Homework 07 (to 4.1.1 – 4.1.6) + Proof exponential continuous RV

นายโสภณ สุขสมบูรณ์
62010631188 sec.1 Commu Math.

## Exponential Continuous RV

Form Theorem CDF of X

$$F_X(x) = \begin{cases} 1-e^{-ax} &, x \geq 0 \\ 0 &, \text{otherwise} \end{cases}$$

เนื่องจาก $f_X(x) = \frac{d}{dx} F_X(x)$

ดังนั้น

$$\left. f_X(x) = \begin{cases} ae^{-ax} &, x \geq 0 \\ 0 &, \text{otherwise} \end{cases} \right\} \text{PDF of } X$$

และ Expected Value

พิจารณา

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_0^{\infty} x(ae^{-ax})\,dx \quad \text{— (1)}$$

ใช้ By Part method : $\int u\,dv = uv - \int v\,du$

พิจารณา $\int x(ae^{-ax})\,dx$

ให้ $u = x \Rightarrow du = dx$ และ $dv = ae^{-ax}dx \Rightarrow v = -e^{-ax}$

จะได้

$$\int x(ae^{-ax})\,dx = -xe^{-ax} + \int e^{-ax}\,dx$$

$$= -xe^{-ax} - \frac{e^{-ax}}{a} \quad \text{— (2)}$$

แทน (2) ลงใน (1) ;

$$E[X] = \left[ \frac{-x}{e^{ax}} - \frac{1}{ae^{ax}} \right]_0^{\infty}$$

เนื่องจากอินทิกรัลนี้เป็นอินทิกรัลไม่ตรงแบบ ;

$$E[X] = \lim_{B\to\infty} \left[ \frac{-x}{e^{ax}} - \frac{1}{ae^{ax}} \right]_0^{B}$$

$$= \lim_{B\to\infty} \left[ \frac{-B}{e^{aB}} - \frac{1}{ae^{aB}} \right] + 0 + \frac{1}{a}$$

$$= \lim_{B\to\infty} \left[ \frac{-1}{ae^{aB}} - 0 \right] + \frac{1}{a} \qquad \text{(L'hopital)}$$

$$= 0 + \frac{1}{a}$$

$$\therefore E[X] = \frac{1}{a} \quad \#$$

และพิจารณาหาค่า Variance ของ X

จากสูตร $\quad Var[X] = E[X^2] - (E[X])^2 \quad —(3)$

พิจารณา $E[X^2] = \int_0^{\infty} x^2 (ae^{-ax})\,dx \quad$ : using by part method

Improper Integral

| u | dv |
|---|---|
| $x^2$ | $+\ ae^{-ax}$ |
| $2x$ | $-\ -e^{-ax}$ |
| $2$ | $+\ \frac{1}{a}e^{-ax}$ |
| $0$ | $-\ \frac{1}{a^2}e^{-ax}$ |

$\int$

$$= \left[ -x^2e^{-ax} - \frac{2x}{a}e^{-ax} - \frac{2}{a^2}e^{-ax} \right]_0^{\infty}$$

$$= \lim_{B\to\infty} \left[ -\frac{x^2}{e^{ax}} - \frac{2x}{ae^{ax}} - \frac{2}{a^2e^{ax}} \right]_0^{B}$$

0 limit $x\to\infty$ 0 0 $\qquad$ limit $x=0$

$$= \lim_{B\to\infty} \left[ \cancel{-\frac{B^2}{e^{aB}}} - \cancel{\frac{2B}{ae^{aB}}} - \cancel{\frac{2}{a^2e^{aB}}} \right] \underbrace{+0+0}+\frac{2}{a^2} \quad —(4)$$

แทน $E[X^2] = \frac{2}{a^2}$ ฯน(3) ;

$$Var[X] = \frac{2}{a^2} - \frac{1}{a^2}$$

$$\therefore Var[X] = \frac{1}{a^2} \quad \#$$

---

**4.1.1** Random variables $X$ and $Y$ have the joint CDF

$$F_{X,Y}(x, y) = \begin{cases} (1 - e^{-x})(1 - e^{-y}) & x \geq 0; \; y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is $P[X \leq 2, Y \leq 3]$?

(b) What is the marginal CDF, $F_X(x)$?

(c) What is the marginal CDF, $F_Y(y)$?

a) $P[X \leq 2, Y \leq 3] = P_{X,Y}(2, 3) = (1-e^{-2})(1-e^{-3})$

จาก ทฤษฎีบท 4.1 ได้ว่า $F_X(x) = F_{X,Y}(x, \infty)$ , $F_Y(y) = F_{X,Y}(\infty, y)$

(b) $$F_X(x) = F_{X,Y}(x, \infty) = \begin{cases} 1 - e^{-x} & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

(c) $$F_Y(y) = F_{X,Y}(\infty, y) = \begin{cases} 1 - e^{-y} & , y \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

**4.1.2** Express the following extreme values of $F_{X,Y}(x, y)$ in terms of the marginal cumulative distribution functions $F_X(x)$ and $F_Y(y)$.

(a) $F_{X,Y}(x, -\infty)$

(b) $F_{X,Y}(x, \infty)$

(c) $F_{X,Y}(-\infty, \infty)$

(d) $F_{X,Y}(-\infty, y)$

(e) $F_{X,Y}(\infty, y)$

$Y \to -\infty \Rightarrow P[Y] = 0$

a) $F_{X,Y}(x,-\infty) = P[X \le x, Y \le -\infty] \le P[Y \le -\infty] = 0$ #

b) $F_{X,Y}(x,\infty) = P[X \le x, Y \le \infty] = P[X \le x] = P_X(x)$ #

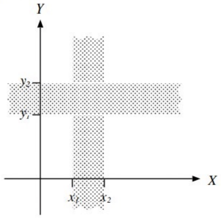
c) $F_{X,Y}(-\infty,\infty) = P[X \le -\infty, Y \le \infty] \le P[X \le -\infty] = 0$ #

d) $F_{X,Y}(-\infty,y) = P[X \le -\infty, Y \le y] \le P[X \le -\infty] = 0$ #

e) $F_{X,Y}(\infty,y) = P[X \le \infty, Y \le y] = P[Y \le y] = P_Y(y)$ #

---

**4.1.3** For continuous random variables $X, Y$ with joint CDF $F_{X,Y}(x, y)$ and marginal CDFs $F_X(x)$ and $F_Y(y)$, find $P[x_1 \le X < x_2 \cup y_1 \le Y < y_2]$. This is the probability of the shaded "cross" region in the following diagram.




** Formula : $P[A \cup B] = P[A] + P[B] - P[A \cap B]$ **

ให้ $P[A \cup B] = P[x_1 \le X \le x_2 \cup y_1 \le Y \le y_2]$

จาก Formula :

$$\underbrace{P[x_1 \le X \le x_2 \cup y_1 \le Y \le y_2]}_{P[A \cup B]} = \underbrace{P[x_1 \le X \le x_2]}_{P[A]} + \underbrace{P[y_1 \le Y \le y_2]}_{P[B]} - \underbrace{P[x_1 \le X \le x_2, y_1 \le Y \le y_2]}_{P[A \cap B]} \quad \text{— (1)}$$

ทฤษฎีบท : $P[x_1 \le X \le x_2, y_1 \le Y \le y_2] = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1)$

ตามทฤษฎีบทก่อนหน้า จะได้ว่า

$$P[x_1 \le X \le x_2 \cup y_1 \le Y \le y_2] = F_X(x_2) - F_X(x_1) + F_Y(y_2) - F_Y(y_1) - F_{X,Y}(x_2,y_2) -$$
$$F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1) \quad ※$$

---

**4.1.4** Random variables $X$ and $Y$ have CDF $F_X(x)$ and $F_Y(y)$. Is $F(x, y) = F_X(x)F_Y(y)$ a valid CDF? Explain your answer.

let $\quad P[x_1 \le X \le x_2,\ y_1 \le Y \le y_2] \ge 0$

* โจทย์กำหนดว่า $F(x,y) = F_X(x)F_Y(y)$ *

$$P[x_1 \le X \le x_2,\ y_1 \le Y \le y_2] = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1)$$

Form Theorem

$$= F_X(x_2)F_Y(y_2) - F_X(x_2)F_Y(y_1) - F_X(x_1)F_Y(y_2) + F_X(x_1)F_Y(y_1)$$

$$= F_X(x_2)[F_Y(y_2) - F_Y(y_1)] + F_X(x_1)[F_Y(y_2) - F_Y(y_1)]$$

$$= [F_X(x_2) - F_X(x_1)][F_Y(y_2) - F_Y(y_1)]$$

$$= F_X(x)F_Y(y) \quad ; x_1 \le x \le x_2 \text{ and } y_1 \le y \le y_2$$

\#

**4.1.5** In this problem, we prove Theorem 4.5.

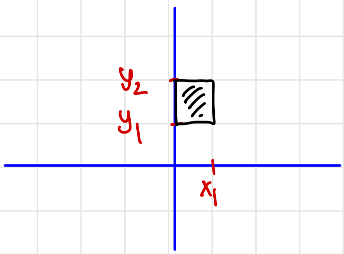
(a) Sketch the following events on the $X, Y$ plane:

$$A = \{X \le x_1, y_1 < Y \le y_2\},$$
$$B = \{x_1 < X \le x_2, Y \le y_1\},$$
$$C = \{x_1 < X \le x_2, y_1 < Y \le y_2\}.$$

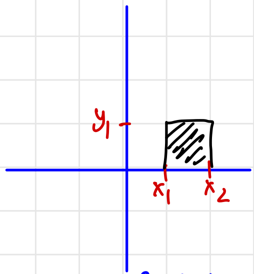
(b) Express the probability of the events $A$, $B$, and $A \cup B \cup C$ in terms of the joint CDF $F_{X,Y}(x, y)$.

(c) Use the observation that events $A$, $B$, and $C$ are mutually exclusive to prove Theorem 4.5.

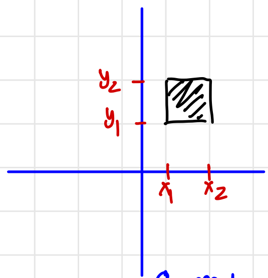
a)




A- region




B-region




C-region

b) $P[A] = F_{x,y}(x_1, y_2) - F_{x,y}(x_1, y_1)$ #

$P[B] = F_{x,y}(x_2, y_1) - F_{x,y}(x_1, y_1)$ #

$P[C] = F_{x,y}(x_2, y_2) - F_{x,y}(x_2, y_1) - F_{x,y}(x_1, y_2) + F_{x,y}(x_1, y_1)$

Disjoint.

$P[A \cup B \cup C] = P[A] + P[B] + P[C] - \boxed{P[A \cap B] - P[A \cap C] - P[B \cap C] - P[A \cap B \cap C]}$

$= \cancel{F_{x,y}(x_1, y_2)} - F_{x,y}(x_1, y_1) + \cancel{F_{x,y}(x_2, y_1)} - \cancel{F_{x,y}(x_1, y_1)} +$

$F_{x,y}(x_2, y_2) - \cancel{F_{x,y}(x_2, y_1)} - \cancel{F_{x,y}(x_1, y_2)} + \cancel{F_{x,y}(x_1, y_1)}$

$\therefore P[A \cup B \cup C] = F_{x,y}(x_2, y_2) - F_{x,y}(x_1, y_1)$ #

c) Theorem 4.5 : $P[x_1 \le X \le x_2, y_1 \le Y \le y_2] = F_{x,y}(x_2, y_2) - F_{x,y}(x_2, y_1) - F_{x,y}(x_1, y_2) + F_{x,y}(x_1, y_1)$

b/c: mutually exculsive : $P[A \cup B \cup C] = P[A] + P[B] + P[C]$

เพื่อหา $P[C] = P[x_1 \leq X \leq x_2,\ y_1 \leq Y \leq y_2]$ ;

$$P[C] = P[A \cup B \cup C] - P[A] - P[B]$$

จากข้อ b) จะได้ว่า

$$P[C] = F_{X,Y}(x_2, y_2) - \cancel{F_{X,Y}(x_1, y_1)} - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1)$$
$$- F_{X,Y}(x_2, y_1) + \cancel{F_{X,Y}(x_1, y_1)}$$

$$\therefore P[C] = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1) \quad \#$$

เป็นไปตามทฤษฎีบทที่ 4.5 #

**4.1.6** Can the following function be the joint CDF of random variables $X$ and $Y$?

$$F(x, y) = \begin{cases} 1 - e^{-(x+y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

CDF $F_X(x)$ & $F_Y(y)$

$$F_X(x) = F_X(x, \infty) = \begin{cases} 1 & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$F_Y(y) = F_Y(\infty, y) = \begin{cases} 1 & , y \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

เมื่อ $X \geq x$ และ $Y \geq y \rightarrow P[X \geq x] = 0,\ P[Y \geq y] = 0$

$$\underbrace{P[X \geq x \cup Y \geq y]}_{F_{X,Y}(x,y)} \leq P[X \geq x] + P[Y \geq y]$$

เพื่อหา $P[X \leq x, Y \leq y] + P[X \geq x \cup Y \geq y] = 1$

$$P[X \geq x \cup Y \geq y] = 1 - 1 + e^{-(x+y)}$$

จะเห็นว่า เมื่อ $X \geq x$ และ $Y \geq y \rightarrow P[\ldots] = e^{-(x+y)} \rightarrow$ จะมีค่าเป็นลบ.

ค่า Prob ไม่สามารถติดลบได้ ดังนั้น CDF ข้างต้นไม่ใช่ Joint CDF #