Homework 07 (to 4.1.1 – 4.1.6) + Proof continuous RV

นายโสภณ สุขสมบูรณ์
62010316371188 sec.1 Commun Math.

Uniform Continuous RV

let PDF of X : $f_X(x) = \begin{cases} \dfrac{1}{b-a} & , a < x \leq b \\ 0 & , \text{otherwise} \end{cases}$

จากความสัมพันธ์

$$f_X(x) = \frac{d}{dx} F_X(x)$$

จะได้ว่า CDF $F_X(x)$ คือ

$$F_X(t) = \int_a^t f_X(x)\,dx \quad , \quad a \leq x < t$$

constant $a \leq t \leq b$

$$= \int_a^t \left(\frac{1}{b-a}\right) dx$$

$$= \frac{x}{b-a}\Bigg|_a^t$$

$$\therefore F_X(t) = \frac{t-a}{b-a} \quad , \quad a < t \leq b$$

ดังนั้น CDF $F_X(x)$ คือ

$$F_X(x) = \begin{cases} 0 & , x \leq a \\ \dfrac{x-a}{b-a} & , a < x \leq b \\ 1 & , x > b \end{cases}$$

let $f_X(x) = \begin{cases} \frac{1}{b-a} & , a < x \leq b \\ 0 & , \text{otherwise} \end{cases}$

พิสูจน์ $E[X]$

จาก Definition : $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$$= \int_a^b x \left(\frac{1}{b-a}\right) dx$$

$$= \left(\frac{1}{b-a}\right) \int_a^b x\,dx$$

$$= \frac{1}{b-a} \cdot \frac{x^2}{2} \Big|_a^b$$

$$= \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}}$$

$\therefore E[X] = \frac{b+a}{2}$ #

พิสูจน์ $\text{Var}[X]$

จาก Definition : $\text{Var}[X] = E[X^2] - (E[X])^2$

$$= \int_a^b x^2 \left(\frac{1}{b-a}\right) dx - \frac{(b+a)^2}{4}$$

$$= \frac{1}{b-a} \cdot \frac{x^3}{3} \Big|_a^b - \frac{(b+a)^2}{4}$$

$$= \frac{b^3-a^3}{3(b-a)} - \frac{(b+a)^2}{4}$$

$$\mathrm{Var}[X] = \frac{b^3 - a^3}{3(b-a)} - \frac{(b+a)^2}{4}$$

$$= \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}} - \frac{(b+a)^4}{4}$$

$$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12}$$

$$= \frac{b^2-2ab+a^2}{12}$$

$$\therefore \mathrm{Var}[X] = \frac{(b-a)^2}{12}$$

Exponential Continuous RV

From Theorem CDF of X

$$F_X(x) = \begin{cases} 1-e^{-ax} & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

เนื่องจาก $f_X(x) = \frac{d}{dx} F_X(x)$

ดังนั้น

$$f_X(x) = \begin{cases} ae^{-ax} & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$ PDF of X

และ Expected Value

พิจารณา $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$$= \int_{0}^{\infty} x(ae^{-ax})\,dx \quad \text{— (1)}$$

ใช้ By Part method : $\int u\,dv = uv - \int v\,du$

พิจารณา $\int x(ae^{-ax})\,dx$

ให้ $u = x \Rightarrow du = dx$ และ $dv = ae^{-ax}dx \Rightarrow v = -e^{-ax}$

จะได้

$$\int x(ae^{-ax})\,dx = -xe^{-ax} + \int e^{-ax}\,dx$$

$$= -xe^{-ax} - \frac{e^{-ax}}{a} \quad \text{— (2)}$$

แทน (2) ใน (1) ;

$$E[X] = \left[\frac{-x}{e^{ax}} - \frac{1}{ae^{ax}} \right]_0^{\infty}$$

เนื่องจาก อินทิกรัลนี้เป็นอินทิกรัลไม่ตรงแบบ ;

$$E[X] = \lim_{B\to\infty} \left[\frac{-x}{e^{ax}} - \frac{1}{ae^{ax}} \right]_0^{B}$$

$$= \lim_{B\to\infty} \left[\frac{-B}{e^{aB}} - \frac{1}{ae^{aB}} \right] + 0 + \frac{1}{a}$$

$$= \lim_{B\to\infty} \left[\frac{-1}{ae^{aB}} - 0 \right] + \frac{1}{a} \qquad \text{(L'hopital)}$$

$$= 0 + \frac{1}{a}$$

$$\therefore \; E[X] = \frac{1}{a} \quad \#$$

และพิจารณาหาค่า Variance ของ X

จากสูตร $\quad Var[X] = E[X^2] - (E[X])^2 \quad$ — (3)

พิจารณา $E[X^2] = \int_0^{\infty} x^2 (ae^{-ax})\,dx \quad$: using by part method

Improper Integral

u	dv
x^2	$+ae^{-ax}$
$2x$	$-e^{-ax}$
2	$+\frac{1}{a}e^{-ax}$
0	$-\frac{1}{a^2}e^{-ax}$

$$= \left[-x^2 e^{-ax} - \frac{2x}{a} e^{-ax} - \frac{2}{a^2} e^{-ax} \right]_0^{\infty}$$

$$= \lim_{B\to\infty} \left[-\frac{x^2}{e^{ax}} - \frac{2x}{ae^{ax}} - \frac{2}{a^2 e^{ax}} \right]_0^{B}$$

0 limit x→∞ 0 limit x=0

$$= \lim_{B\to\infty} \left[\cancel{\frac{-B^2}{e^{aB}}} - \cancel{\frac{2B}{ae^{aB}}} - \cancel{\frac{2}{a^2 e^{aB}}} \right] + 0 + 0 + \underbrace{\frac{2}{a^2}} \quad \text{— (4)}$$

แทน $E[X^2] = \frac{2}{a^2}$ ใน (3) ;

$$Var[X] = \frac{2}{a^2} - \frac{1}{a^2}$$

$$\therefore Var[X] = \frac{1}{a^2} \quad \#$$

4.1.1 Random variables X and Y have the joint CDF

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}) & x \ge 0; \\ & y \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is $P[X \le 2, Y \le 3]$?

(b) What is the marginal CDF, $F_X(x)$?

(c) What is the marginal CDF, $F_Y(y)$?

a) $P[X \le 2, Y \le 3] = P_{X,Y}(2,3) = (1-e^{-2})(1-e^{-3})$

จาก ทฤษฎีบท 4.1 ได้ว่า $F_X(x) = F_{X,Y}(x,\infty)$, $F_Y(y) = F_{X,Y}(\infty, y)$

(b) $$F_X(x) = F_{X,Y}(x,\infty) = \begin{cases} 1-e^{-x} & , x \ge 0 \\ 0 & , \text{otherwise} \end{cases}$$

(c) $$F_Y(y) = F_{X,Y}(\infty, y) = \begin{cases} 1-e^{-y} & , y \ge 0 \\ 0 & , \text{otherwise} \end{cases}$$

4.1.2 Express the following extreme values of $F_{X,Y}(x, y)$ in terms of the marginal cumulative distribution functions $F_X(x)$ and $F_Y(y)$.

(a) $F_{X,Y}(x, -\infty)$

(b) $F_{X,Y}(x, \infty)$

(c) $F_{X,Y}(-\infty, \infty)$

(d) $F_{X,Y}(-\infty, y)$

(e) $F_{X,Y}(\infty, y)$

$Y \to -\infty \Rightarrow P[Y] = 0$

a) $F_{X,Y}(x,-\infty) = P[X \leq x, Y \leq -\infty] \leq P[Y \leq -\infty] = 0$ #

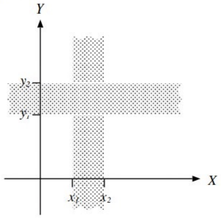
b) $F_{X,Y}(x,\infty) = P[X \leq x, Y \leq \infty] = P[X \leq x] = P_X(x)$ #

c) $F_{X,Y}(-\infty,\infty) = P[X \leq -\infty, Y \leq \infty] \leq P[X \leq -\infty] = 0$ #

d) $F_{X,Y}(-\infty, y) = P[X \leq -\infty, Y \leq y] \leq P[X \leq -\infty] = 0$ #

e) $F_{X,Y}(\infty, y) = P[X \leq \infty, Y \leq y] = P[Y \leq y] = P_Y(y)$ #

4.1.3 For continuous random variables X, Y with joint CDF $F_{X,Y}(x, y)$ and marginal CDFs $F_X(x)$ and $F_Y(y)$, find $P[x_1 \leq X < x_2 \cup y_1 \leq Y < y_2]$. This is the probability of the shaded "cross" region in the following diagram.



** Formula : $P[A \cup B] = P[A] + P[B] - P[A \cap B]$ **

ให้ $P[A \cup B] = P[x_1 \leq X \leq x_2 \cup y_1 \leq Y \leq y_2]$

จาก Formula :

$$\underbrace{P[x_1 \leq X \leq x_2 \cup y_1 \leq Y \leq y_2]}_{P[A \cup B]} = \underbrace{P[x_1 \leq X \leq x_2]}_{P[A]} + \underbrace{P[y_1 \leq Y \leq y_2]}_{P[B]} - \underbrace{P[x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2]}_{P[A \cap B]} \quad —(1)$$

จากทฤษฎีบท : $P[x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2] = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1)$

หากทฤษฎีอยู่บทต้น จะได้ว่า

$$P[x_1 \leq X \leq x_2 \cup y_1 \leq Y \leq y_2] = F_X(x_2) - F_X(x_1) + F_Y(y_2) - F_Y(y_1) - F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1) \quad ※$$

4.1.4 Random variables X and Y have CDF $F_X(x)$ and $F_Y(y)$. Is $F(x, y) = F_X(x)F_Y(y)$ a valid CDF? Explain your answer.

let $\quad P[x_1 \leq X \leq x_2,\ y_1 \leq Y \leq y_2] \geq 0$

* โจทย์ต้องการว่า $F(x,y) = F_X(x)F_Y(y)$ *

$$P[x_1 \leq X \leq x_2,\ y_1 \leq Y \leq y_2] = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1)$$

Form Theorem

$$= F_X(x_2)F_Y(y_2) - F_X(x_2)F_Y(y_1) - F_X(x_1)F_Y(y_2) + F_X(x_1)F_Y(y_1)$$

$$= F_X(x_2)[F_Y(y_2) - F_Y(y_1)] + F_X(x_1)[F_Y(y_2) - F_Y(y_1)]$$

$$= [F_X(x_2) - F_X(x_1)][F_Y(y_2) - F_Y(y_1)]$$

$$= F_X(x)F_Y(y) \quad ; x_1 \leq x \leq x_2 \text{ and } y_1 \leq y \leq y_2$$

\#

Explain your answer.

4.1.5 In this problem, we prove Theorem 4.5.

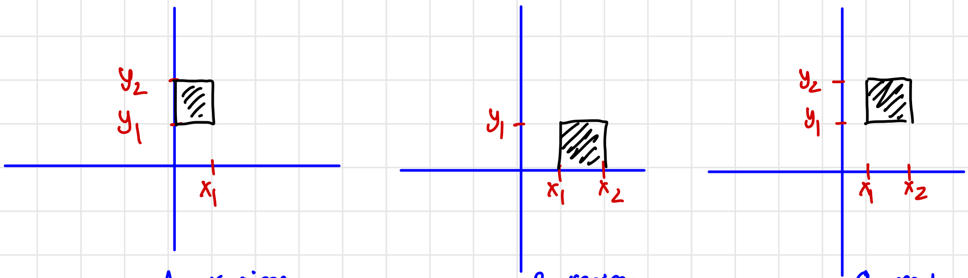
(a) Sketch the following events on the X, Y plane:

$$A = \{X \le x_1, y_1 < Y \le y_2\},$$
$$B = \{x_1 < X \le x_2, Y \le y_1\},$$
$$C = \{x_1 < X \le x_2, y_1 < Y \le y_2\}.$$

(b) Express the probability of the events A, B, and $A \cup B \cup C$ in terms of the joint CDF $F_{X,Y}(x, y)$.

(c) Use the observation that events A, B, and C are mutually exclusive to prove Theorem 4.5.

a)



A- region B-region C-region

b) $P[A] = F_{X,Y}(x_1, y_2) - F_{X,Y}(x_1, y_1)$ #

$P[B] = F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_1)$ #

$P[C] = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1)$

Disjoint.

$P[A \cup B \cup C] = P[A] + P[B] + P[C] - \boxed{P[A \cap B] - P[A \cap C] - P[B \cap C] - P[A \cap B \cap C]}$

$= \cancel{F_{X,Y}(x_1, y_2)} - F_{X,Y}(x_1, y_1) + \cancel{F_{X,Y}(x_2, y_1)} - \cancel{F_{X,Y}(x_1, y_1)} +$

$F_{X,Y}(x_2, y_2) - \cancel{F_{X,Y}(x_2, y_1)} - \cancel{F_{X,Y}(x_1, y_2)} + \cancel{F_{X,Y}(x_1, y_1)}$

$\therefore \; P[A \cup B \cup C] = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_1)$ #

c) Theorem 4.5 : $P[x_1 \le X \le x_2, y_1 \le Y \le y_2] = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1)$

b/c: mutually exculsive : $P[A \cup B \cup C] = P[A] + P[B] + P[C]$

เพื่อหา $P[C] = P[x_1 \le X \le x_2,\ y_1 \le Y \le y_2]$;

$$P[C] = P[A \cup B \cup C] - P[A] - P[B]$$

จากข้อ b) จะได้ว่า

$$P[C] = F_{X,Y}(x_2,y_2) - \cancel{F_{X,Y}(x_1,y_1)} - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1)$$

$$- F_{X,Y}(x_2,y_1) + \cancel{F_{X,Y}(x_1,y_1)}$$

$$\therefore P[C] = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1) \quad \#$$

เป็นตามทฤษฎีบทที่ 4.5 #

4.1.6 Can the following function be the joint CDF of random variables X and Y?

$$F(x, y) = \begin{cases} 1 - e^{-(x+y)} & x \ge 0, y \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

CDF $F_X(x)$ & $F_Y(y)$

$$F_X(x) = F_X(x,\infty) = \begin{cases} 1 & , x \ge 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$F_Y(y) = F_Y(\infty,y) = \begin{cases} 1 & , y \ge 0 \\ 0 & , \text{otherwise} \end{cases}$$

เมื่อ $X \ge x$ และ $Y \ge y$ $\rightarrow$ $P[X \ge x] = 0$, $P[Y \ge y] = 0$

$$P[X \ge x \cup Y \ge y] \le P[X \ge x] + P[Y \ge y]$$

เพื่อหา $\underbrace{P[X \le x, Y \le y]}_{F_{X,Y}(x,y)} + P[X \ge x \cup Y \ge y] = 1$

$$P[X \ge x \cup Y \ge y] = 1 - 1 + e^{-(x+y)}$$

จะเห็นว่า เมื่อ $X \ge x$ และ $Y \ge y$ $\rightarrow$ $P[\ldots] = e^{-(x+y)}$ $\rightarrow$ จะมีค่าเป็นลบ.

ค่า Prob ไม่สามารถติดลบได้ ดังนั้น CDF ข้างต้นไม่ใช่ Joint CDF #