AST3220, spring 2019: Project 2

In addition to the solution of the analytical part, the deliverables are the following:

- The source code in a separate file (so we can run it.)
- All graphs and other output asked for in the following. Note that the plots must be included in your report. It is not enough that your code generates them.
- An explanation of how you thought when constructing your code, including your choice of numerical methods.

The maximum score on the project is 100 points.

In the lectures we looked at the relationship between the neutrino temperature T_ν and the photon temperature T after the neutrinos decoupled. Among the assumptions we made, was that the electrons and positrons become non-relativistic as soon as the temperature drops below $k_B T = m_e c^2$, m_e is the electron (and positron) mass. This is not strictly correct, and you will in the following look at the relation between T_ν and T when the electrons and positrons are treated more accurately. The following expressions for the energy density and pressure of a gas of fermions will be useful:

$$\begin{aligned}
\rho c^2(T) &= \int_0^\infty E(p) n(p,T) dp \\
P(T) &= \int_0^\infty \frac{(pc)^2}{3E(p)} n(p,T) dp \\
n(p,T) &= \frac{4\pi g p^2}{(2\pi\hbar)^3} \frac{1}{\exp[E(p)/k_B T] + 1} \\
E(p) &= \sqrt{p^2 c^2 + m^2 c^4},
\end{aligned}$$

where g is the number of internal degrees of freedom and m is the rest mass of the particle in question. In the lectures we showed that the entropy density is given by

$$s(T) = \frac{\rho c^2 + P}{T},$$

and that entropy conservation implies $a^3 s(T) = \text{constant}$.

1) (10 points) Show that entropy conservation together with the first Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$$

gives

$$t = \int_T^{T_0} \frac{s'(T)dT}{s(T)\sqrt{24\pi G\rho(T)}},$$

where $s'(T) = \frac{ds}{dT}$ and I have defined $t(T_0) = 0$.

2) (10 points) Show that the total entropy density for photons, electrons and positrons can be written as

$$\begin{aligned} s(T) &= \frac{4\pi^2}{45}k_B\left(\frac{k_BT}{\hbar c}\right)^3 \\ &+ \frac{1}{T}\int_0^\infty \frac{16\pi p^2dp}{(2\pi\hbar)^3}\frac{1}{\exp[E_e(p)/k_BT]+1}\left[E_e(p) + \frac{(pc)^2}{3E_e(p)}\right] \\ &= \frac{4\pi^2}{45}k_B\left(\frac{k_BT}{\hbar c}\right)^3 \mathcal{S}\left(x = \frac{m_ec^2}{k_BT}\right), \end{aligned}$$

where $E_e = \sqrt{p^2c^2 + m_e^2c^4}$ and

$$\mathcal{S}(x) = 1 + \frac{45}{2\pi^4}\int_0^\infty y^2\left(\sqrt{y^2+x^2} + \frac{y^2}{3\sqrt{y^2+x^2}}\right)\frac{1}{\exp(\sqrt{y^2+x^2})+1}dy.$$

(Note that $\mathcal{S}(x)$ here has nothing to do with the function we called $\mathcal{S}$ when we discussed distances.)

3) (10 points) I showed in the lectures that after decoupling the neutrinos still follow the Fermi-Dirac distribution with $T_\nu \propto 1/a$. Show that entropy conservation implies

$$a \propto \frac{1}{T\mathcal{S}^{1/3}(x = m_ec^2/k_BT)},$$

and that we therefore have

$$T_\nu = AT\mathcal{S}^{1/3}(x = m_ec^2/k_BT),$$

where A is a constant.

4) (10 points) What is the relation between T and T_ν at very high temperatures ($T \to \infty$)? Use this to find A and show that

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T \mathcal{S}^{1/3} \left(x = \frac{m_e c^2}{k_B T}\right).$$

The following integral may be useful:

$$\int_0^\infty \frac{y^3 dy}{\exp(y) + 1} = \frac{7\pi^4}{120}.$$

5) (10 points) Show that the total energy density can be written as

$$\begin{aligned} \rho c^2 &= \frac{\pi^2}{15} \frac{(k_B T)^4}{(\hbar c)^3} + \frac{7}{8} \frac{\pi^2}{30} 6 \frac{(k_B T_\nu)^4}{(\hbar c)^3} \\ &+ \int_0^\infty \frac{16\pi p^2 dp}{(2\pi\hbar)^3} \frac{\sqrt{p^2c^2 + m_e^2 c^4}}{\exp\left(\sqrt{p^2c^2 + m_e^2c^4}/k_B T\right) + 1} \\ &= \frac{\pi^2}{15} \frac{(k_B T)^4}{(\hbar c)^3} \mathcal{E}(x = m_e c^2 / k_B T), \end{aligned}$$

where

$$\mathcal{E}(x) = 1 + \frac{21}{8}\left(\frac{4}{11}\right)^{4/3} \mathcal{S}^{4/3}(x) + \frac{30}{\pi^4} \int_0^\infty \frac{y^2 \sqrt{y^2 + x^2} dy}{\exp\left(\sqrt{y^2 + x^2}\right) + 1}.$$

6) (10 points) Show that

$$t = \sqrt{\frac{15\hbar^3}{24\pi^3 G m_e^4 c^3}} \int_{m_e c^2/k_B T_0}^{m_e c^2/k_B T} \left(3 - \frac{x \mathcal{S}'(x)}{\mathcal{S}(x)}\right) \mathcal{E}^{-1/2}(x) x dx$$

7) (10 points) Write a code to evaluate $\mathcal{S}(x)$ and plot it. Find by physical arguments what the value of $\mathcal{S}$ in the limits $x = 0$ and $x \gg 1$. Check that your code agrees with these limits.

8) (30 points) Take $T_0 = 10^{11}$ K. Write a code to evaluate $t(T)$ and $T_\nu(T)$, and complete the table on the next page.

T (K)	T_ν/T	t (s)
10^{11}	1.000	0
$6 \cdot 10^{10}$		
$2 \cdot 10^{10}$		
10^{10}		
$6 \cdot 10^{9}$		
$3 \cdot 10^{9}$		
$2 \cdot 10^{9}$		
10^{9}		
$3 \cdot 10^{8}$		
10^{8}		
10^{7}		
10^{6}		