Some Useful Continuous RVs

- Uniform
- Exponential
- Gaussian

Uniform Continuous RV

Definition:

$$f_X(x) = \begin{cases} 1/(b-a) & a \le x < b \\ 0 & \text{Otherwise} \end{cases}$$

where b > a

Uniform Continuous RV

Theorem:

- $$F_X(x) = \begin{cases} 0 & x \le a \\ (x-a)/(b-a) & a < x \le b \\ 1 & x > b \end{cases}$$
- $E[X] = (b + a)/2$
- $Var[X] = (b - a)^2/12$

Exponential Continuous RV

Definition:

$$f_X(x) = \begin{cases} a\,e^{-ax} & x \ge 0 \\ 0 & \text{Otherwise} \end{cases}$$

where $a > 0$

Poisson distribution: เรานับจำนวนความสำเร็จ หรือสิ่งที่สนใจที่เกิดขึ้นในช่วงระยะเวลาหนึ่งที่กำหนดให้

Exponential distribution: แทนช่วงระยะเวลาของการรอคอยจนกระทั่งกิดความสำเร็จเป็นครั้งแรก

Exponential Continuous RV

Theorem:

- $$F_X(x) = \begin{cases} 1 - e^{-ax} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$
- $E[X] = 1/a$
- $Var[X] = 1/a^2$

Example 3.12
Example 3.13
Theorem 3.9 ***
Example 3.14
6.10

Gaussian Random Variables

Definition:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

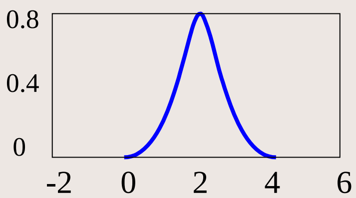
where $\mu \in$ Real, and $\sigma > 0$

• Gaussian RV → **Normal** RV

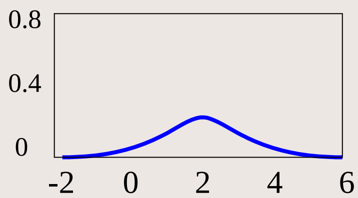
$N[\mu, \sigma^2]$

Normal Mean Variance

Gaussian Random Variables



$\mu = 2, \ \sigma = 1/2$



$\mu = 2, \ \sigma = 2$

$f_X(x)$ → Bell Shape with 2 parameters: μ and σ

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$E[X] = \mu \qquad Var[X] = \sigma^2$

Linear Transformation

Theorem :

If X is a **Gaussian RV** with μ and σ then

$Y = aX + b$ *(also Gaussian)* with $\mathbf{a\mu + b}$ and $\mathbf{a\sigma}$

Standard Normal RV

Definition :

Standard Normal RV Z

is the **Gaussian RV** with $\mu = 0,\ \sigma = 1$

Definition :

Standard Normal CDF is

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{\frac{-u^2}{2}} du$$

Gaussian RV with μ and σ

Transform X → Z

Theorem:

- For a Gaussian RV with μ and σ

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

- For X is in the interval (a,b]

$$P[a < X \le b] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Standard Normal CDF Φ(z) Table

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	0.50	0.6915	1.00	0.8413	1.50	0.9332
0.01	0.5040	0.51	0.6950	1.01	0.8438	1.51	0.9345
0.02	0.5080	0.52	0.6985	1.02	0.8461	1.52	0.9357
0.03	0.5120	0.53	0.7019	1.03	0.8485	1.53	0.9370
0.04	0.5160	0.54	0.7054	1.04	0.8508	1.54	0.9382
0.05	0.5199	0.55	0.7088	1.05	0.8531	1.55	0.9394
0.06	0.5239	0.56	0.7123	1.06	0.8554	1.56	0.9406
0.07	0.5279	0.57	0.7157	1.07	0.8577	1.57	0.9418
0.08	0.5319	0.58	0.7190	1.08	0.8599	1.58	0.9429
0.09	0.5359	0.59	0.7224	1.09	0.8621	1.59	0.9441
0.10	0.5398	0.60	0.7257	1.10	0.8643	1.60	0.9452
0.11	0.5438	0.61	0.7291	1.11	0.8665	1.61	0.9463
0.12	0.5478	0.62	0.7324	1.12	0.8686	1.62	0.9474

for $0 \le z \le 2.99$

Example

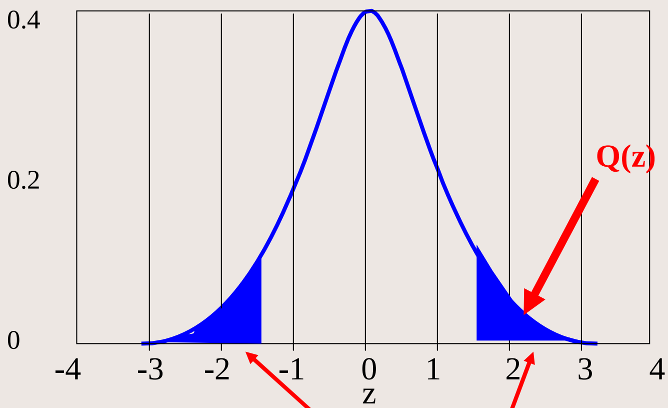
- Use a protocol analyzer to capture packets
- For a packet with size x = 2048 bytes.
- The expected value of packet size is 1024 bytes with the standard deviation 512 bytes

$$z = \frac{x-\mu}{\sigma} = \frac{(2048-1024)}{512} = 2.0$$

$$F_X(2048) = \Phi(2.0) = 0.97725$$

- For x = 768 bytes → z = – 1.5

PDF of $f_Z(z)$ with zero mean



Theorem: $\Phi(-z) = 1 - \Phi(z)$

Standard Normal Complementary CDF Q(z)

Definition :

$$Q(z) = P[Z > z]$$

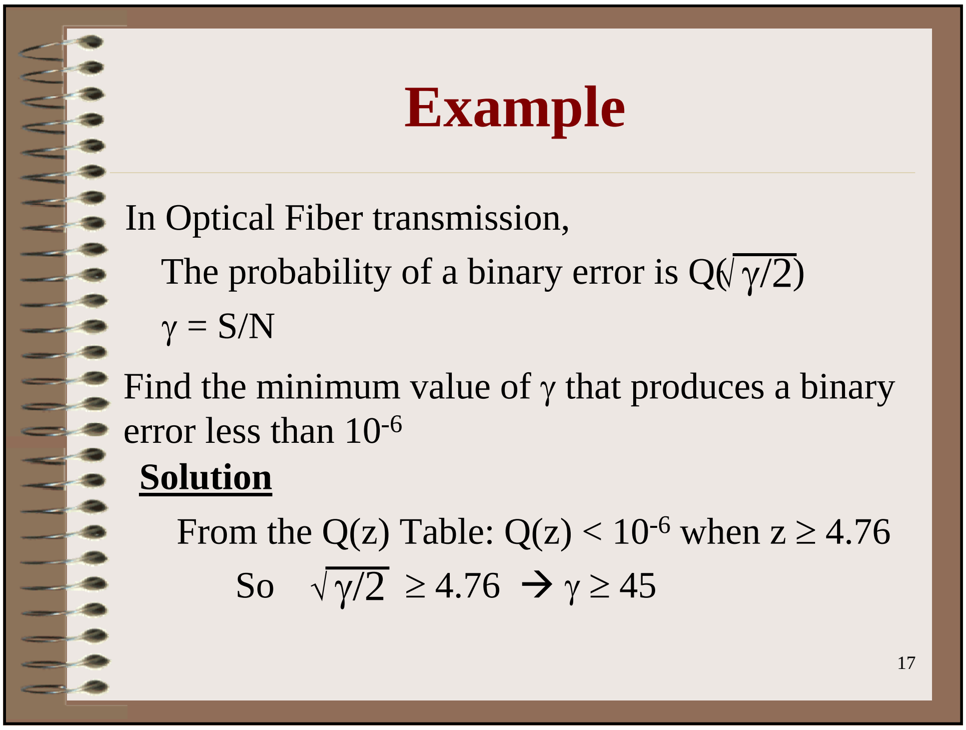
$$= \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{\frac{-u^2}{2}} \, du$$

$$= 1 - \Phi(z)$$

Standard Normal Complementary CDF Q(z) Table

z	$Q(z)$	z	$Q(z)$	z	$Q(z)$	z	$Q(z)$
3.00	$1.35\cdot10^{-3}$	3.40	$3.37\cdot10^{-4}$	3.80	$7.23\cdot10^{-5}$	4.20	$1.33\cdot10^{-5}$
3.01	$1.31\cdot10^{-3}$	3.41	$3.25\cdot10^{-4}$	3.81	$6.95\cdot10^{-5}$	4.21	$1.28\cdot10^{-5}$
3.02	$1.26\cdot10^{-3}$	3.42	$3.13\cdot10^{-4}$	3.82	$6.67\cdot10^{-5}$	4.22	$1.22\cdot10^{-5}$
3.03	$1.22\cdot10^{-3}$	3.43	$3.02\cdot10^{-4}$	3.83	$6.41\cdot10^{-5}$	4.23	$1.17\cdot10^{-5}$
3.04	$1.18\cdot10^{-3}$	3.44	$2.91\cdot10^{-4}$	3.84	$6.15\cdot10^{-5}$	4.24	$1.12\cdot10^{-5}$
3.05	$1.14\cdot10^{-3}$	3.45	$2.80\cdot10^{-4}$	3.85	$5.91\cdot10^{-5}$	4.25	$1.07\cdot10^{-5}$
3.06	$1.11\cdot10^{-3}$	3.46	$2.70\cdot10^{-4}$	3.86	$5.67\cdot10^{-5}$	4.26	$1.02\cdot10^{-5}$
3.07	$1.07\cdot10^{-3}$	3.47	$2.60\cdot10^{-4}$	3.87	$5.44\cdot10^{-5}$	4.27	$9.77\cdot10^{-6}$
3.08	$1.04\cdot10^{-3}$	3.48	$2.51\cdot10^{-4}$	3.88	$5.22\cdot10^{-5}$	4.28	$9.34\cdot10^{-6}$

for $3.00 \le z \le 4.99$


Example
In Optical Fiber transmission,
The probability of a binary error is $Q(\sqrt{\gamma/2})$
$\gamma = S/N$
Find the minimum value of γ that produces a binary error less than 10^{-6}
Solution
From the Q(z) Table: $Q(z) < 10^{-6}$ when $z \geq 4.76$
So $\sqrt{\gamma/2} \geq 4.76 \rightarrow \gamma \geq 45$
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