

# Intro to Robotics

## Exercise 3

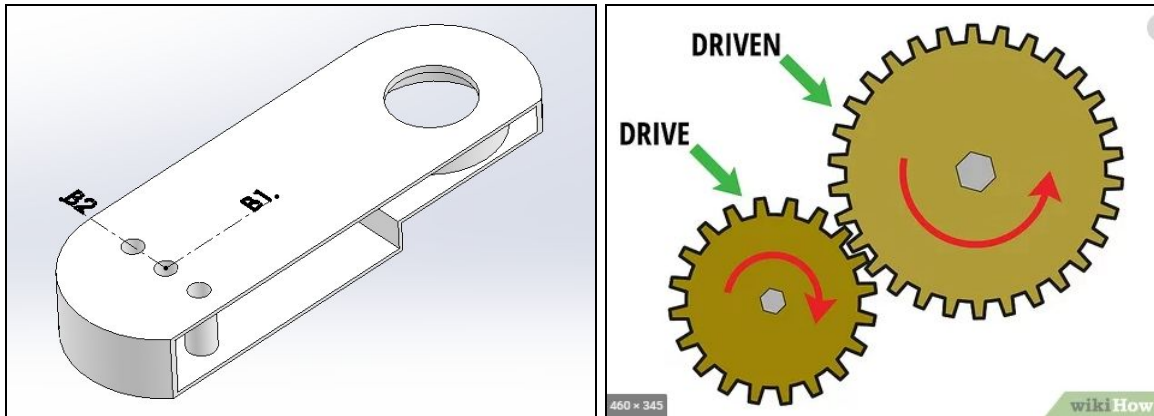
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Samuel Law & Rhys Miller

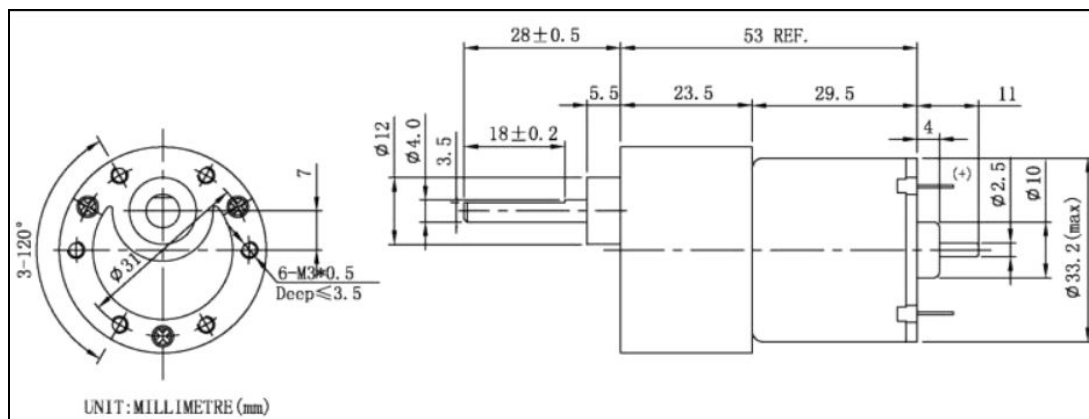
The constraints equation is  $\Delta q_0 = -q_1 g^r$  where the q's are the generalized coordinates and gr is the gear ratio between the motor shaft and the motor output spindle. The constraint equation carries to both the velocity and acceleration terms.

The gear ratio used in the gear constraint was printed on the side of the motor.

# Mass Properties



The mass for the link, body B, was computed using ABS mass properties from SolidWorks applied to the CAD model. The model was hollowed out, and the shell thickness set to 0.90 mm, the same as was used to print the physical model. The internally printed structure was neglected as a sparse infill was used.



The mass of the axle that spins in body A was calculated using the drawing shown above. The axle was assumed to be 10.0 mm in diameter and 53 mm in length. A CAD model was generated, and the mass estimated with steel mass properties applied to the CAD model. The gears were assumed to interact in a manner similar to the ones shown below

The moments of inertia for both bodies were evaluated in the same manner as the mass properties.

Body	Mass [kg]	Izz [kg m <sup>2</sup> ]
Body A	$26.45 \times 10^{-3}$	$3.30 \times 10^{-7}$
Body B	$7.28 \times 10^{-3}$	$6.39 \times 10^{-6}$

# Equations of Motion

Per the assignment, the equations of motion were derived by hand. This is inconsistent with prior exercises, and points should not be taken off for not showing the equations of motion derived computationally. A direct quotation from the instructions is shown below.

Write up this lab as a homework problem including the development of the equations of motion by hand. Make sure to include:

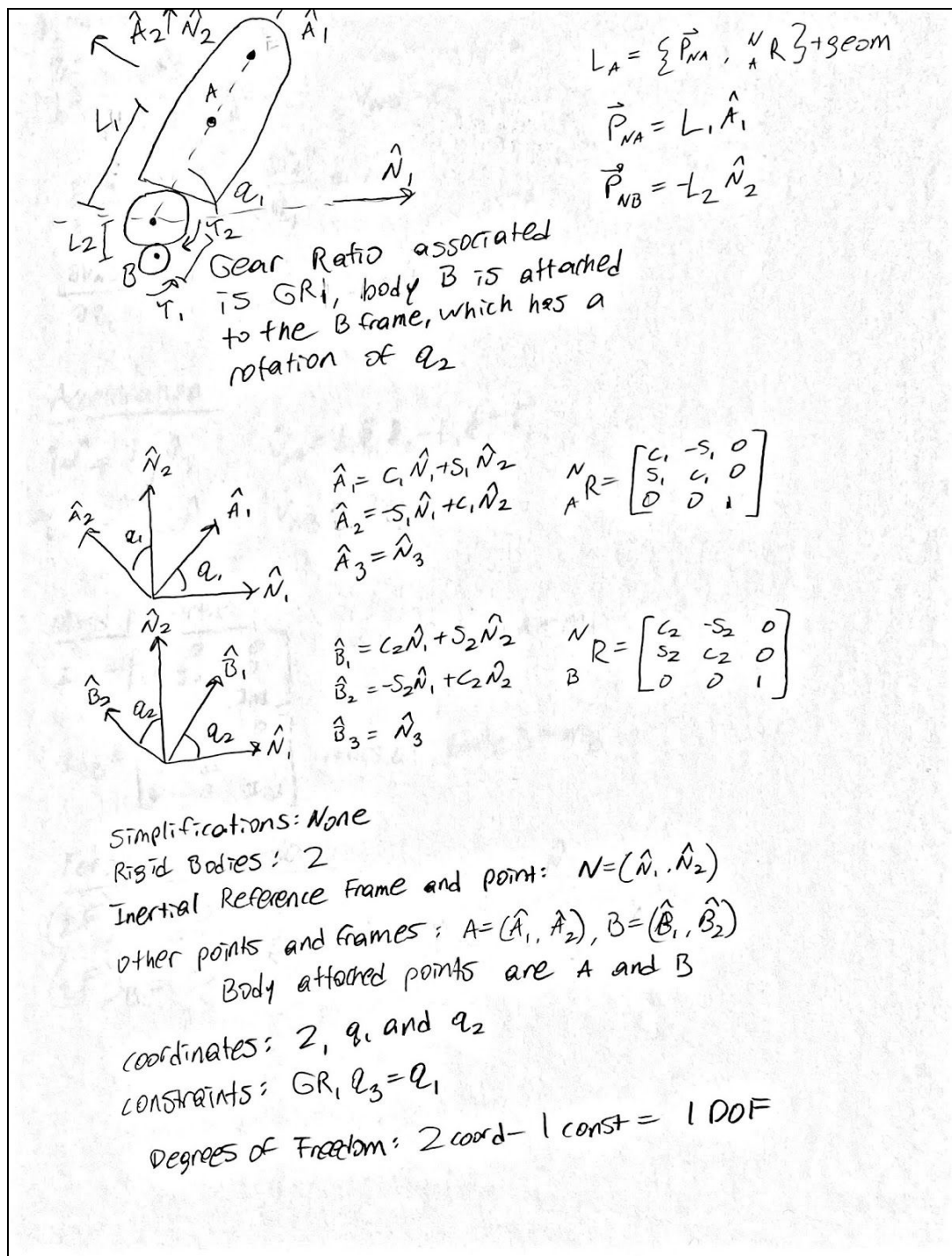


Diagram of a mechanical system showing two bodies, A and B, and a gear ratio constraint. Body A is a rod of length  $L_1$  pivoted at point A. Body B is a rod of length  $L_2$  pivoted at point B. The gear ratio is associated with body B, which is attached to the B frame, which has a rotation of  $q_2$ .

Equations for the system:

$$L_A = \{ \vec{P}_{NA}, {}^N_A R \} + geom$$

$$\vec{P}_{NA} = L_1 \hat{A}_1$$

$$\vec{P}_{NB} = -L_2 \hat{N}_2$$

Coordinate transformations:

$$\begin{aligned} \hat{A}_1 &= c_1 \hat{N}_1 + s_1 \hat{N}_2 \\ \hat{A}_2 &= -s_1 \hat{N}_1 + c_1 \hat{N}_2 \\ \hat{A}_3 &= \hat{N}_3 \end{aligned} \quad {}^N_A R = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \hat{B}_1 &= c_2 \hat{N}_1 + s_2 \hat{N}_2 \\ \hat{B}_2 &= -s_2 \hat{N}_1 + c_2 \hat{N}_2 \\ \hat{B}_3 &= \hat{N}_3 \end{aligned} \quad {}^N_B R = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Simplifications: None

Rigid Bodies: 2

Inertial Reference Frame and point:  $N = (\hat{N}_1, \hat{N}_2)$

Other points and frames:  $A = (\hat{A}_1, \hat{A}_2)$ ,  $B = (\hat{B}_1, \hat{B}_2)$

Body attached points are A and B

Coordinates:  $z$ ,  $q_1$  and  $q_2$

Constraints: GR,  $q_3 = q_1$

Degrees of Freedom:  $2 \text{ coord} - 1 \text{ const} = 1 \text{ DOF}$

### Velocity

$$\begin{aligned}\vec{w}^A &= \dot{q}_1 \hat{N}_3 & v_{NA} &= L_1 \dot{q}_1 \hat{A}_2 \\ \vec{w}^B &= \dot{q}_3 \hat{N}_3 = \frac{\dot{q}_1}{GR_1} \hat{N}_3 & v_{NB} &= 0\end{aligned}$$

$$\frac{\partial v_{NA}}{\partial \dot{q}_1} = L_1 \hat{A}_2 \quad \frac{\partial \vec{w}^A}{\partial \dot{q}_1} = \hat{N}_3$$

$$\frac{\partial v_{NA}}{\partial \dot{q}_2} = 0$$

$$\frac{\partial \vec{w}^A}{\partial \dot{q}_2} = 0$$

$$\frac{\partial v_{NB}}{\partial \dot{q}_1} = 0 \quad \frac{\partial \vec{w}^B}{\partial \dot{q}_1} = \frac{1}{GR_1} \hat{N}_3$$

$$\frac{\partial v_{NB}}{\partial \dot{q}_2} = 0$$

### Acceleration

$$\vec{w}^A = \ddot{q}_1 \hat{N}_3 \quad \dot{v}_{NA} = L_1 \ddot{q}_1 \hat{A}_2 - L_1 \dot{q}_1^2 \hat{A}_1$$

$$\vec{w}^B = \frac{\ddot{q}_1}{GR_1} \hat{N}_3 \quad \dot{v}_{NB} = 0$$

### Mass properties

$$I_{AA} = \begin{bmatrix} I_{A1} & 0 & 0 \\ 0 & I_{A2} & 0 \\ 0 & 0 & I_{A3} \end{bmatrix}$$

mass of body A =  $m_A$

$$I_{BB} = \begin{bmatrix} I_{B1} & 0 & 0 \\ 0 & I_{B2} & 0 \\ 0 & 0 & I_{B3} \end{bmatrix}$$

mass of body B =  $m_B$

### Forces and moments

$$(\sum F)_A = -g m_A \hat{N}_3 \quad (\sum M)_{AA} = -\tau_2 \hat{N}_3$$

$$(\sum F)_B = -g m_B \hat{N}_3 \quad (\sum M)_{BB} = -\tau_1 \hat{N}_3$$

$$\frac{\partial H_{AA}}{\partial t} = I_{AA} \dot{\omega}^A + \omega^A \times I_{AA} \omega^A = \begin{bmatrix} I_{A1} & 0 & 0 \\ 0 & I_{A2} & 0 \\ 0 & 0 & I_{A3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} \times \begin{bmatrix} I_{A1} & 0 & 0 \\ 0 & I_{A2} & 0 \\ 0 & 0 & I_{A3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}$$

$$\frac{\partial H_{AA}}{\partial t} = I_{A3} \ddot{q}_1 \hat{n}_3$$

$$\frac{\partial H_{BB}}{\partial t} = I_{BB} \dot{\omega}^B + \omega^B \times I_{BB} \omega^B = \begin{bmatrix} I_{B1} & 0 & 0 \\ 0 & I_{B2} & 0 \\ 0 & 0 & I_{B3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 / GR_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 / GR_1 \end{bmatrix} \times \begin{bmatrix} I_{B1} & 0 & 0 \\ 0 & I_{B2} & 0 \\ 0 & 0 & I_{B3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 / GR_1 \end{bmatrix}$$

$$\frac{\partial H_{BB}}{\partial t} = \frac{I_{B3} \ddot{q}_1}{GR_1} \hat{n}_3$$

$$F_1 = \left[ \sum F_A \cdot \frac{\partial V_{NA}}{\partial \dot{q}_1} + \sum M_A \cdot \frac{\partial \omega^A}{\partial \dot{q}_1} \right] + \left[ \sum F_B \cdot \frac{\partial V_{NB}}{\partial \dot{q}_1} + \sum M_B \cdot \frac{\partial \omega^B}{\partial \dot{q}_1} \right]$$

$$F_1 = -T_2 + \frac{T_1}{GR_1}$$

$$F_1^* = \left[ m_A \dot{V}_{NA} \cdot \frac{\partial V_{NA}}{\partial \dot{q}_1} + \frac{\partial H_{AA}}{\partial t} \cdot \frac{\partial \omega^A}{\partial \dot{q}_1} \right] + \left[ m_B \dot{V}_{NB} \cdot \frac{\partial V_{NB}}{\partial \dot{q}_1} + \frac{\partial H_{BB}}{\partial t} \cdot \frac{\partial \omega^B}{\partial \dot{q}_1} \right]$$

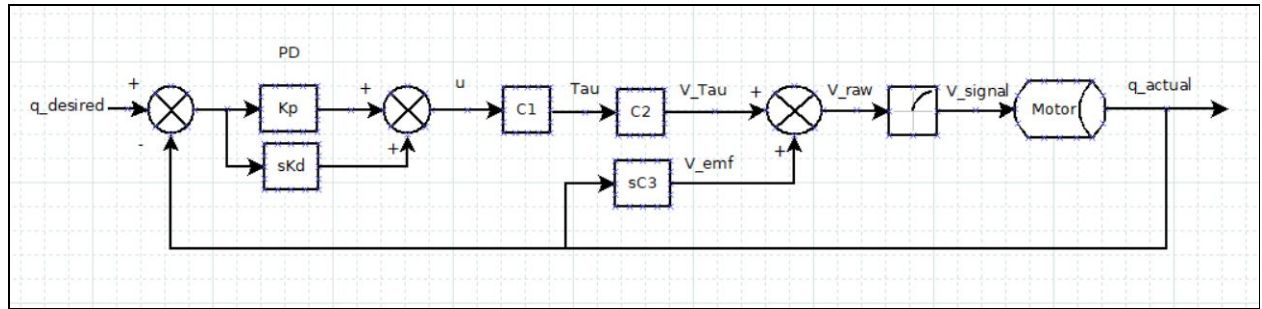
$$F_1^* = \left[ m_A (L_1 \ddot{q}_1 \hat{A}_2 - L_1 \dot{q}_1^2 \hat{A}_1) \cdot L_1 \hat{A}_2 \right] + \left[ I_{A3} \ddot{q}_1 \hat{n}_3 \cdot \hat{n}_3 \right] + \left[ m_B (0) \cdot 0 \right] +$$

$$\left[ \frac{I_{B3} \ddot{q}_1}{GR_1} \hat{n}_3 \cdot \frac{1}{GR_1} \hat{n}_3 \right]$$

$$F_1^* = (I_{A3} + m_A L_1^2) \ddot{q}_1 + \frac{I_{B3}}{GR_1^2} \ddot{q}_1$$

$$F_1 - F_1^* = 0$$

## Block Diagram & Discussion



All simulations were performed using an RK45 integrator. The  $K_p$  value used was calculated based on the CPR of the encoder, which was set to 1000. The equation used is shown below, where  $V_{min}$  is the minimum voltage required for the motor to begin moving.  $V_{min}$  was determined via experimentation. For the joint space model,  $K_v$  was simply set so as to be critically damped. The saturation block between  $V_{raw}$  and  $V_{signal}$  indicates that the voltage signal was constrained between 0 and 12 volts.

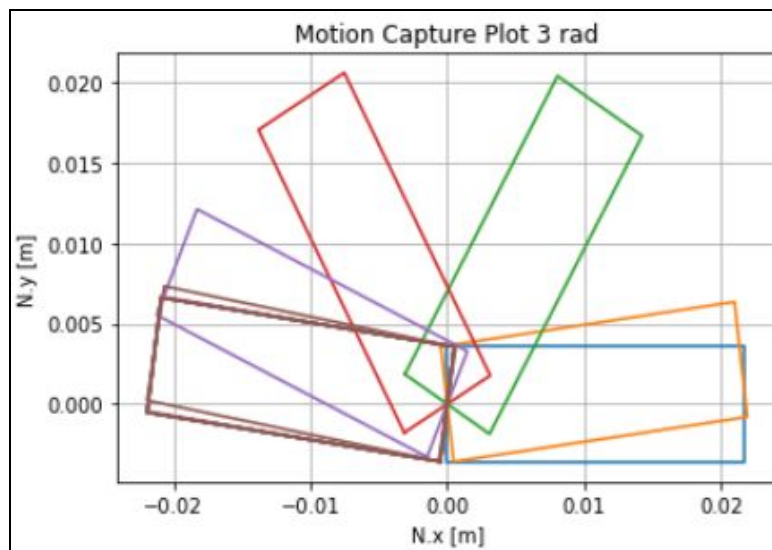
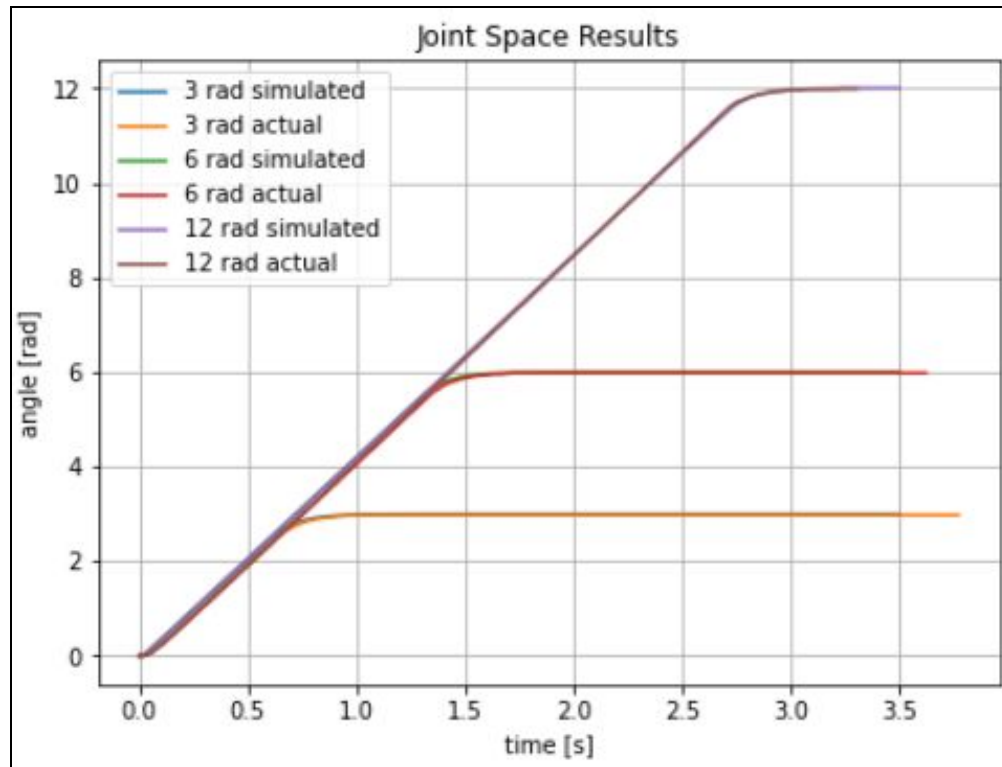
$$V_{min} = K_p \left( \frac{2\pi}{CPR} \right)$$

The only change in the block diagram for the PD controller was that the  $C1$  block was removed, thus eliminating any kind of compensation for mass properties. This meant that the controller had to be tuned in order to work correctly.



# Joint Space Simulation & Experiment Results

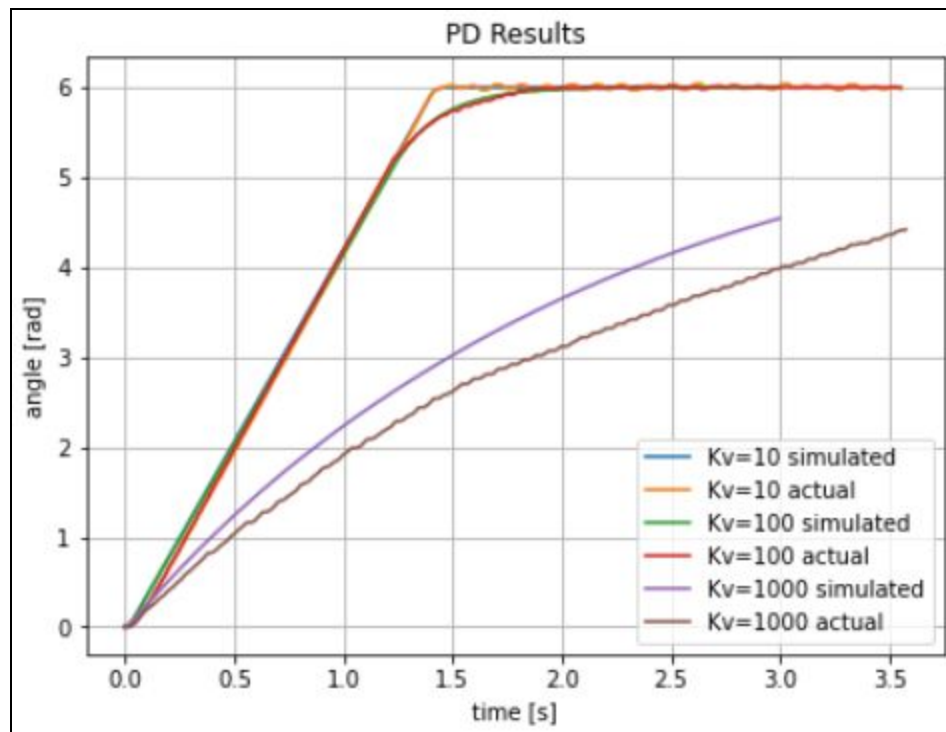
The joint space controller simulation closely matched the experimental results. That being said a slight DC gain of 4.5 had to be applied to the controller signal in order to overcome the static friction in the system and get the motor to not stop short.



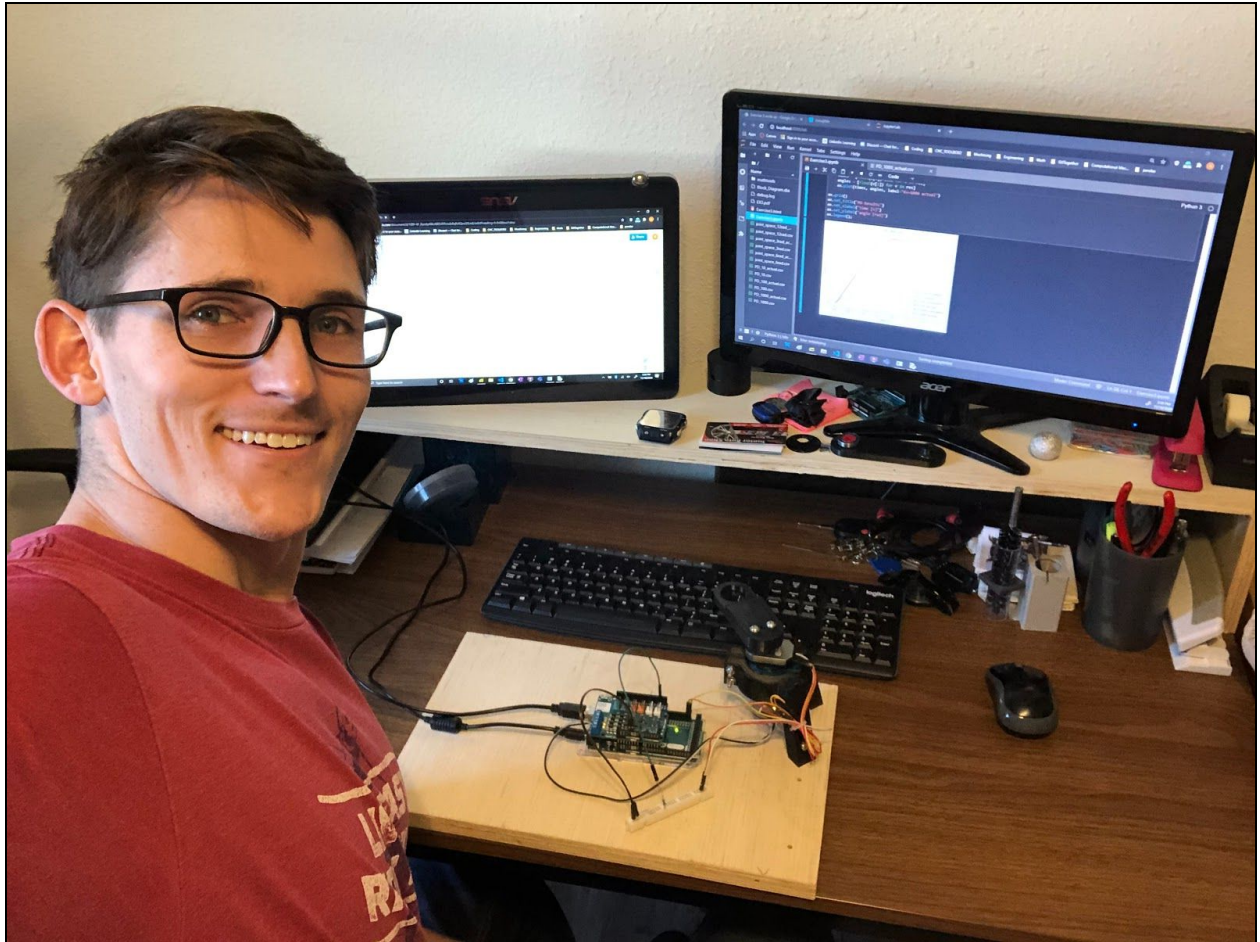


## PD Simulation & Experiment Results

The PD controller did not behave as simulated. Due to the massive increase in the strength of the signal, as  $C_1$  was a very small number and now was absent from the controller, the controller required a DC gain of 0.5 to function in the realm of normal. There was also substantial vibration. A PID controller is recommended as it would allow for lower  $K_p$  and  $K_v$  values, while not undershooting the target.



Team Photo (note, only one person due to COVID)



# References

[gears] <https://www.wikihow.com/Determine-Gear-Ratio>



