HW 10

5.1.3 The random variables $X_1, \ldots, X_n$ have the joint PDF

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \begin{cases} 1 & 0 \le x_i \le 1; \\ & i = 1,\ldots,n, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the joint CDF, $F_{X_1,\ldots,X_n}(x_1,\ldots,x_n)$?

(b) For $n = 3$, what is the probability that $\min_i X_i \le 3/4$?

(a) เราสามารถเขียน Joint CDF ในเทอม Joint PDF ดังนี้

$$F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{X_1,\ldots,X_n}(y_1,\ldots,y_n)\, dy_1 \ldots dy_n$$

เนื่องจาก Joint PDF $f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = 1$ ที่ $0 \le x_i \le 1;\ i = 1,\ldots,n$

นอกเหนือจากกรณีนี้ จะเป็น Joint PDF จะเป็น 0

$$;\ F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \int_0^{\min(1,x_1)} \cdots \int_0^{\min(1,x_n)} 1\ dy_1 \ldots dy_n$$

$$= \min(1,x_1) \times \min(1,x_2) \times \ldots \times \min(1,x_n)$$

$$\therefore\ F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \begin{cases} \prod_{i=1}^{n} & , 0 \le x_i;\ i = 1,2,\ldots,n \\ 0 & , \text{otherwise} \end{cases}$$

(b) For $n = 3$, $Prob[X_i \le 3/4]$ $\quad 0 \le x_i \le 1$

$$P[\min X_i \le 3/4] = 1 - P[\min X_i > 3/4]$$

$$= 1 - P[X_1 > 3/4, X_2 > 3/4, X_3 > 3/4]$$

$$= 1 - \int_{3/4}^{1} \int_{3/4}^{1} \int_{3/4}^{1} dx_1\, dx_2\, dx_3$$

$$\therefore P[\min X_i \le 3/4] = \frac{63}{64}$$ ※

5.4.4 As in Example 5.4, the random vector **X** has PDF

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} 6e^{-\mathbf{a}'\mathbf{x}} & \mathbf{x} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathbf{a} = [1 \quad 2 \quad 3]'$. Are the components of **X** independent random variables?

$$\text{PDF } f_X(x) = \begin{cases} 6e^{-(x_1+2x_2+3x_3)} & , x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

หา Marginal PDFs $f_{X_i}(x_i)$ สำหรับ $x_1 \geq 0$,

$$f_{X_1}(x_1) = \int_0^\infty \int_0^\infty f_X(x)\, dx_2\, dx_3$$

$$= 6e^{-x_1} \int_0^\infty \int_0^\infty e^{-2x_2} \cdot e^{-3x_3}\, dx_2\, dx_3$$

$$= 6e^{-x_1} \lim_{b\to\infty} \frac{e^{-2x_2}}{-2}\Big|_0^b \times \lim_{c\to\infty} \frac{e^{-3x_3}}{-3}\Big|_0^c$$

$$= \cancel{6}e^{-x_1} \times \frac{1}{\cancel{-2}} \times \frac{1}{\cancel{-3}} \times \left[\cancelto{0}{\frac{1}{e^{2b}}} - 1\right]\left[\cancelto{0}{\frac{1}{e^{3c}}} - 1\right]$$

$$= e^{-x_1} \quad , \quad x_1 \geq 0$$

$$\therefore \quad f_{X_1}(x) = \begin{cases} e^{-x_1} & , x_1 \geq 0 \\ 0 & , \text{otherwise} \end{cases} \quad \text{※}$$

พิจารณา $f_{X_2}(x_2)$ & $f_{X_3}(x_3)$ เช่นเดียวกับ $f_{X_1}(x_1)$;

$$f_{X_2}(x) = 6e^{-2x_2} \int_0^\infty \int_0^\infty e^{-x_1} \cdot e^{-3x_3}\, dx_1\, dx_3$$

$$= 6e^{-2x_2} \times \frac{1}{-1} \times \frac{1}{-3} \times \left[\frac{1}{e^{x}}\right]_0^\infty \times \left[\frac{1}{e^{3x}}\right]_0^\infty$$

$$= 2e^{-2x_2} \quad ; \quad x_2 \geq 0$$

$$\therefore f_{X_2}(x) = \begin{cases} 2e^{-2x_2} & , x_2 \geq 0 \\ 0 & , \text{otherwise} \end{cases} \quad \text{※}$$

$$f_{X_3}(x_3) = 6e^{-3x_3} \int_0^\infty \int_0^\infty e^{-x_1} \cdot e^{-2x_2}\, dx_1\, dx_2$$

$$= 6e^{-3x_2} \times \frac{1}{-1} \times \frac{1}{-2} \times \left[\frac{1}{e^x}\right]_0^\infty \times \left[\frac{1}{e^{2x}}\right]_0^\infty$$

$$f_{X_3}(x_3) = 3e^{-3x_2} \quad ; \quad x_3 \geq 0$$

$$\therefore \quad f_{X_3}(x_3) = \begin{cases} 3e^{-3x_2} & , \quad x_3 \geq 0 \\ 0 & , \quad \text{otherwise} \end{cases}$$

5.5.5 In a weekly lottery, each \$1 ticket sold adds 50 cents to the jackpot that starts at \$1 million before any tickets are sold. The jackpot is announced each morning to encourage people to play. On the morning of the ith day before the drawing, the current value of the jackpot J_i is announced. On that day, the number of tickets sold, N_i, is a Poisson random variable with expected value J_i. Thus six days before the drawing, the morning jackpot starts at \$1 million and N_6 tickets are sold that day. On the day of the drawing, the announced jackpot is J_0 dollars and N_0 tickets are sold before the evening drawing.

What are the expected value and variance of J, the value of the jackpot the instant before the drawing? Hint: Use conditional expectations.

5.6.4 The 4-dimensional random vector $\mathbf{X}$ has PDF

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} 1 & 0 \le x_i \le 1, i = 1, 2, 3, 4 \\ 0 & \text{otherwise.} \end{cases} \rightarrow \text{uniform RV มีค่าระหว่าง } (0,1)$$

Find the expected value vector $E[\mathbf{X}]$, the correlation matrix $\mathbf{R_X}$, and the covariance matrix $\mathbf{C_X}$.

Marginal

$$\Rightarrow \quad f_X(x) = f_{X_1}(x_1)\, f_{X_2}(x_2)\, f_{X_3}(x_3)\, f_{X_4}(x_4)$$

$$E[X_i] = \frac{1-0}{2} = \frac{1}{2}, \quad Var[X_i] = \frac{(0-1)^2}{12} = \frac{1}{12}$$

กำหนดให้ X_i และ X_j มี Correlation $i \times j \rightarrow$ เมทริกซ์

$$E[X_i X_j] = E[X_i]\, E[X_j]$$

1) Expected Value

$$E[X] = [\, E[X_1] \quad E[X_2] \quad E[X_3] \quad E[X_4] \,]'$$

$$= [\, 1/2 \quad 1/2 \quad 1/2 \quad 1/2 \,]' \rightarrow \text{Uniform RV} \quad \#$$

2) Correlation matrix

$$R_X = E[XX'] = \begin{bmatrix} E[X_1^2] & E[X_1X_2] & E[X_1X_3] & E[X_1X_4] \\ E[X_2X_1] & E[X_2^2] & E[X_2X_3] & E[X_2X_4] \\ E[X_3X_1] & E[X_3X_2] & E[X_3^2] & E[X_3X_4] \\ E[X_4X_1] & E[X_4X_2] & E[X_4X_3] & E[X_4^2] \end{bmatrix}$$

$$\therefore \quad R_X = \begin{bmatrix} 1/3 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/3 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/3 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/3 \end{bmatrix} \quad \#$$

3) $$Cov[X] = [Cov[X_i, X_j]]_{4\times 4}$$

$$\therefore \quad Cov[X] = \begin{bmatrix} 1/12 & 0 & 0 & 0 \\ 0 & 1/12 & 0 & 0 \\ 0 & 0 & 1/12 & 0 \\ 0 & 0 & 0 & 1/12 \end{bmatrix} \quad \#$$

5.7.4 Let **X** be a Gaussian random vector with expected value $[\mu_1 \quad \mu_2]'$ and covariance matrix

$$C_X = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

Show that **X** has PDF $f_X(x) = f_{X_1,X_2}(x_1, x_2)$ given by the bivariate Gaussian PDF of Definition 4.17.

Gaussian Random vector

$$f_X(x) = \frac{1}{2\pi[\det[C_X]]^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_X)' C_X^{-1} (x-\mu_X)\right)$$

โดยที่

$$\det[C_X] = \sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2$$

$$\det[C_X] = \sigma_1^2\sigma_2^2(1-\rho^2)$$

หา Inverse Matrix of C_X

$$C_X^{-1} = \frac{1}{\det[C]} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_2\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}$$

$$\therefore -\frac{1}{2}(x-\mu_X)' C_X^{-1} (x-\mu_X) = \frac{[x_1-\mu_1 \quad x_2-\mu_2]\left[\frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\right]\begin{bmatrix} \sigma_2^2 & -\rho\sigma_2\sigma_1 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}}{2}$$

$$= \frac{[x_1-\mu_1 \quad x_2-\mu_2]\begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix}\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}}{2(1-\rho^2)}$$

$$= \frac{[x_1-\mu_1 \quad x_2-\mu_2]\begin{bmatrix} \frac{x_1-\mu}{\sigma_1^2} & \frac{-\rho(x_2-\mu_2}{\sigma_1\sigma_2} \\ \frac{-\rho(x_1-\mu_1}{\sigma_1\sigma_2} & \frac{x_2-\mu_2}{\sigma_2^2} \end{bmatrix}}{2(1-\rho^2)}$$

$$\therefore -\frac{1}{2}(x-\mu_X)' C_X^{-1}(x-\mu_X) = \frac{-\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}}{2(1-\rho^2)}$$

$$\therefore f_X(x) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[\frac{-\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}}{2(1-\rho^2)}\right]$$

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