UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in: AST4320 — Cosmology and Extragalactic Astronomy

Day of examination: Thursday 13. October 2016

Examination hours: 09.00 – 12.00

This problem set consists of 5 pages.

Appendices: None

Permitted aids: Formula book by Rottmann.
Calculator that meets standard requirements by university.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Preface: some useful relations:

- $H(a) = \frac{\dot{a}}{a} = H_0\sqrt{\Omega_{\rm m}a^{-3} + \Omega_\Lambda + \Omega_{\rm rad}a^{-4}}$, where H_0 is the Hubble parameter today, Ω's denote the density parameters in matter, dark energy, and radiation.
- $\Omega_{\rm m} = \frac{\bar{\rho}_{\rm m}}{\rho_{\rm crit}}$, where $\rho_{\rm crit} = \frac{3H^2}{8\pi G}$, $G = 6.67 \times 10^{-8}$ cm^3 g^{-1}s^{-2}.
- Einstein-de Sitter Universe: $\Omega_{\rm m} = 1.0$, $\Omega_\Lambda = 0.0$.
- $1M_\odot = 2 \times 10^{33}$ g.

Be brief in your answers. Lengthy derivation are never required.

Problem 1 Jeans Mass (weight 20%)

a Explain briefly what the Jeans mass/length is, and how it is relevant for the formation of structure in the Universe.
Pressure forces prevent perturbations smaller than the Jeans length from collapsing under the influence of gravity, whole perturbations larger than the Jeans length collapse. Th Jeans length thus marks the transition between these two regimes. The Jeans mass $M_{\rm J}$ is related to the Jeans length simply through $M_{\rm J} = \frac{4\pi}{3}\rho\pi(\lambda_{\rm J}/2)^3$ (note that the factor of '2' may be omitted); The Jeans length plays a key role in regulating the growth of baryonic perturbations.

b Show that for a sphere of mass M and radius R in which $\rho \propto r^{-2}$, that $U = -\frac{GM^2}{R}$.

(Continued on page 2.)

Write $\rho(r) = C/r^2$. The gravitational potential at r is given by $u(r) = \frac{GM(<r)}{r}$, $M(<r) = 4\pi Cr$, i.e. $u(r) = G4\pi C$. The total potential energy in a shell of matter at $r \pm dr/2$ equals $dU = u(r)4\pi r^2 dr \rho(r) = G(C4\pi)^2 dr$. The total potential energy $U(<r) = \int_0^r G(C4\pi)^2 dr = G(C4\pi)^2 r = \frac{GM^2(<r)}{r}$.

c In the lecture we derived the so-called 'Virial Theorem' which states that $2K = |U|$. Briefly discuss what this theorem describes, and discuss qualitatively (in one sentence) what would happen if $2K > |U|$, and what would happen if $2K < |U|$.
When $2K > |U|$ the kinetic term 'wins' over gravity, and the system expands. The system contracts when $2K < |U|$.

d Use the qualititive discussion from 1c and that $U = -\frac{GM^2}{R}$ to derive an expression for the Jeans length (in terms of gas temperature and density T and ρ). Use that the total kinetic energy *per particle* is given by $E = 3k_\mathrm{B}T/2$, in which T denotes the temperature of the gas, and $k_\mathrm{B} = 1.38 \times 10^{-16}$ erg K^{-1} is Boltzmann's constant.

The total kinetic energy (times 2) is $2K = 3Nk_\mathrm{B}T$. The total binding energy (up to a constant) is $U = \frac{GM^2}{R}$. The total number of particles is $N = M/m$, where m denotes the mass per particle. If we set equal $2K = |U|$, the this corresponds to

$$3Nk_\mathrm{b}T = \frac{GM^2}{R} \Rightarrow \frac{3k_\mathrm{b}T}{m} = \frac{GM}{R} \Rightarrow \frac{3k_\mathrm{b}T}{m} = \frac{G\frac{4\pi}{3}\rho R^3}{R} \Rightarrow \tag{1}$$
$$R^2 = \frac{9k_\mathrm{b}T}{4\pi G\rho m} \Rightarrow R_\mathrm{J} \equiv \sqrt{\frac{9k_\mathrm{b}}{4\pi Gm}}\frac{T^{1/2}}{\rho^{1/2}}$$
$$\Rightarrow \quad M_\mathrm{J} = \frac{4\pi}{3}\rho\pi(R_\mathrm{J}/2)^3 = C\frac{T^{3/2}}{\rho^{1/2}},$$

where $C = \frac{4\pi}{3}\left(\frac{9k_\mathrm{B}}{4\pi Gm}\right)^{3/2}$ is a numerical constant. Precise numerical values (e.g. '9' and '4') will differ when you adopted for example that $U = 3GM^2/[5R]$ (as appropriated for a uniform medium), and the precise definition of the Jeans mass. The only thing that is important here is that the factors of k_b, G and m are in the right location here.

Problem 2 Spherical Top-Hat Model (weight 20%)

The non-linear evolution of a spherical density perturbation of radius R (with uniform density, enclosing a total mass M) was given by a parameterised solution:

$$R = A(1 - \cos\theta) \tag{2}$$
$$t = B(\theta - \sin\theta),$$

where $A^3 = GMB^2$, in which M denotes the total mass of the perturbation.

a Sketch the time-evolution of a mass-shell within the top-hat model (i.e. radius R vs time t). Indicate 'turn-around', 'virialization', 'collapse',

(Continued on page 3.)

and indicate how the evolution at small R and t is compatible with the result obtained from linear theory (do *not* derive the linear theory result:that would take too long!).

See lecture notes.

b Consider a stationary, spherical object of mass M. Use the parametrised solution to show that it takes a time $t_{\rm coll}$ for this object to collapse into a point, where the collapse time $t_{\rm coll}$

$$t_{\rm coll} = \sqrt{\frac{3\pi}{32G\rho_i}}, \tag{3}$$

where ρ_i denotes the initial density of the object (i.e. the density before it starts collapsing).

Stationary sphere corresponds to top-hat model at maximum expansion radius, $R_{\rm max} = 2A$ at $t = \pi B$. Collapse to a point happens at $t = 2\pi B$. Total time that passed in $2\pi B - \pi B = \pi B$. Using that $A^3 = GMB^2$ we can write $\Delta t = \pi\sqrt{\frac{A^3}{GM}} = \pi\sqrt{\frac{R^3_{\rm max}}{8GM}}$. Using that $\rho_i = 3M/4\pi R^3_{\rm max}$, we get our answer.

c In the spherical top-hat model, we assume that δ =constant up to some radius $R_{\rm TH}$. Now instead assume that $\delta = \delta_0/r^2$. If a mass-shell initially at R_1 collapses at t_1, compute at what time a mass shell at $R_2 = 2R_1$ collapses. Assume an Einstein-de-Sitter Universe in which $\delta \propto a$ in the linear regime. *Hint:* use linear theory results to identify when a mass-shell collapses.

Total enclosed overdensity within any radius grows as $\bar{\delta} \propto a$. Total enclosed overdensity with R, $\bar{\delta} = \frac{3}{4\pi R^3}\int_0^R 4\pi R^2 dR \frac{\delta_0}{R^2} \propto R^{-2}$. In other words, $\bar{\delta}(R_1) = 4\bar{\delta}(R_2)$. In linear theory $\bar{\delta} \propto a \propto t^{2/3}$, and collapse occurs when $\bar{\delta} = 1.69$. If shell enclosing R_1 collapses at R_1 then shell enclosing R_2 must grow by an additional factor of 4 in order to collapse, which proceeds at a rate $\propto t^{2/3}$. We must therefore have $4 = (t_2/t_1)^{2/3} \rightarrow t_2 = 8t_1$.

Problem 3 Gas Cooling (weight 20%)

a Describe three main cooling processes of an astrophysical gas (*Hint*: we referred to one of the three as "bound-bound").
See lecture notes. Bound-bound, bound-free, free-bound, free-free. Any 3 is good.

b Explain the shape of the cooling curve shown in Fig 1.
See lecture notes

c Describe the conditions that allow gas inside a dark matter halo to collapse into its center (*Hint:* use cooling time vs collapse time arguments). Why do we not expect galaxies to form efficiently in dark

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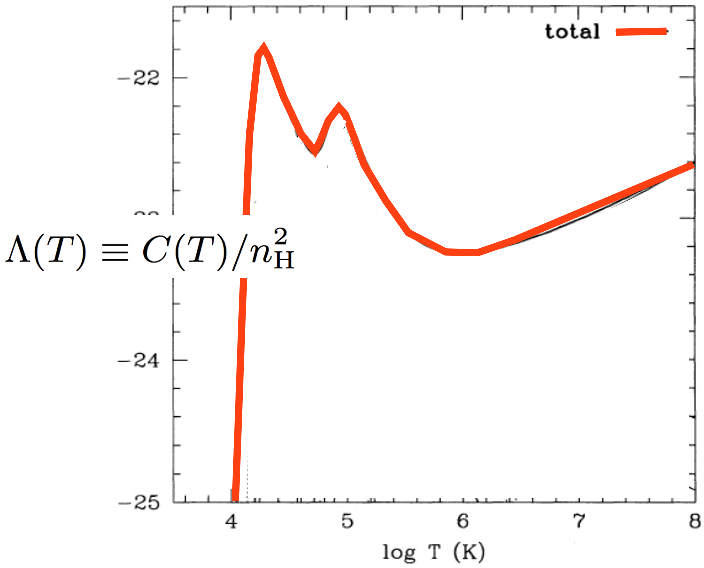

Figure 1:

matter halos less massive than $\sim 10^8 M_\odot$ and more massive than $\sim 10^{14} M_\odot$?

see lecture notes. We must have that the the cooling time $t_{\rm cool} < t_{\rm dyn}$ the dynamical or collapse time.

d A perturbation of mass M virializes at $z = 2$. Use the expression from **Problem 2b** to compute the collapse time of the virialized halo. Assume an Einstein-de Sitter Universe for the background cosmology, with a present-day Hubble parameter $H_0 = 2.27 \times 10^{-18}$ s^{-1}.

$$t_{\rm coll} = \sqrt{\frac{3\pi}{32 G \rho_i}}, \tag{4}$$

. Compute mean density of the Universe at $z = 2$. We know that $\bar{\rho}(z=2) = (3)^3 \times \bar{\rho}(z=0) = 27\Omega\rho_{\rm crit} = 27\frac{3H_0^2}{8\pi G}$. Mean density of the Universe at $z = 2$ is therefore $\bar{\rho}(z=2) = 2.5 \times 10^{-28}$ gr/cm^3. Density at virialization is $\sim 200 times higher$. Plugging in you find $t \sim 2 \times 10^8$ yr.

e The cooling time is given by $t_{\rm cool} = \frac{3k_{\rm b}T}{2n\Lambda(T)}$, n denotes the number density of gas. Estimate the cooling time for gas inside a $M = 10^{12} M_\odot$ halo using that the "virial temperature" is given by

$$T_{\rm vir} \equiv \frac{m_{\rm p} v_{\rm circ}^2}{2k_{\rm B}} \approx 2 \times 10^4 \text{ K} \left(\frac{M}{10^8 \, M_\odot}\right)^{2/3} \left(\frac{1+z}{10}\right). \tag{5}$$

(Continued on page 5.)

How does $t_{\rm cool}$ compare to $t_{\rm coll}$? What does this imply for the future evolution of the gas?

Compute virial temperature from equation. We get $T_{\rm vir} = 3.7 \times 10^6$ K. We can read off that $\log \Lambda \sim -23.2$. Compute $n \sim \rho_{\rm vir}/m_{\rm p} \sim 178 \times \bar{\rho}/m_{\rm p} \sim 0.027$ cm^{-3}. We get $t_{\rm cool} \sim 3 \times 10^8$ yrs. Cooling not quite efficient enough to cause run-away collapse.

Problem 4 Redshift space distortions (weight 20%)

A massive dark matter halo, $M = 2 \times 10^{14} M_\odot$ virializes at $z = 0.01$. At this redshift, the angular diameter distance, the luminosity distance and proper distance are all equal and $d_{\rm A}(z) = d_{\rm L}(z) = r \sim \frac{cz}{H_0}$.

a Compute the angular size of the collapsed dark matter halo. What would a halo of this mass correspond to (a galaxy, or ...)?
Mean density at $z \sim 0$ in EdS is $\bar{\rho} = \frac{3H_0^2}{8\pi G} = 9.2 \times 10^{-30}$ g/cm^3. Mean density of virialized halo is ~ 200 times larger, i.e 1.8×10^{-27} g/cm^3. Virial radius is then $R_{\rm vir} = [3M/(4\pi\rho)]^{1/3} \sim 3.8 \times 10^{24}$ cm. Angular size is $\theta = R_{\rm vir}/r \sim \frac{3.08 \times 10^{24}}{1.3 \times 10^{26}} \sim 0.03$ radians.

b What would the line-of-sight size of the cluster be in real, and redshift space?

In real space: L.O.S size is $\sim 2R_{\rm vir} \sim 8 \times 10^{24}$ cm. In redshift space it would be $2\frac{v_{\rm circ}}{H_0}$, where $v_{\rm circ} = \sqrt{GM/R_{\rm vir}}$. Substituting numbers: $v_{\rm circ} \sim 800$ km/s, which translates to $2v_{\rm circ}/H_0 \sim 8 \times 10^{25}$ cm, ten times longer.

c Sketch the shape of the halo in redshift space. Indicate approximate sizes in Mpc (in both transverse and line-of-sight directions).
Halo appears stretched out by factor of ~ 10 along the L.O.S direction!