In order to verify programs we first need to be able to specify what they should do; the standard language for formal specification is *logic*. The purpose of this lab is to make a bridge with your previous module on *Discrete Structures*, and to revise some of the material on logic you met there.

1 Warming up

Very often books on "logic puzzles" feature narratives like the following:

> Suppose there are three hats, about which I tell John and Mary the following: Two of them have feathers, one does not. Two of them are red, one is blue.
>
> I put one hat on Mary, and one on John. Each can see the other's hat, but neither can see their own. (The spare hat remains hidden.) Both are honest and can reason perfectly with the information they are given.
>
> I ask Mary: "What can you deduce about your hat?" She replies that all she knows is that it is red.
>
> Given that John can see a feathered hat on Mary, can you work out the colours of the hats which are feathered, and who is wearing which?

Exercise 1 While this does require some reasoning, it isn't exactly what we mean by (formal) logic... Nonetheless, you might like to solve the puzzle before proceeding, just as a warming-up exercise.

2 The connectives

The five logical connectives are:

Symbol	Pronounced	Called
$\wedge$	and	conjunction
$\vee$	or	disjunction
$\neg$	not	negation
$\rightarrow$	implies	implication
$\Longleftrightarrow$	if-and-only-if	equivalence

Exercise 2 Let A = "Anthony is dangerous" and let B = "Brian is dangerous". Express each of the following sentences using A, B and the logical connectives.

1. Not more than one of them is dangerous
2. Both of them are dangerous.
3. Neither of them is dangerous.
4. At least one of them is dangerous.
5. Either both or neither of them is dangerous.
6. Exactly one of them is dangerous.

2.1 Conditionals

We use implication to express the standard "if...then..." argument. For example, to express the sentence "If John plays well then he will win" in logic, we might let P="John plays well" and W="John will win" and write $P \rightarrow W$.

Exercise 3 Express the following two sentences in logic, and use truth tables to show that they are equivalent to the statement $P \rightarrow W$:

1. Either John doesn't play well, or he does and he wins
2. This will not happen: John plays well and doesn't win

Exercise 4 Express the following in logic:

1. If the demand has remained constant and prices have been increased, then turnover must have decreased.
2. The sum of two sums is even if and only if either both numbers are even or both numbers are odd.
3. If y is an integer then z is not real, provided that x is a rational number.

2.2 Logical Status

A pair of sentences are:

- *consistent* if it is possible for them both to be true simultaneously
- *contrary* if they cannot both be true simultaneously
- *contradictory* if one, but not both, of them is true

For example, suppose John recently went to a party. We hear the following two reports:

- He either didn't go with his wife or he did go with his wife and he had a good time.
- It's not the case that he went with his wife and didn't have a good time.

Are these reports consistent? Are they actually the same?

Exercise 5 To check this, write out the truth table, letting A = "John went with his wife" and B = "John had a good time"

A	B	$(A \wedge B)$	$\neg A \vee (A \wedge B)$	$(A \wedge \neg B)$	$\neg(A \wedge \neg B)$
F	F				
F	T				
T	F				
T	T				

3 Predicates and Quantifiers

The basic until of logic is the **predicate**, written like: $p(x_1, x_2, \ldots, x_n)$. This is like a boolean-valued function, returning "true" iff the relationship denoted by p hold for its arguments $x_1, x_2, \ldots, x_n$.

Suppose we wanted to check if the following sentences are consistent

- Socrates didn't teach Aristotle
- Either Socrates taught Plato or Plato taught Socrates
- Socrates didn't teach Plato

We might use a binary predicate t to denote the teaching-relationship, where $t(x, y)$ means that "x taught y".

Exercise 6 Fill in the following truth table to find out if the sentences are consistent:

$t(S, A)$	$t(S, P)$	$t(P, S)$	$\neg t(S, A)$	$t(S, P) \vee t(P, S)$	$\neg t(S, P)$
F	F	F			
F	F	T			
F	T	F			
F	T	T			
T	F	F			
T	F	T			
T	T	F			
T	T	T			

3.1 Quantifiers

Our syntax for the quantifiers may look a little different to what you've met before: we're using *typed* first-order logic. Thus we write:

- $\forall\, x : U \;\cdot\; P(x)$ to mean "for all x of type U, $p(x)$ holds"
- $\exists\, x : U \;\cdot\; P(x)$ to mean "there exists an x of type U for which $p(x)$ holds"

To ease the transition from untyped logic, we'll just assume for the moment that there is some set U denoting the "universe", to which all things belong.

Exercise 7 Express the following sentences in logic, defining predicates and introducing quantified variables as appropriate:

1. Every cat washes itself
2. If all dogs are mortal, Fido is mortal
3. Not every scorpion is lethal
4. Every scorpion is non-lethal
5. At least one satellite orbits Mars
6. There is a planet larger than Neptune but cooler than it
7. Every village has at least one church

You might like to use the following predicates:
- **Unary predicates:** is-a-cat, is-a-dog, is-mortal, is-a-scorpion, is-lethal, is-a-satellite, is-a-planet, is-a-village, is-a-church,
- **Binary predicates:** washes, likes, orbits, larger-than, cooler-than, has-a.

Exercise 8 In each of the following the subject is underlined and the remainder is the predicate

- <u>Socrates</u> is a man.
- <u>The world</u> owes everyone a living.
- <u>I</u> write books.

We could write these sentences as follows using the predicates M for is-a-man, O for -owes-everyone-a-living and W for write-books.

- M(Socrates).
- O(World).
- W(I).

Express the following sentences in logic, defining parameterised predicates and introducing quantified variables as appropriate:

1. Not all birds can fly.
2. Anyone can do that.
3. Some people are stupid.
4. There is an integer that is greater than every other integer.