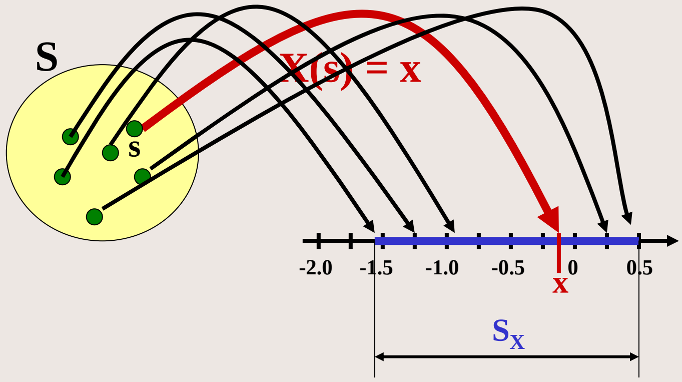
Random Variable

X is a function that maps each outcome, s, in S to a real number $X(s)$, x



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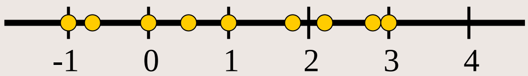
Continuous Sample Space



In **Discrete**: countable set of numbers

$S_X = \{-1,0,1,3,4\}$

$S_Y = \{-1,-0.9,0,0.5,1,1.8,2.25,2.9,3\}$

In **Continuous**: uncountable set of numbers

S_X = **Interval** between 2 limits

$S_X = (x_1,x_2) = (-1,3)$

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Continuous Set Boundary

(x_1,x_2) → (lower limit, upper limit)

- $(x_1,x_2) = \{x \mid x_1 < x < x_2\}$
- $[x_1,x_2] = \{x \mid x_1 \le x \le x_2\}$
- $[x_1,x_2) = \{x \mid x_1 \le x < x_2\}$
- $(x_1,x_2] = \{x \mid x_1 < x \le x_2\}$

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Continuous Random Variables

- Measuring T, the eating time of a student

  $S_T = \{t \mid 0 \le t \le 120\}$
- Measuring V, voltage across a resistor

  $S_V = \{v \mid -\infty < v < \infty\}$
- Measuring P, a program file download time

  $S_P = \{p \mid 0 < p < \infty\}$
- Measuring D, the distance of wireless connection from Access Point

  $S_D = \{d \mid 0 \le d \le 100\}$

T, V, P, D → Continuous Random Variables

Probability of a continuous RV outcome

- Measuring P, a program file download time

  $S_P = \{p \mid 0 < p < \infty\}$
- Guess the download time is (0, 10] minutes
- Guess the download time is [5, 8] minutes
- Guess the download time is [5, 5.5] minutes

  Chance that our guess is correct is decreasing
- Guess the download time is exactly 5.25 min.

Probability of each individual outcome is zero.
The interesting probability is an **interval.**

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Cumulative Distribution Function

- In discrete:

  Probability Mass Function (PMF), $P_X(x)$
- In continuous:

  Cumulative Distribution Function (CDF)

Definition: $F_X(x) = P[X \leq x]$

- Contain complete information about the probability model of the random variable
- PMF ⟺ CDF

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CDF Theorem

Theorem:

- $F_X(-\infty) = 0$
- $F_X(\infty) = 1$
- $P[x_1 < X \leq x_2] = F_X(x_2) - F_X(x_1)$

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Games & Probability

Roulette

Wheel of Fortune

Gamble is a disastrous habit

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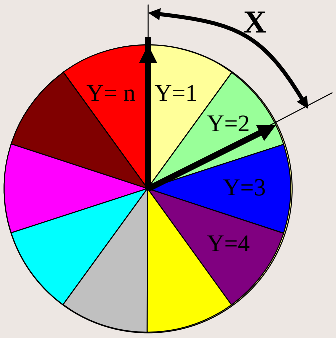
Example



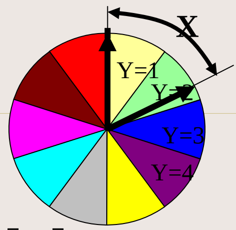
X : Distance, $0 \le X < 1$
$P[X = x] = ?$
$F_X(x) = ?$
Let Y = # of arc that pointer stops
$S_Y = \{1,2,3,\ldots,n\}$

$$P_Y(y) = \begin{cases} 1/n & y = 1,2,\ldots, n \\ 0 & \text{Otherwise} \end{cases}$$

Event $\{X = x\} \subset \{Y = \lceil nx \rceil\}$
For n = 10, x = 0.17 → $\lceil nx \rceil = 2$
$P[X = x] \le P[Y = \lceil nx \rceil] = 1/n$

$n \to \infty$ $\quad$ $P[X = x] = 0$

Example

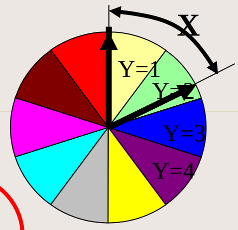


- Event $\{X = x\} \subset \{Y = \lceil nx \rceil\}$
- Event $\{X \le x\} \subset \{Y \le \lceil nx \rceil\}$
- $P\{Y \le \lceil nx \rceil - 1\} < P\{X \le x\} < P\{Y \le \lceil nx \rceil\}$
- $F_Y(\lceil nx \rceil - 1) \; < \; F_X(x) \; < \; F_Y(\lceil nx \rceil)$
- Y: uniform PMF

$$F_Y(y) = \begin{cases} 0, & y < 1 \\ k/n, & k \le y < (k+1), \quad k = 1,2,\ldots, n \\ 1, & y \ge n \end{cases}$$

$$\frac{\lceil nx \rceil - 1}{n} \le F_X(x) \le \frac{\lceil nx \rceil}{n}$$

Example



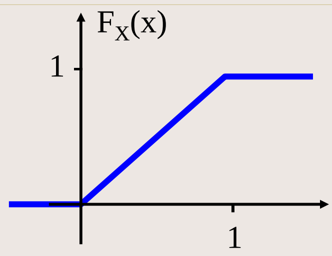
$$\frac{\lceil nx \rceil - 1}{n} \le F_X(x) \le \frac{\lceil nx \rceil}{n}$$

x

As $n \to \infty$,

Example

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$



- **P[X ≤ 0.5] = ?**

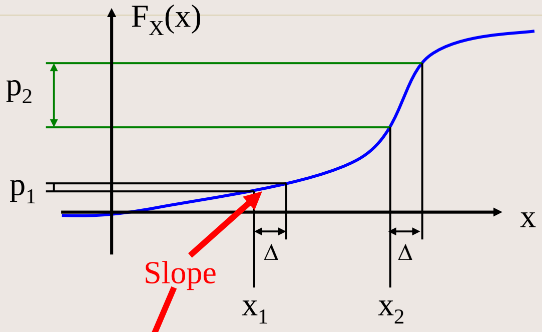
  $= F_X(0.5) \;=\; 0.5$
- **P[X ≤ – 2] = ?**

  $= F_X(-2) \;=\; 0$
- **P[0.3 < X ≤ 0.5] = ?**

  $= F_X(0.5) - F_X(0.3) \;= 0.5 - 0.3$
- **P[X ≥ 0.7] = ?**

  $= 1 - F_X(0.7) \;= 0.3$

Probability Density Function



$p_1 = P[x_1 < X \le x_1 + \Delta]$

$= F_X(x_1 + \Delta) - F_X(x_1)$

$= \dfrac{F_X(x_1 + \Delta) - F_X(x_1)}{\Delta}\Delta$

$p_2 = P[x_2 < X \le x_2 + \Delta]$

$= F_X(x_2 + \Delta) - F_X(x_2)$

$\boxed{\text{For } \Delta \rightarrow 0, \text{ Slope} \rightarrow dF_X(x)/dx \text{ at } x_1}$

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Probability Density Function

- The slope of CDF in a region near x
  - →Probability of random variable X near x
  - →The prob. in a small region(Δ) = slope * Δ
- Slope of CDF → PDF

Definition:

Probability Density Function (PDF) is

$$f_X(x) = \frac{dF_X(x)}{dx}$$

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PDF Theorem

Theorem:

- $f_X(x) \geq 0 \quad$ for all x
- $F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$
- $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

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