UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in: AST4320 — Cosmology and Extragalactic Astronomy

Day of examination: Thursday 8. October 2015

Examination hours: 10.00 – 13.00

This problem set consists of 7 pages.

Appendices: None

Permitted aids: Formula book by Rottmann.
Calculator that meets standard requirements by university.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 Gaussian Random Fields and The Two-Point Correlation Function (weight 20%)

a Describe briefly what a Gaussian random field is, and *qualitatively* what the two-point correlation function $\xi(r)$ describes (I have stressed "qualitatively" because you do *not* need to provide with the equation describing a multi-variate Gaussian).

A GRF is a field of variables, $x_1, x_2, x_3, ...$ each of which obey Gaussian statistics (i.e. each x_i is drawn from a Gaussian distribution). The two-point correlation function $\xi(r)$ fully specifies the statistical properties of the field, i.e. it describes the mean and standard deviation of each field variable x_i.

b Explain what the *galaxy* 2-point correlation function $\xi_{\rm gal}(r)$ is, and why it can be different than the $\xi(r)$ describing the matter density field. A qualitative description suffices.
$\xi_{\rm gal}(r)$ denotes the excess probability over random of finding two galaxies separated by r, i.e. the probability of finding two galaxies in two identical volume elements dV separated by r equals $P = dV^2n^2(1+\xi_{\rm gal}(r))$, where ndV denotes the probability of finding a galaxy in a volume element dV.

Discuss why $\xi_{\rm gal}(r) \rightarrow 0$ when $r \rightarrow \infty$.

Total number of galaxy surrounding galaxy 1 is $4\pi\int_0^\infty r^2dr[1+\xi(r)]\bar{n}$, where $\bar{n}$ is the average number of galaxies. If we make $r \rightarrow \infty$ then we must get that the total number of galaxies surrounding galaxy "1" is just the total number of galaxies in the Universe, i.e. $4\pi\int_0^\infty r^2dr[1+\xi(r)]\bar{n} \rightarrow 4\pi\int_0^\infty r^2dr[1+\xi(r)]\bar{n}$

(Continued on page 2.)

$\xi_{\rm gal}(r)$ denotes the excess probability over random of finding two galaxies separated by r, i.e. the probability of finding two galaxies in two identical volume elements dV separated by r equals $P = dV^2 n^2(1+\xi_{\rm gal}(r))$, where ndV denotes the probability of finding a galaxy in a volume element dV.

c Explain the difference between 'real space' and 'redshift space'.
Real space is ordinary space. In redshift the line-of-sight coordinate of an astrophysical object is inferred entirely from its redshift.

Sketch the shape of the contours of the galaxy 2-point correlation function in redshift space in two-dimensions. Highlight three regimes, and briefly discuss existing cosmological tests within these regimes (if they exist): (*i*) large r where peculiar velocities can be ignored, (*ii*) intermediate r where we cannot ignore them, and (*iii*) small r where peculiar velocities can be very large.

(*i* large scales): Two-point correlation function contours should be circular, due to isotropy of galaxy distribution. Circular contours are only obtained if we have the correct cosmology, which gives us a direct test for cosmology known as the Alcock-Paczynski test; (*ii* intermediate scales): infall onto overdense regions causes a flattening/squeezing of the two-point function along the line-of-sight direction. Flattening constrains "growth of structure", (*iii* smallest scales) random motions inside virialized objects give rise to the Finger-of-God effect.

Problem 2 Non Linear Spherical Collapse & Virialization (weight 25%)

The non-linear evolution of a spherical density perturbation of radius R (with uniform density, enclosing a total mass M) was given by a parameterised solution:

$$\begin{aligned} R &= A(1-\cos\theta) \\ t &= B(\theta - \sin\theta), \end{aligned} \tag{1}$$

where $A^3 = GMB^2$, in which M denotes the total mass of the perturbation.

a Assume that the spherical density perturbation "lives" inside an Einstein-de Sitter Universe ($\Omega_{\rm m} \equiv \rho/\rho_{\rm crit} = 1.0, \Omega_\Lambda = 0.0$, where $\rho_{\rm crit} = 3H^2/[8\pi G]$). Use that $H \equiv \frac{\dot{a}}{a} = H_0\sqrt{\Omega_{\rm m} a^{-3}}$ to show that the mean density of the Universe is $\bar{\rho}_{\rm m} = 1/[6\pi G t^2]$.

We know that

$$\bar{\rho} = \bar{\rho}_0 a^{-3} = \Omega_{\rm m}\rho_{\rm crit,0} a^{-3} = \frac{3H_0^2}{8\pi G} a^{-3} \tag{2}$$

and that

$$\frac{\dot{a}}{a} = H = H_0 a^{-3/2} \Rightarrow \dot{a} = H_0 a^{-1/2} \Rightarrow a(t) = (3H_0 t/2)^{2/3} \tag{3}$$

Combining the two gives the answer.

(Continued on page 3.)

b Show that the average density *contrast*, $\delta \equiv \frac{\rho}{\bar{\rho}} - 1$, of the perturbation equals

$$\delta = \frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1-\cos\theta)^3} - 1 \tag{4}$$

The density contrast $\delta \equiv \delta\rho/\rho_m$ equals:

$$\delta_{\rm m} = \frac{\rho_{\rm sphere}}{\rho_{\rm m}} - 1 = 6\pi G t^2 \frac{M}{\frac{4}{3}\pi R^3} - 1 = \tag{5}$$

$$\frac{9GMt^2}{2R^3} - 1 = \frac{9GM}{2}\frac{B^2(\theta-\sin\theta)^2}{A^3(1-\cos\theta)^3} - 1 = \frac{9}{2}\frac{(\theta-\sin\theta)^2}{(1-\cos\theta)^3} - 1.$$

c Show that the virial theorem, $2T = |U|$ (where U $[T]$ denotes the total gravitational binding [kinetic] energy of the perturbation), states that "virialization" occurs when $R_{\rm vir} = A = 0.5R_{\rm max}$, where $R_{\rm max}$ is the radius of maximum expansion.

In the spherical collapse model virialization occurs at $R_{\rm vir} = 0.5R_{\rm max}$. This is because the total energy in particle inside the perturbation is equal to the potential energy of the perturbation at turn-around (because then the kinetic energy is 0). That is,

$$E_{\rm tot} = U(= -\int_0^M dm \frac{GM(<m)dm}{r(m)} = ... =) - \frac{3GM^2}{5R_{\rm max}}. \tag{6}$$

, where $dm = \rho 4\pi r^2 dr$. When the sphere contracts to $R = 0.5R_{\rm max}$ then the total binding energy is $U_{\rm new} = -\frac{6GM^2}{5R_{\rm max}}$. The additional $\Delta U = -\frac{3GM^2}{5R_{\rm max}}$ that the sphere lost has been transformed into kinetic energy T. We therefore have $T = \frac{3GM^2}{5R_{\rm max}}$ and $U_{\rm new} = -\frac{6GM^2}{5R_{\rm max}}$ which satisfies the virial theorem. The radius at which virialization occurs is the 'virial radius', denoted with $R_{\rm vir}$.

d Show that $\delta = 4.5$ at turn-around, and that $\delta \approx 178$ at "virialization", (i.e. that at virialization the collapsed "halo" is ~ 178 more dense than the Universe as a whole at that moment). *Hint:* use that virial equilibrium is said to be truly reached when the parameterised solution indicates that $R \to 0$.

We know that the sphere virialized $R = 0.5R_{\rm max}$. The density of the sphere ρ is therefore $2^3 = 8\times$ larger than it was at turnaround. Similarly, we need to evaluate $\rho_{\rm m}$ at $t_{\rm collapse} = 2\pi B$ while turn-around happened at $t_{\rm turn-around} = \pi B$. That is $t_{\rm collapse} = 2t_{\rm turn-around}$ and $\rho_{\rm m}$ decreased by a factor of $2^2 = 4$ (because $\rho_{\rm m} \propto t^{-2}$). We therefore have

$$\delta^{\rm collapse}_{\rm non-lin} = \frac{\rho^{\rm virialization}_{\rm sphere}}{\rho^{\rm collapse}_{\rm m}} - 1 = \frac{8\rho^{\rm turn-around}_{\rm sphere}}{\frac{1}{4}\rho^{\rm turnaround}_{\rm m}} - 1 = \tag{7}$$

$$32\frac{\rho^{\rm turn-around}_{\rm sphere}}{\rho^{\rm turnaround}_{\rm m}} - 1 = 32(\delta^{\rm turn-around}_{\rm non-lin} + 1) - 1 \sim 32(5.5) - 1 \sim 178,$$

(Continued on page 4.)

where we used that the non-linear overdensity at turn-around was $\delta = 4.5$. An object virializes with ~ 178 times the mean density of the Universe at that moment.

Briefly discuss qualitatively if you expect this number ($\delta \approx 178$) to be larger or smaller in $\Lambda-$CDM.

We expect Delta to be larger in LCDM for three reasons: (i) Since the Universe expands faster with a cosmological constant present, the critical overdensity required for collapse is expected to be larger – and so also the final overdensity (ii) The accelerated expansion of the Universe compared to an Einstein-de-Sitter (EdS) leads to a smaller surrounding density of the sphere. Therefore, the density *contrast* is expected to be greater. (iii) The energy of the cosmological constant can be modelled as an additional potential energy of the sphere. This means the kinetic energy has for the sphere to be in virialization equilibrium. This means the shells have to gain a greater speed, and, therefore, have to fall further inwards compared to the EdS case leading to a greater density contrast. **Each one of these three answers gives full credit!**.

Problem 3 Dynamical Time (weight 25%)

a Consider a stationary, spherical object of mass M. Use the parametrised solution from **Problem 2** to show that it takes a time $t_{\rm coll}$ for this object to collapse into a point, where the collapse time $t_{\rm coll}$

$$t_{\rm coll} = \sqrt{\frac{3\pi}{32G\rho}} \tag{8}$$

A stationary spherical object of mass M can be seen as the point of maximum expansion in the spherical top-hat model. Maximum expansion occurs at $\theta = \pi$, collapse to a point at $\theta = 2\pi$. The total time difference is given by $t_{\rm coll} = t(\theta = 2\pi) - t(\theta = \pi) = \pi B$. We know that $A^3 = GMB^2$ in the parametrised solution. We also know that $2A = R_{\rm max}$, where $R_{\rm max}$ is now the radius of the cloud when it starts collapsing. So we know that $R^3_{\rm max}/8 = GMB^2$, so $B^2 = R^3_{\rm max}/[8GM]$. We also know that the density inside the cloud when it starts to collapse is $\rho = \frac{3M}{4\pi R^3_{\rm max}}$. Eliminate M from the expression for B to get $B^2 = 3/[32G\pi\rho]$. We therefore get $t_{\rm coll} = \pi B = \sqrt{3\pi/32G\rho}$.

b Assume that at the beginning of collapse, sound waves start propagating through the object. The sound-crossing time, $t_{\rm cross}$, denotes the time it takes to propagate from one edge of the sphere to the other. Discuss what happens when the sound-crossing time $t_{\rm cross} \gg t_{\rm coll}$ (i.e. when it takes much longer for a sound wave to cross the perturbation, than for the whole perturbation to collapse). Also discuss what happens when $t_{\rm cross} \ll t_{\rm coll}$.

When $t_{\rm cross} \gg t_{\rm coll}$ sound waves will not arrive in time to inform the other side of the collapse. The cloud can not respond hydrodynamically to the contraction, and collapse is uninterrupted. When $t_{\rm cross} \ll t_{\rm coll}$

(Continued on page 5.)

sound waves reach the other side of the cloud well before significant structural changes happen. The cloud can respond to the collapse, and collapse is prevented.

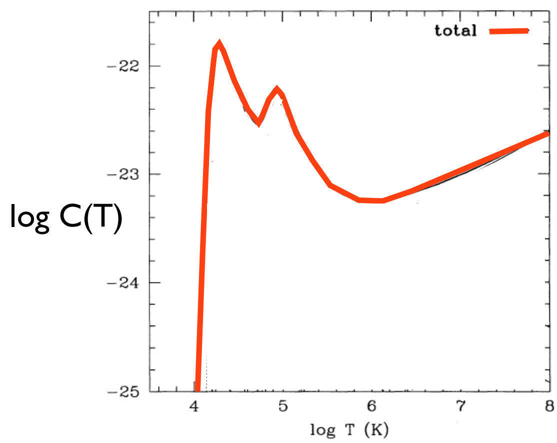
c Show that the condition $t_{\rm cross} = t_{\rm coll}$ translates to a mass M

$$M = \frac{\pi^{5/2}\sqrt{3}}{8\sqrt{32}} \frac{c_{\rm s}^3}{G^{3/2}\rho^{1/2}}. \tag{9}$$

Briefly comment on how this relates to the Jeans Mass.

The distance covered by sound waves over a time $t_{\rm cross}$ is $c_{\rm s}t_{\rm cross} = l_{\rm cross}$. The corresponding mass would be $M = 4\pi\rho l_{\rm cross}^3/3 = 4\pi\rho c_{\rm s}^3 t_{\rm coll}^3/3 = (4\pi/3)c_{\rm s}^3(3\pi/[32G\rho])^{3/2} = \frac{4\times3\times3^{1/2}}{3\times32\times\sqrt{32}} \frac{\pi^{5/2}c_{\rm s}^3}{G^{3/2}\rho^{1/2}}$, and that's that. M *is* the Jeans mass.

Problem 4 Gas Cooling (weight 30%)



a Describe three main cooling processes of an astrophysical gas (*Hint:* we referred to one of the three as "bound-bound"). Bound-bound cooling refers to the process where an electron collides with an atom, which puts the atom in a higher energy state. The atom transitions back down to the lower energy state by a emitting a photon. The energy in the photon is lost from the gas. Bound-free cooling refers to a collision between a high energy electron and an atom. The collision knocks one electron off the atom. The energy it took to release the electron is lost from the gas. Free-free cooling is due to charged particles that accelerate/decelerate each other, causing them to emit photons which cool the gas.

b The Figure shows the cooling curve $\log_{10} C(T)$, where $C(T)$ has units erg cm^3 s^{-1}. Briefly describe qualitatively the processes that determine the shape this function.

(Continued on page 6.)

Peak rises beyond 10^4 because collisional excitation of hydrogen becomes efficient (bound-bound cooling by hydrogen). This cooling mechanism is shut off once hydrogen is ionised away. This causes the first dip at $\sim 2-3\times 10^4$ K. The second peak is again due to bound-bound cooling, this time (ionized) Helium. This cooling process becomes less efficient once Helium is (doubly) ionised away (i.e. there is no Helium left that can cool the gas). This causes the second dip. Then free-free cooling kicks in, and keeps rising with T.

c A perturbation of mass M virializes at $z=2$. Use the expression from **Problem 3a** to compute the collapse time of the virialized halo. Assume an Einstein-de Sitter Universe for the background cosmology, with a present-day Hubble parameter $H_0 = 2.27\times 10^{-18}$ s^{-1}.

We know that $t_{\rm coll} = \sqrt{\frac{3\pi}{32G\rho}}$. At virialization $\rho = 178\bar{\rho}(z=2) = 178(1+z)^3\bar{\rho}(z=0) = 178\times 27\times 3H_0^2/8\pi G \sim 4.4\times 10^{-26}$ g cm^{-3}. The collapse time is then 10^{16} s $\sim 3\times 10^8$ yr.

d The cooling time is given by $t_{\rm cool} = \frac{3k_bT}{2nC(T)}$, where $k_{\rm B} = 1.38\times 10^{-16}$ erg K^{-1} is Boltzmann's constant, n denotes the number density of gas. Estimate the cooling time for gas inside the halo using that the "virial temperature" is given by

$$T_{\rm vir} \equiv \frac{m_{\rm p}v_{\rm circ}^2}{2k_{\rm B}} \approx 2\times 10^4\ {\rm K}\left(\frac{M}{10^8\ M_\odot}\right)^{2/3}\left(\frac{1+z}{10}\right). \tag{10}$$

How does $t_{\rm cool}$ compare to $t_{\rm coll}$? What does this imply for the future evolution of the gas?

Collapse time did not depend on mass. Cooling time does. Pick a mass. Compute $T_{\rm vir}$. Compute $C(T)$ by reading off from the figure, and use the equation to compute the cooling time. If it is less than the collapse time, your halo was free to collapse. If not, it would not collapse freely.

e In the lecture we showed that the density profile of "isothermal" pressure less dark matter is $\rho(r) = Ar^{-2}$, where A is a numerical constant. Show that *isothermal* gas in hydrostatic equilibrium with the dark matter potential also settles to this density profile. The equation describing hydrostatic equilibrium is given by

$$\frac{dP}{dr} = -\frac{GM(<r)\rho_{\rm gas}}{r^2} \tag{11}$$

If $\rho(r) = Ar^{-2}$ and $\rho_{\rm gas} = Cr^{-2}$, then $M(r) = 4\pi\int_0^r x^2dx[\rho(x)+\rho_{\rm gas}(r)] = 4\pi[A+C]r$, and $P = nkT = \rho_{\rm gas}kT/[\mu m_{\rm p}] \equiv B\rho_{\rm gas}(r)$, where B is a new constant that absorbed all constants. The equation becomes

$$\frac{BCdr^{-2}}{dr} = -\frac{G4\pi[A+C]r\times Cr^{-2}}{r^2} \rightarrow \frac{-2BC}{r^3} = -\frac{4G\pi[A+C]C}{r^3} \tag{12}$$

, which is true when $B = 2G\pi[A+C]$. We decided not to count this, unless you solved it correctly (in which case the credit would be extra).

(Continued on page 7.)

f Describe how the "cooling time", and dynamical time evolve as a function of radius, and briefly comment on what this implies for gas cooling and collapse throughout the halo.

Cooling time scales as $\rho^{-1} \propto r^2$, and the collapse time scales as $\rho^{-1/2} \propto r$. The condition that cooling time equals dynamical time for efficient collapse now becomes a function of radius. Gas cooling can become relatively more efficient at small radii where the density is higher. Cooling & galaxy formation may start in the centre of the halo, while it may be prohibited at larger radii.

SLUTT