Exploring extra dimensions with scalar fields

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This paper provides a pedagogical introduction to the physics of extra dimensions by examining the behavior of scalar fields in three landmark models: the ADD, Randall-Sundrum, and DGP spacetimes. Results of this analysis provide qualitative insights into the corresponding behavior of gravitational fields and elementary particles in each of these models. In these "brane world" models, the familiar four dimensional spacetime of everyday experience is called the brane and is a slice through a higher dimensional spacetime called the bulk. The particles and fields of the standard model are assumed to be confined to the brane, while gravitational fields are assumed to propagate in the bulk. For all three spacetimes, we calculate the spectrum of propagating scalar wave modes and the scalar field produced by a static point source located on the brane. For the ADD and Randall-Sundrum models, at large distances, the field looks like that of a point source in four spacetime dimensions, but at short distances, it crosses over to a form appropriate to the higher dimensional spacetime. For the DGP model, the field has the higher dimensional form at long distances rather than short. The behavior of these scalar fields, derived using only undergraduate level mathematics, closely mirror the results that one would obtain by performing the far more difficult task of analyzing the behavior of gravitational fields in these spacetimes. © *2018 American Association of Physics Teachers.*
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I. INTRODUCTION

On the basis of everyday experience and on all scales that have been probed experimentally, it seems the space we inhabit is three dimensional. Nonetheless, it remains a possibility that additional spatial dimensions exist. A fourth spatial dimension was invoked by Kaluza and Klein in the 1920s in order to develop a unified theory of electromagnetism and gravity. Klein[1] introduced the hypothesis that the extra dimension was rolled up, with an estimated circumference of order 10^{-32} m, too small to be directly observed by experiments then and now. The idea of extra dimensions was revived in the 1980s by the discovery that superstring theory is only consistent in ten spacetime dimensions.[2] At first string, theorists presumed that the extra dimensions were rolled up with a circumference on the Planck scale ($\ell_P = 1.6 \times 10^{-35}$ m) and hence unobservable. However, several subsequent models challenged this notion.

In 1998, Arkani-Hamed, Dimopoulos, Dvali (ADD), and Antoniadis showed that extra dimensions could be much larger and yet remain hidden.[3,4] Moreover, ADD found that sufficiently large extra dimensions could explain the hierarchy problem, namely, the weakness of gravity compared to other forces of nature, or, equivalently, the lightness of elementary particles compared to the Planck mass scale (which is inversely proportional to the strength of gravity). ADD showed that large extra dimensions allow gravity to be comparable in strength to other forces on short distance scales but to appear much weaker on scales that are large compared to the extra dimension. Randall and Sundrum (RS) then showed that a warped extra dimension provided a different explanation[5] of the hierarchy problem; remarkably since their extra dimension that warped it could be small and yet

explain the hierarchy problem,[5] or infinite and yet remain hidden.[6] Somewhat later, Dvali, Gabadadze, and Porrati (DGP) invented another model with an infinite extra dimension; however, unlike ADD and RS, the DGP model sought to explain the observed accelerated expansion of the Universe.[7] In all of these models, the familiar four dimensional spacetime inhabited by particles and fields of the standard model is a brane: a slice through a higher dimensional universe called the bulk.

Brane worlds with extra dimensions have engendered tremendous public interest. Randall's *Warped Passages*[8] was a bestseller and a version of the Randall Sundrum model was the scientific basis of the five dimensional world depicted in Christopher Nolan's 2014 movie *Interstellar*.[9] Yet there remains a gap between popular accounts like Refs. 8–10, and review articles, such as Refs. 11–14, which require advanced graduate coursework in general relativity, field theory, and standard model physics. The purpose of this article is to provide a pedagogical introduction to this subject at a level accessible to undergraduate physics majors. The target audience for our paper includes physicists in other fields as well as undergraduate and graduate students. No specialized background is required beyond one semester courses in electromagnetism and quantum mechanics. Indeed, the material in this paper could be incorporated into such courses as well as introductory courses on particle physics or general relativity.

We show that a simple scalar field serves as a proxy for gravitational fields and elementary particles in the ADD, RS, and DGP models. For each of the three models, we calculate the solutions to the scalar wave equation in the absence of sources and the scalar field for a point source located on the brane. For the ADD and RS models, we find that on long

distance scales the field of a point source has the form appropriate to three spatial dimensions; the form appropriate for higher dimensions becomes manifest only on short distance scales. For the DGP model, by contrast, the field of a point source has the form appropriate to three dimensions on short distance scales, and the higher dimensional form becomes manifest on long distance scales. The behavior of the field due to a point source reveals some of the most compelling features of these models, in particular, their solutions to the hierarchy and dark energy problems. We show that these features can be understood without invoking the machinery of tensor calculus or differential geometry required to fully analyze the gravitational field. It is not surprising that scalar fields serve as a useful caricature for more complicated fields. For example, in electromagnetism courses, it is common practice to introduce scalar waves as a prelude to studying electromagnetic waves.[15,16] It is also not surprising that the spatial dependence of the field of a point source changes when the number of spatial dimensions changes: Gauss' law for the gravitational or electrostatic field implies that in d spatial dimensions the field must fall off like $1/r^{d-1}$, so in three spatial dimensions, we get the famous $1/r^2$ behavior, while in four spatial dimensions, the fall off goes like $1/r^3$.

We begin in Sec. II with the familiar four dimensional Minkowski spacetime; we solve the scalar wave equation and calculate the field of a static point source therein. These results lay the groundwork and provide insights into the physics of gravitational fields and elementary particles. In Secs. III–V, we then carry out a parallel analysis in the higher dimensional spacetimes of the ADD, RS, and DGP models. Within each section, we identify the same constructs for each model in turn, labeled with section headings. The geometry of each spacetime is described; the necessary differential geometry background is presented in an appendix available online as supplementary material.[17] The hierarchy problem and its resolution within the ADD and RS models are explained, as are the cosmological implications of the

DGP model. In Sec. VI, we conclude by briefly discussing the experimental implications and tests of the three models. It is worth noting here that the simplest versions of the ADD and DGP models discussed in this paper are ruled out by data; however, descendants of these models as well as the RS model itself remain viable. The key results presented in the paper are summarized in Table I. In order to make our article more useful for students and instructors, we provide a primer on differential geometry in Appendix A, on one dimensional quantum mechanics in Appendix B, and a number of problems in Appendix C; the appendices are available as supplementary material.[17]

II. MASSLESS SCALAR FIELD THEORY IN MINKOWSKI SPACETIME

A. The spacetime interval

Minkowski spacetime is the familiar spacetime from special relativity. The interval between two neighboring events, with coordinates (t, x, y, z) and $(t + dt, x + dx, y + dy, z + dz,)$, is given by

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2, \tag{1}$$

where $-\infty < t, x, y, z < \infty$ and where we have chosen a system of units wherein the speed of light $c = 1$.

B. The wave equation

A scalar wave propagating in Minkowski spacetime obeys the familiar wave equation

$$\Box^2 \phi = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) \phi = s(\mathbf{r}, t), \tag{2}$$

where ϕ is the scalar field and s is its source.

Table I. Comparison and summary of all the models studied in this paper. For the ADD model, the space dimensionality $d = 4, 5, 6, \dots$ In the table, for illustrative purposes, we take $d = 4$.

Model	Mode types	Particle interpretation and dispersion relation	Point source profile	Comments
		Source-free solutions		
Massless scalar field $(3+1)$	Plane waves	Massless with $\omega = k$	$\phi \propto 1/r$ [Eq. (4)]	Ordinary spacetime of special relativity.
Massive scalar field $(3+1)$	Plane waves	Massive with $\omega = \sqrt{k^2 + \mu^2}$	$\phi \propto e^{-\mu r}/r$ [Eq. (6)]	Reduces to massless for $\mu \to 0$.
ADD $(d+1)$	Zero mode	Massless with $\omega = k$	$\phi \propto \begin{cases} 1/r & \text{for } r \gg L \\ 1/r^2 & \text{for } r \ll L \end{cases}$	Solves hierarchy problem via $G_5 = GL/\pi$ and $E_{EW} = E_{P5}$ for single extra dimension of size L.
	KK modes: discrete	Massive with $\omega = \sqrt{k^2 + \kappa_n^2}$	[Eqs. (16) and (17)]	
RS $(4+1)$	Zero mode	Massless with $\omega = k$	$\phi \propto \begin{cases} 1/r & \text{for } r \gg \gamma^{-1} \\ 1/r^2 & \text{for } r \ll \gamma^{-1} \end{cases}$	RS1 solves hierarchy problem via $E_{EW} = E_P e^{-\gamma \ell}$ for extra dimension of size ℓ and curvature γ.
	KK modes: discrete (RS1) continuum (RS2)	Massive with $\omega = \sqrt{k^2 + \mu^2}$	[Eqs. (40) and (41)]	
DGP $(4+1)$	No zero modes but a resonance at $\mu = 0$	Width of peak $\sim 1/\ell$	$\phi \propto \begin{cases} 1/r & \text{for } r \ll \ell \\ 1/r^2 & \text{for } r \gg \ell \end{cases}$	Solves dark energy problem for screening length $\ell \sim$ Hubble length.
	KK modes: continuum	Massive with $\omega = \sqrt{k^2 + \mu^2}$	[Eqs. (49) and (50)]	

C. Source-free solution

If $s = 0$, the wave equation (2) has plane wave solutions $\phi = f(\mathbf{r}; \mathbf{k}) \exp(-i\omega t)$ where

$$f(\mathbf{r}; \mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \exp(i\mathbf{k} \cdot \mathbf{r}), \tag{3}$$

and the frequency ω and wave-vector $\mathbf{k}$ obey the dispersion relation $\omega = k$. We refer to the function $f(\mathbf{r}; \mathbf{k})$ as the mode function.

D. Static point source solution

Consider a static point source $s(\mathbf{r}) = \lambda\delta(\mathbf{r})$; more general static sources can be analyzed by superposition. For a static source, the field ϕ will also be independent of time. Thus, we are just solving Poisson's equation with a delta function source and so the answer can immediately be written down from the known electrostatic potential of a point charge. However, for later use, it is helpful to describe another approach. Taking the Fourier transform of Eq. (2) yields $k^2\tilde{\phi}(\mathbf{k}) = \lambda$, where $\tilde{\phi}(\mathbf{k})$ is the spatial Fourier transform of $\phi(\mathbf{r})$. Inverting the Fourier transform yields.

$$\phi(\mathbf{r}) = \lambda \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{k^2} \exp(i\mathbf{k} \cdot \mathbf{r}) = \frac{\lambda}{4\pi r}. \tag{4}$$

The Fourier integral is most easily evaluated by working in spherical coordinates and integrating over the angular variables first. The radial integral can be performed making use of $\int_0^\infty dx \, (\sin x)/x = \pi/2$ Equations (3) and (4) are analogous to familiar results from electrostatics and gravity: in the absence of sources, Maxwell's equations and Einstein's equations also have plane wave solutions with the same dispersion $\omega = k$, and a static point charge or point mass produces a $1/r$ potential. Thus, we see that solutions to the scalar field equations provide insights into the solutions to Maxwell's equations and Einstein's equations as well.

It is useful here to briefly discuss a variant on the scalar wave equation in Minkowski spacetime, called the Klein–Gordon model or massive scalar field theory. In this model, the scalar field obeys

$$\Box^2\phi + \mu^2\phi = s, \tag{5}$$

where the mass parameter μ has units of inverse length. We may regard the conventional wave equation Eq. (2) as the massless limit ($\mu \to 0$) of the Klein–Gordon model. In the absence of sources, plane waves are still solutions, with f given by Eq. (2), but the frequency and wave-vector are related by the dispersion relation $\omega = \sqrt{k^2 + \mu^2}$.

A key insight is that in quantum field theory the Klein–Gordon model describes massive particles. This can be motivated by multiplying both sides of the dispersion relation by $\hbar$ and recalling that $\hbar\omega$ is the energy and $\hbar k$ is the momentum by Planck's formula and the de Broglie relation, respectively. Then, comparing the dispersion relation to the energy-momentum relation of a particle of mass m in special relativity reveals that the Klein–Gordon model describes a particle of mass $m = \mu\hbar/c$, where we have momentarily restored the factors of c. The wave equation corresponds to $\mu = 0$ and describes a massless particle in quantum field theory, the scalar field analog of the photon and the graviton.

In the Klein–Gordon model also, ϕ will be time-independent for a static source. And again it is sufficient to analyze a point source $s = \lambda\delta(\mathbf{r})$ and to obtain more general static solutions by superposition. Taking the Fourier transform with respect to the spatial dependence in Eq. (5) yields $(k^2 + \mu^2)\tilde{\phi} = \lambda$. Inverting the Fourier transform yields the Yukawa potential

$$\phi(\mathbf{r}) = \lambda \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{k^2 + \mu^2} \exp(i\mathbf{k} \cdot \mathbf{r}) = \frac{\lambda}{4\pi r} \exp(-\mu r). \tag{6}$$

At short distances, $r \ll \mu^{-1}$, the Yukawa potential reproduces the massless $1/r$ behavior but at long distances, $r \gg \mu^{-1}$, the Yukawa potential vanishes exponentially. We will see later that the exponential disappearance of the field is the key to hiding the extra dimensions in the ADD model and in one version of the RS model as well.

It is a good exercise to verify Eq. (6). The Fourier integral is most easily evaluated by working in spherical coordinates and integrating over the angular variables first. The radial integral can be performed by making use of the definite integral

$$\int_0^\infty dk \, \frac{k \sin kr}{k^2 + \mu^2} = \frac{\pi}{2} \exp(-\mu r). \tag{7}$$

III. THE ADD MODEL

A major motivation for considering models with extra dimensions is that they offer an elegant solution to the "hierarchy problem." The hierarchy problem can be simply understood in terms of dimensional analysis. From the three fundamental constants, c, G, and $\hbar$, one can form only one expression with units of energy, the Planck energy, given by $E_P = \sqrt{\hbar c^5/G} = 10^{28}$ eV. Assuming that these are the only parameters that should appear in a fundamental theory of nature, it follows that E_P should be the characteristic energy scale of that fundamental theory. In the standard model of particle physics, an important energy scale is the electroweak scale E_{EW} at which the electromagnetic and weak nuclear forces are unified. In a fundamental theory that subsumes the standard model, the dimensional argument suggests that the electroweak scale E_{EW} should be of the same order as the Planck scale E_P. But in fact the electroweak scale $E_{EW} \approx 10^{13}$ eV $\ll E_P$. The vast discrepancy between these scales is the hierarchy problem. The problem grows more acute in the context of quantum field theory, wherein such a large discrepancy in these scales is unstable. (For an authoritative non-technical discussion placing the hierarchy problem in context of quantum field theory, see Ref. 8.) Also note that as $G \to 0$, $E_P \to \infty$. The weaker gravity is, the worse the hierarchy problem becomes. Thus, we may regard the hierarchy problem as a measure of the weakness of gravity compared to other forces. Any theory that seeks to explain the hierarchy problem must explain the vast gulf between the Planck and electroweak scales; the ADD and RS models do this in different ways by incorporating extra dimensions.

A. The spacetime interval

The ADD model can include any number of extra spatial dimensions; here, we consider the simplest version in which there are four spatial dimensions and one temporal

dimension. The four spatial dimensions are specified by Cartesian coordinates (x, y, z, w) where the extra dimension w is rolled up with a circumference L. More precisely, (x, y, z, w) and $(x, y, z, w + L)$ correspond to the same spatial point. The spacetime interval between two nearby events located at (t, x, y, z, w) and $(t + dt, x + dx, y + dy, z + dz, w + dw)$ is given by

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - dw^2. \quad (8)$$

The four dimensional cross-section of ADD spacetime corresponding to $w = 0$ is called the brane. More generally, branes are lower dimensional hyper-surfaces in a higher dimensional spacetime. Branes are assumed to trap standard model particles and fields so that they effectively only propagate on the brane. A useful analogy to a particle trapped on the brane is a water droplet on a shower curtain.[8] Although the droplet lives in a three dimensional world, it is confined to travel in only two dimensions. In contrast, the gravitational field propagates throughout the five dimensional spacetime (the bulk).

B. The wave equation

By analogy to Eq. (2) the scalar wave equation in ADD spacetime has the form

$$\Box^2 \phi = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial w^2} \right) \phi = s(\mathbf{r}, t), \quad (9)$$

where ϕ is the scalar field and s is its source.

C. Source-free solution: Zero mode and Kaluza–Klein modes

In the absence of sources, the wave equation has plane wave solutions $\phi = f(\mathbf{r}, w; \mathbf{k}, \kappa_n) \exp(-i\omega t)$ where the mode function

$$f(\mathbf{r}, w; \mathbf{k}, \kappa_n) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{L}} \exp(i\mathbf{k} \cdot \mathbf{r}) \exp(i\kappa_n w), \quad (10)$$

and where the frequency ω obeys the dispersion relation $\omega = \sqrt{k^2 + \kappa_n^2}$. In order to ensure the periodicity $f(\mathbf{r}, w + L; \mathbf{k}, \kappa_n) = f(\mathbf{r}, w; \mathbf{k}, \kappa_n)$, the extra dimensional component of the wave-vector κ_n obeys the quantization

$$\kappa_n = \frac{2\pi n}{L}, \quad (11)$$

where n is an integer. Waves with $n = 0$ are called zero modes. Zero modes behave essentially like waves in four dimensional spacetime. They are labelled by a wave vector $\mathbf{k}$ that only has components along the brane and have the massless dispersion $\omega = k$. Modes with non-zero n are called Kaluza–Klein modes. Kaluza–Klein modes have an ω-$\mathbf{k}$ dispersion corresponding to the massive wave equation with a mass parameter $\mu \to \kappa_n$.

D. Static point source solution

Next let us examine the solution to the wave equation (10) for a static source that is confined to the brane. As in Sec. II, we consider a point source $s = \lambda \delta(\mathbf{r}) \delta(w)$; more complicated distributions can be treated by superposition. The resulting field ϕ will be time-independent and has the Fourier expansion

$$\phi(\mathbf{r}, w) = \sum_{n=-\infty}^{\infty} \int d\mathbf{k} \, \tilde{\phi}(\mathbf{k}, \kappa_n) f(\mathbf{r}, w; \mathbf{k}, \kappa_n), \quad (12)$$

where the Fourier amplitudes $\tilde{\phi}(\mathbf{k}, \kappa_n)$ are given by

$$\tilde{\phi}(\mathbf{k}, \kappa_n) = \int d\mathbf{r} \int_0^L dw \, \phi(\mathbf{r}, w) f^*(\mathbf{r}, w; \mathbf{k}, \kappa_n). \quad (13)$$

In Fourier space, the wave equation (10) takes the form $(k^2 + \kappa_n^2)\tilde{\phi} = \lambda/\sqrt{L}(2\pi)^{3/2}$ and hence

$$\phi(\mathbf{r}, w) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\lambda}{k^2 + \kappa_n^2} \exp(i\mathbf{k} \cdot \mathbf{r}) \exp(i\kappa_n w). \quad (14)$$

The $\mathbf{k}$ integral can be performed by comparison to Eq. (6). Hereafter, let us consider only the value of the field on the brane by setting $w = 0$. We obtain

$$\phi(\mathbf{r}, 0) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \frac{\lambda}{4\pi r} \exp\left(-\frac{2\pi |n| r}{L} \right). \quad (15)$$

Equation (15) is the field of a point source that is located on the brane at the origin at a distance r from the source. The sum over modes can be performed exactly but it is more illuminating to consider limiting cases. For $r \gg L$, only the zero mode ($n = 0$) contributes to the field—the contribution of the Kaluza–Klein modes is exponentially suppressed. We obtain

$$\phi(\mathbf{r}, 0) \approx \frac{\lambda}{L} \frac{1}{4\pi r}. \quad (16)$$

At distances that are long compared to the size of the extra dimension, we obtain the $1/r$ behavior characteristic of four spacetime dimensions; the extra dimension is essentially invisible. On the other hand, for $r \ll L$, all modes contribute and we obtain

$$\phi(\mathbf{r}, 0) \approx \frac{\lambda}{4\pi^2} \frac{1}{r^2}. \quad (17)$$

At distances that are short compared to the extra dimension, we obtain the $1/r^2$ behavior characteristic of five spacetime dimensions. An intuitive explanation of these limiting cases is provided by examining the field lines of a point source in ADD spacetime (see Chap. 4 of Ref. 9).

Hierarchy revisited. Standard model fields are confined to the brane in the ADD model. Only gravity propagates through the bulk. Above we have analyzed a scalar field that propagates in the bulk as a proxy for the gravitational field. Now we demonstrate the essence of the solution to the hierarchy problem within the ADD model.

Suppose we were interested in the gravitational field of a point mass m that lives on the brane. Based on the above analysis, at short distances, we expect that the Newtonian potential will have the five dimensional form $G_5 m/r^2$, where G_5, characterizing the gravitational interaction, is the five dimensional version of the Newtonian constant G. On the other hand, at long distances, we expect that the Newtonian potential would have the familiar four dimensional form

Gm/r. Comparing these forms to Eqs. (16) and (17), we infer that $\lambda = 4\pi^2 G_5 m = 4\pi GLm$. Hence, we arrive at the important result

$$G_5 = \frac{GL}{\pi}. \tag{18}$$

Note that except for the factor of π, the form of Eq. (18) follows simply because the dimensions of G_5 have an extra factor of length compared to the dimensions of G. Eq. (18) lies at the heart of the ADD solution to the hierarchy problem. The idea is that the fundamental Planck scale should be determined by $c, \hbar$, and G_5. Simple dimensional analysis shows that the Planck scale should be given by $E_{P5} = (\hbar^2 c^6 / G_5)^{1/3}$. In this expression, we replace $G_5 \to GL$, in accordance with Eq (18), and impose the requirement that $E_{P5} \sim 1$ TeV (the electroweak scale) to obtain $L \sim 10^{13}$ m. Thus, the hierarchy problem is solved if we posit the existence of an extra dimension of this enormous size. Physically, the idea is that gravity is not weaker than other forces; it merely appears weaker on long length scales because gravitational fields spread out into the higher dimensional bulk, while other force fields remain concentrated on the brane.

Similarly, if there were two large extra dimensions, we would have $G_6 \sim GL^2$. On dimensional grounds, the Planck scale would be given by $(\hbar^3 c^7 / G_6)^{1/4}$. Replacing $G_6 \to GL^2$ and imposing the requirement that $E_P \sim 1$ TeV yield $L \sim 1$ mm. Thus, the six dimensional ADD model solves the hierarchy problem if we posit the existence of two extra dimensions on the millimeter scale (which is still enormous compared to the Planck length scale ℓ_P). See Sec. VI for further discussion of the experimental implications of the ADD model and the current experimental constraints on it.

IV. THE RS MODEL

There are two versions of the RS model known as RS1[5] and RS2.[6] Here, for simplicity, we will focus primarily on the RS2 model and relegate the analysis of RS1 to a problem (see Appendix C of the supplementary material).[17] It is worth noting at the outset that in both RS models there is exactly one extra spatial dimension, whereas in the ADD model the number of extra dimensions is arbitrary. Another key difference between the two models is that the RS1 and RS2 spacetimes are curved, whereas the ADD spacetime is flat. Readers who do not have a background in general relativity and curved spacetime may find it useful to read Appendix A of the supplementary material[17] before proceeding with the remainder of this section.

A. The spacetime interval

An event in the RS2 spacetime is specified by (t, x, y, z, w), where $0 \leq w < \infty$ and $-\infty < t, x, y, z < \infty$. The brane is the four dimensional cross section of this spacetime corresponding to $w = 0$. The spacetime interval between neighboring events located at (t, x, y, z, w) and $(t + dt, x + dx, y + dy, z + dz, w + dw)$ is given by

$$ds^2 = e^{-2\gamma w}(dt^2 - dx^2 - dy^2 - dz^2) - dw^2, \tag{19}$$

where γ is a parameter with units of inverse length. For fixed w, the spacetime is essentially flat Minkowski spacetime but with an overall w dependent "warp factor" $e^{-2\gamma w}$. There is no

system of coordinates in which the spacetime interval globally has the flat form of Eq. (8). Randall and Sundrum did not merely assume this form but rather showed that it was compatible with general relativity if they made particular assumptions about the brane. However that analysis is beyond the scope of this article.

Although, in principle, we could continue to work with the coordinates (t, x, y, z, w), it is more convenient to exchange the coordinate w with ζ, where $\zeta = e^{\gamma w}/\gamma$. This gives $\gamma^{-1} \leq \zeta < \infty$ and now the brane is the four dimensional cross section with $\zeta = \gamma^{-1}$. In terms of these coordinates, the spacetime interval in (19) is given by

$$ds^2 = \frac{1}{\gamma^2 \zeta^2} \left(dt^2 - dx^2 - dy^2 - dz^2 - d\zeta^2\right), \tag{20}$$

as the diligent reader should verify. Equation (20) reveals that the RS2 spacetime has the same interval as the ADD spacetime of Eq. (8) up to an overall factor. Since the two spacetime intervals are related by an overall multiplicative factor, they are said to be conformally equivalent. The coordinates (t, x, y, z, ζ) are called conformal coordinates.

B. The wave equation

In conformal coordinates, the scalar wave equation has the form

$$-\frac{\partial^2}{\partial \zeta^2}\phi + \frac{3}{\zeta}\frac{\partial}{\partial \zeta}\phi - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi + \frac{\partial^2}{\partial t^2}\phi$$
$$= \gamma^2 \zeta^2 s. \tag{21}$$

Here, ϕ is the scalar field and s is the source. The reader should verify that Eq. (21) follows from the spacetime interval Eq. (20) following the methods of Appendix A of the supplementary material.[17]

C. Source-free solution: Zero mode and Kaluza–Klein modes

The form of Eq. (21) suggests that the solutions in the absence of sources will have the form of plane waves modulated by a ζ dependent factor $v(\zeta)$, namely, $\phi = v(\zeta) \exp(i\mathbf{k} \cdot \mathbf{r}) \exp(-i\omega t)$. Substitution of this ansatz in Eq. (21) reveals that the modulation factor obeys

$$\frac{d^2 v}{d\zeta^2} - \frac{3}{\zeta}\frac{dv}{d\zeta} + (\omega^2 - k^2)v = 0. \tag{22}$$

We will see below that $\omega^2 \geq k^2$; hence it is convenient to write $\omega^2 - k^2$ as μ^2 in Eq. (22). One solution to Eq. (22) that is immediately apparent is to take v to be a constant independent of ζ if we also choose the massless dispersion $\omega = k$. This solution is the zero-mode. It is an acceptable solution provided we impose Neumann boundary conditions $dv/d\zeta = 0$ on the brane.

1. Schrödinger analogy

Now we systematically construct all solutions to Eq. (22). To this end, following Ref. 6, we first transform to the independent variable ψ defined by $v = \zeta^{3/2}\psi$. Equation (22) then has the form

$$-\frac{1}{2}\frac{d^2}{d\zeta^2}\psi + \frac{15}{8}\frac{1}{\zeta^2}\psi = \frac{\mu^2}{2}\psi. \tag{23}$$

Equation (23) coincides with the time-independent Schrödinger equation for a particle that is confined to the half line $\zeta > \gamma^{-1}$ and that experiences a potential barrier $15/8\zeta^2$. In this analogy, the "energy" of the particle is $\mu^2/2$. Thus, we can draw upon our intuition from non-relativistic quantum mechanics in order to deduce the solutions to Eq. (23). The form of the potential suggests that there should be a continuum of states with $\mu^2 > 0$. These states constitute the Kaluza–Klein modes of the RS2 model. In addition, there is a single bound state which corresponds to the zero-mode of the RS2 model.

2. Zero mode

It may seem surprising that Eq. (23) has a bound state solution since the potential is purely repulsive. However, we still have the freedom to choose the boundary condition we apply to the "wave-function" ψ at the location of the brane $\zeta = \gamma^{-1}$. The boundary condition will have the form $\psi(\gamma^{-1}) = \lambda\psi'(\gamma^{-1})$ where λ is a parameter that we have freedom to specify. The boundary condition effectively functions like a contact interaction which can be repulsive or attractive. By choosing the right boundary condition, we can ensure that there will be a bound state with $\mu = 0$. Indeed, if we set $\mu = 0$, it is easy to see that Eq. (23) has power law solutions of the form $\psi = \zeta^\beta$ with $\beta = -3/2$ or $\beta = 5/2$. We discard the latter on the grounds that it is not normalizable. Normalizing the former we conclude that Eq. (23) has the $\mu = 0$ bound state solution given by

$$\psi_0 = \frac{\sqrt{2}}{\gamma}\zeta^{-3/2}, \tag{24}$$

which satisfies the normalization condition $\int_{\gamma^{-1}}^\infty d\zeta\,|\psi_0|^2 = 1$. The solution (24) satisfies the boundary condition $\psi(\gamma^{-1}) = \lambda\psi'(\gamma^{-1})$ with $\lambda = -(2/3)\gamma^{-1}$ and hence this is the boundary condition we will impose on the scalar field at the brane hereafter Recall that $v = \zeta^{3/2}\psi$ and $\phi = v(\zeta)\exp(i\mathbf{k}\cdot\mathbf{r})\exp(-i\omega t)$. Transforming the boundary condition on ψ into a boundary condition on v, we see that ϕ simply satisfies Neumann boundary conditions on the brane, $\partial\phi/\partial\zeta|_{\zeta\to\gamma^{-1}} = 0$). Undoing the transformation $v = \zeta^{3/2}\psi$, we see that $v_0 = \sqrt{2}/\gamma$, which matches the form for the zero mode we had already deduced following Eq. (22).

3. Kaluza–Klein modes

Determination of the positive energy continuum of solutions to Eq. (23) is a straightforward exercise in one dimensional quantum mechanics. We relegate the details to Appendix B of the supplementary material[17] and only quote the results here. The solutions have the form

$$\psi(\zeta; \mu) = \sqrt{\zeta}[\alpha(\mu)J_2(\mu\zeta) + \beta(\mu)Y_2(\mu\zeta)]. \tag{25}$$

Here, J_2 and Y_2 denote the Bessel and Neumann functions of second order. Equation (25) is a solution to Eq. (23) for arbitrary coefficients α and β. The boundary condition on the brane at $\zeta = \gamma^{-1}$ fixes the ratio of α to β. We find

$$\alpha(\mu) = AY_1\left(\frac{\mu}{\gamma}\right) \quad \text{and} \quad \beta(\mu) = -AJ_1\left(\frac{\mu}{\gamma}\right). \tag{26}$$

We choose the overall constant A by imposing the condition that $\alpha^2 + \beta^2 = \mu$. As shown in Appendix B of the supplementary material,[17] this choice ensures that the continuum solutions satisfy the normalization condition

$$\int_{\gamma^{-1}}^\infty d\zeta\,\psi^*(\zeta; \mu)\psi(\zeta; \mu') = \delta(\mu - \mu'). \tag{27}$$

Recall that solutions to the Schrödinger equation constitute a complete set. For later use, it is worth noting that the continuum solutions derived here together with the zero mode Eq. (24) satisfy the completeness relation

$$\psi_0^*(\zeta)\psi_0(\zeta') + \int_0^\infty d\mu\,\psi^*(\zeta; \mu)\psi(\zeta'; \mu) = \delta(\zeta - \zeta'). \tag{28}$$

4. Summary of source-free modes in RS2

In summary, the scalar wave equation in RS2 spacetime has two kinds of solutions in the absence of sources. The zero-mode solutions are labelled by their three dimensional wave-vector $\mathbf{k}$ and have the form $g(\zeta, \mathbf{r}; \mathbf{k})\exp(-i\omega t)$. The mode function g is given by

$$g(\zeta, \mathbf{r}; \mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\frac{\sqrt{2}}{\gamma}\exp(i\mathbf{k}\cdot\mathbf{r}). \tag{29}$$

Zero modes have the dispersion relation $\omega = k$ characteristic of waves in four spacetime dimensions. The second class of solutions are the Kaluza–Klein modes labelled by their three dimensional wave-vector $\mathbf{k}$ and their mass parameter μ. The Kaluza-Klein modes have the form $f(\zeta, \mathbf{r}; \mathbf{k}, \mu)\exp(-i\omega t)$. The mode function f is given by

$$f(\zeta, \mathbf{r}; \mathbf{k}, \mu) = \frac{1}{(2\pi)^{3/2}}\zeta^{3/2}\psi(\zeta; \mu)\exp(i\mathbf{k}\cdot\mathbf{r}), \tag{30}$$

with $\psi(\zeta; \mu)$ is given by Eq. (25). The Kaluza–Klein modes have the ω–$\mathbf{k}$ dispersion relation $\omega = \sqrt{k^2 + \mu^2}$ corresponding to the massive wave equation with mass parameter μ. In contrast to the ADD model, which had a discrete set of Kaluza–Klein modes with quantized mass parameters, the RS2 model has a continuum of Kaluza–Klein modes and the mass parameter can take on any positive real value.

D. Static point source solution

Finally, let us consider the solution to a static point source confined to the brane. The resulting field ϕ will also be time-independent. We wish to show that, following Ref. 6, the field produced by such a source falls off inversely with distance at distances that are long compared with γ^{-1}. Hence, in the RS2 model too, the extra dimension is hidden on long length scales.

For a point source localized on the brane, $s = \lambda\delta(\mathbf{r})\delta(\zeta - \gamma^{-1})$. With the time-independence of ϕ taken into account, Eq. (21) has the form

$$-\nabla^2_{\text{RS}}\phi = \lambda\delta(\mathbf{r})\delta(\zeta - \gamma^{-1}), \tag{31}$$

where

$$\nabla_{\mathrm{RS}}^2 = \left(\frac{\partial^2}{\partial \zeta^2} - \frac{3}{\zeta} \frac{\partial}{\partial \zeta} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right). \quad (32)$$

In order to solve Eq. (31), we write the solution $\phi(\mathbf{r}, \zeta)$ as a superposition of zero-mode and Kaluza–Klein mode functions. Thus,

$$\phi(\zeta, \mathbf{r}) = \int d\mathbf{k}\, \tilde{\phi}_0(\mathbf{k}) g(\zeta, \mathbf{r}; \mathbf{k})$$
$$+ \int d\mathbf{k} \int_0^\infty d\mu\, \tilde{\phi}(\mathbf{k}, \mu) f(\zeta, \mathbf{r}; \mathbf{k}, \mu). \quad (33)$$

The amplitudes $\tilde{\phi}_0(\mathbf{k})$ and $\tilde{\phi}(\mathbf{k}, \mu)$ are to be determined.

It is now helpful to note that

$$-\nabla_{\mathrm{RS}}^2 g(\zeta, \mathbf{r}; \mathbf{k}) = k^2 g,$$
$$-\nabla_{\mathrm{RS}}^2 f(\zeta, \mathbf{r}; \mathbf{k}, \mu) = (k^2 + \mu^2) f. \quad (34)$$

Equation (34) follows because $g \exp(-i\omega t)$ and $f \exp(-i\omega t)$ obey the scalar wave [Eq. (21)] and from the dispersion relations for the zero and Kaluza–Klein modes, respectively. Using Eqs. (33) and (34), we can write the left hand side of Eq. (31) as

$$-\nabla_{\mathrm{RS}}^2 \phi(\zeta, \mathbf{r}) = \int d\mathbf{k}\, k^2 \tilde{\phi}_0(\mathbf{k}) g(\zeta, \mathbf{r}; \mathbf{k})$$
$$+ \int d\mathbf{k} \int_0^\infty d\mu\, (k^2 + \mu^2) \tilde{\phi}(\mathbf{k}, \mu) f(\zeta, \mathbf{r}; \mathbf{k}, \mu). \quad (35)$$

On the other hand, we can obtain the right hand side of Eq. (31) by substituting $\zeta' \to \gamma^{-1}$ and $\mathbf{r}' \to 0$ in the identity

$$\zeta'^3 \delta(\zeta - \zeta') \delta(\mathbf{r} - \mathbf{r}') = \int d\mathbf{k}\, g^*(\zeta', \mathbf{r}'; \mathbf{k}) g(\zeta, \mathbf{r}; \mathbf{k})$$
$$+ \int d\mathbf{k} \int_0^\infty d\mu\, f^*(\zeta', \mathbf{r}'; \mathbf{k}, \mu) f(\zeta, \mathbf{r}; \mathbf{k}, \mu). \quad (36)$$

The identity is merely a precise statement that the modes g and f are a complete basis. It is easily proved by making use of the explicit forms of g and f in Eqs. (29) and (30), as well as the completeness relation (28) and the standard Fourier integral $\int d\mathbf{k} \exp[i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')] = (2\pi)^3 \delta(\mathbf{r} - \mathbf{r}')$. Making these substitutions in Eq. (36) and equating to Eq. (35) yields

$$\tilde{\phi}_0(\mathbf{k}) = \lambda \frac{\gamma^3}{k^2} g^*\left(\frac{1}{\gamma}, \mathbf{0}; \mathbf{k} \right),$$
$$\tilde{\phi}(\mathbf{k}, \mu) = \lambda \frac{\gamma^3}{k^2 + \mu^2} f^*\left(\frac{1}{\gamma}, \mathbf{0}; \mathbf{k}, \mu \right). \quad (37)$$

Substituting Eq. (37) in Eq. (33) and making use of the explicit forms of g and f given in Eqs. (29) and (30), we obtain

$$\phi(\zeta, \mathbf{r}) = 2\lambda\gamma \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{k^2} \exp(i\mathbf{k} \cdot \mathbf{r})$$
$$+ \lambda\gamma\zeta^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \int_0^\infty d\mu \frac{1}{k^2 + \mu^2} \exp(i\mathbf{k} \cdot \mathbf{r})$$
$$\times \left[\alpha(\mu) J_2\left(\frac{\mu}{\gamma} \right) + \beta(\mu) Y_2\left(\frac{\mu}{\gamma} \right) \right]^2. \quad (38)$$

The first term corresponds to the zero-mode contribution to the field; the second term corresponds to that of the Kaluza–Klein modes. Performing the integrals over $\mathbf{k}$, which are the same as in Eqs. (4) and (6), yields

$$\phi(\zeta, \mathbf{r}) = \frac{\lambda\gamma}{2\pi r} + \frac{\lambda\gamma\zeta^2}{4\pi r} \int_0^\infty d\mu \exp(-\mu r)$$
$$\times \left[\alpha(\mu) J_2\left(\frac{\mu}{\gamma} \right) + \beta(\mu) Y_2\left(\frac{\mu}{\gamma} \right) \right]^2. \quad (39)$$

Equation (39) is the exact expression for the field due to a point source located at the brane. It is instructive to simplify it in the limit of long distance $r \gg \gamma^{-1}$. In this limit, due to the decaying exponential, it is acceptable to approximate the remaining terms in the integrand by their small μ limit. The expansion is tedious by hand but Mathematica readily finds that the square of the term in square brackets in Eq. (39) is approximately equal to μ in the small μ limit. Making this approximation and evaluating the integral over μ, we finally obtain

$$\phi(\gamma^{-1}, \mathbf{r}) \approx \frac{\lambda\gamma}{2\pi r} \left(1 + \frac{1}{2\gamma^2 r^2} + \cdots \right), \quad (40)$$

for $r \gg \gamma^{-1}$. We have also set $\zeta \to \gamma^{-1}$ since we are primarily interested in the field on the brane. Thus, as advertised, at long distances, a point source couples primarily to the zero mode and produces a field that varies as $1/r$; the extra dimension is hidden. The leading correction due to coupling to the Kaluza–Klein modes varies as $1/r^3$ rather than exponentially as in the ADD model.

In principle, we can work out the small r behavior also from Eq. (39). In practice, it is simpler to make the following physical argument. Locally, any space or spacetime looks flat. The expression for the field of a point source in a flat four dimensional space can be obtained by taking the $L \to \infty$ limit of Eq. (15) which is the same as the $r \ll L$ result [Eq. (17)]. Thus, we conclude that

$$\phi(\gamma^{-1}, \mathbf{r}) \approx \frac{\lambda}{4\pi^2 r^2}, \quad (41)$$

for $r \ll \gamma^{-1}$ in the RS model.

Randall and Sundrum assumed that standard model particles and fields were confined to the brane. Only gravity was assumed to propagate through the bulk. Here, we have analyzed a bulk scalar field as a proxy for the gravitational field. Scalar fields are simpler to analyze but have useful parallels to gravitational fields. For example, gravitational waves also have a discrete zero mode and a continuum of Kaluza–Klein modes; indeed, the same Schrödinger analogy and the same Bessel functions arise in both cases.

Suppose now that we are interested in the gravitational field of a point mass m located on the brane. At short distances, we expect its Newtonian gravitational potential will have the five dimensional form $G_5 m / r^2$, where G_5 is the gravitational coupling constant. On the other hand, at long distances, we expect the Newtonian potential to have the four dimensional form Gm/r, where G is the familiar Newtonian constant of gravity (see Fig. 89 of Ref. 8 for a diagram of the field lines of a point source in RS2 spacetime). Comparing these forms to Eqs. (40) and (41), we infer that $\lambda = 4\pi^2 G_5 m = 2\pi Gm/\gamma$ which implies

$$G_5 = \frac{G}{2\pi\gamma}. \tag{42}$$

Equation (42) agrees with Eq. (7) of Ref. 6 and is analogous to Eq. (18) for the ADD model and is the key to resolving the hierarchy problem within this model.

1. RS1 and the hierarchy problem

The model we have analyzed thus far was introduced in Ref. 6 and is sometimes called the RS2 model. In their earlier paper,[5] Randall and Sundrum introduced a slightly different model called RS1 which provides an especially elegant resolution of the hierarchy problem. In this model, the geometry of spacetime is still given by Eq. (19) but the coordinate w only extends over the range $0 \le w \le \ell$. In addition to the brane at $w = 0$, a second brane is located at $w = \ell$. We shall refer to these branes as the left brane and the right brane, respectively. In conformal co-ordinates, the left brane is located at $\zeta = \gamma^{-1}$ and the right brane at $\zeta = \gamma^{-1} \exp(\gamma\ell)$. In RS1, the zero mode (29) remains a solution that satisfies the boundary condition at both branes. The Kaluza–Klein modes become quantized: it is only for particular values of μ that the solutions (30) obey the boundary conditions at both branes. The lowest allowed value of μ is of order $\gamma \exp(-\gamma\ell)$ and the spacing between successive allowed values of μ is also of this order (see problem 5 in Appendix C of the supplementary material[17] for a derivation). In RS1, we choose γ^{-1} and ℓ to be both of order the Planck length, but if we choose the product $\gamma\ell \approx 35$–40, then the lowest allowed μ is 16 orders of magnitude smaller and is of the weak scale. Thus, a large hierarchy can emerge in RS1 due to the exponential warp factor even though the basic parameters of the model are all of Planck scale. Problem 6 in Appendix C of the supplementary material[17] provides further insight into the resolution of the hierarchy problem in RS1. It is shown there that the natural mass for a standard model particle confined to the right brane is of the electroweak scale even though all the parameters in the RS1 model are of Planck scale. See Sec. VI for further discussion of the experimental implications and tests of the RS models.

V. DGP MODEL

According to the current cosmological paradigm about 70% of the total energy density of the universe is in the form of a mysterious component called "dark energy." The first observational evidence for dark energy came from supernovae;[18] distant supernovae appeared dimmer than expected, which could be explained by accelerated cosmic expansion. The DGP model was introduced as an alternative to the idea that dark energy is an actual component of the Universe. In the DGP model, gravity becomes modified on distance scales that are long compared to ℓ, a parameter of the DGP model. On distance scales short compared to ℓ, spacetime appears four-dimensional and gravity obeys Newton's inverse square law. On distance scales long compared to ℓ, the five dimensionality of spacetime is revealed; the gravitational force now falls off much more rapidly, as the inverse cube of the distance, making gravity less effective on cosmological scales. This could help explain the supernova observations but provides no explanation for why the parameter ℓ has the value that is implied by observations. Note that in contrast to the ADD and RS models, in the DGP model, the higher dimensional behavior is manifested on long length scales rather than short.

A. The spacetime interval

In the DGP model, as in ADD and RS, there is a fourth spatial dimension, w. The spacetime interval between nearby events at in DGP is given by

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - dw^2. \tag{43}$$

All coordinates have the same infinite range $-\infty < t$, $x, y, z, w < \infty$. Thus, DGP spacetime is both flat and possesses an infinite extra dimension. As in ADD and RS, the four dimensional section $w = 0$ is the brane; standard model particles and fields are confined to it, while gravity propagates throughout the bulk. Although for simplicity we do not consider this case here, it is consistent to restrict w to the range $w > 0$ as in the Randall–Sundrum model. Indeed, in the original DGP paper, this restriction was effectively imposed in the form of a boundary condition that the solutions had to be symmetric functions of w.

B. The wave equation

In the DGP model, a scalar field that propagates in the bulk is taken to satisfy the wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial w^2}\right)\phi$$
$$+ \ell\delta(w)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)\phi = s(\mathbf{r}, w, t). \tag{44}$$

Equation (44) has the form of the wave equation one would expect from the spacetime interval in Eq. (43) only if we set $\ell = 0$. The term in Eq. (44) that is proportional to ℓ is responsible for the remarkable behavior of the DGP model described above. At the end of this section, we give a physical motivation for the form of this term by an analogy to the electrostatics of a dielectric sheet. The analogy shows that much as a dielectric sheet will screen a point charge, so also in the DGP model gravity is screened by the brane at short distances. The screening leads to behavior at short distances appropriate to three spatial dimensions, but at long distances gravity is unscreened, yielding behavior appropriate to four spatial dimensions.

C. Source free solution

We seek solutions to Eq. (44) of the form $v(w)\exp(i\mathbf{k}\cdot\mathbf{r})\exp(-i\omega t)$. Substituting this ansatz into Eq. (44), we obtain

$$-\frac{1}{2}\frac{d^2}{dw^2}v - \frac{\ell\mu^2}{2}\delta(w)v = \frac{\mu^2}{2}v. \tag{45}$$

Here, as usual we have written $\omega^2 = k^2 + \mu^2$. Superficially, this resembles the Schrödinger equation for a non-relativistic particle moving in one dimension under the influence of an attractive delta function potential. The analogy is not perfect because the strength of the delta potential depends on $(1/2)\mu^2$ which is analogous to the energy. Nonetheless, we can draw upon experience with delta potentials in quantum

mechanics in solving Eq. (45). We expect a solution of the scattering form

$$v_+(w; \mu) = \frac{1}{\sqrt{2\pi}} \exp(i\mu w) + \frac{r(\mu)}{\sqrt{2\pi}} \exp(-i\mu w) \quad \text{for } w < 0$$

$$= \frac{t(\mu)}{\sqrt{2\pi}} \exp(i\mu w) \quad \text{for } w > 0. \quad (46)$$

There should be a second solution, $v_-(w; \mu)$, corresponding to a plane wave incident along the negative w axis. The detailed analysis of these solutions is relegated to the problems. Here, we note that the "transmission" coefficient is found to be

$$|t(\mu)|^2 = \frac{1}{1 + (\mu\ell/2)^2}. \quad (47)$$

There is a peak in the transmission at $\mu = 0$. The peak has a Lorentzian form and a width of $1/\ell$. Remarkably, in the DGP model when there is no source, there is no zero mode. But there is a continuum of massive Kaluza–Klein modes and the mass parameter μ can take any positive real value. Although there is no discrete zero-mode bound to the brane, there is a resonance in the Kaluza–Klein modes at $\mu = 0$.

As discussed in introductory textbooks, a resonance is an almost-bound state in quantum mechanics. As a concrete example, consider a non-relativistic particle moving in one dimension in a double barrier potential. If the barrier walls were infinitely high, there would be bound states between the barriers. For finite barriers, these states are no longer bound and become resonances. They acquire a finite lifetime as the particle can escape by tunneling. If we consider scattering states in which a particle is incident on the double barrier potential from outside, we find sharp Lorentzian peaks in the transmission coefficient as a function of the energy. The energy of the peaks corresponds roughly to the energy of the resonance and the width of the peak to the lifetime.

D. Static point source solution

We now consider the field produced by a static point source localized on the brane. Thus, the source is $s \to \lambda\delta(\mathbf{r})\delta(w)$. The resulting field will also be static and we are primarily interested in the field on the brane, $\phi(\mathbf{r}, w \to 0)$.

We obtain the exact expression for the point source profile from straightforward Fourier analysis of Eq. (44); see Problem 4 of Appendix C in the supplementary material.[17] The result is

$$\phi(\mathbf{r}, 0) = \frac{\lambda}{4\pi^2 r} \int_0^\infty dk \, \frac{\sin(kr)}{\left(1 + \frac{1}{2}\ell k\right)}. \quad (48)$$

At short distances $(r \ll \ell)$

$$\phi(\mathbf{r}, 0) \approx \frac{1}{4\pi} \frac{\lambda}{\ell} \frac{1}{r}, \quad (49)$$

whereas at long distances $(r \gg \ell)$

$$\phi(\mathbf{r}, 0) \approx \frac{\lambda}{4\pi^2} \frac{1}{r^2}. \quad (50)$$

Intuitively, we can understand these results as follows. Evidently, $\phi(\mathbf{r}, 0)$ must depend on r, ℓ, and λ. On dimensional grounds, we expect $\phi(\mathbf{r}, 0) = (\lambda/r^2)f(\ell/r)$ where f is an unknown function. This form suggests that to obtain the $\ell \ll r$ limit it is sufficient to consider the $\ell \to 0$ limit of Eq. (44). In that limit, Eq. (44) reduces to the electrostatics of a point source in four space dimensions. Hence, ϕ has the same form as in the small r limits of the ADD and RS models; compare Eqs. (17), (41), and (50). Conversely, it is plausible that the $\ell \gg r$ limit can be derived by neglecting the first term on the left hand side of Eq. (44). In that case, we are simply solving Poisson's equation in ordinary three dimensional space for the potential of a point charge λ/ℓ, thereby justifying Eq. (49). Remarkably, we obtain behavior appropriate to four spacetime dimensions at short distances and to five spacetime dimensions at long distances. In contrast to the ADD and RS models, here the extra dimension reveals itself at long distances and remains hidden at short.

1. Dielectric sheet analogy

Finally, we motivate the form of the wave equation (44). To this end, we proceed by analogy to the electrostatics of a two dimensional dielectric sheet with a point charge q place at the origin. As a result, the bound charges in the dielectric sheet are polarized. The polarization vector lies in the plane of the sheet and is given by

$$P_x = -\ell \frac{\partial}{\partial x} \phi \, \delta(z) \quad \text{and} \quad P_y = -\ell \frac{\partial}{\partial y} \phi \, \delta(z). \quad (51)$$

Here, ℓ denotes the polarizability of the sheet, which is assumed to lie in the x-y plane, and ϕ is the electrostatic scalar potential. The resulting bound charge density is then given by the negative divergence of the polarization. Thus,

$$\rho_{\text{pol}} = \ell \left(\frac{\partial^2}{\partial x^2} \phi + \frac{\partial^2}{\partial y^2} \phi \right) \delta(z). \quad (52)$$

By Gauss's law $\nabla \cdot \mathbf{E} = q\delta(\mathbf{r}) + \rho_{\text{pol}}$. Thus, Poisson's equation takes the form

$$-\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] \phi - \ell\delta(z) \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] \phi = q\delta(\mathbf{r}). \quad (53)$$

Equation (53) is easily solved using Fourier analysis. The result is that at short distances from the source the field is screened and appears to be that of a line of charge but at long distances it is unscreened and reduces to the field of a point charge. Note that Eq. (44) is a natural generalization of Eq. (53) that incorporates dynamics by including time derivatives and that also incorporates an extra dimension of space.

VI. DISCUSSION

Table I summarizes the main properties of the ADD, RS, and DGP models discussed here. Having demonstrated the utility of scalar fields for providing insights into the physics of these models, we now briefly discuss the motivation for these models and their experimental tests. A valuable resource on experiments is Ref. 12 which provides a lucid and authoritative review that is regularly updated.

The standard model of particle physics is believed to be a low energy approximation to a more fundamental theory that remains to be discovered. Grand unification is a longstanding and ambitious program to develop a unified theory of the strong and electroweak forces.[19] Within this paradigm, the strong and electroweak forces are unified at an energy scale $E_{\text{GUT}} \sim 10^{25}$ eV $\sim 10^{-3} E_P$. This energy scale is beyond the reach of any collider of the near future. Physics on the grand unification scale is only accessible to precision experiments such as proton decay or via cosmology. Crucially with the grand unification paradigm, E_{GUT} and E_P are well separated and gravity is a separate problem not addressed by grand unification.

Extra dimensional models emerged more recently as an alternative to grand unification and represent a very different view of the world. As discussed above, according to the ADD model, the fundamental scale for the unification of all forces, including gravity, is the weak scale, not E_{GUT} or E_P. According to the RS model too, although the fundamental scale for gravity is the Planck scale, quantum gravity effects should be accessible to colliders in the form of TeV scale Kaluza–Klein particles as we discuss below.

Precision tests of Newton's inverse square law with torsion pendulums represent an important class of experimental tests of the ADD model.[20] Both RS and ADD predict that the gravitational force will depart from Newton's inverse square law at small distances; however, for the RS models, this happens at distances that are too small to be observable. For the ADD model, such experiments easily preclude the existence of a single extra dimension of order 10^{13} m. For the case of two extra dimensions, the best upper bound is $L < 60\,\mu$m.[21]

Another important class of experiments aims to detect the Kaluza–Klein modes either directly or indirectly. Taking into account quantum mechanics, in both ADD and RS models, the zero mode corresponds to a massless particle—the graviton. Each Kaluza–Klein mode can be thought of as a separate particle with a mass equal to $\mu\hbar/c$ where μ is the mass parameter of the mode. Thus, both models predict an infinite tower of new particles. The mass of the lightest Kaluza–Klein particle in the ADD model would be as low as 10^{-4} eV assuming two extra dimensions of millimeter scale (since $\mu \sim 1/L$ for the ADD model). This is also the scale of the separation between in mass between successive Kaluza–Klein particles. In RS1, the lightest Kaluza-Klein particle has a much larger mass of order 1 TeV (since $\mu \sim \gamma e^{-\gamma\ell}$) and that is also the order of the spacing between successive Kaluza–Klein particle masses. Furthermore, though this is not evident from our presentation, the Kaluza–Klein particles in RS1 interact strongly with standard model particles on the TeV scale, whereas the interaction with the corresponding ADD particles is only of gravitational strength. Thus, the Kaluza–Klein modes in RS1 would be manifested in collider experiments as well-defined particles with masses on the TeV scale. The Kaluza–Klein modes of the ADD model would not be individually observable, but because they are so numerous their production would be manifested as missing energy in collisions.[12] Experiments at LHC have put an upper bound of 50–100 fm for ADD models with between four to six extra dimensions.[22]

Astrophysical observations are also sensitive to the production of Kaluza–Klein particles in the ADD model. For example, if the production becomes too copious, it would put supernovae models in conflict with data.[12,23]

Observations of neutron stars by the Fermi Gamma Ray Space Telescope provide another astrophysical constraint.[24] For the case of two extra dimensions, the best current astrophysical constraints are of order nanometers. Thus, although indirect these constraints can be quite stringent.

In contrast to the ADD and RS models, the DGP model is concerned with gravity on the cosmological scale. The DGP model predicts modifications of gravity on length scales that are long compared to the Hubble scale, $\ell \sim 10^9$ light years. The natural tests of the DGP model therefore come from cosmology. The DGP model is now disfavored by observations of the accelerated expansion of the Universe,[25] but it has many descendants that remain viable.[13]

Finally, we note that beyond the immediate application to particle physics the RS model has deep connections to string theory. An important development in string theory is the AdS-CFT correspondence that demonstrates a deep relationship between theories defined in the bulk of AdS spacetime and on its boundary.[2] This correspondence further illuminates the RS model. Many interesting applications of the AdS-CFT correspondence are in condensed matter physics.[26] Thus, extra dimensional models sit at a vibrant intersection of theory and experiment and different subfields of physics.

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[17] See supplementary material at https://doi.org/10.1119/1.5024221 for Appendices. Appendix A is a primer on differential geometry needed for

this paper; Appendix B on the normalization of the Kaluza-Klein modes of the RS2 model; and Appendix C contains problems based on the material of this paper.

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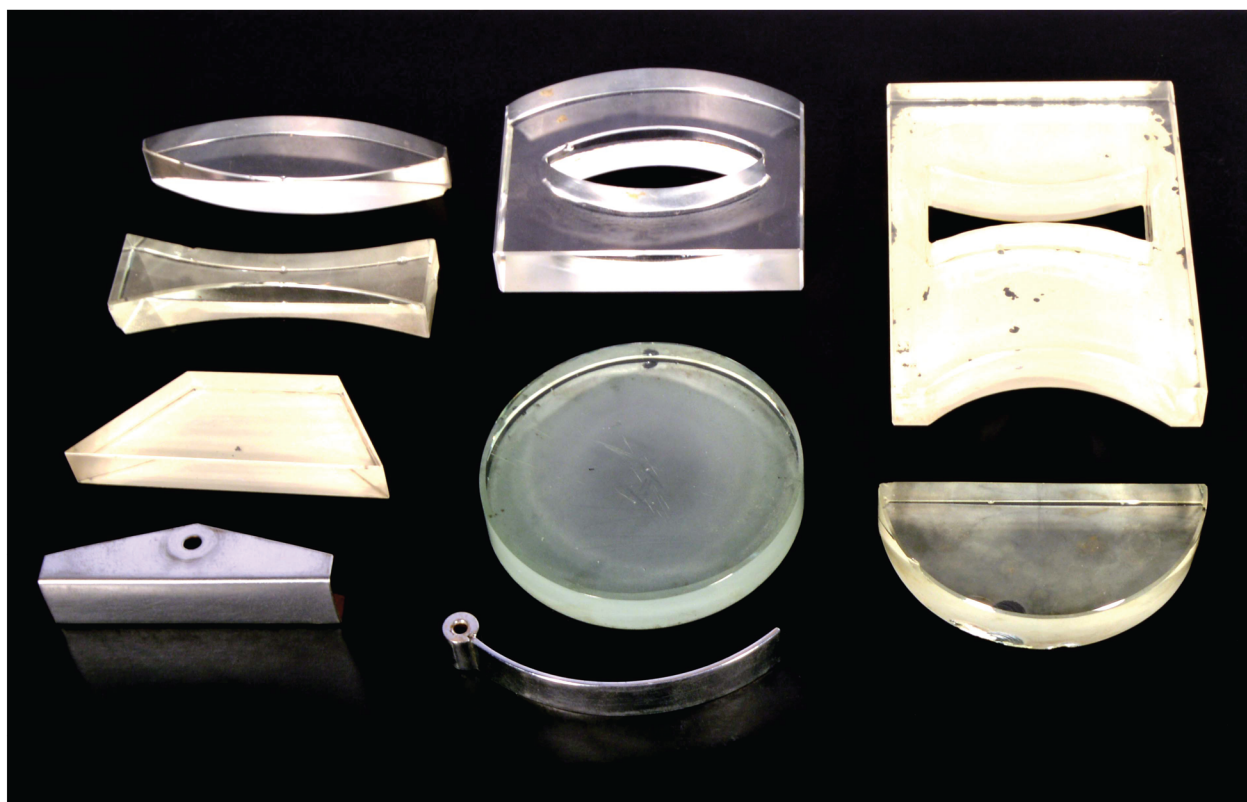
Objects for a Hartl Optical Disk

The Hartl Optical Disk was a staple for instruction in geometrical optics for the first half of the 20th century. These glass objects were held in the middle of the disk, and rays were traced through them. If you are fortunate, you can still find these lenses and reflectors tucked away in corners of undergraduate optics labs, and you can use them for ray tracing experiments using lasers. The two "lenses" on the right-hand side of the top row produce unusual effects; the cut-out that shapes a double-convex air lens is actually a diverging lens, while the double concave aperture acts like a converging lens. These are in the Greenslade Collection, which also has several of the optical disks. (Picture and Notes by Thomas B. Greenslade, Jr., Kenyon College)