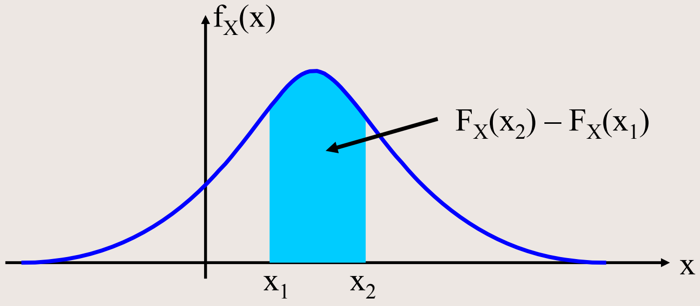
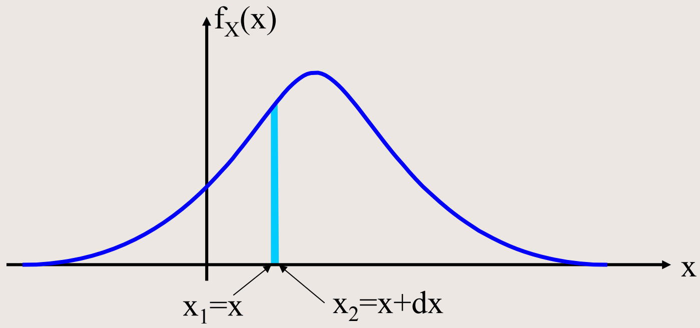
PDF and CDF



$$P[x_1 < X \le x_2] = F_X(x_2) - F_X(x_1)$$

$$P[x_1 < X \le x_2] = \int_{x_1}^{x_2} f_X(x)\, dx$$

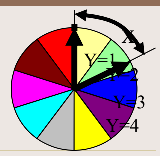
RV X & infinitesimal dx



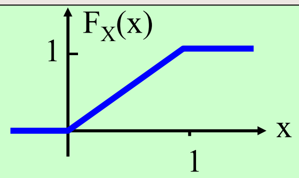
$$P[x_1 < X \le x_2] = \int_{x_1}^{x_2} f_X(x)\, dx$$

Approx: $P[x < X \le x+dx] = f_X(x)\, dx$

Example

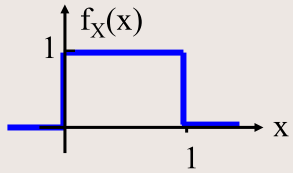


$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$



Find $f_X(x)$ and $P[1/4 < X \le 3/4]$

$$f_X(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{Otherwise} \end{cases}$$



$$P[1/4 < X \le 3/4] = F_X(3/4) - F_X(1/4) = 3/4 - 1/4 = 1/2$$

$$P[1/4 < X \le 3/4] = \int_{1/4}^{3/4} f_X(x)\,dx = \int_{1/4}^{3/4} dx = 1/2$$

- Example 3.5
- Quiz 3.2

Expected Values

For Discrete Random Variable:

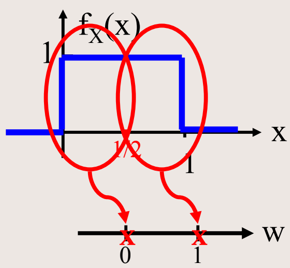
$$E[X] = \sum_{x \in S_X} x\, P_X(x)$$

For Continuous Random Variable:

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

Function of RV

- A **function** of a continuous random variable is also a random variable (not necessary to be continuous)
- Example



$$W = g(X) = \begin{cases} 0 & X \le 1/2 \\ 1 & X > 1/2 \end{cases}$$

W = Discrete RV

$S_W = \{0,1\}$

Expected Values

For a function g(X) of Random Variable X:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

Expected Value & Variance

- Find E[X]

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

- Find $E[X^2]$

$$E[X^2] = \int_{-\infty}^{\infty} x^2\, f_X(x)\, dx$$

- Find Var[X]

$$Var[X] = \int_{-\infty}^{\infty} (x - \mu_X)^2\, f_X(x)\, dx$$

Theorem

- $E[X - \mu_X] = 0$
- $E[aX + b] = aE[X] + b$
- $Var[X] = E[X^2] - (E[X])^2$
- If X always takes value "a", $Var[X] = 0$
- For $Y=X+b \rightarrow Var[Y] = Var[X]$
- For $Y=aX \rightarrow Var[Y] = a^2 Var[X]$

- Example 3.7
- Example 3.10
- Quiz 3.3

Some Useful Continuous RVs

- Uniform
- Exponential
- Gaussian

Uniform Continuous RV

Definition:

$$f_X(x) = \begin{cases} \mathbf{1/(b - a)} & \mathbf{a \le x < b} \\ \mathbf{0} & \textbf{Otherwise} \end{cases}$$

where b > a

Uniform Continuous RV

Theorem:

- $$F_X(x) = \begin{cases} 0 & x \le a \\ (x-a)/(b-a) & a < x \le b \\ 1 & x > b \end{cases}$$
- $E[X] = (b + a)/2$
- $Var[X] = (b - a)^2/12$

Exponential Continuous RV

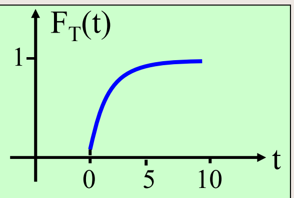
Definition:

$$f_X(x) = \begin{cases} a\, e^{-ax} & x \ge 0 \\ 0 & \text{Otherwise} \end{cases}$$

where $a > 0$

Exponential Example

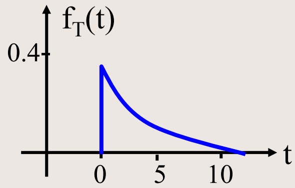
$$F_T(t) = \begin{cases} 1 - e^{-t/3} & t \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$



Find PDF

$$f_T(t) = \frac{dF_T(t)}{dt}$$

$$= \begin{cases} (1/3)\, e^{-t/3} & t \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$



Exponential Example

Find E[T]

$$E[T] = \int_{-\infty}^{\infty} t\, f_T(t)\, dt$$

$$= \int_{0}^{\infty} t\,(1/3)e^{-t/3}\, dt$$

$$= -t\, e^{-t/3} \Big|_{0}^{\infty} + \int_{0}^{\infty} e^{-t/3}\, dt$$

$$= 3$$

Exponential Example

Find Var[T] $\mathrm{Var}[T] = E[T^2] - (E[T])^2$

$$E[T^2] = \int_{-\infty}^{\infty} t^2 f_T(t)\,dt$$

$$= (1/3)\int_0^{\infty} t^2 e^{-t/3}\,dt$$

$$= -t^2 e^{-t/3}\Big|_0^{\infty} + \int_0^{\infty} (2t) e^{-t/3}\,dt$$

$$= 2\int_0^{\infty} t e^{-t/3}\,dt = 2(3E[T]) = 18$$

Exponential Example

$$\mathrm{Var}[T] = E[T^2] - (E[T])^2$$

$$= 18 - 3^2 = 9 \text{ min}$$

$$\sigma_T = \sqrt{\mathrm{Var}[X]} = 3 \text{ min}$$

Find Prob. that call duration is within 1 standard variation

$$P[0 \le T \le 6] = F_T(6) - F_T(0)$$
$$= 1 - e^{-2} = 0.865$$

Exponential Continuous RV

Theorem:

- $$F_X(x) = \begin{cases} 1 - e^{-ax} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$
- $E[X] = 1/a$
- $Var[X] = 1/a^2$

Geometric & Exponential RV

Theorem:
If **X** = Exponential RV with parameter **a**
Then $\mathbf{K} = \lceil \mathbf{X} \rceil$ is a Geometric RV
with parameter $\mathbf{p = 1 - e^{-a}}$

$$\begin{aligned}
P_K(k) &= P[K{=}k] = P[k - 1 < X \leq k] \\
&= F_X(k) - F_X(k-1) \\
&= 1 - e^{-ak} - (1 - e^{-a(k-1)}) \\
&= -e^{-ak} + e^{-a(k-1)} \\
&= e^{-a(k-1)} \left(1 - \frac{e^{-ak}}{e^{-a(k-1)}}\right) \\
&= e^{-a(k-1)} (1 - e^{-a}) \\
&= (1 - p)^{k-1} p \qquad ; p = (1 - e^{-a})
\end{aligned}$$

Example

- Phone Company A:
  - 3 Baht / min.
  - With full min. charge
- Phone Company B:
  - 3 Baht / min.
  - With exact charge
- Let T = duration of call
- T: exponential with a = 1/3

Example

- E[T] = 1/a = 3 min.
- For Company B:

  E[R] = 3 E[T] = 9 Baht/Call
- For Company A:

  E[R] = 3 E[K]

  where $K = \lceil T \rceil \rightarrow$ geometric with $p = 1 - e^{-1/3}$

  E[R] = 3 (1/p) = 3 (3.53) = 10.59 Baht/Call