Supplementary Online Material

Exploring extra dimensions with scalar fields

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Appendix A: Differential Geometry

Curved space. Let us start with ordinary three dimensional Euclidean space. Adopting cartesian co-ordinates the distance between neighboring points is given by the Pythagorean formula $ds^2 = dx^2 + dy^2 + dz^2$. Suppose however we use spherical polar co-ordinates (r, θ, φ) to label points in the same space. The distance between neighboring points in spherical polar co-ordinates is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2. \tag{A1}$$

Equation (A1) follows readily from the transformation that given Cartesian co-ordinates in terms of polar; namely, $z = r \cos\theta$, $x = r \sin\theta \cos\varphi$ and $y = r \sin\theta \sin\varphi$. Regardless of the co-ordinates we use, the geometry of the space is fully specified by the formula for distance between neighboring points.

Thus far we are only describing the same flat space in different co-ordinates. Now let us discuss curved spaces. The simplest example is the surface of a sphere of radius R that is located at the origin of the Cartesian co-ordinate system. Points on the surface of the sphere can be labelled by the co-latitude θ and the longitude φ. The distance between neighboring points on the surface of a sphere is given by

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2 \theta \; d\varphi^2. \tag{A2}$$

The co-ordinates are restricted to lie in the range $0 \leq \theta \leq \pi$ and $0 \leq \varphi < 2\pi$. Although it is helpful to picture the sphere as a surface in three dimensional Euclidean space mathematically the space is fully defined by Eq. (A2) together with a statement about the allowed range of the co-ordinates.

Another example of a curved space is the two dimensional hyperbolic space. Points in this space are labelled by the co-ordinates (θ, φ) which have the ranges $0 \leq \theta < \infty$ and $0 \leq \varphi < 2\pi$. The distance between neighboring points on the surface of hyperbolic space is given by

$$ds^2 = R^2 d\theta^2 + R^2 \sinh^2 \theta \; d\varphi^2. \tag{A3}$$

Hyperbolic space cannot be regarded as a surface in ordinary three dimensional space but it can be embedded in a different flat three dimensional space in which the co-ordinates (x, y, z) have the usual range $-\infty < x, y, z < \infty$ but the distance between neighboring points is given by

$$ds^2 = -dz^2 + dx^2 + dy^2 \tag{A4}$$

This space is flat but it has the peculiar feature that the square of the distance between nearby points can be positive, negative or zero. Mathematically this space is said to have an indefinite metric. The hyperbolic space is the two dimensional surface that satisfies the equation

$$z^2 - x^2 - y^2 = R^2 \tag{A5}$$

together with the condition $z > 0$. It is easy to see that the constraint Eq. (A5) is automatically satisfied if we write

$$z = R\cosh\theta, x = R\sinh\theta\cos\varphi, y = R\sinh\theta\sin\varphi. \tag{A6}$$

Furthermore the entire hyperboloid will be covered by this parametrization if (θ, φ) are allowed the range noted above. It is now a simple exercise to see that the distance formula Eq. (A3) follows when Eq. (A6) is substituted into Eq. (A4).

The Laplacian. Let us now return to ordinary three dimensional space inhabited by a scalar field such as the scalar potential, $\phi(x, y, z)$. It is a familiar result of elementary calculus that the gradient of a scalar field is a vector; and that the divergence of the gradient, which is the Laplacian, is a scalar. In symbols, $\nabla \cdot \nabla\phi = \nabla^2\phi$ is a scalar. In Cartesian co-ordinates the Laplacian may be written as

$$\nabla^2\phi = \frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi + \frac{\partial^2}{\partial z^2}\phi. \tag{A7}$$

It is helpful sometimes to work in a different co-ordinate system such as polar or cylindrical co-ordinates and therefore it is useful to be able to express the Laplacian directly in other co-ordinate systems. Suppose we transform to a system of co-ordinates (ξ, η, ζ) in which the distance between neighboring points is given by

$$ds^2 = h_\xi^2 d\xi^2 + h_\eta^2 d\eta^2 + h_\zeta^2 d\zeta^2. \tag{A8}$$

The distance formula Eq. (A8) can easily be derived from the Pythagorean formula $ds^2 = dx^2 + dy^2 + dz^2$ given (ξ, η, ζ) as functions of (x, y, z). In general the formula might involve cross terms like $d\xi d\eta, d\eta d\zeta$ and $d\zeta d\xi$ but here we will focus only on orthogonal co-ordinates in which such terms are absent. Most co-ordinate systems of interest in mathematical physics are orthogonal. The factors h_ξ, h_η and h_ζ are called scale factors and their product $h = h_\xi h_\eta h_\zeta$ is called the invariant measure. (For readers who have studied general relativity we note, parenthetically, that h is commonly denoted $\sqrt{|g|}$ where g is the determinant of the metric tensor). For example for spherical polar co-ordinates the scale factors are $h_r = 1, h_\theta = r, h_\varphi = r \sin\theta$ and the invariant measure $h = r^2 \sin\theta$. Now it is proved in books on electromagnetism and mathematical physics that the Laplacian in the co-ordinates (ξ, η, ζ) is given by

$$\frac{1}{h}\frac{\partial}{\partial \xi}\left(\frac{h}{h_\xi^2}\frac{\partial}{\partial \xi}\phi\right) + \frac{1}{h}\frac{\partial}{\partial \eta}\left(\frac{h}{h_\eta^2}\frac{\partial}{\partial \eta}\phi\right) + \frac{1}{h}\frac{\partial}{\partial \zeta}\left(\frac{h}{h_\zeta^2}\frac{\partial}{\partial \zeta}\phi\right) \tag{A9}$$

The generalization of this result to more or fewer dimensions should be obvious. It is a good exercise for the reader to use Eq. (A9) construct the Laplacian in familiar cases like spherical polar and cylindrical co-ordinates.

Thus far we have been discussing the mundane subject of writing the Laplacian in different co-ordinate systems in ordinary flat three dimensional space. Remarkably it turns out that in a curved space in which the distance formula has the form Eq. (A8) it is still true that the Laplacian is given by Eq. (A9). This is intuitively plausible and gives us a simple method to construct the Laplacian in curved space without going through the full machinery of differential geometry[1]. It is a good exercise for the reader to use Eq. (A9) to write an expression for the Laplacian on the surface of a sphere and in hyperbolic space.

Curved space-time. The central dogma of special relativity is that an inertial observer using Cartesian co-ordinates can label events by the co-ordinates (t, x, y, z) which lie in the range $-\infty < t, x, y, z < \infty$ and that the interval between two nearby events is given by Eq. (1). Suppose we transform to a system of co-ordinates (τ, ξ, η, ζ) in which the space-time interval between neighboring events is given by

$$ds^2 = h_\tau^2 d\tau^2 - h_\xi^2 d\xi^2 - h_\eta^2 d\eta^2 - h_\zeta^2 d\zeta^2. \tag{A10}$$

The interval Eq. (A10) can easily be derived from Eq. (1) given (τ, ξ, η, ζ) as a function of (t, x, y, z). In general the formula might involve cross terms like $d\tau d\xi$ or $d\xi d\eta$ but here we will focus only on orthogonal co-ordinates in which such terms are absent. The factors h_τ, h_ξ, h_η and h_ζ are called scale factors and their product $h = h_\tau h_\xi h_\eta h_\zeta$ is called the invariant measure. Thus far we are dealing with the Minkowski space-time of special relativity. Although Eq. (A10) looks complicated we know that the underlying space-time is flat and that we can always transform back to Cartesian co-ordinate (t, x, y, z) in which the interval has the simple form Eq. (1). By contrast a curved space-time is one in which the space-time interval might have a form like Eq. (A10) but where it is impossible to find an alternative set of co-ordinates in which the interval globally has the simple flat form given in Eq. (1).

A concrete example of a curved space-time is the Friedman-Walker-Robertson space-time that is believed to describe our expanding universe. It has the space-time interval

$$ds^2 = dt^2 - a^2(t)[dx^2 + dy^2 + dz^2]. \tag{A11}$$

Here $a(t)$ is called the scale factor and it grows with time reflecting the expansion of the Universe. This space-time is called flat FRW. Notwithstanding that name it is undeniably a curved space-time. There is no change of co-ordinates which will bring the space-time interval (A11) to the form Eq. (1).

A second example of a curved space-time is the celebrated Anti-de-Sitter (AdS) space-time. Like the hyperbolic space discussed above, AdS space-time is best understood by embedding it in a space of higher dimensionality. We start with a six dimensional flat space with the interval

$$ds^2 = du^2 + dv^2 - d\alpha^2 - d\beta^2 - d\eta^2 - d\xi^2. \tag{A12}$$

The co-ordinates have the usual range $-\infty < u, v, \alpha, \beta, \eta, \xi < \infty$. AdS is the five dimensional surface defined by the constraint

$$u^2 + v^2 - \alpha^2 - \beta^2 - \eta^2 - \xi^2 = \frac{1}{\gamma^2}. \tag{A13}$$

We now put down new co-ordinates (ζ, t, x, y, z) on the half of AdS space-time that satisfies $v + \xi > 0$. The new co-ordinates are related to the old via

$$v + \xi = \frac{1}{\gamma\zeta},\ u = \frac{t}{\gamma\zeta},\ \alpha = \frac{x}{\gamma\zeta},\ \beta = \frac{y}{\gamma\zeta},\ \eta = \frac{z}{\gamma\zeta}. \tag{A14}$$

The constraint Eq. (A13) is automatically satisfied provided we take

$$v - \xi = \frac{1}{\gamma\zeta}\left(\zeta^2 + x^2 + y^2 + z^2 - t^2\right). \tag{A15}$$

The new co-ordinates have the range $\zeta \geq 0$ and $-\infty < t, x, y, z, < \infty$. Making use of Eq. (A14) and Eq. (A12) it follows after some algebra that the interval on AdS space-time in terms of the new co-ordinate is given by

$$ds^2 = \frac{1}{\gamma^2\zeta^2}\left(dt^2 - d\zeta^2 - dx^2 - dy^2 - dz^2\right), \tag{A16}$$

exactly the form for Randall-Sundrum space-time in conformal co-ordinates.

Scalar waves in curved space-time. A scalar wave in Minkowski space-time obeys the wave Eq. (2). The operator $\Box^2$ defined in Eq. (2) is called the d'Alembertian and is the space-time analog of a Laplacian. If ϕ is a scalar then so is $\Box^2\phi$ (see for example, Ref. 15, vol II, chapter 25, section 3). Suppose we now transform to a system of co-ordinates (τ, ξ, η, ζ) in which the space-time interval between neighboring events is given by Eq. (A10). Evidently in this co-ordinate system the d'Alembertian operator will have the form

$$\begin{aligned}\Box^2\phi \;=\; & \frac{1}{h}\frac{\partial}{\partial\tau}\left(\frac{h}{h_\tau^2}\frac{\partial\phi}{\partial\tau}\right) - \frac{1}{h}\frac{\partial}{\partial\xi}\left(\frac{h}{h_\xi^2}\frac{\partial\phi}{\partial\xi}\right) \\ & - \frac{1}{h}\frac{\partial}{\partial\eta}\left(\frac{h}{h_\eta^2}\frac{\partial\phi}{\partial\eta}\right) - \frac{1}{h}\frac{\partial}{\partial\zeta}\left(\frac{h}{h_\zeta^2}\frac{\partial\phi}{\partial\zeta}\right) \end{aligned} \tag{A17}$$

Thus far we are discussing the mundane subject of writing the d'Alembertian in different co-ordinate systems in ordinary flat Minkowski space-time. Remarkably it turns out that in a curved space-time in which the space-time interval has the form Eq. (A10), it is still true that the d'Alembertian is given by Eq. (A17). This is intuitively plausible and it gives us a simple method to write down the wave equation in curved space-times without going through the full machinery of differential geometry. Although we have written our results specifically for a four dimensional space-time the generalization to higher or lower dimensions is self-evident.

Appendix B: One dimensional quantum mechanics

Normalization. We begin by recalling some useful results from undergraduate quantum mechanics. For derivations readers should consult their favorite textbook. Consider a free particle in one dimension. In a state of definite momentum the state of the particle is described by the wave function

$$\psi(x; k) = \frac{1}{\sqrt{2\pi}} \exp(ikx). \tag{B1}$$

The pre-factor has been chosen to ensure the normalization

$$\int_{-\infty}^{\infty} dx\; \psi^*(x; p)\psi(x; k) = \delta(p - k). \tag{B2}$$

Now suppose that there is a potential barrier $V(x)$ that is localized near the origin. Sufficiently far from the origin the particle is free and we expect to find scattering solutions of the form

$$\begin{aligned}\psi(x;k) &= \frac{1}{\sqrt{2\pi}}\exp(ikx) + \frac{r}{\sqrt{2\pi}}\exp(-ikx) \\ &\qquad\qquad \text{for} \quad x \to -\infty \\ &= \frac{t}{\sqrt{2\pi}}\exp(ikx) \quad \text{for} \quad x \to \infty \end{aligned} \tag{B3}$$

Here the reflection coefficient r and the transmission coefficient t might depend on the wave-vector k and satisfy the unitarity condition $|r|^2 + |t|^2 = 1$. The scattering wave functions still satisfy the normalization condition Eq. (B2). The solution above corresponds to the scattering of a particle that is incident on the barrier from the left. An analogous solution may be written down for particles incident from the right but will not be needed here. Now suppose that the potential $V \to \infty$ as $x \to \infty$ or else that the particle encounters an impenetrable barrier such as a hard wall at the origin. In that case the transmission coefficient $t \to 0$ and the reflection coefficient has magnitude unity and may be written as $r = \exp(i2\delta)$. This defines the scattering phase shift δ which may depend on k. For this circumstance the scattering solution has the form

$$\psi(x;k) = \sqrt{\frac{2}{\pi}}\cos(kx - \delta) \quad \text{for} \quad x \to -\infty. \tag{B4}$$

This form is obtained by substituting $r \to \exp(i2\delta)$ in Eq. (B3) and multiplying the solution by a factor of $\exp(-i\delta)$. The scattering solutions Eq. (B4) still satisfy the normalization condition Eq. (B2).

Bessel solution. We turn now to the solution to Eq. (23) for $\mu > 0$. It is convenient to transform to the dependent variable φ defined by $\psi = \zeta^{1/2}\varphi$ and to change to the independent variable $\xi = \mu\zeta$. In terms of these variables Eq. (23) is revealed to be Bessel's equation of the second order

$$\frac{d^2\varphi}{d\xi^2} + \frac{1}{\xi}\frac{d\varphi}{d\xi} + \left(1 - \frac{4}{\xi^2}\right)\varphi = 0 \tag{B5}$$

with independent solutions $J_2(\xi)$ and $Y_2(\xi)$. Hence the general solution to Eq. (23) has the form given in Eq. (25) with the coefficients α and β at this stage arbitrary. Imposing the boundary condition $\psi = -\frac{2}{3}\gamma^{-1}\psi'$ on ψ at $\zeta = \gamma^{-1}$ determines the ratio α/β,

$$\frac{\alpha}{\beta} = -Y_1\left(\frac{\mu}{\gamma}\right)/J_1\left(\frac{\mu}{\gamma}\right), \tag{B6}$$

and leads to Eq. (26). Here we have used the Bessel function recursion $xZ_1(x) = 2Z_2(x) + xZ_2'(x)$ where Z denotes either J or Y. Finally let us write

$$\alpha = \sqrt{\alpha^2+\beta^2}\cos\Delta \quad \text{and} \quad \beta = \sqrt{\alpha^2+\beta^2}\sin\Delta \tag{B7}$$

which defines Δ. Inserting this form into the solution Eq. (25) and making use of the large argument asymptotics of the Bessel and Neumann functions,

$$J_2(x) \approx \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{5}{4}\pi\right), Y_2(x) \approx \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{5}{4}\pi\right), \tag{B8}$$

we obtain

$$\psi(\zeta) \approx \sqrt{\frac{2(\alpha^2+\beta^2)}{\pi\mu}}\cos\left(\mu\zeta - \frac{5}{4}\pi - \Delta\right) \tag{B9}$$

for $\mu\zeta \gg 1$. Comparing Eq. (B9) to Eq. (B4) we see that our solutions will have the desired normalization, Eq. (27), if we choose $\alpha^2 + \beta^2 = \mu$. This completes the derivation of the continuum solutions to Eq. (23).

For reference we note that A defined in Eq. (26) of the paper is given by

$$A = \sqrt{\mu}/\sqrt{[Y_1(\mu/\gamma)]^2 + [J_1(\mu/\gamma)]^2} \tag{B10}$$

and therefore has the small μ asymptotic behavior $A \approx (\pi/2\gamma)\mu^{3/2}$. Making use of Eq. (26) and Eq. (B10) we can also deduce that $\alpha(\mu) \approx -\sqrt{\mu}$ and $\beta(\mu) \approx -(\pi/4\gamma^2)\mu^{5/2}$ for small μ.

Appendix C: Problems.

Problem 1. What data would you invoke in order to rule out the ADD model with a single extra dimension? (*Answer:* 10^{13} m is comparable to the size of Pluto's orbit. An extra dimension of this size would lead to violations of Newton's inverse square law of gravity throughout the solar system. Tycho Brahe's data, and very likely even Sumerian tablets, would be sufficient to rule out such violations.)

Problem 2. The ADD hierarchy problem. (a) An alternative way to describe the hierarchy problem is in terms of length scales. (i) Construct a length scale out of the three fundamental constants c, G and $\hbar$. This is the Planck length, $\ell_{\rm P}$. (ii) Denote the electroweak scale $\eta = 1$ TeV. From $\eta, \hbar$ and c construct the length scale $\ell_{\rm ew}$ that corresponds to electroweak physics. The conventional hierarchy problem is then the observation that $\ell_{\rm P} \ll \ell_{\rm ew}$. (b) The ADD model solves this hierarchy problem by introducing extra dimensions of characteristic scale L. Compare L to ℓ_{ew} and comment.

Answer 2. (a) $\ell_{\mathrm{P}} = \sqrt{\hbar G/c^3} \sim 10^{-35}$ m. $\ell_{\mathrm{ew}} = \hbar c/\eta \sim 10^{-19}$ m. (b) In section III we found that $L \sim 1$ mm if there are two extra dimensions. Thus the ADD model resolves the hierarchy $\ell_{\mathrm{P}} \ll \ell_{\mathrm{ew}}$ by introducing extra dimensions of length scale L; however it suffers from a hierarchy problem of its own, namely, $\ell_{\mathrm{ew}} \ll L$.

Problem 3. Three dimensional hyperbolic space. Points in a flat four dimensional space can be labelled by the four Cartesian co-ordinates (x, y, z, w) that have the range $-\infty < x, y, z, w < \infty$. The space has an indefinite metric. The distance between neighboring points is given by $ds^2 = dx^2 + dy^2 + dz^2 - dw^2$. The three dimensional hyperbolic space is defined as the set of points that satisfy the constraints $w^2 - z^2 - x^2 - y^2 = R^2$ and $w > 0$. Points on three dimensional hyperbolic space can be labelled by the co-ordinates (ψ, θ, φ) that are related to the Cartesian co-ordinates via

$$\begin{aligned} w &= R\cosh\psi; \\ z &= R\sinh\psi\cos\theta; \\ x &= R\sinh\psi\sin\theta\cos\varphi; \\ y &= R\sinh\psi\sin\theta\sin\varphi. \end{aligned} \tag{C1}$$

Here the co-ordinates have the ranges $0 \le \psi < \infty$, $0 \le \theta \le \pi$ and $0 \le \varphi < 2\pi$. (a) Justify the parametrization (C1) and explain the ranges on the coordinates (ψ, θ, φ). (b) Determine the distance between two nearby points with co-ordinates (ψ, θ, φ) and $(\psi + d\psi, \theta + d\theta, \varphi + d\varphi)$.

Answer 3. (a) The constraints that $w^2 - x^2 - y^2 - z^2 = R^2$ and $w > 0$ together imply that $w > R$. Hence we may write $w = R\cosh\psi$ with $0 \le \psi < \infty$. Adopting this form the constraint then becomes $x^2 + y^2 + z^2 = R^2\sinh^2\psi$. From here we simply put down spherical polar co-ordinates on a sphere of radius $R\sinh\psi$.
(b) $ds^2 = R^2 d\psi^2 + R^2\sinh^2\psi d\theta^2 + R^2\sinh^2\psi\sin^2\theta d\varphi^2$.

Problem 4. DGP Analysis. Here we fill in the steps that lead from Eq. (44) to Eqs. (48), (49) and (50). (a) Let $\tilde{g}(p)$ be the Fourier transform of $g(w)$. Show that the Fourier transform of $g(w)\delta(w)$ is a constant given by

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi}\tilde{g}(p). \tag{C2}$$

Thus it follows that the Fourier transform of $-\delta(w)\nabla^2\phi(x,y,z,w)$ is $k^2\tilde{f}(\mathbf{k})$ where

$$\tilde{f}(\mathbf{k}) = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\,\tilde{\phi}(\mathbf{k},p) \tag{C3}$$

and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. (b) Use the result of part (a) to rewrite Eq. (44) in Fourier space. You should obtain

$$\tilde{\phi}(\mathbf{k},p) + \frac{\ell k^2}{k^2+p^2}\tilde{f}(\mathbf{k}) = \frac{\lambda}{k^2+p^2}. \tag{C4}$$

(c) Use Eqs. (C3) and (C4) to determine $\tilde{f}$. You should obtain

$$\tilde{f}(\mathbf{k}) = \frac{\lambda}{2k}\frac{1}{\left(1+\frac{1}{2}\ell k\right)}. \tag{C5}$$

(d) Show that it is sufficient to know $\tilde{f}$ to determine $\phi(\mathbf{r},0)$. One does not need $\tilde{\phi}$. You should find

$$\phi(\mathbf{r},0) = \int \frac{d\mathbf{k}}{(2\pi)^3}\tilde{f}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r}). \tag{C6}$$

(e) The angular integrals in Eq. (C6) can be exactly evaluated. Perform the angular integration. You should obtain

$$\phi(\mathbf{r},0) = \frac{\lambda}{4\pi^2 r}\int_0^{\infty} dk\,\frac{\sin(kr)}{\left(1+\frac{1}{2}\ell k\right)}. \tag{C7}$$

This is the exact expression for the potential of a point source in the DGP model. It can be rewritten in terms of rather obscure special functions (the cosine-integral and sine-integral functions) but those expressions are not especially edifying. (f) Verify that Eq. (C7) matches Eqs. (49) and (50) in the appropriate limits. Useful asymptotic formulae:

$$\mathcal{I}(\alpha) = \int_0^{\infty} dx\,\frac{\sin x}{1+\alpha x} \tag{C8}$$

has the asymptotic behavior $\mathcal{I}(\alpha) \approx 1 - 2!\alpha^2 + 4!\alpha^4 - 6!\alpha^6 + \ldots$ for small α and $\mathcal{I}(\alpha) \approx \pi/2\alpha + [\gamma_E - 1 + \ln(1/\alpha)](1/\alpha^2) + \ldots$ for large α. Here $\gamma_E = 0.577216\ldots$ is Euler's constant.

Problem 5. RS1 Kaluza-Klein mode analysis. (a) Verify that the zero-mode solution $\phi = g(\zeta,\mathbf{r};\mathbf{k})\exp(i\omega t)$ satisfies the wave equation (21) as well as the boundary condition $\partial\phi/\partial\zeta = 0$ at both $\zeta_l = \gamma^{-1}$ (left brane) and $\zeta_r = \gamma^{-1}\exp(\gamma\ell)$ (right brane). Here g is given by Eq. (29). (b) The Kaluza-Klein mode $\phi = f(\zeta,\mathbf{r};\mathbf{k})\exp(i\omega t)$, with f given by Eq. (30) already satisfies the wave equation (21) as well as the boundary condition on the left brane. Impose the boundary condition $\partial\phi/\partial\zeta = 0$ on the right brane, $\zeta = \gamma^{-1}\exp(\gamma\ell)$ to obtain the quantization condition that must be satisfied by the allowed values of μ. You should find

$$Y_1\left(\frac{\mu}{\gamma}\right)J_1\left(\frac{\mu}{\gamma}e^{\gamma\ell}\right)=J_1\left(\frac{\mu}{\gamma}\right)Y_1\left(\frac{\mu}{\gamma}e^{\gamma\ell}\right). \tag{C9}$$

(c) Simplify Eq. (C9) assuming that $(\mu/\gamma)e^{\gamma\ell} \gg 1$ and $\mu/\gamma \ll 1$. Use the resulting expression to determine approximate quantized values of μ. You should obtain

$$\cot\left(\frac{\mu}{\gamma}e^{\gamma\ell}-\frac{3}{4}\pi\right)\approx 0. \tag{C10}$$

for the approximate quantization condition and $\mu_n = \gamma\exp(-\gamma\ell)(n\pi+\frac{\pi}{4})$ with $n = 0, 1, 2, \ldots$ for the quantized μ values. Rigorously the approximations made in this part are valid only for large n but in fact these μ values are surprisingly accurate for all values of n.

Problem 6. Klein-Gordon field on RS1 right brane. (a) Let us work with the original co-ordinates (t, x, y, z, w) on Randall-Sundrum space-time. In these co-ordinates the space-time interval is given by Eq. (19). As a prelude write the Klein-Gordon equation $\Box^2\phi + \mu^2\phi = 0$ for a scalar field ϕ that lives in the bulk. (b) Now let us consider scalar field that is confined to the right brane. This scalar field is meant to represent a standard model particle. The space-time interval between neighboring points on the right brane is given by

$$ds^2 = e^{-2\gamma\ell}(dt^2 - dx^2 - dy^2 - dz^2). \tag{C11}$$

Use Eq. (C11) to write the Klein-Gordon equation $\Box_R^2\xi + m^2\xi = 0$ for a field ξ confined to the right brane. You should obtain

$$e^{2\gamma\ell}\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}\right)\xi + m^2\xi = 0. \tag{C12}$$

If we introduce the rescaled field $\overline{\xi} = \xi\exp(2\gamma\ell)$ we can bring the derivative terms in Eq. (C12) to the canonical form of a Klein-Gordon equation with mass parameter $me^{-\gamma\ell}$. Choosing $m\hbar/c$ to be of order a Planck mass and choosing $\gamma\ell \approx$35-40 we can arrange for standard model particles to have a mass of order 1 TeV, the electroweak scale, even though the fundamental scale in the RS1 model is the Planck scale.

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[1] A full course in differential geometry derives the expression for a Laplacian by first generalizing the notions of scalars, vectors and tensors to curved spaces, developing the notion of a covariant derivative, and then defining the Laplacian as the scalar obtained by contracting a second covariant derivative of a scalar field.