e(\delta u_h^n)\|_A^2 + h \sum_{n=1}^{N} h D_h^n \leq \frac{C}{2\eta} \sum_{n=1}^{N} h \|\ell(t_n)\|_{H^{-1}}^2.$$

Using the coercivity of the different energies, and the fact that the terms D_h^n are non-negative, we obtain the desired result. $\square$

Lemma 3.3 does not yet furnish a bound on $\nabla \delta v_h^n$. This can be achieved by exploiting the strong coupling between $\nabla \delta u_h^n$ and $\nabla \delta v_h^n$ in the variational inequality (3.3), as in the following lemma.

Lemma 3.4. *There exists a constant C_2, independent of h, such that*

$$\sum_{n=1}^{N_f} h \|\delta v_h^n\|_{H^1}^2 \leq C_2. \tag{3.14}$$

Proof. Since $v_h^n \leq v_h^{n-1}$, we obtain from (3.3), with indices shifted by 1, that

$$(|e(u_h^{n-1})|_A^2 v_h^{n-1}, \delta v_h^n) + (2\varepsilon)^{-1}(v_h^{n-1} - 1, \delta v_h^n) + (2\varepsilon)(\nabla v_h^{n-1}, \nabla \delta v_h^n) \geq 0. \tag{3.15}$$

Subtracting (3.15) from (3.6) gives

$$\|(\delta v_h^n)\, |e(u_h^n)|_A\|^2 + (2\varepsilon)^{-1} \|\delta v_h^n\|^2 + (2\varepsilon) \|\nabla \delta v_h^n\|^2 \\ \leq \frac{1}{h} \int_\Omega \big(|e(u_h^{n-1})|_A^2 - |e(u_h^n)|_A^2\big) v_h^{n-1} \delta v_h^n \mathrm{d}x. \tag{3.16}$$

The fact that $0 \leq v_h^n \leq v_h^{n-1} \leq 1$ (cf. Lemma 3.1), and thereby $|v_h^n - v_h^{n-1}| \leq 1$, gives $|\delta v_h^n| \leq 1/h$. With this in mind, we can rewrite and estimate the right-hand side of (3.16), using also the fact that $\delta v_h^n \leq 0$, by

$$\begin{aligned} &\frac{1}{h} \int_\Omega A\big(e(u_h^{n-1}) - e(u_h^n)\big) : \big(e(u_h^{n-1}) + e(u_h^n)\big) v_h^{n-1} \delta v_h^n \,\mathrm{d}x \\ &\quad = \frac{1}{h} \int_\Omega \big|\big(e(u_h^{n-1}) - e(u_h^n)\big)\big|_A^2 v_h^{n-1} \delta v_h^n \,\mathrm{d}x \\ &\qquad + \frac{2}{h} \int_\Omega A\big(e(u_h^{n-1}) - e(u_h^n)\big) : e(u_h^n) v_h^{n-1} \delta v_h^n \,\mathrm{d}x \\ &\quad \leq \frac{2}{h} \int_\Omega (|e(u_h^n)|_A \, |\delta v_h^n|) \, (|v_h^{n-1}| \, |e(u_h^n) - e(u_h^{n-1})|_A) \mathrm{d}x \\ &\quad \leq 2 \|(\delta v_h^n) |e(u_h^n)|_A\| \, \|(v_h^{n-1}) |e(\delta u_h^n)|_A\|. \end{aligned}$$

Through an application of Cauchy's inequality we obtain from (3.16) that

$$(2\varepsilon)^{-1} \|\delta v_h^n\|^2 + 2\varepsilon \|\nabla \delta v_h^n\|^2 \leq \|(v_h^{n-1}) e(\delta u_h^n)\|_A^2, \quad n = 1, \ldots, N_f. \tag{3.17}$$

The stated *a priori* bound (3.14) now follows from Lemma 3.3 and the fact that $0 \le v_h^n \le 1$ for all n. □

As we remarked above, the bound (3.7) is, in some sense, the natural *a priori* bound for (2.2). However, the structure of the coefficient a admits additional regularity in time for the family $(u_h^n)_{n=1}^{N_f}$, and, as in Lemma 3.4, also for the family $(v_h^n)_{n=1}^{N_f}$.

Lemma 3.5. *There exists a constant C_3, independent of h, such that*

$$\sum_{n=1}^{N_f} h\|\delta^2 u_h^n\|^2 + \max_{n=1,\dots,N_f}\left(\|e(\delta u_h^n)\|_A^2 + \|\nabla \delta v_h^n\|^2\right) \le C_3. \tag{3.18}$$

Proof. Testing (3.1) with $\varphi = h\delta w_h^n$, where $w_h^n = u_h^n + \delta u_h^n$, gives

$$\begin{aligned} & h\|\delta^2 u_h^n\|^2 + (\delta u_h^n - \delta u_h^{n-1}, \delta u_h^n) \\ & \qquad + (a_h^{n-1} A e(w_h^n), e(w_h^n) - e(w_h^{n-1})) = \langle \ell(t_n), w_h^n - w_h^{n-1}\rangle. \end{aligned} \tag{3.19}$$

Using (3.11), but with u_h^n replaced by w_h^n, the third term on the left-hand side can be estimated by

$$\begin{aligned} & (a_h^{n-1} A e(w_h^n), e(w_h^n) - e(w_h^{n-1})) \\ & \quad \ge \frac{1}{2}\|(a_h^n)^{1/2}\,|e(w_h^n)|_A\|^2 - \frac{1}{2}\|(a_h^{n-1})^{1/2}\,|e(w_h^{n-1})|_A\|^2 \\ & \qquad + \frac{1}{2}\|(a_h^{n-1})^{1/2}\,|e(w_h^n - w_h^{n-1})|_A\|^2 + \frac{1}{2}\int_\Omega (a_h^{n-1} - a_h^n)|e(w_h^n)|_A^2\,\mathrm{d}x. \end{aligned}$$

Since $v_h^n \le v_h^{n-1}$, we have $a_h^{n-1} - a_h^n \ge 0$, and therefore

$$\begin{aligned} & (a_h^{n-1} A e(w_h^n), e(w_h^n) - e(w_h^{n-1})) \\ & \quad \ge \frac{1}{2}\|(a_h^n)^{1/2}\,|e(w_h^n)|_A\|^2 - \frac{1}{2}\|(a_h^{n-1})^{1/2}\,|e(w_h^{n-1})|_A\|^2 \\ & \qquad + \frac{1}{2}\eta\|e(w_h^n - w_h^{n-1})\|_A^2 + \frac{1}{2}h\int_\Omega |\delta a_h^n||e(w_h^n)|_A^2\,\mathrm{d}x. \end{aligned} \tag{3.20}$$

Inserting (3.10) and (3.20) into (3.19), and discarding the non-negative numerical dissipation terms, yields

$$\begin{aligned} & h\|\delta^2 u_h^n\|^2 + \frac{1}{2}\|\delta u_h^n\|^2 - \frac{1}{2}\|\delta u_h^{n-1}\|^2 + \frac{1}{2}\|(a_h^n)^{1/2}\,|e(w_h^n)|_A\|^2 \\ & \quad - \frac{1}{2}\|(a_h^{n-1})^{1/2}\,|e(w_h^{n-1})|_A\|^2 + \frac{1}{2}\eta\|e(w_h^n - w_h^{n-1})\|_A^2 \\ & \quad + \frac{1}{2}h\int_\Omega |\delta a_h^n||e(w_h^n)|_A^2\,\mathrm{d}x \le \langle \ell(t_n), w_h^n - w_h^{n-1}\rangle. \end{aligned}$$

Summing this estimate over $n = 1, \dots, N$, and using the fact that $u_h^0 = u_0$ and $\delta u_h^0 = u_1$, gives, for any $N \in \{1, \dots, N_f\}$,

$$\sum_{n=1}^{N} h\|\delta^2 u_h^n\|^2 + \frac{1}{2}\|\delta u_h^N\|^2 + \frac{1}{2}\|(a_h^n)^{1/2} e(w_h^N)\|_A^2 + \frac{1}{2}\sum_{n=1}^{N} h\int_\Omega |\delta a_h^n||e(w_h^n)|_A^2\,\mathrm{d}t$$
$$\leq \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|(a_h^0)^{1/2} e(u_0 + u_1)\|_A^2 + \sum_{n=1}^{N}\langle \ell(t_n), w_h^n - w_h^{n-1}\rangle.$$

To bound the final term on the right-hand side, we use summation by parts to reorder the sum as follows:

$$\begin{aligned}\sum_{n=1}^{N}\langle \ell(t_n), w_h^n - w_h^{n-1}\rangle &= -\sum_{n=0}^{N-1}\langle \ell(t_{n+1}) - \ell(t_n), w_h^n\rangle + \langle \ell(t_N), w_h^N\rangle - \langle \ell(t_0), w_h^0\rangle \\ &\leq \sum_{n=0}^{N-1} h\|h^{-1}(\ell(t_{n+1}) - \ell(t_n))\|_{H^{-1}}\|\nabla w_h^n\| \\ &\quad + \|\ell(t_N)\|_{H^{-1}}\|\nabla w_h^N\| + \|\ell(t_0)\|_{H^{-1}}\|\nabla w_h^0\| \\ &\leq C\sum_{n=0}^{N-1} h\|h^{-1}(\ell(t_{n+1}) - \ell(t_n))\|_{H^{-1}}\|e(w_h^n)\|_A \\ &\quad + C\|\ell(t_N)\|_{H^{-1}}\|e(w_h^N)\|_A + C\|\ell(t_0)\|_{H^{-1}}\|e(w_h^0)\|_A,\end{aligned}$$

where, in the transition to the last line, we used Korn's inequality. Using the assumption that $\ell \in C^1(H^{-1})$ and $w_h^0 = u_0 + u_1$, we obtain the stated result. $\square$

3.4. *Discrete energy inequality*

Starting from (3.13), we deduce an energy inequality for the time-discretization (3.1), (3.2).

We remark that (3.21) is in fact an equality up to the numerical dissipation $h\sum_{n=1}^{N} hD_h^n$, which we would expect to be of order $\mathcal{O}(h)$. However, we will not require this in our analysis, and only use the fact that the terms D_h^n are non-negative.

Lemma 3.6. *For all $N \in \{1, \dots, N_f\}$, the following discrete energy inequality holds*:

$$\mathcal{F}(t_N; u_h^N, \delta u_h^N, v_h^N) \leq \mathcal{F}(0; u_0, \delta u_0, v_0) - \sum_{n=1}^{N} h\{\|(a_h^{n-1})^{1/2}|e(\delta u_h^n)|_A\|^2 + \langle \delta\ell_h^n, u_h^{n-1}\rangle\}. \tag{3.21}$$

Proof. Identity (3.13) gives, for $1 \leq N \leq N_f$,

$$\begin{aligned}&\mathcal{F}(t_N; u_h^N, \delta u_h^N, v_h^N) - \mathcal{F}(0; u_0, \delta u_0, v_0) \\ &\quad = -\langle \ell(t_N), u_h^N\rangle + \langle \ell(0), u_0\rangle + \sum_{n=1}^{N}\langle \ell(t_n), u_h^n - u_h^{n-1}\rangle \\ &\qquad - \sum_{n=1}^{N} h\|(a_h^{n-1})^{1/2}\,|e(\delta u_h^n)|_A\|^2 - h\sum_{n=1}^{N} hD_h^n.\end{aligned}$$

We reorder the sum over the forcing terms as follows:

$$-\langle \ell(t_N), u_h^N \rangle + \langle \ell(0), u_0 \rangle + \sum_{n=1}^{N} \langle \ell(t_n), u_h^n - u_h^{n-1} \rangle = -\sum_{n=1}^{N} \langle \ell(t_n) - \ell(t_{n-1}), u_h^{n-1} \rangle.$$

Hence, we obtain the discrete energy inequality (3.21). □

3.5. *Passage to the limit*

The purpose of this section is to show that, as $h \to 0$, weakly convergent sequences of the families $(u_h^n, v_h^n)_{n=1}^{N_f}$, $h > 0$, can be extracted. To this end, we will first define a number of useful interpolants of these families and then apply the *a priori* estimates from Sec. 3.3 to extract convergent subsequences of these interpolants.

Let u_h denote the piecewise affine interpolant of the sequence $(u_h^n)_{n=0}^{N_f}$, defined as

$$u_h(t) = u_h^n + (t - t_n)\delta u_h^n, \quad t \in [t_{n-1}, t_n], \quad n = 1, \ldots, N_f. \tag{3.22}$$

In the same way, we define v_h to be the piecewise affine interpolant of $(v_h^n)_{n=0}^{N_f}$ and u_h' that of $(\delta u_h^n)_{n=0}^{N_f}$. Let $a_h = v_h^2 + \eta$. Furthermore, we define the backward interpolant u_h^+ and the forward interpolant a_h^-:

$$\begin{aligned} u_h^+(\cdot, t) &= u_h^n, \qquad t \in (t_{n-1}, t_n], \\ a_h^-(\cdot, t) &= a_h^{n-1}, \quad t \in [t_{n-1}, t_n), \end{aligned}$$

with analogous definitions of v_h^+, v_h^-, ℓ_h^+. Finally, we define u_h'' to be the backward interpolant of $(\delta^2 u_h^n)_{n=1}^{N_f}$. We emphasize that, while $u_h'' = \dot{u}_h'$, u_h' is *not* the derivative of u_h. Instead, $\dot{u}_h$ is the backward interpolant of $(\delta u_h^n)_{n=1}^{N_f}$.

With this notation, (3.1) reads

$$\begin{aligned} (u_h''(t), \varphi) + \big(a_h^-(t) A e(u_h^+(t) + \dot{u}_h(t)), e(\varphi)\big) &= \langle \ell_h^+(t), \varphi \rangle \\ \forall\, \varphi \in H_D^1, \quad \forall\, t \in (0, T_f]&. \end{aligned} \tag{3.23}$$

We will pass to the limit in this formulation. As a first step, we extract a weakly convergent subsequence of the interpolants.

Lemma 3.7. *There exists a subsequence* $h_j \searrow 0$ (*not relabeled*; *we write* h *instead of* h_j *throughout*) *and* $u \in H^2(L^2) \cap W^{1,\infty}(H_D^1)$ *and* $v \in W^{1,\infty}(1 + H_D^1)$, *such that* $0 \leq v(\cdot, t) \leq 1$ *and* $\dot{v}(\cdot, t) \leq 0$ *a.e. in* Ω, *and for all* $t \in (0, T_f]$, *and such that*

$$u_h \overset{*}{\rightharpoonup} u \quad \text{in } W^{1,\infty}(H^1) \tag{3.24}$$

and

$$v_h \overset{*}{\rightharpoonup} v \quad \text{in } W^{1,\infty}(H^1). \tag{3.25}$$

Moreover, the following convergence results hold:

$$u_h(t) \rightharpoonup u(t) \quad \text{and} \quad v_h(t) \rightharpoonup v(t) \quad \text{in } H^1 \quad \forall\, t \in [0, T_f], \tag{3.26}$$

$$u_h^+ \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(H^1) \quad \text{and} \quad u_h^+(t) \rightharpoonup u(t) \quad \text{in } H^1 \quad \forall\, t \in (0, T_f], \tag{3.27}$$

$$u_h'' \rightharpoonup \ddot{u} \quad \text{in } L^2(L^2). \tag{3.28}$$

Proof. Since $\dot{u}_h$, $\dot{v}_h$ and u_h'' are the backward interpolants of, respectively, $(\delta u_h^n)_{n=0}^{N_f}$, $(\delta v_h^n)_{n=0}^{N_f}$ and $(\delta^2 u_h^n)_{n=0}^{N_f}$, the *a priori* bound (3.18) and Korn's inequality imply that

$$\|u_h\|_{W^{1,\infty}(H^1)} + \|v_h\|_{W^{1,\infty}(H^1)} + \|u_h''\|_{L^2(L^2)} \le C. \tag{3.29}$$

(We remark that only the bounds on $\|\dot{u}_h\|_{L^\infty(H^1)}$ and on $\|\dot{v}_h\|_{L^\infty(H^1)}$ are required to deduce this.) Furthermore, $0 \le v_h(\cdot, t) \le 1$ and $\dot{v}_h(\cdot, t) \le 0$ a.e. on Ω and for all $t \in (0, T_f]$. This ensures weak-* pre-compactness of the family $(u_h, v_h)_{h>0}$ in $W^{1,\infty}(H^1) \times W^{1,\infty}(H^1)$ and thus gives (3.24) and (3.25).

To prove (3.26), we note that (2.1) implies that $u_h(t) \to u(t)$ in L^2, for every $t \in [0, T_f]$. Since the sequences $(u_h(t))_{h>0}$ and $(v_h(t))_{h>0}$ (for fixed t) are bounded in H^1, the convergence is also weak in H^1, for every $t \in [0, T_f]$. This also shows that the pointwise bounds on v_h stated in the lemma hold for all t.

The convergence (3.27) follows from the fact that $u_h(t) = u_h^+(t) + (t - t_n)\dot{u}_h$, which implies

$$\|u_h - u_h^+\|_{L^\infty(H^1)} \le h\|\dot{u}_h\|_{L^\infty(H^1)} \le hC. \tag{3.30}$$

Combining (3.26) and (3.30), we deduce (3.27).

Next, we show that $u \in H^2(L^2)$. Similarly as in the previous paragraph we obtain (note that u_h' is the piecewise affine interpolant and $\dot{u}_h$ is the backward interpolant of $(\delta u_h^n)_{n=1}^{N_f}$)

$$\|u_h'(t) - \dot{u}_h(t)\| \le h\|u_h''(t)\|, \quad \forall\, t \in (0, T_f].$$

The estimate (3.29) gives an *a priori* bound on $\|u_h''\|_{L^2(L^2)}$, which implies, using also (3.24) to deduce that $\dot{u}_h \rightharpoonup \dot{u}$ weakly in $L^2(L^2)$, that

$$u_h' \rightharpoonup \dot{u} \quad \text{in } L^2(L^2).$$

Furthermore, it implies that $\|u_h'\|_{H^1(L^2)}$ is bounded, which shows that, in fact,

$$u_h' \rightharpoonup \dot{u} \quad \text{in } H^1(L^2).$$

Hence, we deduce that $u \in H^2(L^2)$ and that (3.28) holds. □

Using the various convergence results stated in Lemma 3.7, it is not too difficult now to pass to the limit in (3.23), and obtain the following result.

Lemma 3.8. *The accumulation point* (u, v) *from the statement of Lemma* 3.7 *satisfies the wave equation* (2.5).

Proof. Since v_h is uniformly bounded in $W^{1,\infty}(H^1)$, (2.1) implies that $v_h \to v$ in $C(L^2)$. Since $0 \le v_h(t) \le 1$ a.e. in Ω, for all t, this convergence also implies that

$$a_h \to a \quad \text{in } C(L^2) \quad \text{and} \quad a_h^\pm \to a \quad \text{in } L^\infty(L^2), \tag{3.31}$$

where $a = v^2 + \eta$. The latter convergence follows from estimate (3.18) and an argument similar to the one given in (3.30).

We are now in a position to take the limit $h \searrow 0$ in (3.23). For any fixed $t_1, t_2 \in [0, T_f]$, and $\varphi \in H^1_D$, we have that

$$\begin{aligned}
&\int_{t_1}^{t_2} \big[(u_h'', \varphi) + \big(a_h^- Ae(u_h^+ + \dot{u}_h), e(\varphi)\big) - \langle \ell_h^+, \varphi\rangle\big] \mathrm{d}t \\
&\quad = \int_{t_1}^{t_2} \big[(u_h'', \varphi) + \big(aAe(u_h^+ + \dot{u}_h), e(\varphi)\big) - \langle \ell, \varphi\rangle\big] \mathrm{d}t \\
&\qquad + \int_{t_1}^{t_2} \big((a_h^- - a)Ae(u_h^+ + \dot{u}_h), e(\varphi)\big) \mathrm{d}t + \int_{t_1}^{t_2} \langle \ell_h^+ - \ell, \varphi\rangle \mathrm{d}t.
\end{aligned} \tag{3.32}$$

We bound the third term on the right-hand side of (3.32) by

$$\int_{t_1}^{t_2} \langle \ell_h^+ - \ell, \varphi\rangle \mathrm{d}t \le h(t_2 - t_1)\|\dot{\ell}\|_{C(H^{-1})}\|\varphi\|_{H^1}.$$

For the second term on the right-hand side, we use the Cauchy–Schwarz inequality to obtain

$$\int_{t_1}^{t_2} \big((a_h^- - a)Ae(w_h), e(\varphi)\big) \mathrm{d}t \le \|(a_h^- - a)|e(\varphi)|_A\|_{L^2(L^2)} \||e(w_h)|_A\|_{L^2(L^2)},$$

where $w_h = u_h^+ + \dot{u}_h$. Since $(a_h^- - a) \to 0$ a.e. in $\Omega \times (0, T_f)$, and $|a_h^- - a|^2 \le 4$, and since $|\nabla\varphi|^2 \in L^1(\Omega \times (0, T_f))$, Lebesgue's dominated convergence theorem implies

$$\begin{aligned}
&\lim_{h \searrow 0} \int_{t_1}^{t_2} \big|\big((a_h^- - a)Ae(w_h), e(\varphi)\big)\big| \mathrm{d}t \\
&\quad \le C \lim_{h \searrow 0} \|w_h\|_{L^2(H^1)} \left(\int_{t_1}^{t_2}\int_\Omega |a_h^- - a|^2 |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} = 0.
\end{aligned}$$

Finally, noting that $a \in L^\infty(L^\infty)$, we can simply take the (weak) limits as $h \searrow 0$ in each component of the first term on the right-hand side of (3.32), using (3.28), (3.27) and (3.24), to deduce that

$$\int_{t_1}^{t_2} \big[(\ddot{u}, \varphi) + \big(aAe(u + \dot{u}), e(\varphi)\big) - \langle \ell, \varphi\rangle\big] \mathrm{d}t = 0 \quad \forall\, t_1, t_2 \in [0, T_f].$$

The wave equation (2.5) follows immediately from this identity and from Lebesgue's differentiation theorem. $\square$

3.6. *Strong convergence*

To obtain strong convergence of u_h to u, we bound the truncation error and then use a discrete stability estimate followed by an application of a discrete Gronwall inequality.

For $h = T_f/N_f$ and for $n = 1, 2, \ldots$, we define $U_h^n = u(nh)$, $U_h^0 = u_0$, and $U_h^{-1} = u_0 - hu_1$. Furthermore, let U_h denote the piecewise affine interpolant and U_h^+ the backward piecewise constant interpolant of the samples $(U_h^n)_{n=0}^{N_f}$, and let U_h'' denote the backward piecewise constant interpolant of $(\delta^2 U_h^n)_{n=1}^{N_f}$. The same notation is used for interpolants of other discrete functions.

With this notation, and recalling from Theorem 3.1 that $u \in H^2(L^2) \cap W^{1,\infty}(H_D^1)$, we have the following result.

Lemma 3.9. *The following convergence results hold*:

$$U_h \to u \quad \text{in } H^1(H^1), \tag{3.33}$$

$$U_h^+ \to u \quad \text{in } L^2(H^1) \quad \text{and} \tag{3.34}$$

$$U_h'' \to \ddot{u} \quad \text{in } L^2(L^2). \tag{3.35}$$

Proof. We begin by showing (3.33). Since $u \in H^1(H^1) \subset C(H^1)$, it follows that $I_h u := U_h$ is correctly defined as an element of $H^1(H^1)$. If we had $u \in C^2(H^1)$ then (3.33) would trivially follow. However, we can approximate u by a sequence $(u_\delta)_{\delta>0} \subset C^2(H^1)$ of such smooth functions and use the uniform boundedness of the linear operator $I_h - I : H^1(H^1) \to H^1(H^1)$, where I is the identity operator, to deduce (3.33).

The uniform boundedness of $I_h - I : H^1(H^1) \to H^1(H^1)$ is shown as follows. First, observe that

$$|I_h w|_{H^1(H^1)} \le |w|_{H^1(H^1)}, \quad \text{and thus } |I_h w - w|_{H^1(H^1)} \le 2|w|_{H^1(H^1)} \quad \forall\, w \in H^1(H^1).$$

Furthermore,

$$\|I_h w - w\|_{L^2(H^1)} \le \pi^{-1} h |w|_{H^1(H^1)} \quad \forall\, w \in H^1(H^1),$$

and therefore, for any $w \in H^1(H^1)$,

$$\|I_h w - w\|_{H^1(H^1)} \le (4 + \pi^{-2} h^2)^{1/2} |w|_{H^1(H^1)} \le (4 + \pi^{-2} T_f^2)^{1/2} \|w\|_{H^1(H^1)}. \tag{3.36}$$

Hence, we have that

$$\begin{aligned}\|U_h - u\|_{H^1(H^1)} &= \|I_h u - u\|_{H^1(H^1)} \\ &\le \|I_h u_\delta - u_\delta\|_{H^1(H^1)} + \|(I_h - I)(u - u_\delta)\|_{H^1(H^1)}.\end{aligned}$$

For $\delta > 0$ fixed, the first term on the right-hand side tends to zero as $h \searrow 0$, while the second term, by (3.36) with $w = u - u_\delta$, is bounded by a constant multiple of

$\|u - u_\delta\|_{H^1(H^1)}$, which, in turn, can be made arbitrarily small by letting $\delta \searrow 0$; this implies (3.33). The convergence result (3.34) can be deduced exactly as in (3.30).

The same argument can be employed for proving (3.35). One proves, first, that the interpolation operator $u \mapsto U_h''$ is bounded from $H^2(L^2)$ to $L^2(L^2)$, and then repeats the regularization argument. □

To simplify the notation, we define $e_h^n := U_h^n - u_h^n$. From the definition of e_h^n and from (3.1) it follows that

$$(\delta^2 e_h^n, \varphi) + (a_h^{n-1} A e(e_h^n + \delta e_h^n), e(\varphi)) = T_h^n(\varphi) \quad \forall \varphi \in H_D^1, \tag{3.37}$$

where

$$T_h^n(\varphi) := (\delta^2 U_h^n, \varphi) + (a_h^{n-1} A e(U_h^n + \delta U_h^n), e(\varphi)) - \langle \ell(t_n), \varphi \rangle. \tag{3.38}$$

It is obvious from the definition that $T_h^n \in H^{-1}$ for $n = 1, \dots, N_f$. In the next step, we use Lemma 3.9 to estimate the *truncation error* $(T_h^n)_{n=1}^{N_f}$.

Lemma 3.10. *Let T_h^n, $n = 1, \dots, N_f$, be defined by (3.38), and let $(\varphi_h^n)_{n=1}^{N_f} \subset H^1$, then*

$$\left| \sum_{n=1}^{N_f} h T_h^n(\varphi_h^n) \right| \le \tilde{T}_h \left(\sum_{n=1}^{N_f} h \|e(\varphi_h^n)\|_A^2 \right)^{1/2},$$

where $\tilde{T}_h$ is defined as follows:

$$\begin{aligned} \tilde{T}_h = C\bigg(\int_0^{T_f} \{ &\|U_h'' - \ddot{u}\|_{L^2}^2 + \|e(U_h^+) - e(u)\|_A^2 + \|e(\dot{U}_h) - e(\dot{u})\|_A^2 \\ &+ \|\ell - \ell_h^+\|_{-1}^2 + \|(a_h^- - a)\, |e(u + \dot{u})|_A\|^2 \} \mathrm{d}t \bigg)^{1/2}. \end{aligned} \tag{3.39}$$

In particular, we have that $\tilde{T}_h \to 0$ as $h \to 0$.

Proof. The sum $\sum_{n=1}^{N_f} h|T_h^n(\varphi_h^n)|$ can be rewritten as

$$\sum_{n=1}^{N} h|T_h^n(\varphi_h^n)| = \int_0^T \left| (U_h'', \varphi_h^+) + \big(a_h^- A e(U_h^+ + \dot{U}_h), e(\varphi_h^+)\big) - \langle \ell_h^+, \varphi_h^+ \rangle \right| \mathrm{d}t.$$

Testing (2.5) with $\varphi = \varphi_h^+(t)$, and applying Korn's inequality, we obtain

$$\begin{aligned} \sum_{n=1}^{N} h|T_h^n(\varphi_h^n)| &= \int_0^T \big| (U_h'' - \ddot{u}, \varphi_h^+) + (a_h^- A e(U_h^+ - u + \dot{U}_h - \dot{u}), e(\varphi_h^+)) \\ &\qquad + \langle \ell - \ell_h^+, \varphi_h^+ \rangle + \big((a_h^- - a) A e(u + \dot{u}), e(\varphi_h^+)\big) \big| \,\mathrm{d}t \\ &\le C\bigg(\int_0^{T_f} \{ \|U_h'' - \ddot{u}\|_{L^2}^2 + \|e(U_h^+) - e(u)\|_A^2 + \|e(\dot{U}_h) - e(\dot{u})\|_A^2 \\ &\qquad + \|\ell - \ell_h^+\|_{-1}^2 + \|(a_h^- - a)\, |e(u + \dot{u})|_A\|^2 \} \mathrm{d}t \bigg)^{1/2} \left(\sum_{n=1}^{N} h \|e(\varphi_h^n)\|_A^2 \right)^{1/2} \\ &=: \tilde{T}_h \left(\sum_{n=1}^{N} h \|e(\varphi_h^n)\|_A^2 \right)^{1/2}, \end{aligned}$$

where $\tilde{T}_h$ is defined by (3.39).

Using (3.33)–(3.35) and the assumption that $\ell \in C^1(H^{-1})$, it follows that all terms in (3.39), except for the term $\|(a_h^- - a)\,|e(u+\dot{u})|_A\|$, tend to zero as $h \to 0$.

To treat the last term, we first extract a further subsequence for which the upper limit

$$\limsup_{h\to 0} \int_0^T \|(a_h^- - a)|e(u+\dot{u})|_A\|^2 \mathrm{d}t$$

is attained *and* for which $a_h^- \to a$ pointwise a.e. in $\Omega \times (0, T_f)$. Since $|a_h^- - a|^2 \le 4$ and $|e(u+\dot{u})|_A^2 \in L^1$, Lebesgue's dominated convergence theorem implies that

$$\|(a_h^- - a)\,|e(u+\dot{u})|_A\|_{L^2(L^2)}^2 \to 0 \quad \text{as } h \to 0. \tag{3.40}$$

This concludes the proof of the lemma. □

Using stability arguments similar to those in Sec. 3.3 and applying the truncation error estimate of the previous lemma, we obtain the following strong convergence result.

Lemma 3.11. *Let (u, v) be the accumulation point from the statement of Lemma 3.7; then*

$$u_h \to u \quad \textit{in } H^1(H^1), \quad \textit{and} \tag{3.41}$$

$$u_h^+ \to u \quad \textit{in } L^\infty(H^1). \tag{3.42}$$

(*We recall, however, that both* (3.41) *and* (3.42) *are understood in the sense of a subsequence, which we had previously extracted in Lemma* 3.7.)

Proof. We test (3.37) with $\varphi = w_h^n = e_h^n + \delta e_h^n$, and sum over n, to obtain, for $1 \le N \le N_f$,

$$\sum_{n=1}^N h(\delta^2 e_h^n, w_h^n) + \sum_{n=1}^N h\big(a_h^{n-1} A e(e_h^n + \delta e_h^n), e(w_h^n)\big) = \sum_{n=1}^N h T_h^n(w_h^n). \tag{3.43}$$

Using summation by parts and the fact that $\delta e_h^n = 0$, the first term on the left-hand side of (3.43) is rewritten in the form

$$\sum_{n=1}^N h(\delta^2 e_h^n, e_h^n + \delta e_h^n) = (\delta e_h^N, e_h^N + \delta e_h^N) - \sum_{n=1}^N h(\delta e_h^{n-1}, \delta e_h^n + \delta^2 e_h^n).$$

Using the same argument as in (3.10) to estimate the first term on the right-hand side, and a Cauchy inequality to estimate the second term on the left-hand side, we obtain

$$\begin{aligned}\sum_{n=1}^N h(\delta^2 e_h^n, e_h^n + \delta e_h^n) \ge \|\delta e_h^N\|^2 &+ \frac{1}{2h}(\|e_h^N\|^2 - \|e_h^{N-1}\|^2) \\ &- \frac{1}{2}\sum_{n=1}^N h(\|\delta e_h^n\|^2 + \|\delta e_h^{n-1}\|^2) - \sum_{n=1}^N h(\delta e_h^{n-1}, \delta^2 e_h^n).\end{aligned}$$

Using Cauchy's inequality again, and the fact that $\delta e_h^0 = 0$, we have

$$-\sum_{n=1}^{N} h(\delta e_h^{n-1}, \delta^2 e_h^n) \geq -\frac{1}{2}\sum_{n=1}^{N}(\|\delta e_h^n\|^2 - \|\delta e_h^{n-1}\|^2) = -\frac{1}{2}\|\delta e_h^N\|^2,$$

and hence, we arrive at

$$\sum_{n=1}^{N} h(\delta^2 e_h^n, e_h^n + \delta e_h^n) \geq \frac{1}{2}\|\delta e_h^N\|^2 + \frac{1}{2h}(\|e_h^N\|^2 - \|e_h^{N-1}\|^2) - \sum_{n=1}^{N} h\|\delta e_h^n\|^2. \quad (3.44)$$

For the second term on the left-hand side of (3.43), we have

$$\begin{aligned}\sum_{n=1}^{N} h\|(a_h^{n-1})^{1/2}\,|e(e_h^n + \delta e_h^n)|_A\|^2 &\geq \eta\sum_{n=1}^{N} h\|e(e_h^n + \delta e_h^n)\|_A^2 \\ &\geq \frac{1}{2}\eta\sum_{n=1}^{N} h\|e(e_h^n + \delta e_h^n)\|_A^2 + \frac{1}{2}\eta\sum_{n=1}^{N} h\|e(\delta e_h^n)\|_A^2 \\ &\quad + \eta\sum_{n=1}^{N} h(Ae(e_h^n), e(\delta e_h^n)).\end{aligned}$$

Since

$$\sum_{n=1}^{N} h(Ae(e_h^n), e(\delta e_h^n)) \geq \sum_{n=1}^{N}\left(\frac{1}{2}\|e(e_h^n)\|_A^2 - \frac{1}{2}\|e(e_h^{n-1})\|_A^2\right) = \frac{1}{2}\|e(e_h^N)\|_A^2,$$

we therefore obtain

$$\begin{aligned}\sum_{n=1}^{N} h\|(a_h^{n-1})^{1/2}\,|e(e_h^n + \delta e_h^n)|_A\|^2 &\geq \frac{1}{2}\eta\sum_{n=1}^{N} h\|e(e_h^n + \delta e_h^n)\|_A^2 \\ &\quad + \frac{1}{2}\eta\sum_{n=1}^{N} h\|e(\delta e_h^n)\|_A^2 + \frac{1}{2}\eta\|e(e_h^N)\|_A^2. \quad (3.45)\end{aligned}$$

Combining (3.44) and (3.45), and applying Lemma 3.10 with $\varphi_h^n = w_h^n$, we deduce that

$$\begin{aligned}&\|\delta e_h^N\|^2 + \frac{1}{h}(\|e_h^N\|^2 - \|e_h^{N-1}\|^2) + \sum_{n=1}^{N} h\|e(\delta e_h^n)\|_A^2 + \|e(e_h^N)\|_A^2 \\ &\quad\leq C\left\{\sum_{n=1}^{N} h\|\delta e_h^n\|^2 + \tilde{T}_h^2\right\}, \quad (3.46)\end{aligned}$$

where $\tilde{T}_h$ is defined in (3.38). Multiplying this inequality by h and summing the result over $N = 1, \ldots, M$, where $1 \leq M \leq N_f$, leads to

$$E_M \leq C\left\{\sum_{N=1}^{M} hE_N + \tilde{T}_h^2\right\},$$

where

$$E_M = \|e_h^M\|^2 + \sum_{N=1}^{M} h\left\{ \|\delta e_h^N\|^2 + \|e(e_h^N)\|_A^2 + \sum_{n=1}^{N} h\|e(\delta e_h^n)\|_A^2 \right\}.$$

An application of a discrete Gronwall inequality, for example Lemma 10.5 in Thomée,[18] and recalling from Lemma 3.10 that $\tilde{T}_h \to 0$, we obtain

$$\max_{M=1,\dots,N_f} E_M \le C\tilde{T}_h^2 \to 0 \quad \text{as } h \searrow 0. \tag{3.47}$$

In terms of the piecewise affine and backward interpolants of $(e_h^n)_{n=1}^{N_f}$, after applying Korn's inequality, this implies

$$\|e_h^+\|_{L^\infty(L^2)} + \|\dot{e}_h\|_{L^2(L^2)} + \|\nabla e_h^+\|_{L^2(L^2)} \to 0 \quad \text{as } h \searrow 0. \tag{3.48}$$

Note that, in (3.48), we have neglected the final term in the definition of E_M, which would have appeared in the form of a double-integral and not have given the desired result. To obtain convergence of $\nabla \dot{e}_h$, we first revert to (3.46) from which we can now deduce, using (3.47),

$$\begin{aligned} \|\delta e_h^N\|^2 + \sum_{n=1}^{N} h\|e(\delta e_h^n)\|_A^2 &\le C\tilde{T}_h^2 - \frac{1}{h}(\|e_h^N\|^2 - \|e_h^{N-1}\|^2) \\ &\le C\tilde{T}_h^2 - (e_h^N + e_h^{N-1}, \delta e_h^N) \\ &\le C\tilde{T}_h^2 + \frac{1}{4}\|e_h^N + e_h^{N-1}\|^2 + \|\delta e_h^N\|^2. \end{aligned}$$

Canceling the first term on the left-hand side and the third term on the right-hand side of the inequality, and using (3.47) again to estimate $\|e_h^N + e_h^{N-1}\| \le C\tilde{T}_h$, we arrive at

$$\sum_{n=1}^{N} h\|e(\delta e_h^n)\|_A^2 \le C\tilde{T}_h^2 \to 0 \quad \text{as } h \searrow 0.$$

After taking the supremum over N, applying Korn's inequality, and recalling that e_h is the piecewise affine interpolant of $(e_h^n)_{n=0}^{N_f}$, this may also be read as

$$\|\nabla \dot{e}_h\|_{L^2(L^2)}^2 \le C\tilde{T}_h^2 \to 0 \quad \text{as } h \searrow 0.$$

Thus, we finally arrive at

$$e_h = U_h - u_h \to 0 \quad \text{in } H^1(H^1) \quad \text{as } h \searrow 0.$$

Using the interpolation error estimate (3.33), and the fact that $\|u_h^+(t) - u_h(t)\|_{H^1} \le h\|\dot{u}_h(t)\|_{H^1}$ for all $t \in [0, T_f]$, we obtain (3.41) and (3.42). $\square$

3.7. *Minimality of v*

We use the strong convergence result (3.42) to establish the unilateral minimality of v.

Lemma 3.12. *Let (u, v) be the accumulation point from the statement of Lemma 3.7; then (2.3) is satisfied for every $t \in (0, T_f]$.*

Proof. Fix $t \in (0, T_f]$ and recall as in (3.3) that the variational inequality associated with (2.3) is

$$\partial_v \mathcal{E}(u(t), v(t))[\psi - v(t)] + \mathcal{H}'(v(t))[\psi - v(t)] \geq 0 \quad \forall\, \psi \in 1 + H^1_D, \quad \psi \leq v(t), \tag{3.49}$$

or equivalently, upon substituting $\chi = \psi - v(t) \leq 0$,

$$\partial_v \mathcal{E}(u(t), v(t))[\chi] + \mathcal{H}'(v(t))[\chi] \geq 0 \quad \forall\, \chi \in H^1_D, \quad \chi \leq 0. \tag{3.50}$$

Since $v_h^+(t)$ is minimal among $\psi \leq v_h^-(t)$, it is also minimal among $\psi \leq v_h^+(t)$, which again gives

$$\partial_v \mathcal{E}(u_h^+(t), v_h^+(t))[\chi] + \mathcal{H}'(v_h^+(t))[\chi] \geq 0 \quad \forall\, \chi \in H^1_D, \quad \chi \leq 0.$$

Since $\mathcal{H}'(v)[\chi]$ is linear in v, and since $v_h^+(t) \rightharpoonup v(t)$ weakly in H^1 for every t, it follows that

$$\mathcal{H}'(v_h^+(t))[\chi] \to \mathcal{H}'(v(t))[\chi],$$

where $\chi \in H^1_D, \chi \leq 0$ is held fixed. Furthermore, for every fixed $t \in [0, T_f]$, $\nabla u_h^+(t) \to \nabla u(t)$ strongly in L^2, which implies that $|e(u_h^+(t))|_A^2 \to |e(u(t))|_A^2$ strongly in L^1. After extracting a subsequence $h' \subset h$ for which $v_{h'}^+(t) \to v(t)$ pointwise a.e. in Ω, for all $t \in [0, T_f]$, Lebesgue's dominated convergence theorem implies that

$$\int_\Omega v_h^+ \chi |e(u_h^+)|_A^2 \,\mathrm{d}x \to \int_\Omega v\chi |e(u)|_A^2 \,\mathrm{d}x \quad \forall\, \chi \in L^\infty.$$

Thus, we have shown that (3.50) holds for all $\chi \in L^\infty \cap H^1$, $\chi \leq 0$. Since the only reasonable competitors ψ for the energy satisfy $0 \leq \psi \leq v$, this is sufficient to deduce unilateral minimality of v and thus concludes the proof of (3.49), and equivalently of (2.3), for any $t \in (0, T_f]$. □

3.8. *Energy balance*

Lemmas 3.7, 3.8, 3.11 and 3.12 establish all the results contained in Theorem 3.1 except for the strong convergence of v_h in H^1, and the claim that the trajectory (u, v) satisfies the energy balance formula (2.4). The purpose of this section is to prove this. The following lemma and its corollary below therefore conclude the proof of Theorem 3.1, which immediately implies Theorem 2.1 as well.

Lemma 3.13. *Let (u, v) be the accumulation point from the statement of Lemma 3.7; then it satisfies the energy balance condition (2.4).*

Proof. Testing (2.5) with $\varphi = \dot{u}$ gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{u}\|^2 + \|a^{1/2}|e(\dot{u})|_A\|^2 + (aAe(u), e(\dot{u})) = \langle \ell, \dot{u}\rangle. \tag{3.51}$$

Using the fact that $\ell \in C^1(H^{-1})$, the left-hand side can be rewritten as

$$\langle \ell, \dot{u}\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle \ell, u\rangle - \langle \dot{\ell}, u\rangle. \tag{3.52}$$

In what follows, we will find a way to bypass the technically subtle product rule formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|a^{1/2}|e(u)|_A\|^2 = \frac{1}{2}(\dot{a}Ae(u), e(u)) + (aAe(u), e(\dot{u})),$$

which, formally, would quickly lead to the energy balance condition (2.4).

First, we use the discrete energy inequality (3.21) to deduce a corresponding result for the limit. Using only the weak convergence of v_h and u_h and the strong convergence of a_h^- in $L^2(L^2)$, as well as a standard lower-semicontinuity property of convex integrands we obtain

$$\begin{aligned}
&\mathcal{F}(T, u(T), \dot{u}(T), v(T)) + \int_0^T \|a^{1/2}|e(\dot{u})|_A\|^2\,\mathrm{d}t + \int_0^T \langle \dot{\ell}(t), u\rangle \mathrm{d}t \\
&\quad\le \liminf_{h\searrow 0}\Big\{\mathcal{F}(T, u_h(T), \dot{u}_h(T), v_h(T)) + \int_0^T \|(a_h^-)^{1/2}|e(\dot{u}_h)|_A\|^2\mathrm{d}t \\
&\qquad\qquad + \int_0^T \langle \dot{\ell}_h(t), u_h\rangle \mathrm{d}t\Big\} \\
&\quad\le \limsup_{h\searrow 0}\Big\{\mathcal{F}(T, u_h(T), \dot{u}_h(T), v_h(T)) + \int_0^T \|(a_h^-)^{1/2}|e(\dot{u}_h)|_A\|^2\mathrm{d}t \\
&\qquad\qquad + \int_0^T \langle \dot{\ell}_h(t), u_h\rangle \mathrm{d}t\Big\} \\
&\quad\le \mathcal{F}(0, u_0, \dot{u}_0, v_0),
\end{aligned} \tag{3.53}$$

i.e.

$$\begin{aligned}
\mathcal{F}(T, u(T), \dot{u}(T), v(T)) + \int_0^T \|a^{1/2}|e(\dot{u})|_A\|^2\,\mathrm{d}t + \int_0^T \langle \dot{\ell}(t), u\rangle \mathrm{d}t& \\
\le \mathcal{F}(0, u(0), \dot{u}(0), v(0)).&
\end{aligned} \tag{3.54}$$

It remains to prove the reverse inequality,

$$\begin{aligned}
\mathcal{F}(T, u(T), \dot{u}(T), v(T)) + \int_0^T \|a^{1/2}|e(\dot{u})|_A\|^2\,\mathrm{d}t + \int_0^T \langle \dot{\ell}(t), u\rangle \mathrm{d}t& \\
\ge \mathcal{F}(0, u(0), \dot{u}(0), v(0)).&
\end{aligned} \tag{3.55}$$

By integrating (3.51) and using (3.52), we find that (3.55) is equivalent to

$$\mathcal{E}(u(T), v(T)) + \mathcal{H}(v(T)) - \int_0^T (aAe(u), e(\dot{u}))\mathrm{d}t \geq \mathcal{E}(u(0), v(0)) + \mathcal{H}(v(0)),$$

which can be rearranged as

$$\begin{aligned} &\mathcal{E}(u(T), v(T)) - \mathcal{E}(u(0), v(0)) \\ &\qquad \geq -\mathcal{H}(v(T)) + \mathcal{H}(v(0)) + \int_0^T \big(aAe(u), e(\dot{u})\big)\mathrm{d}t. \end{aligned} \tag{3.56}$$

We prove (3.56) by a time-discretization. In order to avoid confusion with the earlier time-step discretization, let $M \in \mathbb{N}$, let $\tau = \tau_M = T/M$, and set $s_i = i\tau$, $i = 0, \ldots, M$. For each i we write

$$\begin{aligned} &\mathcal{E}(u(s_i), v(s_i)) - \mathcal{E}(u(s_{i-1}), v(s_{i-1})) \\ &\quad = \frac{1}{2}\int_\Omega \big(a(s_i)|e(u(s_i))|_A^2 - a(s_{i-1})|e(u(s_{i-1}))|_A^2\big)\mathrm{d}x \\ &\quad = \frac{1}{2}\int_\Omega \big(a(s_i) - a(s_{i-1})\big)|e(u(s_{i-1}))|_A^2 \mathrm{d}x \\ &\qquad + \frac{1}{2}\int_\Omega a(s_i)\big(|e(u(s_i))|_A^2 - |e(u(s_{i-1}))|_A^2\big)\mathrm{d}x \\ &\quad =: \mathbf{A}_i + \mathbf{B}_i. \end{aligned} \tag{3.57}$$

We bound the terms $\mathbf{A}_i$ and $\mathbf{B}_i$ separately.

Thanks to the unilateral minimality of $v(s_{i-1})$, we have

$$\begin{aligned} &\frac{1}{2}\int_\Omega a(s_i)|e(u(s_{i-1}))|_A^2\, \mathrm{d}x + \mathcal{H}(v(s_i)) \\ &\qquad \geq \frac{1}{2}\int_\Omega a(s_{i-1})|e(u(s_{i-1}))|_A^2\, \mathrm{d}x + \mathcal{H}(v(s_{i-1})) \end{aligned}$$

and hence

$$\mathbf{A}_i = \frac{1}{2}\int_\Omega \big(a(s_i) - a(s_{i-1})\big)|e(u(s_{i-1}))|_A^2\, \mathrm{d}x \geq \mathcal{H}(v(s_{i-1})) - \mathcal{H}(v(s_i)). \tag{3.58}$$

To bound the term $\mathbf{B}_i$ we note first that

$$\begin{aligned} \mathbf{B}_i &= \frac{1}{2}\int_\Omega a(s_i)Ae(u(s_i) + u(s_{i-1})) : e(u(s_i) - u(s_{i-1}))\mathrm{d}x \\ &= \int_{s_{i-1}}^{s_i}\int_\Omega a_\tau^-(s)Ae(\bar{u}_\tau(s)) : e(\dot{u}(s))\mathrm{d}x\, \mathrm{d}s, \end{aligned} \tag{3.59}$$

where

$$a_\tau^-(s) = a(s_{i-1})$$

and

$$\bar{u}_\tau(s) = \frac{1}{2}(u(s_{i-1}) + u(s_i)) \quad \text{for } s \in (s_{i-1}, s_i), \quad i = 1, \ldots, M.$$

Due to the regularity of u and v, it follows that

$$\begin{aligned} a_\tau^- &\to a \quad \text{strongly in } L^\infty(L^2), \quad \text{and} \\ \bar{u}_\tau &\to u \quad \text{strongly in } L^\infty(H^1). \end{aligned}$$

Summing (3.59) over $i = 1, \ldots, M$ gives

$$\begin{aligned} \sum_{i=1}^M \mathbf{B}_i = \int_0^T (a_\tau^- Ae(\bar{u}_\tau), e(\dot{u}))\mathrm{d}s &= \int_0^T (aAe(u), e(\dot{u}))\mathrm{d}s \\ &+ \int_0^T (a_\tau^- Ae(\bar{u}_\tau - u), e(\dot{u}))\mathrm{d}s + \int_0^T ((a_\tau^- - a)Ae(u), e(\dot{u}))\mathrm{d}s. \end{aligned}$$

The second term on the right-hand side clearly converges to zero as $\tau \to 0$. For the third term on the right-hand side, we use again Lebesgue's dominated convergence theorem to prove that, after extracting a suitable subsequence, so that $a_\tau^-(t) \to a(t)$ pointwise a.e. in $(0, T)$, the third term tends to zero as well. Summing (3.57) and (3.58) over $i = 1, \ldots, M$ as well, we have shown that

$$\begin{aligned} \mathcal{E}(u(T), v(T)) - \mathcal{E}(u(0), v(0)) &= \limsup_{\tau \searrow 0} \sum_{i=1}^M (\mathbf{A}_i + \mathbf{B}_i) \\ &\geq \mathcal{H}(v(0)) - \mathcal{H}(v(T)) + \int_0^T (aAe(u), e(\dot{u}))\mathrm{d}t, \end{aligned}$$

which, as we have argued above, implies (3.55). Together with (3.54), we have shown that the limit indeed satisfies the energy conservation condition (2.4). □

Corollary 3.1. *For every* $t \in [0, T_f]$ *we have*

$$v_h(t) \to v(t) \quad \textit{strongly in } H^1. \tag{3.60}$$

Proof. We begin by noting that, since (u, v) satisfies (2.4), we have in fact equality in all inequalities in the chain of estimates in (3.53), that is, for any $T \in (0, T_f]$,

$$\begin{aligned} &\mathcal{F}\Big(t, u_h(t), \dot{u}_h(T), v_h(T)\Big) + \int_0^T \big\{\|(a_h^-)^{1/2}|e(\dot{u}_h)|_A\|^2 + \langle \dot{\ell}_h(t), u_h\rangle\big\}\mathrm{d}t \\ &\quad \to \mathcal{F}\big(t, u(t), \dot{u}(T), v(T)\big) + \int_0^T \big\{\|a^{1/2}|e(\dot{u})|_A\|^2 + \langle \dot{\ell}(t), u\rangle\big\}\mathrm{d}t, \end{aligned}$$

as $h \to 0$.

Using the strong convergence result from Lemma 3.11, and repeating the argument used to show (3.40), it is straightforward to show that all terms, except for the term $\mathcal{H}(v_h(t))$ contained in $\mathcal{F}(t, u_h(t), \dot{u}_h(t), v_h(t))$, converge separately. However, since their sum converges we obtain also that

$$\mathcal{H}(v_h(t)) \to \mathcal{H}(v(t)) \quad \forall\, t \in (0, T_f].$$

Since weak convergence together with convergence of the norm implies strong convergence, this gives the desired result for all $t \in (0, T_f]$. For $t = 0$, $v_h(0) = v_h^0 = v_0 = v(0)$, so that the result holds trivially. □

4. Conclusions

In this paper, we have developed the first steps of a theory for a regularized model of dynamic crack propagation based on the time-discrete model put forward in Bourdin, Larsen & Richardson.[4] By proving convergence of a time-discretization, as the time-step tends to zero, we have established the existence of solutions to a time-continuous formulation as well as balance of total energy of the system. We stress, once again, that this model and our theory do not require any *a priori* assumptions on the crack topology and is in particular dimension-independent.

Of course, a number of questions remain open. For example, we were unable to perform our analysis without the damping term. We believe that it should be possible to establish existence of solutions in the absence of the damping term, however, we could not then see the way for ensuring energy balance.

Second, taking the limit as $\varepsilon \searrow 0$ poses a formidable challenge. Note, for example, that the unilateral minimality of the v variable has no obvious counterpart in a sharp-interface model. While some possibilities have been proposed by Larsen,[16] proof that any of these is the limiting model is open.

Third, we were unable to establish a sufficiently strong notion of maximal dissipation. Intuitively, it seems reasonable to us that unilateral minimality of v and energy balance could provide such a condition.

Finally, a fascinating question is to rescale time, and to investigate the quasi-static limit of our dynamic model. Here, the presence of the damping is crucial. It would be interesting to see whether one can, at all, recover the model of Francfort & Marigo,[12] as well as discover situations in which the model of Francfort & Marigo[12] does not give the limiting model.

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