Control Systems Engineering

Chapter 4: Time Response

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Chapter 4: Time Response

Learning Outcomes:

- Use poles and zeros of transfer functions to determine the time response of a control system (Sections 4.1-4.2)
- Describe quantitatively the transient response of first-order systems (Section 4.3)
- Write the general response of second-order systems given the pole location (Section 4.4)
- Find the damping ratio and natural frequency of a second-order system (Section 4.5)

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Learning Outcomes (cont.):

- Find the settling time, peak time, percent overshoot, and rise time for an underdamped second-order system (Section 4.6)
- Approximate higher-order systems and systems with zeros as first- or second-order systems (Sections 4.7-4.8)
- Describe the effects of nonlinearities on the system time response (Section 4.9)

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Case Study :

Given the antenna azimuth position control system shown on the front endpapers, you will be able to

(1) predict, by inspection, the form of the open-loop angular velocity response of the load to a step voltage input to the power amplifier;

(2) describe quantitatively the transient response of the open-loop system;

(3) derive the expression for the open-loop angular velocity output for a step voltage input;

(4) obtain the open-loop state-space representation;

(5) plot the open-loop velocity step response using a computer simulation.

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4.1 Introduction:

- After the engineer obtains a mathematical representation of a subsystem, *the subsystem is analyzed for its transient and steady-state responses to see if these characteristics yield the desired behavior.* This chapter is devoted to the analysis of system transient response.
- This chapter demonstrates *applications of the system representation by evaluating the transient response from the system model.* (Since the engineer may indeed want to evaluate the response of a subsystem prior to inserting it into the closed-loop system.)

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4.1 Introduction (cont.):

- After describing a valuable analysis and design tool, poles and zeros, we begin analyzing our models to find the step response of first- and second-order systems.

Note: The order refers to the order of the equivalent differential equation representing the system—the order of the denominator of the transfer function after cancellation of common factors in the numerator.

$$G(s) = \frac{s^2 + 2s + 1}{s^2 + 3s + 2}$$

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4.1 Introduction (cont.):

- After describing a valuable analysis and design tool, poles and zeros, we begin analyzing our models to find the step response of first- and second-order systems.

Note: The order refers to the order of the equivalent differential equation representing the system—the order of the denominator of the transfer function after cancellation of common factors in the numerator.

$$G(s) = \frac{s^2 + 2s + 1}{s^2 + 3s + 2} \rightarrow \frac{(s+1)^2}{(s+1)(s+2)} \Rightarrow G(s) = \frac{(s+1)}{(s+2)}$$

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4.2 Poles, Zeros, and System Response:

- The <u>output response of a system </u>is the sum of two responses: the forced response and the natural response.
- Many techniques, such as solving a differential equation or taking the inverse Laplace transform, enable us to evaluate this output response.
- Productivity is aided by analysis and design techniques that yield results in a minimum of time.
- If the technique is so rapid that we feel we derive the desired result <u>by inspection</u>, we sometimes use the attribute <u>*qualitative*</u> to describe the method.
- The use of poles and zeros and their relationship to the time response of a system is such a technique.

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4.2 Poles, Zeros, and System Response:

Note:

- The concept of poles and zeros, fundamental to the analysis and design of control systems, simplifies the evaluation of a system's response.
- The reader is encouraged to master the concepts of poles and zeros and their application to problems throughout this book.

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(4.2) Poles of a Transfer Function

- The **poles** of a transfer function are (1) the values of the Laplace transform variable, s, that cause the transfer function to become infinite or (2) any roots of the denominator of the transfer function that are not common to roots of the numerator.
- In control systems, we often refer to the root of the canceled factor in the denominator as a pole even though the transfer function will not be infinite at this value.

$$G(s) = \frac{s^2 + 2s + 1}{s^2 + 3s + 2} \rightarrow \frac{(s+1)}{(s+2)}$$

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(4.2) Zeros of a Transfer Function

- The **zeros** of a transfer function are (1) the values of the Laplace transform variable, s, that cause the transfer function to become zero, or (2) any roots of the numerator of the transfer function that are not common to roots of the denominator.
- In control systems, we often refer to the root of the canceled factor in the numerator as a zero even though the transfer function will not be zero at this value.

$$G(s) = \frac{s^2 + 2s + 1}{s^2 + 3s + 2} \rightarrow \frac{(s+1)}{(s+2)}$$

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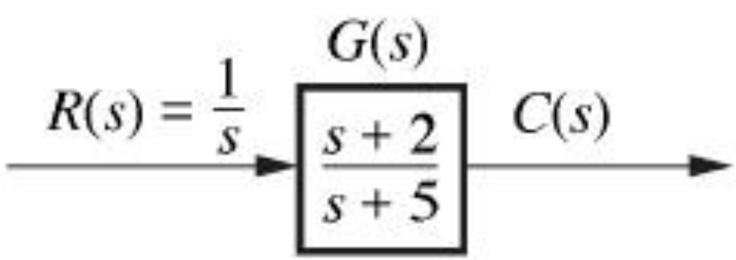
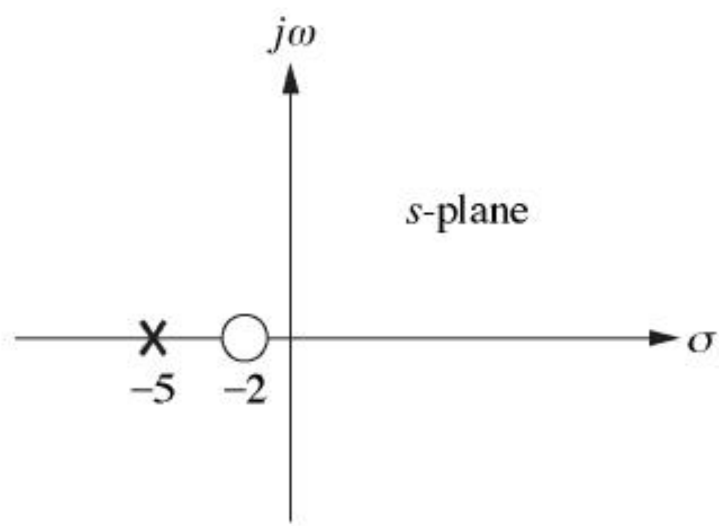
(4.2) Poles and Zeros of a First-Order System: An Example

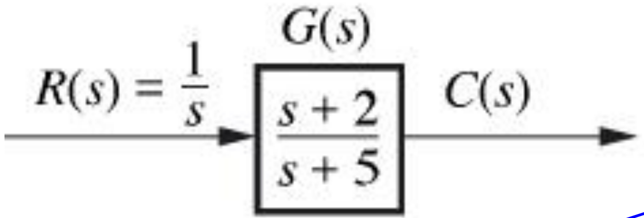
- Given the transfer function G(s) in Figure 4.1(a), a pole exists at s = - 5 , and a zero exists at -2. These values are plotted on the complex s-plane in Figure 4.1(b), using an x for the pole and a O for the zero.
- To show the properties of the poles and zeros, let us find the **unit step** response of the system.





(4.2) Poles and Zeros of a First-Order System: An Example

- Multiplying the transfer function by a step function yields



$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5}$$

$$A = \left.\frac{(s+2)}{(s+5)}\right|_{s\to 0} = \frac{2}{5}$$

$$B = \left.\frac{(s+2)}{s}\right|_{s\to -5} = \frac{3}{5}$$

$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

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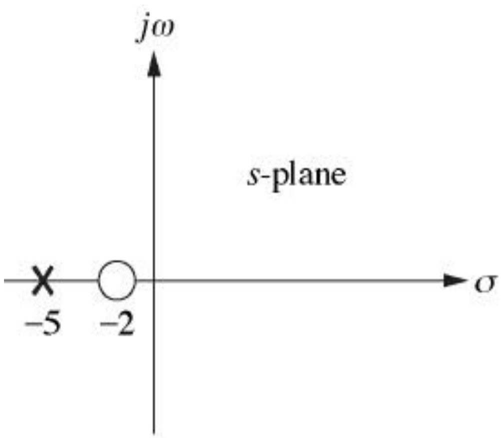
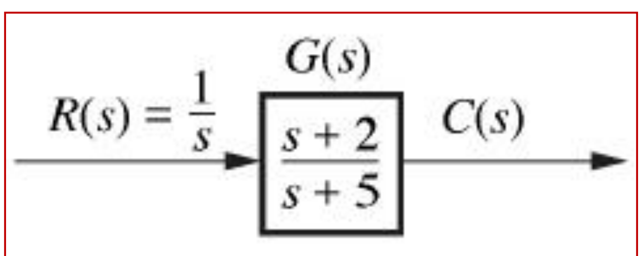
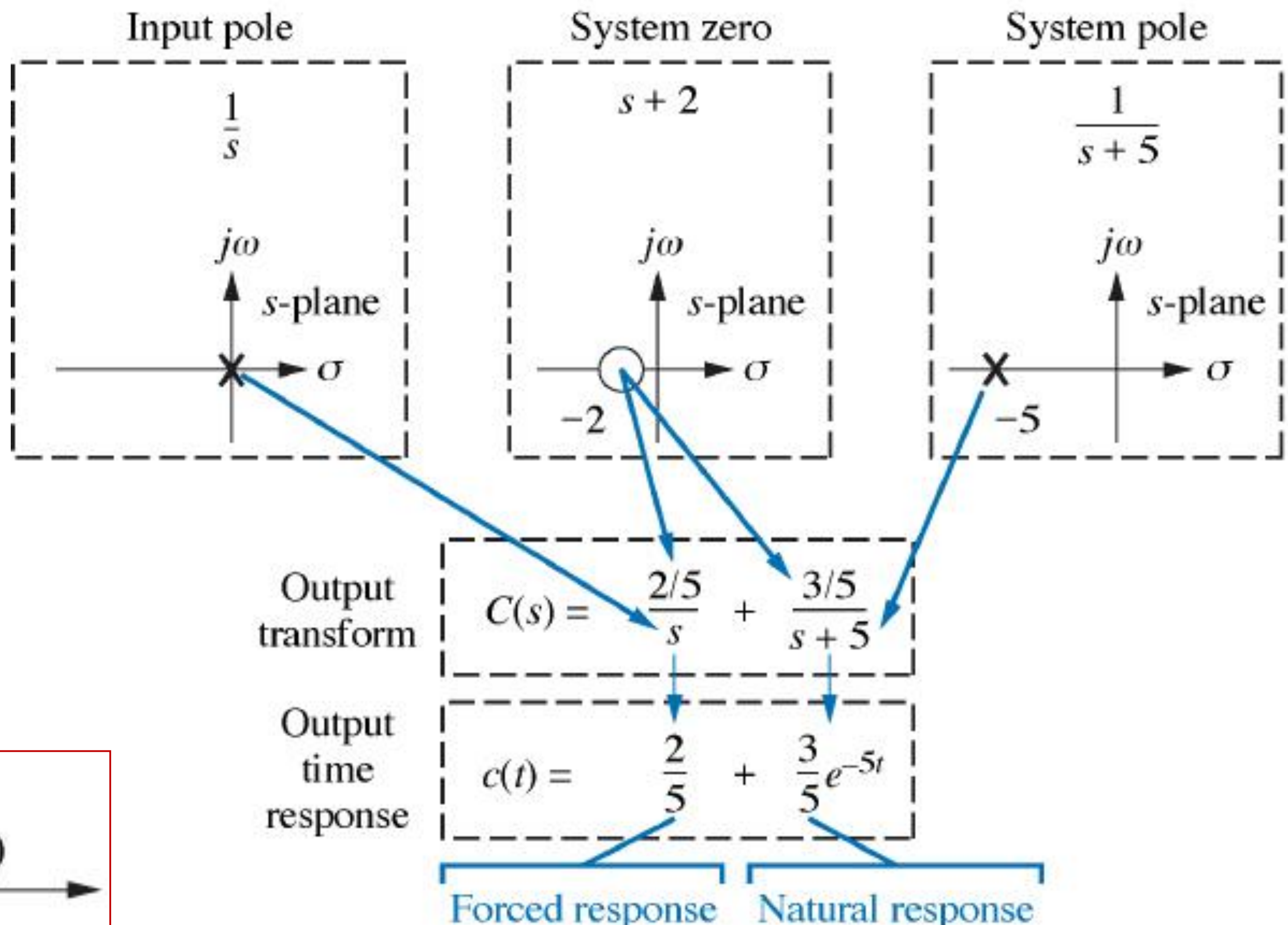
(4.2) Poles and Zeros of a First-Order System: An Example

- Summary







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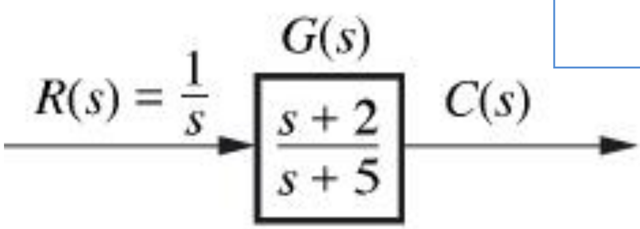
(4.2) Poles and Zeros of a First-Order System: An Example



$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5}$$

$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

<u>Conclusion</u>

- 1. A pole of the input function generates the form of the <u>*forced response*</u> *(that is, the* pole at the origin generated a step function at the output).
- 2. A pole of the transfer function generates the form of the <u>*natural response*</u> *(that is,* the pole at - 5 generated e^{-5t}).

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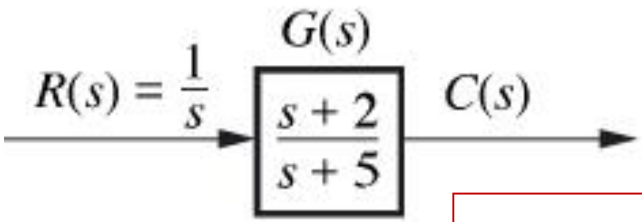
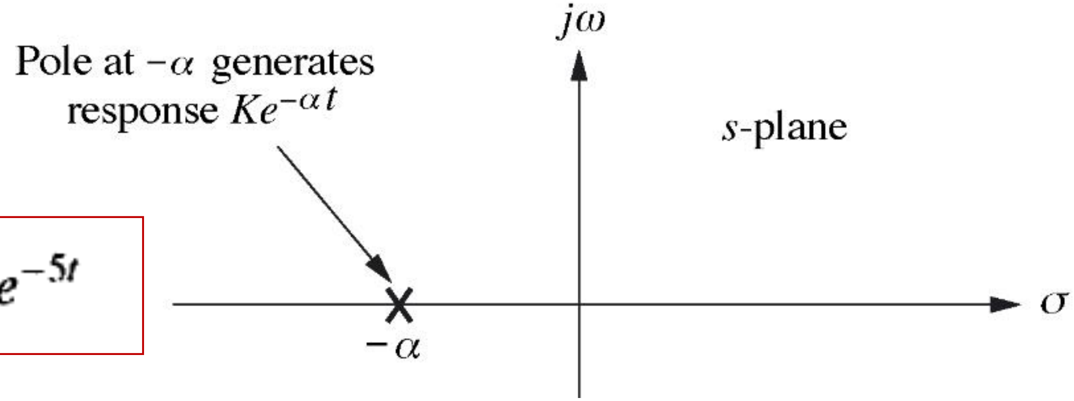
(4.2) Poles and Zeros of a First-Order System: An Example



$$\boxed{c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}}$$



Conclusion

3. A pole on the real axis generates an *exponential response of the form $e^{-\alpha t}$, where -α is the pole location on the real axis. Thus, the farther to the left a pole is on the* negative real axis, the faster the exponential transient response will decay to zero.

4. The zeros and poles generate the *amplitudes for both the forced and natural* responses.

Note: We will learn to write the form of the response by inspection.

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Example 4.1: Given the system write the output, *c(t), in general terms.*

$$R(s) = \frac{1}{s} \longrightarrow \boxed{\frac{(s+3)}{(s+2)(s+4)(s+5)}} \longrightarrow C(s)$$

SOLUTION: By inspection, each system pole generates an exponential as part of the natural response. The input's pole generates the forced response. Thus,

$$C(s) \equiv \underbrace{\frac{K_1}{s}}_{\substack{\text{Forced}\\ \text{response}}} + \underbrace{\frac{K_2}{s+2} + \frac{K_3}{s+4} + \frac{K_4}{s+5}}_{\substack{\text{Natural}\\ \text{response}}}$$

Taking the inverse Laplace transform, we get

$$c(t) \equiv \underbrace{K_1}_{\substack{\text{Forced}\\ \text{response}}} + \underbrace{K_{2e^{-2t}} + K_{3e^{-4t}} + K_{4e^{-5t}}}_{\substack{\text{Natural}\\ \text{response}}}$$

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PROBLEM: A system has a transfer function, $G(s) = \frac{10(s+4)(s+6)}{(s+1)(s+7)(s+8)(s+10)}$.
Write, by inspection, the output, $c(t)$, in general terms if the input is a unit step.

ANSWER: $c(t) \equiv A + Be^{-t} + Ce^{-7t} + De^{-8t} + Ee^{-10t}$

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<u>Conclusion</u>:

- In this section, we learned that poles determine the nature of the time response: Poles of the input function determine the form of the forced response, and poles of the transfer function determine the form of the natural response.
- Zeros and poles of the input or transfer function contribute to the amplitudes of the component parts of the total response.
- Finally, poles on the real axis generate exponential responses.

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4.3 First-Order Systems:

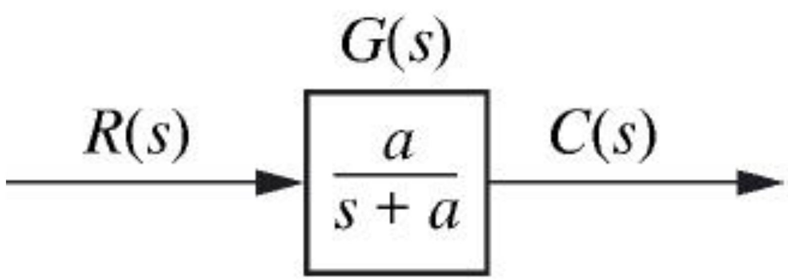
- We now discuss first-order systems without zeros to define a performance specification for such a system. (We will discuss about the first-order systems with zeros later.)

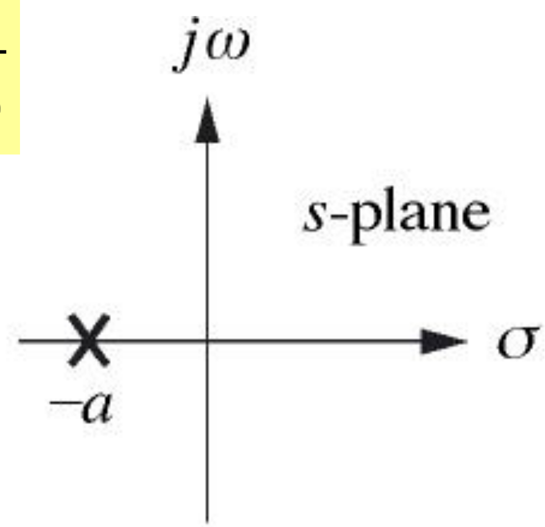
$$G(s) = \frac{2}{(s+2)}$$

$$G(s) = \frac{(s+1)}{(s+2)}$$

$$G(s) = \frac{3(s+1)}{(s+2)}$$

$$G(s) = \frac{3(2)}{(s+2)}$$





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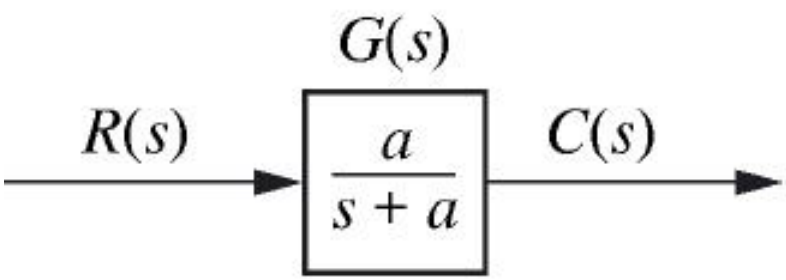
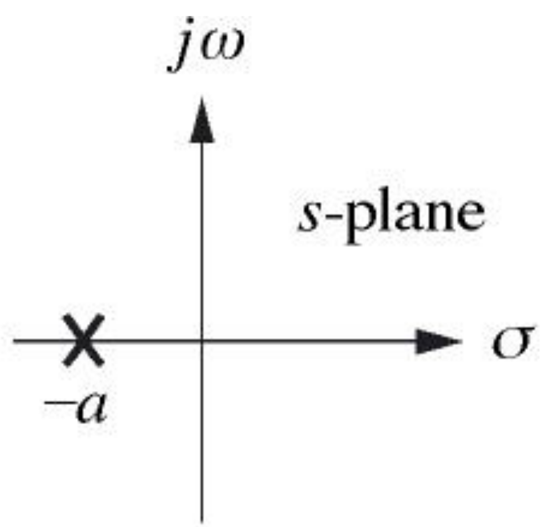
4.3 First-Order Systems:

- If the input is a unit step, where *R(s) = 1/s, the Laplace* transform of the step response is *C(s), where*

$$\boxed{C(s) = R(s)G(s) = \frac{1}{s}\frac{a}{(s+a)}}$$

$$\boxed{c(t) = c_f(t) + c_n(t) = 1 - e^{-at}}$$





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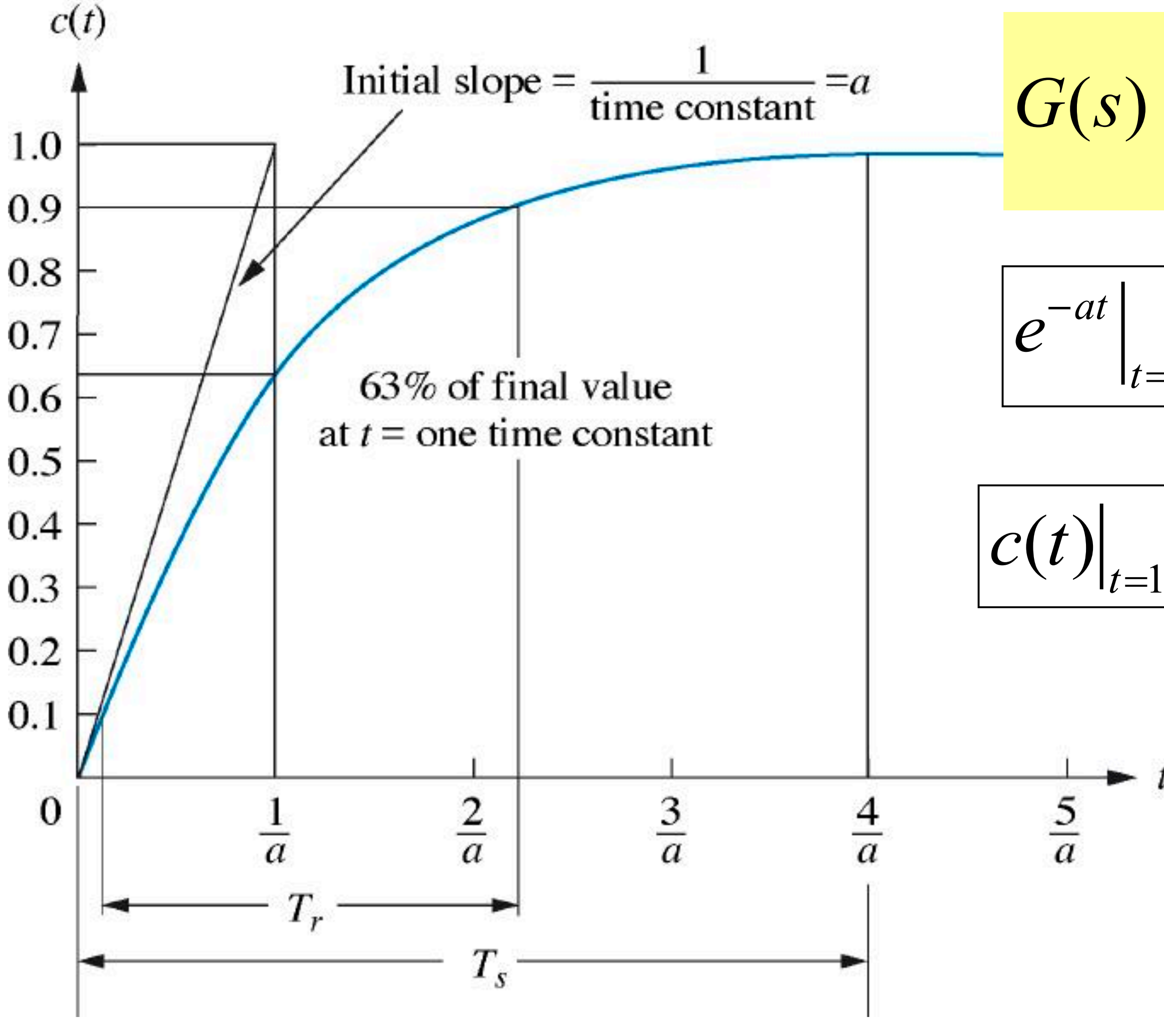
4.3 First-Order Systems:

- Let us examine the significance of parameter *a.*



$$G(s) = \frac{a}{(s+a)}$$

$$c(t) = 1 - e^{-at}$$

$$e^{-at}\Big|_{t=1/a} = e^{-1} = 0.37$$

$$c(t)\Big|_{t=1/a} = 1 - 0.37 = 0.63$$

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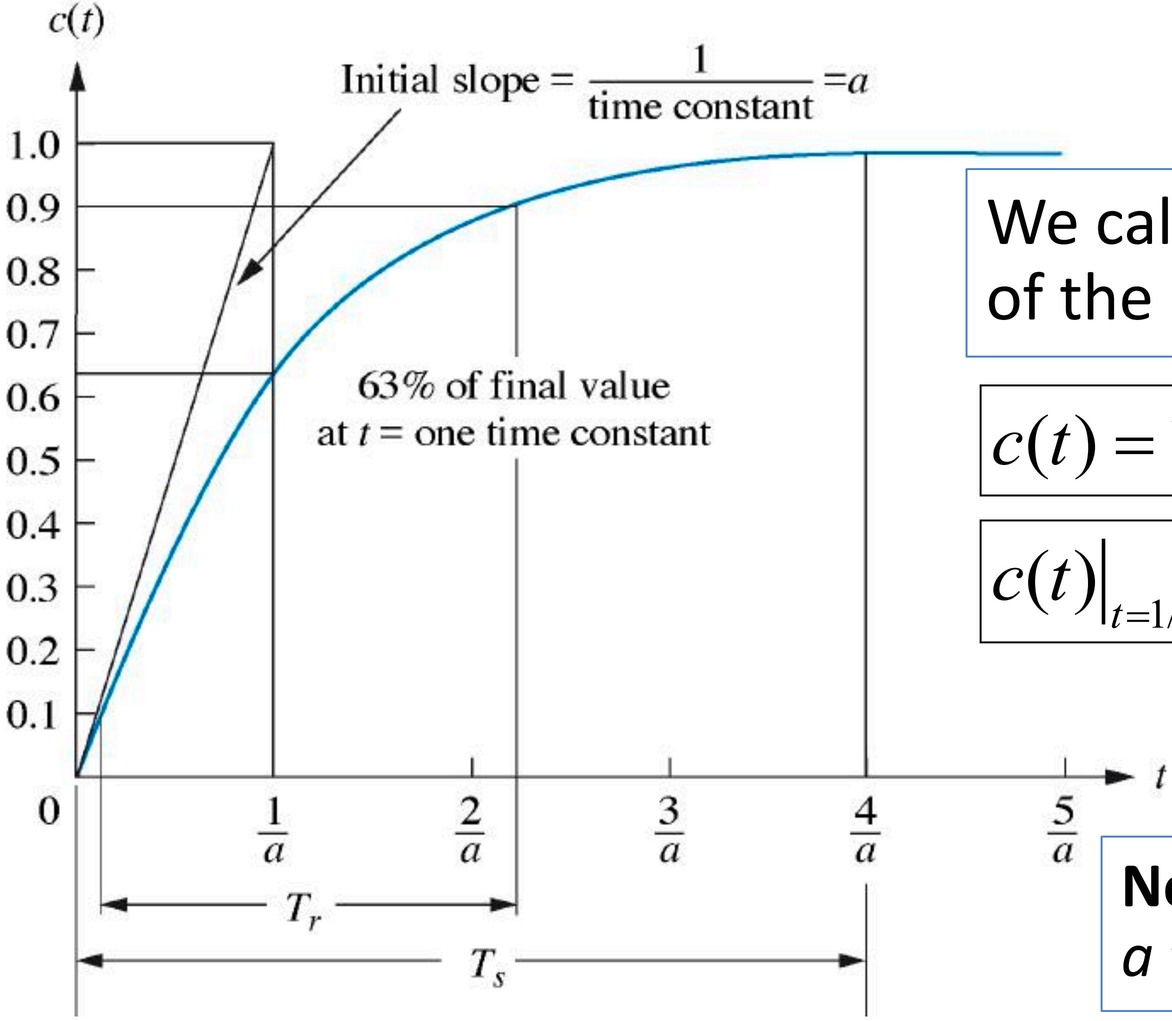
4.3 First-Order Systems: **Time Constant**

- **Time constant** is *the time for e^{-at} to decay to 37% of its initial.*



We call *1/a* the time constant of the response.

$$c(t) = 1 - e^{-at}$$

$$c(t)\big|_{t=1/a} = 1 - 0.37 = 0.63$$

Note: we call the parameter *a the exponential frequency.*

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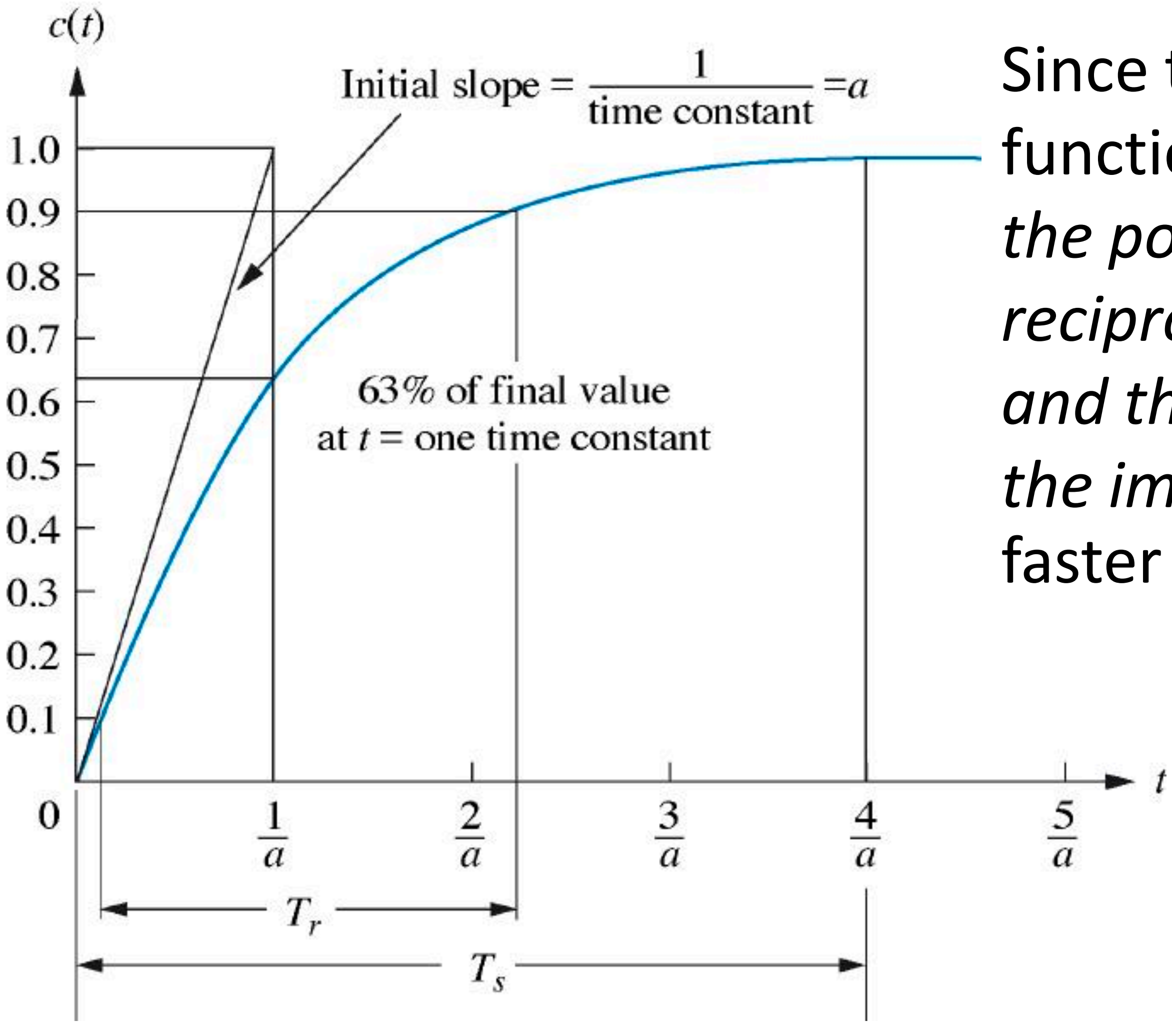
4.3 First-Order Systems: Time Constant

- The time constant can also be evaluated from the pole plot.



Since the pole of the transfer function is at -*a, we can say the pole is located at the reciprocal of the time constant, and the farther the pole from the imaginary axis, the* faster the transient response.

$$\boxed{c(t) = 1 - e^{-at}}$$

$$G(s) = \frac{a}{(s+a)}$$

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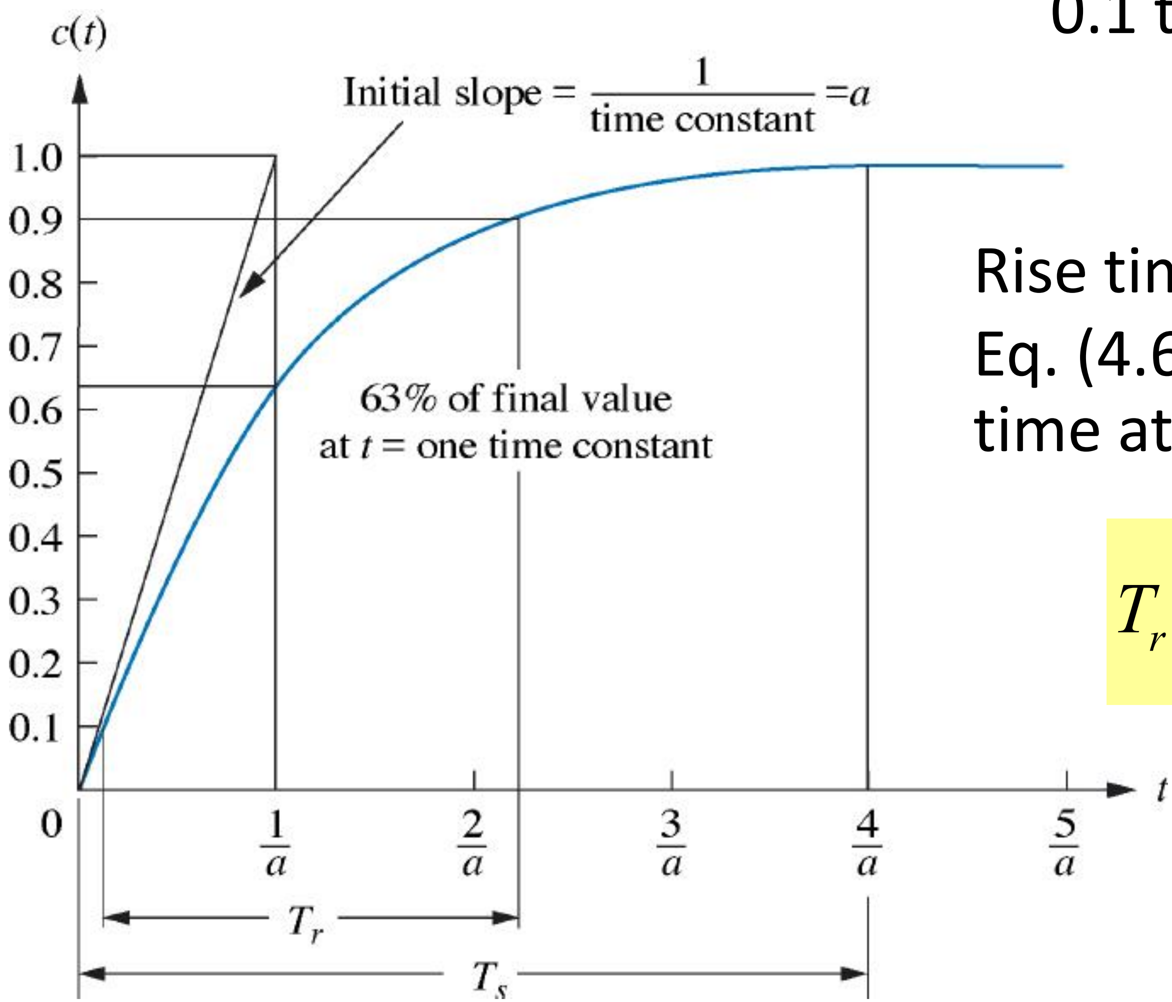
4.3 First-Order Systems: Rise Time, T_r

- *Rise time* is defined as the time for the waveform to go from 0.1 to 0.9 of its final value.



Rise time is found by solving Eq. (4.6) for the difference in time at *c(t) = 0.9* and *c(t) = 0.1.*

$$T_r = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}$$

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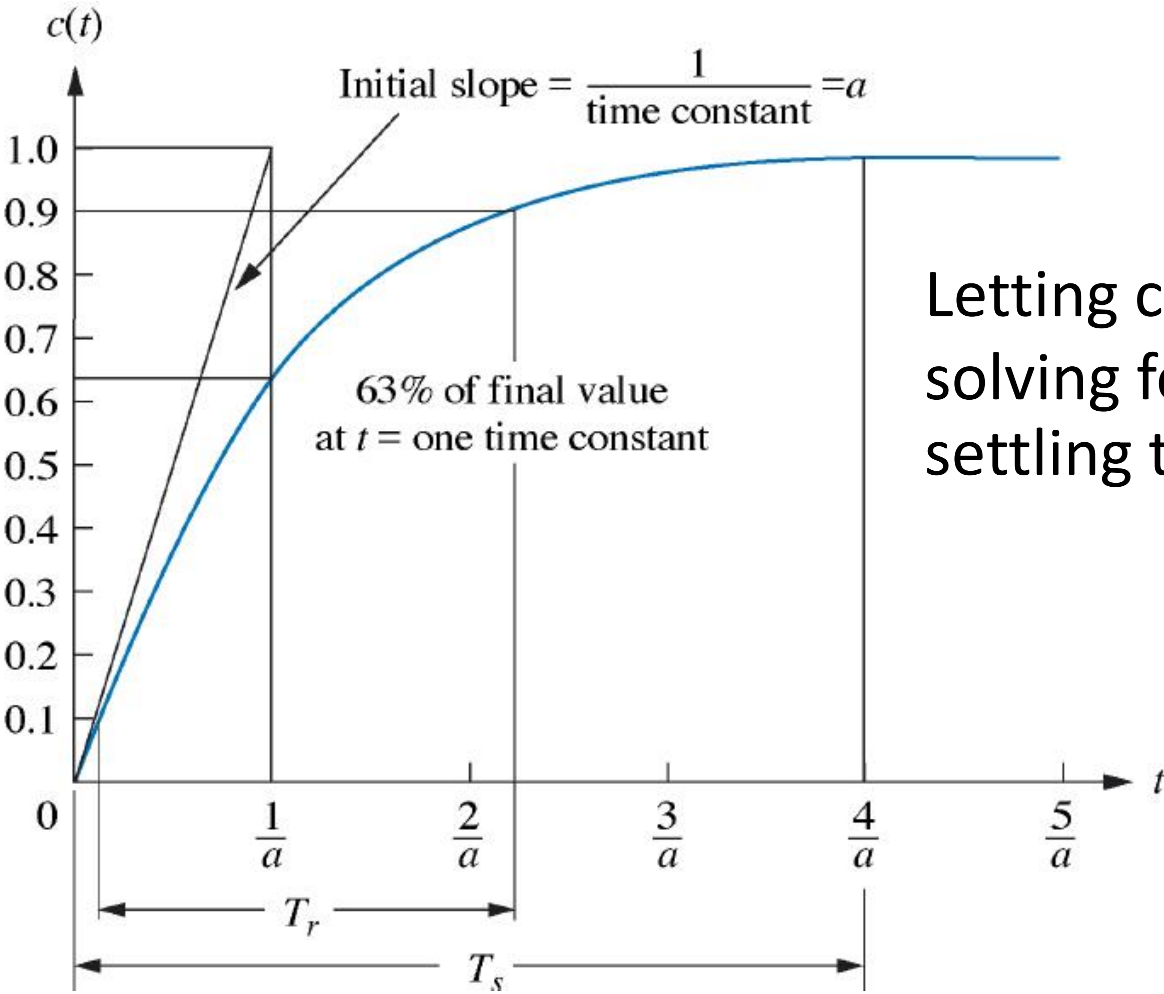
4.3 First-Order Systems: Settling Time, T_s

- *Settling time* is defined as the time for the response to reach, and stay within, 2% of its final value.



Letting c(t) = 0.98 in Eq. (4.6) and solving for time, t, we find the settling time to be.

$$T_s = \frac{4}{a}$$

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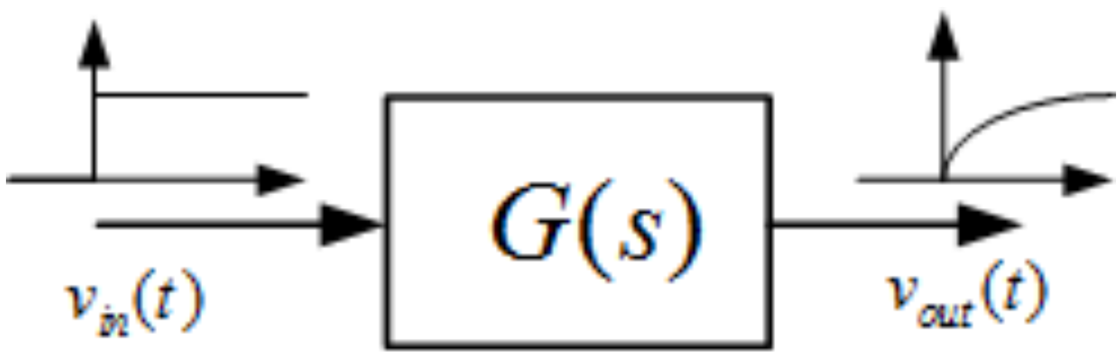
(4.3) First-Order Transfer Functions via Testing

- Often it is not possible or practical to obtain a system's transfer function analytically.
- Perhaps the system is closed, and the component parts are not easily identifiable.
- Since the transfer function is a representation of the system from input to output, <u>the system's step response can lead to a representation even though the inner construction is not known.</u>(This method can give a rough model of the system.)



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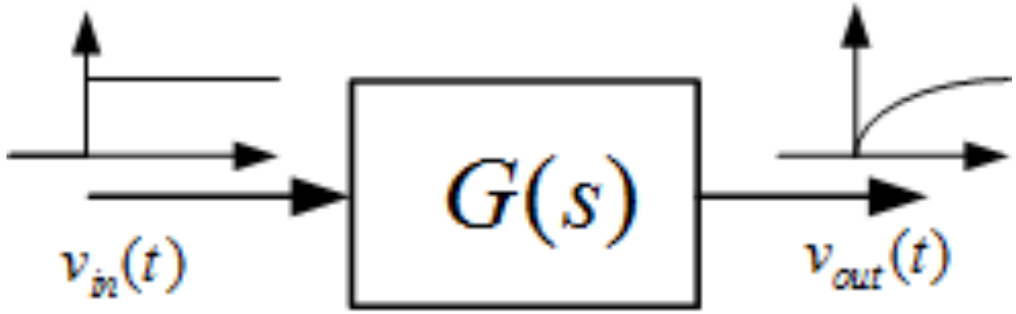
(4.3) First-Order Transfer Functions via Testing

- With a step input, we can measure the time constant and the steady-state value, from which the transfer function can be calculated.



$$G(s) = \frac{K}{(s+a)}$$

$$C(s) = \frac{K}{(s+a)}\frac{1}{s} = \frac{K/a}{s} - \frac{K/a}{(s+a)}$$

- If we can identify *K* and *a* from laboratory testing, we can obtain the transfer function of the system.

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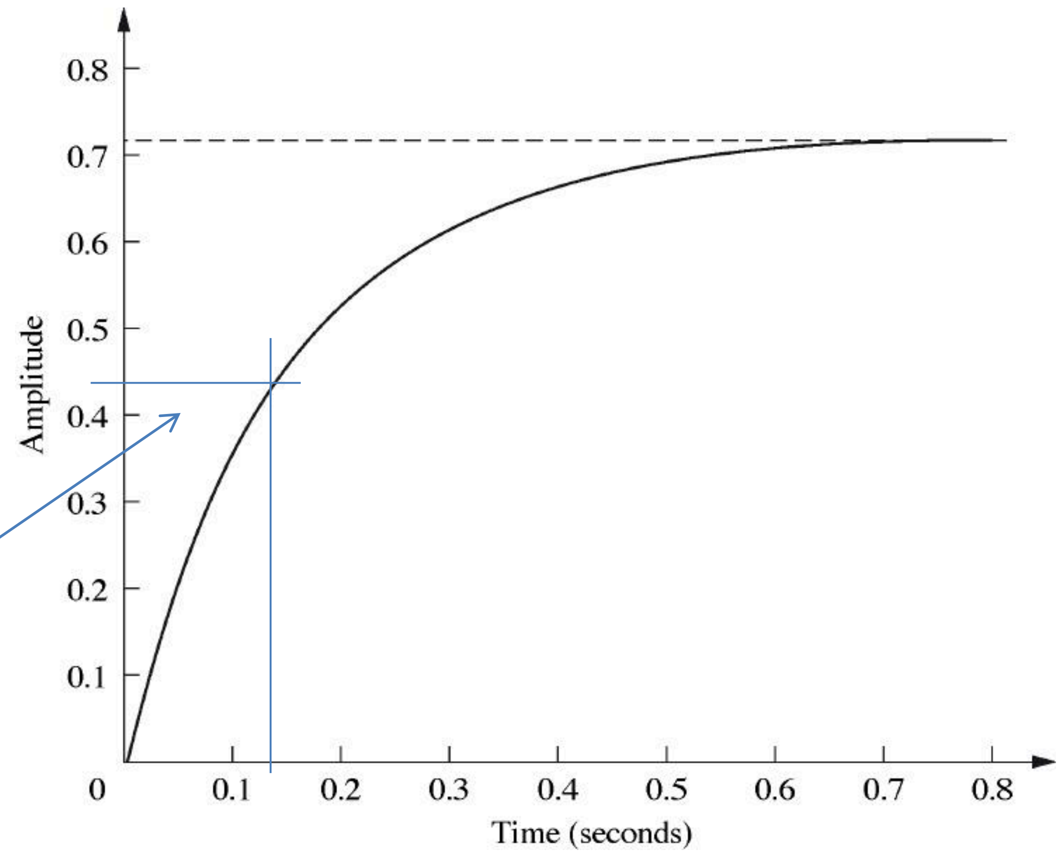
(4.3) First-Order Transfer Functions via Testing

- For example, assume the unit step response given in Figure 4.6. We determine that it has the first-order characteristics we have seen thus far, such as no overshoot and nonzero initial slope.

From the response, we measure the time constant, that is, the time for the amplitude to reach 63% of its final value.



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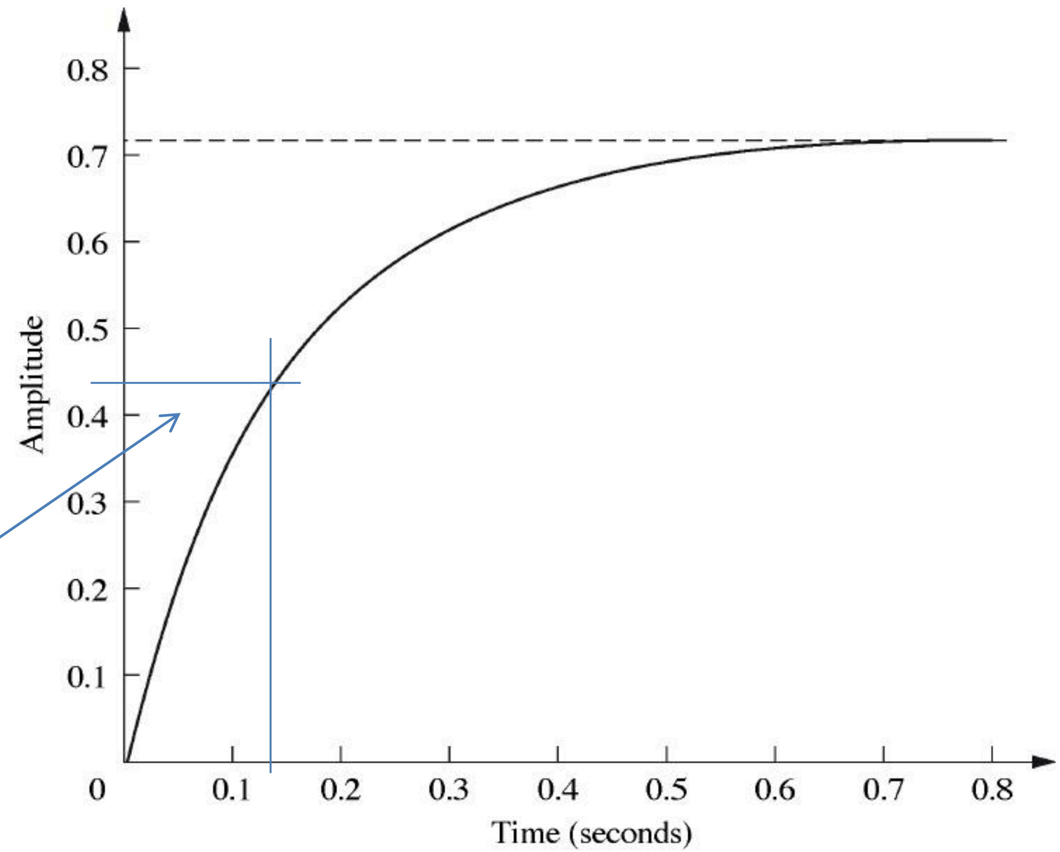
(4.3) First-Order Transfer Functions via Testing

- Since the final value is about 0.72, the time constant is evaluated where the curve reaches 0.63 x 0.72 = 0.45, or about 0.13 second. Hence, a = 1/0.13 = 7.7.

From the response, we measure the <u>time constant</u>, that is, the time for the amplitude to reach 63% of its final value.



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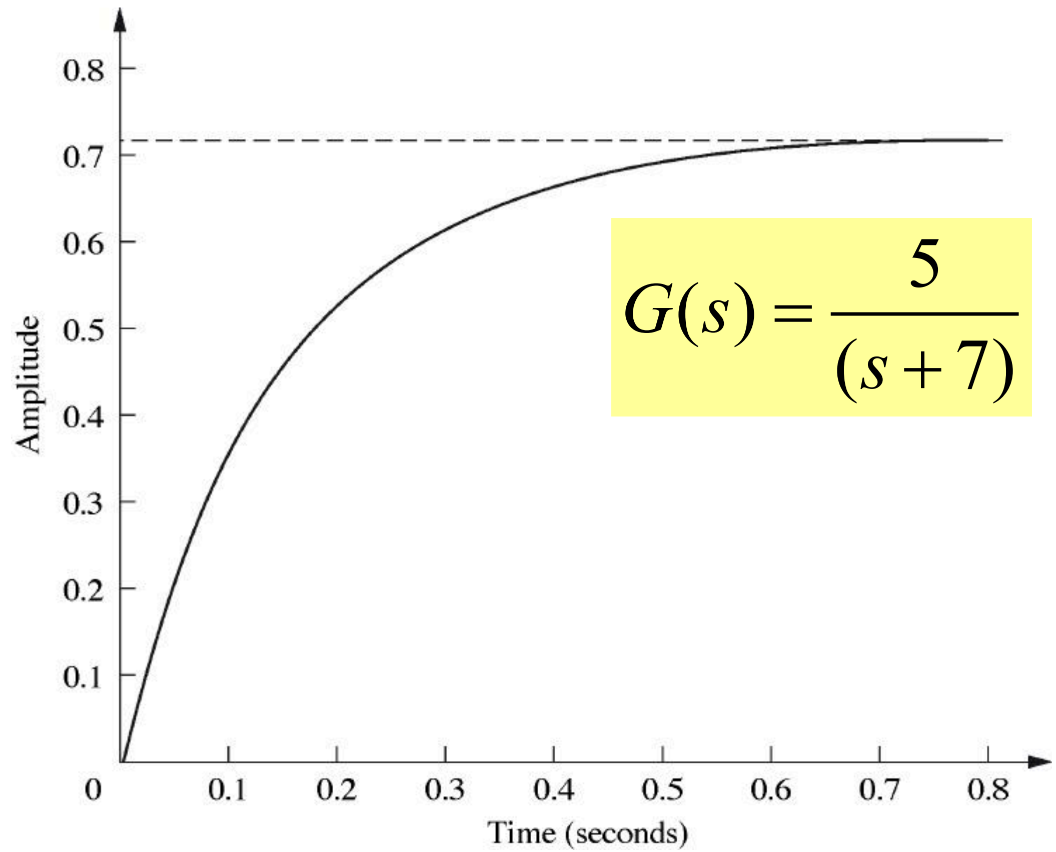
(4.3) First-Order Transfer Functions via Testing

- To find K, we realize from Eq. (4.11) that the forced response reaches a steady state value of K/*a* = 0.72. Substituting the value of a, we find K = 5.54.

Thus, the transfer function for the system is G(s) = 5.54/(s + 7.7).



$$G(s) = \frac{5}{(s+7)}$$

$$G_1(s) = \frac{K}{(\tau s + 1)}$$

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Exercise 4.2

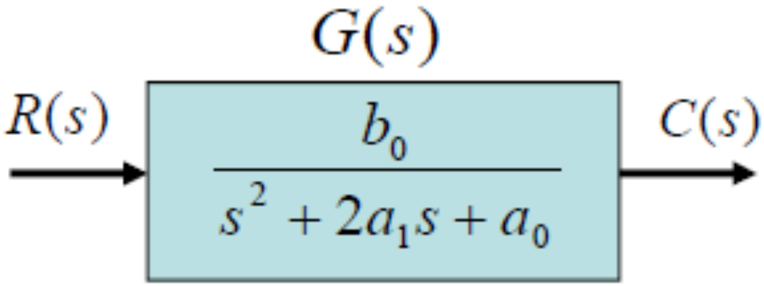
- PROBLEM: A system has a transfer function, $G(s) = \dfrac{50}{(s+50)}$

Find the time constant, T_c, settling time, T_s, and rise time, T_r.

ANSWER: T_c *= 0.02 s,* T_s *= 0.08 s, and* T_r *= 0.044 s.*

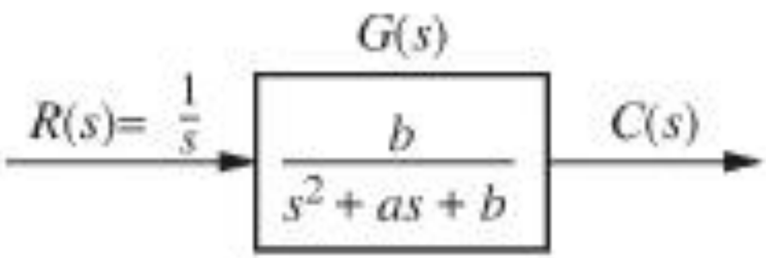
4.4 Second-Order Systems: Introduction

- Let us now extend the concepts of poles and zeros and transient response to second order systems.
- Compared to the simplicity of a first-order system, a second-order system exhibits a <u>wide range of responses</u> that must be analyzed and described.



General

$$G(s) = \frac{b_0}{(s^2 + 2a_1 s + a_0)} = \frac{C(s)}{R(s)}$$



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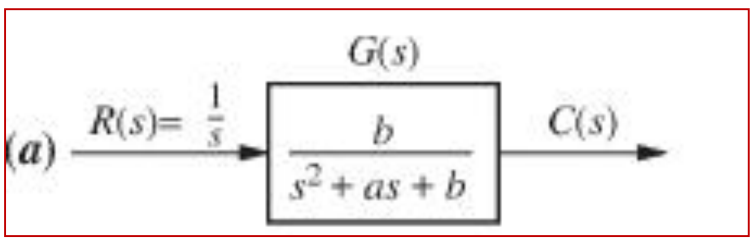
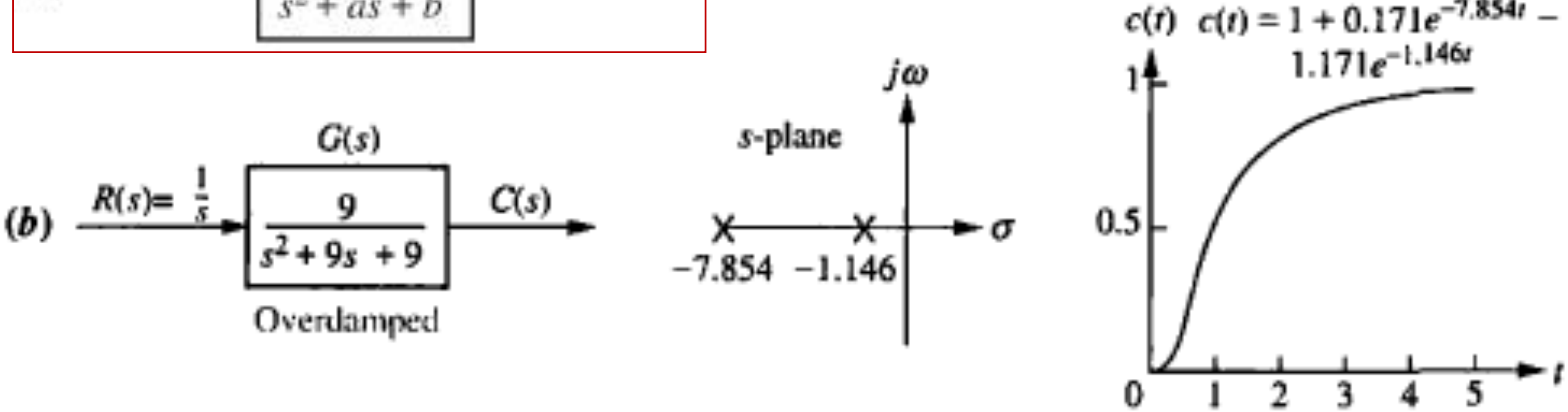
4.4 Second-Order Systems: Introduction

- To become familiar with the wide range of responses before formalizing our discussion in the next section, we take a look at numerical examples of the second order system, which has two finite poles and no zeros, shown in Figure 4.7 (a).
- By assigning appropriate values to parameters *a and b, we can show all possible second-order transient responses.*





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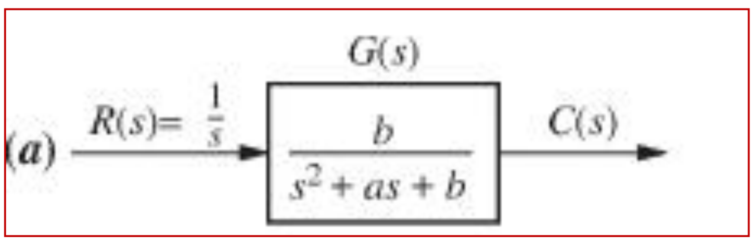
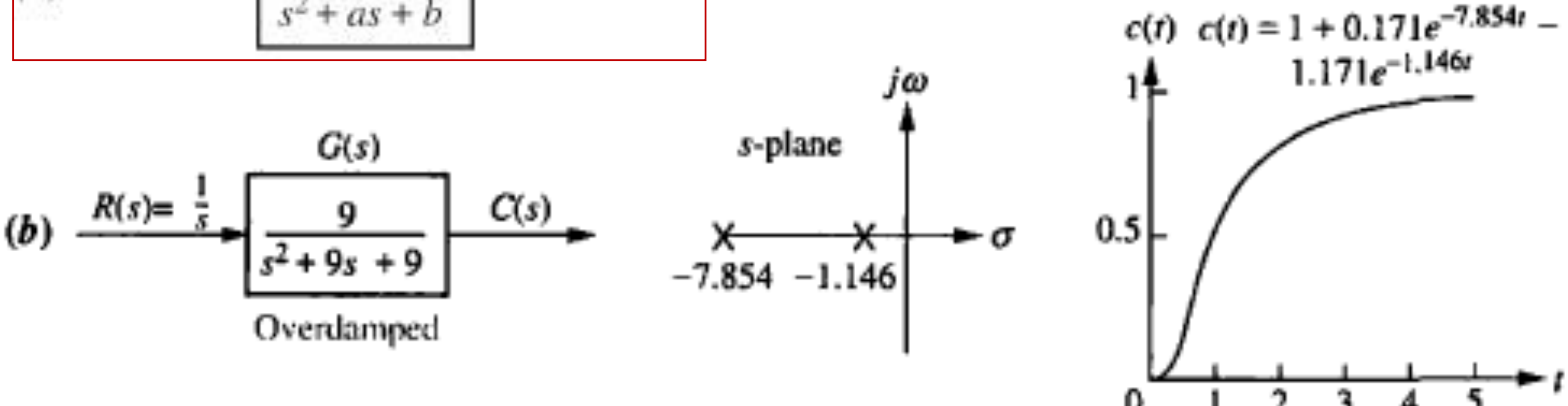
4.4 Second-Order Systems: Introduction

- The unit step response then can be found using *C(s) = R(s)G(s), where R(s) = 1/s,* followed by a partial-fraction expansion and the inverse Laplace transform.
- We now explain each response and show how we can use the poles to determine the nature of the response ***without*** going through the procedure of inverse Laplace transform.





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4.4 Second-Order Systems: Overdamped Response

- By assigning the parameters *a = 9 and b = 9*, we will have.

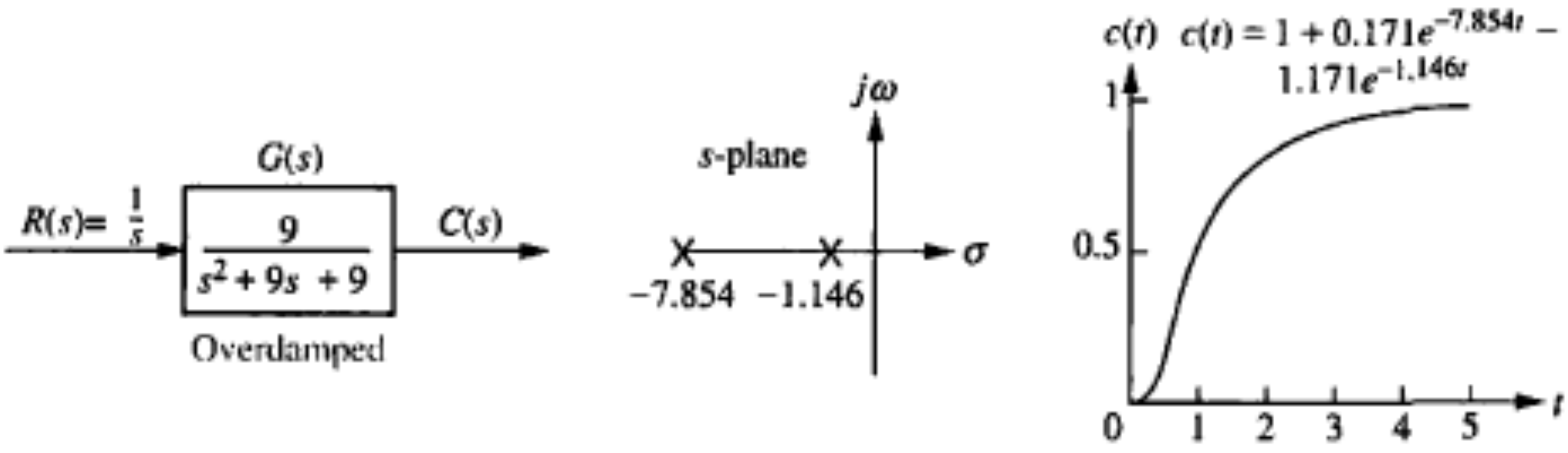
$$C(s) = \frac{9}{s(s^2+9s+9)} = \frac{9}{s(s+7.854)(s+1.146)}$$

- This function has a pole at the origin that comes from the unit step input and two real poles that come from the system.

$$c(t) = K_1 + K_2 e^{-7.854t} + K_3 e^{-1.146t}$$

- We see that the poles tell us the form of the response without the tedious calculation of the inverse Laplace transform.

(b)



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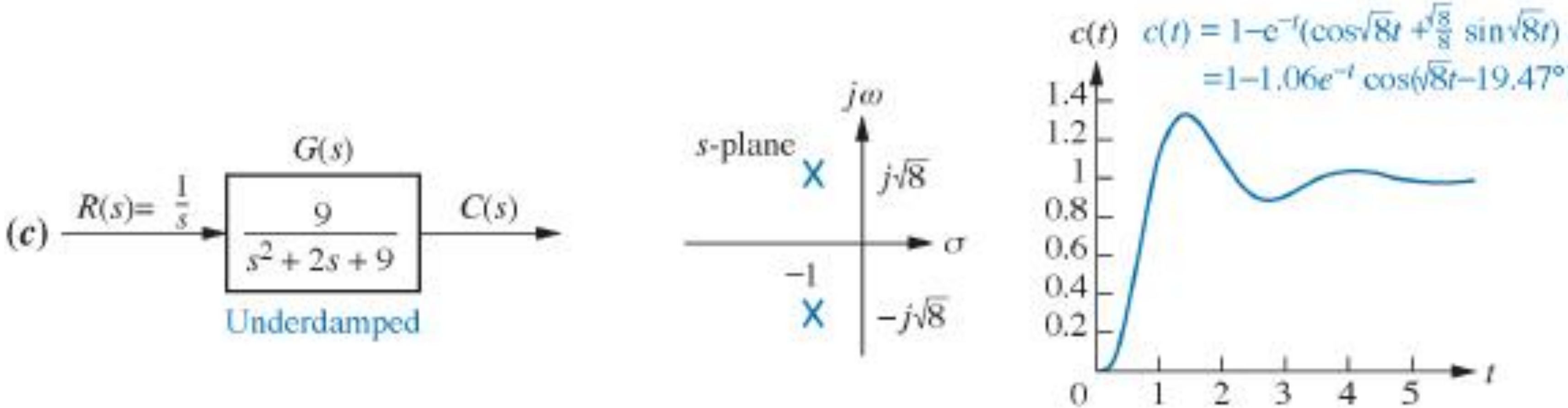
4.4 Second-Order Systems: Underdamped Response

- By assigning the parameters *a = 2 and b = 9*, we will have.

$$C(s) = \frac{9}{s(s^2 + 2s + 9)}$$

- This function has a pole at the origin that comes from the unit step input and two complex poles at $s = -1 \pm j\sqrt{8}$ that come from the system. **Comparing these values to *c(t) in the same figure***, *we see that the real part of the pole* matches the exponential decay frequency of the sinusoid's amplitude, while the imaginary part of the pole matches the frequency of the sinusoidal oscillation.



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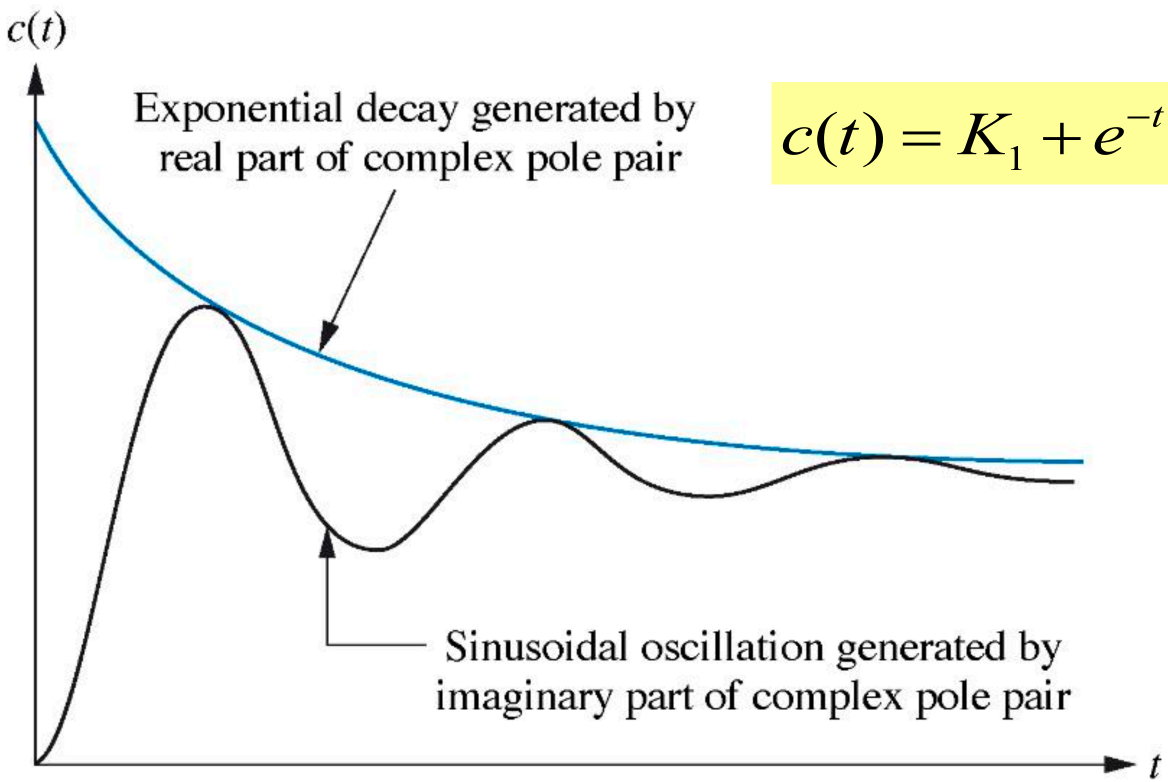
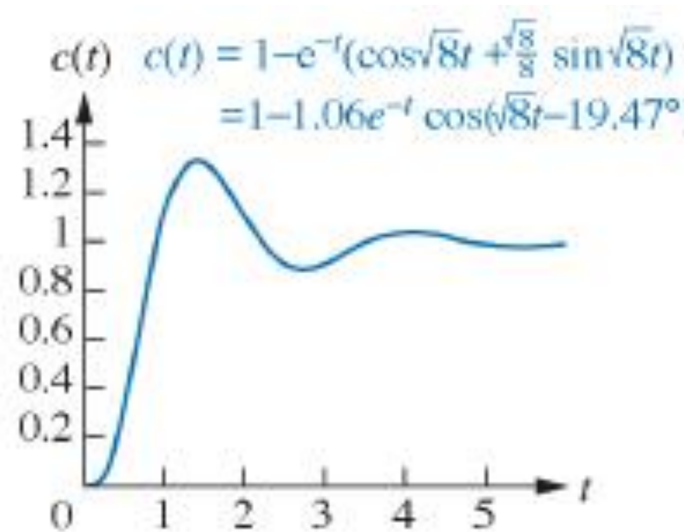
4.4 Second-Order Systems: Underdamped Response

- The time constant of the exponential decay is equal to the reciprocal of the real part of the system pole. The value of the imaginary part is the actual frequency of the sinusoid, as depicted in Figure 4.8. (system poles are at s = -1 $\pm\sqrt{8}$)

$$C(s) = \frac{9}{s(s^2 + 2s + 9)}$$

$$c(t) = K_1 + e^{-t}(K_2 \cos\sqrt{8}t + K_3 \sin\sqrt{8}t)$$

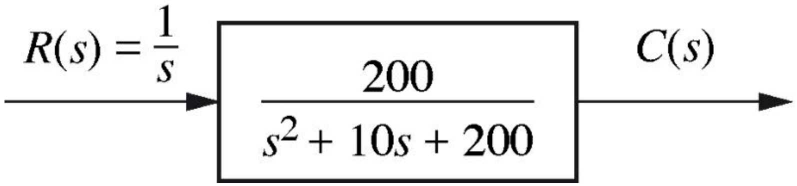




Chapter 4: Time Response

4.4 Second-Order Systems: Underdamped Response

- **Example 4.2** By inspection, write the form of the step response of the system in Figure 4.9.



- SOLUTION:
  First we determine the pole of the system and then the form of the response. Factoring the denominator of the transfer function we find the poles to be $s = -5 \pm j\sqrt{13.23}$.

$$c(t) = K_1 + e^{-5t}(K_2 \cos\sqrt{13.23}t + K_3 \sin\sqrt{13.23}t)$$

$$c(t) = K_1 + K_4 e^{-5t}(\cos\sqrt{13.23}t - \phi)$$

$$\text{where } \phi = tan^{-1}\frac{K_3}{K_2}, K_4 = \sqrt{K_2^2 + K_3^2}$$

Chapter 4: Time Response

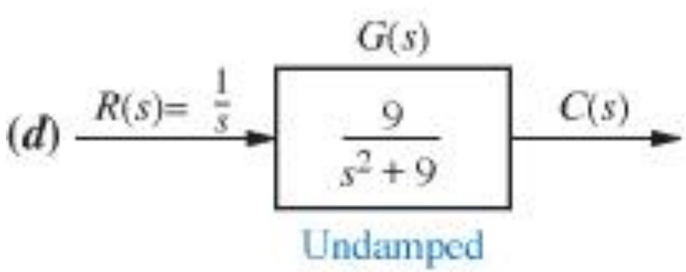
4.4 Second-Order Systems: Undamped Response

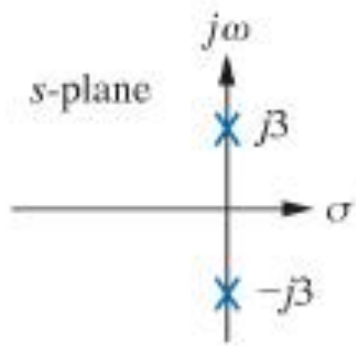
- By assigning the parameters *a = 0 and b = 9*, we will have.

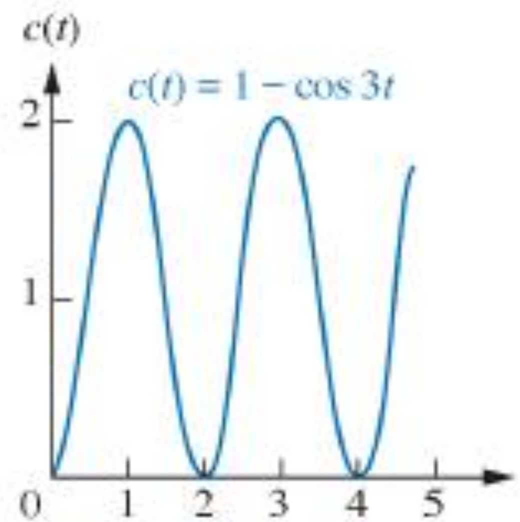
$$C(s) = \frac{9}{s(s^2+9)}$$

- This function has a pole at the origin that comes from the unit step input and two imaginary poles at s = ± 3, that come from the system. Hence, the output can be estimated as:

$$c(t) = K_1 + K_4(\cos 3t + \phi)$$







Chapter 4: Time Response

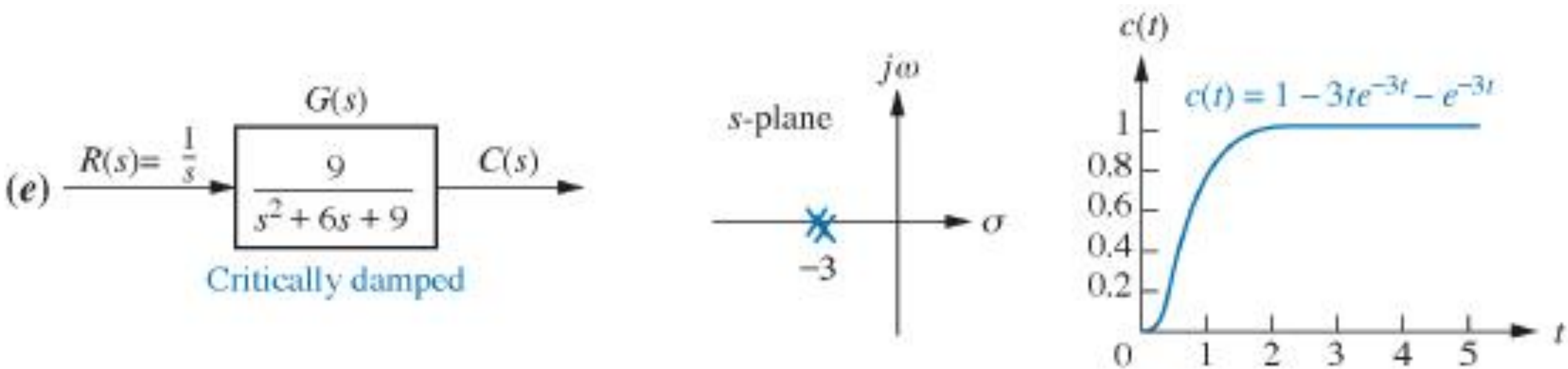
4.4 Second-Order Systems: Critically damped Response

- By assigning the parameters *a = 6 and b = 9*, we will have.

$$C(s) = \frac{9}{s(s^2 + 6s + 9)} = \frac{9}{s(s+3)^2}$$

- This function has a pole at the origin that comes from the unit step input two multiple real poles that come from the system.
- Hence, the output can be estimated as

$$c(t) = K_1 + K_2 e^{-3t} + tK_3 e^{-3t}$$



Chapter 4: Time Response

4.4 Second-Order Systems:

- We now summarize our observations.

1. *Overdamped responses*

   Poles: Two real at $-\sigma_1, \ -\sigma_2$

   Natural response: Two exponentials with time constants equal to the reciprocal of the pole locations, or

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

2. *Underdamped responses*

   Poles: Two complex at $-\sigma_d \pm j\omega_d$

   Natural response: Damped sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole's real part. The radian frequency of the sinusoid, the damped frequency of oscillation, is equal to the imaginary part of the poles, or

$$c(t) = A e^{-\sigma_d t} \cos(\omega_d t - \phi)$$

Chapter 4: Time Response

4.4 Second-Order Systems:

- We now summarize our observations.

3. *Undamped responses*

Poles: Two imaginary at $\pm j\omega_1$

Natural response: Undamped sinusoid with radian frequency equal to the imaginary part of the poles, or

$$c(t) = A\cos(\omega_1 t - \phi)$$

4. *Critically damped responses*

Poles: Two real at $-\sigma_1$

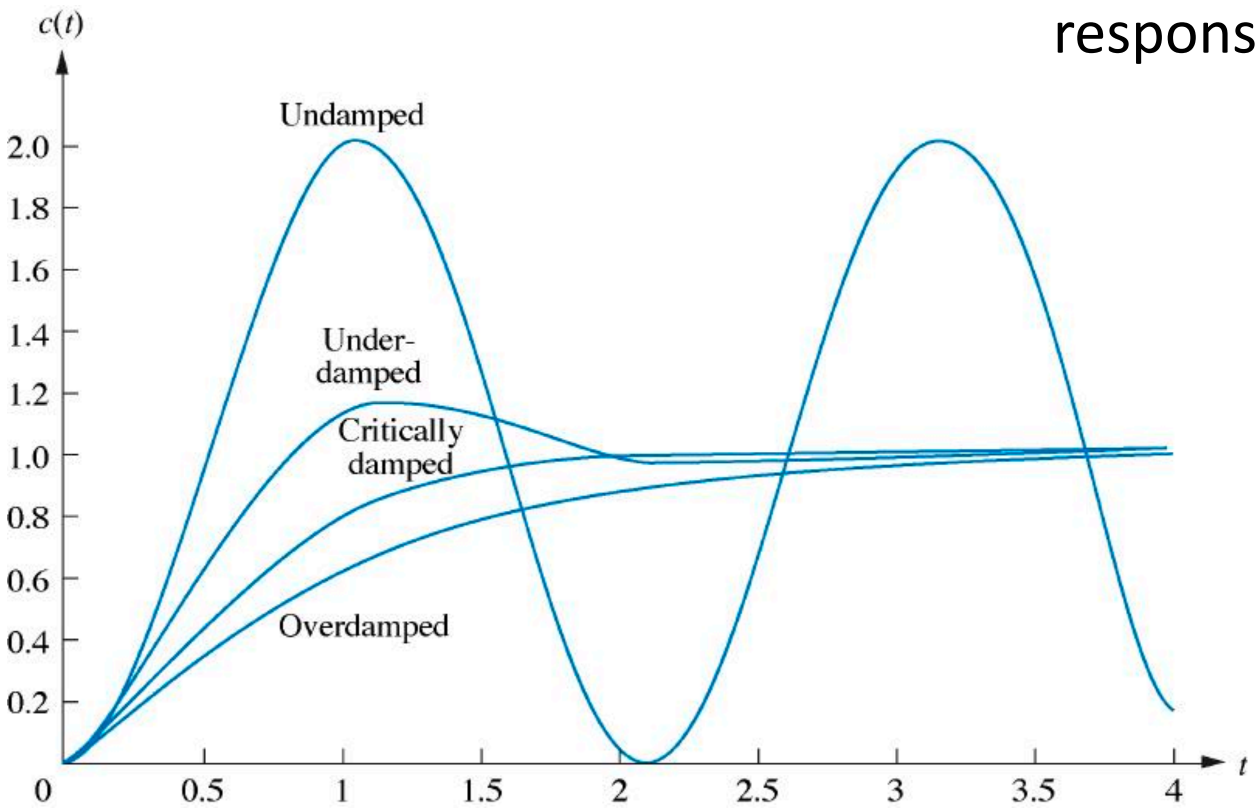
Natural response: One term is an exponential whose time constant is equal to the reciprocal of the pole location. Another term is the product of time, t, and an exponential with time constant equal to the reciprocal of the pole location, or

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$$

Chapter 4: Time Response

4.4 Second-Order Systems:

- The step responses for the four cases of damping discussed in this section are superimposed in Figure 4.10.
- Notice that the critically damped case is the division between the overdamped cases and the underdamped cases and is the fastest response without overshoot.



Chapter 4: Time Response

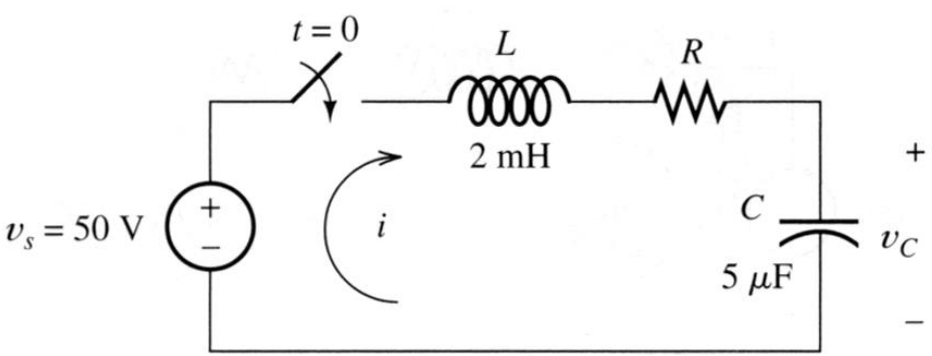
4.5 The General Second-Order System :

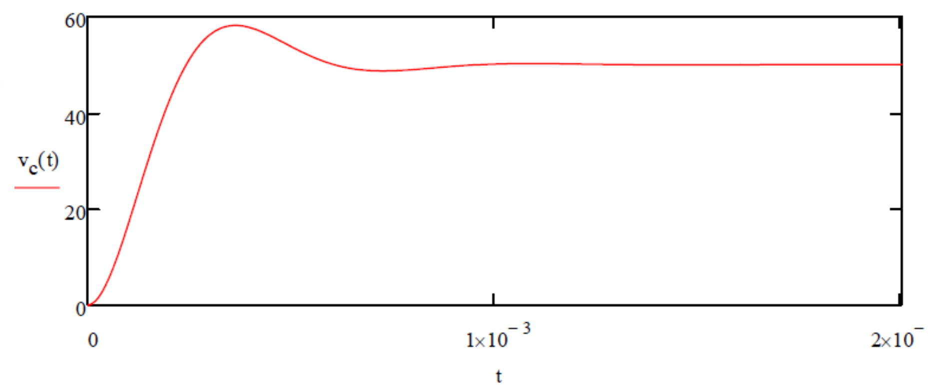
- Now that we have become familiar with second-order systems and their responses, we generalize the discussion and establish quantitative specifications defined in such a way that the response of a second-order system can be described to a designer *without the need for sketching the response*.
- In this section, we define two physically meaningful specifications for second-order systems. These quantities can be used to describe the characteristics of the second-order transient response just as time constants describe the first-order system response.
- The two quantities are called <u>*natural frequency*</u> and <u>*damping ratio*</u>.

4.5 The General Second-Order System :

Natural Frequency, ω_n

- The *natural frequency of a second-order system is the frequency of oscillation of the* system without damping.
- For example, the frequency of oscillation of a series *RLC* circuit with the resistance shorted would be the natural frequency.





$i(0) = 0 \quad v_C(0) = 0$

$$\omega_n := \sqrt{\frac{1}{L_1 \cdot C_1}}$$

$$\zeta := \frac{R_1}{2} \cdot \sqrt{\frac{C_1}{L_1}}$$

$$v_c(t) := u_o \cdot \left(1 - \frac{e^{-\zeta \cdot \omega_n \cdot t}}{\sqrt{1-\zeta^2}} \cdot \sin(\omega_d \cdot t + \varphi)\right)$$

$$\omega_d := \omega_n \cdot \sqrt{1-\zeta^2}$$

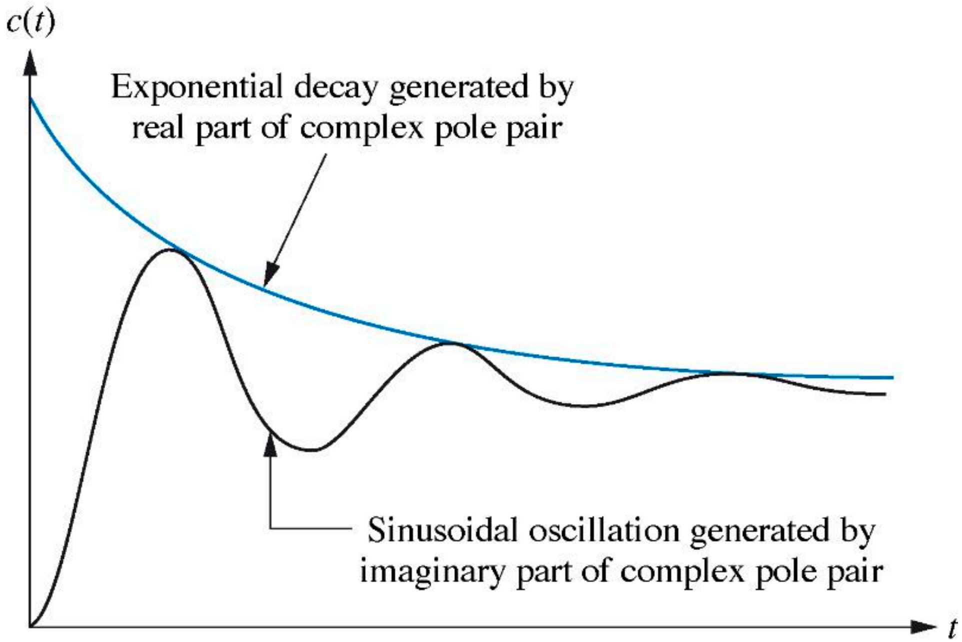
Chapter 4: Time Response

4.5 The General Second-Order System :

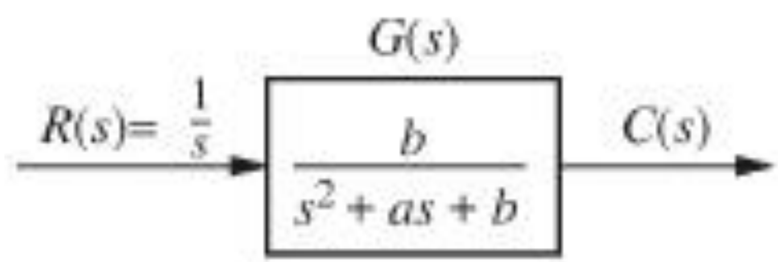
Damping Ratio, ζ

- A viable definition for this quantity is one that compares the exponential decay frequency of the envelope to the natural frequency. We define the *damping ratio, ζ, to be*

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/second)}}$$



This ratio is constant regardless of the time scale of the response.



$$s = \frac{1}{2}(-a \pm \sqrt{a^2 - 4b})$$

4.5 The General Second-Order System :

- The general second-order system shown in Figure 4.7(a) can be transformed to show the quantities ζ and ω_n.
- *Consider the system*

$$G(s) = \frac{b}{(s^2 + 2as + b)}$$ ⟸ General Form

- Without damping, the poles would be on the jω-axis, and the response would be an undamped sinusoid.
- For the poles to be purely imaginary, *a = 0.*

$$G(s) = \frac{b}{(s^2 + b)}$$

poles are at $s = \pm\sqrt{b}$

$$\therefore \omega_n = \sqrt{b} \Rightarrow b = \omega_n^{\ 2}$$

$$\boxed{\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/second)}}}$$

$$\therefore \zeta = \frac{a/2}{\omega_n} \Rightarrow a = 2\zeta\omega_n$$

$$G(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$ ⟸ Standard Form

Chapter 4: Time Response

4.5 The General Second-Order System :

- Now that we have defined ζ*, and* ω_n*, let us relate these quantities to the pole* location. Solving for the poles of the transfer function in standard form yields:

$$G(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \Rightarrow s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Example 4.3

Finding ζ and ω_n For a Second-Order System

PROBLEM: Given the transfer function of Eq. (4.23), find ζ and ω_n.

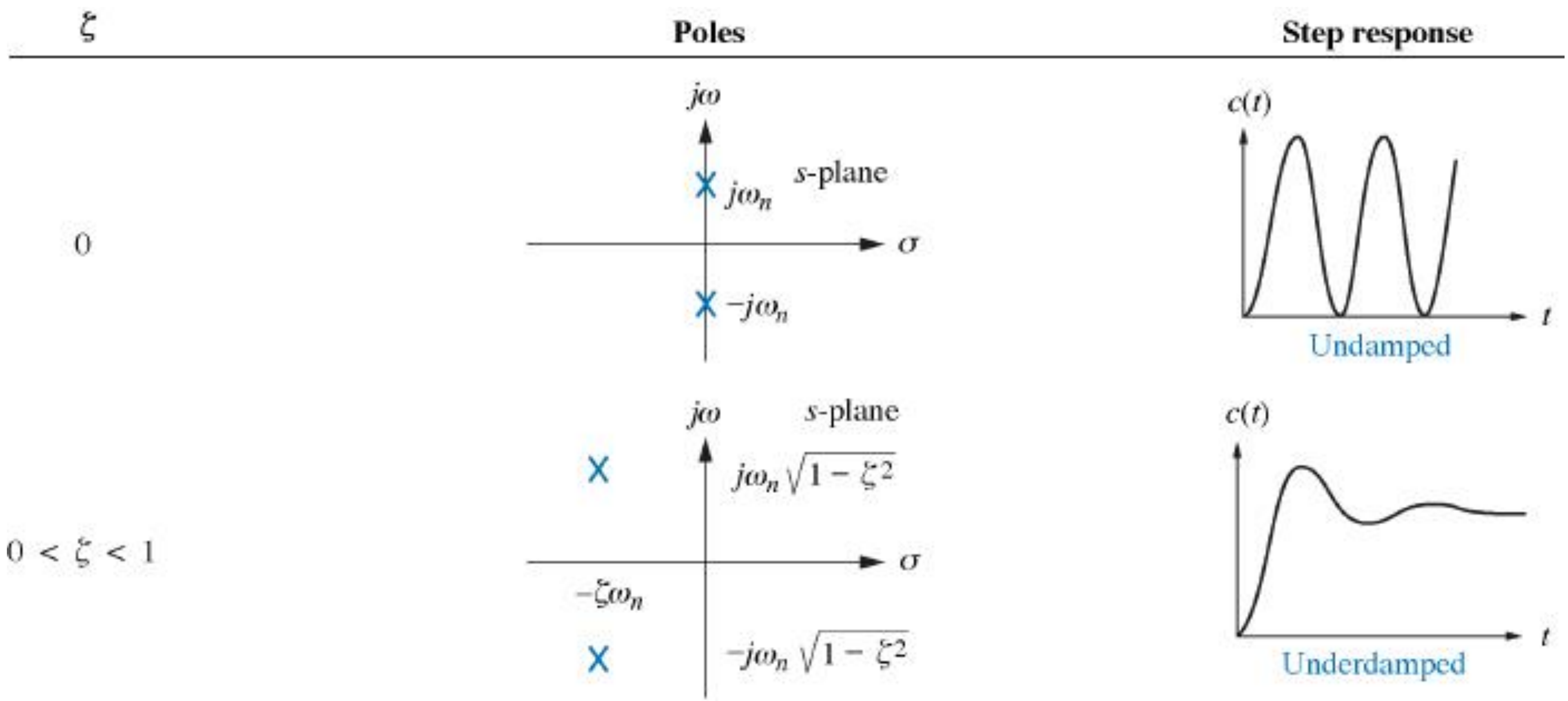
$$G(s) = \frac{36}{s^2 + 4.2s + 36} \tag{4.23}$$

SOLUTION: Comparing Eq. (4.23) to (4.22), $\omega_n^2 = 36$, from which $\omega_n = 6$. Also, $2\zeta\omega_n = 4.2$. Substituting the value of ω_n, $\zeta = 0.35$.

Chapter 4: Time Response

4.5 The General Second-Order System :

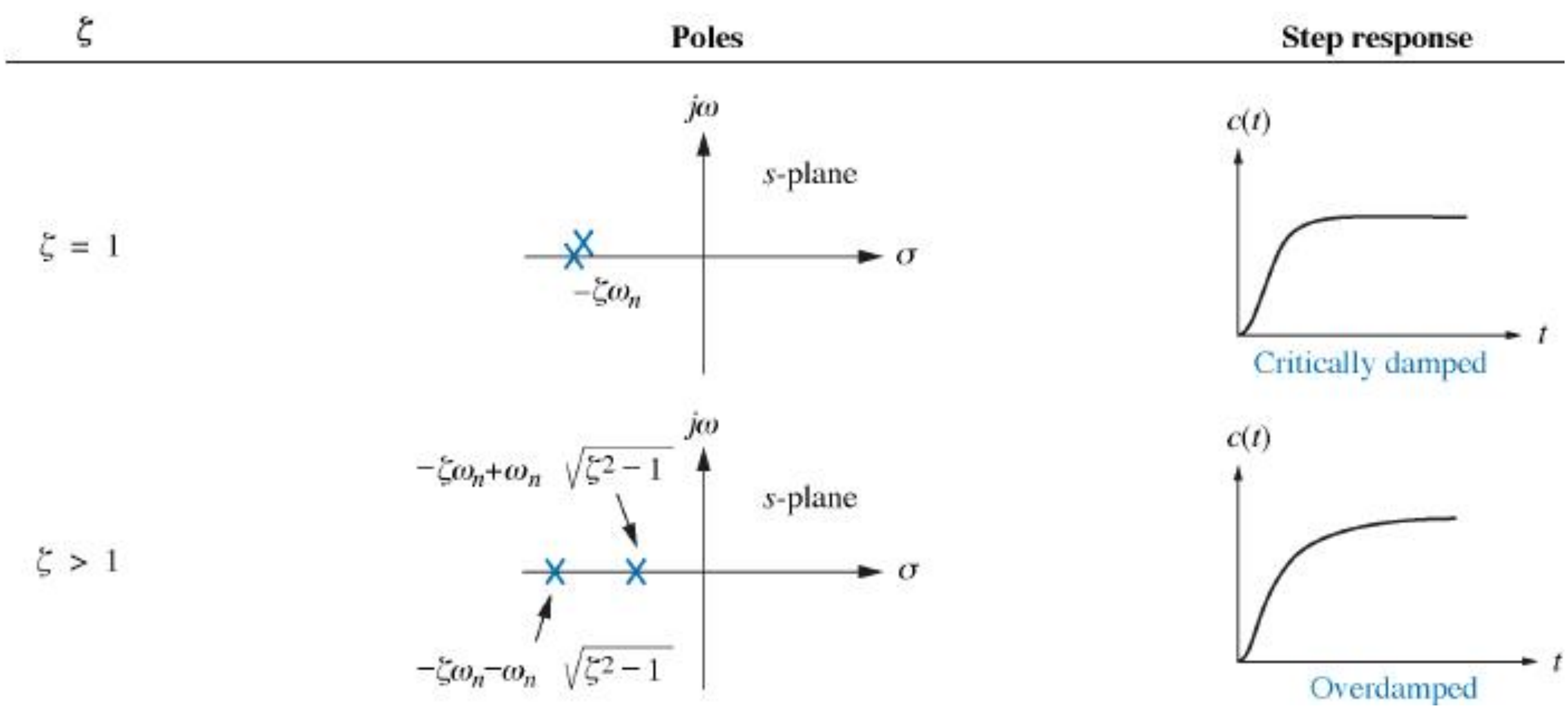
- We see that the various cases of second-order response are a function of ζ.



Chapter 4: Time Response

4.5 The General Second-Order System :

- We saw that the nature of the response obtained was related to the value of ζ . *Variations of damping ratio alone yield the complete range of overdamped, critically damped, underdamped, and undamped responses.*



Chapter 4: Time Response

4.5 The General Second-Order System :

Characterizing Response from the Value of ζ

PROBLEM: For each of the systems shown in Figure 4.12, find the value of ζ and report the kind of response expected.

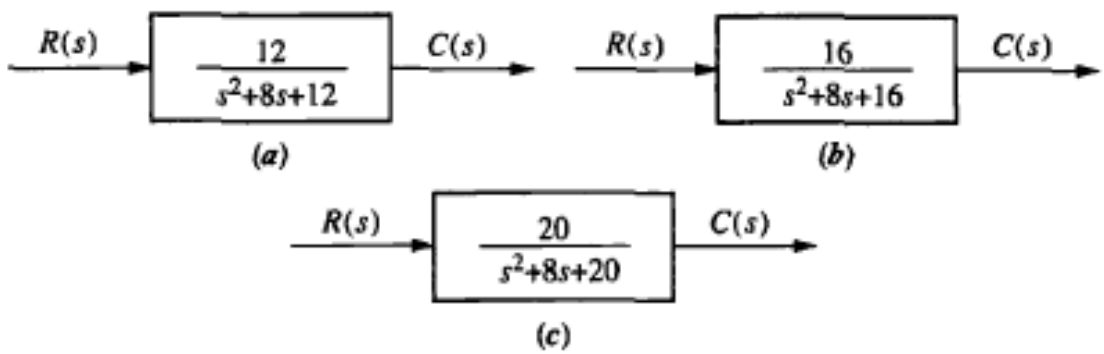


FIGURE 4.12 Systems for Example 4.4

SOLUTION: First match the form of these systems to the forms shown in Eqs. (4.16) and (4.22). Since $a = 2\zeta\omega_n$ and $\omega_n = \sqrt{b}$,

$$\zeta = \frac{a}{2\sqrt{b}} \tag{4.25}$$

Using the values of a and b from each of the systems of Figure 4.12, we find $\zeta = 1.155$ for system (a), which is thus overdamped, since $\zeta > 1$; $\zeta = 1$ for system (b), which is thus critically damped; and $\zeta = 0.894$ for system (c), which is thus underdamped, since $\zeta < 1$.

Chapter 4: Time Response

4.6 Underdamped Second-Order Systems:

- Now that we have generalized the second-order transfer function in terms of ζ and ω_n, let us analyze the step response of an underdamped second-order system. Not only will this response be found in terms of ζ and ω_n, *but more specifications* indigenous to the underdamped case will be defined.
- <u>Our first objective is to define transient specifications associated with underdamped responses</u>. Next we relate these specifications to the pole location, drawing an association between pole location and the form of the underdamped second-order response. Finally, we tie the pole location to system parameters, thus closing the loop: Desired response generates required system components.

Chapter 4: Time Response

4.6 Underdamped Second-Order Systems:

- Let us begin by finding the step response for the general second-order system.

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}t\right)$$

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}t - \phi)$$

Chapter 4: Time Response

4.6 Underdamped Second-Order Systems:

- Let us begin by finding the <u>step response for the general second-order</u> system.

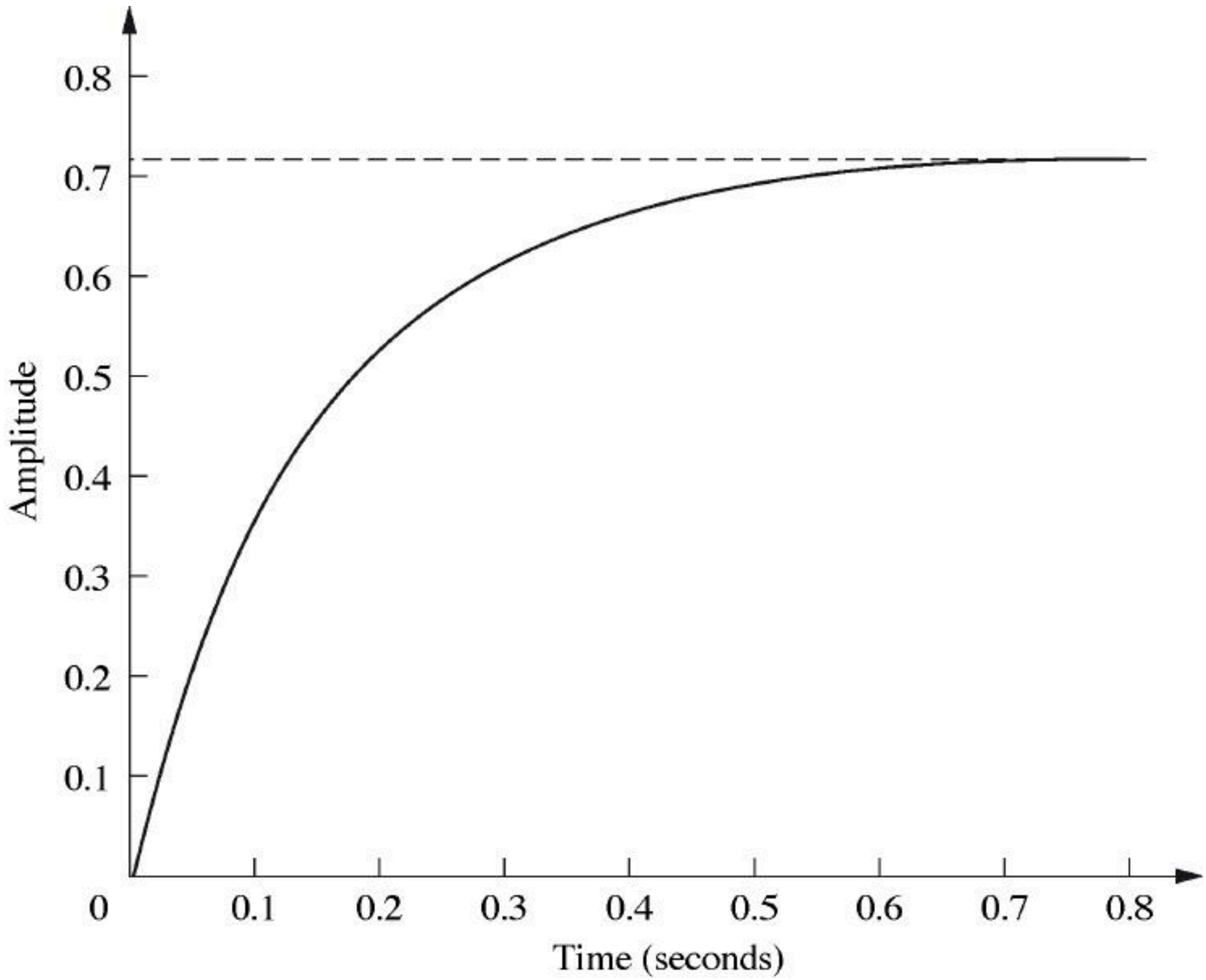
$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}t\right)$$

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}t - \phi)$$

where $\phi = \tan^{-1}(\zeta/\sqrt{1-\zeta^2})$.



Amplitude
0.8
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0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
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0.4
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0.6
0.7
0.8
Time (seconds)