# Basics of Determinants and Related Rules

Mathematics Primer

## Introduction to Determinants

The determinant is a scalar value that can be computed from the elements of a square matrix. It is a fundamental concept in linear algebra with applications in solving systems of linear equations, finding the area or volume in geometry, and more.

## 1 Definition of Determinants

For a square matrix $A$, the determinant is denoted by $\det(A)$ or $|A|$. Determinants are defined recursively:

- The determinant of a $1 \times 1$ matrix $A = [a]$ is $\det(A) = a$.
- For a $2 \times 2$ matrix:
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(A) = ad - bc.$$
- For larger matrices, the determinant is computed using more complex rules, such as expansion by minors or other algorithms like Sarrus' Rule (for $3 \times 3$ matrices).

## 2 Laplace Expansion

Laplace expansion, also known as expansion by minors, is a method to compute the determinant of an $n \times n$ matrix. It is defined as:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij},$$

where:

- $a_{ij}$ is the element in the $i$-th row and $j$-th column of $A$.
- $M_{ij}$ is the minor of $a_{ij}$, obtained by removing the $i$-th row and $j$-th column from $A$ and computing the determinant of the resulting $(n-1) \times (n-1)$ matrix.

Laplace expansion can be performed along any row or column of the matrix.

## 3 Sarrus' Rule for $3 \times 3$ Matrices

Sarrus' Rule provides a shortcut for calculating the determinant of a $3 \times 3$ matrix:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

The determinant is given by:

$$\det(A) = aei + bfg + cdh - ceg - bdi - afh.$$

This rule involves summing the products of the diagonals extending from the top-left to the bottom-right and subtracting the products of the diagonals extending from the bottom-left to the top-right.

## 4 Properties of Determinants

- **Linearity:** The determinant is linear with respect to the rows and columns of the matrix.
- **Effect of Row/Column Swaps:** Swapping two rows or columns of a matrix multiplies the determinant by $-1$.
- **Effect of Scalar Multiplication:** Multiplying a row or column by a scalar $k$ multiplies the determinant by $k$.
- **Effect of Adding Rows/Columns:** Adding a multiple of one row (or column) to another does not change the determinant.
- **Singular Matrices:** If a matrix is singular (i.e., it has linearly dependent rows or columns), then its determinant is zero.

## 5 Applications of Determinants

- Solving linear systems using Cramer's Rule.
- Checking the invertibility of matrices (a matrix is invertible if and only if its determinant is non-zero).
- Finding the area or volume of geometric shapes defined by vectors.
- Understanding transformations in linear algebra, such as scaling and rotation.

## Conclusion

The determinant is a versatile and essential tool in mathematics. From basic properties to advanced computational techniques, it serves as a gateway to understanding linear transformations, systems of equations, and geometric concepts.