UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in: AST4320 — Cosmology and Extragalactic Astronomy

Day of examination: Monday 8. October 2018

Examination hours: 9:00 – 12:00

This problem set consists of 13 pages.

Appendices: None

Permitted aids: Formula book by Rottmann.
Calculator that meets standard requirements by university.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Preface: some useful relations and constants

- $H(a) = \frac{\dot{a}}{a} = H_0\sqrt{\Omega_{\rm m}a^{-3} + \Omega_\Lambda + \Omega_{\rm rad}a^{-4}}$, where H_0 is the Hubble parameter today, Ω's denote the density parameters in matter, dark energy, and radiation.
- $a = \frac{1}{1+z}$.
- $\Omega_{\rm m} = \frac{\rho_{\rm m}}{\rho_{\rm crit}}$, $\Omega_{\rm b} = \frac{\rho_{\rm b}}{\rho_{\rm crit}}$, where $\rho_{\rm m}$ is the mean matter density of the Universe, $\rho_{\rm b}$ is the mean baryon density of the Universe, $\rho_{\rm crit} = \frac{3H^2}{8\pi G}$
- Einstein-de Sitter Universe: $\Omega_{\rm m} = 1.0$, $\Omega_\Lambda = 0.0$.
- Planck cosmology parameters: $h = 0.678$, $\Omega_{\rm m} = 0.308$, $\Omega_\Lambda = 0.692$, $\Omega_{\rm b} = 0.048$.
- Hubble parameter at z = 0: $H_0 = 67.8$ km s^{-1} Mpc^{-1} = 2.20×10^{-18}s^{-1}, critical density at z = 0: $\rho_{\rm crit,0} = 8.64 \times 10^{-30}$g cm^{-3}
- Units and constants: $1M_\odot = 2 \times 10^{33}$ g, 1pc $= 3.086 \times 10^{18}$ cm, 1 year $= 3.15 \times 10^7$ s, gravitational constant $G = 6.67 \times 10^{-8}$ cm^3 g^{-1}s^{-2}, mass of a hydrogen atom $m_H = 1.67 \times 10^{-24}$ g, Boltzmann's constant $k_B = 1.38 \times 10^{-16}$ cm^{-2}g s^{-2}K^{-1}, speed of light $c = 3.0 \times 10^{10}$cm s^{-1}.

Please show important intermediate steps in your answers, but try to be concise. Lengthy descriptions are never required.

Problem 1 Jeans Length, Jeans Mass and Fragmentation (weight 25%)

a (3 points) In lectures we derived the Jeans length $\lambda_J = c_s\sqrt{\frac{\pi}{G\rho}}$ from Newtonian linear perturbation theory. Write down an expression for

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the Jeans mass (M_J). Explain briefly what the Jeans length and Jeans mass are, and how they are relevant for the formation of structure in the Universe.

Solution: Physically, we can understand the existence of this critical length scale by considering the balance between gravity and thermal pressure. If we take a small part of the pre- galactic gas and perturb it adiabatically, its density and temperature will increase. It will therefore be over-pressured relative to the surrounding gas, and the pressure gradients that we have created will try to smooth out the perturbation. Our perturbation will survive and grow only if its self-gravity, i.e. the gravitational force acting on the perturbation due to the perturbations own mass is larger than the pressure forces acting to smooth out the perturbation. It should be plain that for very small perturbations, with very low masses, pressure will overcome gravity. Similarly, it should be clear that on very large scales, gravity will win. There must therefore be some intermediate scale at which we go from being pressure-dominated to being gravity-dominated. This critical scale is just the Jeans length:

$$\lambda = c_s \sqrt{\frac{\pi}{G\rho}}$$

And the Jeans mass is:

$$M_J = \frac{4\pi}{3}\rho(\lambda/2)^3 = \frac{1}{6}\pi^{5/2}c_s^3 G^{-3/2}\rho^{-1/2}$$

Grading: 1.5 point for expression of M_J and 1.5 point for the explanation.

b (4 points) Briefly describe what recombination is, what happens to the baryonic matter (i.e. gas) and radiation before and after recombination, and how it affects Jeans length, Jeans Mass and structure formation. A *Qualitative* description is sufficient.

Solution: Before recombination, the baryons and photons where coupled together via Thompson scattering of free electrons. Recombination occurs when the universe have cooled enough ($T \propto a^{-1}$) for it to be energetically favorable for the free electrons and ions to combine into neutral hydrogen. This causes a massive drop in free electrons which consequentially means that the photons are no longer coupled to the baryons. The sound speed of the gas will thus drop rapidly and follow the classical ideal gas law ($c_s^2 = \frac{kT}{\mu m_H}$) instead of the coupled baryon fluid (see eq 1.). The Jeans length and mass decrease significantly after the recombination. **Grading: 2 points for explaining recombination, 2 points for describing the decrease of the sound speed, Jeans length and mass**.

c (9 points) After the Big Bang, the energy density of the Universe is initially dominated by radiation, then matter starts to dominate. As we discussed in the lectures, this occurs at the radiation - matter equality

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($z_{eq} \sim 3300$), and it is earlier than recombination ($z_{rec} \sim 1100$). Derive expressions for the Jeans length and Jeans Mass as a function of the scale factor a for 1) radiation dominated era; 2) matter dominated era, before recombination; 3) matter dominated era, after recombination; *Hint:* the sound speed of the epoch in 2) is given by:

$$c_s = \frac{c}{\sqrt{3}}(\frac{3\rho_m}{4\rho_\gamma} + 1)^{-1/2}, \tag{1}$$

where c is the speed of light, ρ_m and ρ_γ are the average density of matter and radiation, respectively. You can assume $\frac{\rho_m}{\rho_\gamma} \gg 1$ for matter domination.

Solution: Before matter-radiation equilibrium the universe was radiation/relativistic dominated and $\rho \propto a^{-4}$ and $c_s = \frac{c}{\sqrt{3}}$. We plug this into the equations from 1a).

$$\lambda \propto a^2$$

$$M_J \propto a^2 \qquad t \leq t_{eq}$$

After matter-radiation equilibrium but before decoupling, we have a coupled photon-baryon fluid which is matter dominated. The correct speed of sound to use for this fluid is:

$$c_s = \frac{c}{\sqrt{3}}\left(\frac{3}{4}\frac{\bar{\rho}_m}{\bar{\rho}_\gamma} + 1\right)^{-1/2}$$

As we can see, if we have radiation dominated we get back what we had before, however if its matter dominated we will have the following relation:

$$c_s \propto a^{-1/2}$$

Together with $\rho \propto a^{-3}$ we have the following Jeans length relation:

$$\lambda \propto a$$

$$M_J \propto a^0 = constant \qquad t_{eq} \leq t \leq t_{rec}$$

After recombination the radiation decouples and the speed of sound will follow the ideal gas law:

$$c_s = \sqrt{\frac{k_b T}{\mu m_p}}$$

$$\lambda = \sqrt{\frac{k_b T_0}{\mu m_p}}\sqrt{\frac{\pi}{G\rho_0}}a^{1/2}$$

For non-relativistic gas, $T \propto a^{-2}$. Calculating the Jeans mass we can see that the minimum mass is magnitudes larger before recombination then after.

$$\lambda \propto a^{1/2}$$

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$$M_J \propto a^{3/2} \qquad t \geq t_{rec}$$

Grading: 3 points for each era, just derive how λ_J and M_J depend on a (i.e. the exponents) should be sufficient. Partially credits are given if you did not express $T \propto a^{-2}$ in part 3).

d (4 points) For non-relativistic ideal gas, the temperature and density follows the relationship $T \propto \rho^{\gamma-1}$. Here γ is the adiabatic index. Show that the Jeans length and mass follow the relationships:

$$\lambda_J \propto \rho^{\frac{1}{2}(\gamma-2)} \tag{2}$$
$$M_J \propto \rho^{\frac{3}{2}(\gamma-\frac{4}{3})}$$

Solution: We know that the jeans length and mass are proportional to:

$$\lambda \propto c_s \rho^{-1/2}$$

$$M_J \propto \lambda^3 \rho$$

The speed of sound of an non-relativistic ideal gas is given in previous exercise:

$$c_s \propto T^{1/2} \propto \rho^{\frac{1}{2}(\gamma-1)}$$

Which by inserting into the jeans length and mass gives the requested relations. **Grading: 2 points for λ_J, 2 points for M_J.**

e (5 points) When gas in galaxies (in the form of giant molecular clouds) exceeds the Jeans mass and collapse, it usually does not collapse into a single star. What instead happens is that sub regions within the cloud will start to **locally** collapse and fragment. This fragmentation is what eventually leads to formation of star clusters with individual stars much less massive than the cloud. This fragmentation is directly linked to the Jeans mass, explain why and for what values of γ fragmentation can occur in a collapsing cloud.

Solution: During a collapse the density increases, how the Jeans mass will change depends on the adiabatic index γ. For a $\gamma < 4/3$ the Jeans mass will decrease during collapse, which will allow smaller regions within the cloud to surpass their Jeans mass and collapse on their own, leading to fragmentation. One could also add that sub regions will have a shorter t_{ff} which means that these sub-regions collapse faster. For $\gamma > 4/3$ the Jeans mass will increase and thus we will have no fragmentation.

Grading: indicating $\gamma < 4/3$ for fragmentation and $\gamma > 4/3$ for non-fragmentation 2 points, explain why 3 points.

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Problem 2 Structure Growth in the Relativistic Case (weight 23%)

During the radiation-dominated era, the primordial perturbations are in a radiation-dominated, ultrarelativistic plasma. In this case, the energy-momentum tensor must be used in the fluid equations, and the relativistic equation of state $p = \frac{1}{3}\rho c^2$ (where c is the speed of light) should be used. Consider perturbations on scales much smaller than the horizon scale. It is shown in an article by Coles and Lucchin (1995) that the overdensity growth in the relativistic plasma, $\delta \equiv \delta\rho/\bar{\rho}$ follows the equation below:

$$\frac{d^2\delta}{dt^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{d\delta}{dt} = \delta\left(32\pi G\bar{\rho}/3 - k^2c_s^2\right) \tag{3}$$

This is very similar to the evolution of overdensity in non-relativistic case.

a (4 points) From Equation (3), derive an expression for the Jeans length ($\lambda_J = \frac{2\pi}{k_J}$) as a function of the speed of light c and average density $\bar{\rho}$ for relativistic plasma.

Solution: We know that the jeans length represent the scale in which we go from being pressure-dominated to being gravity-dominated. Looking at equation 3 we can see that this occurs when the right expression is equal to zero (if right term is positive we have increase in overdensity otherwise we have oscillations in the overdensity):

$$32\pi G\bar{\rho}/3 = k_J^2 c_s^2 \rightarrow$$

$$\rightarrow \lambda_J = \frac{2\pi}{k_J} = c\sqrt{\frac{\pi}{8G\rho}}$$

We are in radiation-dominated era so we have used $c_s = c/\sqrt{3}$

Grading: 2 points for indicating $32\pi G\bar{\rho}/3 = k_J^2 c_s^2$ for Jeans wavenumber, 3 points for deriving the correct expression.

b (4 points) Using the time evolution of the scale factor in a flat Universe:

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = H_0^2\left[\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_\Lambda\right] \tag{4}$$

show that during the radiation domination (i.e. $\Omega_r = 1$, $\Omega_m = 0$ and $\Omega_\Lambda = 0$), the scale factor grows in time as:

$$a(t) = (2H_0t)^{1/2} \tag{5}$$

Solution: From the Friedmann equation:

$$H = \frac{\dot{a}}{a} = H_0 a^{-2} \rightarrow a = (2H_0t)^{1/2}$$

Grading: show the correct Friedmann equation 2 points, integration 2 points.

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c (11 points) If the perturbation scale is much larger than the Jeans length, the second term in the bracket of Equation (3) can be neglected. In this case the density perturbation will grow. Show that during the radiation-dominated era, the overdensity growth is also algebraic, and

$$\delta \equiv \frac{\delta\rho}{\rho} \propto a^2 \tag{6}$$

(*Hint*: in radiation domination the average density of the plasma $\bar{\rho} = \bar{\rho}_0\ a^{-4}$ where $\bar{\rho}_0$ is the same as critical density of the Universe today.)

Solution: One can solve it analytically, similar to what we did in assignment 1. However a much easier way is just to use the information of the previous two exercises. We know that the jeans length is very large, such that ($32\pi G\bar{\rho}/3 >> k_J^2 c_s^2$). We also have an analytical time expression for a (from equation 5.), which means that we know that $\frac{\dot{a}}{a} = (2t)^{-1}$. We also know that we are in radiation-dominated era, which means that $\bar{\rho} = \bar{\rho}_{crit,0}a^{-4} = (\frac{3H_0^2}{8\pi G})((2H_0 t)^{-2}$ (given from hint also). Plugging this info into equation 3 gives:

$$\frac{d^2\delta}{dt^2} + t^{-1}\frac{d\delta}{dt} = t^{-2}\delta$$

Here we can of course solve the differential equation in anyway we seem fit. We do however know that $\delta \propto a^2 \propto t$, so we can just test that in the differential equation, which we can see satisfies the equation. The second solution to the differential equation can also very easily be seen from just looking at the equation $\delta \propto t^{-1}$(this term will quickly be damped and not play a role).

Grading: 3 points for express $\frac{\dot{a}}{a}$ as function of t, 3 points for express $\bar{\rho}$ as a function of t, 2 points for re-write the equation as function of time only, and 3 points for solving it correctly.

d (4 points) Derive an expression of the particle horizon size in the radiation dominated era as a function of the scale factor a. Discuss why in reality the condition described in **Problem 2c** never happens, i.e., one can hardly have a perturbation in the baryon-photon plasma smaller than the horizon but much larger than the Jeans length.

Solution: The particle horizon is the maximum distance in which particles could have traveled to the observer at time t from the beginning of the universe. Or in other words, the region which have been in casual contact with the observer (however the region which the observer can casually effect in the future is set by the cosmic horizon). The general formula to calculate the particle horizon at time t is given by:

$$r_{\rm H}(t) = a(t)\int_0^t c\ a(\tau)^{-1}d\tau.$$

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We have from equation 5 the relation between a and time. Inserting and solving we get:

$$r_H = 2ct = \frac{c}{H_0}a^2$$

Comparing the horizon with the Jeans expression from a)

$$\frac{\lambda_J}{r_H} = \sqrt{\frac{H_0^2\pi}{8G\rho_{crit,0}}} = \sqrt{\frac{\pi^2}{3}} \approx \sqrt{\pi}$$

Which means that the Jeans length is always larger than the horizon.

Grading: 2 points for deriving the particle horizon size, 2 points for comparing with the Jeans length and indicating that the horizon size is larger.

Problem 3 Gas Virialization, Cooling and Galaxy Formation (weight 27%)

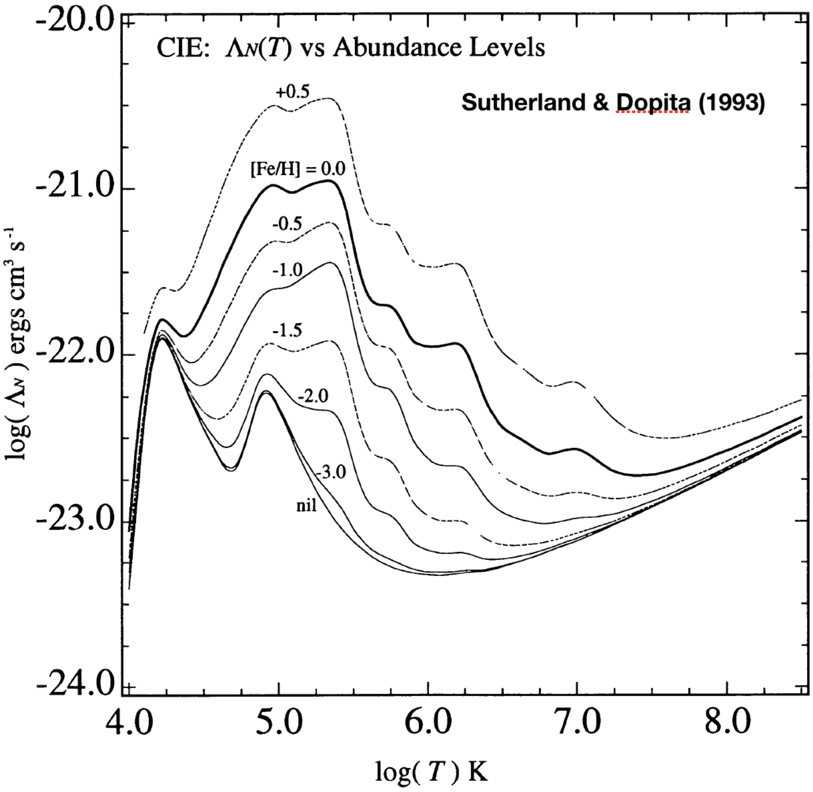

Figure 1: Cooling rate as a function of temperature for gas with different metallicities under the collisional ionisation equilibrium. Adapted from Sutherland and Dopita (1993).

a (3 points) Briefly describe what virialization is, and how dark matter and gas behave differently during virialization.

Solution: Virialization represent the equilibrium end-state of a density perturbation that was allowed to collapse under the influence of

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gravity. Basically dark matter is collisionless such that it is not halted by pressure forces. It instead reaches an equilibrium configuration(dark matter halo) which is solely depended on the gravitational potential. Collapsing gas will however be very different as it cannot freely fall due to pressure. Instead the infalling gas will crash into the stationary gas which heats the gas up to the virial temperature. The evolution will then depend on the cooling of the gas, as to collapse it will need to cool faster then the cloud can reestablish hydrostatic support.

Grading: 1 point for the concept of virialization and 2 points for the behaviours of dark matter and gas.

b (4 points) We have leant in the lectures that the infall velocity of gas accreting onto a dark matter halo with mass M and virial radius R_{vir} is $v_{infall} = \sqrt{\frac{GM}{R_{vir}}}$. This is also the so called "free-fall" speed (v_{ff}) of gas collapsing inside the halo. Assuming gas remains a uniform distribution inside the halo and collapse with a constant speed, show that the free-fall timescale for gas collapsing into the centre is

$$t_{ff} = \sqrt{\frac{3}{4\pi G\rho_i}} \tag{7}$$

where ρ_i denotes the initial density of the object. We can use t_{ff} as an approximation of the dynamical time.

Solution: Free fall time from R_{vir}:

$$t_{ff} = \frac{R_{vir}}{v_{inf}} = \sqrt{\frac{R_{vir^3}}{GM}}$$

$$M = \frac{4\pi}{3}\rho_i R_{vir}^3$$

Which gives

$$t_{ff} = \sqrt{\frac{3}{4\pi G\rho_i}}$$

Grading: 2 points for the equation $t_{ff} = \frac{R_{vir}}{v_{inf}}$ and 2 points for derive the correct answer.

c (4 points) Primordial gas contains only hydrogen (H) and helium (He). Explain the shape of the **primordial** cooling curve shown in Figure 1 (i.e. the curve labeled with “nil” in the figure). In particular, explain what physical processes cause the local peaks and minima in the curve and the increase of cooling rate at high temperature.

Solution: When $T < 10^4K$ the gas is mostly neutral and cooling very inefficient. As we increase the temperature the gas becomes energetic enough to start excite and ionize the gas. Cooling from excitation will be dominant and will peak at around 2×10^4K, when most of the hydrogen have been ionized. Past this the cooling will decrease rapidly from both excitation and ionization, until around $3-4\times10^4K$ in which this decrease is counteracted by the excitation and ionization of ionized

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helium. This eventually also peaks(around $10^5 K$), when most of the helium is doubly ionized. The decrease in cooling is eventually stopped by the increased cooling from the free-free process, as the gas becomes fully ionized.

Grading: 1 points for $T < 10^4$ K, 2 points for the physical processes cause the peaks and troughs, and 1 point to the free-free emission at high temperatures.

d (2 points) Describe the conditions that allow gas inside a dark matter halo to collapse into its center.

Solution: To collapse, the gas needs to cool so fast that the gas cannot re-establish hydrostatic equilibrium. In other words, we require that the cooling time for the gas is less than the dynamical time-scale:

$$t_{cool} < t_{dyn}$$

e (5 points) Massive disk galaxies like the Milky Way are very efficient in star formation. Consider a perturbation of mass similar to a Milky Way-like galaxy $M = 10^{12}\mathrm{M}_\odot$ virializes at $z = 0$. Use the expression from **Problem 3b** to compute the collapse time of the virialized halo (in Gyrs $\equiv 10^9$ years). Assume **Planck cosmology** and use the parameters and the values of H_0 and $\rho_{crit,0}$ given at the beginning of the test sheet. Use overdensity $\delta = 178$ for virialized halos.

Solution: We begin by calculating the mass density of the universe at z in Planck cosmology:

$$\rho_m = \Omega_m \rho_{c,0}(1+z)^3$$

We are given that virilization occurs for an overdenisty of 178 at redshift z=0:

$$\rho_i = 178\Omega_m \rho_{c,0}$$

Which gives the collapse time:

$$t_{coll} = \sqrt{\frac{3}{4\pi G \cdot 178\Omega_m \rho_{c,0}}}$$

Enter values from beginning of test sheet and we get:

$$t_{coll} \approx 2.76 Gyr$$

Grading: 3 points for the correct expression of collapse time, and 2 points for correct numerical value.

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f (5 points) The cooling time is given by $t_{\rm cool} = \frac{3k_{\rm b}T}{2n\Lambda(T)}$, n denotes the number density of **gas**. Gas in galaxies has the virial temperature given by:

$$T_{\rm vir} \equiv \frac{m_{\rm p}v_{\rm circ}^2}{2k_{\rm B}} \approx 2\times10^4\ {\rm K}\left(\frac{M}{10^8\ M_\odot}\right)^{2/3}\left(\frac{1+z}{10}\right). \tag{8}$$

How is the cooling time t_{cool} compare to the collapse time (i.e. the free-fall time t_{ff}) for the $M = 10^{12}$ M$_\odot$ halo? What does this imply for the future evolution of the gas? Use **Planck cosmology** and note that the *average* gas density of the Universe is $\rho_{\rm b} = \Omega_{\rm b}\rho_{\rm crit}$, and $n_{\rm gas} = \rho_{\rm gas}/(\mu m_H)$, where $\rho_{\rm gas}$ is the gas mass density, $\mu = 0.6$ is the mean molecular weight of the gas, and m_H is the mass of the hydrogen atom.

Solution: Basically, use equation 8 to figure out the virial temperature for the given mass (z=0), $T_{vir} = 2\times10^4 K(10^{12}/10^8)^{2/3}\times0.1 = 9.28\times10^5 K$. Then extract Λ from the nil cooling curve figure $log(\Lambda) \approx -23.25$. The gas density is given by the baryon content times the overdensity required for virialization:

$$n_{\rm gas} = 178\Omega_b\rho_{c,0}/(\mu m_H) = 7.4\times10^{-5}cm^{-3}$$

We then have everything to calculate the cooling time:

$$t_{\rm cool} = \frac{3k_{\rm b}T}{2n_{gas}\Lambda(T)} = 14.72Gyr$$

Which is larger than the collapse time we got from the previous question. So the gas will not be able to collapse.

Grading: 1 point for T_{vir}, 1 point for reading off Λ, 1 point for calculating n_{gas}, 1 point for the correct cooling time and 1 point for explaining whether the halo will collapse or not.

g (4 points) In reality, the halo gas in a $M = 10^{12}$M$_\odot$ halo is not primordial, but enriched with metals (i.e. elements having higher atomic number than helium). Metals are mostly made by fusion process within stars, and ejected outside galaxies through energetic feedback processes. In Figure 1, various curves indicate how the amount of metals (i.e. the metallicity) can change the cooling rates. Now assume that the collapsed gas has the metallicity same as observed in the solar neighbourhood (i.e. the thick-line curve labeled with [Fe/H] = 0.0 in the Figure). How is the cooling time t_{cool} compare to the collapse time (i.e. the free-fall time t_{ff}) in this case? Does this result change your conclusion from **Problem 3f**?

Solution: We redo the calculation in the previous question, but with $\Lambda(T)$ given from the thick-line curve labeled with [Fe/H] = 0.0 in the Figure $log(\Lambda) \approx -22$. We then get:

$$t_{\rm cool} = 0.83Gyr$$

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Which is smaller than 2.73Gyr and the gas can collapse.

Grading: 1 point for reading off Λ from the figure, 2 points for the correct cooling time and 1 point for explaining whether the halo will collapse or not.

Problem 4 Two-Point Correlation Function, Bias and Matter Power Spectrum (weight 25%)

a (5 points) Describe briefly what a Gaussian random field is, and **qualitatively** what the two-point correlation function $\xi(r)$ describes (I have stressed "qualitatively" because you do not need to provide with the equation describing a multi-variate Gaussian).

Solution: A Gaussian Random Field (GRF) is a field of variables, $x_1, x_2, x_3, ...$ each of which obey Gaussian statistics (i.e. each x_i is drawn from a Gaussian distribution). The two point correlation function $\xi(r)$ fully specifies the statistical properties of the field, i.e. it describes the standard deviation of each field variable x_i.

Grading: 3 points for explaining GRF and 2 points for two-point correlation function.

b (5 points) Explain what the galaxy 2-point correlation function $\xi_{gal}(r)$ is, and why it can be different than the $\xi(r)$ describing the matter density field. The difference is called "bias". Describe how the "bias" behaves for galactic halos in different mass regimes.

Solution: $\xi_{gal}(r)$ denotes the excess probability over random of finding two galaxies separated by r, i.e. the probability of finding two galaxies in two identical volume elements dV separated by r equals to $P = dV^2 n^2(1 + \xi_{gal}(r))$, where ndV denotes the probability of finding a galaxy in a volume element dV. Galaxies forms in collapsed objects, so $\xi_{gal}(r)$ is related to the 2-point correlation of collapsed matter. Because not all matters in the GRF are part of collapsed objects, $\xi_{gal}(r)$ differs from the underlying matter 2-point correlation function $\xi_(r)$.

Using the Press-Schechter Formalism, $\xi_{gal}(r) = b_E^2 \xi(r)$, where $b_E \equiv 1 + \frac{\nu 2-1}{\sigma(M)\nu}$ is the Eulerian bias. $\nu \equiv \delta_{crit}/\sigma(M)$ and $\sigma(M)^2$ is the variance of the density filed. There are four regimes:

(a) $\nu = 1 \Rightarrow b_{\rm E} = 1$. Collapsed objects with the characteristic mass of the PS mass-function have the same 2-pt function $\xi(r)$ as all matter.

(b) $\nu < 1 \Rightarrow b_{\rm E} < 1$. Collapsed objects with $\nu < 1$ (i.e. objects in the power-law low mass end of the PS function) have a 2-pt function $\xi(r)$ that is suppressed to that of all matter. These objects 'cluster less', or said to be 'anti-biased' tracers.

(c) $\nu > 1 \Rightarrow b_{\rm E} > 1$. Collapsed objects with $\nu > 1$ (i.e., objects in the exponential tail of the halo mass function) have a 2-pt function

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$\xi(r)$ that is enhanced to that of all matter. These objects 'cluster stronger', or said to be 'biased' tracers.

(d) $\nu \gg 1 \Rightarrow b_{\rm E} \sim \frac{\nu}{\sigma}$. Note that $\nu \equiv \delta_{\rm crit}/\sigma(M) \gg 1$. i.e. $\sigma(M) \ll 1$. Therefore $b_{\rm E} \gg \nu \gg 1$. The rarest objects are very strongly clustered.

Note: You do not need to write down explicitly the mathematical form of the bias. Just say 'halos with mass lower than the characteristic mass cluster less (anti-biased) and halos with mass lower than the characteristic mass cluster more (biased)' is sufficient.

Grading: 3 points for explaining what $\xi_{gal}(r)$ is and how it differs from $\xi(r)$, 2 points for bias as a function of halo mass.

c (5 points) Explain what the acoustic peak is, and briefly describe the physics of the formation of this peak.

Solution:
$https://www.cfa.harvard.edu/\ deisenst/acousticpeak/acoustic_physics.html$
$https\ :\ //scienceblogs.com/startswithabang/2008/04/25/cosmic - sound - waves - rule$

Grading: 2 points for describing what the acoustic peak is and 3 points for explaining how it formed.

d (5 points) Assume that the Universe is presently dominated by cold dark matter and that the primordial density fluctuations were generated by quantum fluctuations during inflation. Sketch a plot of the present-day linear power spectrum $P(k)$ as a function of wavenumber k. Indicate the approximate scale where the "turn-over" occurs, and the power law slope at scales smaller and larger than this turn-over.

Solution: The power spectrum looks like Figure 2. You need to indicate the turn-over scale (or wavenumber) k_0, and $P(k) \propto k$ for $k \ll k_0$ and $P(k) \propto k^{-3}$ for $k \gg k_0$.

Grading: 1 point for sketching the shape of the power spectrum, 2 points for the correct slopes at small and large scales, and 2 points for indicating the turn-over scale.

e (5 points) Explain the shape of $P(k)$. Discuss what sets the "turn-over", and discuss qualitatively the difference in $P(k)$ at scales smaller and larger than this turn-over radius.

Solution: For large scales - i.e. small k - we have the 'primordial' power spectrum $P(k) = Ak^n$. The slope has been inferred to be $n = 1$ (from the CMB). For small scales (i.e. large k) we have $P(k) = Ak^nT^2(k)$ where $T(k) \propto k^{-2}$ denotes the transfer function. The transfer function quantifies how the 'Meszaros' effect suppresses

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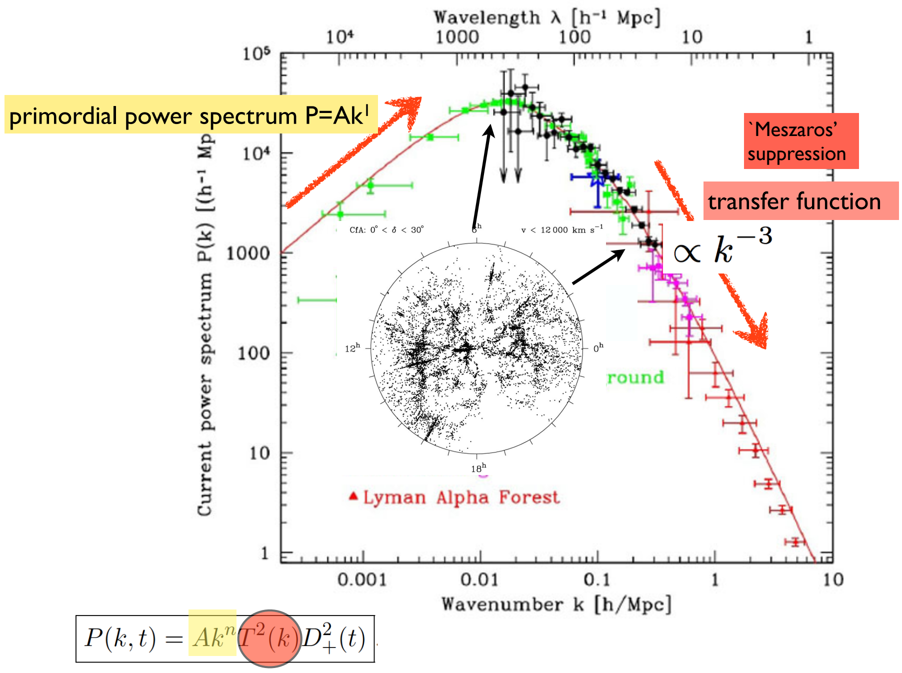

Figure 2: .

the growth of small scale perturbations: these perturbations entered the horizon prior to matter radiation equality. The Meszaros effect suppressed the growth of these perturbations during radiation domination. The turn-over in the power spectrum occurs on a scale which corresponds to the size of a perturbation that entered the horizon at matter-radiation equality.

Grading: discuss what sets the turn-over scale, 3 points; discuss why the power law for large scale is 1 and smaller scale is -3, 2 points.