Comprehensive Guide to Series and Convergence Tests

Introduction to Series

A series is the sum of the terms of a sequence $\{a_n\}$:

$$S = \sum_{n=1}^{\infty} a_n.$$

Key questions about a series are whether it converges and under what conditions. This document covers the fundamental concepts, conditions, and various convergence tests.

1 Necessary Condition for Convergence

A necessary condition for the convergence of a series $\sum a_n$ is:

$$\lim_{n\to\infty} a_n = 0.$$

If this condition is not met, the series diverges. Note that this condition is not sufficient for convergence.

How to Apply

1. Take the general term a_n of the series.
2. Compute $\lim_{n\to\infty} a_n$.
3. If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.
4. If $\lim_{n\to\infty} a_n = 0$, further tests are needed to check convergence.

Examples

1. $a_n = \frac{1}{n}$:
$$\lim_{n\to\infty} a_n = 0,$$
but $\sum \frac{1}{n}$ diverges (harmonic series).
2. $a_n = \frac{1}{n^2}$:
$$\lim_{n\to\infty} a_n = 0,$$
and $\sum \frac{1}{n^2}$ converges (p-series with $p > 1$).
3. $a_n = \frac{\sin(n)}{n}$:
$$\lim_{n\to\infty} a_n = 0,$$
but additional tests are required to determine convergence.

2 Absolutely and Conditionally Convergent Series

Absolute convergence: $\sum |a_n|$ converges.
Conditional convergence: $\sum a_n$ converges, but $\sum |a_n|$ diverges.

How to Apply

1. Compute $\sum |a_n|$: Replace all terms a_n with their absolute values.
2. If $\sum |a_n|$ converges, the series is absolutely convergent.
3. If $\sum a_n$ converges but $\sum |a_n|$ diverges, the series is conditionally convergent.

Examples

1. $a_n = \frac{(-1)^n}{n}$: $\sum a_n$ converges conditionally (alternating harmonic series).
2. $a_n = \frac{(-1)^n}{n^2}$: $\sum a_n$ converges absolutely (p-series with $p > 1$).
3. $a_n = \frac{(-1)^n}{\sqrt{n}}$: $\sum a_n$ converges conditionally.

3 Convergence Tests

3.1 Direct Comparison Test

This test determines the convergence or divergence of a series by comparing it to a known benchmark series. For a series $\sum a_n$:

- If $0 \leq a_n \leq b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ also converges.
- If $a_n \geq b_n \geq 0$ for all n and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

How to Apply

1. Identify a comparison series $\sum b_n$ that is simpler and whose convergence is known.
2. Check that $a_n \leq b_n$ for all n.
3. Determine whether $\sum b_n$ converges or diverges.
4. Conclude the same for $\sum a_n$.

Examples

1. $a_n = \frac{1}{n^2+1}$: Compare to $b_n = \frac{1}{n^2}$:

$$0 \leq \frac{1}{n^2+1} \leq \frac{1}{n^2}.$$

   Since $\sum \frac{1}{n^2}$ converges, $\sum \frac{1}{n^2+1}$ also converges.

2. $a_n = \frac{1}{n}$: Compare to $b_n = \frac{1}{\sqrt{n}}$:

$$\frac{1}{n} \geq \frac{1}{\sqrt{n}} > 0.$$

Since $\sum \frac{1}{\sqrt{n}}$ diverges, $\sum \frac{1}{n}$ also diverges.

3. $a_n = \frac{\ln(n)}{n^2}$: Compare to $b_n = \frac{1}{n^2}$:

$$\frac{\ln(n)}{n^2} \leq \frac{1}{n^2},$$

and since $\sum \frac{1}{n^2}$ converges, $\sum \frac{\ln(n)}{n^2}$ also converges.

3.2 Ratio Test (d'Alembert's Criterion)

This test examines the ratio of consecutive terms:

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

- If $L < 1$, the series converges absolutely.
- If $L > 1$ or $L = \infty$, the series diverges.
- If $L = 1$, the test is inconclusive.

How to Apply

1. Compute $\frac{a_{n+1}}{a_n}$ for the general term.
2. Take the limit $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$.
3. If $L < 1$, conclude absolute convergence. If $L > 1$, conclude divergence.
4. If $L = 1$, use another test.

Examples

1. $a_n = \frac{1}{n!}$:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 < 1,$$

hence the series converges absolutely.

2. $a_n = \frac{1}{2^n}$:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2} < 1,$$

hence the series converges absolutely.

3. $a_n = \frac{n}{n^2+1}$:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1,$$

test is inconclusive.

3.3 Alternating Series Test (Leibniz's Criterion)

This test applies to series of the form $\sum(-1)^n a_n$, where a_n are positive terms.

- The terms $|a_n|$ must decrease monotonically.
- $\lim_{n\to\infty} a_n = 0$.

How to Apply

1. Verify that $|a_n|$ is decreasing for all n.
2. Check that $\lim_{n\to\infty} a_n = 0$.
3. If both conditions are met, conclude convergence.

Examples

1. $a_n = \frac{1}{n}$: Alternating harmonic series.
2. $a_n = \frac{1}{n^2}$: Converges absolutely.
3. $a_n = \frac{1}{\sqrt{n}}$: Converges conditionally.

3.4 Root Test (Cauchy's Criterion)

This test examines the limit of the n-th root of the terms of the series:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

- If $L < 1$, the series converges absolutely.
- If $L > 1$, the series diverges.
- If $L = 1$, the test is inconclusive.

How to Apply

1. Compute $\sqrt[n]{|a_n|}$ for the general term.
2. Take the limit $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$.
3. If $L < 1$, conclude absolute convergence. If $L > 1$, conclude divergence.
4. If $L = 1$, use another test.

Examples

1. $a_n = \frac{1}{2^n}$:
$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{2} < 1,$$
hence the series converges absolutely.

2. $a_n = \frac{1}{n}$:
$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1,$$
test is inconclusive.

3. $a_n = \frac{1}{n^n}$:
$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0 < 1,$$
hence the series converges absolutely.

3.5 Raabe's Test

This test refines the ratio test by considering:

$$R = n\left(\frac{a_n}{a_{n+1}} - 1\right).$$

- If $R > 1$, the series converges absolutely.
- If $R < 1$, the series diverges.
- If $R = 1$, the test is inconclusive.

How to Apply

1. Compute $\frac{a_n}{a_{n+1}} - 1$ for the general term.
2. Multiply by n and simplify.
3. Take the limit $\lim_{n\to\infty} R$.
4. If $R > 1$, conclude absolute convergence. If $R < 1$, conclude divergence.

Examples

1. $a_n = \frac{1}{n!}$: Converges absolutely.
2. $a_n = \frac{1}{n^2}$: Converges absolutely.
3. $a_n = \frac{1}{\ln(n)n}$: Diverges.

3.6 Cauchy Condensation Test

This test applies to positive, decreasing sequences a_n:

$$\sum a_n \text{ converges if and only if } \sum 2^n a_{2^n} \text{ converges.}$$

How to Apply

1. Identify the sequence a_n and compute $2^n a_{2^n}$.
2. Determine whether the new series $\sum 2^n a_{2^n}$ converges or diverges.
3. Conclude the same for $\sum a_n$.

Examples

1. $a_n = \frac{1}{n^2}$: Condensation yields $\sum \frac{1}{2^n}$, which converges.
2. $a_n = \frac{1}{n}$: Condensation yields $\sum 1$, which diverges.
3. $a_n = \frac{1}{n \ln(n)}$: Condensation shows divergence.

3.7 Integral Test

The integral test compares the series $\sum a_n$ with the improper integral $\int_1^\infty f(x)dx$:

- If $f(x)$ is positive, continuous, and decreasing for $x \geq 1$, then the convergence of $\sum a_n$ and $\int_1^\infty f(x)dx$ is the same.

How to Apply

1. Define $f(x) = a_n$ and verify that $f(x)$ is positive, continuous, and decreasing for $x \geq 1$.
2. Compute $\int_1^\infty f(x)dx$.
3. If the integral converges, conclude that $\sum a_n$ converges. If it diverges, conclude that $\sum a_n$ diverges.

Examples

1. $a_n = \frac{1}{n^2}$:
$$\int_1^\infty \frac{1}{x^2}dx = 1,$$
hence the series converges.
2. $a_n = \frac{1}{n}$: Harmonic series diverges.
3. $a_n = \frac{1}{n \ln(n)}$: Diverges.

4 Conclusion

This document provides a detailed overview of series and convergence tests, with examples ranging from basic to advanced. Mastering these concepts and methods is essential for understanding the behavior of series in mathematical analysis.