HW07 (4.2.1 - 4.2.8)

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Commu Eng Math.

4.2.1 Random variables X and Y have the joint PMF

$$P_{X,Y}(x,y) = \begin{cases} cxy & x = 1, 2, 4; \quad y = 1, 3, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of the constant c?

(b) What is $P[Y < X]$?

(c) What is $P[Y > X]$?

(d) What is $P[Y = X]$?

(e) What is $P[Y = 3]$?

a) From Theorem : $\sum_{x \in S_X} \sum_{y \in S_Y} P_{X,Y}(x,y) = 1$

พิจารณา $\sum_{x=1,2,4} \sum_{y=1,3} cxy = 1$

$$c \sum_{x=1,2,4} \sum_{y=1,3} xy = 1$$

$$c[\underbrace{1}_{x}(\overbrace{1+3}^{y}) + \underbrace{2}_{x}(\overbrace{1+3}^{y}) + \underbrace{4}_{x}(\overbrace{1+3}^{y})] = 1$$

$$\therefore c = \frac{1}{28}$$

b)

$$P[Y<X] = \sum_{x \in S_X} \sum_{y<x} P_{X,Y}(x,y)$$

$$= \sum_{x \in 1,2,4} \sum_{y<x} \left(\frac{1}{28}\right) xy$$

$$= \frac{1}{28}\left[P_{X,Y}(2,1) + P_{X,Y}(4,1) + P_{X,Y}(4,3)\right]$$

$$= \frac{1}{28}\left[2 + 4 + 12\right]$$

$$\therefore P[Y<X] = \frac{18}{28}$$

c) $P[Y > X] = \sum_{x \in S_X} \sum_{y > x} cxy$

$= \frac{1}{28} \sum_{x=1,2,4} \sum_{y>x} xy$

$= \frac{1}{28} \left[P_{X,Y}(1,3) + P_{X,Y}(2,3) \right]$

$= \frac{1}{28} [3 + 6]$

$\therefore P[Y>X] = \frac{9}{28}$ $\cancel{\times}$

d) $P[Y = X] = \sum_{x \in S_X} \sum_{y = x} cxy$

$= \frac{1}{28} \sum_{x=1,2,4} \sum_{y=x} xy$

$= \frac{1}{28} \left[P_{X,Y}(1,1) \right]$

$\therefore P[Y=X] = \frac{1}{28}$

e) $P[Y=3] = \sum_{x \in S_X} \sum_{y=3} cxy$

$= \frac{1}{28} \sum_{x=1,2,4} \sum_{y=3} xy$

$= \frac{1}{28} \left[P_{X,Y}(1,3) + P_{X,Y}(2,3) + P_{X,Y}(4,3) \right]$

$\therefore P[Y=3] = \frac{21}{28}$ $\cancel{\times}$

4.2.2 Random variables X and Y have the joint PMF

$$P_{X,Y}(x,y) = \begin{cases} c|x+y| & x=-2,0,2; \\ & y=-1,0,1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of the constant c?

(b) What is $P[Y < X]$?

(c) What is $P[Y > X]$?

(d) What is $P[Y = X]$?

(e) What is $P[X < 1]$?

a) From Theorem : $\displaystyle\sum_{x\in S_X}\sum_{y\in S_Y} P_{X,Y}(x,y) = 1$

$$** \quad |x+y| = \begin{cases} x+y & , \ x+y \geq 0 \\ -(x+y) & , \ x+y < 0 \end{cases} \quad **$$

พิจารณา $\displaystyle\sum_{x=-2,0,2}\ \sum_{y=-1,0,1} c|x+y| = 1$

$$c\Big[\overset{3}{-(-2-1)} \overset{2}{-(-2+0)} \overset{1}{-(-2+1)} \overset{1}{-(0-1)} \overset{0}{+0} \overset{1}{+(0+1)} \overset{1}{+(2-1)} \overset{2}{+(2+0)} \overset{3}{+(2+1)}\Big] = 1$$

$$\therefore \ c = \frac{1}{14} \quad \#$$

b) $\displaystyle P[Y<X] = \sum_{x\in S_X}\sum_{y<x} c|x+y|$

$$= \frac{1}{14}\sum_{x=-2,0,2}\ \sum_{y<x} |x+y|$$

$$= \frac{1}{14}\left[P_{X,Y}(0,-1) + P_{X,Y}(2,1) + P_{X,Y}(2,0) + P_{X,Y}(2,-1)\right]$$

$$= \frac{1}{14}\left[1+3+2+1\right]$$

$$\therefore \ P[Y<X] = \frac{7}{14} \quad \#$$

c) $P[Y>X] = \sum_{x=-2,0,2} \sum_{y>x} c|x+y|$

$= \frac{1}{14}\left[P_{X,Y}(-2,-1) + P_{X,Y}(-2,0) + P_{X,Y}(-2,1) + P_{X,Y}(0,1) \right]$

$= \frac{1}{14}[3+2+1+1]$ $\qquad \because P[Y>X] = 1 - P[Y<X]$

$\therefore P[Y>X] = \frac{7}{14}$ ※

d) $P[Y=X] = \sum_{x=-2,0,2} \sum_{y=x} c|x+y|$

$= \frac{1}{14} P_{X,Y}(0,0)$

$\therefore P[Y=X] = 0$ ※

e) $P[X<1] = \sum_{x=-2,0} \sum_{y=-1,0,1} c|x+y|$

$= \frac{1}{14}\left[P_{X,Y}(-2,-1) + P_{X,Y}(-2,0) + P_{X,Y}(-2,1) + P_{X,Y}(0,-1) + P_{X,Y}(0,0) + P_{X,Y}(0,1) \right]$

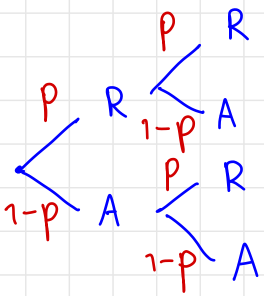
$= \frac{1}{14}\left[3+2+1+1+0+1 \right]$

$\therefore P[X<1] = \frac{8}{14}$ ※

4.2.3 Test two integrated circuits. In each test, the probability of rejecting the circuit is p. Let X be the number of rejects (either 0 or 1) in the first test and let Y be the number of rejects in the second test. Find the joint PMF $P_{X,Y}(x,y)$.

ให้ R แทน เหตุการณ์ที่ทดสอบวงจรแล้ว Rejected

A แทน เหตุการณ์ที่ทดสอบวงจรแล้ว Accepted



p^2 $X = 1,\ Y = 1$

$p(1-p)$ $X = 1,\ Y = 0$

$(1-p)p$ $X = 0,\ Y = 1$

$(1-p)^2$ $X = 0,\ Y = 0$

ดังนั้น Joint PMF $P_{X,Y}(x,y)$ คือ

$$P_{X,Y}(x,y) = \begin{cases} p^2 & x=1,\ y=1 \\ p(1-p) & x=1,\ y=0 \\ p(1-p) & x=0,\ y=1 \\ (1-p)^2 & x=0,\ y=0 \\ 0 & \text{otherwise} \end{cases}$$

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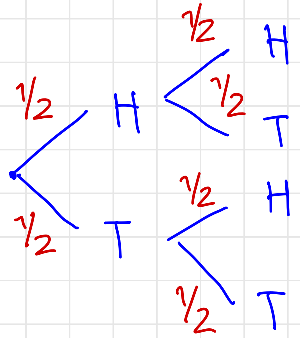
4.2.4 For two flips of a fair coin, let X equal the total number of tails and let Y equal the number of heads on the last flip. Find the joint PMF $P_{X,Y}(x, y)$.

* fair coin

* two flip

กำหนดให้ H แทน เหตุการณ์ที่โยนเหรียญแล้วออกหัว

T แทน เหตุการณ์ที่โยนเหรียญแล้วออกก้อย.



$\frac{1}{4}$ HH $X = 0, Y = 1$

$\frac{1}{4}$ HT $X = 1, Y = 0$

$\frac{1}{4}$ TH $X = 1, Y = 1$

$\frac{1}{4}$ TT $X = 2, Y = 0$

ดังนั้น Joint PMF $P_{X,Y}(x,y)$ คือ

$$P_{X,Y}(x,y) = \begin{cases} \frac{1}{4} & x=1,\ y=1 \\ \frac{1}{4} & x=0,\ y=1 \\ \frac{1}{4} & x=1,\ y=0 \\ \frac{1}{4} & x=2,\ y=0 \\ 0 & \text{otherwise} \end{cases}$$

4.2.5 In Figure 4.2, the axes of the figures are labeled X and Y because the figures depict possible values of the random variables X and Y. However, the figure at the end of Example 4.1 depicts $P_{X,Y}(x, y)$ on axes labeled with lowercase x and y. Should those axes be labeled with the uppercase X and Y? Hint: Reasonable arguments can be made for both views.

ในกราฟ

ตอบ ไม่จำเป็น เนื่องจาก x, y พิมพ์เล็กเป็นการระบุค่าที่กำหนดด้วยตัวแปร X, Y พิมพ์ใหญ่เช่นกัน เนื่องจาก $x, y \in X, Y$ ตามลำดับ x, y คือสมาชิกใน Set X, Y นั่นเอง

4.2.6 As a generalization of Example 4.1, consider a test of n circuits such that each circuit is acceptable with probability p, independent of the outcome of any other test. Show that the joint PMF of X, the number of acceptable circuits, and Y, the number of acceptable circuits found before observing the first reject, is

$$P_{X,Y}(x, y) = \begin{cases} \binom{n-y-1}{x-y} p^x (1-p)^{n-x} & 0 \le y \le x < n, \\ p^n & x = y = n, \\ 0 & \text{otherwise.} \end{cases}$$

Hint: For $0 \le y \le x < n$, show that

$$\{X = x, Y = y\} = A \cap B \cap C,$$

where

A: The first y tests are acceptable.

B: Test $y + 1$ is a rejection.

C: The remaining $n - y - 1$ tests yield $x - y$ acceptable circuits

ให้ A แทนเหตุการณ์ที่วงจร Accept ครั้งที่ y
B แทน เหตุการณ์ที่วงจร Rejected ครั้งหน้า
C แทน เหตุการณ์ที่วงจรจะ accept ครั้งที่ $x-y$ จากทั้งหมด $n-y-1$ ครั้ง

$$P_{X,Y}(x,y) = P[ABC]$$

$$= P[A]\,P[B]\,P[C]$$

p แทน acceptable

$$= p^y (1-p) \binom{n-y-1}{x-y} p^{x-y} (1-p)^{n-x-1}$$

$$= \binom{n-y-1}{x-y} p^x (1-p)^{n-x}$$

Joint PMF $P_{X,Y}$ คือ

$$\begin{cases} \binom{n-y-1}{x-y} p^x (1-p)^{n-x}, & x < n,\ 0 \le y \le x \\ p^n, & x = y = n \\ 0, & \text{otherwise} \end{cases}$$

4.2.7 Each test of an integrated circuit produces an acceptable circuit with probability p, independent of the outcome of the test of any other circuit. In testing n circuits, let K denote the number of circuits rejected and let X denote the number of acceptable circuits (either 0 or 1) in the last test. Find the joint PMF $P_{K,X}(k,x)$.

ให้ A แทนการ Acceptable ; $P[A] = p$

p = acce

$(1-p)$

จาก Pascal RV ;

$$P_N(n) = \begin{cases} \binom{n-1}{a-1} p^a (1-p)^{n-a} & , n = a, a+1, \ldots \\ 0 \end{cases}$$

เนื่องจาก K แทนจำนวนวงจรที่ rejected
X แทน —น— acceptable

กรณีที่ 1 ถ้าวงจรไม่ acceptable ทุกกรณี $(X=0)$

$$P_{K,X}(k,0) = \binom{n-1}{k-1} (1-p)^k (p)^{n-k} \quad , k = 1, 2, \ldots, n$$

กรณีที่ 2 ถ้าวงจร acceptable 1 ครั้ง $(X = 1, k \leq n-1)$

$$P_{K,X}(k,1) = \binom{n-1}{k} (1-p)^{k-1} p^{n-k-1} \cdot \underbrace{p(1-p)}_{\text{acceptable 1 ครั้ง}} \quad , k = 0, 1, \ldots, n-1$$

จะได้ว่า Joint PMF $P_{K,X}(k,x)$ คือ

$$P_{K,X}(k,x) = \begin{cases} \binom{n-1}{k-1} (1-p)^k p^{n-k} & x=0, k=1,2,\ldots,n \\ \binom{n-1}{k-1} (1-p)^k p^{n-k} & x=1, k=1,2,\ldots,n \\ 0 & \text{otherwise} \end{cases}$$

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4.2.8 Each test of an integrated circuit produces an acceptable circuit with probability p, independent of the outcome of the test of any other circuit. In testing n circuits, let K denote the number of circuits rejected and let X denote the number of acceptable circuits that appear before the first reject is found. Find the joint PMF $P_{K,X}(k,x)$.

ให้ A แทนเหตุการณ์แรก x ตัววงจร Acceptable

B แทนเหตุการณ์ที่วงจรจะ Rejected

C แทนเหตุการณ์ที่การทดสอบจะพบ Rejected

$k-1$ ครั้งจาก $n-1$ ครั้ง → Pascal

→ Independent , $x+k \le n$, $x \ge 0$, $k \ge 0$

$$P_{K,X}(k,x) = P[ABC]$$

$$= P[A]P[B]P[C]$$

$$= p^{x}(1-p)\binom{n-x-1}{k-1}(1-p)^{k-1}p^{n-x-1-(k-1)}$$

$$= \binom{n-x-1}{k-1}(1-p)^{k}p^{n-k}$$

$\therefore$ Joint PMF $P_{K,X}(k,x)$ คือ

$$P_{K,X}(k,x) = \begin{cases} \binom{n-x-1}{k-1}(1-p)^{k}p^{n-k} & , n+k \le n,\ x \ge 0, k \ge 0 \\ 0 & , \text{otherwise} \end{cases}$$