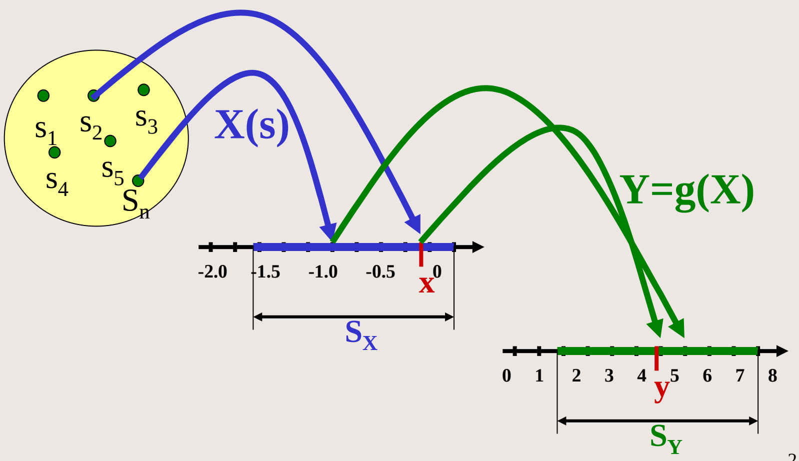
Lecture #6
Discrete Random Variable (3)

Derived Random Variable



Why do we need a Derived Random Variable?

- From sample values of the random variable, use these values to compute other quantities.
- Example:
  - – Find a decibel value from signal-to-noise ratio
- $Y = g(X)$

Example-1

- Random Variable X = # pages in one fax
- $P_X(x)$ = number of pages in each fax
- Charging plan
  - 1st page = 10 Baht
  - 2nd page = 9 Baht
  - 3rd page = 8 Baht
  - 4th page = 7 Baht
  - 5th page = 6 Baht
  - 6 – 10 pages = 50 Baht
- Find the charge in Baht for sending one fax

Example-1

- Random Variable Y = the charge in Baht for sending one fax

$$Y = g(X) = \begin{cases} 10.5X - 0.5X^2 & 1 \le X \le 5 \\ 50 & 6 \le X \le 10 \end{cases}$$

PMF of Y

Theorem:

$$P_Y(y) = \sum_{x:g(x)=y} P_X(x)$$

P[Y=y] = Σ of all outcomes X = x for which Y = y

Example-2

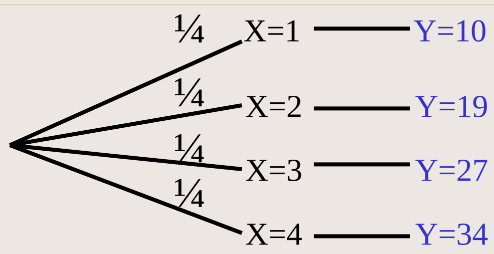
$$Y = g(X) = \begin{cases} 10.5X - 0.5X^2 & 1 \le X \le 5 \\ 50 & 6 \le X \le 10 \end{cases}$$

- Suppose all faxes contain 1,2,3, or 4 pages with equal probability
- Find PMF and E[Y]

$$P_X(x) = \begin{cases} 1/4 & x = 1,2,3,4 \\ 0 & \text{Otherwise} \end{cases}$$

- $S_X = \{1,2,3,4\}$
- $S_Y = \{10,19,27,34\}$

Example-2



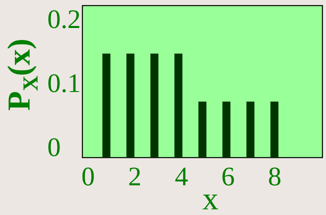
$$P_Y(y) = \begin{cases} 1/4, & y = 10,19,27,34 \\ 0, & \text{otherwise} \end{cases}$$

E[Y] = (1/4)(10 + 19 + 27 + 34) = 22.7 Baht

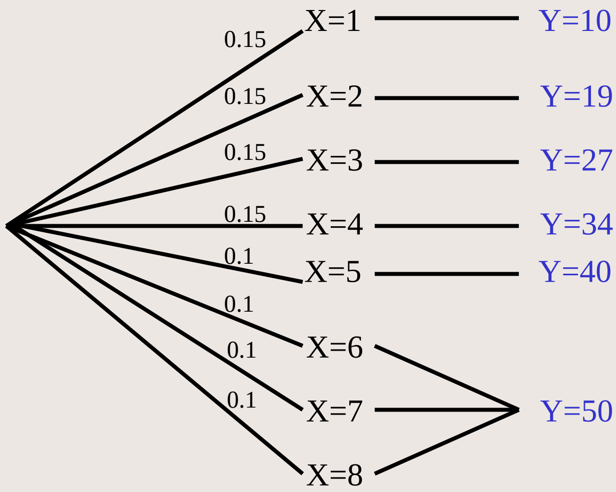
Example-3

$$P_X(x) = \begin{cases} 0.15, & x = 1,2,3,4 \\ 0.1, & x = 5,6,7,8 \\ 0, & \text{otherwise} \end{cases}$$

- Find PMF and E[Y]



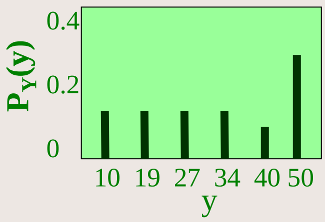
Example-3



Example-3

From Theorem

$$P_Y(y) = \begin{cases} 0.15 & y = 10,19,27,34 \\ P_X(5) = 0.1 & y = 40 \\ P_X(6)+P_X(7)+P_X(8) = 0.3 & y = 50 \\ 0 & \text{Otherwise} \end{cases}$$



$$\begin{aligned} E[Y] &= 0.15(10+19+27+34) \\ &\quad + 0.1(40) + 0.3(50) \\ &= 32.5 \end{aligned}$$

Quiz 2.6

Example 2.30
Quiz 2.6

Expected Value of a Derived Random Variable

- To Find E[Y]
  - → Find $P_Y(y)$
  - → Find E[Y]
- In case of interesting only E[Y]

Theorem:

$$E[Y] = E[g[X]] = \sum_{x \in S_X} g[x] \, P_X(x)$$

Important Property of E[Y]

Let $Y = g(X) = X - E[X] = X - \mu_X$

$$E[Y] = E[g(X)]$$

$$= \sum_{x \in S_x} (x - \mu_X) \, P_X(x)$$

$$= \sum_{x \in S_x} x \, P_X(x) \quad - \quad \sum_{x \in S_x} \mu_X \, P_X(x)$$

$$= \quad \mu_X \quad - \quad \mu_X \sum_{x \in S_x} P_X(x)$$

$$= 0$$

Important Property of E[Y]

Theorem: Form any random variable X

$$E[X - \mu_x] = 0$$

Important Property of E[Y]

Theorem: For $P_X(x)$,

$$E[aX + b] = aE[X] + b$$

Note:
- Linear Transformation
- scale change of quantity (change the unit)
  Ex. Celsius → Fahrenheit
- Adding score to every one
  → new E[X] = old E[X] + adding value
- $Y = X^2 \rightarrow E[Y] \neq (E[X])^2 \rightarrow E[g(X)] \neq g(E[X])$

Example

$$P_R(r) = \begin{cases} 1/4 & r = 0 \\ 3/4 & r = 2 \\ 0 & \text{otherwise} \end{cases}$$

Find E[Y] for $Y = g(R) = 2R + 4$

$E[R] = (0)\, P_R(0) + 2P_R(2) = (1/4)(0) + (3/4)(2)$
$= 3/2$

$E[Y] = E[g(R)]$
$= E[2R + 4]$
$= 2E[R] + 4$
$= 2(3/2) + 4 = 7$

$E[Y] = E[g(R)] = \Sigma\, g(r)P_R(r)$
$= g(0)(1/4) + g(2)(3/4)$
$= (2{*}0+4)(1/4) + (2{*}2+4)(3/4)$
$= 1 + 6 = 7$

Quiz 2.7

- Number of memory chip M depends on number of application programs, A
- M = 4 chips for 1 program
  4 chips for 2 programs
  6 chips for 3 programs
  8 chips for 4 programs

$P_A(a) = 0.1(5-a)$ $a=1,2,3,4$
0 otherwise

1) E[A]
2) M = g(A)
3) Show E[M]=E[g(A)]
4) Show E[M] is not equal g(E[A])

Variance & Standard Deviation

- We knew average, E[X],

  Why do we need these Variance & Standard Variation?
- How far from the average?
- $T = X - \mu_x$

  $E[T] = E[X - \mu_x]$

  $= 0$

Variance & Standard Variation

- The useful measurement is **E[|T|]**
- $E[T^2] = E[(X - \mu_x)^2]$ → **Variance**

Definition:

$$Var[X] = E[(X - \mu_x)^2]$$

Variance & Standard Variation

Definition:

Sigma X

$$\sigma_X = \sqrt{Var[X]}$$

- σ_X tells how much RV deviates from mean.
- Ex: $\sigma_X = 15$, Score + 6 from mean
  → OK. Middle of class
- Ex: $\sigma_X = 3$, Score + 6 from mean
  → V.Good In Top class group

Variance of X

$$\begin{aligned}
Var[X] &= \sigma_x^2 \\
&= E[(X - \mu_x)^2] \\
&= \sum_{x \in S_x} (x - \mu_X)^2 P_X(x) \\
&= \sum_{x \in S_x} x^2 P_X(x) - \sum_{x \in S_x} 2\mu_x x P_X(x) + \sum_{x \in S_x} \mu_x^2 P_X(x) \\
&= E[X^2] - 2\mu_x \sum_{x \in S_x} x P_X(x) + \mu_x^2 \sum_{x \in S_x} P_X(x) \\
&= E[X^2] - 2\mu_x^2 + \mu_x^2
\end{aligned}$$

$$Var[X] = E[X^2] - \mu_x^2 = E[X^2] - (E[X])^2$$

Variance of X

Theorem:

- If X is a constant,

  $\mathrm{Var}[X] = 0$
- If $Y = X+b$,

  $\mathrm{Var}[Y] = \mathrm{Var}[X]$
- If $Y = aX$,

  $\mathrm{Var}[Y] = a^2\, \mathrm{Var}[X]$

Variance

(a) Bernoulli – p → $\mathrm{Var}[X] = p(1 – p)$

(b) Geometric – p → $\mathrm{Var}[X] = (1 – p)/p^2$

(c) Binomial – n,p → $\mathrm{Var}[X] = np(1 – p)$

(d) Pascal – k,p → $\mathrm{Var}[X] = k(1 – p)/p^2$

(e) Poisson – α → $\mathrm{Var}[X] = \alpha$

(f) Discrete uniform – k,l

→ $\mathrm{Var}[X] = (l – k)\,(l – k+2)/12$

Quiz 2.8

- Example 2.36
- Quiz 2.8

Conditional Probability Mass Function

$$P[A|B] = P[X = x|B]$$

Definition: Given event B, P[B] > 0

$$\mathbf{P_{X|B}(x) = P[X=x|B]}$$

Theorem: $P[A] = \sum_{i=1}^{n} P[A|B_i]P[B_i]$

Theorem: $P_X(x) = \sum_{i=1}^{n} P_{X|B_i}(x)P[B_i]$

Quiz 2.9

- Example 2.38
- Example 2.39
- Example 2.41
- Quiz 2.9

Quiz 2.9

- On the Internet, data is transmitted in packets. N is the number of packets needed to transmit a web page depends on whether the page has a graphic images.
- If the page has image (event I), then N is uniform distribution between 1-50
- If the page has only text (event T), then N is uniform distribution between 1-5
- Assume a page has images with prob ¼ , find

1) $P_{N|I}(n)$
2) $P_{N|T}(n)$
3) $P_N(n)$
4) $P_{N|N<=10}(n)$
5) $E[N|N<=10]$
6) $Var[N|N <= 10]$

Homework

1) 2.2.4
2) 2.2.6
3) 2.3.4
4) 2.3.8
5) 2.4.3
6) 2.5.2
7) 2.5.9
8) 2.6.6
9) 2.7.7
10) 2.8.1
11) 2.9.3
12) 2.9.7