Control Systems Engineering

Chapter 2: Modeling in the Frequency Domain

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Chapter 2: Modeling in the Frequency Domain

Learning Outcomes:

- Find the Laplace transform of time functions and the inverse Laplace transform (Sections 2.1-2.2)
- Find the <u>transfer function</u> from a differential equation and solve the differential equation using the transfer function (Section 2.3)
- Find the transfer function for
  - linear, time-invariant electrical networks (Section 2.4)
  - linear, time-invariant translational mechanical systems
  - time-invariant rotational mechanical systems
  - gear systems with no loss and for gear systems with loss

Chapter 2: Modeling in the Frequency Domain

Learning Outcomes: (cont.)

- Find the transfer function for
  - linear, time-invariant electrical networks (Section 2.4)
  - linear, time-invariant translational mechanical systems
  - time-invariant rotational mechanical systems
  - gear systems with no loss and for gear systems with loss
  - electromechanical systems
  - Produce analogous electrical and mechanical circuits
  - Linearize a nonlinear system in order to find the transfer function

Chapter 2: Modeling in the Frequency Domain

Case Study:

- Given the antenna azimuth position control system shown on the front endpapers*, you will be able to find the <u>transfer function</u> of each subsystem.*
- Given a model of a human leg or a nonlinear electrical circuit, you will be able to linearize the model and then find the transfer function.

Modeling in the Frequency Domain

What is "*Transfer Function*" ?

- Transfer function is the ratio of the Laplace transform of the output signal to the input signal with the initial conditions as zero.

$$\boxed{\frac{C(s)}{R(s)} = G(s)}$$

Modeling in the Frequency Domain

Example 2.4: Transfer Function for a Differential Equation

PROBLEM: Find the transfer function represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t) \tag{2.55}$$

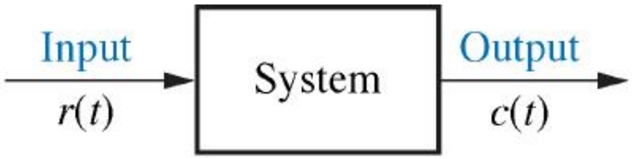
SOLUTION: Taking the Laplace transform of both sides, assuming zero initial conditions, we have

$$sC(s) + 2C(s) = R(s) \tag{2.56}$$

The transfer function, $G(s)$, is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2} \tag{2.57}$$

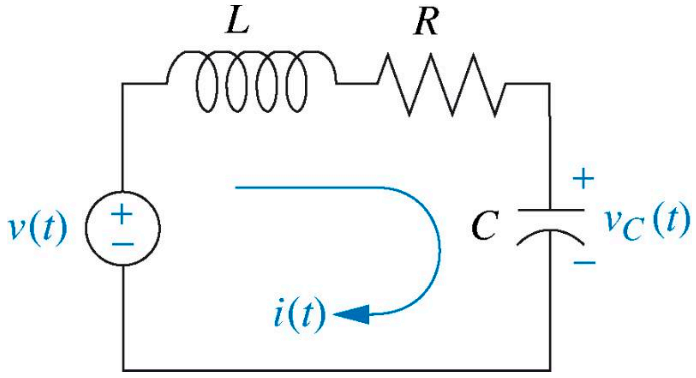
Note:
$$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$$
$$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$$



$$\frac{C(s)}{R(s)} = G(s)$$

Modeling in the Frequency Domain

Example 2.6: Find the transfer function relating the capacitor voltage, $V_C(s)$, *to* the input voltage, *V(s).*



Solution*: Assuming all initial conditions = 0*

- Apply KVL

$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int_0^t i(\tau)d\tau=v(t)$$

$$L\frac{di(t)}{dt}+Ri(t)+v_C(t)=v(t)$$

$$\boxed{i_C(t)=C\frac{dv_C(t)}{dt}=i(t)}$$

$$LC\frac{d^2v_C(t)}{dt^2}+RC\frac{dv_C(t)}{dt}+v_C(t)=v(t)$$

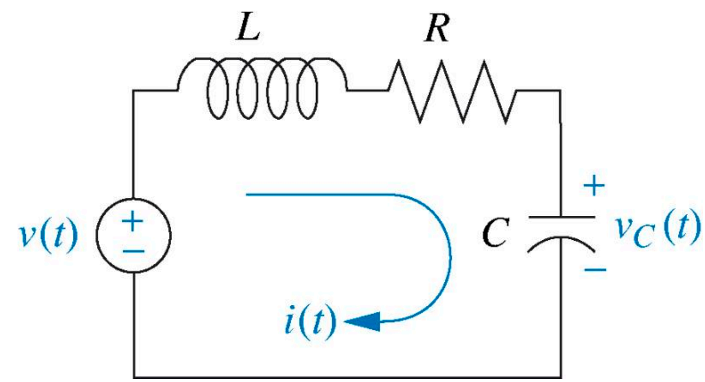
➡ Differential Eq.

$$LCs^2V_C(s)+RCsV_C(s)+V_C(s)=V(s)$$

➡ Algebraic Eq.

Modeling in the Frequency Domain

Example 2.6: Find the transfer function relating the capacitor voltage, *$V_C(s)$, to* the input voltage, *$V(s)$.*



<u>Solution</u>*: Assuming all initial conditions = 0*

- Apply KVL

$$L\frac{di(t)}{dt}+Ri(t)+v_C(t)=v(t)$$

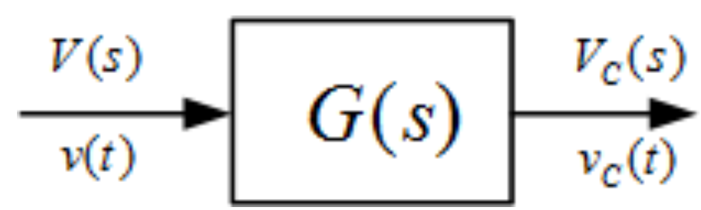
$$i_C(t)=C\frac{dv_C(t)}{dt}=i(t)$$

$$LC\frac{d^2v_C(t)}{dt^2}+RC\frac{dv_C(t)}{dt}+v_C(t)=v(t)$$

$$LCs^2V_C(s)+RCsV_C(s)+V_C(s)=V(s)$$

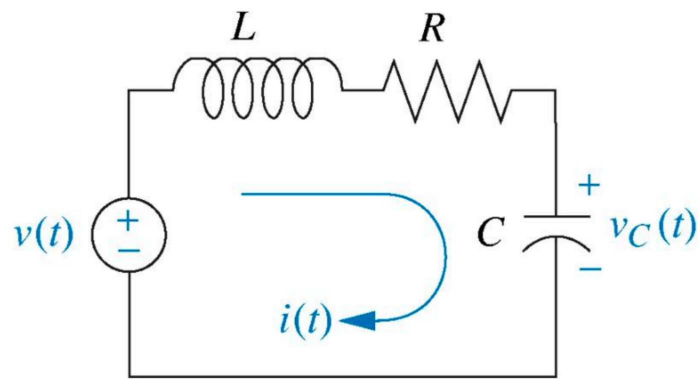
$$(LCs^2+RCs+1)V_C(s)=V(s)$$

$$\frac{V_C(s)}{V(s)}=\frac{1}{(LCs^2+RCs+1)}$$



Modeling in the Frequency Domain

Example 2.6: Find the transfer function relating the capacitor voltage, $V_C(s)$, *to* the input voltage, $V(s)$.



Solution*: Assuming all initial conditions = 0*

- Apply KVL

$$L\frac{di(t)}{dt} + Ri(t) + v_C(t) = v(t)$$

$$LC\frac{d^2 v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t)$$

1. Write the differential equations for the system.

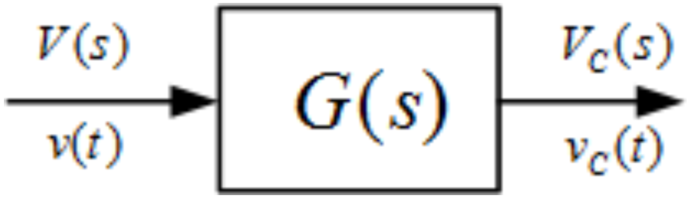
$$LCs^2V_C(s) + RCsV_C(s) + V_C(s) = V(s)$$

2. Apply Laplace Transform

$$(LCs^2 + RCs + 1)V_C(s) = V(s)$$

3. Solve the equations for the Transfer function.

$$\frac{V_C(s)}{V(s)} = \frac{1}{(LCs^2 + RCs + 1)}$$



Modeling in the Frequency Domain

Electrical Network Transfer Functions

- Let us now develop a technique for simplifying the solution for future problems. Assuming **zero initial conditions**.

TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

Modeling in the Frequency Domain

Electrical Network Transfer Functions

- Let us now develop a technique for simplifying the solution for future problems. Assuming zero initial conditions.

For the capacitor,

$$\boxed{V(s) = \frac{1}{Cs}I(s)} \tag{2.67}$$

For the resistor,

$$\boxed{V(s) = RI(s)} \tag{2.68}$$

For the inductor,

$$\boxed{V(s) = LsI(s)} \tag{2.69}$$

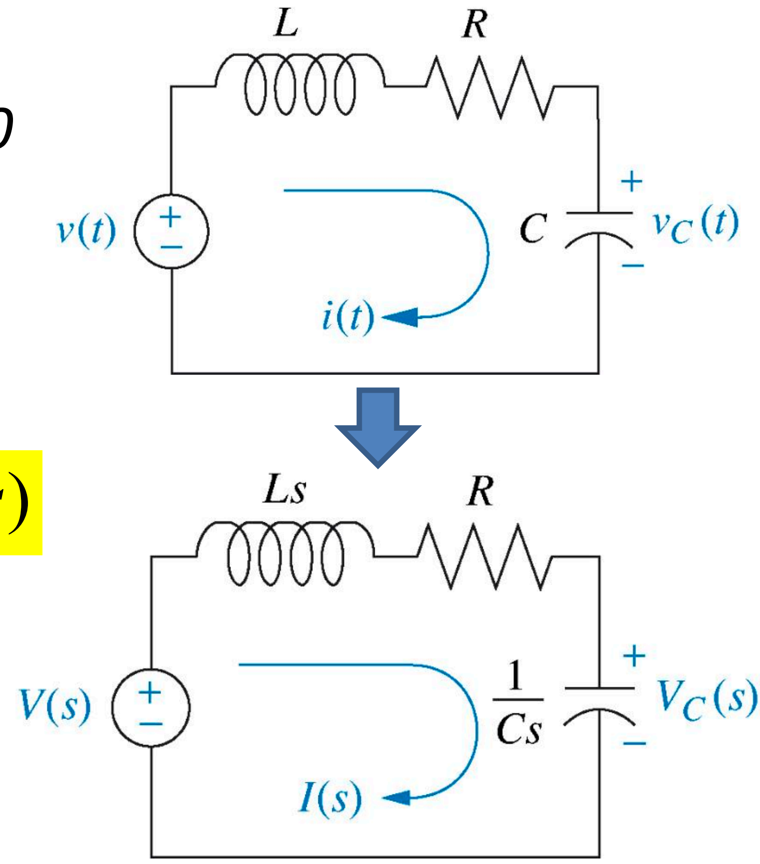
Now define the following transfer function:

All of the transfer functions have this form.

$$\boxed{\frac{V(s)}{I(s)} = Z(s)} \tag{2.70}$$

Modeling in the Frequency Domain

Example 2.6: Find the transfer function relating the capacitor voltage, $V_C(s)$, *to* the input voltage, $V(s)$.



Solution: *Assuming all initial conditions = 0*

- Apply KVL

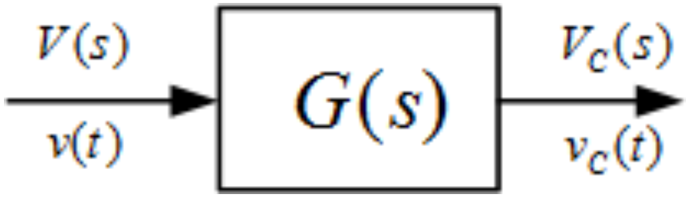
$$sLI(s) + RI(s) + \frac{1}{sC}I(s) = V(s)$$

$$I(s) = sCV_C(s)$$

$$LCs^2V_C(s) + RCsV_C(s) + V_C(s) = V(s)$$

$$(LCs^2 + RCs + 1)V_C(s) = V(s)$$

$$\frac{V_C(s)}{V(s)} = \frac{1}{(LCs^2 + RCs + 1)}$$

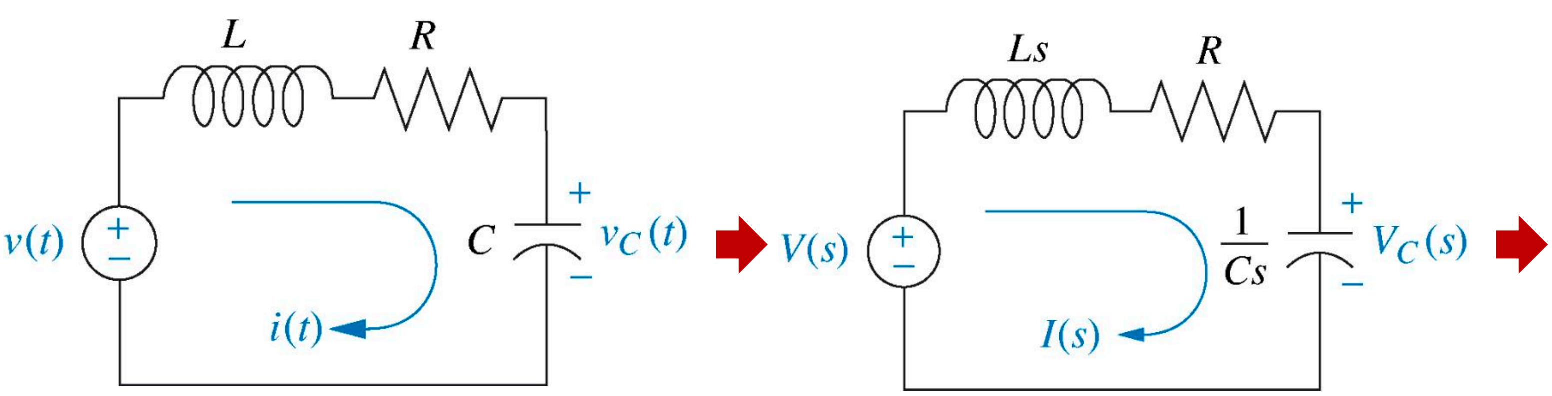


Modeling in the Frequency Domain

Electrical Network Transfer Functions

- We summarize the steps as follows:

1. Redraw the original network showing all time variables, such as *v(t), i(t), and $v_c(t)$, as Laplace transforms V(s), I(s), and $V_c(s)$, respectively.* <u>Assuming zero initial conditions.</u>
2. <u>Replace the component values with their impedance</u> values. This replacement is similar to the case of dc circuits, where we represent resistors with their resistance values.
3. Solve the equations for the transfer function.



$$\frac{V_C(s)}{V(s)} = G(s)$$

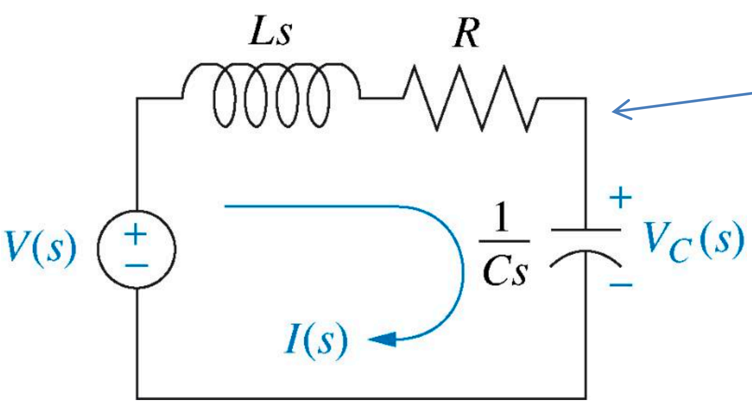
Modeling in the Frequency Domain

Electrical Network Transfer Functions

- <u>Note</u>: Not only the KVL and KCL, we can also use the Mesh and Nodal analysis to determine the transfer function.

Example 2.8: Repeat Example 2.6 using **nodal analysis** and <u>*without writing a differential equation.*</u>

<u>Solution</u>: We assume that currents leaving the node are positive and currents entering the node are negative. Hence,



$$\frac{V_C(s)}{1/sC} + \frac{VC(s) - V(s)}{R + Ls} = 0$$

$$\boxed{\frac{V_C(s)}{V(s)} = \frac{1}{(LCs^2 + RCs + 1)}}$$

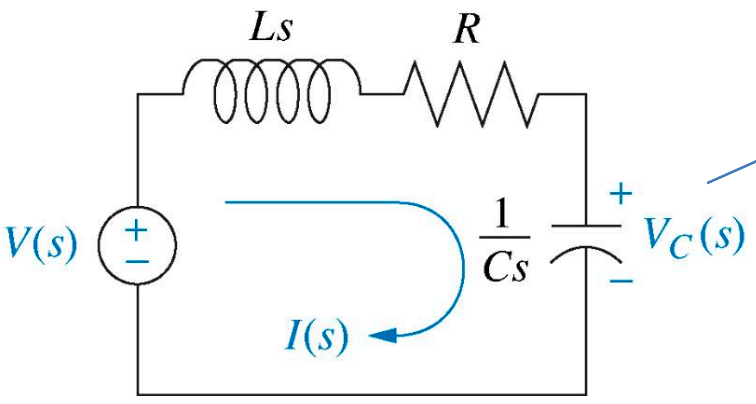
Modeling in the Frequency Domain

Electrical Network Transfer Functions

- For some simple circuit, the transfer function can be determined via voltage division.

- **Example 2.9:** Repeat Example 2.6 using voltage division and the transformed circuit.

Solution: We assume that currents leaving the node are positive and currents entering the node are negative. Hence,



$$V_C(s) = \frac{1/sC}{(R + Ls + 1/sC)} V(s)$$

$$\frac{V_C(s)}{V(s)} = \frac{1}{(LCs^2 + RCs + 1)}$$

Modeling in the Frequency Domain

Electrical Network Transfer Functions

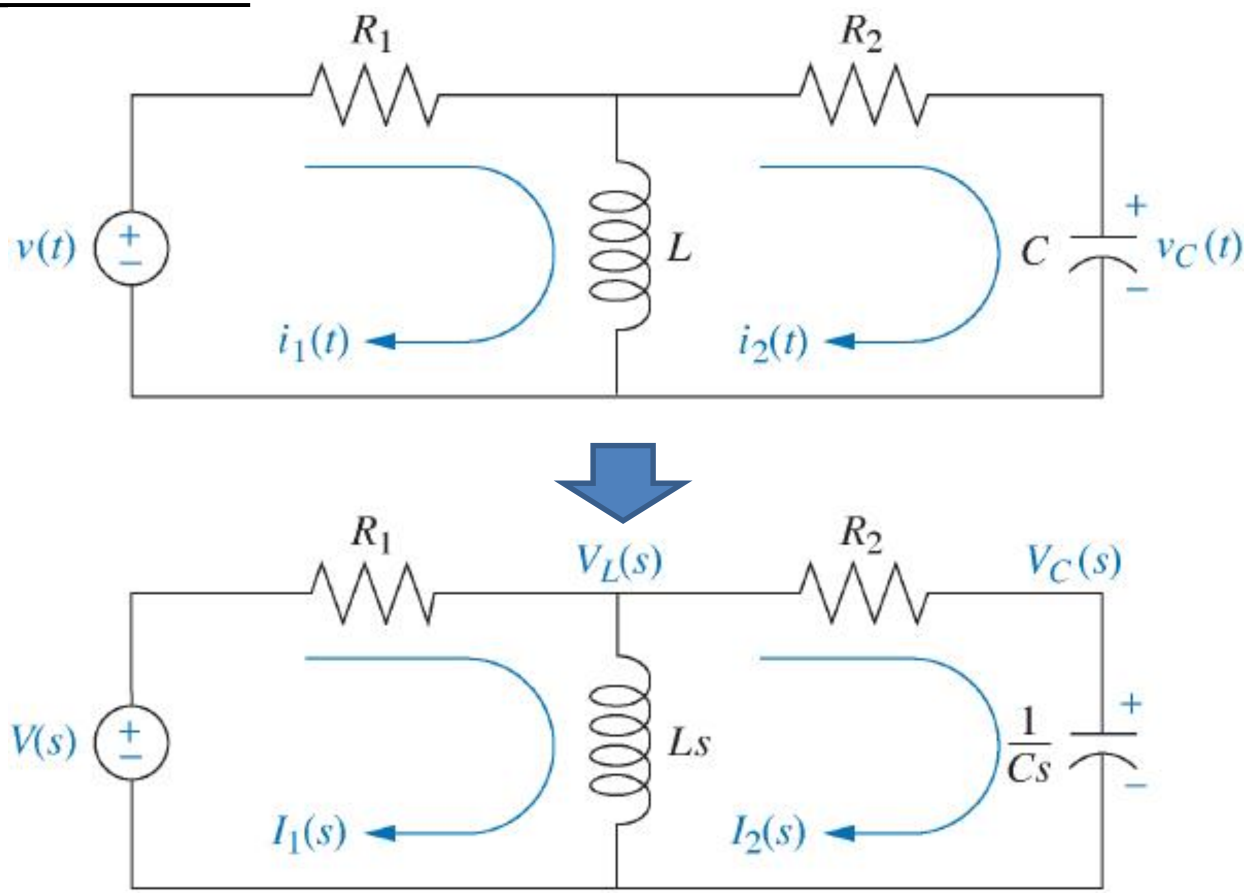
- To solve **complex electrical networks**—those with multiple loops and nodes — using **mesh analysis**, we can perform the following steps:

1. Replace passive element values with their impedances.
2. Replace all sources and time variables with their Laplace transform.
3. Assume a transform current and a current direction in each mesh.
4. Write Kirschoff's voltage law around each mesh.
5. Solve the simultaneous equations for the output.
6. Form the transfer function.

Modeling in the Frequency Domain

- **Example 2.10:** Given the network of Figure 2.6(a), find the transfer function, $I_2(s)/V(s)$.

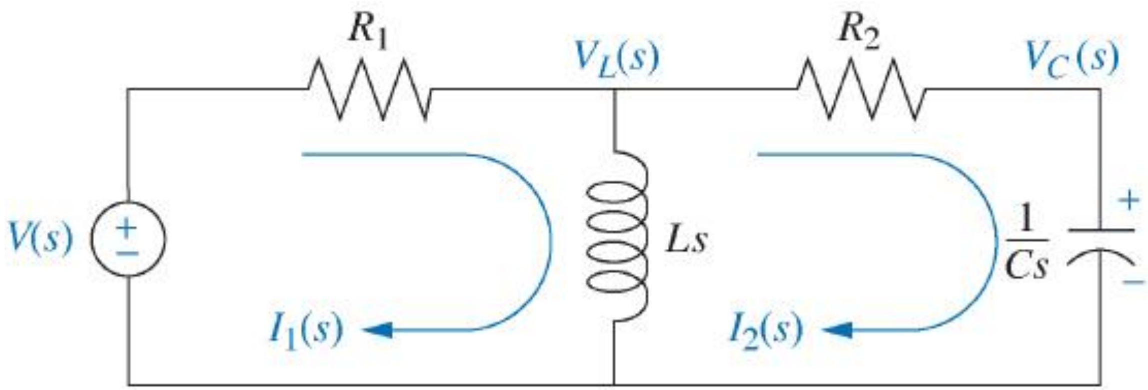
Solution: ***Assuming all initial conditions = 0****, and draw the transformed circuit.*



Modeling in the Frequency Domain

- **Example 2.10:** Given the network of Figure 2.6(a), find the transfer function, $I_2(s)/V(s)$.

<u>Solution</u>*:* Write Kirchoff's voltage law around each mesh.



$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s) \qquad (1)$$

$$LsI_2(s) + R_2 I_2(s) + \frac{1}{sC} I_2(s) - LsI_1(s) = 0 \qquad (2)$$

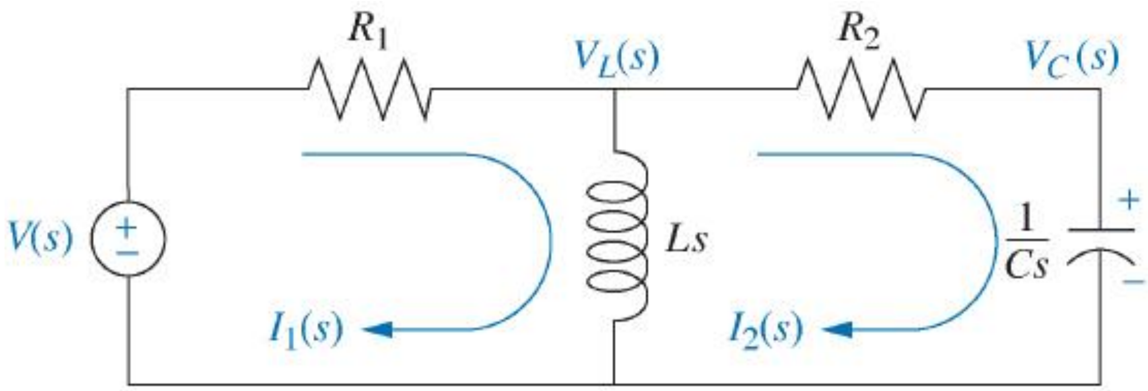
$$\frac{I_2(s)}{V(s)} = ?$$

Note : we have to eleminate $I_1(s)$ from the equations.

Modeling in the Frequency Domain

- **Example 2.10:** Given the network of Figure 2.6(a), find the transfer function, $I_2(s)/V(s)$.

Solution: Write Kirschoff's voltage law around each mesh.



$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s) \Rightarrow (R_1 + Ls)I_1(s) = V(s) + LsI_2(s)$$

$$LsI_2(s) + R_2 I_2(s) + \frac{1}{sC} I_2(s) - LsI_1(s) = 0$$

$$I_1(s) = \frac{V(s) + LsI_2(s)}{(R_1 + Ls)}$$

$$\Downarrow$$

$$LsI_2(s) + R_2 I_2(s) + \frac{1}{sC} I_2(s) - Ls\left(\frac{V(s) + LsI_2(s)}{R_1 + Ls}\right) = 0$$

Modeling in the Frequency Domain

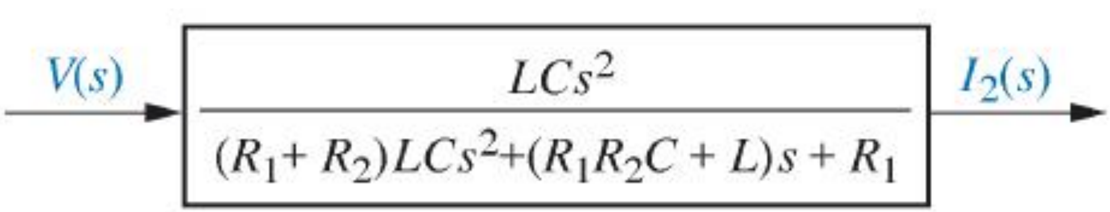
- **Example 2.10:** Given the network of Figure 2.6(a), find the transfer function,*$I_2(s)/V(s)$.*

<u>Solution</u>*:* Write Kirchoff's voltage law around each mesh.

$$LsI_2(s) + R_2I_2(s) + \frac{1}{sC}I_2(s) - Ls\left(\frac{V(s) + LsI_2(s)}{R_1 + Ls}\right) = 0$$

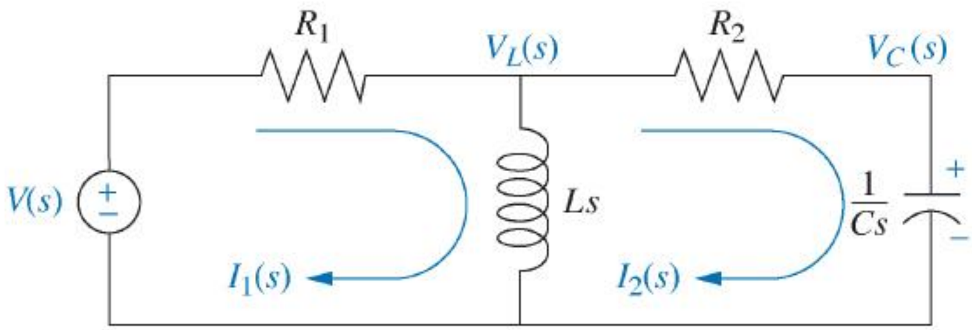
$$(R_1 + Ls)\left(LsI_2(s) + R_2I_2(s) + \frac{1}{sC}I_2(s)\right) = Ls(V(s) + LsI_2(s))$$

$$(R_1 + Ls)\left(R_2I_2(s) + \frac{1}{sC}I_2(s)\right) + R_1LsI_2(s) = LsV(s)$$



Modeling in the Frequency Domain

- **Example 2.10:** Before leaving the example, we notice a pattern first illustrated by Eq. (2.72). The form that Eq. (2.80) take is.



$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s)$$

$$LsI_2(s) + R_2 I_2(s) + \frac{1}{sC} I_2(s) - LsI_1(s) = 0$$

$$\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around Mesh 1} \end{bmatrix} I_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to the} \\ \text{two meshes} \end{bmatrix} I_2(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 1} \end{bmatrix}$$

$$-\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to the} \\ \text{two meshes} \end{bmatrix} I_1(s) + \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around Mesh 2} \end{bmatrix} I_2(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 2} \end{bmatrix}$$

Modeling in the Frequency Domain

Electrical Network Transfer Functions

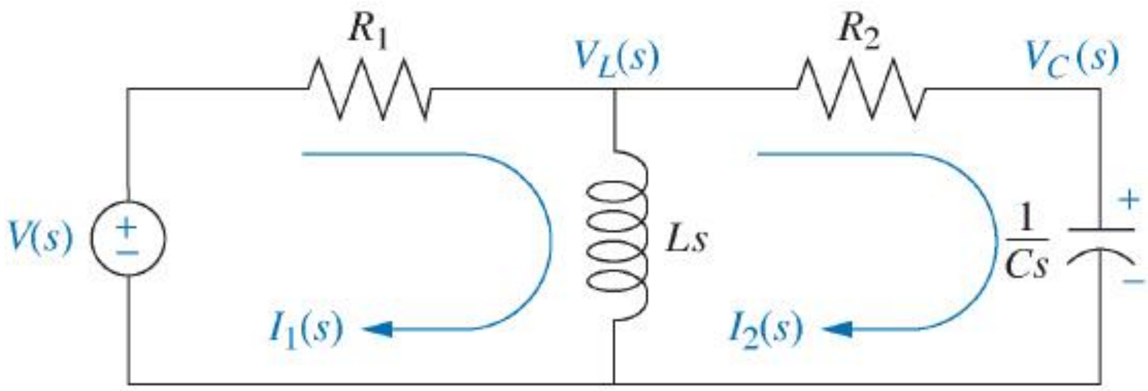
- In order to handle multiple-node electrical networks using **nodal analysis**, we can perform the following steps:

1. Replace passive element values with their admittances.
2. Replace all sources and time variables with their Laplace transform.
3. Assume a transform voltage for each node.
4. Write **Kirschoff's current law** at each node.
5. Solve the simultaneous equations for the output.
6. Form the transfer function.

Modeling in the Frequency Domain

- **Example 2.11:** Find the transfer function, *$V_c(s)/V(s)$, for the circuit in Figure 2.6(b).* Use nodal analysis.

Solution*:* Write Kirschoff's current law for each node.



$G_1 = 1/R_1$ $G_2 = 1/R_2$

$$\frac{V_L(s) - V(s)}{R_1} + \frac{V_L(s)}{Ls} + \frac{V_L(s) - V_C(s)}{R_2} = 0$$

$$CsV_C(s) + \frac{V_C(s) - V_L(s)}{R_2} = 0$$

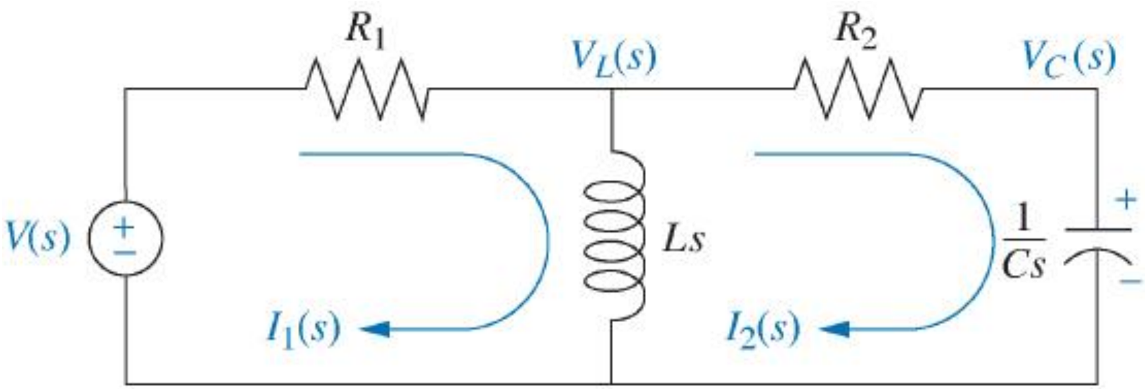
$$\left(G_1 + G_2 + \frac{1}{Ls}\right)V_L(s) \qquad - G_2V_C(s) = V(s)G_1$$

$$-G_2V_L(s) + (G_2 + Cs)V_C(s) = 0$$

Modeling in the Frequency Domain

- **Example 2.11:** Find the transfer function, $V_c(s)/V(s)$*, for the circuit in Figure 2.6(b).* Use nodal analysis.

Solution*:* Write Kirschoff's current law for each node.



$G_1 = 1/R_1$ $G_2 = 1/R_2$

$$\left(G_1 + G_2 + \frac{1}{Ls}\right)V_L(s) - G_2 V_C(s) = V(s)G_1$$

$$-G_2 V_L(s) + (G_2 + Cs)V_C(s) = 0$$

$$\frac{V_C(s)}{V(s)} = \frac{\frac{G_1 G_2}{C}s}{(G_1 + G_2)s^2 + \frac{G_1 G_2 L + C}{LC}s + \frac{G_2}{LC}}$$

$$V(s) \longrightarrow \boxed{\frac{\frac{G_1 G_2}{C}s}{(G_1 + G_2)s^2 + \frac{G_1 G_2 L + C}{LC}s + \frac{G_2}{LC}}} \longrightarrow V_C(s)$$

Modeling in the Frequency Domain

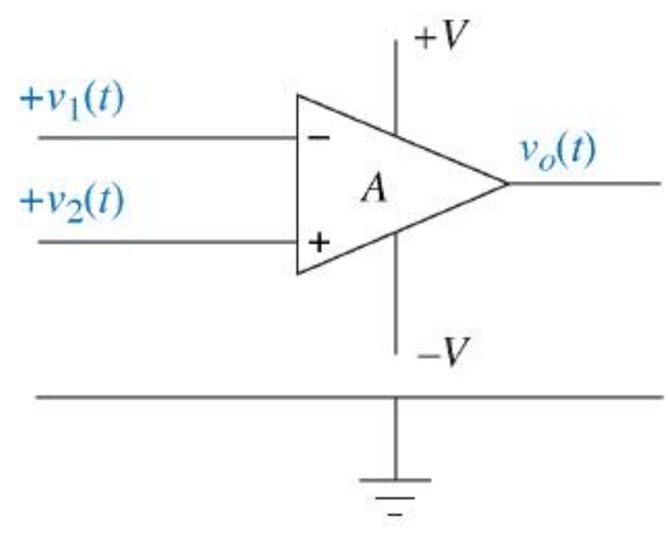
Transfer Function : Operational Amplifiers

- An *operational amplifier, pictured in Figure 2.10(a), is an electronic amplifier* *used as* a basic building block to implement transfer functions. It has the following characteristics:

1. Differential input, $v_2(t) - v_1(t)$
2. High input impedance, $Z_i = \infty$ (ideal)
3. Low output impedance, $Z_o 0 = 0$ *(ideal)*
4. High constant gain amplification, $A = \infty$ *(ideal)*

- The output, $v_o(t)$, *is given by*

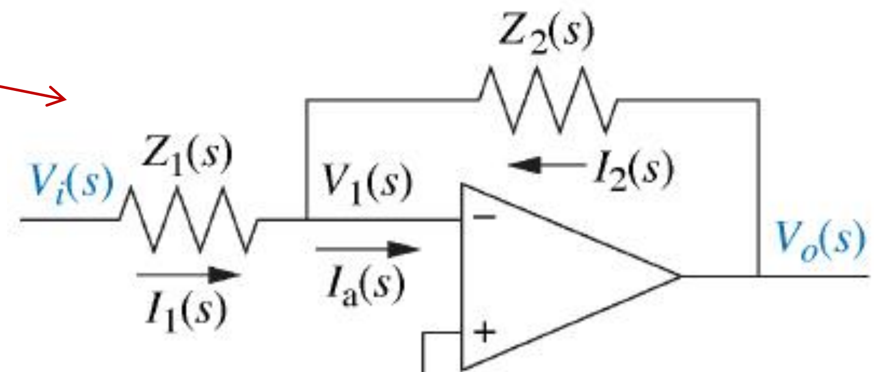
$$v_o(t)=A(v_2(t)-v_1(t))$$



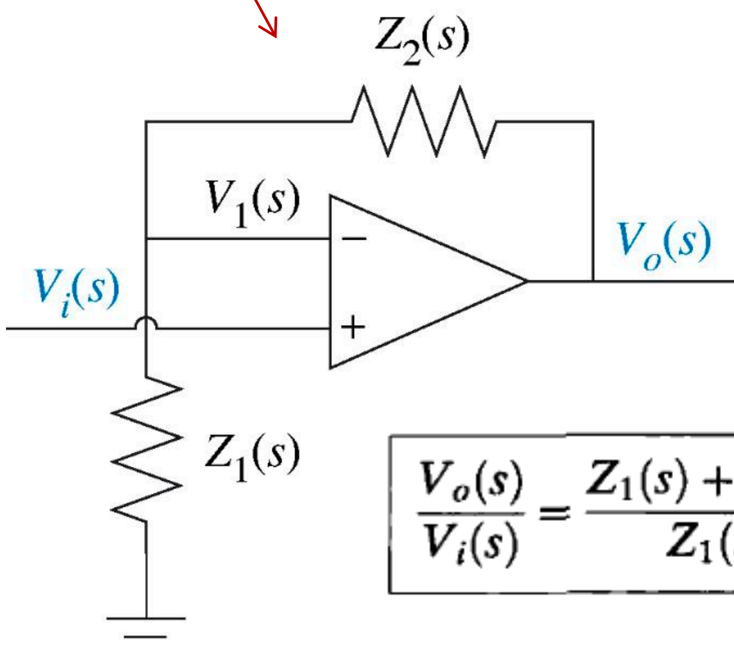
Modeling in the Frequency Domain

Transfer Function : Operational Amplifiers

- To build the transfer function, the operational amplifiers are mostly applied in the following circuit :
  - **Inverting Amplifier**
  - **Non inverting Amplifier**



$$\boxed{\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}}$$

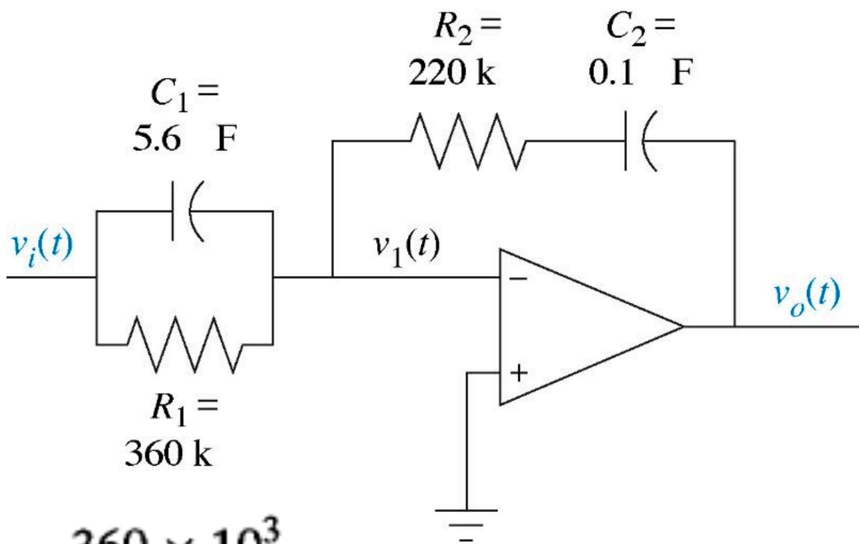


$$\boxed{\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}}$$

Modeling in the Frequency Domain

Transfer Function : Inverting Operational Amplifier Circuit

- **Example 2.14** Find the transfer function, *$V_o(s)/V_i(s)$, for the circuit given in* Figure 2.11.
- <u>Solution</u>



$$\boxed{\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}}$$

$$Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}} = \frac{1}{5.6 \times 10^{-6} s + \frac{1}{360 \times 10^3}} = \frac{360 \times 10^3}{2.016 s + 1}$$

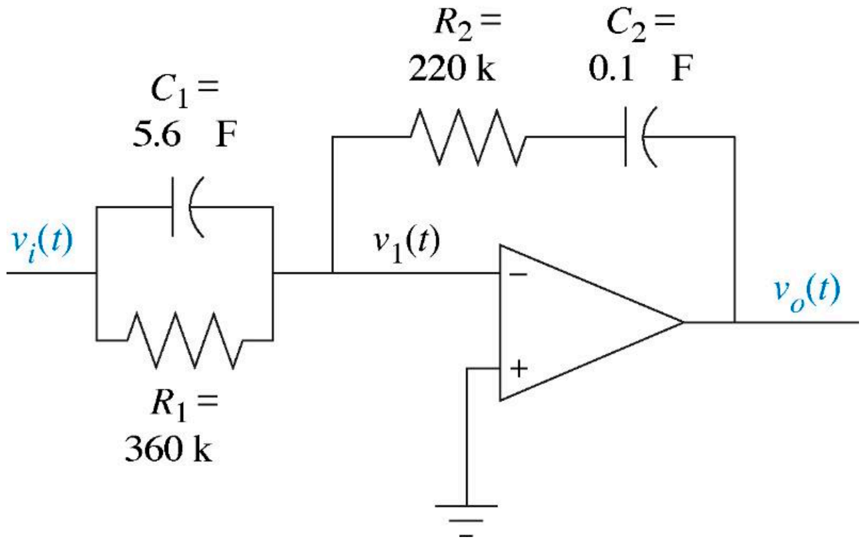
$$Z_2(s) = R_2 + \frac{1}{C_2 s} = 220 \times 10^3 + \frac{10^7}{s}$$

$$\boxed{\frac{V_o(s)}{V_i(s)} = -1.232 \frac{s^2 + 45.95 s + 22.55}{s}}$$

Modeling in the Frequency Domain

Transfer Function : Inverting Operational Amplifier Circuit

- **Example 2.14** Find the transfer function, $V_o(s)/V_i(s)$, *for the circuit given in* Figure 2.11.
- <u>Solution</u>



$$\boxed{\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}}$$

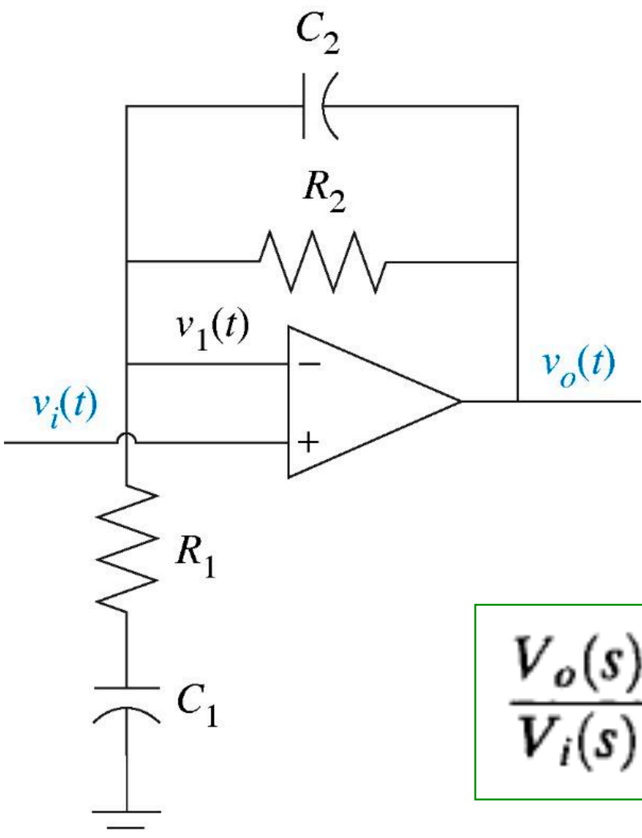
$$\boxed{\frac{V_o(s)}{V_i(s)} = -1.232\frac{s^2 + 45.95s + 22.55}{s}}$$

<u>Note</u>: The resulting circuit is called a <u>PID controller</u> and can be used to improve the performance of a control system.

Modeling in the Frequency Domain

Transfer Function : Noninverting Operational Amplifier Circuit

- **Example 2.15** Find the transfer function, $V_o(s)/V_i(s)$, *for the circuit given in* Figure 2.13.
- <u>Solution</u>



$$\boxed{\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}}$$

$$Z_1(s) = R_1 + \frac{1}{C_1 s}$$

$$Z_2(s) = \frac{R_2(1/C_2 s)}{R_2 + (1/C_2 s)}$$

$$\boxed{\frac{V_o(s)}{V_i(s)} = \frac{C_2C_1R_2R_1s^2 + (C_2R_2 + C_1R_2 + C_1R_1)s + 1}{C_2C_1R_2R_1s^2 + (C_2R_2 + C_1R_1)s + 1}}$$

Modeling in the Frequency Domain

Electrical Network Transfer Functions

Conclusion:

- In this section, we found transfer functions for multiple-loop and multiple-node electrical networks, as well as operational amplifier circuits.
- In the next section we begin our work with mechanical systems. We will see that **many of the concepts applied to electrical networks can also be applied to mechanical systems**.
- This revelation will give you the confidence to move beyond this textbook and study systems not covered here, such as hydraulic or pneumatic systems.

Modeling in the Frequency Domain

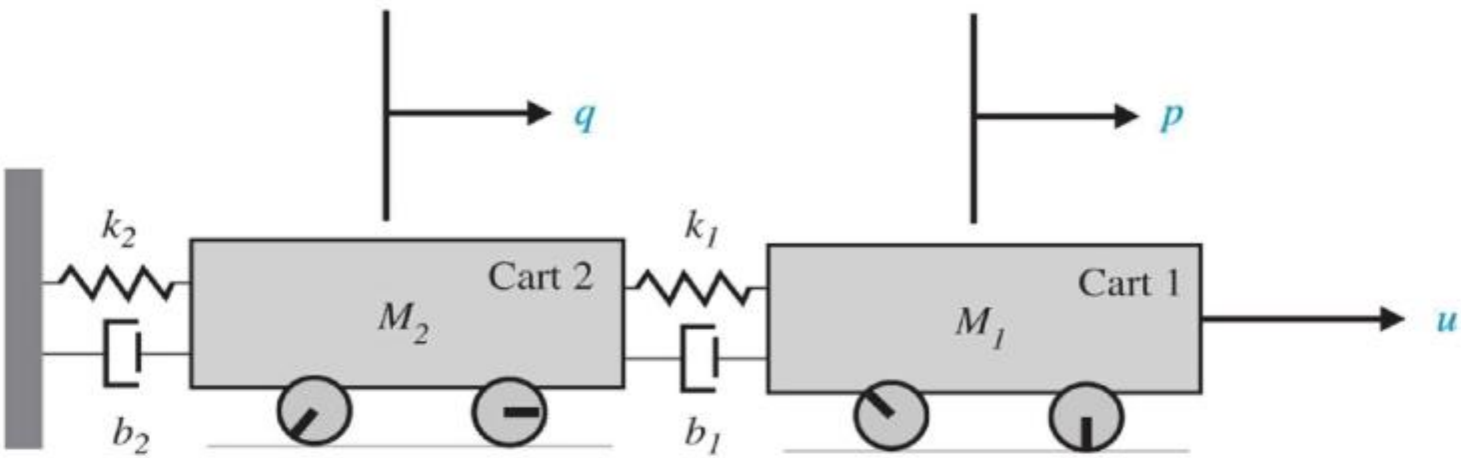
Mechanical System Transfer Functions

- Since there are two types of the movement in mechanical system, this topic is separated in two groups:

  1. Translational Mechanical System Transfer Functions

  2. Rotational Mechanical System Transfer Functions

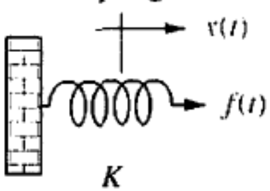
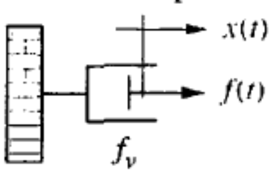
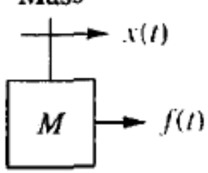
- However, both of them are based on the same concept.



Modeling in the Frequency Domain

Mechanical System Transfer Functions

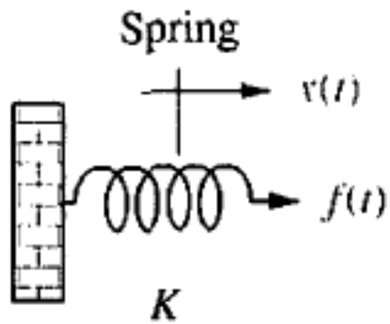
- The mechanical systems are mostly composed of three components, namely: **Mass**, **Spring** and **Damper**.
- The properties or the components are listed in the table 2.4.

Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Modeling in the Frequency Domain

Mechanical System Transfer Functions

- The **spring constant *K*** determines the relationship between the displacement x(t) and the force f(t).

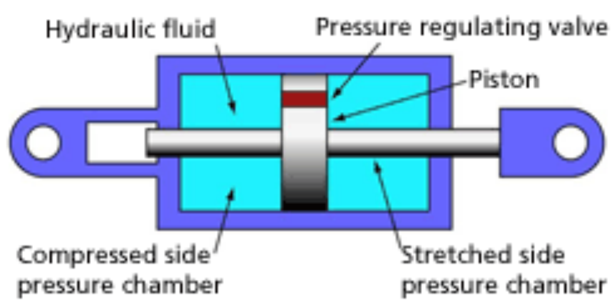
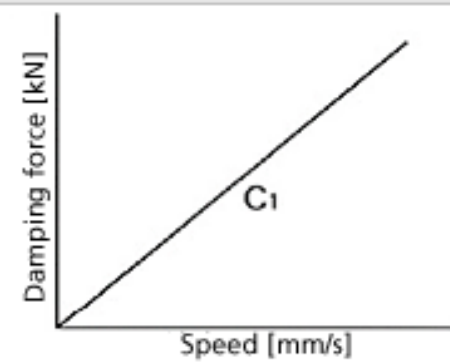


$$f(t) = Kx(t)$$

Modeling in the Frequency Domain

Mechanical System Transfer Functions

- The **damping constant *or*** *coefficient of viscous friction* f_v determines the relationship between the velocity v(t) and the force f(t).

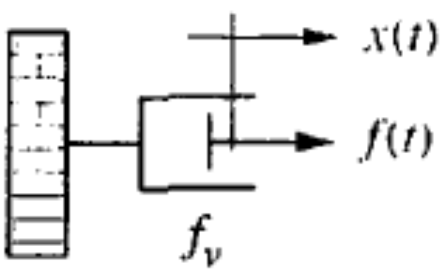
Feature	Damper structure	Damping force-speed characteristics
An oil damper that is equipped with a sole pressure regulating valve and delivers damping force as represented by the F = CV straight line.		

Modeling in the Frequency Domain

Mechanical System Transfer Functions

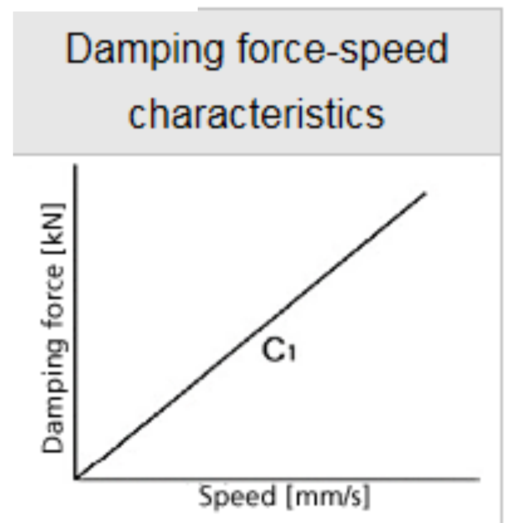
- The **damping constant *or*** *coefficient of viscous friction* f_v determines the relationship between the velocity v(t) and the force f(t).

Viscous damper



$$f(t) = f_v v(t)$$

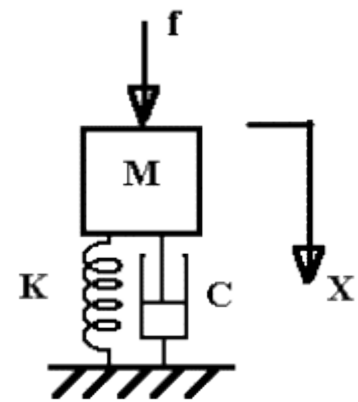
$$f(t) = f_v \frac{dx(t)}{dt}$$

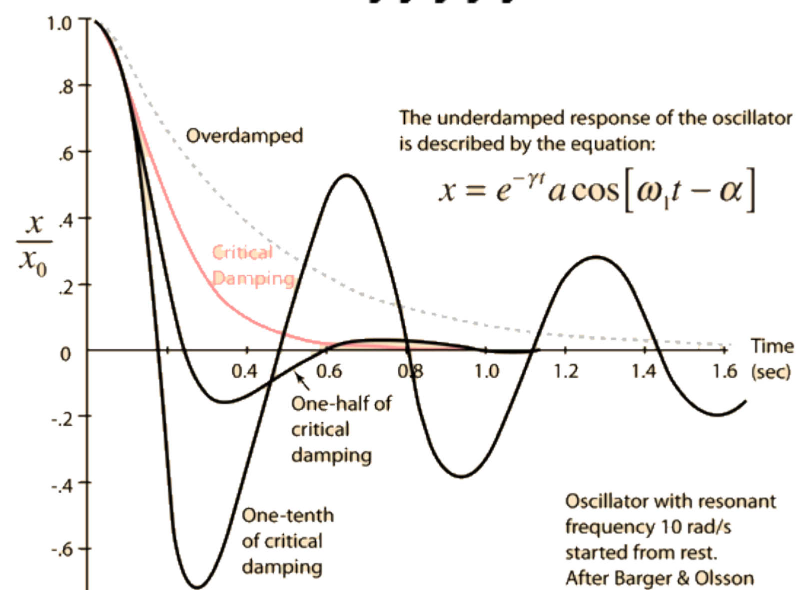


Modeling in the Frequency Domain

Mechanical System Transfer Functions

- The damper are always used along with the spring, to provide good oscillation !

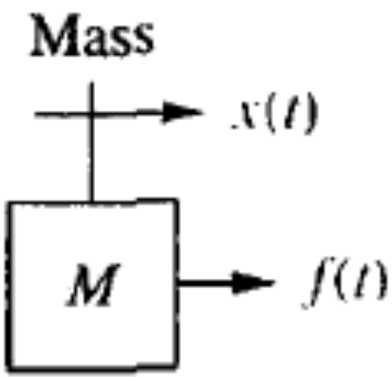




Modeling in the Frequency Domain

Mechanical System Transfer Functions

- The relationship between the displacement x(t) and the force f(t) are determined according to Newton's 2nd Law.



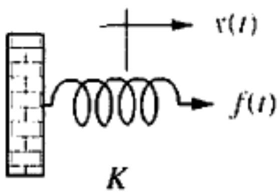
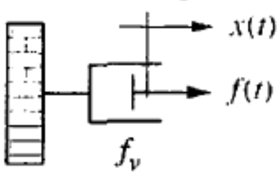
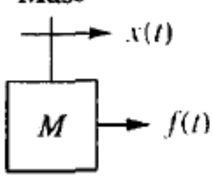
$$f(t) = M\frac{dv(t)}{dt}$$

$$f(t) = M\frac{d^2x(t)}{dt^2}$$

Modeling in the Frequency Domain

Mechanical System Transfer Functions

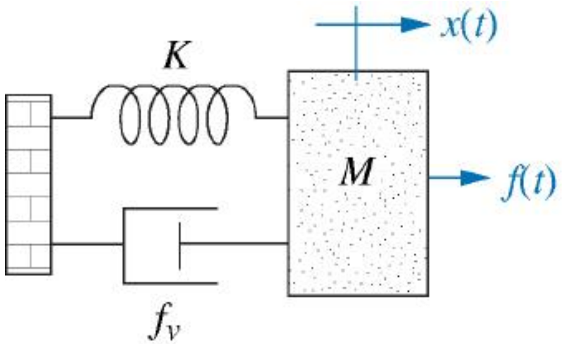
- The force-displacement relationships for springs, viscous dampers, and mass are listed in **the table 2.4**.

Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K \int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \dfrac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M \dfrac{dv(t)}{dt}$	$f(t) = M \dfrac{d^2x(t)}{dt^2}$	Ms^2

Modeling in the Frequency Domain

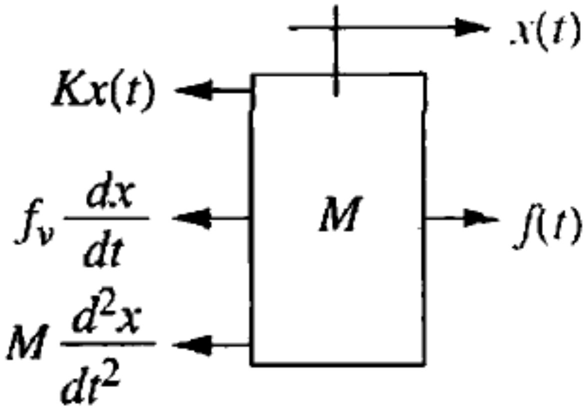
Transfer Function—One Equation of Motion

- **Example 2.11:** Find the transfer function, X(s)/F(s), for the system in the figure. (Ignore the force due to the gravity)



<u>Solution</u>*:*

1. Draw the free-body diagram
2. Place on the mass all forces felt by the mass
3. Write the differential equation of motion using Newton's law to sum to zero all of the forces shown on the mass



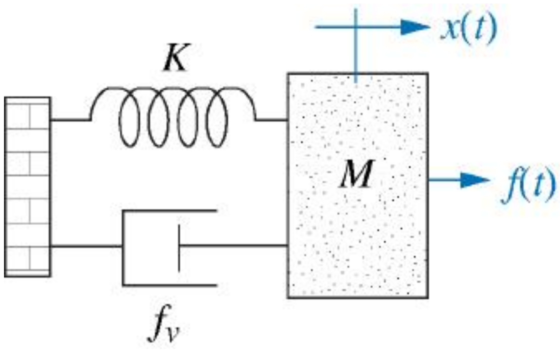
$$\sum f(t) = m\frac{d^2x(t)}{dt^2}$$

$$M\frac{d^2x(t)}{dt^2} + f_v\frac{dx(t)}{dt} + Kx(t) = f(t)$$

Modeling in the Frequency Domain

Transfer Function—One Equation of Motion

- **Example 2.11:** Find the transfer function, X(s)/F(s), for the system in the figure. (Ignore the force due to the gravity)



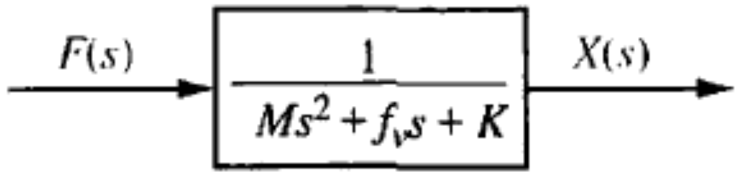
Solution: (cont.)

4. Taking the Laplace transform, assuming zero initial conditions

5. Solve the equation for the transfer function yields

$$\boxed{M\frac{d^2x(t)}{dt^2} + f_v\frac{dx(t)}{dt} + Kx(t) = f(t)} \Rightarrow Ms^2X(s) + f_v sX(s) + KX(s) = F(s)$$

$$(Ms^2 + f_v s + K)X(s) = F(s)$$



$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

Modeling in the Frequency Domain

Concept of *Mechanical Impedance*

Now can we parallel our work with electrical networks by circumventing the writing of differential equations and by defining impedances for mechanical components? If so, we can apply to mechanical systems the problem-solving techniques learned in the previous section. Taking the Laplace transform of the force-displacement column in Table 2.4, we obtain for the spring,

$$\boxed{F(s) = KX(s)} \tag{2.112}$$

for the viscous damper,

$$\boxed{F(s) = f_v sX(s)} \tag{2.113}$$

and for the mass,

$$\boxed{F(s) = Ms^2X(s)} \tag{2.114}$$

If we define impedance for mechanical components as

$$\boxed{Z_M(s) = \frac{F(s)}{X(s)}} \tag{2.115}$$

Modeling in the Frequency Domain

Concept of Mechanical Impedance

If we define impedance for mechanical components as

$$\boxed{Z_M(s) = \frac{F(s)}{X(s)}} \tag{2.115}$$

and apply the definition to Eqs. (2.112) through (2.114), we arrive at the impedances of each component as summarized in Table 2.4 (*Raven, 1995*).[7]

Replacing each force in Figure 2.16(a) by its Laplace transform, which is in the format

$$\boxed{F(s) = Z_M(s)X(s)} \tag{2.116}$$

we obtain Figure 2.16(b), from which we could have obtained Eq. (2.109) immediately without writing the differential equation. From now on we use this approach.

Note:

Finally, notice that Eq. (2.110) is of the form

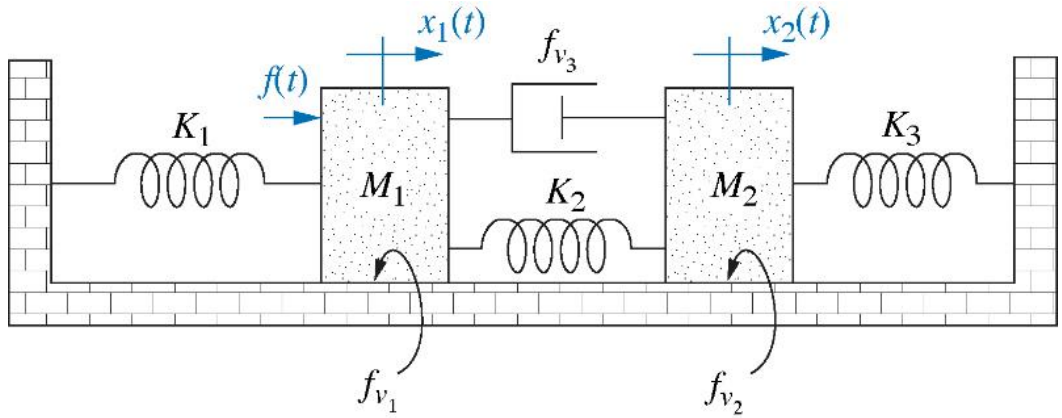
$$[\text{Sum of impedances}]X(s) = [\text{Sum of applied forces}] \tag{2.117}$$

which is similar, but not analogous, to a mesh equation (see footnote 7).

Modeling in the Frequency Domain

Transfer Function—Two Degrees of Freedom

Example 2.17: Find the transfer function, $X_2(s)/F(s)$, for the system.



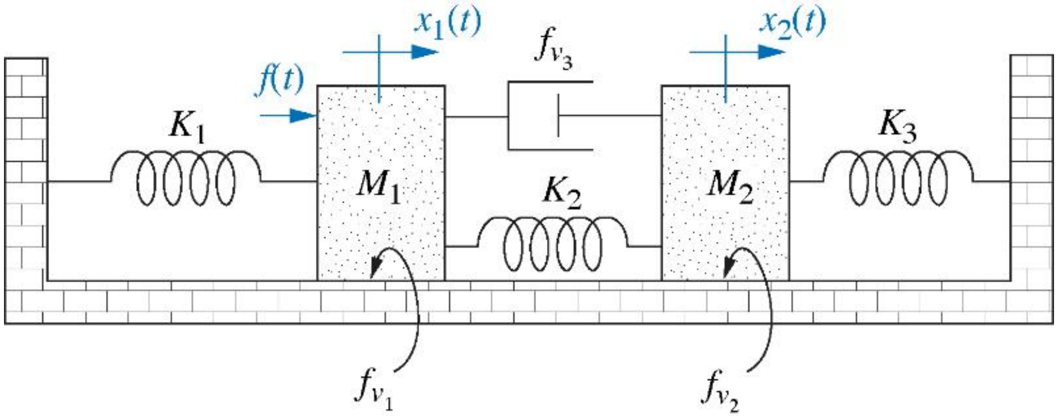
Solution:

- The system has two degrees of freedom, since each mass can be moved in the horizontal direction while the other is held still.
- Thus, two simultaneous equations of motion will be required to describe the system.

Modeling in the Frequency Domain

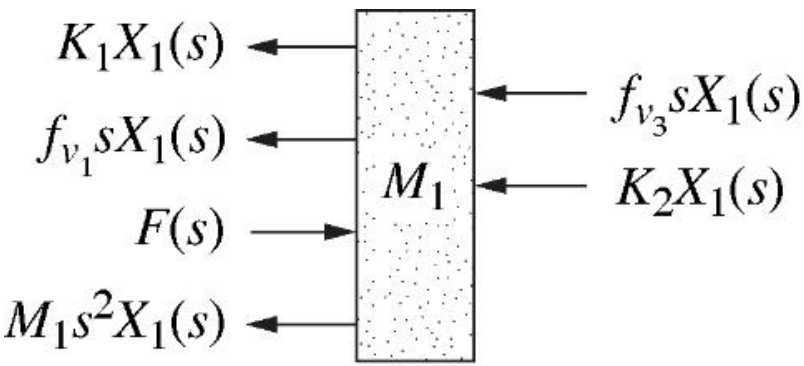
Transfer Function—Two Degrees of Freedom

Example 2.17: Find the transfer function, $X_2(s)/F(s)$, for the system.

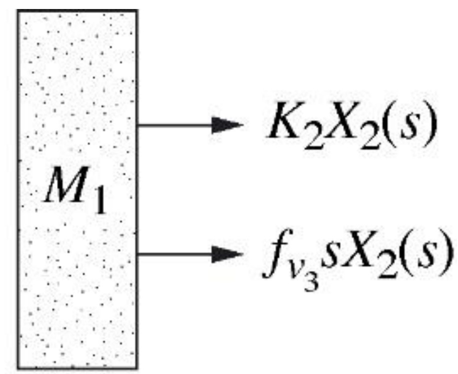


<u>Solution:</u> (For the M_1) If we hold *M_2 still and move M_1 to the right.*

Then hold *M_1 still and move M_2 to the right*



(a)



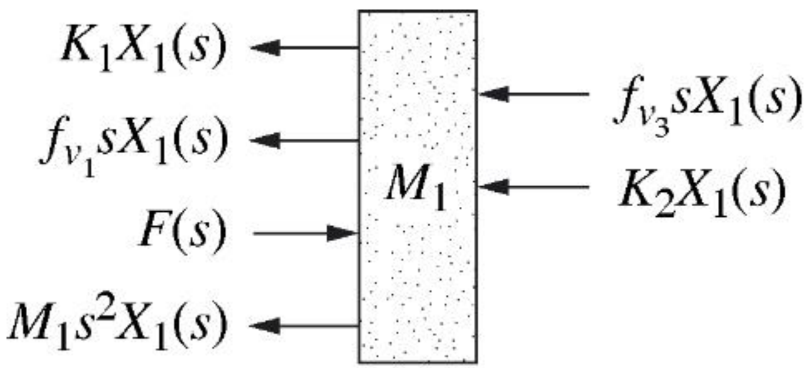
(b)

Modeling in the Frequency Domain

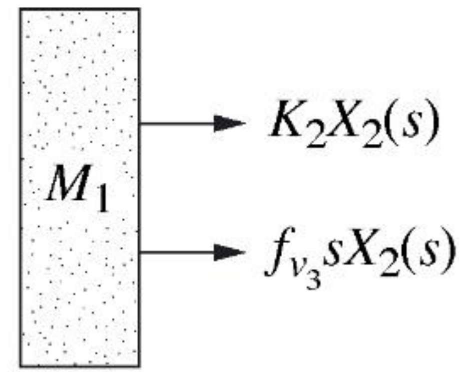
Transfer Function—Two Degrees of Freedom

Example 2.17: Find the transfer function, $X_2(s)/F(s)$, for the system.

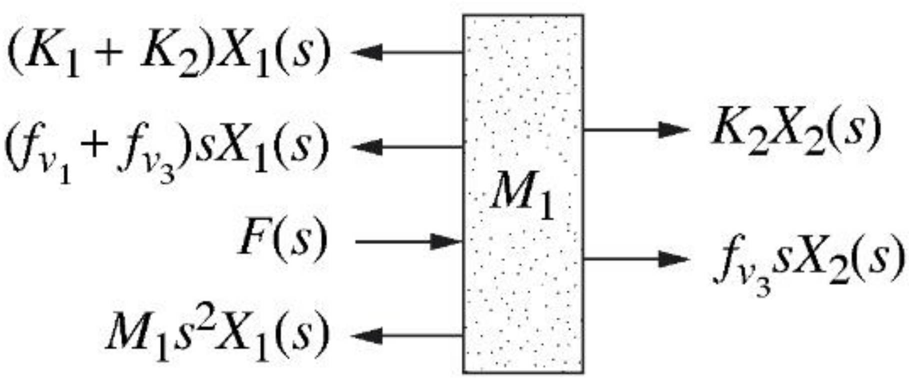
<u>Solution:</u> The total force on M_1 *is the superposition.*



(*a*)



(*b*)

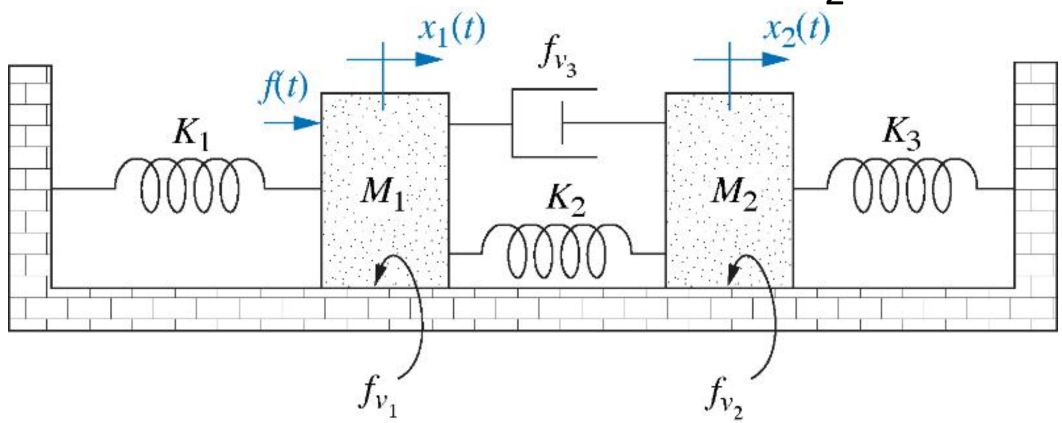


$$\boxed{[M_1s^2 + (f_{v1} + f_{v3})s + (K_1 + K_2)]X_1(s) - (f_{v3}s + K_2)X_2(s) = F(s)}$$

Modeling in the Frequency Domain

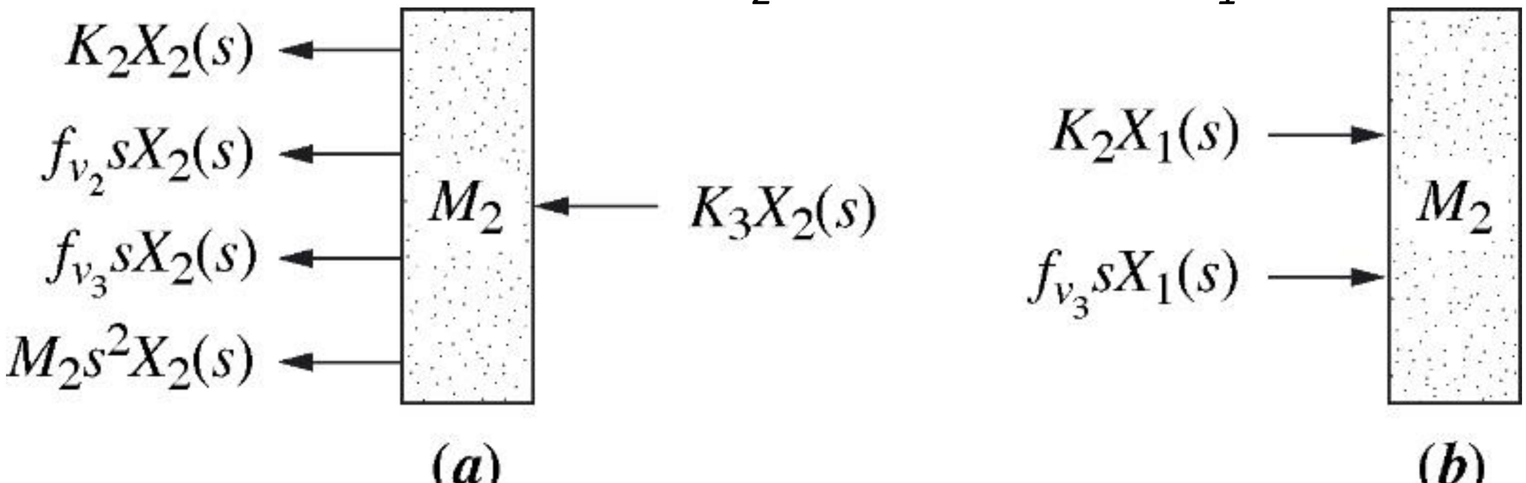
Transfer Function—Two Degrees of Freedom

Example 2.17: Find the transfer function, $X_2(s)/F(s)$, for the system.



Solution: (For the M_2) If we hold *M_1 still and move M_2 to the right.*

Then hold *M_2 still and move M_1 to the right*

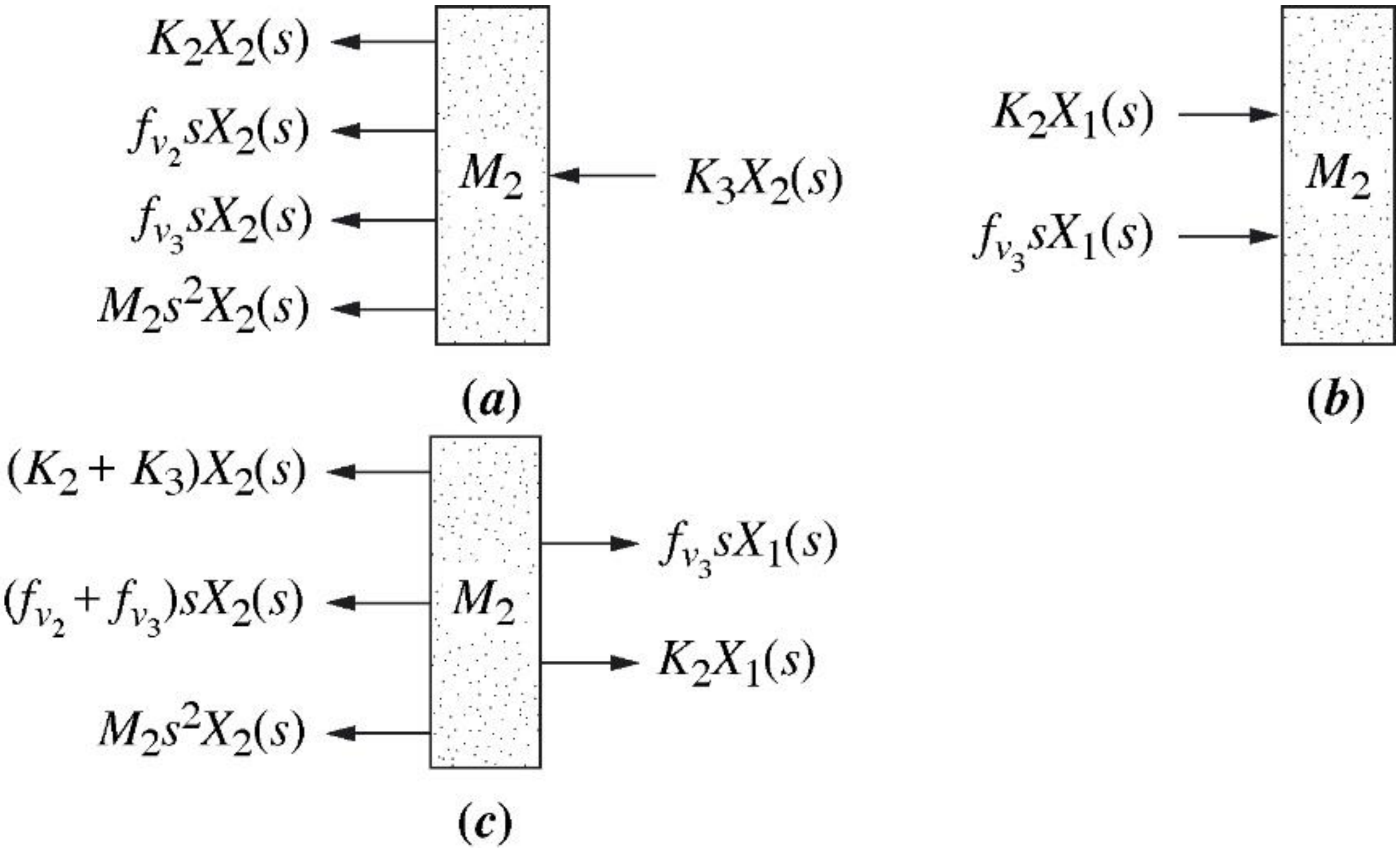


Modeling in the Frequency Domain

Transfer Function—Two Degrees of Freedom

Example 2.17: Find the transfer function, $X_2(s)/F(s)$, for the system.

<u>Solution:</u> The total force on M_2 *is the superposition.*



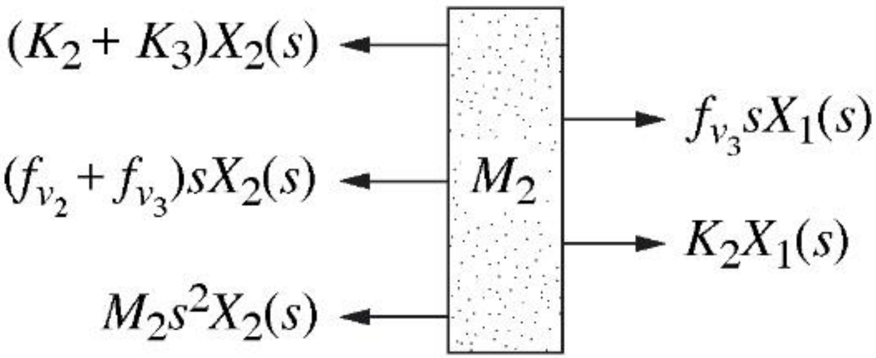
$$\boxed{-(f_{v_3}s + K_2)X_1(s) + \left[M_2s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)\right]X_2(s) = 0}$$

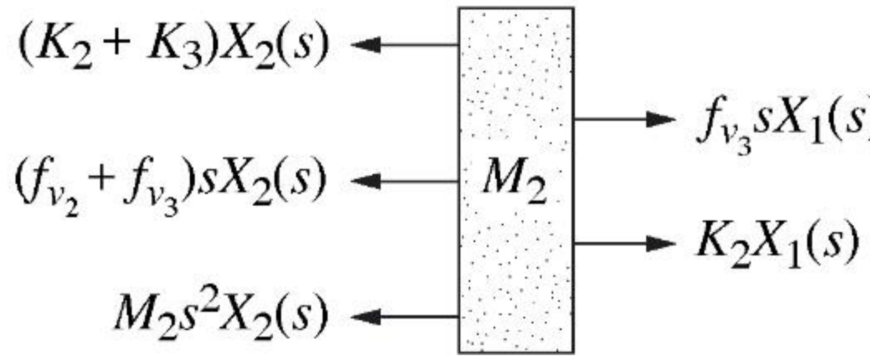
Modeling in the Frequency Domain

Transfer Function—Two Degrees of Freedom

Example 2.17: Find the transfer function, $X_2(s)/F(s)$, for the system.

<u>Solution:</u> The total force on M_2 *is the superposition.*

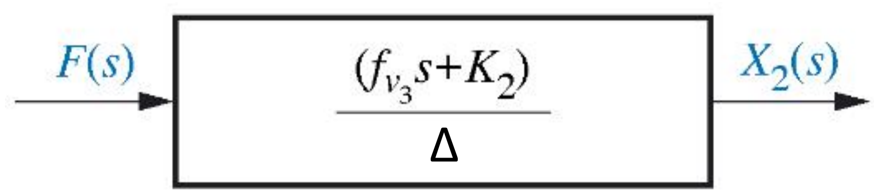




$$\boxed{[M_1s^2+(f_{v1}+f_{v3})s+(K_1+K_2)]X_1(s)-(f_{v3}s+K_2)X_2(s)=F(s)}$$

$$\boxed{-(f_{v_3}s+K_2)X_1(s)+\left[M_2s^2+(f_{v_2}+f_{v_3})s+(K_2+K_3)\right]X_2(s)=0}$$

$$\frac{X_2(s)}{F(s)}=G(s)=\frac{(f_{v_3}s+K_2)}{\Delta}$$

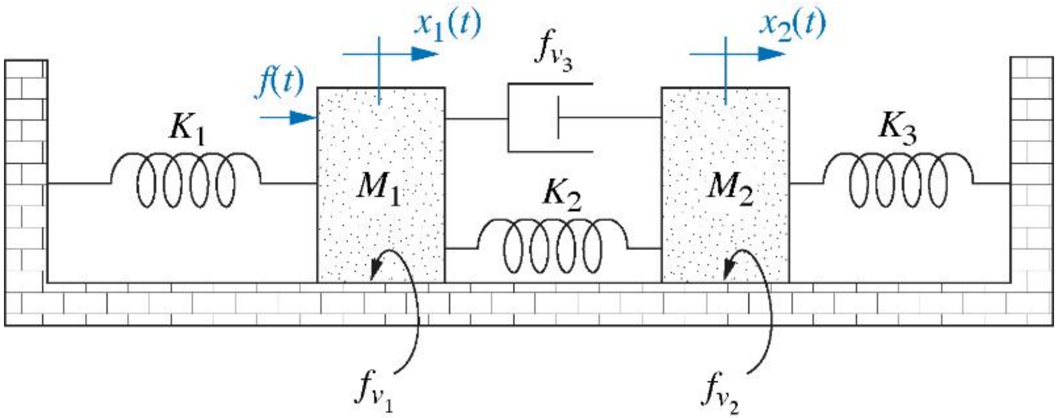


Modeling in the Frequency Domain

Transfer Function—Two Degrees of Freedom

Example 2.17: Find the transfer function, $X_2(s)/F(s)$, for the system.

<u>Solution:</u> The total force on *Mass is the superposition.*



$$\boxed{[M_1 s^2 + (f_{v1} + f_{v3})s + (K_1 + K_2)]X_1(s) - (f_{v3}s + K_2)X_2(s) = F(s)}$$

<u>Note</u>: The form of the equations is similar to electrical mesh equations:

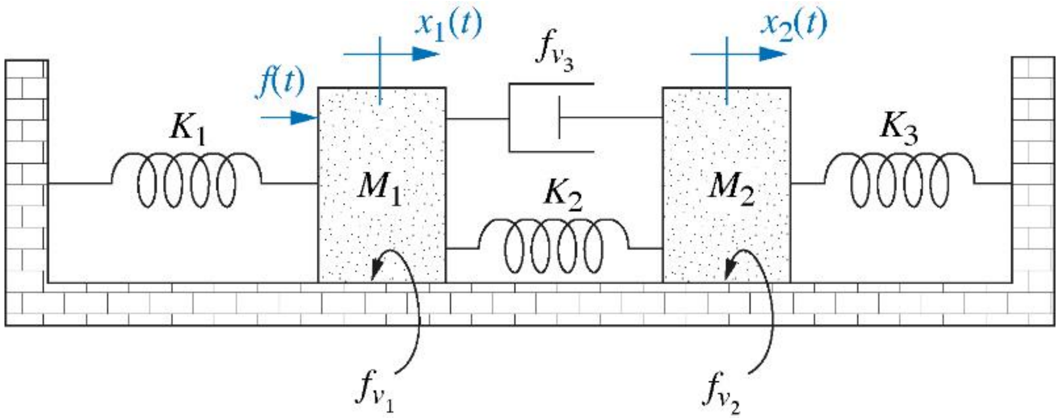
$$\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_1 \end{bmatrix} X_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_2 \end{bmatrix} X_2(s) = \begin{bmatrix} \text{Sum of} \\ \text{applied forces} \\ \text{at } x_1 \end{bmatrix}$$

Modeling in the Frequency Domain

Transfer Function—Two Degrees of Freedom

Example 2.17: Find the transfer function, $X_2(s)/F(s)$, for the system.

Solution: The total force on *Mass is the superposition.*



$$\boxed{-(f_{v_3}s + K_2)X_1(s) + \left[M_2s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)\right]X_2(s) = 0}$$

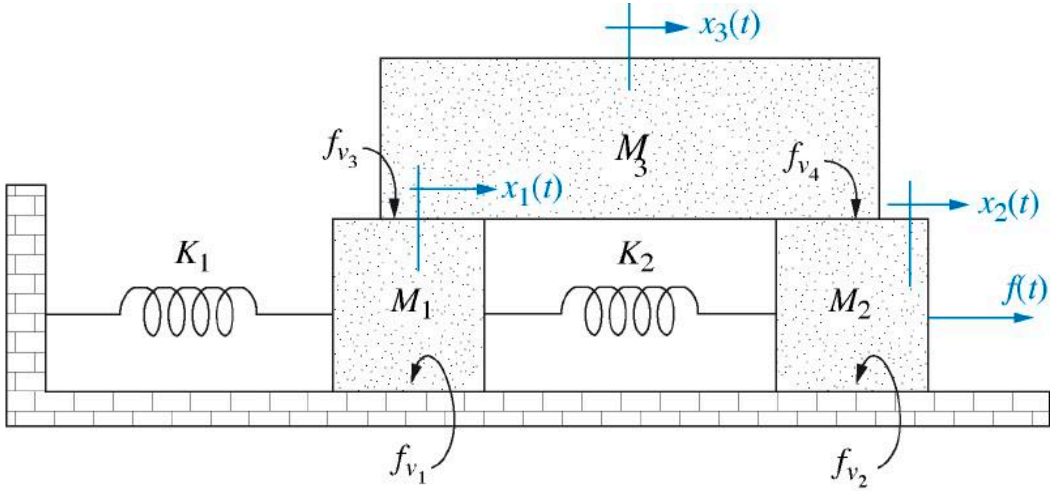
Note: The form of the equations is similar to electrical mesh equations:

$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_1 \text{ and } x_2\end{bmatrix} X_1(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } x_2\end{bmatrix} X_2(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied forces}\\ \text{at } x_2\end{bmatrix}$$

Modeling in the Frequency Domain

Example 2.18 :*Equations of Motion by Inspection*

Write the equations of motion for the mechanical network.



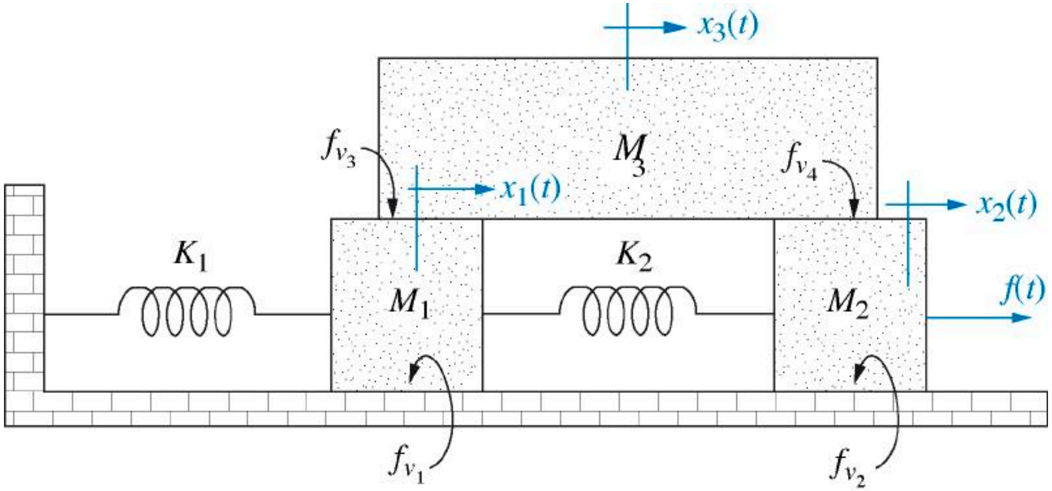
$$\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_1 \end{bmatrix} X_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_2 \end{bmatrix} X_2(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_3 \end{bmatrix} X_3(s) = \begin{bmatrix} \text{Sum of} \\ \text{applied forces} \\ \text{at } x_1 \end{bmatrix}$$

$$[M_1 s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)]X_1(s) - K_2 X_2(s) - f_{v_3} s X_3(s) = 0$$

Modeling in the Frequency Domain

Example 2.18 :*Equations of Motion by Inspection*

Write the equations of motion for the mechanical network.



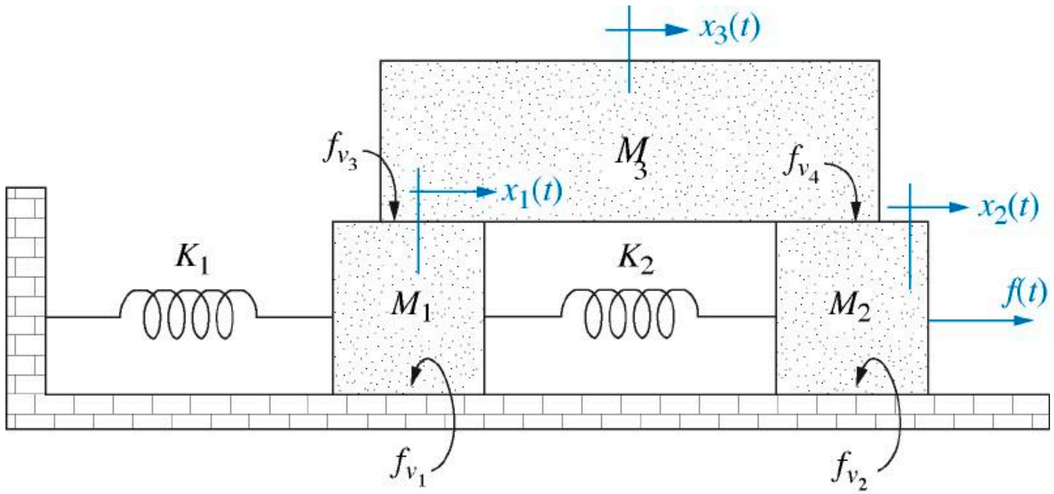
$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_1 \text{ and } x_2\end{bmatrix} X_1(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } x_2\end{bmatrix} X_2(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_2 \text{ and } x_3\end{bmatrix} X_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied forces}\\ \text{at } x_2\end{bmatrix}$$

$$-K_2X_1(s) + [M_2s^2 + (f_{v_2} + f_{v_4})s + K_2]X_2(s) - f_{v_4}sX_3(s) = F(s)$$

Modeling in the Frequency Domain

Example 2.18 :*Equations of Motion by Inspection*

Write the equations of motion for the mechanical network.



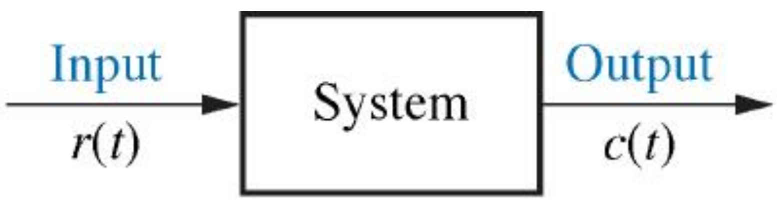
$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_1 \text{ and } x_3\end{bmatrix} X_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_2 \text{ and } x_3\end{bmatrix} X_2(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } x_3\end{bmatrix} X_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied forces}\\ \text{at } x_3\end{bmatrix}$$

$$-f_{v_3}sX_1(s) - f_{v_4}sX_2(s) + \left[M_3s^2 + (f_{v_3} + f_{v_4})s\right]X_3(s) = 0$$

Modeling in the Frequency Domain

2.10 Nonlinearities

- The systems can be roughly classified in 2 groups namely: **Linear** and **Nonlinear.** If the system does not belong to the first group, it will be a nonlinear system.
- So far we have studied only about the linear time invariant system (LTI), and we loosely called it linear system.



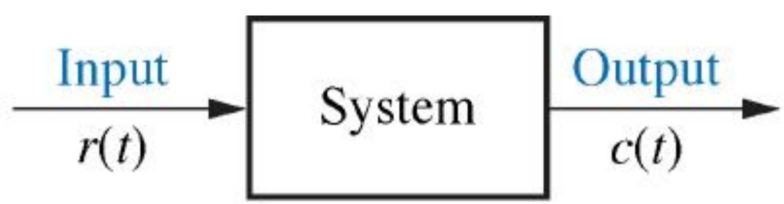
$$\boxed{\frac{C(s)}{R(s)} = G(s)}$$

Note: A *linear time invariant system* can be modeled by a transfer function.

Modeling in the Frequency Domain

2.10 Nonlinearities

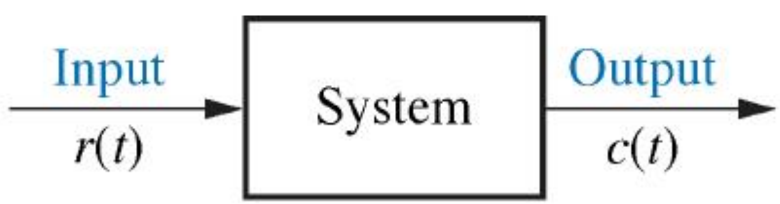
- *A linear system possesses two properties:* ***superposition*** *and* ***homogeneity***.
- *Superposition means that the output response of a system to the sum of* inputs is the sum of the responses to the individual inputs. Thus, if an input of $r_1(t)$ yields an output of $c_1(t)$ *and an input of* $r_2(t)$ *yields an output of* $c_2(t)$, *then an input of* $r_1(t) + r_2(t)$ *yields an output of* $c_1(t) + c_2(t)$.



Modeling in the Frequency Domain

2.10 Nonlinearities

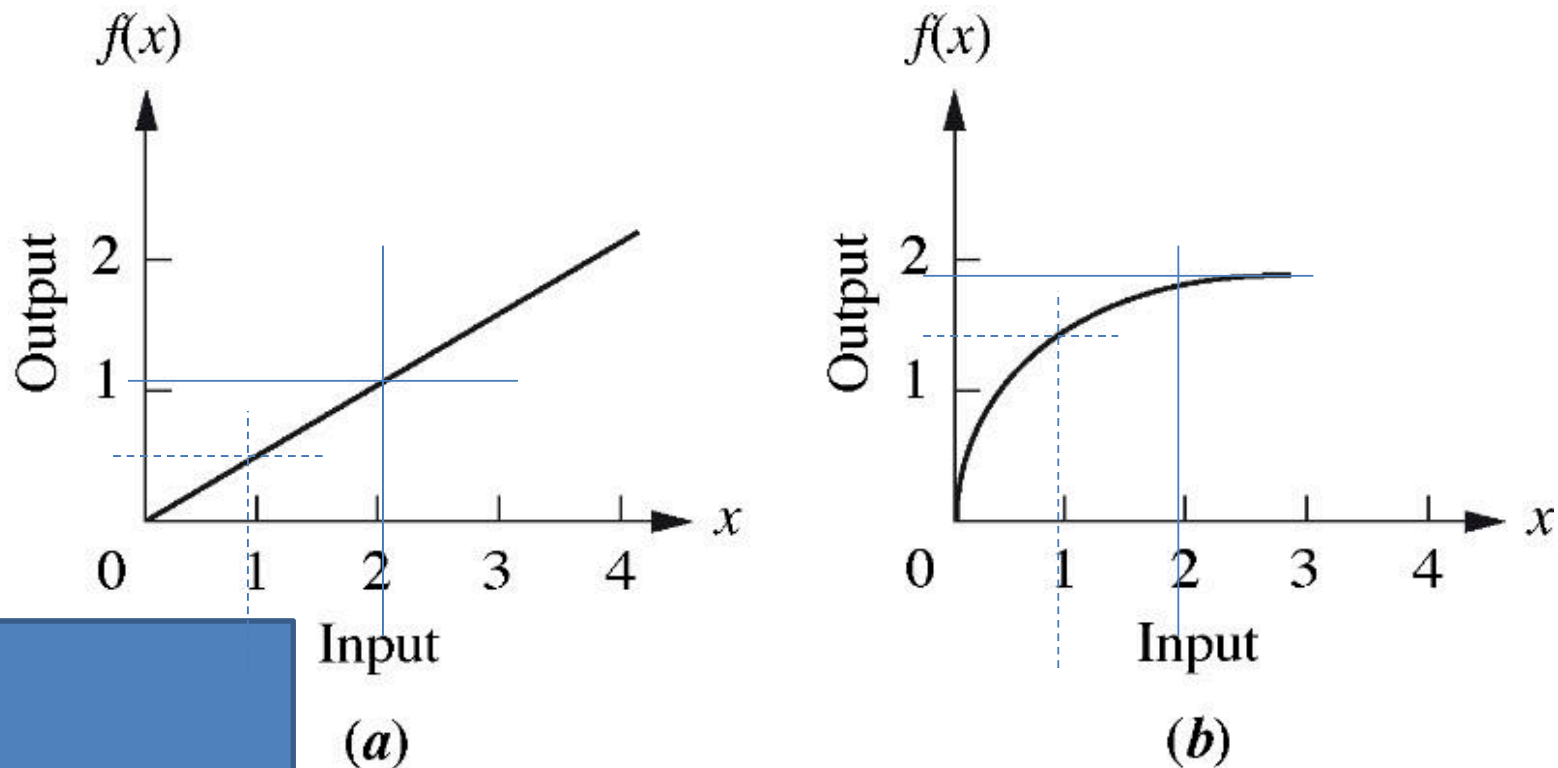
- *A* <u>*linear system possesses*</u> *two properties:* ***superposition*** *and* ***homogeneity****.*
- ***Homogeneity*** *describes* the response of the system to a multiplication of the input by a scalar.
- A system is said to be a homogeneity system, if for an input of $r_1(t)$ that yields an output of $c_1(t)$, *and input of* $Ar_1(t)$ *yields an output of* $Ac_1(t)$.
- *That means,* multiplication of an input by a scalar yields a response that is multiplied by the same scalar.



Modeling in the Frequency Domain

2.10 Nonlinearities

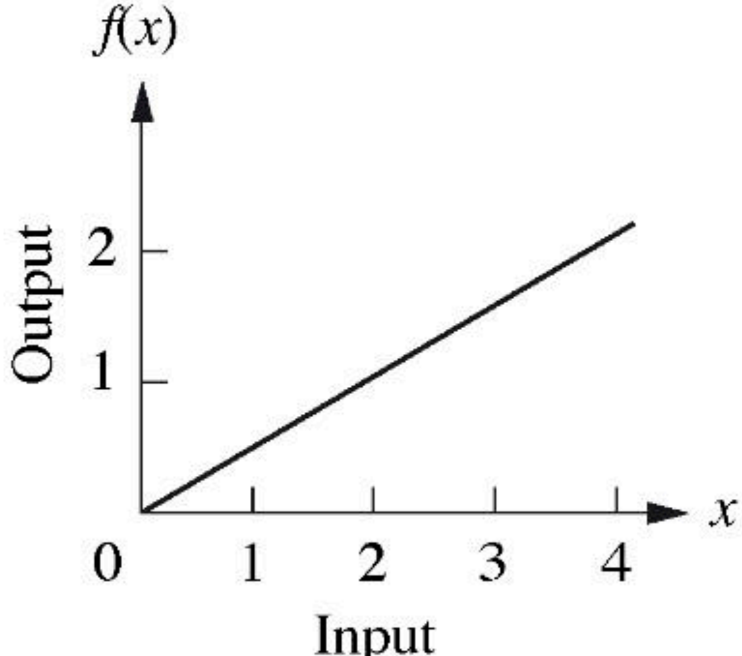
- *A linear system possesses two properties:* ***superposition*** *and* ***homogeneity***.



Modeling in the Frequency Domain

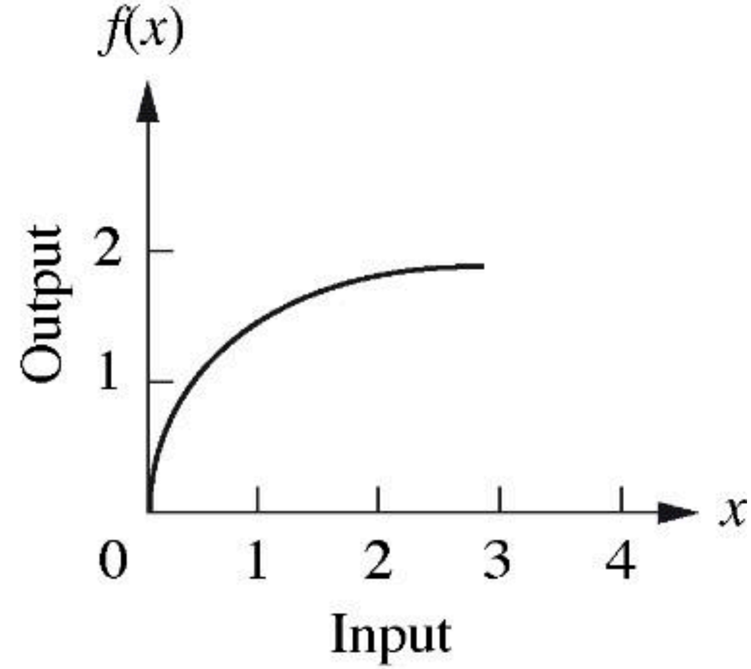
2.10 Nonlinearities

- *A linear system possesses two properties:* ***superposition*** *and* ***homogeneity****.*



$f(x) = 0.5\,x$

(*a*)



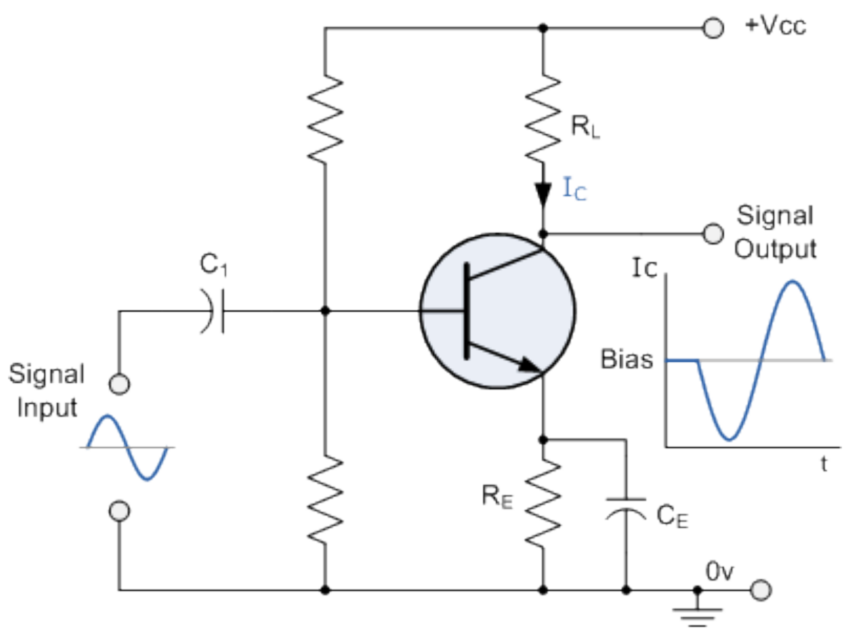
(*b*)

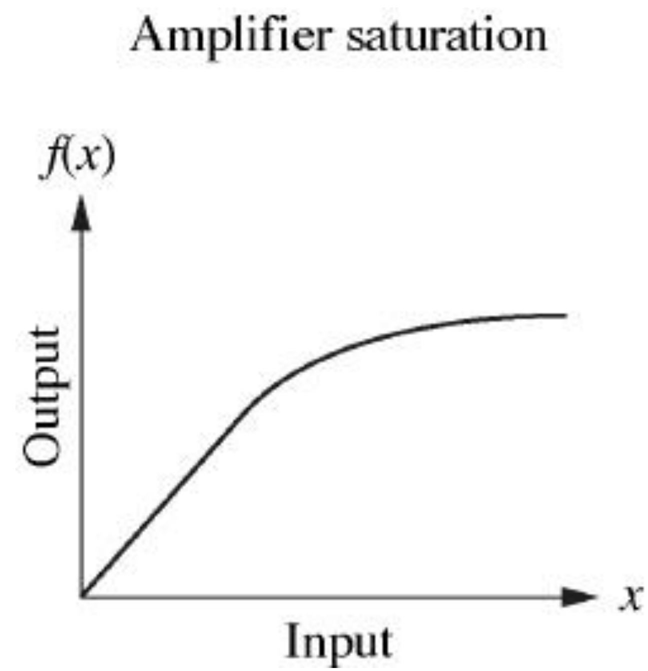
Modeling in the Frequency Domain

2.10 Nonlinearities

<u>Examples of physical nonlinearities</u>:

- An electronic amplifier is linear over a specific range but exhibits the nonlinearity called <u>*saturation*</u> at high input voltages.





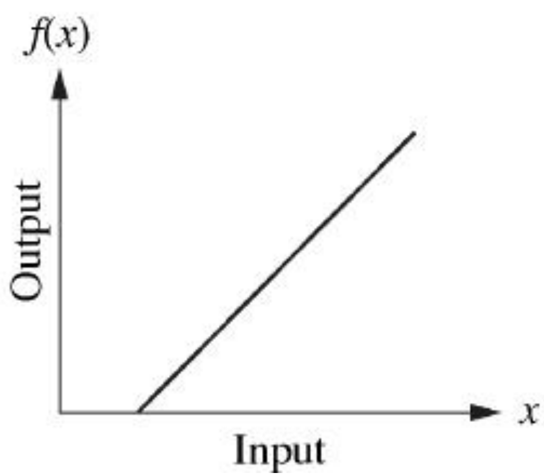
Modeling in the Frequency Domain

2.10 Nonlinearities

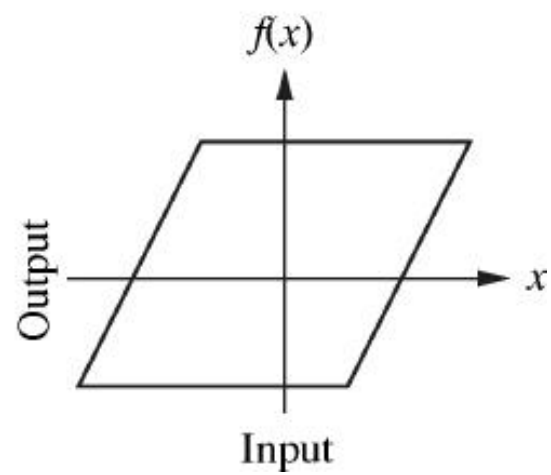
Examples of physical nonlinearities:

- A motor that does not respond at very low input voltages due to frictional forces exhibits a nonlinearity called *dead zone.*
- *Gears that do not fit* tightly exhibit a nonlinearity called *backlash.*

Motor dead zone



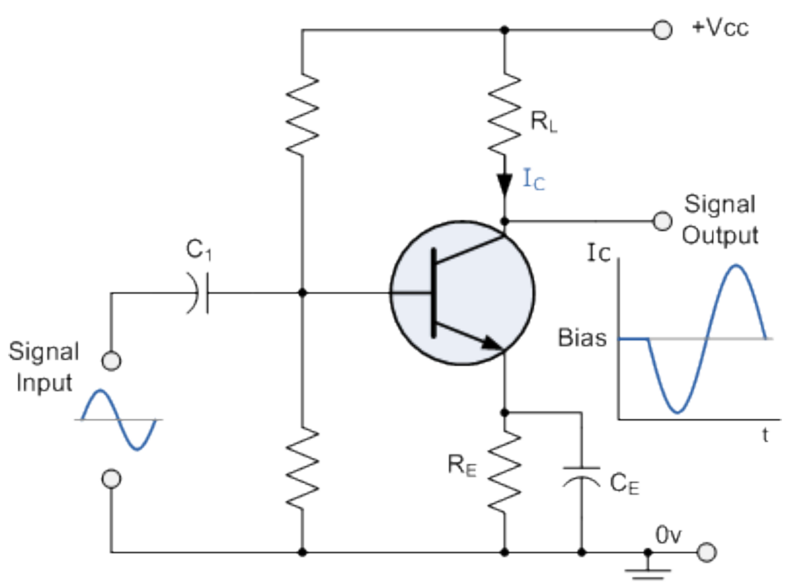
Backlash in gears

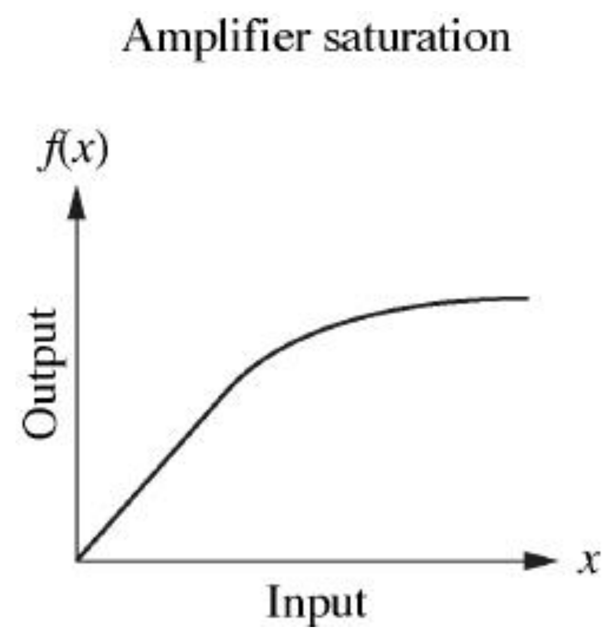


Modeling in the Frequency Domain

2.10 Nonlinearities

- Actually most of the physical systems are nonlinear!
- But a designer can often make a linear approximation to a nonlinear system. Linear approximations simplify the analysis and design of a system and are used as long as the results yield a good approximation to reality.





Modeling in the Frequency Domain

2.10 Nonlinearities

- For example, a linear relationship can be established at a point on the nonlinear curve if the range of input values about that point is small and the origin is translated to that point. Electronic amplifiers are an example of physical devices that perform linear amplification with small excursions about a point.