Poisson Process

Definition: A Counting Process $N(t)$ is a Poisson Process $N(t)$ if

- # of arrivals in $(t_0,t_1]$, $N(t_1) - N(t_0)$, is a Poisson RV with expected value $\lambda(t_1-t_0)$
- # of arrivals in each non-overlapping interval are independent random variable

Poisson Process

- Process rate $(\lambda) = E[N(t)] / t$
- $M = N(t_1) - N(t_0) =$ Poisson RV

$$P_M(m) = \begin{cases} \dfrac{[\lambda(t_1-t_0)]^m}{m!} e^{-\lambda(t_1-t_0)} & m = 0,1,2,.. \\ 0 & \text{Otherwise} \end{cases}$$

Joint PMF

Theorem: Poisson Process N(t) of rate λ,
Joint PMF of $N(t_1),\ldots,N(t_k)$, $t_1 <\ldots< t_k$

$$P_{N(t_1)N(t_2)N(t_3)\ldots N(t_k)}(n_1,n_2,n_3,\ldots,n_k)$$

$$=\begin{cases} \dfrac{\alpha_1^{\,n_1}e^{-\alpha_1}}{n_1!}\dfrac{\alpha_2^{\,(n_2-n_1)}e^{-\alpha_2}}{(n_2-n_1)!}\cdots\dfrac{\alpha_k^{\,(n_k-n_{k-1})}e^{-\alpha_k}}{(n_k-n_{k-1})!}, & 0\le n_1\le n_2\le\ldots\le n_k \\ 0, & \text{otherwise} \end{cases}$$

$$\alpha_i=\lambda(t_i-t_{i-1})$$

Joint PMF

$$P_{N(t_1)N(t_2)N(t_3)\ldots N(t_k)}(n_1,n_2,n_3,\ldots,n_k)$$

$$= P_{N(t_1)}(n_1)\ P_{N(t_2)|N(t_1)}(n_2\,|\,n_1)\ P_{N(t_3)|N(t_2)N(t_1)}(n_3\,|\,n_1,n_2)\cdots$$

$$P_{N(t_k)|N(t_1)N(t_2)\ldots N(t_{k-1})}(n_k\,|\,n_1,n_2\ldots,n_{k-1})$$

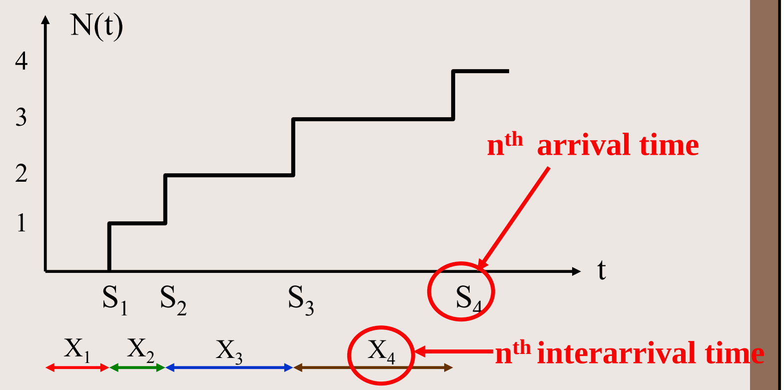
$$= P_{N(t_1)}(n_1)\ P_{N(t_2)|N(t_1)}(n_2\,|\,n_1)\ P_{N(t_3)|N(t_2)}(n_3\,|\,n_2)\cdots$$

$$P_{N(t_k)|N(t_{k-1})}(n_k\,|\,n_{k-1})$$

$$=\begin{cases} \dfrac{\alpha_1^{\,n_1}e^{-\alpha_1}}{n_1!}\dfrac{\alpha_2^{\,(n_2-n_1)}e^{-\alpha_2}}{(n_2-n_1)!}\cdots\dfrac{\alpha_k^{\,(n_k-n_{k-1})}e^{-\alpha_k}}{(n_k-n_{k-1})!}, & 0\le n_1\le n_2\le\ldots\le n_k \\ 0, & \text{otherwise} \end{cases}$$

$$\alpha_i=\lambda(t_i-t_{i-1})$$

Counting Process



Example

A mobile station transmits data packet as Poisson process with rate 12 packets/sec

– Find PMF # of packets transmitted in the k^{th} hour

– Find Joint PMF of # of packets transmitted in the k^{th} hour and z^{th} hour

Example

- Let N_k = # of packets transmitted in k^{th} hour
- # packets in each hour is IID

$$P_{Nk}(n) = \begin{cases} \dfrac{[12(3600-0)]^n \; e^{-12\,(3600-0)}}{n!} & n = 0,1,2,.. \\ 0 & \text{Otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{[43200]^n \; e^{-43200}}{n!} & n = 0,1,2,.. \\ 0 & \text{Otherwise} \end{cases}$$

Example

- Joint PMF of # of packets transmitted in the k^{th} hour and z^{th} hour

$$P_{N_k,N_z}(n_k,n_z) = \begin{cases} \dfrac{\alpha_k^{n_k} e^{-\alpha_k}}{n_k!} \; \dfrac{\alpha_z^{n_z} e^{-\alpha_z}}{n_z!} & n_k = 0,1,\ldots \;\; n_z = 0,1,\ldots \\ 0 & \text{Otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{\alpha^{(n_k+n_z)}}{n_k!\,n_z!} \; e^{-2\alpha} & n_k = 0,1,\ldots \;\; n_z = 0,1,\ldots \\ 0 & \text{Otherwise} \end{cases}$$

$\alpha = \alpha_k = \alpha_z = \lambda\, T = 12(3600\text{-}0)] = 43200$

Interarrival Time

Theorem: Poisson Process of rate λ, the interarrival times $X_1, X_2, \ldots$ are an iid random sequence with **Exponential PDF**

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

Interarrival Time

Theorem: A Counting Process with ***independent exponential interarrival time*** $X_1, X_2, \ldots$ with $E[X_i] = 1/\lambda$ is a ***Poisson Process*** of rate λ

Brownian Motion Process

- A continuous time, continuous value random process

Definition: A Brownian Motion Process $X(t)$ has the property that

- $X(0) = 0$
- For $\tau > 0$, $X(t + \tau) - X(t)$ = Gaussian RV
  with $E[X(t+\tau) - X(t)] = 0$
  and $Var[X(t+\tau) - X(t)] = \alpha\tau$
  that is independent of $X(t')$ for $t' \leq t$

Brownian Motion Process

Gaussian RV

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \, e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Joint PDF

- **Theorem:** For the Brownian motion process $X(t)$, joint PDF of $X(t_1),\ldots,X(t_k)$

$$f_{X(t_1),\ldots,X(t_k)}(x_1,\ldots,x_k) = \prod_{n=1}^{k} \frac{1}{\sqrt{2\pi\alpha(t_n - t_{n-1})}} e^{\frac{-(x_n - x_{n-1})^2}{2\,\alpha(t_n - t_{n-1})}}$$

Expected Value

$$\mathbf{X(t):}\ X(t_1) \rightarrow f_{X(t_1)}(x) \rightarrow E[X(t_1)]$$

Definition: The expected value of a stochastic process $X(t)$ is the deterministic function

$$\mu_x(t) = E[X(t)]$$

Covariance of X and Y

Definition: $Cov[X,Y] = E[(X-\mu_x)(Y-\mu_Y)]$

$$\begin{aligned} Cov[X,Y] &= E[XY - \mu_x Y - \mu_Y X + \mu_x \mu_y] \\ &= E[XY] - \mu_x E[Y] - \mu_Y E[X] + \mu_x \mu_y \\ &= E[XY] - \mu_x \mu_y - \mu_x \mu_y + \mu_x \mu_y \end{aligned}$$

Theorem: $Cov[X,Y] = E[XY] - \mu_x \mu_y$

- High covariance = an observation of X provides an accurate indication of Y

Autocovariance

Definition: The autocovariance function of a stochastic process X(t) is

$$C_X(t,\tau) = Cov[X(t),X(t+\tau)]$$

<u>Note</u>

- For $\tau = 0 \Rightarrow C_X(t,0) = Var[X(t)]$

Autocovariance

- Auto = self = same process
- For a same process X(t), in 2 different times $t_1 = t$ and $t_2 = t + \tau$
- For high covariance = a sample function, is unlikely to change in τ interval
- For near zero covariance = rapid change

How much the sample function is likely to change in the τ interval after t

Correlation

Definition: The correlation of X and Y is $r_{X,Y}$

$$r_{X,Y} = E[XY]$$

Theorem: $\mathrm{Cov}[X,Y] = r_{X,Y} - \mu_x \mu_y$

Autocorrelation

- **Definition**: The autocorrelation function of a stochastic process X(t) is

$$R_X(t,\tau) = E[\, X(t)\, X(t+\tau)\,]$$

Autocovariance & Autocorrelation

Theorem: The autocovariance and autocorrelation functions of a process X(t) satisfy

$$C_X(t,\tau) = R_X(t,\tau) - \mu_x(t)\, \mu_x(t+\tau)$$

Note:

Autocovariance ➔ use X(t) to predict a future $X(t+\tau)$

Autocorrelation ➔ describe the power of a random signal

Random Sequence

For a discrete time process, the sample function is described by the ordered sequence of random variable $X_n = X(nT)$

- **Definition**: A random sequence X_n is an ordered sequence of random variable $X_0, X_1, \ldots$

Autocovariance & Autocorrelation of a Random Sequence

Definition: The autocovariance of a random sequence X_n is

$$C_X[m,k] = \mathrm{Cov}[X_m, X_{m+k}]$$

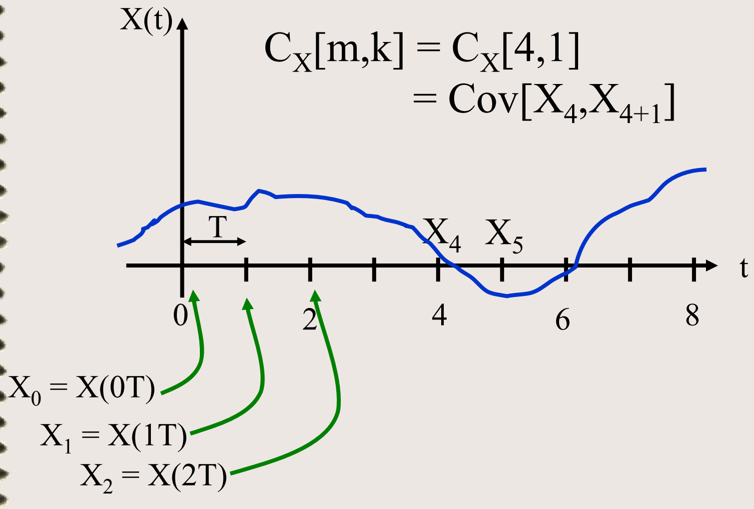
Definition: The autocorrelation of a random sequence X_n is

$$R_X[m,k] = E[X_m X_{m+k}]$$

m and k are integers

Autocovariance & Autocorrelation of a Random Sequence

$C_X[m,k] = C_X[4,1]$
$= Cov[X_4, X_{4+1}]$



Stationary Process

- For a general random process X(t), normally,
  at t_1: $X(t_1)$ has pdf = $f_{X(t_1)}(x)$ [depends on t_1]
- For a stationary random process X(t),
  at t_1: $X(t_1)$ has pdf = $f_{X(t_1)}(x)$ [not depend on t_1]

Stationary Process

= same random variable at all time
= no statistical properties change with time

$$f_{X(t_1)}(x) = f_{X(t_1+\tau)}(x) = f_X(x)$$

Stationary Process

Definition: A stochastic process X(t) is stationary iif for all sets of time $t_1,\ldots,t_m$ and any time different τ,

$$f_{X(t_1),\ldots,X(t_m)}(x_1,\ldots,x_m) = f_{X(t_1+\tau),\ldots,X(t_m+\tau)}(x_1,\ldots,x_m)$$

Stationary Random Sequence

Definition: A random sequence X_n is stationary iif for any finite sets of time instants $n_1,\ldots,n_m$ and any time different k,

$$f_{X_{n_1},\ldots,X_{n_m}}(x_1,\ldots,x_m) = f_{X_{n_1+k},\ldots,X_{n_m+k}}(x_1,\ldots,x_m)$$

Stationary Process

Theorem: A stationary process X(t),

$$\mu_X(t) = \mu_X$$
$$R_X(t,\tau) = R_X(0,\tau) = R_X(\tau)$$
$$C_X(t,\tau) = R_X(0,\tau) - \mu^2_X = C_X(\tau)$$

Stationary Random Sequence

Theorem: A stationary random sequence X_n, for all m

$$E[X_m] = \mu_X$$
$$R_X[m.k] = R_X[0,k] = R_X[k]$$
$$C_X[m,k] = R_X[0,k] - \mu^2_X = C_X[k]$$

Wide Sense Stationary

Definition: $X(t)$ is a wide sense stationary random process iff for all t,

$$E[X(t)] = \mu_X$$
$$R_X(t,\tau) = R_X(0,\tau) = R_X(\tau)$$

Definition: X_n is a wide sense stationary random sequence iff for all n,

$$E[X_n] = \mu_X$$
$$R_X[n,k] = R_X[0,k] = R_X[k]$$

Wide Sense Stationary

- For every **stationary** process or sequence, it is also **wide sense stationary**.
- However, if it is a **wide sense stationary** it may or may not be stationary.

Example

- For n = even
  $X_n = \pm 1$ with prob = ½ (n = even)
- For n = odd
  $X_n = -1/3$ with prob = 9/10
  $X_n = 3$ with prob = 1/10
- Stationary ?
  - No
- Wide sense stationary ?
  - Mean = 0 for all n
  - $C_X(t,\tau) = 0$ for $\tau > 0$
  - $C_X(t,\tau) = 1$ for $\tau = 0$
  - Yes , it's wide sense stationary

Wide Sense Stationary

Theorem: For a wide sense stationary process X(t),

$$R_X(0) \geq 0$$
$$R_X(\tau) = R_X(-\tau)$$
$$|R_X(\tau)| \leq R_X(0)$$

Wide Sense Stationary

Theorem: For a wide sense stationary sequence X_n,

$$R_X[0] \geq 0$$

$$R_X[k] = R_X[-\tau]$$

$$|R_X[k]| \leq R_X[0]$$

Average Power

- From Ohm's Law : $V = IR$
- For v(t), i(t), R Ω, the instantaneous power dissipated, P(t),

$$P(t) = v^2(t)/R = i^2(t)R$$

- For R = 1 Ω, $P(t) = v^2(t) = i^2(t)$
- For a voltage or current is a sample function of random process, x(t,s)

➔ P across 1 Ω resistor = $x^2(t,s)$

Average Power

- Define $x^2(t,s)$ as the instantaneous power of $x(t,s)$
- For a $X(t)$, $X^2(t)$ is the instantaneous of power $X(t)$

Definition:

For a wide sense stationary process $X(t)$,

$$R_X(0) = E[X^2(t)]$$

Homework

- 6.2.2
- 6.3.3
- 6.4.1
- 6.5.3
- 6.7.2
- 6.8.2