Joint CDF

- Pairs of Random Variables
- Discrete:

  Joint PMF $P_{X,Y}(x,y) = P[X=x, Y=y]$
- Continuous:

  $P_{X,Y}(x,y) = 0$ $(P_X(x) = 0, P_Y(y) = 0)$

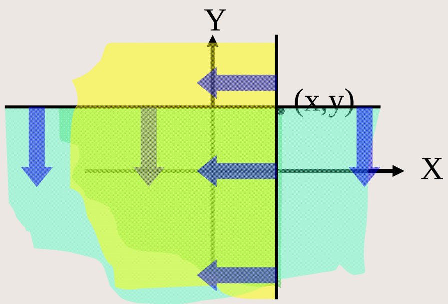
  For 1 RV → interval on real axis

  For 2 RVs → area in a plane

Joint CDF

Definition: Joint CDF of X and Y

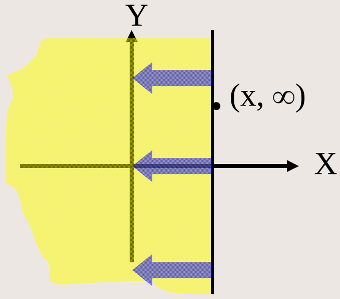
$$F_{X,Y}(x,y) = P[X \le x, Y \le y]$$



Interesting Properties

- For Event $\{X \leq x\}$

$$F_X(x) = P[X \leq x]$$
$$= P[X \leq x,\ Y \leq \infty]$$
$$= \lim_{y \to \infty} F_{X,Y}(x,y)$$
$$= F_{X,Y}(x,\ \infty)$$



Joint CDF

Theorem :

(a) $0 \leq F_{X,Y}(x,y) \leq 1$

(b) $F_X(x) = F_{X,Y}(x,\ \infty)$

(c) $F_Y(y) = F_{X,Y}(\infty,\ y)$

(d) $F_{X,Y}(-\infty,\ y) = F_{X,Y}(x,\ -\infty) = 0$

(e) If $x_1 \geq x$ and $y_1 \geq y$

then $F_{X,Y}(x_1,y_1) \geq F_{X,Y}(x,y)$

(f) $F_{X,Y}(\infty,\ \infty) = 1$

Joint PDF

Definition: Joint PDF of X and Y is satisfied

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)\, dv\, du$$

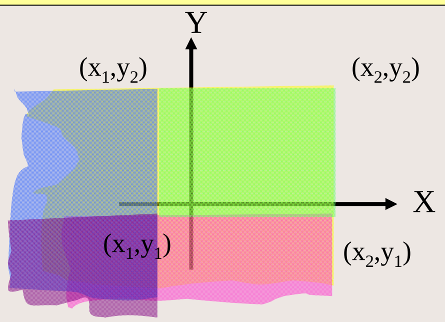
Theorem:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y}$$

Joint CDF

Theorem:

$P[x_1 < X \le x_2, y_1 < Y \le y_2]$
$= F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1)$



Joint PDF

Theorem:

(a) $f_{X,Y}(x,y) \geq 0$ for all (x,y)

(b) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$

Theorem:

$$P[A] = \iint_{A} f_{X,Y}(x,y)\, dx\, dy$$

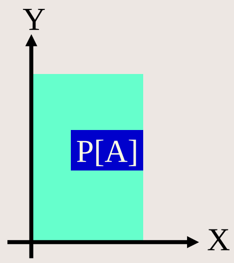
Example

$$f_{X,Y}(x,y) = \begin{cases} c & 0 \leq x \leq 3,\ 0 \leq y \leq 5 \\ 0 & \text{Otherwise} \end{cases}$$

Find constant c

$$\int_0^3 \int_0^5 c\, dx\, dy = 15c = 1$$

$\rightarrow$ c = 1/15



Find P[A] = P[$1 \leq X \leq 3$, $2 \leq Y \leq 3$]

$$P[A] = \int_1^3 \int_2^3 1/15\, dy\, dx = 2/15$$

Marginal PDF

Theorem:

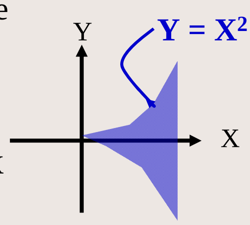
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

Example

$$f_{X,Y}(x,y) = \begin{cases} cx & 0 \le x \le 1,\ |y| < x^2 \\ 0 & \text{Otherwise} \end{cases}$$

Find constant c



$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = \int_0^1 \left(\int_{-x^2}^{x^2} cx\, dy \right) dx$$

$$= \int_0^1 cx\,(2x^2)\, dx = \left. \frac{cx^4}{2} \right|_0^1$$

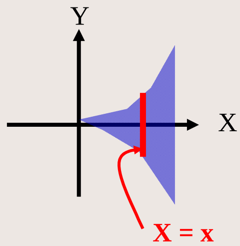
$$= \frac{c}{2} = 1 \quad \rightarrow c = 2$$

Example

Find the marginal PDF $f_X(x)$ and $f_Y(y)$

Fixed x **(X = x)** then integrate all y

$$f_X(x) = \int_{-x^2}^{x^2} 2x \, dy = 4x^3$$

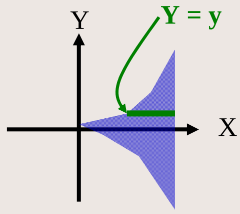


$$f_X(x) = \begin{cases} 4x^3 & 0 \le x \le 1 \\ 0 & \text{Otherwise} \end{cases}$$

Example

Fixed y **(Y = y)** then integrate all x

$$f_{Y,}(y) = \int_{\sqrt{|y|}}^{1} 2x \, dx = 1 - |y|$$



$$f_Y(y) = \begin{cases} 1 - |y| & -1 \le y \le 1 \\ 0 & \text{Otherwise} \end{cases}$$

Functions of 2 RVs

Example:

Wireless base station with 2 antennas. X and Y are RVs of the signal

- Find the strongest signal

  $W = X$ if $|X| > |Y|$ or $W = Y$ otherwise

- Find the addition of 2 signals

  $W = X + Y$

- Find the addition of 2 signals with weight

  $W = aX + bY$

Functions of 2 RVs

$$F_W(w) = P[W \le w] = \iint_{g(x,y) \le w} f_{X,Y}(x,y)\, dx\, dy$$

Example

$$f_{X,Y}(x,y) = \begin{cases} 1/15 & 0 \le x \le 3,\ 0 \le y \le 5 \\ 0 & \text{Otherwise} \end{cases}$$

Find PDF of W = max(X,Y)

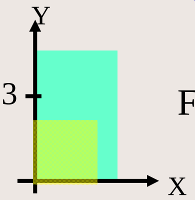
For W = max(X,Y) → $\{W \le w\} = \{X \le w,\ Y \le w\}$

$$F_W(w) = P[X \le w,\ Y \le w]$$

$$= \int_{-\infty}^{w}\int_{-\infty}^{w} f_{X,Y}(x,y)\, dx\, dy$$

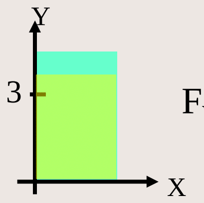
Example

We can divide into 2 cases



$0 \le w \le 3$

$$F_W(w) = \int_0^w\int_0^w 1/15\, dx\, dy = w^2/15$$

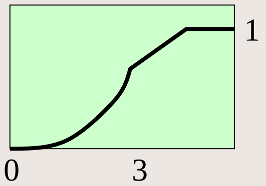


$3 \le w \le 5$

$$F_W(w) = \int_0^w \left(\int_0^3 1/15\, dx \right) dy = w/5$$

Example

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w^2/15 & 0 \le w \le 3 \\ w/5 & 3 < w \le 5 \\ 1 & w > 5 \end{cases}$$



$$f_W(w) = \begin{cases} 2w/15 & 0 \le w \le 3 \\ 1/5 & 3 < w \le 5 \\ 0 & \text{Otherwise} \end{cases}$$