# HUST

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ONE LOVE. ONE FUTURE.

# PLANNING OPTIMIZATION

## Integer Linear Programming

# CONTENT

- Relaxation and Bound
- Branch and Bound
- Cutting plane
- Integer Rounding
- Gomory Cut

# RELAXATION AND BOUND

- Given an Integer Program (IP)
- Find decreasing sequence of upper bounds

$$\overline{z}_1 > \overline{z}_1 > \ldots > \overline{z}_s \geq z$$

- Find increasing sequence of lower bounds

$$\underline{z}_1 < \underline{z}_1 < \ldots < \underline{z}_t \leq z$$

- Algorithm stop when $\overline{z}_s - \underline{z}_t \leq \varepsilon$

# RELAXATION AND BOUND

- Primal bounds
  - Every feasible solution $x^* \in X$ provides a lower bound of the maximization problem: $\underline{z} = cx^* \leq z$
  - Example: in TSP, every closed tour is a upper bound of the objective function (as TSP is a minimization problem)

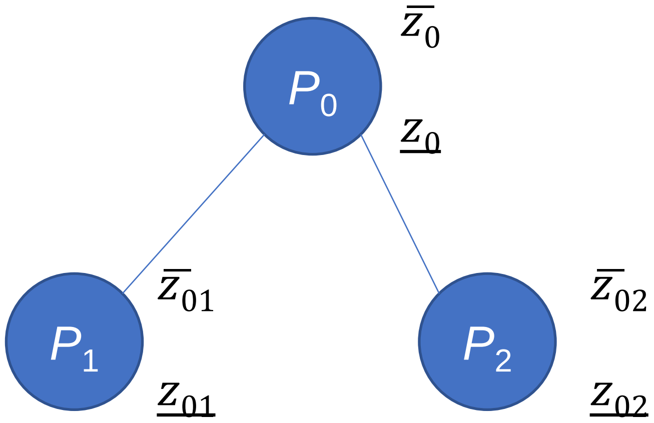
# RELAXATION AND BOUND

- Dual bounds
  - Finding upper bounds for a maximization problem (or lower bounds for a minimization problem) gives dual bounds of the objective
- **Definition** A problem ($RP$) $z^R = \max\{f(x): x \in T \subseteq R^n\}$ is a relaxation of ($IP$) $z = \max\{c^T x: x \in X \subseteq Z^n\}$ if:
  - $X \subseteq T$
  - $f(x) \geq c^T x,\ \forall\ x \in X$

# RELAXATION AND BOUND

- Linear Relaxation
  - $Z^{LP} = \max\{c^T x : x \in P\}$ with $P = \{x \in R^n : Ax \leq b\}$ is a linear relaxation program of the (IP) $\max\{c^T x : x \in P \cap Z^n\}$

- Feasible region of $P_0$ is divided into feasible regions of $P_1$ and $P_2$: $X(P_0) = X(P_1) \cup X(P_2)$



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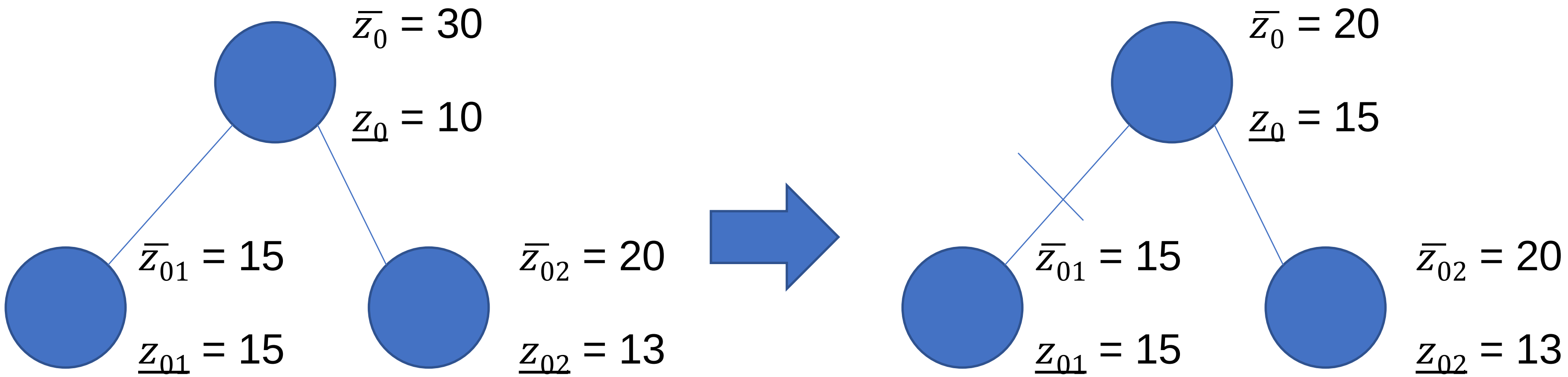

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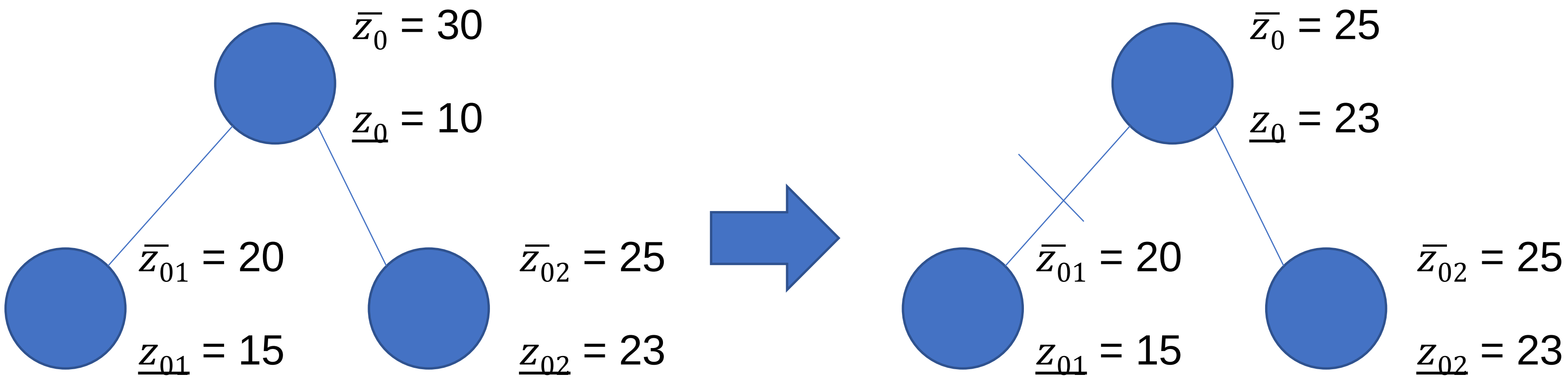


# BRANCH AND BOUND

- Branch and Bound schema

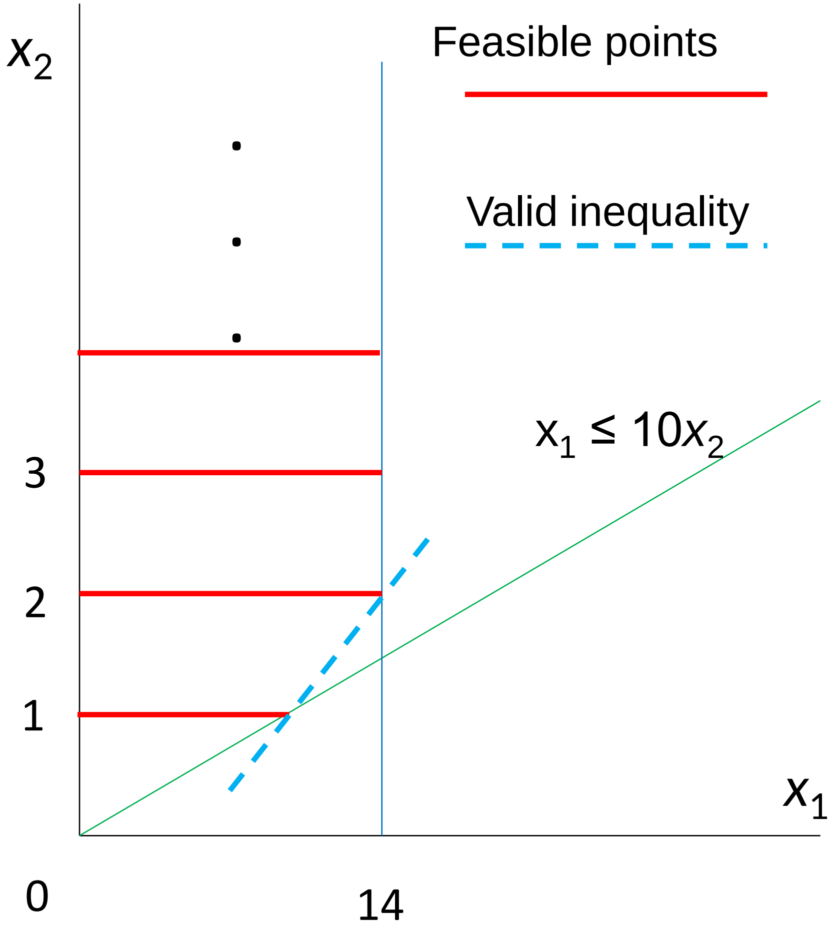
```
Initial problem S with formulation P on a list L;
Incumbent x* is initialized with primal bound z = -INF;
while L not empty do {
  Select a problem S^i with formulation P^i from L;
  Solve LP relaxation over P^i got dual bound z̄^i and solution x^i(LP);
  if z̄^i ≤ z then continue; // prune by dual bound;
  if x^i(LP) integer then{
     z = z̄^i ;
     x* = x^i(LP);
  }else{
     select a component x_i of x^i(LP) whose value λ_i is fractional;
     P^i_1 = P^i ∪ (x_i ≤ ⌊λ_i⌋), P^i_2 = P^i ∪ (x_i ≥ ⌈λ_i⌉);
     add P^i_1 and P^i_2 to L;
  }
}
return x*;
```

# CUTTING PLANE

- Given a (MIP) $\max\{c^T z: z \in X\}$
- Inequality $\pi z \leq \pi_0$ is called a valid inequality if $\pi z \leq \pi_0$ is true for all $z \in X$
- Finding valid inequalities allows us to narrow the search space, transform the (MIP) to a corresponding (LP) in which an optimal solution to (LP) is an optimal solution to the original (MIP)

# CUTTING PLANE

- Example, consider a MIP with $X = \{(x_1,x_2): x_1 \le 10x_2,\ 0 \le x_1 \le 14,\ x_2 \in Z_+^1\}$
- Red lines represent $X$
- $x_1 \le 6 + 4x_2$ is a valid inequality (dashed line)



- Example: **Integer Rounding**
  - Consider feasible region $X = P \cap Z^3$ where $P = \{x \in R^3_+ : 5x_1 + 9x_2 + 13x_3 \geq 19\}$
  - From $5x_1 + 9x_2 + 13x_3 \geq 19$ we have $x_1 + \frac{9}{5}x_2 + \frac{13}{5}x_3 \geq \frac{19}{5}$

  → $x_1 + 2x_2 + 3x_3 \geq \frac{19}{5}$
  - As $x_1, x_2, x_3$ are integers, so we have

$$x_1 + 2x_2 + 3x_3 \geq \left\lceil \frac{19}{5} \right\rceil = 4 \text{ (this is a valid inequality for } X\text{)}$$

# GOMORY CUT

- (IP) max {$cx$: $Ax = b$, $x \geq 0$ and integer}
- Solve corresponding linear programming relaxation ($LP$) max {$cx$: $Ax = b$, $x \geq 0$}
- Suppose with an optimal basis, the (LP) is rewritten in the form

$$\overline{a_{00}} + \sum_{j \in JN} \overline{a_{0j}} x_j \rightarrow \max$$

$$x_{B_u} + \sum_{j \in JN} \overline{a_{uj}} x_j = \overline{a_{u0}},\ u = 1, 2, \ldots, m$$

$$x \geq 0 \text{ and integer}$$

with $\overline{a_{0j}} \leq 0$ (as these coefficients corresponds to a maximizer), and $\overline{a_{u0}} \geq 0$

# GOMORY CUT

- (IP) max {*cx*: *Ax* = *b*, *x* $\geq$ 0 and integer}
- Solve corresponding linear programming relaxation (*LP*) max {*cx*: *Ax* = *b*, *x* $\geq$ 0}
- Suppose with an optimal basis, the (LP) is rewritten in the form

$$\overline{a_{00}} + \sum_{j \in JN} \overline{a_{0j}} x_j \rightarrow \max$$

$$x_{B_u} + \sum_{j \in JN} \overline{a_{uj}} x_j = \overline{a_{u0}}, \; u = 1, 2, \ldots, m$$

$$x \geq 0 \text{ and integer}$$

with $\overline{a_{0j}} \leq 0$ (as these coefficients corresponds to a maximizer), and $\overline{a_{u0}} \geq 0$

- If the basic optimal solution $x^*$ is not integer, then there exists some row u with $\overline{a_{u0}}$ is not integer

→ Create a Gomory cut $x_{B_u} + \sum_{j \in JN} \lfloor \overline{a_{uj}} \rfloor x_j \leq \lfloor \overline{a_{u0}} \rfloor$ (1)

# GOMORY CUT

- **Example**. Consider the Integer Linear Program (ILP)

$$
\begin{aligned}
f(x_1, x_2, x_3, x_4, x_5) = \quad & x_1 + x_2 \rightarrow \max \\
& 2x_1 + x_2 + x_3 = 8 \\
& 3x_1 + 4x_2 + x_4 = 24 \\
& x_1 - x_2 + x_5 = 2 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0 \text{ and integer}
\end{aligned}
$$

# GOMORY CUT

- **Solve the corresponding** (LP)

$$f(x_1, x_2, x_3, x_4, x_5) = x_1 + x_2 \rightarrow \max$$

$$2x_1 + x_2 + x_3 = 8$$

$$3x_1 + 4x_2 + x_4 = 24$$

$$x_1 - x_2 + x_5 = 2$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

→ Optimal solution (1.6, 4.8, 0, 0, 5.2) with $J_B$ = (1,2,5) and $J_N$ = (3, 4)

Rewrite the original (ILP)

$$f(x_1, x_2, x_3, x_4, x_5) = 6.4 - 0.2x_3 - 0.2x_4 \rightarrow \max$$

$$x_1 + 0.8x_3 - 0.2x_4 = 1.6$$

$$x_2 - 0.6x_3 + 0.4x_4 = 4.8$$

$$x_5 - 1.4x_3 + 0.6x_4 = 5.2$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0 \text{ and integer}$$

# GOMORY CUT

- Add gomory cut
  - $x_1 + 0.8x_3 - 0.2x_4 = 1.6 \rightarrow$ add gomory cut: $x_1 + 0x_3 - x_4 \leq 1$
  - $x_2 - 0.6x_3 + 0.4x_4 = 4.8 \rightarrow$ add gomory cut: $x_2 - x_3 + 0x_4 \leq 4$
  - $x_5 - 1.4x_3 + 0.6x_4 = 5.2 \rightarrow$ add gomory cut: $x_5 - 2x_3 + 0x_4 \leq 5$

# BRANCH AND CUT [Wolsey, 98]

INITIALIZATION $z = \max\{cx: x \in X\}$ with formulation P, $\underline{z}$ = -INF, incumbent $x^*$ empty, init Nodelist

NODE if Nodelist empty, goto EXIT else choose and remove node $i$ from Nodelist and goto RESTORE

RESTORE formulation $P_i$ of the set $X^i$, set $k = 1$ and $P^{i,1} = P^i$

LP RELAXATION Iteration k. Solve $\bar{z}^{i,k} = \max\{cx: x \in P^{i,k}\}$. If infeasible, prune and goto NODE Else solution $x^{i,k}$ and goto CUT

CUT Iteration k. Try to cut off $x^{i,k}$. If not cuts found, goto PRUNE Else add cuts to $P^{i,k}$ giving $P^{i,k+1}$. Increase k by 1, and goto LP RELAXATION

PRUNE If $\bar{z}^{i,k} \leq \underline{z}$, goto NODE. If $x \in X$, set $\underline{z} = \bar{z}^{i,k}$, update incumbent $x^* = x^{i,k}$ and goto NODE Else goto BRANCHING

BRANCHING Create two or more new problems and add them to the Nodelist

EXIT Incumbent $x^*$ with optimal value $\underline{z}$

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# THANK YOU !

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