Lecture #5
Discrete Random Variable (2)

Quiz 2.3

- Each time modem transmits one bit, the receiver analyzes whether the bit is 0 or 1.
- The transmitted bit is error with Prob = p
  - If transmission until receiving the 1st error
  - p=0.1, P[X=x]=?
  - Transmitting 100 bits, and number of error is equal to y bits
  - p=0.01, P[Y<=2]=?
  - Transmission continue until find 3 errors
  - p=0.25, P[Z=12]=?

Poisson Random Variable

- Occur randomly in a time period
- Known the average number of occurrences per unit time
- Example:
  - Arrival of packets at each station
  - Initiation of telephone calls
  - Query rate in Search Engine

Poisson Random Variable

Definition: X is a Poisson Random Variable if the PMF of X, $P_X(x)$, has the form:

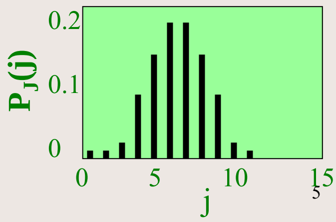
$$P_X(x) = \begin{cases} \dfrac{(\lambda T)^x e^{-(\lambda T)}}{x!} & x = 0,1,2\ldots \\ 0 & \text{Otherwise} \end{cases}$$

λ= average arrival rate in a time interval
T= time interval

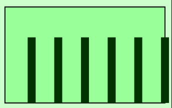
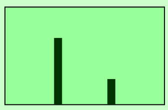
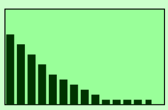
Poisson RV Example

- Call arrive at the telephone office at rate of 0.25 call per second.
- Find the PMF of the number of calls that arrive in any 20 second interval
- **$\lambda T = 0.25 * 20 = 5$**

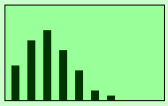
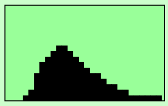
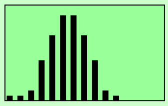
$$P_J(j) = \begin{cases} \dfrac{(5)^j e^{-(5)}}{j!} & j = 0,1,\ldots \\ 0 & \text{Otherwise} \end{cases}$$



Discrete RV Summary

Uniform Equiprobable outcomes	$\begin{cases} 1/(j-k+1) & x = k,k+1,k+2,..,j \\ 0 & \text{Otherwise} \end{cases}$	
Bernoulli Pass/Fail	$\begin{cases} 1 - p & x = 0 \\ p & x = 1 \\ 0 & \text{Otherwise} \end{cases}$	
Geometric # tests until fail	$\begin{cases} p(1 - p)^{x-1} & x = 1,2,3,\ldots \\ 0 & \text{Otherwise} \end{cases}$	

Discrete RV Summary

Binomial # fails in n tests	$\begin{cases} \binom{n}{x} p^x(1-p)^{n-x} & x=1,2,\ldots,n \\ 0 & \text{Otherwise} \end{cases}$	
Pascal # tests until k fails	$\begin{cases} \binom{x-1}{k-1} p^k(1-p)^{x-k} & x=k,k+1,\ldots \\ 0 & \text{Otherwise} \end{cases}$	
Poisson occurrence in a period	$\begin{cases} \dfrac{(\lambda T)^x e^{-(\lambda T)}}{x!} & x=0,1,2\ldots \\ 0 & \text{Otherwise} \end{cases}$	

Cumulative Distribution Function (CDF)

- Definition:

$$F_X(x) = P[X \le x]$$

- Contain complete information about the probability model of the random variable
- PMF ⟺ CDF

CDF Theorem

Theorem: For a discrete random variable X

with $S_X=\{x_1,x_2,\ldots\}$ & $\mathbf{x_1 \le x_2 \le \ldots}$

1) $F_x(-\infty) = 0$ and $F_x(\infty) = 1$ → **From 0 to 1**

2) $\forall x' \ge x,\ F_x(x') \ge F_x(x)$ → **Monotonic Increasing**

3) For $x_i \in S_x$ and ε = +small number

$F_x(x_i) - F_x(x_i - \varepsilon) = P_x(x_i)$ → **Discontinuity = $P_X(x)$**

4) $F_x(x) = F_x(x_i) \quad \forall x,\ x_i \le x < x_{i+1}$ → **Horizon line**

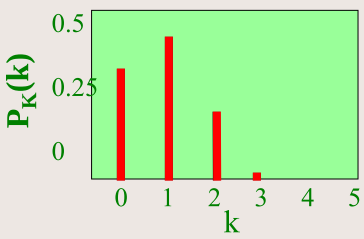
CDF Example

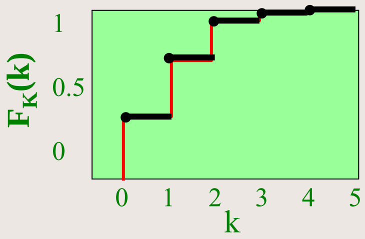
- For a binomial RV, # of fail programs in 5 tests with p = 0.2

$$P_K(k) = \begin{cases} \binom{5}{k}(0.2)^k(0.8)^{5-k} & k = 0,1,2,\ldots,5 \\ 0 & \text{Otherwise} \end{cases}$$

CDF Example

k	$P_K(k)$	k	$P_K(k)$
0	0.33	3	0.05
1	0.41	4	0.01
2	0.20	5	0





Note: $F_K(k)$ is continuous from right

More CDF Theorem

$$\forall b \geq a,$$
$$F_X(b) - F_X(a) = P[a < X \leq b]$$

Difference of the CDF is the probability that RV takes on the value between two points

Quiz 2.4

Average

- Study RV → average
- What is the average of an RV?
  - A single number that describes the RV
  - An example of statistic
- What is Statistic?
  - Numbers that collect all information of things under our interesting
  - Averages: mean, mode, and median

Average

- Mean:
  - Sum / #terms
- Mode:
  - Most common value
  - $P_X(x_{mod}) \geq P_X(x) \quad \forall x$
- Median:
  - The middle of the data set
  - $P[X < x_{med}] = P[X > x_{med}]$

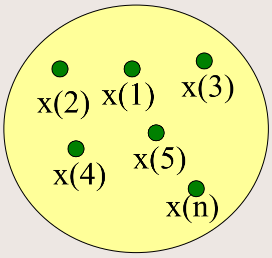
Mean → Expected Value

- Adding all measurements / #terms

$$E[X] = \mu_X = \sum_{x \in S_X} x P_X(x)$$

Expected Value

- Experiment → Random Variable X
- Perform n independent trials
- The value X takes on i^{th} trial → x(i)

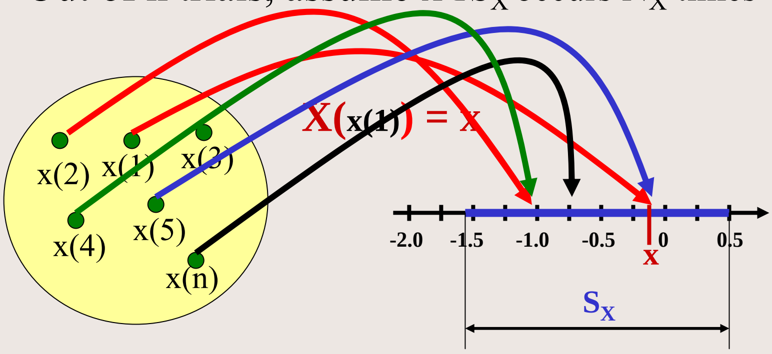


The average

$$m_n = \frac{1}{n}\sum_{i=1}^{n} x(i)$$

Expected Value

- Each x(i) take values in the set S_x
- Out of n trials, assume $x \in S_X$ occurs N_X times



Expected Value

$$m_n = \frac{1}{n}\sum_{i=1}^{n} x(i)$$

$$m_n = \frac{1}{n}\sum_{x \in S_x} N_x x \quad = \sum_{x \in S_x} \frac{N_x}{n} x$$

$$P[A] = \lim_{n \to \infty} \frac{N_A}{n} \quad \Rightarrow \quad P_X(x) = \lim_{n \to \infty} \frac{N_X}{n}$$

$$\lim_{n \to \infty} m_n = \sum_{x \in S_x} x P_X(x)$$

$$\boxed{E[X] = \sum_{x \in S_X} x P_X(x)}$$

Expected Value

$$\boxed{E[X] = \sum_{x \in S_X} x P_X(x)}$$

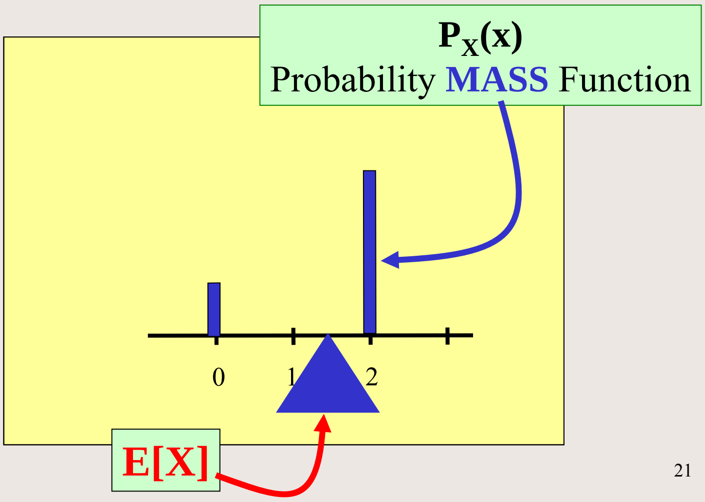
- Example:

$$P_T(t) = \begin{cases} 1/4 & t = 0 \\ 3/4 & t = 2 \\ 0 & \text{Otherwise} \end{cases}$$

- $E[T] = ?$

$$= 0(1/4) + 2(3/4) \quad = 3/2$$

Expected Value



Bernoulli Expected Value

$$P_X(x) = \begin{cases} 1 - p, & x = 0 \\ p, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = \sum_{x \in S_X} x P_X(x)$$

$$S_X = \{0,1\}$$
$$E[X] = 0(1-p) + 1(p)$$
$$= p$$

Geometric Expected Value

$$P_X(x) = \begin{cases} p(1-p)^{x-1}, & x = 1,2,3,\ldots \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = \sum_{x \in S_X} x P_X(x)$$

$$= \sum_{x=1}^{\infty} xp(1-p)^{x-1} \qquad = \sum_{x=1}^{\infty} xpq^{x-1}$$

$$= p\sum_{x=1}^{\infty} xq^{x-1} \; ? \quad \Rightarrow \quad \boxed{E[X] = \frac{1}{p}}$$

Geometric Series

$$S_n = \sum_{x=1}^{n} xq^{x-1}$$

$$\begin{aligned}
S_n &= 1 + 2q + 3q^2 + 4q^3 + \ldots + nq^{n-1} \\
q\,S_n &= \phantom{1 +{}} q + 2q^2 + 3q^3 + \ldots + (n-1)q^{n-1} + nq^n \\
(1-q)\,S_n &= 1 + q + q^2 + q^3 + \ldots + q^{n-1} - nq^n \\
q(1-q)\,S_n &= \phantom{1 +{}} q + q^2 + q^3 + \ldots + q^n - nq^{n+1} \\
(1-q)^2\,S_n &= 1 - (n+1)q^n + nq^{n+1}
\end{aligned}$$

$$S_n = \frac{1}{(1-q)^2} - \frac{(n+1)q^n}{(1-q)^2} + \frac{nq^{n+1}}{(1-q)^2}$$

$$S_\infty = \frac{1}{p^2} \quad \Rightarrow \quad \boxed{E[X] = p\sum_{x=1}^{\infty} xq^{x-1} = \frac{1}{p}}$$

Geometric Expected Value

- **From example**:
  Find the number of tests until find a fail program
- We have p = 0.2
  → 2/10 → 1/5
- E[X] = 1/p = 5
- Intuitively, on average, we will find the fail program after 5 tests.

Poisson Expected Value

$$P_X(x) = \begin{cases} \dfrac{\alpha^x e^{-\alpha}}{x!}, & x = 0,1,2\ldots \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = \sum_{x=0}^{\infty} x P_X(x) = \sum_{x=0}^{\infty} x \frac{\alpha^x}{x!} e^{-\alpha}$$

(x=0) → 0

$$= \alpha \sum_{x=1}^{\infty} \frac{\alpha^{x-1}}{(x-1)!} e^{-\alpha} = \alpha \underbrace{\sum_{k=0}^{\infty} \frac{\alpha^k}{k!}}_{e^{\alpha}} e^{-\alpha}$$

$$= \alpha e^{\alpha} e^{-\alpha} \Rightarrow \boxed{E[X] = \alpha}$$

Discrete RV Summary

Uniform Equiprobable outcomes	$\begin{cases} 1/(j-k+1) & x = k,k+1,k+2,..,j \\ 0 & \text{Otherwise} \end{cases}$	$E[X] = \frac{(j+k)}{2}$
Bernoulli Pass/Fail	$\begin{cases} 1-p & x = 0 \\ p & x = 1 \\ 0 & \text{Otherwise} \end{cases}$	$E[X] = p$
Geometric # tests until fail	$\begin{cases} p(1-p)^{x-1} & x = 1,2,3,\ldots \\ 0 & \text{Otherwise} \end{cases}$	$E[X] = 1/p$

Discrete RV Summary

Binomial # fails in n tests	$\begin{cases} \binom{n}{x} p^x(1-p)^{n-x} & x=1,2,\ldots,n \\ 0 & \text{Otherwise} \end{cases}$	$E[X] = np$
Pascal # tests until k fails	$\begin{cases} \binom{x-1}{k-1} p^k(1-p)^{x-k} & x = k,k+1,\ldots \\ 0 & \text{Otherwise} \end{cases}$	$E[X] = k/p$
Poisson occurrence in a period	$\begin{cases} \frac{(\lambda T)^x e^{-(\lambda T)}}{x!} & x = 0,1,2\ldots \\ 0 & \text{Otherwise} \end{cases}$	$E[X] = \alpha$ $\alpha = \lambda T$

Binomial → Poisson

Theorem: Let $p = \alpha/n$ $(\alpha > 0 \text{ and } n > \alpha)$

Binomial PMF → Poisson PMF (parameter α)

$$P_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x=0,1,2,\dots,n \\ 0 & \text{Otherwise} \end{cases}$$

Binomial → Poisson

$$P_X(x) = \begin{cases} \binom{n}{x} \left(\frac{\alpha}{n}\right)^x \left(1-\frac{\alpha}{n}\right)^{n-x} & x=0,1,2,\dots,n \\ 0 & \text{Otherwise} \end{cases}$$

$$= \frac{n(n-1)\dots(n-x+1)}{n^x} \frac{\alpha^x}{x!} \left(1-\frac{\alpha}{n}\right)^{n-x}$$

$$\boxed{\lim_{n\to\infty} \frac{(n-j)}{n} = 1}$$

$$\boxed{\frac{\left(1-\frac{\alpha}{n}\right)^n}{\left(1-\frac{\alpha}{n}\right)^x} \quad \left(1-\frac{\alpha}{n}\right)^n \to e^{-\alpha}, \quad \left(1-\frac{\alpha}{n}\right)^x \to 1}$$

$$\lim_{n\to\infty} P_X(x) = \begin{cases} \frac{\alpha^x e^{-\alpha}}{x!}, & x = 0,1,2\dots \\ 0, & \text{Otherwise} \end{cases}$$

Quiz 2.5