

# Methods of Mathematical Proofs

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# 1 Introduction

Proofs are the foundation of mathematics. They provide a rigorous way to demonstrate the truth of mathematical statements. In this document, we will explore various proof methods, including direct proofs, proof by contradiction, proof by induction, and proof by contrapositive. Each method will be illustrated with examples of varying difficulty levels: easy, medium, and advanced.

## 2 Direct Proof

A direct proof involves starting from the given assumptions and using logical deductions to arrive at the conclusion. This method is straightforward and commonly used in mathematics.

### 2.1 Example 1: Easy

**Statement:** If  $n$  is an even integer, then  $n^2$  is also even.

**Proof:**

- Assume  $n$  is even. By definition,  $n = 2k$  for some integer  $k$ .
- Calculate  $n^2$ :  $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$ .
- Since  $2k^2$  is an integer,  $n^2$  is even. □

### 2.2 Example 2: Medium

**Statement:** For any two even integers  $a$  and  $b$ , their sum  $a + b$  is even.

**Proof:**

- Assume  $a$  and  $b$  are even. By definition,  $a = 2k$  and  $b = 2m$  for some integers  $k$  and  $m$ .
- Calculate  $a + b$ :  $a + b = 2k + 2m = 2(k + m)$ .
- Since  $k + m$  is an integer,  $a + b$  is even. □

### 2.3 Example 3: Advanced

**Statement:** For any integers  $a$  and  $b$ , if  $a$  and  $b$  are both divisible by 4, then  $a^2 + b^2$  is divisible by 16.

**Proof:**

- Assume  $a$  and  $b$  are divisible by 4. By definition,  $a = 4k$  and  $b = 4m$  for some integers  $k$  and  $m$ .
- Calculate  $a^2 + b^2$ :  $a^2 + b^2 = (4k)^2 + (4m)^2 = 16k^2 + 16m^2 = 16(k^2 + m^2)$ .
- Since  $k^2 + m^2$  is an integer,  $a^2 + b^2$  is divisible by 16. □

### 3 Proof by Contradiction

Proof by contradiction involves assuming the negation of the statement to be proved and then showing that this assumption leads to a contradiction.

#### 3.1 Example 1: Easy

**Statement:**  $\sqrt{2}$  is irrational.

**Proof:**

- Assume the negation:  $\sqrt{2}$  is rational. Then  $\sqrt{2} = \frac{p}{q}$ , where  $p$  and  $q$  are integers with  $\gcd(p, q) = 1$ .
- Square both sides:  $2 = \frac{p^2}{q^2}$ , so  $p^2 = 2q^2$ .
- This implies  $p^2$  is even, so  $p$  is even. Let  $p = 2k$ . Then  $p^2 = 4k^2$ .
- Substitute:  $4k^2 = 2q^2$ , so  $q^2 = 2k^2$ . Thus,  $q$  is also even.
- Both  $p$  and  $q$  are even, contradicting  $\gcd(p, q) = 1$ . □

#### 3.2 Example 2: Medium

**Statement:** There is no largest prime number.

**Proof:**

- Assume the negation: There is a largest prime number, say  $p_n$ .
- Consider the number  $P = p_1 p_2 \cdots p_n + 1$ , where  $p_1, p_2, \dots, p_n$  are all the primes up to  $p_n$ .
- $P$  is not divisible by any  $p_i$  (remainder is 1).
- Hence,  $P$  is either prime or divisible by a prime greater than  $p_n$ , contradicting the assumption. □

#### 3.3 Example 3: Advanced

**Statement:** The decimal expansion of  $\pi$  is non-repeating.

**Proof:**

- Assume the negation:  $\pi$  has a repeating decimal expansion, so  $\pi$  is rational.
- By definition,  $\pi = \frac{p}{q}$  for integers  $p$  and  $q$ .
- However,  $\pi$  is known to be transcendental, which means it is not algebraic (not the root of any polynomial with integer coefficients).
- This contradicts the assumption that  $\pi$  is rational. □

## 4 Proof by Induction

Proof by induction is used to prove statements about integers. It consists of a base case and an inductive step.

### 4.1 Example 1: Easy

**Statement:**  $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$  for all  $n \geq 1$ .

**Proof:**

- Base case ( $n = 1$ ):  $1 = \frac{1(1+1)}{2}$ . True.
- Inductive step: Assume true for  $n = k$ :  $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ .
- Prove for  $n = k + 1$ :  $1 + 2 + \cdots + k + (k + 1) = \frac{k(k+1)}{2} + (k + 1)$ .
- Simplify:  $\frac{k(k+1)}{2} + (k + 1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$ .
- True for  $n = k + 1$ . By induction, true for all  $n \geq 1$ . □

### 4.2 Example 2: Medium

**Statement:**  $2^n > n^2$  for all  $n \geq 5$ .

**Proof:**

- Base case ( $n = 5$ ):  $2^5 = 32 > 25 = 5^2$ . True.
- Inductive step: Assume true for  $n = k$ :  $2^k > k^2$ .
- Prove for  $n = k + 1$ :  $2^{k+1} = 2 \cdot 2^k > 2 \cdot k^2$  (by assumption).
- Need  $2k^2 > (k + 1)^2$ :

$$\begin{aligned} 2k^2 - (k + 1)^2 &= 2k^2 - (k^2 + 2k + 1) \\ &= k^2 - 2k - 1. \end{aligned}$$

- For  $k \geq 5$ ,  $k^2 - 2k - 1 > 0$ . True for  $n = k + 1$ . □

### 4.3 Example 3: Advanced

**Statement:**  $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$  for all  $n \geq 1$ .

**Proof:**

- Base case ( $n = 1$ ):  $1^3 = \left(\frac{1(1+1)}{2}\right)^2 = 1$ . True.
- Inductive step: Assume true for  $n = k$ :  $\sum_{i=1}^k i^3 = \left(\frac{k(k+1)}{2}\right)^2$ .
- Prove for  $n = k + 1$ :  $\sum_{i=1}^{k+1} i^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k + 1)^3$ .
- Simplify:  $\left(\frac{k(k+1)}{2}\right)^2 + (k + 1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$  (details omitted for brevity).
- True for  $n = k + 1$ . By induction, true for all  $n \geq 1$ . □

## 5 Conclusion

In this document, we have explored various methods of mathematical proofs, providing clear, step-by-step examples of each method with increasing levels of difficulty. These examples illustrate the power and utility of mathematical reasoning in establishing truths across diverse domains.