Mixed Random Variable

- Discrete RV → PMF & Summation
- Continuous RV → PDF & Integral
- Combination of Discrete and Continuous RV
  - →**Unit impulse function**
  - →**Can use same formulas to describe both RVs**

Unit Impulse Function

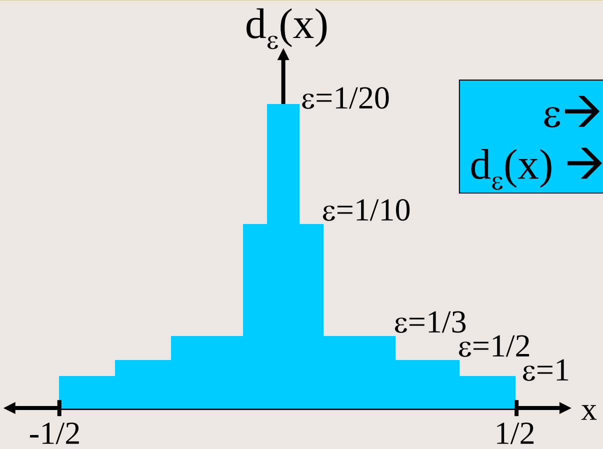
- Delta Function : $\delta(x)$

Definition:

Let $$d_\varepsilon(x) = \begin{cases} 1/\varepsilon & -\varepsilon/2 \le x \le \varepsilon/2 \\ 0 & \text{Otherwise} \end{cases}$$

Then $$\delta(x) = \lim_{\varepsilon \to 0} d_\varepsilon(x)$$

Delta Function



$\varepsilon \to 0$

$d_\varepsilon(x) \to \delta(x)$

No mathematically meaning **but very useful**

Delta Function

$$\int_{-\infty}^{\infty} d_\varepsilon(x)\,dx = \int_{-\varepsilon/2}^{\varepsilon/2} \frac{1}{\varepsilon}\,dx = 1$$

As $\varepsilon \to 0$, $d_\varepsilon(x) \to \delta(x)$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

Special case of

1 0

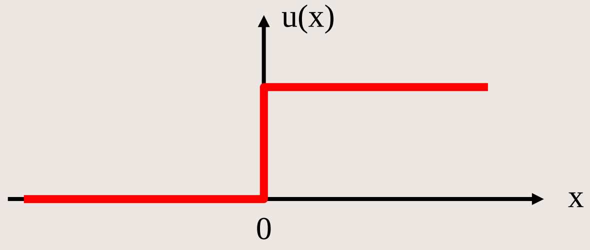
Theorem: (Sifting Property)

$$\int_{-\infty}^{\infty} g(x)\,\delta(x-x_0)\,dx = g(x_0)$$

Unit Step Function

Definition:

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$



Unit Step Function

$$\int_{-\infty}^{-x} d_\varepsilon(v)\, dv = 0 \qquad \int_{-\infty}^{x} d_\varepsilon(v)\, dv = 1, \quad x > 0$$

For $x \neq 0$, $\varepsilon \to 0$

$$\int_{-\infty}^{x} d_\varepsilon(v)\, dv = u(x)$$

Theorem:

$$\int_{-\infty}^{x} \delta(v)\, dv = u(x)$$

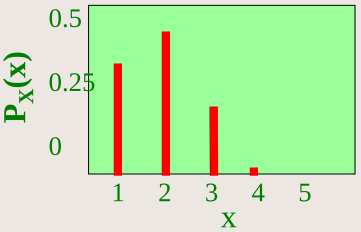
For x = 0 ??

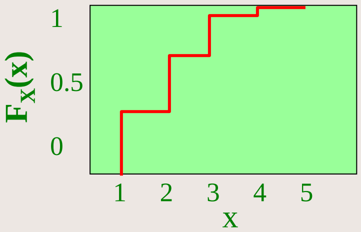
Not exist

$$\delta(x) = \frac{d\,u(x)}{dx}$$

PMF → PDF

$$F_X(x) = \sum_{x_i \in S_X} P_X(x_i)\, u(x-x_i)$$

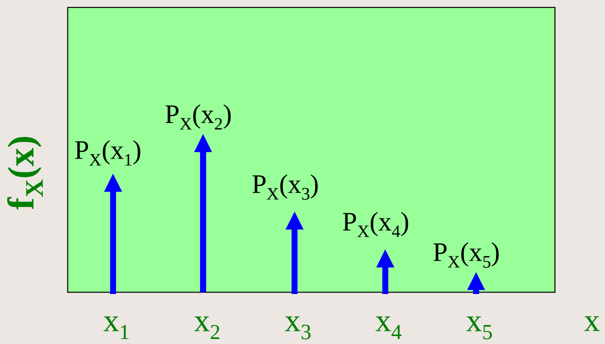




$u(x-x_i)$ → $u(x)$ shift to x_i

PMF → PDF

$$f_X(x) = \sum_{x_i \in S_X} P_X(x_i)\, \delta(x-x_i)$$



PMF → PDF

$$f_X(x) = \sum_{x_i \in S_X} P_X(x_i)\, \delta(x-x_i)$$

$$E[X] = \int_{\infty}^{-\infty} x \sum_{x_i \in S_X} P_X(x_i)\, \delta(x-x_i)\, dx$$

$$= \sum_{x_i \in S_X} \int_{\infty}^{-\infty} x\, P_X(x_i)\, \delta(x-x_i)\, dx$$

$$= \sum_{x_i \in S_X} x_i\, P_X(x_i)$$

PMF ←→ PDF

Theorem :

- $P[X = x_0] = q$
- $P_X(x_0) = q$
- $F_X(x_0^+) - F_X(x_0^-) = q$ Discontinuity at x_0
- $f_X(x_0) = q\, \delta(0)$

Mixed Random Variable

Definition: X is a mixed RV Iff

$\mathbf{f_x(x)}$ = both impulses and nonzero, finite values

Example

- Observe the period of telephone call
  - 1/3 of calls : never begin (no answer/busy)
  - For the success call, with probability of 2/3, call is uniformly [0,3]
- Find PDF, CDF and Mean of call holding time

Example

- Y: call holding time
- A: phone was answered → A^c: not answered
- $0 \le y \le 3$
- $F_Y(y) = P[Y \le y]$

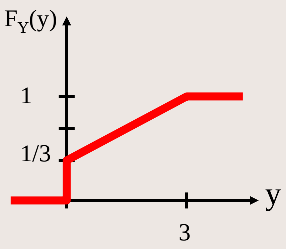
$$= \underbrace{P[Y\le y|A^c]}_{1}\,\underbrace{P[A^c]}_{1/3} + \underbrace{P[Y\le y|A]}_{y/3}\,\underbrace{P[A]}_{2/3}$$

$$= (1)(1/3) + (y/3)(2/3)$$

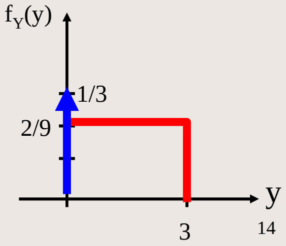
$$= 1/3 + 2y/9$$

Example

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 1/3 + 2y/9 & 0 \le y \le 3 \\ 1 & Y >= 3 \end{cases}$$



$$f_Y(y) = \begin{cases} \delta(y)/3 + 2/9 & 0 \le y \le 3 \\ 0 & \text{Otherwise} \end{cases}$$



Example

$$E[Y] = \int_{-\infty}^{\infty} y\,(1/3)\delta(y)\,dy + \int_{0}^{3} y\,(2/9)\,dy$$

$$= 0 + 1 \quad = 1$$

Derived Random Variable

Y = aX, a > 0

$$F_Y(y) = P[aX \le y] = P[X \le y/a] = F_X(y/a)$$

$$f_Y(y) = \frac{d\,F_Y(y)}{dy} = (1/a)\,f_X(y/a)$$

Theorem :

- $F_Y(y) = F_X(y/a)$
- $f_Y(y) = (1/a)\,f_X(y/a)$

Derive Random Variable

$Y = X + b$

$$F_Y(y) = P[X + b \le y] = P[X \le y - b] = F_X(y - b)$$

$$f_Y(y) = \frac{d\,F_Y(y)}{dy} = f_X(y - b)$$

Theorem :

- $F_Y(y) = F_X(y - b)$
- $f_Y(y) = f_X(y - b)$

Conditioning a continuous RV

$$P[A|B] = P[AB] / P[B]$$

$$P[x_1 < X \le x_2] = \int_{x_1}^{x_2} f_X(x)\,dx$$

Approx: $P[x < X \le x+dx] = f_X(x)\,dx$

$$f_{X|B}(x)\,dx = P[x < X \le x+dx\,|B] = \frac{P[x < X \le x+dx\,,B]}{P[B]}$$

$$= \frac{P[x < X \le x+dx]}{P[B]} \qquad \leftarrow x \in B,\ x+dx \in B$$

$$= \frac{f_X(x)\,dx}{P[B]}$$

Conditioning a continuous RV

$$f_{X|B}(x)\,\cancel{dx} = \frac{f_X(x)\,\cancel{dx}}{P[B]}$$

Definition:

$$f_{X|B}(x) = \begin{cases} \dfrac{f_X(x)}{P[B]} & x \in B \\ 0 & \text{Otherwise} \end{cases}$$

Definition:

$$E[g(X)|B] = \int_{-\infty}^{\infty} g(x)\, f_{X|B}(x)\, dx$$

Example

- Observe the period of telephone call (T) is an **exponential RV** with expected value 3 min.
- Find E[T|T>2]
- **Solution**:

$$f_T(t) = \begin{cases} (1/3)\, e^{-t/3} & t \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

$$P[T > 2] = \int_2^{\infty} f_T(t)\, dt = e^{-2/3}$$

Example

$$f_{T|T>2}(t) = \begin{cases} f_T(t) \,/\, P[T > 2] & t \geq 2 \\ 0 & \text{Otherwise} \end{cases}$$

$$= \begin{cases} (1/3)\, e^{-(t-2)/3} & t \geq 2 \\ 0 & \text{Otherwise} \end{cases}$$

$$E[T \mid T > 2] = \int_2^{\infty} t\,(1/3)\, e^{-(t-2)/3}\, dx$$

$= 5$ min.

Homework

- 3.1.2
- 3.2.2
- 3.3.6
- 3.4.6
- 3.5.5
- 3.6.8
- 3.7.6
- 3.8.3