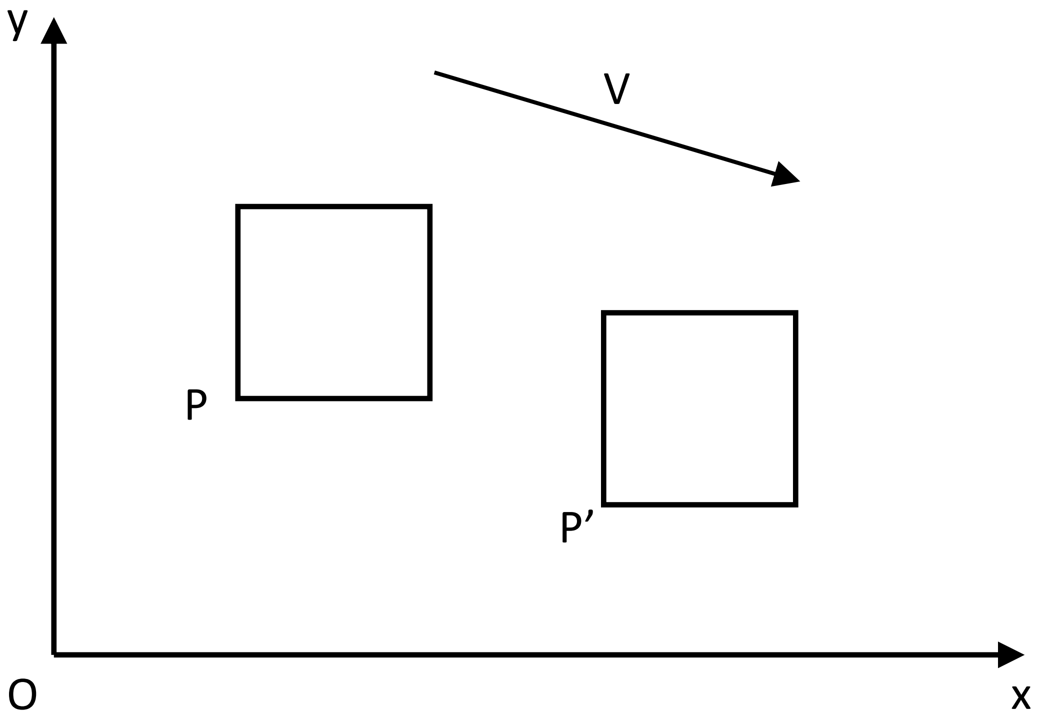
102architect

B-MAT-100

Geometric transformations

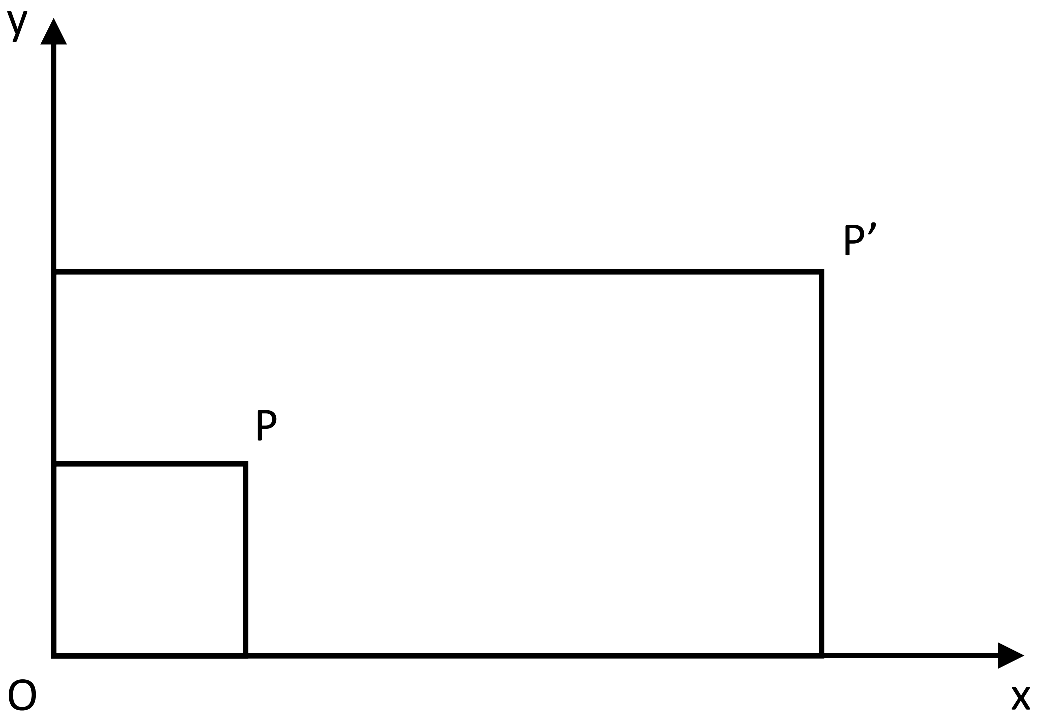
Translation



Translation along vector $V(i, j)$

$$P'(x) = P(x) + i$$
$$P'(y) = P(y) + j$$

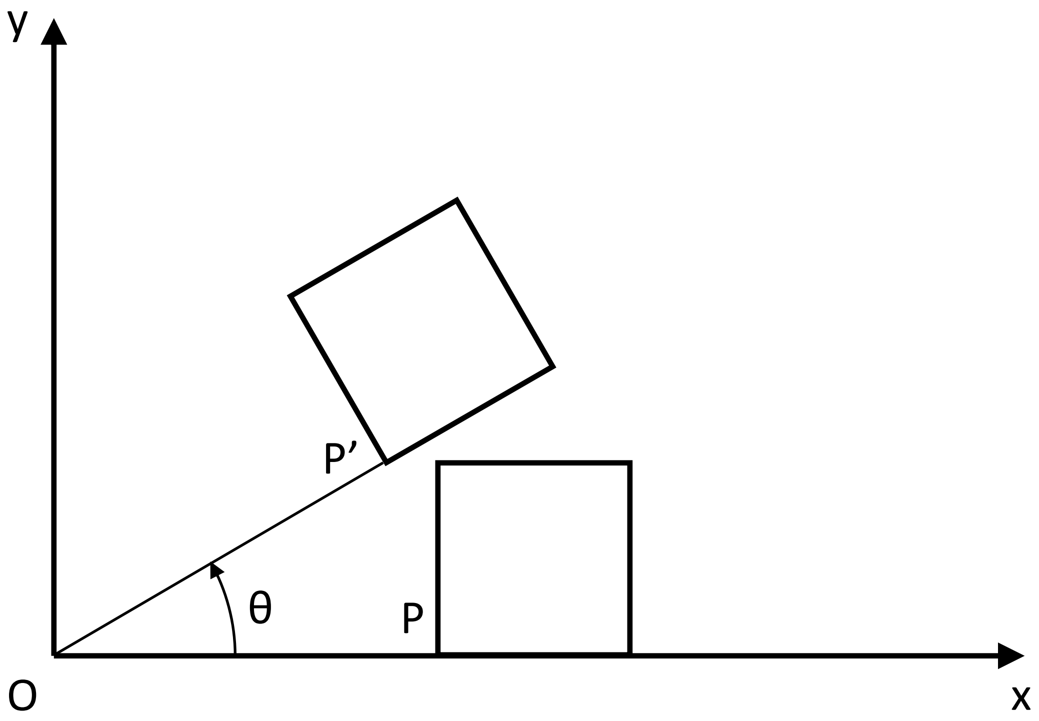
Scaling



Scaling by factors m and n

$$P'(x) = m \times P(x)$$
$$P'(y) = n \times P(y)$$

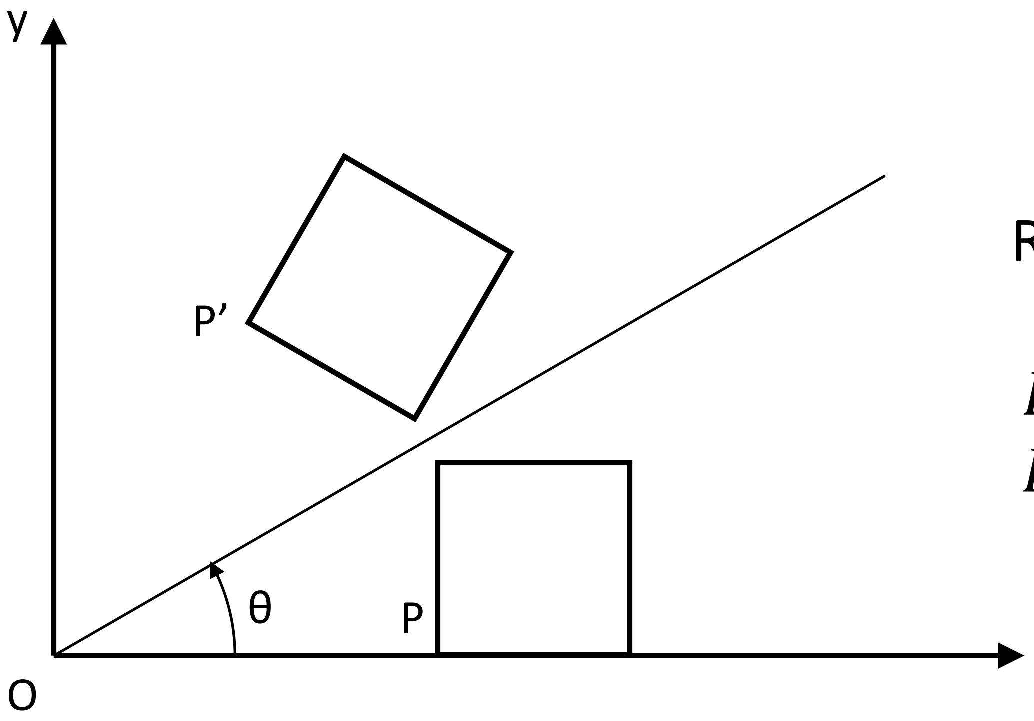
Rotation



Rotation centered in O

$$P'(x) = \cos(\theta) \times P(x) - \sin(\theta) \times P(y)$$
$$P'(y) = \sin(\theta) \times P(x) + \cos(\theta) \times P(y)$$

Reflection



Reflection over an axis

$$P'(x) = \cos(2\theta) \times P(x) + \sin(2\theta) \times P(y)$$
$$P'(y) = \sin(2\theta) \times P(x) - \cos(2\theta) \times P(y)$$

Matrices

- 2-dimensional array of values

$$\begin{pmatrix} 42 & 12 & 34 & 2 \\ 36 & 2 & 1 & 15 \\ 3 & 23 & 17 & 9 \end{pmatrix}$$

- Can be represented in 2D $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$, 3D $\begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}$ and much more…

Matrix addition

- They MUST have same dimensions

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 5 \\ 6 & 2 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 9 & 6 \\ 3 & 8 \end{pmatrix}$$

- The resulting matrix has the same dimensions

Multiplication of 2 matrices

- Two matrices can be multiplied if:
  - Numbers of columns of the 1st one = number of rows of the 2nd one

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 5 & 4 & 8 \\ 6 & 2 & 7 & 6 \\ 2 & 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 23 & 21 & 30 & 39 \\ 18 & 18 & 27 & 31 \end{pmatrix}$$

- The resulting matrix has the number of rows of the 1st one and the number of columns of the 2nd one
- The multiplication is NOT commutative ! $A \times B \neq B \times A$

Multiplication of 2 matrices

$$\begin{pmatrix} \boxed{\begin{matrix} 3 \\ 6 \\ 2 \end{matrix}} & \begin{matrix} 5 \\ 2 \\ 2 \end{matrix} & \begin{matrix} 4 \\ 7 \\ 4 \end{matrix} & \begin{matrix} 8 \\ 6 \\ 3 \end{matrix} \end{pmatrix}$$

$$\begin{pmatrix} \boxed{3 \quad 2 \quad 1} \\ 2 \quad 1 \quad 3 \end{pmatrix} \Rightarrow \begin{pmatrix} \boxed{23} & 21 & 30 & 39 \\ 18 & 18 & 27 & 31 \end{pmatrix}$$

$$(3 * 3) + (2 * 6) + (1 * 2) = 9 + 12 + 2 = 23$$

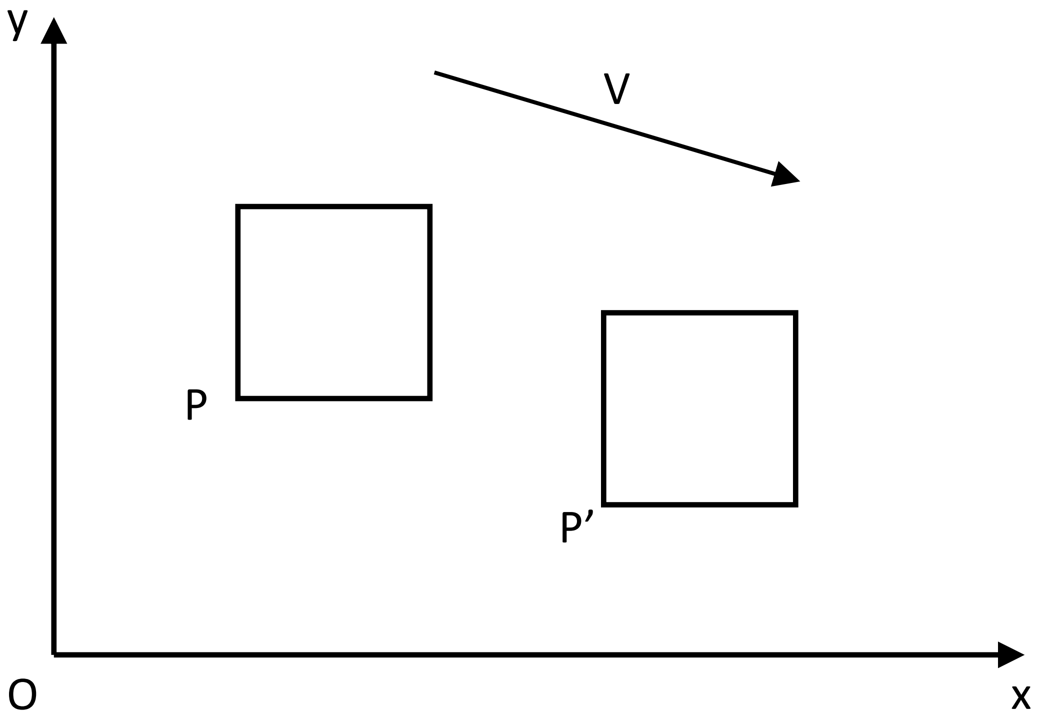
Homogeneous coordinates

- In graphics programming, we use homogeneous coordinates to easily compute transformations.
- A point (x, y) is represented by a 3x1 matrix with a 3rd « coordinate » equal to 1 :

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- We can now use matrix multiplication to compute transformations!

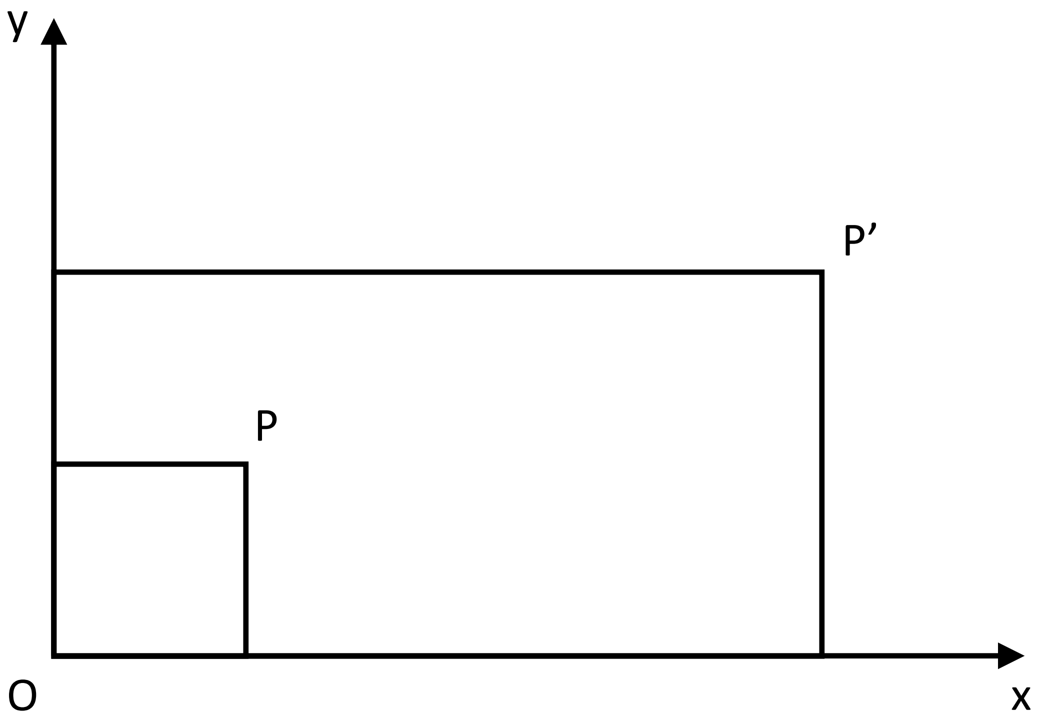
Translation



Translation along vector $V(i, j)$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & j \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

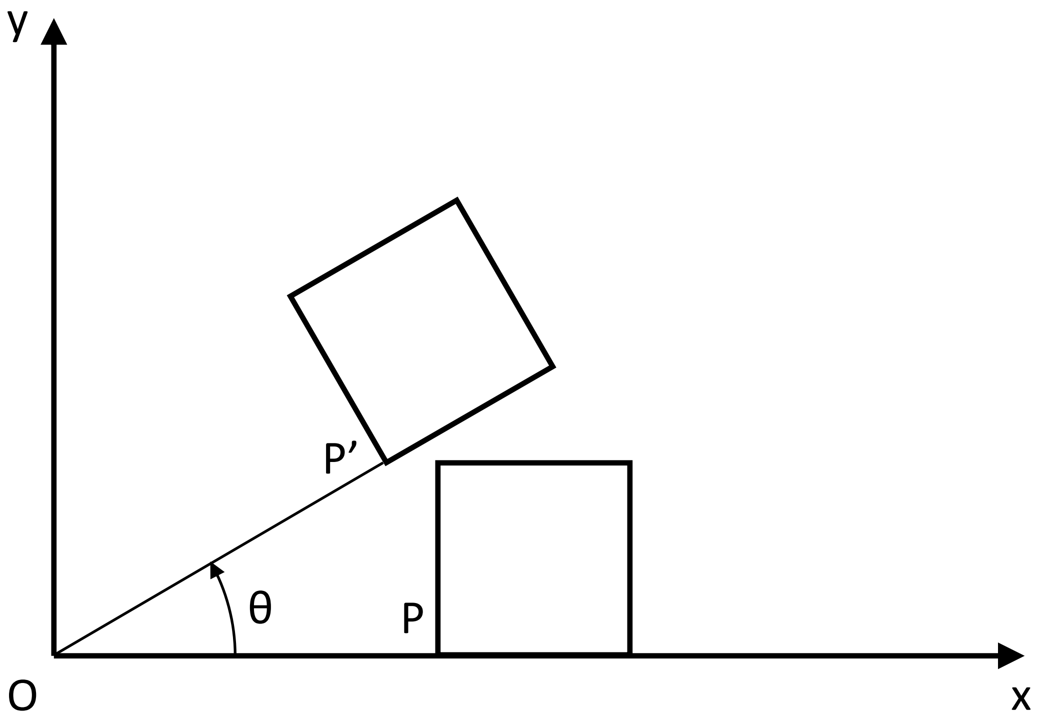
Scaling



Scaling by factors m and n

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} m & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

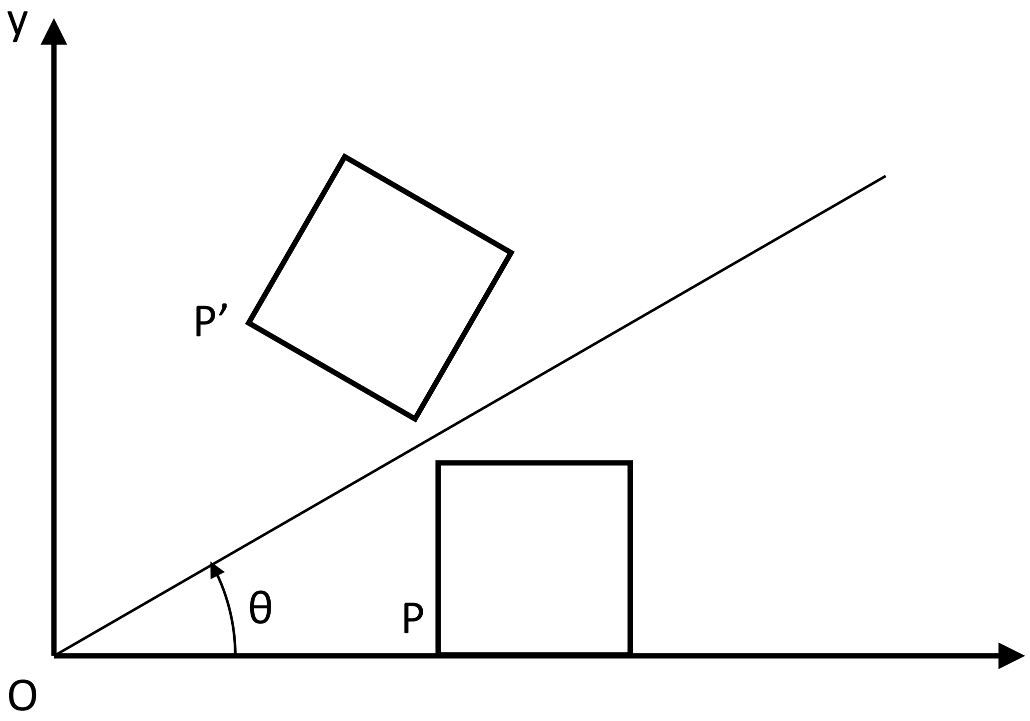
Rotation



Rotation centered in O

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Reflection



Reflection over an axis

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Composed transformations

- T is a translation matrix, and R a rotation matrix
$$P' = T \times P$$
$$P'' = R \times P' = R \times (T \times P) = (R \times T) \times P$$
- $M = R \times T$ is the transformation matrix of a translation followed by a rotation
- This works for as many consecutive transformations as you want!

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- Inputs
  - x, y: original coordinates
  - At least one transformation
- Outputs
  - The matrix of the (composed) transformation
  - The resulting transformed coordinates
- Using a matrix calculus library is considered as cheating

Boni ideas

- Graphical interface showing transformation of several points/figures
- Additional transformations
  - Shear mapping
  - Projection
  - …
- 3D