UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in: AST4320 — Cosmology and Extragalactic Astronomy

Day of examination: Tuesday 7. October 2014

Examination hours: 15.00 – 18.00

This problem set consists of 6 pages.

Appendices: None

Permitted aids: Formula book by Rottmann.
Calculator that meets standard requirements by university.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 Jeans Mass (weight 20%)

a Explain what the Jeans mass/length is, and how it is relevant for the formation of structure in the Universe.

Pressure forces prevent perturbations smaller than the Jeans length from collapsing under the influence of gravity, whole perturbations larger than the Jeans length collapse. Th Jeans length thus marks the transition between these two regimes. The Jeans mass $M_{\rm J}$ is related to the Jeans length simply through $M_{\rm J} = \frac{4\pi}{3}\rho\pi(\lambda_{\rm J}/2)^3$ (note that the factor of '2' may be omitted); The Jeans length plays a key role in regulating the growth of baryonic perturbations.

b In the lecture we derived the so-called 'Virial Theorem' which states that (the average over time) of the total kinetic energy, K, of a stable system consisting of N particles, bound by gravitational forces is related to the the total potential energy, U, via $2K = |U|$. Briefly discuss qualitatively what would happen if $2K > |U|$ and what would happen if $2K < |U|$.

When $2K > |U|$ the kinetic term 'wins' over gravity, and the system expands. The system contracts when $2K < |U|$.

c Use the discussion above to derive an expression for the Jeans length. You can assume that the total kinetic energy per particle is given by $E = \frac{3}{2}k_{\rm B}T$, in which T denotes the temperature of the gas, and $k_{\rm B} = 1.38 \times 10^{-16}$ erg K^{-1} is Boltzmann's constant.

The total kinetic energy (times 2) is $2K = 3Nk_{\rm B}T$. The total binding energy (up to a constant) is $U = \frac{GM^2}{R}$. The total number of particles is $N = M/m$, where m denotes the mass per particle. If we set equal

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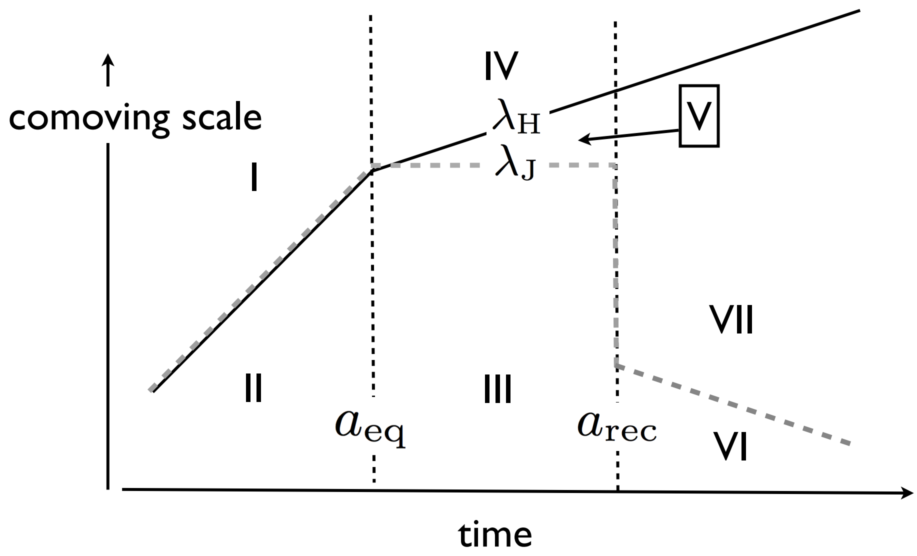
$2K = |U|$, the this corresponds to

$$3Nk_bT = \frac{GM^2}{R} \Rightarrow \frac{3k_bT}{m} = \frac{GM}{R} \Rightarrow \frac{3k_bT}{m} = \frac{G\frac{4\pi}{3}\rho R^3}{R} \Rightarrow \tag{1}$$
$$R^2 = \frac{9k_bT}{4\pi G\rho m} \Rightarrow R_J \equiv \sqrt{\frac{9k_b}{4\pi Gm}}\frac{T^{1/2}}{\rho^{1/2}}$$
$$\Rightarrow M_J = \frac{4\pi}{3}\rho\pi(R_J/2)^3 = C\frac{T^{3/2}}{\rho^{1/2}},$$

where $C = \frac{4\pi}{3}\left(\frac{9k_B}{4\pi Gm}\right)^{3/2}$ is a numerical constant. Precise numerical values (e.g. '9' and '4') will differ when you adopted for example that $U = 3GM^2/[5R]$ (as appropriated for a uniform medium), and the precise definition of the Jeans mass. The only thing that is important here is that the factors of k_b, G and m are in the right location here.

Problem 2 Growth of Structure (weight 25%)

The Figure below shows a diagram that contains the age of the Universe (phrased differently: the time since the Big Bang) on the x-axis, and the comoving scale of a perturbation on the $y-$axis. Also shown in this Figure are the horizon scale λ_H (*solid line*) and the Jeans length λ_J (*dashed line*).



a Discuss the time-dependence of the horizon size λ_H.
$\lambda_H \equiv a(t)\int_0^t \frac{cdt'}{a(t')}$. During radiation domination $a(t) = kt^{1/2}$, in which case $\lambda_H \propto t \propto a^2$. During matter domination $a(t) \propto t^{2/3}$, and $\lambda_H \propto t^{2/3} \propto a$.

b Like above, explain the time-dependence of the Jeans length λ_J. $\lambda_J \propto T^{3/2}\rho^{-1/2}$. **1.** [This question may be too tricky!; The derived

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expression for Jeans mass does not apply!] Prior to matter radiation equality $\lambda_{\rm J} \propto c_{\rm s}^2 \rho^{-1/2}$. During radiation domination the sound speed is $c/\sqrt{3}$, and $\rho \propto a^{-4}$. Hence $\lambda_{\rm J} \propto a^2$.. **2.** During matter domination - but prior to decoupling - $T \propto a^{-1}$, $\rho_{\rm m} \propto a^{-3}$, i.e. $\lambda_{\rm J}$ = constant. **3.** During recombination, the sound speed drops by 5 orders of magnitude. As a result, the Jeans length drops by orders of magnitude. **4.** Following recombination, $T \propto a^{-2}$, and $\rho_{\rm m} \propto a^{-3}$, i.e. $\lambda_{\rm J} \propto a^{-3/2}$.

c Explain the expected time evolution of both the dark matter and the baryons in the 7 places (marked 'I', 'II', 'III', ..., and 'VII') in the diagram. For example, region 'I' corresponds to a perturbation before matter-radiation equality (which occurs at '$a_{\rm eq}$'), and that exceeds the horizon scale at that epoch.

I. Perturbations exceed horizon size, $\delta \propto a^2$ for both dark matter & baryons
II. Perturbations smaller than horizon size, during radiation domination: $\delta_{\rm DM} \sim$ constant (Meszaros effect, more accurately $\delta_{\rm DM} \propto \log a$). Baryonic perturbations smaller than Jeans length, and $\delta_{\rm b}$ = constant.
III. Perturbations smaller than horizon size during matter domination, $\delta_{\rm DM} \propto a$; $\delta_{\rm b}$ = constant (Jeans suppression, does not apply to dark matter)
IV. Perturbations larger than horizon size during matter domination, both dark matter and baryons grow as $\delta \propto a$.
V. Perturbations smaller than horizon size, but larger than Jeans length. Perturbations grow as in IV
VI. Perturbations smaller than both Jeans & horizon size. Baryonic perturbations do not grow ($\delta_{\rm b} = constant$), dark matter perturbations grow as $\delta_{\rm DM} \propto a$.
VII. Both baryonic and dark matter perturbations grow as $\delta_{\rm DM} \propto \delta_{\rm b} \propto a$.

Problem 3 Linear Growth of Structure with Λ (weight 15%)

In the standard cosmological model the matter density decreases as $\rho_{\rm m} \propto a^{-3}$, while the energy density in the vacuum is a constant ($\rho_\Lambda = C$, where C is a constant). We therefore expect the relative contribution of matter to the Universal energy density to decrease with cosmic time.

a Derive the expected time evolution of the mean matter density in the Universe, under the assumption that the cosmological constant dominates the Universal energy density and the expansion of the Universe (i.e. $\rho_\Lambda \gg \rho_{\rm m}$).

First, not that $\rho_{\rm m} \propto a^{-3}$. Next obtain time denWhen $\rho_\Lambda \gg \rho_{\rm m}$, then $H = \frac{\dot{a}}{a} = [8\pi G\rho_\Lambda/3]^{1/2}$ = Constant$\equiv H_0$. In other words, $\frac{da}{a} = H_0 dt$, and $a = \exp(H_0 t)$. We therefore have $\rho_{\rm m} \propto a^{-3} \propto \exp[-3H_0 t]$.

b Under the same assumption, derive the expected time evolution of a density perturbation δ, using that $\ddot{\delta} + 2H\dot{\delta} = 4\pi G\rho_{\rm m}\delta$ in which

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$H \equiv \dot{a}/a = [8\pi G\rho_{\rm tot}/3]^{1/2}$. Comment on this result. Divide left and right side of the differential equation by H^2. We are left with

$$\frac{\ddot{\delta}}{H^2} + \frac{2}{H}\dot{\delta} = \frac{12\delta\rho_{\rm m}}{8\rho_{\rm tot}} = \frac{12\delta\rho_{\rm m}}{8[\rho_{\rm m}+\rho_\Lambda]} \underset{\rho_\Lambda \gg \rho_{\rm m}}{\approx} 0. \tag{2}$$

In other words,

$$\ddot{\delta} + 2H\dot{\delta} = 0, \tag{3}$$

where H is a constant. Because H is a constant, we have $\dot{\delta}+2H\delta = C_1$, where C_1 is a constant. The solution is $\delta = A\exp(-2H_0t)+C_1$, where A is an integration constant. This solution states that in a Universe dominated by dark energy, over densities do not grow, but instead are damped exponentially until they reach $\delta = C_1$, in which C_1 is an integration constant. In the limit that $t\rightarrow\infty$, we expect all density fluctuations to be washed out and $C_1 = 0$.

Problem 4 The Two-Point Correlation Function (weight 15%)

a Explain what the galaxy 2-point correlation function $\xi(r)$ is.

$\xi(r)$ is the *excess* probability -over random - of finding two galaxies in two infinitesimal small volume elements separated by a distance r.

b Explain what the acoustic peak is, and (briefly) describe the physics of the formation of this peak.

The acoustic peak is a feature in the two-point function $\xi(r)$ of mass and galaxies. This feature lies at the 'sound-horizon' at the redshift where recombination happened. Consider a point-like perturbation in density at $t = 0$. The gravitational potential of this perturbation attracts nearby dark matter, and causes it to flow towards it. In contrast, the high radiation pressure in the photon-baryon fluid generates pressure/sound waves that propagate outward at $c/\sqrt{3}$. This sound wave - which corresponds to an overdensity in the baryons - propagates at $c/\sqrt{3}$ until the Universe recombines. At this point, the pressure wave stalls. The overdensity in baryons now gravitationally attracts dark matter, and the acoustic peak is imprinted in the two point function.

c Explain the difference between 'real space' and 'redshift space'.

The location of an astronomical object is our Universe fully determined by 3 coordinates: 2 sky coordinates, and 1 'line-of-sight' coordinate that measures the distance to that object. 'Real' space refers to ordinary space. 'Redshift' space describes the space in which the line-of-sight position of an astronomical object is inferred entirely from its observed redshift (and therefore ignores the possibility that peculiar velocities can contribute to the observed redshift).

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d Sketch the shape of the contours of the galaxy 2-pt function in redshift space (in 2D). Highlight three regimes: (*i*) large r where peculiar velocities can be ignored, (*ii*) intermediate r where we cannot ignore them, and (*iii*) small r where peculiar velocities can be very large.

We plot the two-point function as a function of 2 coordinates: (*1*) the line-of-sight coordinate $r_I I$, and (*2*) the transverse coordinate $r_\perp$ .The two-point function $\xi(r_{||}, r_\perp)$ has the following shape: *i* at large distances isotropy dictates that contours of constant ξ are circular in the $r_{||} - r_\perp$-plane (Alcock-Paczynski test); (*ii*) at intermediate distances the two-point function is compressed along the $r_{||}$ direction (i.e. elongated along $r_\perp$ direction, i.e. redshift space distortions); (*iii*) at small distances the two-point function is elongated along the r_{II} direction, due to virial motions (Finger-of-God effect)

Problem 5 The Isothermal Sphere (weight 15%)

Consider an isothermal sphere, $\rho = Cr^{-2}$ in which C is a constant.

a Compute the total potential energy U of the sphere. Compare with that of a uniform sphere.

The gravitational potential at r is given by $u(r) = \frac{GM(<r)}{r}$, $M(<r) = 4\pi Cr$, i.e. $u(r) = G4\pi C$. The total potential energy in a shell of matter at $r \pm dr/2$ equals $dU = u(r)4\pi r^2 dr\rho(r) = G(C4\pi)^2 dr$. The total potential energy $U(<r) = \int_0^r G(C4\pi)^2 dr = G(C4\pi)^2 r = \frac{GM^2(<r)}{r}$.
This is the almost the same as what we obtained in the lecture for a uniform density, namely $U = \frac{3GM^2(<r)}{5r}$. The isothermal sphere has a slightly more negative binding energy: it is more bound. This is because the matter is more centrally concentrated, which makes it more difficult to unbind.

b Apply virial theorem and compute the 3D velocity dispersion σ of particles within the cloud.
Use that $2K = |U|$. That is $M\sigma^2 = \frac{GM^2}{r} \Rightarrow \sigma = \sqrt{\frac{GM(<r)}{r}}$.

Problem 6 Non Linear Spherical Collapse (weight 10%)

In the lectures we showed that the non-linear evolution of a spherical density perturbation of radius R (and uniform density) was given by a parameterised solution:

$$R = A(1 - \cos\theta) \tag{4}$$
$$t = B(\theta - \sin\theta),$$

where $A^3 = GMB^2$, in which M denotes the total mass of the perturbation. Compute the maximum infall velocity the moment the sphere 'virializes' (i.e. when R first reaches the virial radius $R_{\rm vir} = 0.5R_{\rm max}$, in which $R_{\rm max}$ is the maximum radius) in two ways:

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a In the first, use the parameterised solutions given above.

$$v = \frac{dR}{dt} = \frac{dR}{d\theta}\left(\frac{dt}{d\theta}\right)^{-1} = \frac{A\sin\theta}{B(1-\cos\theta)}. \tag{5}$$

Virialization happens when $\theta = 3\pi/2$, i.e. $v = \frac{-A}{B}$. Note that $A = R_{\rm vir}$, and $B^2 = \frac{R_{\rm vir}^3}{GM}$. Substituting gives us $v = -\sqrt{GM/R_{\rm vir}}$.

b In the second, use energy considerations.
The total energy of a the outermost mass shell 'dm' at turnaround is $E = K + U = U = \frac{-GM(<R_{\rm max})dm}{R_{\rm max}}$. When this mass shell reaches the virial radius, its new binding energy is $U_{\rm new} = \frac{-GM(<R_{\rm max})dm}{0.5R_{\rm max}} = 2U$. Energy conservation dictates that $K = |U|$, and therefore that $\frac{1}{2}dmv^2 = \frac{-GM(<R_{\rm max})dm}{R_{\rm max}}$. We can cancel out dm on both sides, and replace $R_{\rm max} = 2R_{\rm vir}$ to get $v^2 = GM/R_{\rm vir}$.

c How does the infall velocity compare to the 'circular velocity' of the halo, which is defined as $v_{\rm circ}^2 = GM/R_{\rm vir}$?
It is the same.

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