AST4320 COSMOLOGY AND EXTRAGALACTIC ASTRONOMY ASSIGNMENT 1

Deadline: Friday, September 11

1. Exercises to Support Lecture 1-5.

Exercise 1 Concepts (Brief answers with bullet points are sufficient. No need for lengthy discussions. Try not to read your notes or look at Exercise 2 below when you answer these questions)

(1) How are the three equations describing the dynamics of an ideal fluid in a gravitational field called, and what do they state?

(2) In the lecture we defined the quantities of the unperturbed field as $\rho = \bar{\rho}, \vec{v} = \vec{v}_0, p = p_0$ and $\phi = \phi_0$. How are these quantities defined for a perturbed field?

(3) Briefly outline the main steps necessary to derive the perturbed equations. If there are different steps necessary for the different equations, highlight these differences (Note: there is no need to derive anything, just state the steps).

(4) Using Newtonian perturbation theory, we derived the growth of small perturbation should follow the equation:

$$\frac{d^2\delta}{dt^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{d\delta}{dt} = \delta\left(4\pi G\bar{\rho} - k^2 c_s^2\right)$$

Describe the physical effect of each term: $2\left(\frac{\dot{a}}{a}\right)$, $\delta 4\pi G\bar{\rho}$ and $-\delta k^2 c_s^2$, what are their impact on the structure growth?

(5) Explain briefly what the Jeans length and Jeans mass are, and how they are relevant for the formation of structures in the Universe.

(6) The fluctuations in the Cosmic Microwave Background (CMB) indicates that density perturbation is of order $\delta\rho/\rho \sim 10^{-5}$, explain why dark matter is needed for the formation of large-scale structures, galaxies and galaxy clusters as we observe today.

(7) In lectures we derived the Jeans length using linear perturbation theory, but it can be derived simply by comparing the gravitational force to the pressure force. Consider a portion of gas with unit mass at radius r of a sphere with total mass M and uniform density ρ_0 and pressure p, write down an expression for the outward acceleration due to pressure, and an expression for the inward acceleration from gravitational force. Ignore the constant come out of derivatives or integrations (e.g. $\nabla p = dp/dr \sim p/r$), show that the portion of the gas needs to be at $r > \lambda_J \sim c_s/(G\rho_0)^{1/2}$ to be part of the collapsing sphere.

Exercise 2 Let the unperturbed quantities $\bar{\rho}$ (density), ϕ_0 (gravitational potential), $\mathbf{v}_0$ (velocity) and p_0 (pressure) obey the following equations:

$$\frac{d\bar{\rho}}{dt} + \bar{\rho}\nabla\cdot\mathbf{v}_0 = 0 \tag{1}$$

$$\frac{d\mathbf{v}_0}{dt} = -\frac{1}{\bar{\rho}}\nabla p_0 - \nabla\phi_0$$

$$\nabla^2\phi_0 = 4\pi G\bar{\rho},$$

where $\frac{d}{dt}$ denotes the 'total' derivative, which is defined as $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$.

(1) Using the continuity equation, show that in a Universe that is undergoing Hubble expansion (i.e. $\mathbf{v} = H\mathbf{r}$), that $\bar{\rho}(t) = \bar{\rho}(t = t_0)a^{-3}$, where t_0 denotes the age of the Universe today.

(2) In the lecture I introduced perturbed $\rho \equiv \bar{\rho}+\delta\rho$ (density), $\phi \equiv \phi_0+d\phi$ (gravitational potential), $\mathbf{v} \equiv v_0 + \delta v$ (velocity) and $p_0 + \delta p$ (pressure), and showed how the perturbed overdensity $\delta \equiv \delta\rho/\bar{\rho}$ became:

$$\frac{d\delta}{dt} = -\nabla\cdot\delta\mathrm{v},$$

where $\delta \equiv \delta\rho/\bar{\rho}$.

Derive the expressions for the Perturbed Poisson & Euler equations (see Eq 29 in the lecture notes 1 and 2), by substituting perturbed quantities into them, and simplifying as much as possible.

Exercise 3 In the lecture I sketched how we can get a second order differential equation that describes $\delta(t)$

$$\frac{d^2\delta}{dt^2} + 2\frac{\dot{a}(t)}{a(t)}\frac{d\delta}{dt} = \delta(4\pi G\bar{\rho} - k^2c_\mathrm{s}^2),$$

where $k \equiv 2\pi/\lambda$ denotes the wavenumber of the perturbation.

(1) Use the lecture notes to obtain an expression for the term $\frac{\dot{a}}{a}$ from the Friedmann equations for a cosmology with $(\Omega_\mathrm{m}, \Omega_\Lambda) = (1.0, 0.0)$, $(\Omega_\mathrm{m}, \Omega_\Lambda) = (0.3, 0.7)$, and with $(\Omega_\mathrm{m}, \Omega_\Lambda) = (0.8, 0.2)$.

(2) Insert this expression into the differential equation. Solve and plot the time evolution of $\log\delta(a)$ vs $\log a$ in both cosmologies, by numerically integrating forward in time the diferential equation. For the boundary conditions, you can assuming that $\log\delta = -3.0$ at $\log a = -3.0$, and that the perturbation of interest is much

larger than the Jeans length (i.e. ignore the pressure term $k^2c_s^2$). For the boundary condition for $\dot{\delta}$ at $\log a = -3.0$, assume for simplicity that $\delta \propto a$ at early times, and derive the resulting boundary condition for $\dot{\delta}$ from this.

(3) The growth factor f is defined as $f \equiv \frac{d\ln\delta}{d\ln a}$. Plot f as a function of redshift z.

Exercise 4 In the lecture we derived expressions for the Jeans length & mass. After decoupling (which occurred at $z = 1090$) the temperatures of the cosmic radiation background and the gas evolved differently.

(1) Derive & plot the time-evolution of the gas and radiation temperatures within the range $\log a = -4.0 - 0.0$, assuming that both fluids evolve adiabatically during the expansion of the Universe.

(2) Obtain expressions for the $z-$dependence of the cosmological Jeans mass & length. Compare these to their evolution (and amplitude) before decoupling/recombination.

Exercise 5 We followed the non-linear time-evolution of a spherical overdensity inside an Einstein-de-Sitter [$\Omega_{\rm m} = 1.0$ and $\Omega_\Lambda = 0.0$]. We assumed that this overdensity was confined to a sphere of radius $R(t)$. Birkhoff's law states that this overdensity effectively behaves as a closed-universe with $\Omega_{\rm m} > 1.0$. We showed in the lecture that initially $R(t) \sim a(t)$, where $a(t)$ denotes the scale factor of the background Universe. The acceleration of the radius of the sphere is given by

$$\ddot{R} = -\frac{GM}{R^2},$$

where M is the total enclosed mass, and is therefore $M = \frac{4\pi}{3}\rho_{\rm m,0}R_0^3 = \frac{4\pi}{3}\rho_{\rm m,0}$. Here, $\rho_{\rm m,0}$ and R_0 is the initial mass density and radius of the sphere, respectively. Show that the following parameterised solutions satisfy the above equation.

$$\begin{aligned} R &= A(1-\cos\theta) \\ t &= B(\theta - \sin\theta) \\ A^3 &= GMB^2. \end{aligned}$$

Exercise 6 Using the parametrised solution for the collapse of a spherical top-hat, compute the infall velocity v when material first reaches the virial radius. Phrase your answer in terms of G, M and R.