An Introduction to Limits in Mathematics

1 Introduction to Limits

Limits form a fundamental concept in calculus and mathematical analysis. They describe the behavior of a function as its input approaches a particular point or infinity. Limits help formalize ideas like continuity, derivatives, and integrals.

1.1 Definition of a Limit

The limit of a function $f(x)$ as x approaches a is denoted as:

$$\lim_{x \to a} f(x) = L$$

This means that as x gets arbitrarily close to a, $f(x)$ gets arbitrarily close to L. For a formal definition: For every $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

1.2 Intuition

Limits capture the trend of a function, even if it does not reach the value at a certain point. For instance, $f(x) = \frac{1}{x}$ does not reach any finite value as $x \to 0$, but its limit behavior is clear: $f(x) \to \infty$ or $-\infty$, depending on the direction.

2 Calculating Limits

2.1 Limit at a Point (x_0)

Using the definition of limits, we calculate the value of a function as x approaches a specific point x_0.

2.1.1 Example: A Simple Function

$$\lim_{x \to 2}(x^2 - 3x + 2)$$

Substituting directly:

$$2^2 - 3(2) + 2 = 4 - 6 + 2 = 0.$$

2.2 Overall Limit: The Derivative

The derivative of a function at a point is defined using limits:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

For example, if $f(x) = x^2$, the derivative at x is:

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0}(2x + h) = 2x.$$

3 Special Techniques and Formulas for Limits

3.1 Greatest x Limits Rule

For polynomials or rational functions, the term with the highest power dominates as $x \to \infty$ or $x \to -\infty$:

$$\lim_{x \to \infty} \frac{2x^3 + 5x}{3x^3 - 4} = \frac{2}{3}.$$

This is because the highest degree terms $2x^3$ and $3x^3$ dominate the numerator and denominator respectively.

3.2 Conjugate Rule for Square Root Expressions

Used for limits involving square roots, this rule simplifies the expression by multiplying by the conjugate:

$$\lim_{x \to 1} \frac{\sqrt{x+3} - 2}{x - 1}$$

Multiply numerator and denominator by the conjugate:

$$\frac{\sqrt{x+3} - 2}{x - 1} \cdot \frac{\sqrt{x+3} + 2}{\sqrt{x+3} + 2} = \frac{(x+3) - 4}{(x-1)(\sqrt{x+3} + 2)} = \frac{x - 1}{(x-1)(\sqrt{x+3} + 2)}.$$

Cancel the common term $x - 1$:

$$\frac{1}{\sqrt{x+3} + 2} \to \frac{1}{4} \quad \text{as } x \to 1.$$

3.3 Euler's Limit

A fundamental limit that defines the base of natural logarithms e:

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

This result is central in calculus and appears in compound interest and growth models.

3.4 Logarithmic and Exponential Limits

$$\lim_{x \to 0^+} x \ln(x) = 0.$$

$$\lim_{x \to \infty} \frac{\ln(x)}{x} = 0.$$

3.5 Arithmetic and Geometric Sequence Limits

The sum of an arithmetic sequence:

$$S_n = \frac{a_1 + a_n}{2} \cdot n.$$

The sum of a geometric sequence:

$$S_n = a_1 \cdot \frac{1 - q^n}{1 - q}, \quad |q| < 1.$$

3.6 Limits Involving Trigonometric Functions

For small angles (in radians):

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1, \quad \lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2}.$$

4 Applications of Limits

4.1 Science and Engineering

Limits are crucial in physics (e.g., defining instantaneous velocity), economics (e.g., marginal costs), and computer science (e.g., asymptotic analysis).

4.2 Understanding Asymptotic Behavior

In mathematics, limits help in analyzing the growth and decay of functions, studying infinite series, and solving differential equations.