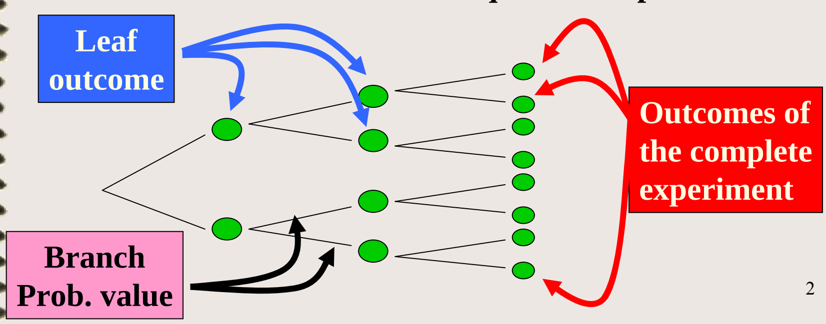
Lecture #3
Introduction to Probability (3)

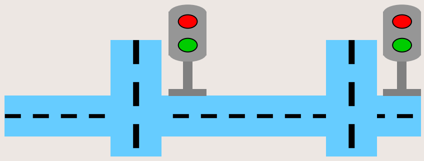
(Sequential Experiment)

Sequential Experiments

- Experiment: in sequence
  sequential → subexperiments
- Each subexp. may depend on the previous one
- Represented by a **Tree Diagram**
- **Model Conditional Prob. → Sequential Experiment**

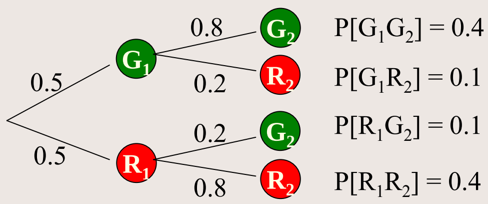


Sequential Example



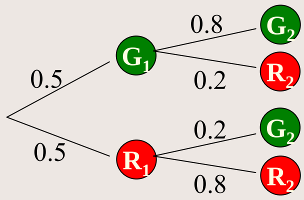
- Timing coordination of 2 traffic lights
  - P[the second light is the same color as the first when the first light is given] = 0.8
  - Assume 1st light is equally likely to be green or red
- Find P[The second light is green] ?
- Find P[wait for at least one red light] ?

Sequential Example



- $P[G_1] = P[R_1] = 0.5$
- $P[G_2G_1] = P[G_2|G_1]P[G_1] = (0.8)(0.5) = 0.4$

Sequential Example



$P[G_1G_2] = 0.4$

$P[G_1R_2] = 0.1$

$P[R_1G_2] = 0.1$

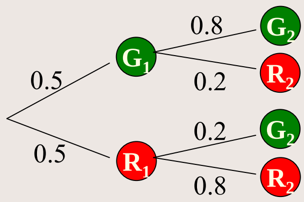
$P[R_1R_2] = 0.4$

P[The second light is green] ?

$$P[G_2] = P[G_2G_1] + P[G_2R_1] \;= 0.4 + 0.1 = 0.5$$

$$P[G_2] = P[G_2|G_1]P[G_1] + P[G_2|R_1]P[R_1]$$

Sequential Example



$P[G_1G_2] = 0.4$

$P[G_1R_2] = 0.1$

$P[R_1G_2] = 0.1$

$P[R_1R_2] = 0.4$

P[wait for at least one red light]?

$$W = G_1R_2 \cup R_1G_2 \cup R_1R_2$$

$$P[W] = P[G_1R_2] + P[R_1G_2] + P[R_1R_2]$$

$$= 0.1 + 0.1 + 0.4 \;= 0.6$$

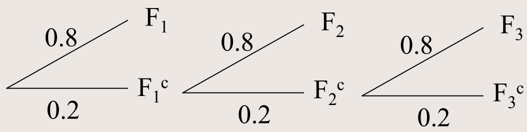
Example

- There are two coins: one bias, one fair
- Coin 1 (C_1) is bias. Prob of head is ¾
- Suppose pick a coin randomly and flip
- Let C_i be the event the coin i^{th} is picked
- Possible outcome is H and T
- What are $P[C_1|H]$ and $P[C_1|T]$?

Quiz 1.7

- Mobile phone must be paged to receive the phone call (the paging system may not be succeeded)
- System must page a phone up to 3 times before giving up
- If a single paging attempt succeeds with probability 0.8
- Sketch the probability tree and
- Find P[F] that the phone is found

Sol: Let *Fi* denote the event that that the user is found on page *i* . The tree for the experiment is



The user is found unless all three paging attempts fail. Thus the probability the user is found is

- $P[F] = 1 - P[F^c_1 F^c_2 F^c_3] = 1 - (0.2)^3 = 0.992$

More example

เครื่องบินโบอิ้ง ประกอบด้วยระบบการทำงาน K1, K2, K3และ K4

ถ้า K1 เสีย ระบบจะทำ K2

ถ้า K2 เสีย ระบบจะทำ K3

ถ้า K3 เสีย ระบบจะทำ K4

ถ้าแต่ละองค์ประกอบทำงานเป็นอิสระต่อกัน และ ความน่าจะเป็นในการเสียของแต่ละองค์ประกอบมีค่าเท่ากับ 0.15 จงหาความน่าจะเป็นที่ระบบจะไม่เสีย

Principle of Counting Method

If experiment A has **n** possible outcomes,
and experiment B has **k** possible outcomes,

→Then there are **nk** possible outcomes when you perform both experiments

Principle of Counting Method

Example: Shuffle a deck and select 3 cards in order.
How many outcomes?

1st draw: select 1 out of 52 → 52 outcomes
2nd draw: select 1 out of 51 (one card has been drawn) → 51 outcomes
3rd draw: select 1 out of 50 → 50 outcomes

Total outcomes = (52)(51)(50)

k-permutations

Theorem:
The number of **k**-permutations
(ordered sequence) of **n** distinguishable objects is

$$(n)_k = n(n-1)(n-2)\ldots(n-k+1)$$
$$= n(n-1)(n-2)\ldots(n-k+1)\frac{(n-k)!}{(n-k)!}$$
$$= \frac{n(n-1)(n-2)\ldots(n-k+1)\,(n-k)(n-k-1)\ldots(1)}{(n-k)!}$$

$$\boxed{(n)_k = \frac{n!}{(n-k)!}}$$

Choose with replacement

Theorem: Given **n** distinguishable objects,
There are n^k ways to choose with replacement
a sample of **k** objects

k-combination

Theorem:

The number of ways to choose **k** objects out of **n** distinguishable objects is

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}$$

Order is irrelevant

2 subexperiments: $\binom{n}{k}$ then $(k)_k$

$\binom{n}{k}.\ (k)_k \quad = \quad (n)_k$

Multinomial Coefficient

Theorem:

For n repetitions of subexperiment with $S = \{s_0, s_1, \ldots, s_{m-1}\}$, the number of length $n = n_0 + n_1 + \ldots + n_{m-1}$ observation sequences with s_i appearing n_i times is

$$\binom{n}{n_0, \ldots, n_{m-1}} = \frac{n!}{n_0! n_1! \ldots n_{m-1}!}$$

Example

- สมมติต้องการเขียนเลขจำนวน ซึ่งแต่ละจำนวนประกอบด้วยตัวเลข 10 ตัวจาก 1,1,2,2,2,3,5,5,6,9

$$\frac{10!}{2!3!2!}$$

- โยนเหรียญ1 อัน 5 ครั้ง จงหาจำนวนผลลัพธ์ที่เป็นไปได้ทั้งหมดที่จะออกหัว 3 ครั้งและออกก้อย 2 ครั้ง

$$\frac{5!}{3!2!}$$

Independent Trials

- Perform repeated trials
- **p** = a success probability
- **(1-p)** = a failure probability
- Each trial is independent
- $\mathbf{S_{k,n}}$ = the event that **k** successes in **n** trials

$$P[S_{k,n}] = \binom{n}{k} p^k (1-p)^{n-k}$$

Independent Trials: Example

- 3 trials, each of success or failure
- 000 001 010 011 100 101 110 111
- How many way to obtain 2 successes out of 3 trials

  $= \binom{n}{k} = 3$
- What is the probability of 2 successes for each way ?
- $p^2 (1-p)$

$$\boxed{P[S_{2,3}] = \binom{3}{2} p^2(1-p)^{3-2}}$$

Independent Trials

<u>Example:</u> In the first round of a programming contest, probability that a program will pass the test is 0.8 . From 10 candidates, what is the probability that x candidates will pass? P[x = 8]?

<u>Solution:</u>

A = {program pass the test}, P[A] = 0.8

Testing a program is an independent trial

$$P[A_{x,10}] = \binom{10}{x} (0.8)^x(1-0.8)^{10-x}$$

$$P[A_{8,10}] = (45)(0.1678)(0.04) = 0.3$$

Example 1.42

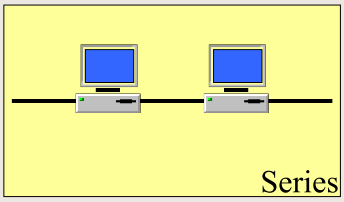
Each call arrives at a telephone switch
Prob of voice call is 7/10
Prob of fax call is 2/10
Prob of data call is 1/10
Observe 100 calls, Find prob of

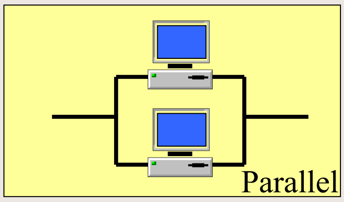
"# voice calls = v times and # fax calls = f times"

Quiz 1.9

- Data packet = 100 bits
- Probability of receiving a bit in error, e = 0.01
- Each error is independent
- If # of error bits <= 3, receiver can correct
- If # of error bits > 3, packet is considered errors
- Find
  - $P[S_{k,100}]$ where k = #of error bits
  - Values of $P[S_{k,100}]$ for k = 0,1,2,3
  - C is event that a packet is decoded correctly. P[C]=?

Independent Trials: Reliability





Let probability that a computer works = p

Series: $P[A] = P[A_1A_2] = p^2$

Parallel: $P[B] = ?$

$$
\begin{aligned}
P[B] &= 1 - P[B^c] \\
&= 1 - P[B_1^cB_2^c] \\
&= 1 - (1 - p)^2
\end{aligned}
$$

Quiz 1.10

- A memory module has 9 chips (work even one of the chips is defective)
- Each chip has n transistors (work if n of them work)
- Each transistor works with probability p
- $P[C]$ = probability of a chip works = ?
- $P[M]$ = probability of a memory module works = ?

Summary

- Probability meaning
- Sample space, Event, Outcome
- Set Theory
- Probability measurement
- Conditional Probability
- Independence
- Sequential experiments → tree diagram
- Counting Methods
- Independent Trials.

Homework #1

1) 1.3.2
2) 1.4.3
3) 1.4.5
4) 1.5.4
5) 1.6.6
6) 1.7.7
7) 1.8.3
8) 1.9.2
9) 1.9.4