Methods of Mathematical Proofs

Your Name

December 20, 2024

Contents

1 Introduction

Proofs are the foundation of mathematics. They provide a rigorous way to demonstrate the truth of mathematical statements. In this document, we will explore various proof methods, including direct proofs, proof by contradiction, proof by induction, and proof by contrapositive. Each method will be illustrated with examples of varying difficulty levels: easy, medium, and advanced.

2 Direct Proof

A direct proof involves starting from the given assumptions and using logical deductions to arrive at the conclusion. This method is straightforward and commonly used in mathematics.

2.1 Example 1: Easy

Statement: If n is an even integer, then n^2 is also even.

Proof:

- Assume n is even. By definition, $n = 2k$ for some integer k.
- Calculate n^2: $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.
- Since $2k^2$ is an integer, n^2 is even. $\square$

2.2 Example 2: Medium

Statement: For any two even integers a and b, their sum $a + b$ is even.

Proof:

- Assume a and b are even. By definition, $a = 2k$ and $b = 2m$ for some integers k and m.
- Calculate $a + b$: $a + b = 2k + 2m = 2(k + m)$.
- Since $k + m$ is an integer, $a + b$ is even. $\square$

2.3 Example 3: Advanced

Statement: For any integers a and b, if a and b are both divisible by 4, then $a^2 + b^2$ is divisible by 16.

Proof:

- Assume a and b are divisible by 4. By definition, $a = 4k$ and $b = 4m$ for some integers k and m.
- Calculate $a^2 + b^2$: $a^2 + b^2 = (4k)^2 + (4m)^2 = 16k^2 + 16m^2 = 16(k^2 + m^2)$.
- Since $k^2 + m^2$ is an integer, $a^2 + b^2$ is divisible by 16. $\square$

3 Proof by Contradiction

Proof by contradiction involves assuming the negation of the statement to be proved and then showing that this assumption leads to a contradiction.

3.1 Example 1: Easy

Statement: $\sqrt{2}$ is irrational.

Proof:

- Assume the negation: $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{p}{q}$, where p and q are integers with $\gcd(p, q) = 1$.
- Square both sides: $2 = \frac{p^2}{q^2}$, so $p^2 = 2q^2$.
- This implies p^2 is even, so p is even. Let $p = 2k$. Then $p^2 = 4k^2$.
- Substitute: $4k^2 = 2q^2$, so $q^2 = 2k^2$. Thus, q is also even.
- Both p and q are even, contradicting $\gcd(p, q) = 1$. □

3.2 Example 2: Medium

Statement: There is no largest prime number.

Proof:

- Assume the negation: There is a largest prime number, say p_n.
- Consider the number $P = p_1 p_2 \cdots p_n + 1$, where $p_1, p_2, \ldots, p_n$ are all the primes up to p_n.
- P is not divisible by any p_i (remainder is 1).
- Hence, P is either prime or divisible by a prime greater than p_n, contradicting the assumption. □

3.3 Example 3: Advanced

Statement: The decimal expansion of π is non-repeating.

Proof:

- Assume the negation: π has a repeating decimal expansion, so π is rational.
- By definition, $\pi = \frac{p}{q}$ for integers p and q.
- However, π is known to be transcendental, which means it is not algebraic (not the root of any polynomial with integer coefficients).
- This contradicts the assumption that π is rational. □

4 Proof by Contrapositive

Proof by contrapositive involves proving a statement by showing that its contrapositive is true. Recall that a statement "if P, then Q" is logically equivalent to "if not Q, then not P."

4.1 Example 1: Easy

Statement: If n^2 is even, then n is even.

Proof:

- Contrapositive: If n is odd, then n^2 is odd.
- Assume n is odd. By definition, $n = 2k + 1$ for some integer k.
- Calculate n^2: $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.
- Since $2k^2 + 2k$ is an integer, n^2 is odd. $\square$

4.2 Example 2: Medium

Statement: If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

Proof:

- Contrapositive: If $a \neq 0$ and $b \neq 0$, then $a \cdot b \neq 0$.
- Assume $a \neq 0$ and $b \neq 0$. By definition, both a and b are nonzero numbers.
- The product of two nonzero numbers is nonzero: $a \cdot b \neq 0$. $\square$

4.3 Example 3: Advanced

Statement: If a number is not divisible by 3, then its square is not divisible by 3.

Proof:

- Contrapositive: If a number's square is divisible by 3, then the number is divisible by 3.
- Assume n^2 is divisible by 3. Then $n^2 = 3k$ for some integer k.
- If n^2 is divisible by 3, then n must also be divisible by 3 (since 3 is prime).
- Thus, if n^2 is divisible by 3, then n is divisible by 3. $\square$

5 Proof by Induction

Proof by induction is used to prove statements about integers. It consists of a base case and an inductive step.

5.1 Example 1: Easy

Statement: $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \geq 1$.

Proof:

- Base case ($n = 1$): $1 = \frac{1(1+1)}{2}$. True.
- Inductive step: Assume true for $n = k$: $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$.
- Prove for $n = k + 1$: $1 + 2 + \cdots + k + (k + 1) = \frac{k(k+1)}{2} + (k + 1)$.
- Simplify: $\frac{k(k+1)}{2} + (k + 1) = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$.
- True for $n = k + 1$. By induction, true for all $n \geq 1$. $\square$

5.2 Example 2: Medium

Statement: $2^n > n^2$ for all $n \geq 5$.

Proof:

- Base case ($n = 5$): $2^5 = 32 > 25 = 5^2$. True.
- Inductive step: Assume true for $n = k$: $2^k > k^2$.
- Prove for $n = k + 1$: $2^{k+1} = 2 \cdot 2^k > 2 \cdot k^2$ (by assumption).
- Need $2k^2 > (k + 1)^2$:

$$\begin{aligned} 2k^2 - (k+1)^2 &= 2k^2 - (k^2 + 2k + 1) \\ &= k^2 - 2k - 1. \end{aligned}$$

- For $k \geq 5$, $k^2 - 2k - 1 > 0$. True for $n = k + 1$. $\square$

5.3 Example 3: Advanced

Statement: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all $n \geq 1$.

Proof:

- Base case ($n = 1$): $1^3 = \left(\frac{1(1+1)}{2}\right)^2 = 1$. True.
- Inductive step: Assume true for $n = k$: $\sum_{i=1}^{k} i^3 = \left(\frac{k(k+1)}{2}\right)^2$.
- Prove for $n = k + 1$: $\sum_{i=1}^{k+1} i^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k + 1)^3$.
- Simplify: $\left(\frac{k(k+1)}{2}\right)^2 + (k + 1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$ (details omitted for brevity).
- True for $n = k + 1$. By induction, true for all $n \geq 1$. $\square$

6 Conclusion

In this document, we have explored various methods of mathematical proofs, providing clear, step-by-step examples of each method with increasing levels of difficulty. These examples illustrate the power and utility of mathematical reasoning in establishing truths across diverse domains.