Lecture #4
Discrete Random Variable (1)

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Random Variable

Experiment (Physical Model)

→ Compose of procedure & observation

→ From observation, we get outcomes

→ From all outcomes, we get a (mathematical) probability model called "Sample space"

→ From the model, we get P[A], $A \subset S$

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Random Variable

From a probability model

- Ex.: 2 traffic lights, observe the seq. of light

  $S = \{R_1R_2, R_1G_2, G_1R_2, G_1G_2\}$

- If assign a number to each outcome in S, each number that we observe is called "**Random Variable**"
- Observe the number of red lights

  $S_X = \{0,1,2\}$

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Random Variable

$S = \{R_1R_2, R_1G_2, G_1R_2, G_1G_2\}$	$S_X = \{0,1,2\}$
From observation → outcome	From observation (number) → Random variable
	X = name of a Random variable (Cap. Letter)
S = Sample space (Domain of the RV) s = each outcome of S	S_X = Range of X x = each value of X (small Letter)

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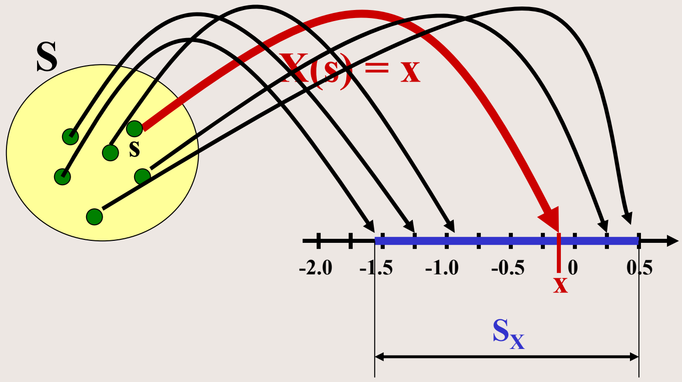
Random Variable

X is a function that maps each outcome, s, in S to a real number X(s), x



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2 types of Random Variable

- Discrete Random Variable

  Example:

  X = # of data packets arrive at a station in a second

  Y = # of shuttle-cocks used in one badminton game

- Continuous Random Variable

  Example:

  Z = The length of a movie in minutes

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Discrete Random Variable

Definition:

- **X** is a **discrete random variable** if the range of **X** is countable

  $S_{\mathbf{x}} = \{x_1,x_2,\ldots\}$

- **X** is a **finite random variable** if all values with nonzero probability are in the finite set

  $S_{\mathbf{x}} = \{x_1,x_2,\ldots,x_n\}$

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Why do we need a Random Variable?

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Random Variable Example

- **Experiment:**

  In our probability class, observe each student's grade
- **Sample space**

  $S = \{F, D, C, C^+, B, B^+, A\}$
- Let **G** be a **finite random variable** to transform the letter grade to the number

  → G maps each letter grade to a value

  G(A) = 4.0 G(B+) = 3.5 G(B) = 3.0 G(C+)= 2.5
  G(C) = 2.0 G(D) = 1.0 G(F) = 0

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Random Variable Example

$S_G = \{ 0, 1.0, 2.0, 2.5, 3.0, 3.5, 4.0 \}$

Why do we need to map the letter grade to the value?

→Calculate the GPA

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Why do we need a Random Variable?

- For a probability model (experiment), the outcome in S can be in arbitrary form
- If we implement a Random Variable, we can calculate the average !
- In Probability, the average is called "**expected value**" of a random variable

Probability Mass Function

- For a (discrete) probability model, **P[A]** = [0,1]
- For a discrete random variable, the probability model is called a "**Probability Mass Function (PMF)**"

Probability Mass Function

$$P_X(x) = P[X=x]$$

PMF of random variable X

Event consisting of all outcomes s in S which X(s) = x

Random variable **X**

x is a dummy variable

$P_X(a)$ or $P_X(z)$ or $P_X(☺)$

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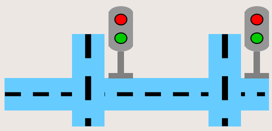
PMF Example



Example:

- There are 2 traffic lights
- Observe the seq. of the lights

  $S = \{ R_1R_2 , R_1G_2 , G_1R_2 , G_1G_2 \}$
- Suppose the outcomes are equally likely.
- **Find PMF of T, the number of red light**

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PMF Example

- T is a random variable of # of red lights

→ Find $P_T(t)$

→ $P_T(t) = P[T = t]$

→ $S_T = \{0,1,2\}$

→ First, find probability for each t

→ Each outcome is equally likely → 1/4

$P[T=0] = P[\{G_1G_2\}] = 1/4$

$P[T=1] = P[\{R_1G_2 , G_1R_2 \}] = 2/4 = 1/2$

$P[T=2] = P[\{R_1R_2\}] = 1/4$

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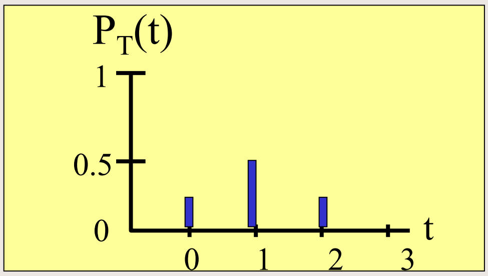
PMF Example

$$P_T(t) = \begin{cases} 1/2, & t = 1 \\ 1/4, & t = 0, 2 \\ 0, & \text{Otherwise} \end{cases}$$



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PMF Theorem

Theorem: For a discrete random variable X with PMF $P_X(x)$ and Range S_X:

1) For any x, $P_X(x) \geq 0$
2) $\sum_{x \in S_X} P_X(x) = 1$
3) For event $B \subset S_X$, the probability P[B] that X is in the set B is

$$\mathbf{P[B] = \sum_{x \in B} P_X(x)}$$

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Random Variables

- From the experiment → probability model
- → Random Variable → PMF
- In practical applications, some random variables frequently appear
- → General forms of Random Variables with only the parameter differences

$$P_T(t) = \begin{cases} 1/2 & t = 1 \\ 1/4 & t = 0, 2 \\ 0 & \text{Otherwise} \end{cases}$$

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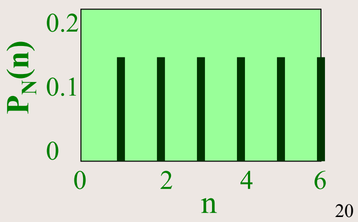
Some Useful Discrete Random Variables

- Discrete Uniform Random Variable
  Bernoulli Random Variable
- Geometric Random Variable
- Binomial Random Variable
- Pascal Random Variable
- Poisson Random Variable

Uniform RV Example

- Roll a fair die
- Let **N** be **the number of spots**
- Find the PMF of **N**

$$P_N(n) = \begin{cases} 1/6 & n = 1,2,3,\ldots,6 \\ 0 & \text{Otherwise} \end{cases}$$



Uniform Random Variable

Definition: X is a Discrete Uniform Random Variable if the PMF of X, $P_X(x)$, has the form:

$$P_X(x) = \begin{cases} 1/(j-k+1), & x = k, k+1, k+2, \ldots, j \\ 0, & \text{otherwise,} \end{cases}$$

where j and k are integer, k < j

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Bernoulli RV Example

- Suppose we test a program, probability that a program fails is 0.2
- Let **Y** be **the number of failed programs in one test**.
- Find the PMF of Y

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Bernoulli RV Example

Solution:

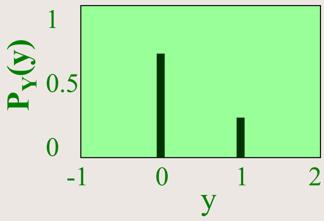
Only 2 outcomes in S

Let p = probability that a program fails

Y = 1 with probability p,

Y = 0 with (1-p)

$$P_Y(y) = \begin{cases} \mathbf{0.8,} & \mathbf{y = 0} \\ \mathbf{0.2,} & \mathbf{y = 1} \\ \mathbf{0,} & \mathbf{otherwise} \end{cases}$$



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Bernoulli Random Variable

Definition: X is a Bernoulli Random Variable if the PMF of X, $P_X(x)$, has the form:

$$P_X(x) = \begin{cases} \mathbf{1 - p,} & \mathbf{x = 0} \\ \mathbf{p,} & \mathbf{x = 1} \\ \mathbf{0,} & \mathbf{otherwise} \end{cases}$$

where p = (0,1)

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Geometric RV Example

- Suppose we test a program, probability that a program fails is 0.2
- Let **Z** be **the number of tests until find a failed program (include the failed one).**
- Find the PMF of Z

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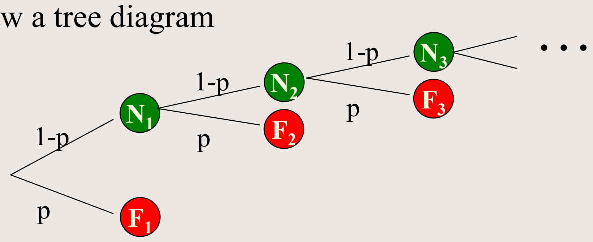
Geometric RV Example

Solution: Let p be probability that program fails

Draw a tree diagram



From the tree,

P[Z=1] = p

P[Z=2] = p(1-p)

P[Z=3] = p(1-p)^2

…

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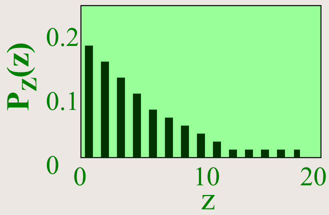
Geometric RV Example

- In general: **$P[Z=z] = p\,(1-p)^{z-1}$**

$$P_Z(z) = \begin{cases} p(1-p)^{z-1}, & z = 1,2,3,\ldots \\ 0, & \text{otherwise} \end{cases}$$

$$P_Z(z) = \begin{cases} 0.2(0.8)^{z-1} & z = 1,2,3,\ldots \\ 0 & \text{Otherwise} \end{cases}$$



Geometric Random Variable

Definition: X is a Geometric Random Variable if the PMF of X, $P_X(x)$, has the form:

$$P_X(x) = \begin{cases} p(1-p)^{x-1}, & x = 1,2,3,\ldots \\ 0, & \text{otherwise} \end{cases}$$

where p = (0,1)

Binomial RV Example

- Suppose we test a program, probability that a program fails is 0.2
- Let **K** be **the number of failed programs in 10 tests.**
- Find the PMF of **K**

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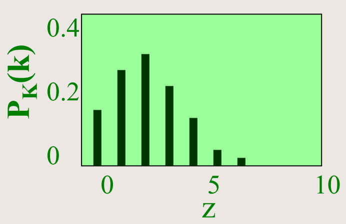
Binomial RV Example

- In general:

$$P_K(k) = \begin{cases} \binom{10}{k} (0.2)^k (0.8)^{10-k}, & k = 0,1,2,\ldots,10 \\ 0, & \text{otherwise} \end{cases}$$



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Binomial Random Variable

Definition: X is a Binomial Random Variable if the PMF of X, $P_X(x)$, has the form:

$$P_X(x) = \begin{cases} \binom{n}{x} p^x(1-p)^{n-x}, & x = 0,1,2,\ldots,n \\ 0, & \text{otherwise} \end{cases}$$

where p = (0,1) and n is an integer that $n \geq 1$

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Pascal RV Example

- Suppose we test a program, probability that a program fails is 0.2
- Let **J** be **the number of tests until k programs fail.**
- Find the PMF of **J**

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Pascal RV Example

- Test until find k failed programs
- A = {The last test (j) is the failed program}
- B = {All tests before the last, there are k–1 failed programs in the j–1 test}
- A and B are independent
- P[A] = p
- $P[B] = \binom{j-1}{k-1} p^{k-1}(1-p)^{(j-1)-(k-1)}$
- P[AB] = P[J = j] = P[A] P[B]

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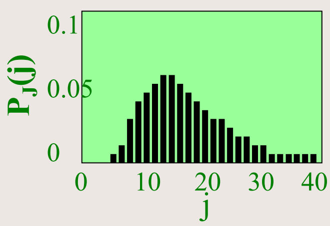
Pascal RV Example

- Finding # of tests until 4 fails

$$P_J(j) = \begin{cases} \binom{j-1}{3} (0.2)^4(0.8)^{j-4}, & j = 4,5,\ldots \\ 0, & \text{otherwise} \end{cases}$$



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Pascal Random Variable

Definition: X is a Pascal Random Variable if the PMF of X, $P_X(x)$, has the form:

$$P_X(x) = \begin{cases} \binom{x-1}{k-1} p^k(1-p)^{x-k}, & x = k, k+1, \ldots \\ 0, & \text{otherwise} \end{cases}$$

where p = (0,1) and k is an integer that $k \geq 1$

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Quiz 2.3

- Each time, modem transmits one bit, the receiver analyzes whether the bit is 0 or 1.
- The transmitted bit is error with Prob = p
  - If transmission until receiving the 1st error
  - p=0.1, P[X=x]=?
  - Transmitting 100 bits, and the number of errors is equal to Y bits
  - p=0.01, P[Y<=2]=?
  - Transmission continues until find 3 errors
  - p=0.25, P[Z=12]=?, where Z is the number of transmitted bits.

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Poisson Random Variable

- Occur randomly in a time period
- Known the average number of occurrences per unit time
- Example:
  - Arrival of packets at each station
  - Initiation of telephone calls
  - Query rate in Search Engine

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Poisson Random Variable

Definition: X is a Poisson Random Variable if the PMF of X, $P_X(x)$, has the form:

$$P_X(x) = \begin{cases} \dfrac{(\lambda T)^x e^{-(\lambda T)}}{x!} & x = 0,1,2\ldots \\ 0 & \text{Otherwise} \end{cases}$$

λ= average arrival rate (number/unit time)
T= time interval

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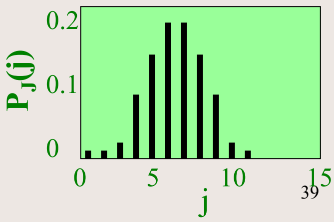
Poisson RV Example

- Call arrive at the telephone office at rate of 0.25 call per second.
- Find the PMF of the number of calls that arrive in any 20 second interval
- **$\lambda T = 0.25 * 20 = 5$**

$$P_J(j) = \begin{cases} \dfrac{(5)^j e^{-(5)}}{j!} & j = 0,1,\ldots \\ 0 & \text{Otherwise} \end{cases}$$



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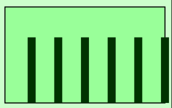
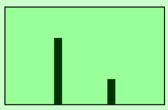
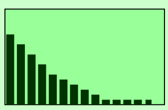
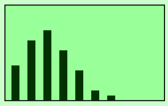
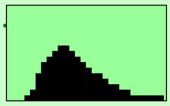
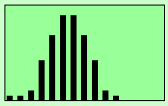
Discrete RV Summary

Uniform Equiprobable outcomes	$\begin{cases} 1/(j-k+1) & x = k,k+1,k+2,..,j \\ 0 & \text{Otherwise} \end{cases}$	
Bernoulli Pass/Fail	$\begin{cases} 1-p & x = 0 \\ p & x = 1 \\ 0 & \text{Otherwise} \end{cases}$	
Geometric # tests until fail	$\begin{cases} p(1-p)^{x-1} & x = 1,2,3,\ldots \\ 0 & \text{Otherwise} \end{cases}$	

Discrete RV Summary

Binomial # fails in n tests	$\begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x=1,2,\ldots,n \\ 0 & \text{Otherwise} \end{cases}$	
Pascal # tests until k fails	$\begin{cases} \binom{x-1}{k-1} p^k (1-p)^{x-k} & x = k,k+1,\ldots \\ 0 & \text{Otherwise} \end{cases}$	
Poisson occurrence in a period	$\begin{cases} \dfrac{(\lambda T)^x e^{-(\lambda T)}}{x!} & x = 0,1,2\ldots \\ 0 & \text{Otherwise} \end{cases}$	

Cumulative Distribution Function (CDF)

- Definition:

$$F_X(x) = P[X \leq x]$$

- Contain complete information about the probability model of the random variable
- PMF $\Longleftrightarrow$ CDF

CDF Theorem

Theorem: For a discrete random variable X

with $S_X=\{x_1,x_2,\ldots\}$ & $\mathbf{x_1 \leq x_2 \leq \ldots}$

1) $F_x(-\infty) = 0$ and $F_x(\infty) = 1$ → **From 0 to 1**

2) $\forall x' \geq x,\ F_x(x') \geq F_x(x)$ → **Monotonic Increasing**

3) For $x_i \in S_x$ and ε = +small number

$F_x(x_i) - F_x(x_i - \varepsilon) = P_x(x_i)$ → **Discontinuity = $P_X(x)$**

4) $F_x(x) = F_x(x_i) \quad \forall x,\ x_i \leq x < x_{i+1}$ → **Horizon line**

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CDF Example

- For a binomial RV, # of fail programs in 5 tests with p = 0.2

$$P_K(k) = \begin{cases} \binom{5}{k}(0.2)^k(0.8)^{5-k} & k = 0,1,2,\ldots,10 \\ 0 & \text{Otherwise} \end{cases}$$

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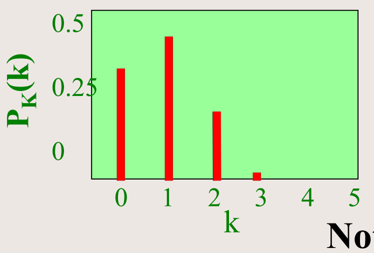
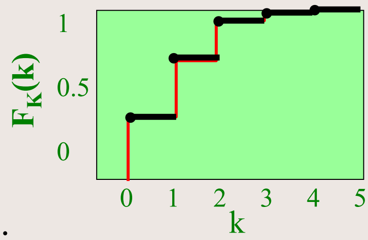
CDF Example

k	$P_K(k)$	k	$P_K(k)$
0	0.33	3	0.05
1	0.41	4	0.01
2	0.20	5	0





Note:

$F_K(k)$ is continuous for variable x from left to right

More CDF Theorem

$$\forall b \geq a,$$
$$F_X(b) - F_X(a) = P[a < X \leq b]$$

Difference of the CDF is the probability that RV takes on the value between two points

Average

- Study RV → average
- What is the average of an RV?
  - A single number that describes the RV
  - An example of statistic
- What is Statistic?
  - Numbers that collect all information of things under our interesting
  - Averages: mean, mode, and median

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Average

- Mean:
  - Sum / #terms
- Mode:
  - Most common value
  - $P_X(x_{mod}) \geq P_X(x) \quad \forall x$
- Median:
  - The middle of the data set
  - $P[X < x_{med}] = P[X > x_{med}]$

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Mean → Expected Value

- Adding all measurements / #terms

$$E[X] = \mu_X = \sum_{x \in S_X} x P_X(x)$$

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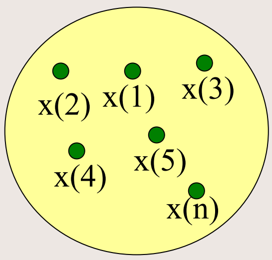
Expected Value

- Experiment → Random Variable X
- Perform n independent trials
- The value X takes on i^{th} trial → x(i)



The average

$$m_n = \frac{1}{n} \sum_{i=1}^{n} x(i)$$

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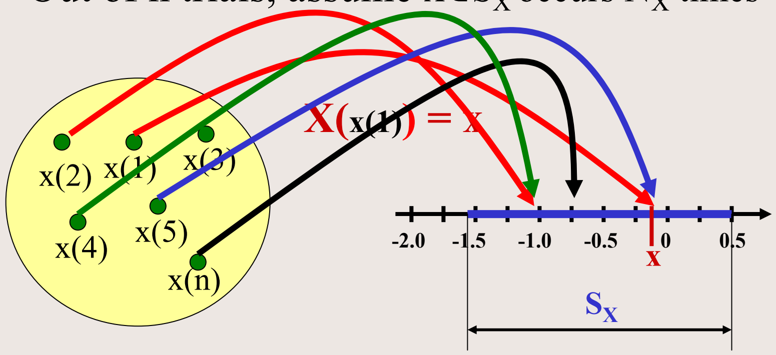
Expected Value

- Each x(i) take values in the set S_x
- Out of n trials, assume $x \in S_X$ occurs N_X times



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Expected Value

$$m_n = \frac{1}{n}\sum_{i=1}^{n} x(i)$$

$$m_n = \frac{1}{n}\sum_{x \in S_x} N_x x \quad = \sum_{x \in S_x} \frac{N_X}{n} x$$

$$P[A] = \lim_{n \to \infty} \frac{N_A}{n} \quad \Rightarrow \quad P_X(x) = \lim_{n \to \infty} \frac{N_X}{n}$$

$$\lim_{n \to \infty} m_n = \sum_{x \in S_x} x P_X(x)$$

$$\boxed{E[X] = \sum_{x \in S_X} x P_X(x)}$$

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Expected Value

$$E[X] = \sum_{x \in S_X} x P_X(x)$$

- Example:

$$P_T(t) = \begin{cases} 1/4 & t = 0 \\ 3/4 & t = 2 \\ 0 & \text{Otherwise} \end{cases}$$

- $E[T] = ?$

$$= 0(1/4) + 2(3/4) \ = 3/2$$

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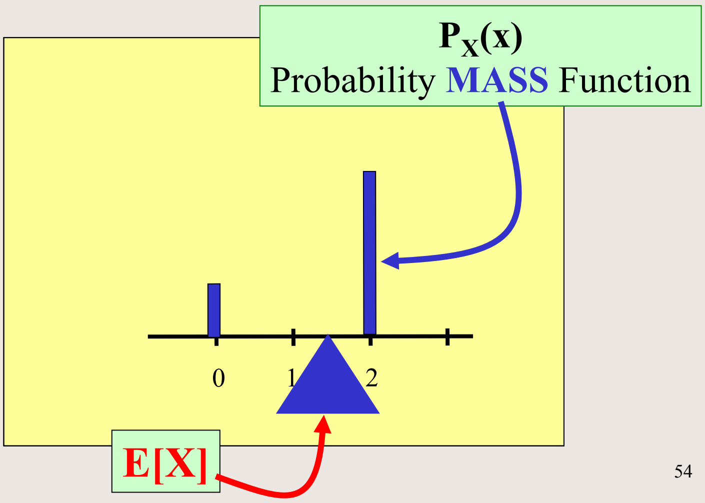
Expected Value



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