

# Robust Learning Rate Selection for Stochastic Optimization via Splitting Diagnostic

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## Abstract

This paper proposes SplitSGD, a new stochastic optimization algorithm with a dynamic learning rate selection rule. This procedure decreases the learning rate for better adaptation to the local geometry of the objective function whenever a *stationary* phase is detected, that is, the iterates are likely to bounce around a vicinity of a local minimum. The detection is performed by splitting the single thread into two and using the inner products of the gradients from the two threads as a measure of stationarity. This learning rate selection is provably valid, *robust* to initial parameters, easy-to-implement, and essentially does not incur additional computational cost. Finally, we illustrate the robust convergence properties of SplitSGD through extensive experiments.

## 1 Introduction

Many machine learning problems boil down to finding a minimizer  $\theta^* \in \mathbb{R}^d$  of a risk function taking the form

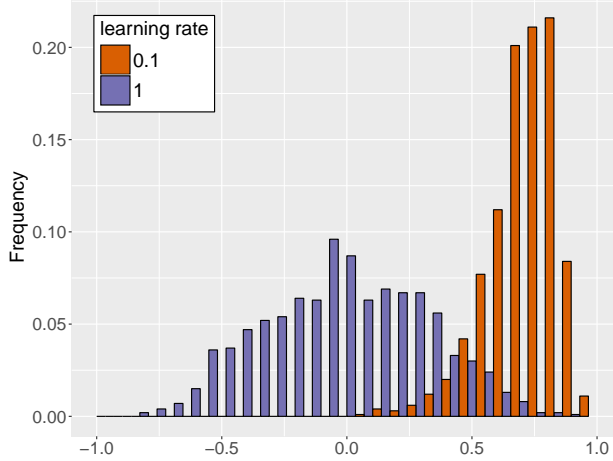
$$F(\theta) = \mathbb{E}[f(\theta, Z)], \quad (1.1)$$

where  $f$  denotes a loss function,  $\theta$  is the model parameter, and the *random* data point  $Z = (X, y)$  contains a feature vector  $X$  and its label  $y$ . In the case of a finite population, for example, this problem is reduced to the empirical minimization problem. The touchstone method for minimizing (1.1) is stochastic gradient descent (SGD). Starting from an initial point  $\theta_0$ , SGD updates the iterates according to

$$\theta_{t+1} = \theta_t - \eta_t \cdot g(\theta_t, Z_{t+1}) \quad (1.2)$$

for  $t \geq 0$ , where  $\eta_t$  is the learning rate,  $\{Z_t\}_{t=1}^\infty$  are i.i.d. copies of  $Z$  and  $g(\theta, Z)$  is the (sub-) gradient of  $f(\theta, Z)$  with respect to  $\theta$ . The noisy gradient  $g(\theta, Z)$  is an unbiased estimate for the true gradient  $\nabla F(\theta)$  in the sense that  $\mathbb{E}[g(\theta, Z)] = \nabla F(\theta)$  for any  $\theta$ .

The convergence rate of SGD crucially depends on the *learning rate*—often recognized as “the single most important hyper-parameter” in training deep neural networks (Bengio, 2012)—and, accordingly, there is a vast literature on how to *decrease* this fundamental tuning parameter for improved convergence performance. In the pioneering work of Robbins and Monro (1951), the learning rate  $\eta_t$  is set to  $O(1/t)$  for convex objectives. Later, it was recognized that a slowly decreasing learning rate in conjunction with iterate averaging leads to a faster rate of convergence for



**Figure 1:** Normalized dot product of averaged noisy gradients over 100 iterations. Stationarity depends on the learning rate:  $\eta = 1$  corresponds to stationarity, while  $\eta = 0.1$  corresponds to non stationarity. Details in Section 2.

strongly convex and smooth objectives (Ruppert, 1988; Polyak and Juditsky, 1992). More recently, extensive effort has been devoted to incorporating preconditioning into learning rate selection rules (Duchi et al., 2011; Dauphin et al., 2015; Tan et al., 2016). Among numerous proposals, a simple yet widely employed approach is to repeatedly halve the learning rate after performing a *pre-determined* number of iterations (see, for example, Bottou et al., 2018).

In this paper, we introduce a new variant of SGD that we term *SplitSGD* with a novel learning rate selection rule. At a high level, our new method is motivated by the following fact: an optimal learning rate should be adaptive to the *informativeness* of the noisy gradient  $g(\theta_t, Z_{t+1})$ . Roughly speaking, the informativeness is higher if the true gradient  $\nabla F(\theta_t)$  is relatively large compared with the noise  $\nabla F(\theta_t) - g(\theta_t, Z_{t+1})$  and vice versa. On the one hand, if the learning rate is too small with respect to the informativeness of the noisy gradient, SGD makes rather slow progress. On the other hand, if the learning rate is too large with respect to the informativeness, the iterates would bounce around a region of an optimum of the objective. The latter case corresponds to a stationary phase in stochastic optimization (Murata, 1998; Chee and Toulis, 2018), which necessitates the *reduction* of the learning rate for better convergence.

SplitSGD differs from other stochastic optimization procedures in its *robust* stationarity phase detection, which we refer to as the *Splitting Diagnostic*. In short, this diagnostic runs two SGD threads initialized at the same iterate using *independent* data points (refers to  $Z_{t+1}$  in (1.2)), and then performs hypothesis testing to determine whether or not the learning rate leads to a stationary phase. The effectiveness of the Splitting Diagnostic is illustrated in Figure 1, which reveals different patterns of dependence between the two SGD threads with difference learning rates. Loosely speaking, in the stationary phase (in purple), the two SGD threads behave as if they are independent due to a large learning rate, and SplitSGD subsequently decreases the learning rate by some factor. In contrast, strong positive dependence is exhibited in the non-stationary phase (in orange) and, thus, the learning rate remains the same after the diagnostic. In essence, the robustness of the Splitting Diagnostic is attributed to its adaptivity to the *local geometry* of the objective, thereby making SplitSGD a *tuning-insensitive* method for stochastic optimization.



**Figure 2:** The architecture of SplitSGD. The initial learning rate is  $\eta$  and the length of the first single thread is  $t_1$ . If the diagnostic does not detect stationarity, the length and learning rate of the next thread remain unchanged. If stationarity is observed, we decrease the learning rate by a factor  $\gamma$  and proportionally increase the length.

## 1.1 Related work

There is a long history of detecting stationarity or non-stationarity in stochastic optimization to improve convergence rates (Yin, 1989; Pflug, 1990; Delyon and Juditsky, 1993; Murata, 1998; Le Roux et al., 2013). Perhaps the most relevant work in this vein to the present paper is Chee and Toulis (2018), which builds on Pflug (1990) for general convex functions. Specifically, this work uses the running sum of the inner products of successive stochastic gradients for stationarity detection. However, this approach does not take into account the strong correlation between consecutive gradients and, moreover, is not sensitive to the local curvature of the current iterates due to unwanted influence from prior gradients. In contrast, the splitting strategy, which is akin to HiGrad (Su and Zhu, 2018), allows our SplitSGD to concentrate on the current gradients and leverage the regained independence of gradients to test for stationarity. Recently, Yaida (2019) and Lang et al. (2019) derived a stationarity detection rule that is based on gradients of a mini-batch to tune the learning rate in SGD with momentum. To develop theoretical guarantees for SplitSGD with momentum is an interesting direction for future work, but is not the topic of the current paper.

From a different angle, another related line of work is concerned with the relationship between the informativeness of gradients and the mini-batch size (Keskar et al., 2016; Yin et al., 2017; Li et al., 2017; Smith et al., 2017). Among others, it has been recognized that the optimal mini-batch size should be adaptive to the local geometry of the objective function and the noise level of the gradients, delivering a growing line of work that leverages the mini-batch gradient variance for learning rate selection (Byrd et al., 2012; Balles et al., 2016; Balles and Hennig, 2017; De et al., 2017; Zhang and Mitliagkas, 2017; McCandlish et al., 2018).

## 2 The SplitSGD algorithm

In this section, we first develop the Splitting Diagnostic for stationarity detection, followed by the introduction of the SplitSGD algorithm in detail.

### 2.1 Diagnostic via splitting

Intuitively, the stationarity phase occurs when two independent threads with the same starting point are *no longer* moving along the same direction. This intuition is the motivation for our Splitting Diagnostic, which is presented in Algorithm 1 and described in what follows. We call  $\theta_0$  the initial value, even though later it will often have a different subscript based on the number of iterations already computed before starting the diagnostic. From the starting point, we run two SGD threads, each consisting of  $w$  windows of length  $l$ . For each thread  $k = 1, 2$ , we define  $g_t^{(k)} = g(\theta_t^{(k)}, Z_{t+1}^{(k)})$

and the iterates are

$$\theta_{t+1}^{(k)} = \theta_t^{(k)} - \eta \cdot g_t^{(k)}, \quad (2.1)$$

where  $t \in \{0, \dots, wl - 1\}$ . On every thread we compute the average noisy gradient in each window, indexed by  $i = 1, \dots, w$ , which is

$$\bar{g}_i^{(k)} := \frac{1}{l} \sum_{j=1}^l g_{(i-1) \cdot l + j}^{(k)} = \frac{\theta_{(i-1) \cdot l + 1}^{(k)} - \theta_{i \cdot l}^{(k)}}{l \cdot \eta}. \quad (2.2)$$

The length  $l$  of each window has the same function as the mini-batch parameter in mini-batch SGD (Li et al., 2014), in the sense that a larger value of  $l$  aims to capture more of the true signal by averaging out the errors. At the end of the diagnostic, we have stored two vectors, each containing the average noisy gradients in the windows in each thread.

**Definition 2.1.** For  $i = 1, \dots, w$ , we define the gradient coherence with respect to the starting point of the Splitting Diagnostic  $\theta_0$ , the learning rate  $\eta$ , and the length of each window  $l$ , as

$$Q_i(\theta_0, \eta, l) = \langle \bar{g}_i^{(1)}, \bar{g}_i^{(2)} \rangle. \quad (2.3)$$

We will drop the dependence from the parameters and refer to it simply as  $Q_i$ .

The gradient coherence expresses the relative position of the average noisy gradients, and its sign indicates whether the SGD updates have reached stationarity. In fact, if in the two threads the noisy gradients are pointing on average in the same direction, it means that the signal is stronger than the noise, and the dynamic is still in its transient phase. In contrast, when the gradient coherence is on average very close to zero, and it also assumes negative values thanks to its stochasticity, this indicates that the noise component in the gradient is now dominant, and stationarity has been reached. Of course these values, no matter how large  $l$  is, are subject to some randomness. Our diagnostic then considers the signs of  $Q_1, \dots, Q_w$  and returns a result based on the number of negative  $Q_i$ . One output is a boolean value  $T_D$ , defined as follows:

$$T_D = \begin{cases} S & \text{if } \sum_{i=1}^w (1 - \text{sign}(Q_i))/2 \geq q \\ N & \text{if } \sum_{i=1}^w (1 - \text{sign}(Q_i))/2 < q. \end{cases} \quad (2.4)$$

where  $T_D = S$  indicates that stationarity has been detected. The parameter  $q$  controls the tightness of this guarantee, being the smallest number of negative  $Q_i$  required to declare stationarity. In addition to  $T_D$ , we also return the average last iterate of the two threads as a starting point for the following iterations. We call it  $\theta_D := (\theta_{w \cdot l}^{(1)} + \theta_{w \cdot l}^{(2)})/2$ .

## 2.2 The algorithm

The Splitting Diagnostic can be employed in a more sophisticated SGD procedure, which we call SplitSGD. We start by running the standard SGD with constant learning rate  $\eta$  for  $t_1$  iterations. Then, starting from  $\theta_{t_1}$ , we use the Splitting Diagnostic to verify if stationarity has been reached. If stationarity is not detected, the next single thread has the same length  $t_1$  and learning rate  $\eta$  as the previous one. In contrast, if  $T_D = S$ , we decrease the learning rate by a factor  $\gamma \in (0, 1)$  and increase the length of the thread by  $1/\gamma$ , as suggested by Bottou et al. (2018) in their SGD<sup>1/2</sup>

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**Algorithm 1** SplittingDiagnostic( $\eta, w, l, q, \theta^{in}$ )

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```
1:  $\theta_0^{(1)} = \theta_0^{(2)} = \theta^{in}$ 
2: for  $i = 1, \dots, w$  do
3:   for  $k = 1, 2$  do
4:     for  $j = 0, \dots, l - 1$  do
5:        $\theta_{(i-1) \cdot l + j + 1}^{(k)} = \theta_{(i-1) \cdot l + j}^{(k)} - \eta \cdot g_{(i-1) \cdot l + j}^{(k)}$ 
6:     end for
7:      $\bar{g}_i^{(k)} = \frac{\theta_{(i-1) \cdot l + 1}^{(k)} - \theta_{i \cdot l}^{(k)}}{l \cdot \eta}$ 
8:   end for
9:    $Q_i = \langle \bar{g}_i^{(1)}, \bar{g}_i^{(2)} \rangle$ 
10: end for
11: if  $\sum_{i=1}^w (1 - \text{sign}(Q_i))/2 \geq q$  then
12:   return  $\left\{ \theta_D = (\theta_{w \cdot l}^{(1)} + \theta_{w \cdot l}^{(2)})/2, T_D = S \right\}$ 
13: else
14:   return  $\left\{ \theta_D = (\theta_{w \cdot l}^{(1)} + \theta_{w \cdot l}^{(2)})/2, T_D = N \right\}$ 
15: end if
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procedure. Figure 2 illustrates what happens when the first diagnostic does not detect stationarity, but the second one does. SplitSGD puts together two crucial aspects: it employs the Splitting Diagnostic at deterministic times, but it does not deterministically decrease the learning rate. We will see in Section 5 how both of these features come into play in the comparison with other existing methods. A detailed explanation of SplitSGD is presented in Algorithm 2.

### 3 Theoretical guarantees for stationarity detection

This section develops theoretical guarantees for the validity of our learning rate selection. Specifically, in the case of a relatively small learning rate, we can imagine that, if the number of iterations is fixed, the SGD updates are not too far from the starting point, so the stationary phase has not been reached yet. On the other hand, when  $t \rightarrow \infty$  and the learning rate is fixed, we would like the diagnostic to tell us that we have reached stationarity, since we know that in this case the updates will oscillate around  $\theta^*$ . Our first assumption concerns the convexity of the function  $F(\theta)$ . It will not be used in Theorem 1, in which we focus our attention on a neighborhood of  $\theta_0$ .

**Assumption 3.1.** *The function  $F$  is strongly convex, with convexity constant  $\mu > 0$ . Thus, for all  $\theta_1, \theta_2$ ,*

$$F(\theta_1) \geq F(\theta_2) + \langle \nabla F(\theta_2), \theta_1 - \theta_2 \rangle + \frac{\mu}{2} \|\theta_1 - \theta_2\|^2$$

*and also  $\|\nabla F(\theta_1) - \nabla F(\theta_2)\| \geq \mu \cdot \|\theta_1 - \theta_2\|$ .*

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**Algorithm 2** SplitSGD( $\eta, w, l, q, B, t_1, \theta_0, \gamma$ )

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```
1:  $\eta_1 = \eta$ 
2:  $\theta_1^{in} = \theta_0$ 
3: for  $b = 1, \dots, B$  do
4:   Run SGD with constant step size  $\eta_b$  for  $t_b$  steps, starting from  $\theta_b^{in}$ 
5:   Let the last update be  $\theta_b^{last}$ 
6:    $D_b = \text{SplittingDiagnostic}(\eta_b, w, l, q, \theta_b^{last})$ 
7:    $\theta_{b+1}^{in} = \theta_{D_b}$ 
8:   if  $T_{D_b} = S$  then
9:      $\eta_{b+1} = \gamma \cdot \eta_b$  and  $t_{b+1} = \lfloor t_b / \gamma \rfloor$ 
10:  else
11:     $\eta_{b+1} = \eta_b$  and  $t_{b+1} = t_b$ 
12:  end if
13: end for
```

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**Assumption 3.2.** The function  $F$  is smooth, with smoothness parameter  $L > 0$ . That is, for all  $\theta_1, \theta_2$ ,

$$\|\nabla F(\theta_1) - \nabla F(\theta_2)\| \leq L \cdot \|\theta_1 - \theta_2\|.$$

We said before that the noisy gradient is an unbiased estimate of the true gradient. The next assumption that we make is on the distribution of the errors.

**Assumption 3.3.** We define the error in the evaluation of the gradient in  $\theta_{t-1}$  as

$$\epsilon_t := \epsilon(\theta_{t-1}, Z_t) = g(\theta_{t-1}, Z_t) - \nabla F(\theta_{t-1}) \quad (3.1)$$

and the filtration  $\mathcal{F}_t = \sigma(Z_1, \dots, Z_t)$ . Then  $\epsilon_t \in \mathcal{F}_t$  and  $\{\epsilon_t\}_{t=1}^\infty$  is a martingale difference sequence with respect to  $\{\mathcal{F}_t\}_{t=1}^\infty$ , which means that  $\mathbb{E}[\epsilon_t | \mathcal{F}_{t-1}] = 0$ . The covariance of the errors satisfies

$$\sigma_{\min} \cdot I \preceq \mathbb{E}[\epsilon_t \epsilon_t^T | \mathcal{F}_{t-1}] \preceq \sigma_{\max} \cdot I, \quad (3.2)$$

where  $0 < \sigma_{\min} \leq \sigma_{\max} < \infty$  for any  $\theta$ .

Our last assumption is on the noisy functions  $f(\theta, Z)$  and on an upper bound on the moments of their gradient. We do not specify  $m$  here since different values are used in the two theorems.

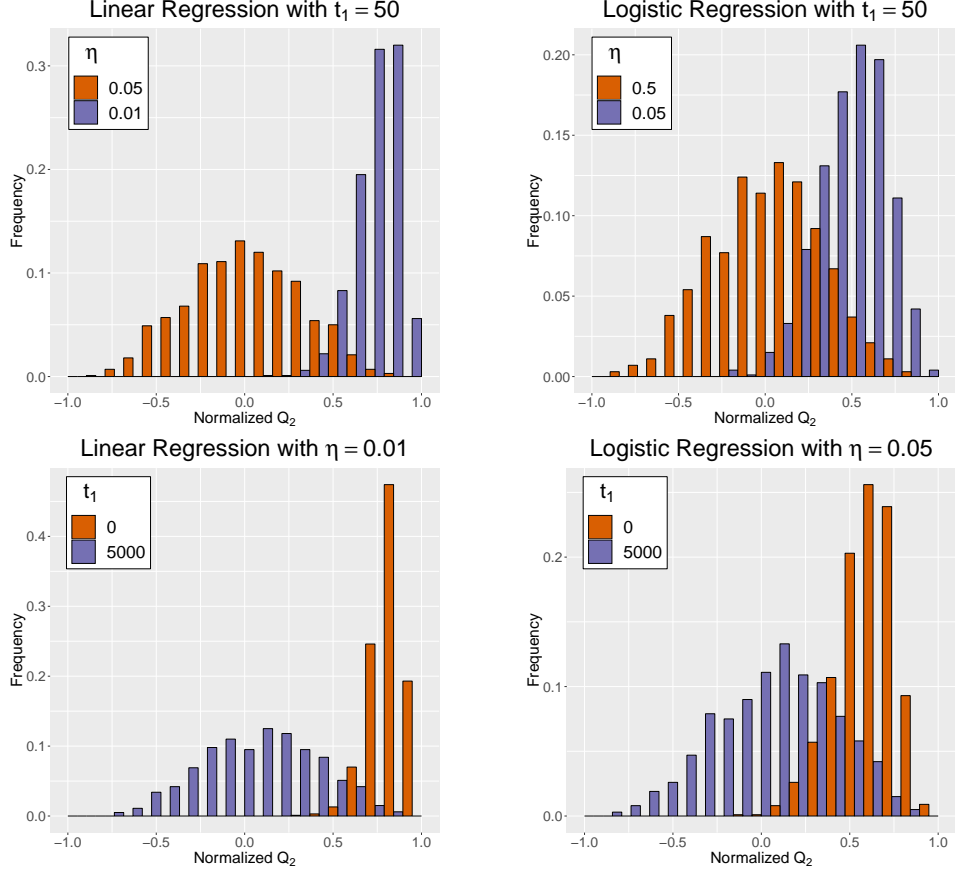
**Assumption 3.4.** Each function  $f(\theta, Z)$  is convex, and there exists a constant  $G$  such that  $\mathbb{E}[\|g(\theta_t, Z_{t+1})\|^m | \mathcal{F}_t] \leq G^m$  for any  $\theta_t$ .

In the following theorem, we show that there exists a learning rate sufficiently small such that the standard deviation of any gradient coherence  $Q_i$  is arbitrarily small compared to its expectation.

**Theorem 1.** If Assumptions 3.2, 3.3, and 3.4 with  $m = 4$  hold, and we run  $t_1$  iterations before the Splitting Diagnostic, then for any  $i \in \{1, \dots, w\}$  we can set  $\eta$  small enough to guarantee that

$$\text{sd}(Q_i) \leq C_1(\eta, l) \cdot \mathbb{E}[Q_i],$$

where  $C_1(\eta, l) = O(1/\sqrt{l}) + O(\sqrt{\eta(t_1 + l)})$ .



**Figure 3:** Histogram of the gradient coherence  $Q_2$  (normalized) of the Splitting Diagnostic for linear and logistic regression. The two top panels show the behavior described in Theorem 1, the two bottom panels the one in Theorem 2. We used  $\theta_0 = (0, \dots, 0)$ , and the settings are as described in Sections 4 and 5.

The intuition behind this proof is that, when  $\eta$  is small,  $\nabla F(\theta_t) = \nabla F(\theta_0) + O(\eta t)$ , for any SGD iterate  $\theta_t$ , thanks to the smoothness of  $F$ . If we assume that  $\eta(t_1 + l)$  is small, we can prove that  $\mathbb{E}[Q_i] \geq \|\nabla F(\theta_0)\|^2 + O(\eta(t_1 + l))$  and  $\text{sd}(Q_i) \lesssim O(1/\sqrt{l}) + O(\sqrt{\eta(t_1 + l)})$ . For an appropriately large  $l$  and sufficiently small  $\eta$ , the output of the splitting diagnostic is  $T_D = N$ , because all the mass of the distribution of  $Q_i$  is concentrated on positive values. This behavior is shown in the two top panels of Figure 3. Note that to obtain this result we do not need to use the strong convexity Assumption 3.1 since, when  $\eta(t_1 + l)$  is small,  $\theta_{t_1+l}$  is not very far from  $\theta_0$ . In the next Theorem we show that, if we let the SGD thread before the diagnostic run for long enough and the learning rate is not too big, then the splitting diagnostic output is  $T_D = S$  with high probability. This is consistent with the fact that, as  $t_1 \rightarrow \infty$ , the iterates will start oscillating in a neighborhood of  $\theta^*$ .

**Theorem 2.** *If Assumptions 3.1, 3.2, 3.3, and 3.4 with  $m = 2$  hold, then for any  $\eta \leq \frac{\mu}{L^2}$ ,  $l \in \mathcal{N}$  and  $i \in \{1, \dots, w\}$ , as  $t_1 \rightarrow \infty$  we have*

$$|\mathbb{E}[Q_i]| \leq C_2(\eta) \cdot \text{sd}(Q_i),$$

where  $C_2(\eta) = C_2 \cdot \eta + o(\eta)$ .

The result of this theorem is confirmed by what we see in the bottom panels of Figure 3. There, most of the mass of  $Q_2$  is on positive values if  $t_1 = 0$ , since the learning rate is sufficiently small. But when we let the first thread run for longer, we see that the distribution of  $Q_2$  is now centered around zero, with an expectation that is much smaller than its standard deviation. An appropriate choice of  $w$  and  $q$  makes the probability that  $T_D = S$  arbitrarily big. In the proof of Theorem 2, we make use of a lemma that is contained in Moulines and Bach (2011) and then subsequently improved in Needell et al. (2014).

**Lemma 3.5.** *If Assumptions 3.1, 3.2, 3.3, and 3.4 with  $m = 2$  hold, and  $\eta \leq \frac{\mu}{L^2}$ , then for any  $t \geq 0$*

$$\mathbb{E} [\|\theta_t - \theta^*\|^2] \leq (1 - 2\eta(\mu - L^2\eta))^t \cdot \mathbb{E} [\|\theta_0 - \theta^*\|^2] + \frac{G^2\eta}{\mu - L^2\eta}.$$

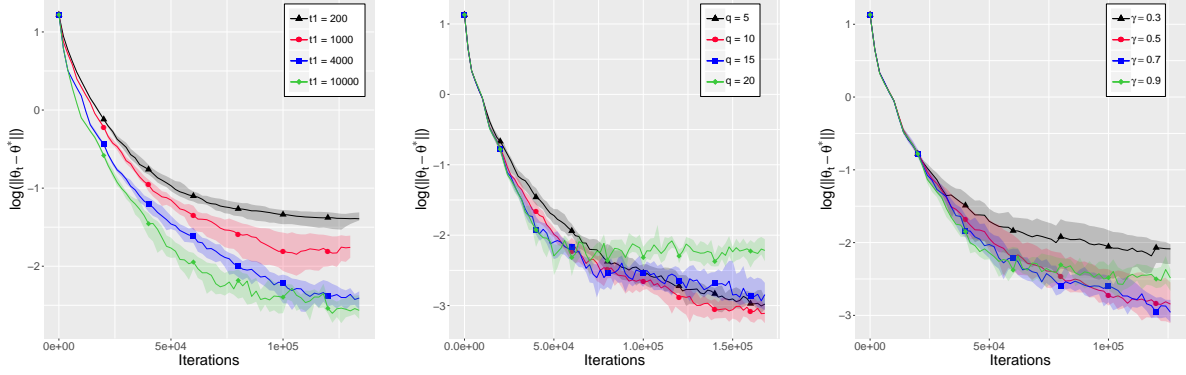
This lemma represents the dynamic of SGD with constant learning rate, where the dependence from the starting point vanishes exponentially fast, but there is a term dependent on  $\eta$  that is not vanishing even for large  $t$ . The simulations in Figure 3 show us that, once stationarity is reached, the distribution of the gradient coherence is fairly symmetric and centered around zero, so its sign will be approximately a coin flip. In this situation, if  $l$  is large enough, the count of negative gradient coherences is approximately distributed as a Binomial with  $w$  number of trials, and 0.5 probability of success. Then we can set  $q$  to control the probability of making a type I error – rejecting stationarity after it has been reached – by making  $\frac{1}{2^w} \sum_{i=0}^{q-1} \binom{w}{i}$  sufficiently small. Notice that a very small value for  $q$  makes the type I error rate decrease but makes it easier to think that stationarity has been reached too early. In the appendix we provide a simple visual interpretation to understand why this trade-off gets weaker as  $w$  becomes larger. We discuss the sensitivity to the parameters for SplitSGD in the next section.

## 4 Sensitivity analysis

For most of the parameters involved – the number of windows used  $w$ , their length  $l$ , the initial length of the single thread  $t_1$ , and the number of diagnostics used  $K$  – no tuning is required. We would like them to be as large as possible, with the only trade-off being the total computational cost of the procedure. We perform a sensitivity analysis for some relevant parameters on logistic regression, using as a baseline  $w = 30, l = 100, t_1 = 4000, q = 14$ , and  $\gamma = 0.7$  (the values that we will also use in Sections 5.1 and 5.2). In the literature, a common choice for  $\gamma$  is  $1/2$ , as it is done in Chee and Toulis (2018) and Bottou et al. (2018), but we decided differently since smaller values can become a problem if stationarity is erroneously detected too early. In this way, the learning rate gets approximately halved every two reductions. In Figure 4 we generate the data using a feature matrix  $X$  with  $n = 1000$  observations and  $d = 10$  features, each one being an independent standard normal. The response  $y_i$  is a Bernoulli random variable generated with probability  $[1 + \exp(\sum_j X_{ij})]^{-1}$ .

In the left plot of Figure 4, we notice that, once the length of the diagnostic is fixed, it is convenient to set  $t_1$  to be at least as long as each of the diagnostic threads. Very large values help speed up the initial convergence but increase the risk of not being able to promptly detect the stationary phase once it is reached. In the middle plot we see how sensitive the performance of SplitSGD is with respect to  $q$ . It was easy to imagine that if  $q$  is much larger than  $w/2$  then





**Figure 4:** Sensitivity analysis for the parameters  $t_1$  (left),  $q$  (center) and  $\gamma$  (right) in SplitSGD applied to logistic regression. The baseline is  $w = 30, l = 100, t_1 = 4000, q = 14$ , and  $\gamma = 0.7$ . Each experiment is repeated 20 times. The solid line is the average log distance from the minimizer, together with 50% confidence bands. All the logarithms here and in the next plots are in the natural base.

stationarity is never detected and the learning rate stays constant. However, it is hard to decide a priori what a good value can be. It is then reassuring to see that SplitSGD is not very sensitive to this parameter once it is set in a reasonable range. Finally, as expected, we got evidence that it is better to set  $\gamma$  around or slightly larger than 0.5, since small values can cause a too aggressive decay schedule for the learning rate, while values close to 1 make the learning rate to stay nearly constant.

## 5 Experiments

We now compare SplitSGD with other optimization techniques. In Section 5.1, we contrast the accuracy in detecting stationarity of our Splitting Diagnostic with the **pflug** Diagnostic introduced in Chee and Toulis (2018). Then, in Section 5.2, we highlight the robustness of SplitSGD against some classic non-adaptive SGD methods in convex settings. Finally, in Section 5.3, we consider an image classification task and show that the SplitSGD adaptive schedule for decreasing the learning rate works effectively compared with popular adaptive procedures like AdaGrad (Duchi et al., 2011) and Adam (Kingma and Ba, 2014). Notice that, in all the experiments, we count the number of data points that are really used, which means that each thread in the diagnostic is considered separately and contributes for  $lw$  iterations. This is why in the bottom panels of Figure 5 we see that in its initial phase SplitSGD is slower than SGD with constant learning rate in approaching stationarity. However, if the two threads of the diagnostic are parallelized, the actual computational time is the same.

### 5.1 Comparison with pflug diagnostic

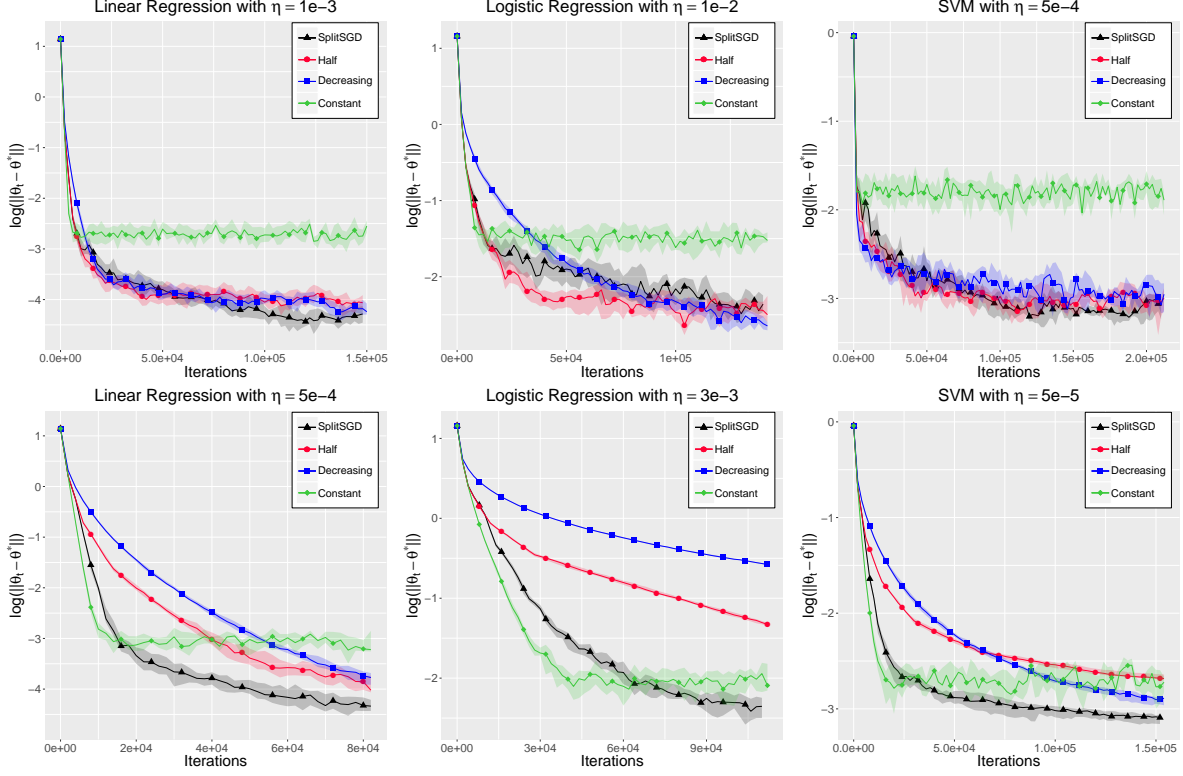
In Table 1 we compare the Splitting Diagnostic with the **pflug** Diagnostic proposed in Chee and Toulis (2018). The feature matrix is described in Section 4, and is of dimension  $1000 \times 10$ . For linear regression we set  $y_i = \sum_j X_{ij} + \epsilon_i$ , where  $\epsilon_i \sim N(0, 1)$ . For support vector machines (SVM) (Boser et al., 1992) we generate the data in the same way as for logistic regression. The starting point of the procedures is  $\theta_0 = (s + N(0, \sigma), \dots, s + N(0, \sigma))$ , where  $\sigma = 0.1$  and  $s \in \{0, 0.9, 1\}$  for linear and

**Table 1:** Comparison between Splitting and **pflug** Diagnostics. As soon as the starting point  $\theta_0 = (s + N(0, 0.1), \dots, s + N(0, 0.1))$  gets far from  $\theta^*$ , the **pflug** Diagnostic heavily overestimates the number of iterates necessary to reach stationarity. The Splitting Diagnostic instead tends to slightly underestimate it. Each entry has to be multiplied by 1000, and the parameter values for the Splitting Diagnostic are listed in Sections 4 and 5.

		Eyeballing			pflug Diagnostic			Splitting Diagnostic		
Linear	$\eta \backslash s$	1	0.9	0	1	0.9	0	1	0.9	0
	0.001	4.0	4.0	5.0	4.7	8.9	717.6	6.1	5.7	10.3
	0.0005	8.0	8.0	10.0	7.9	22.7	985.9	6.5	7.5	12.8
	0.0001	30.0	30.0	50.0	65.3	170.6	1000.0	14.6	20.3	47.1
Logistic	0.01	5.0	5.0	10.0	0.8	4.2	51.5	15.7	15.8	17.1
	0.005	10.0	15.0	30.0	2.6	3.3	146.2	14.2	15.9	21.7
	0.001	30.0	50.0	100.0	3.5	15.9	452.2	20.1	20.9	57.2
SVM	$\eta \backslash s$	0	0.1	1	0	0.1	1	0	0.1	1
	0.0001	10.0	10.0	20.0	165.9	134.2	201.0	9.5	9.0	16.1
	0.00005	15.0	15.0	30.0	304.0	182.9	345.6	13.0	12.0	27.7
	0.00001	100.0	100.0	150.0	915.1	766.7	920.0	46.3	39.0	109.0

logistic regression, and  $s \in \{0, 0.1, 1\}$  for SVM. We consider three choices for the learning rate  $\eta$  and repeat each experiment 100 times, reporting the average number of updates before detecting stationarity. In the left part of the table, we look at SGD trajectories with constant learning rate and provide an approximate value for when we can assume that stationarity has been reached, based on the distance between the iterates  $\theta_t$  and  $\theta^*$ . The value we report is the “elbow” that we see in Figure 5, evaluated for all of the combinations of  $s$  and  $\eta$ . When using the **pflug** Diagnostic we set the burn-in to be 100 and the maximum number of iterations 1,000,000. When the averages are close to the upper bound it means that most of the time the maximum number of iterations has been reached before convergence was detected. In the right part of the table, we show how the Splitting Diagnostic performs. For logistic regression, the parameters used are reported in Section 4. For linear regression and SVM we choose  $l = 50, t_1 = 2000$  and  $q = 12$ , since Table 1 confirms that, with this value of  $q$ , stationarity is detected in a number of iterates that is of the correct order of magnitude. In this experiment we do not need to set the parameter  $\gamma$ , since the learning rate is never decreased. Later we will maintain the standard choice of  $\gamma = 0.5$ , which is a good trade-off between making the learning rate smaller, which increases accuracy once stationarity is reached, and not making the procedure last too long when the single thread is repeatedly increased by a factor  $1/\gamma$ .

We declare that stationarity has been detected at the first time that a diagnostic gives result  $T_D = S$ , and we output the number of data points used up to that time. We can see that the result of the Splitting Diagnostic is always close to the truth in all the situations considered. But in SVM, and when the starting point is not very close to  $\theta^*$ , the **pflug** Diagnostic incurs the risk of running up to its maximum number of iterations, because the initial dot products of consecutive noisy gradients are positive and large compared to the negative increments when stationarity is reached.

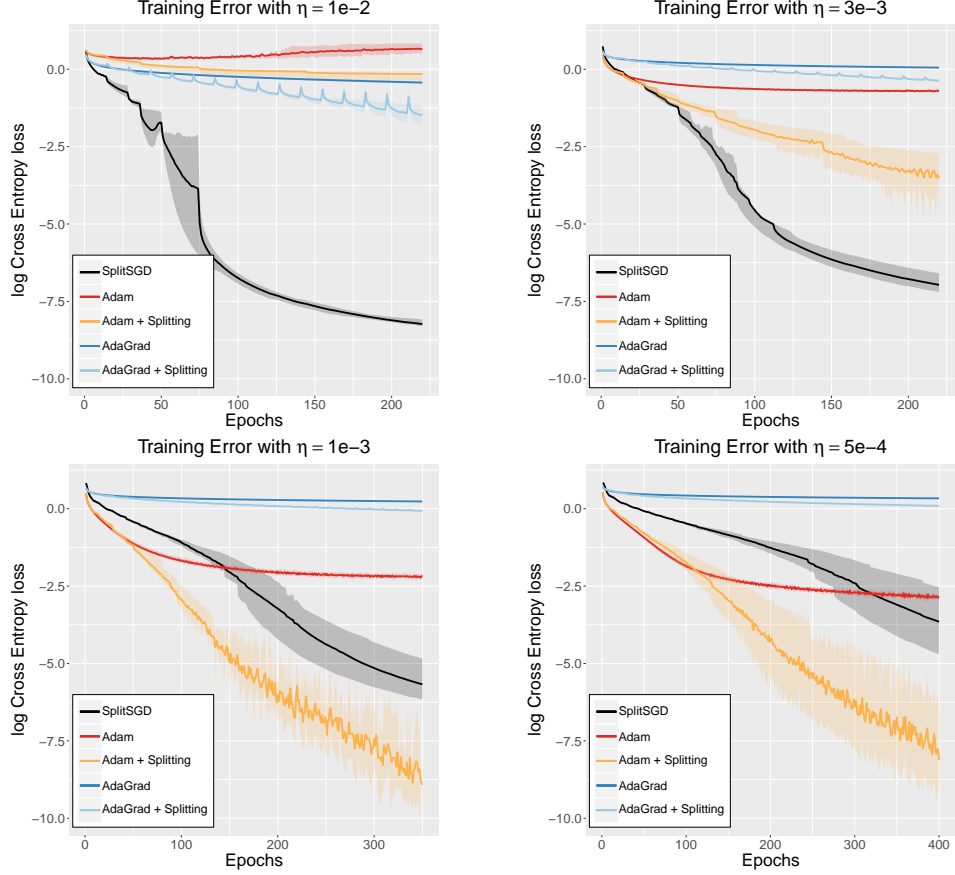


**Figure 5:** Comparison between SplitSGD,  $\text{SGD}^{1/2}$  (Half) and the classic SGD with constant or decreasing learning rate. The robustness of SplitSGD is clear compared to the other methods, which cannot adapt their learning rate to the data. As before each trajectory is run 20 times.

The Splitting Diagnostic does not have this problem, as a checkpoint is set every  $t_1 + 2lw$  iterations. The previous computations are then discarded, and only the new learning rate and starting point are stored. This gives the idea that one could improve the `pflug` procedure by adding checkpoints to reset the sum of the dot products, avoiding the behavior that we see in Table 1. We develop this idea in the appendix.

## 5.2 Comparison with other SGD methods

We now compare SplitSGD with other optimization procedures in convex settings (linear regression, logistic regression and SVM). In Figure 5, we present some popular non-adaptive methods that perform stochastic gradient descent: the classic SGD with decreasing learning rate  $\eta_t \propto 1/\sqrt{t}$ , SGD with constant learning rate, and the procedure  $\text{SGD}^{1/2}$  proposed in Bottou et al. (2018), where the learning rate is halved deterministically and the length of the next thread is double that of the previous one. For this last method, the length of the initial thread is set to be  $t_1$ . In all the situations the robustness of SplitSGD emerges. In fact we see that the methods that deterministically decrease the learning rate can perform well if the initial  $\eta$  is carefully tuned, but are not robust even to a small decrease of the initial learning rate. SplitSGD, on the contrary, maintains the initial learning rate constant for the necessary amount of iterations and achieves better results.



**Figure 6:** Training error for different methods on the CIFAR-10 dataset when the initial learning rate is between  $10^{-2}$  and  $5 \cdot 10^{-4}$ . SplitSGD shows remarkable robustness even in a non-convex setting.

### 5.3 Comparison with AdaGrad and Adam on a deep neural network

The last experiment that we consider is an image classification task on the CIFAR-10 dataset (Krizhevsky et al., 2009). We use a simple neural network to showcase the performance of SplitSGD in the non-convex setting. Specifically, it consists of two convolution layers, each followed by a max pooling layer and connected to three fully connected layers. The total number of weights is about 62,000. Here, instead of using the simple SGD with a constant learning rate inside the SplitSGD procedure, we adopt SGD with momentum (Qian, 1999), where the momentum parameter is set to 0.9. SGD with momentum is a popular choice in training deep neural networks (Sutskever et al., 2013), and when the learning rate is constant, it still exhibits both a transient and stationary phase. Since each layer is a separate unit of the networks, we consider a dot product for each set of weights and biases, and consider one epoch to be the unit measure for updates. Using  $w = 3$ , we then have a vector of 30 dot products for each diagnostic. Empirically, a smaller  $q$  is required, so we use  $q = 3$ . In addition, we set  $l = 1$ , thus each window corresponds to one epoch and  $t_1 = 8$ .

The other methods used are two state-of-the-art procedures in non-convex optimization, AdaGrad and Adam. The former, proposed by Duchi et al. (2011), has an adaptive learning rate defined

at step  $t$  as  $\eta_t = \eta \cdot (G_t^2 + \epsilon)^{-1/2} \in \mathbb{R}^d$  where  $G_t^2 = \sum_{i=1}^t g(\theta_{i-1}, Z_i)^2$  is the running sum of the squared gradients (notice that each operation is performed componentwise). The latter, introduced by Kingma and Ba (2014), improves on AdaGrad by leveraging the idea from RMSProp (Tieleman and Hinton, 2012) of using a weighted sum to keep track of the squared gradients, thereby avoiding a fast decay of the learning rate. In addition, Adam also stores a weighted average of the gradients, which has a function similar to the momentum term. For a discussion on these two methods, see Ruder (2016).

The results of this experiment are reported in Figure 6. After noticing that both AdaGrad and Adam were reaching a stationary phase, we applied the Splitting Diagnostic to them. It was interesting to notice that for AdaGrad, even if the learning rate was never decreased, the diagnostic was still beneficial, allowing the method to forget the running sum at the denominator and alleviate the effect of saturation. Likewise, Adam benefited from the diagnostic, but in this case the learning rate was reduced as expected, making its convergence improve greatly. The performance of SplitSGD on neural networks is similar as in the setting of convex objectives. That is, this method is more robust than the other methods to the initial choice of the learning rate, though its convergence slows down if the initial learning rate is set too small.

## 6 Discussion

We have developed an efficient optimization method called SplitSGD, by splitting the SGD thread for stationarity detection. Extensive simulation studies show that this method is robust to the choice of the initial learning rate in a variety of optimization tasks, compared to classic non-adaptive methods. In particular, SplitSGD on certain deep neural network architectures outperforms AdaGrad and Adam in terms of the convergence performance, which are arguably more complex methods. As the critical element underlying SplitSGD, the Splitting Diagnostic is a simple yet effective strategy that can possibly be incorporated into many optimization methods beyond SGD.

Several directions for future work are open. In light of the promising empirical results in Section 5.3, we intend to develop solid theoretical support for the important use case of the Splitting Diagnostic in the momentum SGD and Adam. Moreover, it would be interesting to possibly boost the convergence of SplitSGD in its initial phases by allowing for different learning rate selection strategies across different layers of the neural networks.

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## A Lemmas

**Lemma A.1** (3.5 in main text). *If Assumptions 3.1, 3.2, 3.3 and 3.4 with  $m = 2$  hold, and  $\eta \leq \frac{\mu}{L^2}$ , then for any  $t = 1, 2, \dots$*

$$\mathbb{E} [\|\theta_t - \theta^*\|^2] \leq (1 - 2\eta(\mu - L^2\eta))^t \cdot \mathbb{E} [\|\theta_0 - \theta^*\|^2] + \frac{G^2\eta}{\mu - L^2\eta}$$

*Proof.* This proof can be easily adapted from Moulines and Bach (2011). From the recursive definition of  $\theta_t$  one has

$$\mathbb{E} [\|\theta_t - \theta^*\|^2] \leq (1 - 2\eta(\mu - L^2\eta)) \cdot \mathbb{E} [\|\theta_{t-1} - \theta^*\|^2] + 2G^2\eta^2.$$

This inequality can be recursively applied to obtain the desired result

$$\begin{aligned} \mathbb{E} [\|\theta_t - \theta^*\|^2] &\leq (1 - 2\eta(\mu - L^2\eta))^t \cdot \mathbb{E} [\|\theta_0 - \theta^*\|^2] + 2G^2\eta^2 \sum_{j=0}^{t-1} (1 - 2\eta(\mu - L^2\eta))^j \\ &\leq (1 - 2\eta(\mu - L^2\eta))^t \cdot \mathbb{E} [\|\theta_0 - \theta^*\|^2] + \frac{G^2\eta}{\mu - L^2\eta} \end{aligned}$$

□

**Lemma A.2.** *If Assumption 3.4 with  $m = 4$  holds, then for any  $t, i \in \mathcal{N}$  one has*

$$\mathbb{E} [\|\theta_{t+i} - \theta_t\|^4 \mid \mathcal{F}_t] \leq \eta^4 i^4 G^4$$

*Proof.* For any  $j = 1, \dots, l$ , let  $x_j$  be a vector of length  $n$ . Applying Cauchy-Schwarz inequality twice, we get

$$\begin{aligned} \left\| \sum_{j=1}^l x_j \right\|^4 &= \left\| \sum_{j=1}^l x_j \right\|^2 \cdot \left\| \sum_{j=1}^l x_j \right\|^2 \leq \left( l \cdot \sum_{j=1}^l \|x_j\|^2 \right)^2 \\ &= l^2 \left( \sum_{j=1}^l \|x_j\|^2 \right)^2 \leq l^3 \cdot \sum_{j=1}^l \|x_j\|^4 \end{aligned} \tag{A.1}$$

Since

$$\theta_{t+i} = \theta_t - \eta \sum_{j=0}^{i-1} g(\theta_{t+j}, Z_{t+j+1}),$$



then we can use the fact that  $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$  for any  $k$ , together with Assumption 3.4 and (A.1), to get that

$$\begin{aligned}
\mathbb{E} [\|\theta_{t+i} - \theta_t\|^4 \mid \mathcal{F}_t] &= \eta^4 \cdot \mathbb{E} \left[ \left\| \sum_{j=0}^{i-1} g(\theta_{t+j}, Z_{t+j+1}) \right\|^4 \mid \mathcal{F}_t \right] \\
&\leq \eta^4 i^3 \sum_{j=0}^{i-1} \mathbb{E} [\|g(\theta_{t+j}, Z_{t+j+1})\|^4 \mid \mathcal{F}_t] \\
&= \eta^4 i^3 \sum_{j=0}^{i-1} \mathbb{E} \left[ \underbrace{\mathbb{E} [\|g(\theta_{t+j}, Z_{t+j+1})\|^4 \mid \mathcal{F}_{t+j}]}_{\leq G^4} \mid \mathcal{F}_t \right] \\
&\leq \eta^4 i^4 G^4
\end{aligned}$$

Note that this is a bound that considers the worst case in which all the noisy gradient updates point in the same direction and are of norm  $G$ .  $\square$

*Remark A.3.* We can obviously use the same bound for the unconditional squared norm, since

$$\mathbb{E} [\|\theta_{t+i} - \theta_t\|^4] = \mathbb{E} [\mathbb{E} [\|\theta_{t+i} - \theta_t\|^4 \mid \mathcal{F}_t]] \leq \eta^4 i^4 G^4.$$

**Lemma A.4.** *If Assumption 3.2 and 3.4 with  $m = 2$  hold, then for any  $i = 1, \dots, l$  and  $k = 1, 2$  we have that*

$$\mathbb{E} [\|\nabla F(\theta_{t+i}^{(k)}) - \nabla F(\theta_0)\|^2 \mid \mathcal{F}_t] \leq (L\|\theta_t - \theta_0\| + L\eta Gi)^2$$

*Proof.* By adding and subtracting  $\nabla F(\theta_t)$ , and by Lemma A.2, we get.

$$\begin{aligned}
\mathbb{E} [\|\nabla F(\theta_{t+i}^{(k)}) - \nabla F(\theta_0)\|^2 \mid \mathcal{F}_t] &\leq \mathbb{E} [\|\nabla F(\theta_{t+i}^{(k)}) - \nabla F(\theta_t) + \nabla F(\theta_t) - \nabla F(\theta_0)\|^2 \mid \mathcal{F}_t] \\
&\leq \|\nabla F(\theta_t) - \nabla F(\theta_0)\|^2 + \mathbb{E} [\|\nabla F(\theta_{t+i}^{(k)}) - \nabla F(\theta_t)\|^2 \mid \mathcal{F}_t] \\
&\quad + 2\|\nabla F(\theta_t) - \nabla F(\theta_0)\| \cdot \mathbb{E} [\|\nabla F(\theta_{t+i}^{(k)}) - \nabla F(\theta_t)\| \mid \mathcal{F}_t] \\
&\leq L^2\|\theta_t - \theta_0\|^2 + L^2 \mathbb{E} [\|\theta_{t+i}^{(k)} - \theta_t\|^2 \mid \mathcal{F}_t] + 2L^2\|\theta_t - \theta_0\| \cdot \mathbb{E} [\|\theta_{t+i}^{(k)} - \theta_t\| \mid \mathcal{F}_t] \\
&\leq L^2\|\theta_t - \theta_0\|^2 + L^2\eta^2 G^2 i^2 + 2L^2\|\theta_t - \theta_0\|\eta Gi \\
&= (L\|\theta_t - \theta_0\| + L\eta Gi)^2
\end{aligned}$$

$\square$

*Remark A.5.* When we consider the unconditional distance of the gradients, we can simply use smoothness and Remark A.3 to get

$$\mathbb{E} [\|\nabla F(\theta_{t+i}^{(k)}) - \nabla F(\theta_0)\|^2] \leq L^2 \mathbb{E} [\|\theta_{t+i}^{(k)} - \theta_0\|^2] \leq L^2\eta^2 G^2 (t+i)^2$$

which is the same result that we obtain from Lemma A.4 if at the end we bound  $\|\theta_t - \theta_0\|$  with its expectation, and use the fact that  $\mathbb{E} [\|\theta_t - \theta_0\|] \leq \eta Gt$ .

**Lemma A.6.** *If Assumption 3.2 and 3.4 with  $m = 2$  hold, then for any  $i = 1, \dots, l$  and  $k = 1, 2$  we have that*

$$i) \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(k)})\|^2 \mid \mathcal{F}_t \right] \leq (\|\nabla F(\theta_0)\| + L\|\theta_t - \theta_0\| + L\eta Gi)^2$$

$$ii) \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(k)})\|^2 \mid \mathcal{F}_t \right] \leq (L\|\theta_t - \theta^*\| + L\eta Gi)^2$$

*Proof.* We add and subtract  $\nabla F(\theta_t)$  to the gradient on the left hand side, and apply Lemma A.2.

$$\begin{aligned} \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(k)})\|^2 \mid \mathcal{F}_t \right] &= \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(k)}) - \nabla F(\theta_t) + \nabla F(\theta_t)\|^2 \mid \mathcal{F}_t \right] \\ &\leq \|\nabla F(\theta_t)\|^2 + \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(k)}) - \nabla F(\theta_t)\|^2 \mid \mathcal{F}_t \right] \\ &\quad + 2\|\nabla F(\theta_t)\| \cdot \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(k)}) - \nabla F(\theta_t)\| \mid \mathcal{F}_t \right] \\ &\leq \|\nabla F(\theta_t)\|^2 + L^2 \mathbb{E} \left[ \|\theta_{t+i}^{(k)} - \theta_t\|^2 \mid \mathcal{F}_t \right] + 2L\|\nabla F(\theta_t)\| \cdot \mathbb{E} \left[ \|\theta_{t+i}^{(k)} - \theta_t\| \mid \mathcal{F}_t \right] \\ &\leq \|\nabla F(\theta_t)\|^2 + L^2 \eta^2 G^2 i^2 + 2\|\nabla F(\theta_t)\| \cdot L\eta Gi \end{aligned} \tag{A.2}$$

To get part *i)* we repeat the same trick, this time adding and subtracting  $\nabla F(\theta_0)$  to the terms that contain  $\nabla F(\theta_t)$ .

$$\begin{aligned} (A.2) &\leq \|\nabla F(\theta_0)\|^2 + \|\nabla F(\theta_t) - \nabla F(\theta_0)\|^2 + 2\|\nabla F(\theta_0)\| \cdot \|\nabla F(\theta_t) - \nabla F(\theta_0)\| \\ &\quad + L^2 \eta^2 G^2 i^2 + 2\|\nabla F(\theta_0)\| \cdot L\eta Gi + 2\|\nabla F(\theta_t) - \nabla F(\theta_0)\| \cdot L\eta Gi \\ &\leq \|\nabla F(\theta_0)\|^2 + L^2 \|\theta_t - \theta_0\|^2 + 2L\|\nabla F(\theta_0)\| \cdot \|\theta_t - \theta_0\| \\ &\quad + L^2 \eta^2 G^2 i^2 + 2\|\nabla F(\theta_0)\| \cdot L\eta Gi + 2\|\theta_t - \theta_0\| \cdot L^2 \eta Gi \\ &= (\|\nabla F(\theta_0)\| + L\|\theta_t - \theta_0\| + L\eta Gi)^2 \end{aligned}$$

To get part *ii)*, instead, we can add  $\nabla f(\theta^*)$  and get

$$\begin{aligned} (A.2) &\leq L^2 \|\theta_t - \theta^*\|^2 + L^2 \eta^2 G^2 i^2 + 2\|\theta_t - \theta^*\| \cdot L^2 \eta Gi \\ &= (L\|\theta_t - \theta^*\| + L\eta Gi)^2 \end{aligned}$$

□

*Remark A.7.* For the unconditional squared norm of the gradient we again obtain the same bound as if in Lemma A.6 we were considering  $\mathbb{E}[\|\theta_t - \theta_0\|] \leq \eta Gt$  instead of just the argument of the expectation.

$$\begin{aligned} \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(k)})\|^2 \right] &= \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(k)}) - \nabla F(\theta_0) + \nabla F(\theta_0)\|^2 \right] \\ &\leq \|\nabla F(\theta_0)\|^2 + \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(k)}) - \nabla F(\theta_0)\|^2 \right] \\ &\quad + 2\|\nabla F(\theta_0)\| \cdot \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(k)}) - \nabla F(\theta_0)\| \right] \\ &\leq \|\nabla F(\theta_0)\|^2 + L^2 \eta^2 G^2 (t+i)^2 + 2\|\nabla F(\theta_0)\| L\eta G(t+i) \\ &= (\|\nabla F(\theta_0)\| + L\eta G(t+i))^2 \end{aligned}$$

## B Proof of Theorem 1

*Proof.* To slightly simplify the notation, we consider only  $Q_1$ . For the following windows, the calculations are equal and just involve some more terms, that are negligible if  $\eta$  is small enough. We assume that the Splitting Diagnostic starts after  $t$  iterations have already been made. We use the idea that, for a fixed  $t$ , if the learning rate is sufficiently small, the SGD iterate  $\theta_t$  and  $\theta_0$  will not be very far apart. In particular we will use  $\eta$  small enough such that  $\eta \cdot (t + l)$  is small, making every term of order  $O(\eta^k(t + l)^k)$  negligible for  $k > 1$ . Thanks to the conditional independence of the errors, the expectation of  $Q_1$  can be written only in terms of the true gradients.

$$\begin{aligned}
\mathbb{E}[Q_1] &= \frac{1}{l^2} \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} \mathbb{E} \left[ \langle g(\theta_{t+i}^{(1)}), g(\theta_{t+j}^{(2)}) \rangle \right] \\
&= \frac{1}{l^2} \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} \mathbb{E} \left[ \langle \nabla F(\theta_{t+i}^{(1)}) + \epsilon(\theta_{t+i}^{(1)}), \nabla F(\theta_{t+j}^{(2)}) + \epsilon(\theta_{t+j}^{(2)}) \rangle \right] \\
&= \frac{1}{l^2} \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} \mathbb{E} \left[ \langle \nabla F(\theta_{t+i}^{(1)}), \nabla F(\theta_{t+j}^{(2)}) \rangle \right] \tag{B.1}
\end{aligned}$$

We now add and subtract  $\nabla F(\theta_0)$ , and use L-smoothness and Remark A.3 to provide a lower bound for  $\mathbb{E}[Q_1]$ . From (B.1) we get

$$\begin{aligned}
\mathbb{E}[Q_1] &= \frac{1}{l^2} \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} \left\{ \langle \nabla F(\theta_0), \nabla F(\theta_0) \rangle + \mathbb{E} \left[ \langle \nabla F(\theta_{t+i}^{(1)}) - \nabla F(\theta_0), \nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0) \rangle \right] \right. \\
&\quad \left. + \mathbb{E} \left[ \langle \nabla F(\theta_0), \nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0) \rangle \right] + \mathbb{E} \left[ \langle \nabla F(\theta_{t+i}^{(1)}) - \nabla F(\theta_0), \nabla F(\theta_0) \rangle \right] \right\} \\
&\geq \|\nabla F(\theta_0)\|^2 - \frac{1}{l^2} \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(1)}) - \nabla F(\theta_0)\| \cdot \|\nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0)\| \right] \\
&\quad - \frac{1}{l} \sum_{j=0}^{l-1} \mathbb{E} \left[ \|\nabla F(\theta_0)\| \cdot \|\nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0)\| \right] - \frac{1}{l} \sum_{i=0}^{l-1} \mathbb{E} \left[ \|\nabla F(\theta_0)\| \cdot \|\nabla F(\theta_{t+i}^{(1)}) - \nabla F(\theta_0)\| \right] \\
&\geq \|\nabla F(\theta_0)\|^2 - \frac{L^2}{l^2} \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} \sqrt{\mathbb{E} \left[ \|\theta_{t+i}^{(1)} - \theta_0\|^2 \right] \cdot \mathbb{E} \left[ \|\theta_{t+j}^{(2)} - \theta_0\|^2 \right]} \\
&\quad - \frac{2L}{l} \sum_{i=0}^{l-1} \|\nabla F(\theta_0)\| \cdot \mathbb{E} \left[ \|\theta_{t+i}^{(1)} - \theta_0\| \right] \\
&\geq \|\nabla F(\theta_0)\|^2 - L^2 \eta^2 G^2(t+l)^2 - 2L \|\nabla F(\theta_0)\| \eta G(t+l) \\
&= \|\nabla F(\theta_0)\|^2 - 2L \|\nabla F(\theta_0)\| \eta G(t+l) + O(\eta^2(t+l)^2) \tag{B.2}
\end{aligned}$$

Notice that, in the extreme case where  $\eta = 0$ , we simply have  $\mathbb{E}[Q_1] \geq \|\nabla F(\theta_0)\|^2$  which is actually an equality, since we would have  $\theta_t = \theta_0$  and the noisy gradient at step  $t$  would be  $g(\theta_0, Z_t)$ , whose expectation is just  $\nabla F(\theta_0)$ . We now expand the second moment, and there are a lot of terms to be

considered separately.

$$\begin{aligned}
l^4 \cdot \mathbb{E} [Q_1^2] &= \mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} g(\theta_{t+i}^{(1)}), \sum_{j=0}^{l-1} g(\theta_{t+j}^{(2)}) \right\rangle^2 \right] \\
&= \mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} \left( \nabla F(\theta_{t+i}^{(1)}) + \epsilon(\theta_{t+i}^{(1)}) \right), \sum_{j=0}^{l-1} \left( \nabla F(\theta_{t+j}^{(2)}) + \epsilon(\theta_{t+j}^{(2)}) \right) \right\rangle^2 \right] \\
&= \underbrace{\mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} \nabla F(\theta_{t+i}^{(1)}), \sum_{j=0}^{l-1} \nabla F(\theta_{t+j}^{(2)}) \right\rangle^2 \right]}_I + \underbrace{\mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} \nabla F(\theta_{t+i}^{(1)}), \sum_{j=0}^{l-1} \epsilon(\theta_{t+j}^{(2)}) \right\rangle^2 \right]}_{II} \\
&\quad + \underbrace{\mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} \epsilon(\theta_{t+i}^{(1)}), \sum_{j=0}^{l-1} \nabla F(\theta_{t+j}^{(2)}) \right\rangle^2 \right]}_{III} + \underbrace{\mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} \epsilon(\theta_{t+i}^{(1)}), \sum_{j=0}^{l-1} \epsilon(\theta_{t+j}^{(2)}) \right\rangle^2 \right]}_{IV} \\
&\quad + 2 \underbrace{\mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} \nabla F(\theta_{t+i}^{(1)}), \sum_{j=0}^{l-1} \nabla F(\theta_{t+j}^{(2)}) \right\rangle \cdot \left\langle \sum_{h=0}^{l-1} \nabla F(\theta_{t+h}^{(1)}), \sum_{k=0}^{l-1} \epsilon(\theta_{t+k}^{(2)}) \right\rangle \right]}_V \\
&\quad + 2 \underbrace{\mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} \nabla F(\theta_{t+i}^{(1)}), \sum_{j=0}^{l-1} \nabla F(\theta_{t+j}^{(2)}) \right\rangle \cdot \left\langle \sum_{h=0}^{l-1} \epsilon(\theta_{t+h}^{(1)}), \sum_{k=0}^{l-1} \nabla F(\theta_{t+k}^{(2)}) \right\rangle \right]}_{VI} \\
&\quad + 2 \underbrace{\mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} \nabla F(\theta_{t+i}^{(1)}), \sum_{j=0}^{l-1} \nabla F(\theta_{t+j}^{(2)}) \right\rangle \cdot \left\langle \sum_{h=0}^{l-1} \epsilon(\theta_{t+h}^{(1)}), \sum_{k=0}^{l-1} \epsilon(\theta_{t+k}^{(2)}) \right\rangle \right]}_{VII} \\
&\quad + 2 \underbrace{\mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} \nabla F(\theta_{t+i}^{(1)}), \sum_{j=0}^{l-1} \epsilon(\theta_{t+j}^{(2)}) \right\rangle \cdot \left\langle \sum_{h=0}^{l-1} \epsilon(\theta_{t+h}^{(1)}), \sum_{k=0}^{l-1} \nabla F(\theta_{t+k}^{(2)}) \right\rangle \right]}_{VIII} \\
&\quad + 2 \underbrace{\mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} \nabla F(\theta_{t+i}^{(1)}), \sum_{j=0}^{l-1} \epsilon(\theta_{t+j}^{(2)}) \right\rangle \cdot \left\langle \sum_{h=0}^{l-1} \epsilon(\theta_{t+h}^{(1)}), \sum_{k=0}^{l-1} \epsilon(\theta_{t+k}^{(2)}) \right\rangle \right]}_{IX} \\
&\quad + 2 \underbrace{\mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} \epsilon(\theta_{t+i}^{(1)}), \sum_{j=0}^{l-1} \nabla F(\theta_{t+j}^{(2)}) \right\rangle \cdot \left\langle \sum_{h=0}^{l-1} \epsilon(\theta_{t+h}^{(1)}), \sum_{k=0}^{l-1} \epsilon(\theta_{t+k}^{(2)}) \right\rangle \right]}_X
\end{aligned}$$

In the squared terms  $I$  to  $IV$ , the errors are independent from the other argument of the dot product, conditional on  $\mathcal{F}_t$ , since they are evaluated on different threads. However, in the double

products ( $V$  to  $X$ ), some errors are used to generate the subsequent values of the SGD iterates on the same thread. This means that we cannot just ignore them, but we instead have to carefully find an upper bound for each one.

- In  $I$  we use the Cauchy-Schwarz inequality and Lemma A.6, after exploiting the independence of the two threads conditional on  $\mathcal{F}_t$ .

$$\begin{aligned}
\mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} \nabla F(\theta_{t+i}^{(1)}), \sum_{j=0}^{l-1} \nabla F(\theta_{t+j}^{(2)}) \right\rangle^2 \right] &\leq l^4 \cdot \max_{i,j} \mathbb{E} \left[ \left\langle \nabla F(\theta_{t+i}^{(1)}), \nabla F(\theta_{t+j}^{(2)}) \right\rangle^2 \right] \\
&\leq l^4 \cdot \max_{i,j} \mathbb{E} \left[ \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(1)})\|^2 \mid \mathcal{F}_t \right] \cdot \mathbb{E} \left[ \|\nabla F(\theta_{t+j}^{(2)})\|^2 \mid \mathcal{F}_t \right] \right] \\
&\leq l^4 \cdot \mathbb{E} \left[ (\|\nabla F(\theta_0)\| + L\|\theta_t - \theta_0\| + L\eta G l)^4 \right] \\
&\lesssim l^4 \cdot \mathbb{E} \left[ \|\nabla F(\theta_0)\|^4 + 4L\|\nabla F(\theta_0)\|^3 \cdot \|\theta_t - \theta_0\| + 4\|\nabla F(\theta_0)\|^3 \cdot L\eta G l + O(\eta^2(t+l)^2) \right] \\
&\lesssim l^4 \cdot (\|\nabla F(\theta_0)\|^4 + 4L\eta G \|\nabla F(\theta_0)\|^3(t+l) + O(\eta^2(t+l)^2))
\end{aligned}$$

In the first approximate inequality denoted by  $\lesssim$ , we have included most of the terms of the expansion in the  $O(\eta^2(t+l)^2)$ , even if technically we could have done it only after taking the expected value. Notice that here it was important to have a bound in Remark A.3 up to the fourth order.

- Terms  $II$  and  $III$  are equal, since the two threads are identically distributed, and the errors in one thread are a martingale difference sequence independent from the updates in the other thread. We will use the bound for the error norm

$$\mathbb{E} [\|\epsilon_t\|^2 \mid \mathcal{F}_t] = \mathbb{E} [\epsilon_t^T \epsilon_t \mid \mathcal{F}_t] = \mathbb{E} [\text{tr}(\epsilon_t \epsilon_t^T) \mid \mathcal{F}_t] \leq d \cdot \sigma_{\max} \quad (\text{B.3})$$

which is a consequence of Assumption 3.3, and condition on  $\mathcal{F}_t$  to use independence of the errors. In the last line we use Remark A.7.

$$\begin{aligned}
\mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} \nabla F(\theta_{t+i}^{(1)}), \sum_{j=0}^{l-1} \epsilon_{t+j}^{(2)} \right\rangle^2 \right] &= \sum_{j=0}^{l-1} \mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} \nabla F(\theta_{t+i}^{(1)}), \epsilon_{t+j}^{(2)} \right\rangle^2 \right] \\
&\leq l^2 \max_i \sum_{j=0}^{l-1} \mathbb{E} \left[ \left\| \nabla F(\theta_{t+i}^{(1)}) \right\|^2 \cdot \|\epsilon_{t+j}^{(2)}\|^2 \right] \\
&= l^3 \cdot \max_i \mathbb{E} \left[ \mathbb{E} [\|\epsilon_t^{(2)}\|^2 \mid \mathcal{F}_t] \cdot \mathbb{E} [\left\| \nabla F(\theta_{t+i}^{(1)}) \right\|^2 \mid \mathcal{F}_t] \right] \\
&\leq l^3 \cdot d\sigma_{\max} \cdot \max_i \mathbb{E} [\|\nabla F(\theta_{t+i}^{(1)})\|^2] \\
&\lesssim l^3 \cdot d\sigma_{\max} \cdot (\|\nabla f(\theta_0)\|^2 + 2\|\nabla f(\theta_0)\|LG\eta(t+l) + O(\eta^2(t+l)^2))
\end{aligned}$$

- In  $IV$ , we use the conditional independence of the two threads, and the fact that the errors are a martingale difference sequence, to cancel out all the cross products. An upper bound is then

$$\mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} \epsilon_{t+i}^{(1)}, \sum_{j=0}^{l-1} \epsilon_{t+j}^{(2)} \right\rangle^2 \right] = \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} \mathbb{E} \left[ \left\langle \epsilon_{t+i}^{(1)}, \epsilon_{t+j}^{(2)} \right\rangle^2 \right]$$

$$\begin{aligned}
&\leq \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} \mathbb{E} \left[ \|\epsilon_{t+i}^{(1)}\|^2 \cdot \|\epsilon_{t+j}^{(2)}\|^2 \right] \\
&= \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} \mathbb{E} \left[ \mathbb{E} \left[ \|\epsilon_{t+i}^{(1)}\|^2 \mid \mathcal{F}_t \right] \cdot \mathbb{E} \left[ \|\epsilon_{t+j}^{(2)}\|^2 \mid \mathcal{F}_t \right] \right] \\
&\leq l^2 d^2 \sigma_{max}^2
\end{aligned}$$

Now we start dealing with the double products. The problem here is that these terms are not all null, since the errors are used in the subsequent updates in the same thread, and they are then not independent.

- $V$  and  $VI$  are distributed in the same way. We can cancel out some terms using the conditional independence given  $\mathcal{F}_t$ , and use the conditional version of Cauchy-Schwarz inequality separately on the two threads.

$$\begin{aligned}
&\mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} \nabla F(\theta_{t+i}^{(1)}), \sum_{j=0}^{l-1} \nabla F(\theta_{t+j}^{(2)}) \right\rangle \cdot \left\langle \sum_{h=0}^{l-1} \nabla F(\theta_{t+h}^{(1)}), \sum_{k=0}^{l-1} \epsilon(\theta_{t+k}^{(2)}) \right\rangle \right] \\
&= \sum_{i,j,h,k=0}^{l-1} \mathbb{E} \left[ \left\langle \nabla F(\theta_{t+i}^{(1)}), \nabla F(\theta_{t+j}^{(2)}) \right\rangle \cdot \left\langle \nabla F(\theta_{t+h}^{(1)}), \epsilon(\theta_{t+k}^{(2)}) \right\rangle \right] \\
&= \sum_{i,j,h,k=0}^{l-1} \mathbb{E} \left[ \left\langle \nabla F(\theta_0) + \left( \nabla F(\theta_{t+i}^{(1)}) - \nabla F(\theta_0) \right), \nabla F(\theta_0) + \left( \nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0) \right) \right\rangle \times \right. \\
&\quad \left. \times \left\langle \nabla F(\theta_0) + \left( \nabla F(\theta_{t+h}^{(1)}) - \nabla F(\theta_0) \right), \epsilon(\theta_{t+k}^{(2)}) \right\rangle \right] \\
&= \sum_{i,j,h,k=0}^{l-1} \mathbb{E} \left[ \left\langle \nabla F(\theta_0), \nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0) \right\rangle \cdot \left\langle \nabla F(\theta_0), \epsilon(\theta_{t+k}^{(2)}) \right\rangle \right] \\
&\quad + \sum_{i,j,h,k=0}^{l-1} \mathbb{E} \left[ \left\langle \nabla F(\theta_0), \nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0) \right\rangle \cdot \left\langle \nabla F(\theta_{t+h}^{(1)}) - \nabla F(\theta_0), \epsilon(\theta_{t+k}^{(2)}) \right\rangle \right] \\
&\quad + \sum_{i,j,h,k=0}^{l-1} \mathbb{E} \left[ \left\langle \nabla F(\theta_{t+i}^{(1)}) - \nabla F(\theta_0), \nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0) \right\rangle \cdot \left\langle \nabla F(\theta_0), \epsilon(\theta_{t+k}^{(2)}) \right\rangle \right] \\
&\quad + \sum_{i,j,h,k=0}^{l-1} \mathbb{E} \left[ \left\langle \nabla F(\theta_{t+i}^{(1)}) - \nabla F(\theta_0), \nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0) \right\rangle \times \right. \\
&\quad \left. \times \left\langle \nabla F(\theta_{t+h}^{(1)}) - \nabla F(\theta_0), \epsilon(\theta_{t+k}^{(2)}) \right\rangle \right] \\
&\leq l^2 \|\nabla F(\theta_0)\|^2 \sum_{j,k=0}^{l-1} \mathbb{E} \left[ \|\nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0)\| \cdot \|\epsilon(\theta_{t+k}^{(2)})\| \right] \\
&\quad + l \|\nabla F(\theta_0)\| \sum_{j,h,k=0}^{l-1} \mathbb{E} \left[ \|\nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0)\| \cdot \|\nabla F(\theta_{t+h}^{(1)}) - \nabla F(\theta_0)\| \cdot \|\epsilon(\theta_{t+k}^{(2)})\| \right]
\end{aligned}$$

$$\begin{aligned}
& + l \|\nabla F(\theta_0)\| \sum_{i,j,k=0}^{l-1} \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(1)}) - \nabla F(\theta_0)\| \cdot \|\nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0)\| \cdot \|\epsilon(\theta_{t+k}^{(2)})\| \right] \\
& + \sum_{i,j,h,k=0}^{l-1} \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(1)}) - \nabla F(\theta_0)\| \cdot \|\nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0)\| \times \right. \\
& \quad \left. \times \|\nabla F(\theta_{t+h}^{(1)}) - \nabla F(\theta_0)\| \cdot \|\epsilon(\theta_{t+k}^{(2)})\| \right]
\end{aligned}$$

We bound the four pieces separately. For the first, we can just apply Cauchy-Schwarz and  $L$ -smoothness, together with Remark A.3

$$\begin{aligned}
\mathbb{E} \left[ \|\nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0)\| \cdot \|\epsilon(\theta_{t+k}^{(2)})\| \right] & \leq L \sqrt{\mathbb{E} \left[ \|\theta_{t+j}^{(2)} - \theta_0\|^2 \right]} \cdot \mathbb{E} \left[ \|\epsilon(\theta_{t+k}^{(2)})\|^2 \right] \\
& \leq \sqrt{d\sigma_{\max}} \cdot L\eta G(t+l)
\end{aligned}$$

The bound for the second and third term is equal. We use the conditional independence of the two threads and Lemma A.4.

$$\begin{aligned}
& \mathbb{E} \left[ \|\nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0)\| \cdot \|\nabla F(\theta_{t+h}^{(1)}) - \nabla F(\theta_0)\| \cdot \|\epsilon(\theta_{t+k}^{(2)})\| \right] = \\
& = \mathbb{E} \left[ \mathbb{E} \left[ \|\nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0)\| \cdot \|\epsilon(\theta_{t+k}^{(2)})\| \mid \mathcal{F}_t \right] \cdot \mathbb{E} \left[ \|\nabla F(\theta_{t+h}^{(1)}) - \nabla F(\theta_0)\| \mid \mathcal{F}_t \right] \right] \\
& \leq \mathbb{E} \left[ \sqrt{\mathbb{E} \left[ \|\nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0)\|^2 \mid \mathcal{F}_t \right]} \cdot \mathbb{E} \left[ \|\epsilon(\theta_{t+k}^{(2)})\|^2 \mid \mathcal{F}_t \right] \times \right. \\
& \quad \left. \times \mathbb{E} \left[ \|\nabla F(\theta_{t+h}^{(1)}) - \nabla F(\theta_0)\| \mid \mathcal{F}_t \right] \right] \\
& \leq \sqrt{d\sigma_{\max}} \cdot \mathbb{E} \left[ (L\|\theta_t - \theta_0\| + L\eta G l)^2 \right] \\
& \leq \sqrt{d\sigma_{\max}} \cdot L^2 \eta^2 G^2 (t+l)^2
\end{aligned}$$

The last term again makes use of conditional independence and Lemma A.4.

$$\begin{aligned}
& \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(1)}) - \nabla F(\theta_0)\| \cdot \|\nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0)\| \cdot \|\nabla F(\theta_{t+h}^{(1)}) - \nabla F(\theta_0)\| \cdot \|\epsilon(\theta_{t+k}^{(2)})\| \right] = \\
& = \mathbb{E} \left[ \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(1)}) - \nabla F(\theta_0)\| \cdot \|\nabla F(\theta_{t+h}^{(1)}) - \nabla F(\theta_0)\| \mid \mathcal{F}_t \right] \times \right. \\
& \quad \left. \times \mathbb{E} \left[ \|\nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0)\| \cdot \|\epsilon(\theta_{t+k}^{(2)})\| \mid \mathcal{F}_t \right] \right] \\
& \leq \mathbb{E} \left[ \sqrt{\mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(1)}) - \nabla F(\theta_0)\|^2 \mid \mathcal{F}_t \right]} \cdot \mathbb{E} \left[ \|\nabla F(\theta_{t+h}^{(1)}) - \nabla F(\theta_0)\|^2 \mid \mathcal{F}_t \right] \times \right. \\
& \quad \left. \times \sqrt{\mathbb{E} \left[ \|\nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0)\|^2 \mid \mathcal{F}_t \right]} \cdot \mathbb{E} \left[ \|\epsilon(\theta_{t+k}^{(2)})\|^2 \mid \mathcal{F}_t \right] \right] \\
& \leq \sqrt{d\sigma_{\max}} \cdot \mathbb{E} \left[ (L\|\theta_t - \theta_0\| + L\eta G l)^3 \right] \\
& \leq \sqrt{d\sigma_{\max}} \cdot L^3 \eta^3 G^3 (t+l)^3
\end{aligned}$$

The last inequality follows from the use of Remark A.3 to bound the moments of  $\|\theta_t - \theta_0\|$  up to order three.

- The upper bound for *VII* and *VIII* is the same, even if the error terms are in different positions. Again we invoke conditional independence to get rid of the dot products that only contain  $\nabla F(\theta_0)$ , and subsequently apply Cauchy-Schwarz inequality.

$$\begin{aligned}
& \mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} \nabla F(\theta_{t+i}^{(1)}), \sum_{j=0}^{l-1} \nabla F(\theta_{t+j}^{(2)}) \right\rangle \cdot \left\langle \sum_{i=0}^{l-1} \epsilon(\theta_{t+i}^{(1)}), \sum_{j=0}^{l-1} \epsilon(\theta_{t+j}^{(2)}) \right\rangle \right] \\
&= \sum_{i,j,h,k=0}^{l-1} \mathbb{E} \left[ \left\langle \nabla F(\theta_0) + \left( \nabla F(\theta_{t+i}^{(1)}) - \nabla F(\theta_0) \right), \nabla F(\theta_0) + \left( \nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0) \right) \right\rangle \times \right. \\
&\quad \left. \times \left\langle \epsilon(\theta_{t+h}^{(1)}), \epsilon(\theta_{t+k}^{(2)}) \right\rangle \right] \\
&= \sum_{i,j,h,k=0}^{l-1} \mathbb{E} \left[ \left\langle \nabla F(\theta_{t+i}^{(1)}) - \nabla F(\theta_0), \nabla F(\theta_{t+j}^{(2)}) - \nabla F(\theta_0) \right\rangle \cdot \left\langle \epsilon(\theta_{t+h}^{(1)}), \epsilon(\theta_{t+k}^{(2)}) \right\rangle \right] \\
&\leq L^2 \sum_{i,j,h,k=0}^{l-1} \mathbb{E} \left[ \|\theta_{t+i}^{(1)} - \theta_0\| \cdot \|\theta_{t+j}^{(2)} - \theta_0\| \cdot \|\epsilon(\theta_{t+h}^{(1)})\| \cdot \|\epsilon(\theta_{t+k}^{(2)})\| \right] \\
&\leq L^2 \sum_{i,j,h,k=0}^{l-1} \mathbb{E} \left[ \mathbb{E} \left[ \|\theta_{t+i}^{(1)} - \theta_0\| \cdot \|\epsilon(\theta_{t+h}^{(1)})\| \mid \mathcal{F}_t \right] \cdot \mathbb{E} \left[ \|\theta_{t+j}^{(2)} - \theta_0\| \cdot \|\epsilon(\theta_{t+k}^{(2)})\| \mid \mathcal{F}_t \right] \right] \\
&\leq L^2 \sum_{i,j,h,k=0}^{l-1} \mathbb{E} \left[ \sqrt{\mathbb{E} \left[ \|\theta_{t+i}^{(1)} - \theta_0\|^2 \mid \mathcal{F}_t \right] \cdot \mathbb{E} \left[ \|\epsilon(\theta_{t+h}^{(1)})\|^2 \mid \mathcal{F}_t \right]} \times \right. \\
&\quad \left. \times \sqrt{\mathbb{E} \left[ \|\theta_{t+j}^{(2)} - \theta_0\|^2 \mid \mathcal{F}_t \right] \cdot \mathbb{E} \left[ \|\epsilon(\theta_{t+k}^{(2)})\|^2 \mid \mathcal{F}_t \right]} \right] \\
&\leq l^4 L^2 \eta^2 G^2(t+l)^2 d\sigma_{max}
\end{aligned}$$

- Also the upper bounds for *IX* and *X* are equal. In the first one, when  $k \neq j$  we can condition on  $\mathcal{F}_{t+l}^{(1)}$  and  $\mathcal{F}_{t+\max\{k,j\}}^{(2)}$  to get that the expectation is null. Then we are only left with a sum on three indexes  $i, j, h$  and  $k = j$ . In the last passage we again condition on the appropriate  $\sigma$ -algebras to bound separately the two threads.

$$\begin{aligned}
& \mathbb{E} \left[ \left\langle \sum_{i=0}^{l-1} \nabla F(\theta_{t+i}^{(1)}), \sum_{j=0}^{l-1} \epsilon(\theta_{t+j}^{(2)}) \right\rangle \cdot \left\langle \sum_{h=0}^{l-1} \epsilon(\theta_{t+h}^{(1)}), \sum_{k=0}^{l-1} \epsilon(\theta_{t+k}^{(2)}) \right\rangle \right] \\
&= \sum_{i,j,h=0}^{l-1} \mathbb{E} \left[ \left\langle \nabla F(\theta_0) + \left( \nabla F(\theta_{t+i}^{(1)}) - \nabla F(\theta_0) \right), \epsilon(\theta_{t+j}^{(2)}) \right\rangle \cdot \left\langle \epsilon(\theta_{t+h}^{(1)}), \epsilon(\theta_{t+j}^{(2)}) \right\rangle \right] \\
&= \sum_{i,j,h=0}^{l-1} \mathbb{E} \left[ \left\langle \nabla F(\theta_{t+i}^{(1)}) - \nabla F(\theta_0), \epsilon(\theta_{t+j}^{(2)}) \right\rangle \cdot \left\langle \epsilon(\theta_{t+h}^{(1)}), \epsilon(\theta_{t+j}^{(2)}) \right\rangle \right] \\
&\leq \sum_{i,j,h=0}^{l-1} \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(1)}) - \nabla F(\theta_0)\| \cdot \|\epsilon(\theta_{t+j}^{(2)})\|^2 \cdot \|\epsilon(\theta_{t+h}^{(1)})\| \right]
\end{aligned}$$



$$\begin{aligned}
&\leq \sum_{i,j,h=0}^{l-1} \mathbb{E} \left[ \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(1)}) - \nabla F(\theta_0)\| \cdot \|\epsilon(\theta_{t+h}^{(1)})\| \mid \mathcal{F}_t \right] \cdot \mathbb{E} \left[ \|\epsilon(\theta_{t+j}^{(2)})\|^2 \mid \mathcal{F}_t \right] \right] \\
&\leq l^3 L \eta G(t+l) (d\sigma_{\max})^{3/2}
\end{aligned}$$

We put together all these upper bounds, leaving in extended form all the terms that are more significant than  $O(\eta^2(t+l)^2)$ . We get

$$\begin{aligned}
\text{Var}(Q_1) &= \mathbb{E}[Q_1^2] - \mathbb{E}[Q_1]^2 \\
&\lesssim \frac{2\|\nabla F(\theta_0)\|^2 d\sigma_{\max}}{l} + \frac{d^2 \sigma_{\max}^2}{l^2} \\
&\quad + \eta \cdot \left( \frac{4d\sigma_{\max}\|\nabla F(\theta_0)\| LG(t+l)}{l} + \frac{2LG(t+l)(d\sigma_{\max})^{3/2}}{l} \right) \\
&\quad + \eta \cdot \left( 8LG\|\nabla F(\theta_0)\|^3(t+l) + 2\|\nabla F(\theta_0)\|^2 LG(t+l)\sqrt{d\sigma_{\max}} \right) + O(\eta^2(t+l)^2)
\end{aligned}$$

which immediately translates to a bound for the standard deviation of the following form

$$\begin{aligned}
\text{sd}(Q_1) &\lesssim \frac{\|\nabla F(\theta_0)\|\sqrt{2d\sigma_{\max}}}{\sqrt{l}} + \frac{d\sigma_{\max}}{l} \\
&\quad + \sqrt{\eta} \cdot \left( 8LG\|\nabla F(\theta_0)\|^3(t+l) + 2\|\nabla F(\theta_0)\|^2 LG(t+l)\sqrt{d\sigma_{\max}} \right)^{1/2} \\
&\quad + \sqrt{\eta} \cdot \left( \frac{4d\sigma_{\max}\|\nabla F(\theta_0)\| LG(t+l)}{l} + \frac{2LG(t+l)(d\sigma_{\max})^{3/2}}{l} \right)^{1/2} + O(\eta(t+l))
\end{aligned} \tag{B.4}$$

We combine (B.4) with the fact, consequence of (B.2), that  $\mathbb{E}[Q_1]/\|\nabla F(\theta_0)\|^2 \gtrsim 1 + O(\eta(t+l))$ , to get the desired inequality

$$\text{sd}(Q_1) \lesssim C_1(\eta, l) \cdot \mathbb{E}[Q_1]$$

where

$$\begin{aligned}
C_1(\eta, l) &= \frac{1}{\|\nabla F(\theta_0)\|^2} \cdot \left\{ \frac{\|\nabla F(\theta_0)\|\sqrt{2d\sigma_{\max}}}{\sqrt{l}} + \frac{d\sigma_{\max}}{l} \right. \\
&\quad + \sqrt{\eta} \cdot \left( 8LG\|\nabla F(\theta_0)\|^3(t+l) + 2\|\nabla F(\theta_0)\|^2 LG(t+l)\sqrt{d\sigma_{\max}} \right)^{1/2} \\
&\quad \left. + \sqrt{\eta} \cdot \left( \frac{4d\sigma_{\max}\|\nabla F(\theta_0)\| LG(t+l)}{l} + \frac{2LG(t+l)(d\sigma_{\max})^{3/2}}{l} \right)^{1/2} \right\}
\end{aligned}$$

This confirms that  $C_1(\eta, l) = O(1/\sqrt{l}) + O(\sqrt{\eta(t+l)})$ .  $\square$

## C Proof of Theorem 2

*Proof.* As before, we only consider  $Q_1$  for simplicity. To provide an upper bound for  $|\mathbb{E}[Q_1]|$ , we use the fact that  $\nabla F(\theta^*) = 0$  together with Assumption 3.2. Starting from (B.1) we have

$$|\mathbb{E}[Q_1]| = \frac{1}{l^2} \left| \sum_{j=0}^{l-1} \sum_{k=0}^{l-1} \mathbb{E} \left[ \langle \nabla F(\theta_{t+j}^{(1)}), \nabla F(\theta_{t+k}^{(2)}) \rangle \right] \right|$$

$$\begin{aligned}
&\leq \frac{1}{l^2} \sum_{j=0}^{l-1} \sum_{k=0}^{l-1} \mathbb{E} \left[ \|\nabla F(\theta_{t+j}^{(1)}) - \nabla F(\theta^*)\| \cdot \|\nabla F(\theta_{t+k}^{(2)}) - \nabla F(\theta^*)\| \right] \\
&\leq \frac{L^2}{l^2} \sum_{j=0}^{l-1} \sum_{k=0}^{l-1} \mathbb{E} \left[ \|\theta_{t+j}^{(1)} - \theta^*\| \cdot \|\theta_{t+k}^{(2)} - \theta^*\| \right] \\
&\leq \frac{L^2}{l^2} \sum_{j=0}^{l-1} \sum_{k=0}^{l-1} \sqrt{\mathbb{E} \left[ \|\theta_{t+j}^{(1)} - \theta^*\|^2 \right] \cdot \mathbb{E} \left[ \|\theta_{t+k}^{(2)} - \theta^*\|^2 \right]}
\end{aligned}$$

Now we can use Lemma 3.5 that states that, for  $\eta \leq \frac{\mu}{L^2}$ ,

$$\mathbb{E} [\|\theta_t - \theta^*\|^2] \leq (1 - 2\eta(\mu - L^2\eta))^t \cdot \mathbb{E} [\|\theta_0 - \theta^*\|^2] + \frac{G^2\eta}{\mu - L^2\eta}. \quad (\text{C.1})$$

As  $t \rightarrow \infty$  we have that  $\mathbb{E} [\|\theta_t - \theta^*\|^2] \lesssim \frac{G^2\eta}{\mu - L^2\eta}$ .  $L$ -smoothness combined with (C.1) also gets

$$\mathbb{E} [\|\nabla F(\theta_t)\|^2] \lesssim \frac{L^2 G^2 \eta}{\mu - L^2 \eta} \quad \text{as } t \rightarrow \infty. \quad (\text{C.2})$$

Since the first term of (C.1) is decreasing in  $t$ , our bound on the expectation of  $Q_1$  is

$$|\mathbb{E} [Q_1]| \leq L^2 \cdot \left( (1 - 2\eta(\mu - L^2\eta))^t \cdot \mathbb{E} [\|\theta_0 - \theta^*\|^2] + \frac{G^2\eta}{\mu - L^2\eta} \right) \quad (\text{C.3})$$

To deal with the second moment, we introduce the notation

$$S_k := \sum_{i=0}^{l-1} g(\theta_{t+i}^{(k)}, Z_{t+i+1}^{(k)}) = \sum_{i=0}^{l-1} \nabla F(\theta_{t+i}^{(k)}) + \sum_{i=0}^{l-1} \epsilon(\theta_{t+i}^{(k)}) =: G_k + e_k.$$

where  $G_k$  is the true signal in the first window of thread  $k$  and  $e_k$  the related noise. Conditional on  $\mathcal{F}_t$ , the random variables  $S_1$  and  $S_2$  are independent and identically distributed. Then we can write

$$\begin{aligned}
l^4 \cdot \mathbb{E}[Q_1^2] &= \mathbb{E} [\langle S_1, S_2 \rangle^2] = \mathbb{E} [S_2^T S_1 S_1^T S_2] \\
&= \mathbb{E} [\text{Tr}(S_2^T S_1 S_1^T S_2)] = \mathbb{E} [\text{Tr}(S_1 S_1^T S_2 S_2^T)] \\
&= \text{Tr} (\mathbb{E} [S_1 S_1^T S_2 S_2^T]) = \text{Tr} (\mathbb{E} \{ \mathbb{E} [S_1 S_1^T \mid \mathcal{F}_t] \cdot \mathbb{E} [S_2 S_2^T \mid \mathcal{F}_t] \}) \\
&= \text{Tr} (\mathbb{E} \{ \mathbb{E} [S_1 S_1^T \mid \mathcal{F}_t]^2 \})
\end{aligned}$$

The goal is now to show that the matrix  $\mathbb{E} [S_1 S_1^T \mid \mathcal{F}_t]$  is positive definite, and provide a lower bound for its second moment using the fact that if  $A \succeq \lambda I$  for  $\lambda \geq 0$ , then  $A^2 \succeq \lambda^2 I$ . We can write

$$\begin{aligned}
\mathbb{E} [S_1 S_1^T \mid \mathcal{F}_t] &= \mathbb{E} [(G_1 + e_1)(G_1 + e_1)^T \mid \mathcal{F}_t] \\
&= \mathbb{E} [G_1 G_1^T \mid \mathcal{F}_t] + \mathbb{E} [G_1 e_1^T \mid \mathcal{F}_t] + \mathbb{E} [e_1 G_1^T \mid \mathcal{F}_t] + \mathbb{E} [e_1 e_1^T \mid \mathcal{F}_t]
\end{aligned}$$

We immediately have that  $\mathbb{E} [G_1 G_1^T \mid \mathcal{F}_t] \succeq 0$ , because, for any  $x \in \mathbb{R}^d$ ,

$$x^T \mathbb{E} [G_1 G_1^T \mid \mathcal{F}_t] x = \mathbb{E} [x^T G_1 G_1^T x \mid \mathcal{F}_t] = \mathbb{E} [\|x^T G_1\|^2 \mid \mathcal{F}_t] \geq 0.$$

Moreover we can also find an easy lower bound for the error term using Assumption 3.3,

$$\begin{aligned}\mathbb{E}[e_1 e_1^T | \mathcal{F}_t] &= \mathbb{E} \left[ \left( \sum_{i=0}^{l-1} \epsilon(\theta_{t+i}^{(1)}) \right) \left( \sum_{j=0}^{l-1} \epsilon(\theta_{t+j}^{(1)}) \right)^T \middle| \mathcal{F}_t \right] \\ &= \sum_{i=0}^{l-1} \mathbb{E} \left\{ \mathbb{E} \left[ \epsilon(\theta_{t+i}^{(1)}) \epsilon(\theta_{t+i}^{(1)})^T \middle| \mathcal{F}_{t+i-1} \right] \right\} \\ &\succeq l \cdot \sigma_{\min} \cdot I\end{aligned}$$

To lower bound the remaining terms we introduce a simple Lemma.

**Lemma C.1.** *If  $u, v \in \mathbb{R}^d$ , then  $uv^T + vu^T \succeq -2\|u\| \cdot \|v\| \cdot I$*

*Proof.* We apply the Cauchy-Schwarz inequality and get, for any  $x \in \mathbb{R}^d$ ,

$$\begin{aligned}x^T(uv^T + vu^T + 2\|u\| \cdot \|v\| \cdot I)x &= x^T uv^T x + x^T vu^T x + 2\|u\| \cdot \|v\| \cdot x^T x \\ &= \langle x, u \rangle \langle v, x \rangle + \langle x, v \rangle \langle u, x \rangle + 2\|u\| \cdot \|v\| \cdot \|x\|^2 \geq 0\end{aligned}$$

□

Using Lemma C.1, and Lemma A.6 ii) in the last inequality, we immediately get that

$$\begin{aligned}\mathbb{E}[G_1 e_1^T | \mathcal{F}_t] + \mathbb{E}[e_1 G_1^T | \mathcal{F}_t] &\succeq -2 \mathbb{E}[\|G_1\| \cdot \|e_1\| | \mathcal{F}_t] \cdot I \\ &\succeq -2 \sum_{i=1}^l \sum_{j=1}^l \mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(1)})\| \cdot \|\epsilon(\theta_{t+j}^{(1)})\| \middle| \mathcal{F}_t \right] \cdot I \\ &\succeq -2 \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} \sqrt{\mathbb{E} \left[ \|\nabla F(\theta_{t+i}^{(1)})\|^2 \middle| \mathcal{F}_t \right] \cdot \mathbb{E} \left[ \|\epsilon(\theta_{t+j}^{(1)})\|^2 \middle| \mathcal{F}_t \right]} \cdot I \\ &\succeq -2l^2 \cdot \sqrt{d\sigma_{\max}} \cdot (L\|\theta_t - \theta^*\| + L\eta Gl) \cdot I\end{aligned}$$

Notice that we could improve the bound using the fact that  $\epsilon(\theta_{t+j}^{(1)})$  is independent from  $\nabla F(\theta_{t+i}^{(1)})$  for any  $j \geq i$ . Putting the pieces together we get that

$$\begin{aligned}\mathbb{E}[S_1 S_1^T | \mathcal{F}_t] &\succeq \left( l\sigma_{\min} - 2l^2 \cdot \sqrt{d\sigma_{\max}} \cdot (L\|\theta_t - \theta^*\| + L\eta Gl) \right) \cdot I \\ \Rightarrow \quad \mathbb{E}[S_1 S_1^T | \mathcal{F}_t]^2 &\succeq \left( l\sigma_{\min} - 2l^2 \cdot \sqrt{d\sigma_{\max}} \cdot (L\|\theta_t - \theta^*\| + L\eta Gl) \right)^2 \cdot I \\ &\succeq \left\{ l^2 \sigma_{\min}^2 + 4l^4 d\sigma_{\max} \cdot (L\|\theta_t - \theta^*\| + L\eta Gl)^2 \right. \\ &\quad \left. - 4l^3 \sigma_{\min} \sqrt{d\sigma_{\max}} \cdot (L\|\theta_t - \theta^*\| + L\eta Gl) \right\} \cdot I\end{aligned}$$

and then, using the asymptotic bound in (C.1),

$$\mathbb{E} \left[ \mathbb{E}[S_1 S_1^T | \mathcal{F}_t]^2 \right] \succeq \left\{ l^2 \sigma_{\min}^2 - 4l^3 \sigma_{\min} \sqrt{d\sigma_{\max}} \cdot (L \cdot \mathbb{E}[\|\theta_t - \theta^*\|] + L\eta Gl) \right\} \cdot I$$

$$\stackrel{t \rightarrow \infty}{\underset{\sim}{\geq}} \left\{ l^2 \sigma_{min}^2 - 4l^3 \sigma_{min} \sqrt{d\sigma_{max}} \cdot \left( \frac{LG\sqrt{\eta}}{\sqrt{\mu - L^2\eta}} + L\eta Gl \right) \right\} \cdot I$$

which finally gives the bound on the second moment, which is

$$\begin{aligned} l^4 \cdot \mathbb{E}[Q_1^2] &\gtrsim d \cdot \left( l^2 \sigma_{min}^2 - 4l^3 \sigma_{min} \sqrt{d\sigma_{max}} LG\sqrt{\eta} \cdot \left( \frac{1}{\sqrt{\mu - L^2\eta}} + l\sqrt{\eta} \right) \right) \\ &\geq dl^2 \sigma_{min}^2 - K_1 l^3 \sqrt{\eta} - K_2 l^4 \eta \end{aligned}$$

Using the fact shown before, that

$$\mathbb{E}[Q_1]^2 \lesssim \frac{L^4 G^4 \eta^2}{(\mu - L^2 \eta)^2} \quad \text{as } t \rightarrow \infty,$$

we can bound the variance of  $Q_1$  from below with

$$Var(Q_1) = \mathbb{E}[Q_1^2] - \mathbb{E}[Q_1]^2 \geq \frac{d\sigma_{min}^2}{l^2} - \frac{K_1 \sqrt{\eta}}{l} - K_2 \eta - \frac{L^4 G^4 \eta^2}{(\mu - L^2 \eta)^2}$$

and then

$$Var(Q_1) \gtrsim \left( \frac{d\sigma_{min}^2}{l^2} - \frac{K_1 \sqrt{\eta}}{l} + O(\eta) \right) \cdot \frac{\mathbb{E}[Q_1]^2 (\mu - L^2 \eta)^2}{L^4 G^4 \eta^2}.$$

The desired inequality is finally

$$|\mathbb{E}[Q_1]| \lesssim C_2(\eta) \cdot \text{sd}(Q_1)$$

with

$$C_2(\eta) = \frac{L^2 G^2 \eta}{(\mu - L^2 \eta)} \cdot \left( \frac{d\sigma_{min}^2}{l^2} - \frac{K_1 \sqrt{\eta}}{l} + O(\eta) \right)^{-1/2} = C_2 \cdot \eta + o(\eta).$$

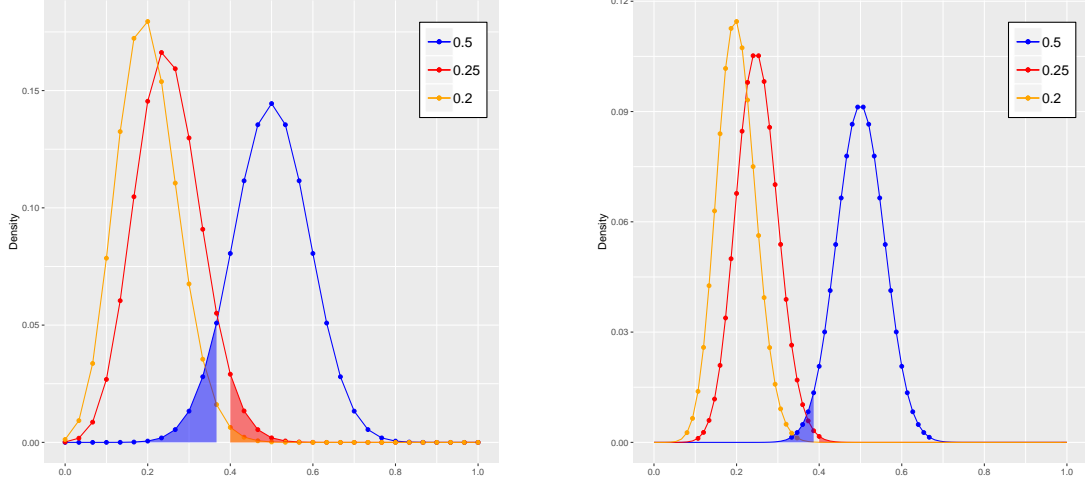
□

## D Experimental setting and data generation

In this Section we describe in more details the settings in which our simulations are performed. As we say in the main text, we would like  $w, l, t_1$  and  $K$  to be as large as possible. The tolerance  $q$ , instead, goes hand in hand with the choice of  $w$ . In Theorem 2 and Figure 3 we shown that, as  $t_1 \rightarrow \infty$ , the distribution of the sign of the gradient coherence is approximately a coin flip, provided that  $\eta$  is small enough. This means that, once stationarity is reached, we want  $q$  not to be too big, so that we will not observe a number of negative gradient coherences smaller than  $q$  just by chance too often (and erroneously think that stationarity has not been reached yet). If we were then to assume independence between the  $Q_i$ , we should set  $q$  to control the probability of a type I error, which is

$$\frac{1}{2^w} \sum_{i=0}^{q-1} \binom{w}{i} \leq \alpha$$

However, if we set  $q$  to be too small, then in the initial phases of the procedure we might think that we have already reached stationarity only because by chance we observed a number of negative dot



**Figure 7:** Continuous representation of the probability mass function of Binomial distributions. On the left we set  $w = 30$  and  $q = 12$ , on the right  $w = 75$  and  $q = 30$ , for both the probability of success (observing a negative gradient coherence) is  $p \in \{0.2, 0.25, 0.5\}$ . When  $p = 0.5$  (stationarity) the type I error happens with probability approximated by the shaded blue region. When  $p < 0.5$  (non stationarity) we erroneously declare stationarity with probability approximated by the shaded red and orange region.

products larger than  $q$ . This trade-off, represented in Figure 7, is particularly relevant if we cannot afford a large number of windows  $w$ , but it loses importance as  $w$  grows.

The convex settings considered in our simulations are linear regression, logistic regression and support vector machines. The feature matrix used in each has  $n = 1000$  rows (observations) and  $d = 10$  columns (variables). The respective loss functions are defined below.

- **Linear Regression:** in linear regression the outcome is a vector  $y \in \mathbb{R}^n$  given by  $y = X^T \theta_* + \epsilon$ , where  $\epsilon_i \sim N(0, 1)$  are independent. The empirical loss has the form

$$l(\theta, X, y) = \frac{1}{n} \sum_{i=1}^n (y_i - X_i \theta)^2$$

Thanks to strong convexity, in linear regression a very large  $l$  is not necessary, and we choose to select  $l = 50$ . We also use  $t_1 = 2000$  so that the length of the single thread and of the diagnostic are comparable from the beginning. Finally, we set  $q = 12$ , since Table 1 confirms that, with this value, stationarity is detected in a number of iterates that is of the correct order of magnitude. For the parameter  $\gamma$ , we stick with the standard choice of  $\gamma = 0.5$ , which is a good tradeoff between making the learning rate smaller, which increases accuracy once stationarity is reached, and not making the procedure last too long when the single thread is repeatedly increased by a factor  $1/\gamma$ .

- **Logistic Regression:** here the outcome takes value in  $\{0, 1\}$  and is defined as

$$y_i \sim \text{Bernoulli} \left( \frac{e^{X_i \theta_*}}{1 + e^{X_i \theta_*}} \right),$$

the empirical loss is

$$l(\theta, X, y) = \frac{1}{n} \sum_{i=1}^n \left( -y_i X_i \theta + \log \left( 1 + e^{X_i \theta} \right) \right)$$

We set  $l = 100$  and  $t_1 = 4000$ , since the loss function is only convex, but not strongly convex (which implies a weaker signal to noise ratio than in linear regression). To avoid the diagnostic detecting stationarity too early by chance, we set  $q = 14$ . We see in Figure 5 that the convergence in this setting is significantly slower than for linear regression, and we also notice in Table 1 that when the learning rate gets small we often underestimate the number of iterations necessary to reach convergence. For these reasons, we set  $\gamma = 0.7$ , since smaller values can become a problem if stationarity is erroneously detected too early. In this way the learning rate is decreased more slowly and it gets approximately halved every two reductions.

- **Support Vector Machines:** In SVM (Boser et al., 1992), we call the parameter of interest  $w \in \mathbb{R}^{d+1}$ , for consistency with existing literature. The vectors  $X_i$  have been augmented adding a 1 in the  $d + 1$  dimension to include the intercept. The output is generated as in logistic regression but with labels in  $\{-1, 1\}$ . The empirical loss is

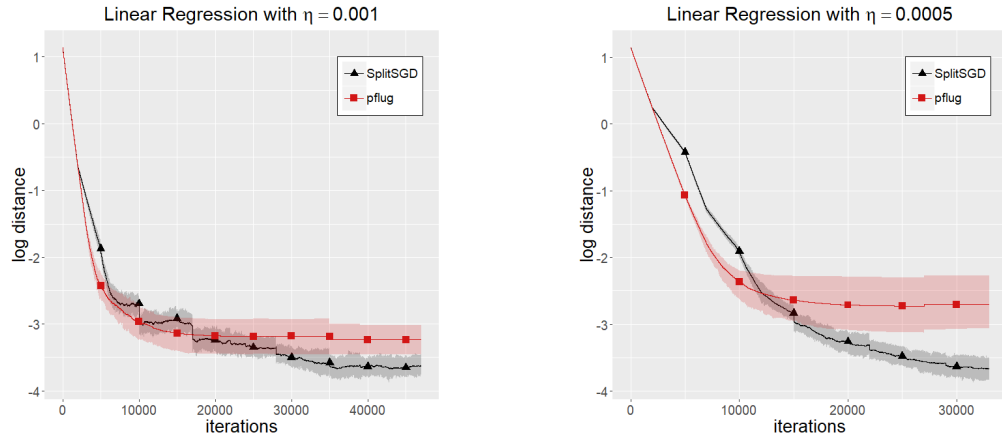
$$l(w, X, y) = \frac{1}{2} \|w\|^2 + \frac{C}{n} \sum_{i=1}^n [1 - y_i X_i w]_+$$

where the parameter  $C$  represents the trade-off between correct classification and increasing the width of the margin, and  $[x]_+ = \max(0, x)$  is the Hinge loss function. We use the same set of parameters as in linear regression.

## E An extension of the pflug diagnostic

As we see in Table 1 in the main text, an obvious problem of the **pflug** Diagnostic is that, when the starting point  $\theta_0$  is far from the minimizer  $\theta^*$ , the first dot products of consecutive gradients are very large and positive. When stationarity is reached, the sum  $S_t = \sum_{i=1}^t \langle g(\theta_{i-1}, Z_i), g(\theta_i, Z_{i+1}) \rangle$  starts getting smaller, but the magnitude of the negative increments is not enough to quickly balance the initial positive ones, so the number of iterations necessary to have  $S_t < 0$  can be very big. A possible simple solution to this problem would be to introduce checkpoints after which, if the sum  $S_t$  is still positive, the procedure restart with  $S_t = 0$ . A new burn-in period is then necessary.

We test this new procedure against SplitSGD in linear regression, and see in Figure 8 that with this modified **pflug** Diagnostic used in a  $\text{SGD}^{1/2}$  procedure the elbow is detected accurately. In our simulation we set a checkpoint every 2000 iterations, with burn-in 100. A new problem though emerges from this simulation. Once we get very close to the minimizer, the **pflug** Diagnostic is now detecting stationarity very fast, and the learning rate is decreased too quickly, slowing down convergence. We suspect that this is caused by the fact that in the neighborhood of  $\theta^*$  the signal to noise ratio in the noisy gradient is very weak, making the sum  $S_t$  to become negative very quickly. We look forward to test if some sort of averaging could further improve this method.



**Figure 8:** Comparison in the linear regression framework of SplitSGD with the version of  $\text{SGD}^{1/2}$  that uses the modified **pflug** Diagnostic described in Section E.