Expected Value

Theorem:

Let

W = g(X,Y)

then

$$E[W] = E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

Expected Value

Theorem:

For $g(X,Y) = g_1(X,Y) + \ldots + g_n(X,Y)$

$$E[g(X,Y)] = E[g_1(X,Y)] + \ldots + E[g_n(X,Y)]$$

Expected Value

Theorem:

$$E[\,X+Y\,] = E[X] + E[Y]$$

Find E[X] → From $f_{X,Y}(x,y)$ **Not necessary**

→ Can find from Marginal PDF $f_X(x)$

So, we can find Var[X+Y], Cov, $\rho_{X,Y}$

Conditioning Joint PDF by Event

Definition:

$$f_{X,Y|B}(x,y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{P[B]} & (x,y) \in B \\ 0 & \text{Otherwise} \end{cases}$$

Conditional PDF

Definition:

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Theorem:

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)\, f_X(x) = f_{X|Y}(x|y)\, f_Y(y)$$

Conditional Expected Value

Definition: for $f_Y(y) > 0$

$$E[X|Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx$$

Definition: for $f_Y(y) > 0$

$$E[g(X,Y)|Y=y] = \int_{-\infty}^{\infty} g(x,y)\, f_{X|Y}(x|y)\, dx$$

Independent RVs

Definition: X and Y are independent iff

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

Example:

$$f_{X,Y}(x,y) = \begin{cases} 4xy & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{Otherwise} \end{cases}$$

Are X and Y independent ?

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{Otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{Otherwise} \end{cases}$$

For **all pairs** are true as definition →X and Y are independent

Independent RVs

Theorem: for independent rv X and Y

$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$

$Cov\,[X,Y] = 0$

$Var\,[X+Y] = Var[X] + Var[Y]$

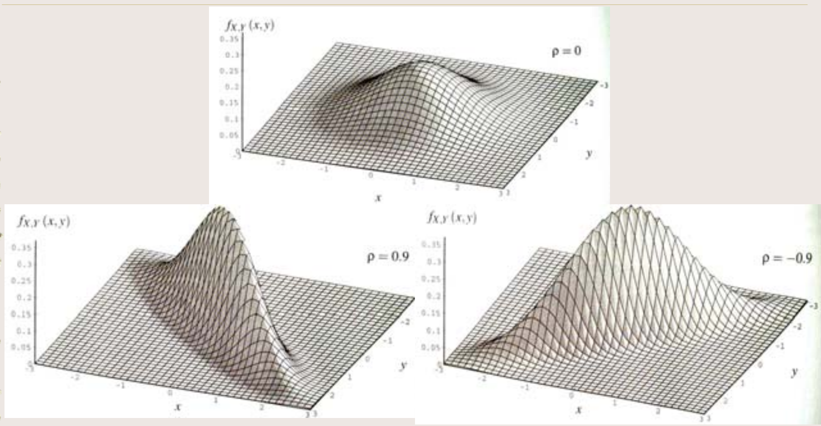
Jointly Gaussian RV

Definition: Bivariate Gaussian RV

$$f_{X,Y}(x,y) = \frac{\exp\left[-\dfrac{\left(\dfrac{x-\mu_1}{\sigma_1}\right)^2 - \dfrac{2\rho\,(x-\mu_1)\,(y-\mu_2)}{\sigma_1\,\sigma_2} + \left(\dfrac{y-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)}\right]}{2\pi\,\sigma_1\,\sigma_2\sqrt{1-\rho^2}}$$

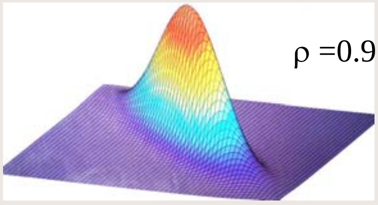
μ_1 and $\mu_2 \in$ real number, $\sigma_1 > 0$, $\sigma_2 > 0$ and $-1 \le \rho \le 1$

Jointly Gaussian RV



For $\mu1 = \mu2 = 0$, $\sigma1 = \sigma2 = 1$ and $\rho = 0, 0.9, -0.9$

Jointly Gaussian RV



$\rho = 0.9$

$\rho = 0$	Uncorrelated
$\rho > 0$	If X $\Uparrow$ (relative to mean) $\rightarrow$ Y $\Uparrow$ If X $\Downarrow$ (relative to mean) $\rightarrow$ Y $\Downarrow$
$\rho < 0$	If X $\Uparrow$ (relative to mean) $\rightarrow$ Y $\Downarrow$ If X $\Downarrow$ (relative to mean) $\rightarrow$ Y $\Uparrow$

Jointly Gaussian RV

$$f_{X,Y}(x,y) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{\frac{-(x-\mu_1)^2}{2\sigma_1^2}} \frac{1}{\tilde{\sigma}_2\sqrt{2\pi}} e^{\frac{-(y-\tilde{\mu}_2(x))^2}{2\tilde{\sigma}_2^2}}$$

Marginal $f_X(x)$ = ?

$$f_X(x) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{\frac{-(x-\mu_1)^2}{2\sigma_1^2}} \overbrace{\int_{-\infty}^{\infty} \frac{1}{\tilde{\sigma}_2\sqrt{2\pi}} e^{\frac{-(y-\tilde{\mu}_2(x))^2}{2\tilde{\sigma}_2^2}}\, dy}^{1}$$

Jointly Gaussian RV

$$f_{X,Y}(x,y) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{\frac{-(x-\mu_1)^2}{2\sigma_1^2}} \frac{1}{\tilde{\sigma}_2\sqrt{2\pi}} e^{\frac{-(y-\tilde{\mu}_2(x))^2}{2\tilde{\sigma}_2^2}}$$

Theorem:

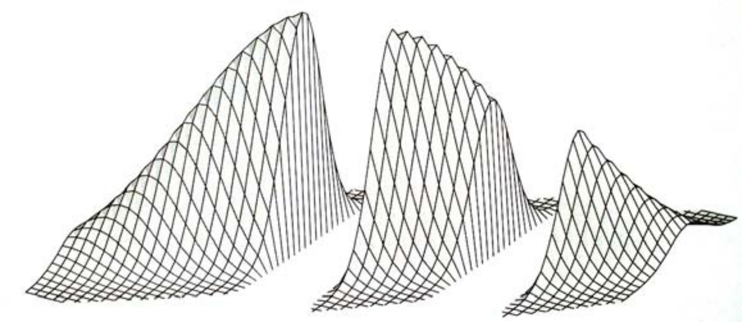
$$f_X(x) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{\frac{-(x-\mu_1)^2}{2\sigma_1^2}} \qquad f_Y(y) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{\frac{-(y-\mu_2)^2}{2\sigma_2^2}}$$

Bivariate Gaussian RV

Theorem: Conditional PDF of Y given X

$$f_{Y|X}(y|x) = \frac{1}{\tilde{\sigma}_2\sqrt{2\pi}} e^{\frac{-(y-\tilde{\mu}_2(x))^2}{2\tilde{\sigma}_2^2}}$$

Joint Gaussian PDF



For $\mu 1 = \mu 2 = 0$, $\sigma 1 = \sigma 2 = 1$ and $\rho = 0.9$

$f_{Y|X}(y|x)$ = Gaussian → Bell shape cross section

More Than 2 RVs

- 2 RVs → **Bivariate** Joint PDF
- > 2 RVs → **Multivariate** Joint PDF

Homework

- 5.1.3
- 5.2.3
- 5.3.5
- 5.4.4
- 5.5.5
- 5.6.4
- 5.7.4
- 5.8.4
- 5.9.2