HUST

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ONE LOVE. ONE FUTURE.

PLANNING OPTIMIZATION

Linear Programming

CONTENT

- Linear programs
- Geometric approach
- Simplex method
- Two-phase simplex method

- Standard form

$$f(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \max$$

$$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n \le b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n \le b_2$$

$$\ldots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n \le b_m$$

$$x_1, x_2, \ldots x_n \in R, x_1, x_2, \ldots x_n \ge 0$$

LINEAR PROGRAMS

- Standardize general linear programs
  - $f(x) \rightarrow \min \Leftrightarrow -f(x) \rightarrow \max$
  - $g(x) \geq b \Leftrightarrow -g(x) \leq -b$
  - $A = B \Leftrightarrow (A \leq B)$ and $(-A \leq -B)$
  - A variable $x_j \in R$ can be represented by $x_j = x_j^+ - x_j^-$ where $x_j^+, x_j^- \geq 0$

LINEAR PROGRAMS

- Example: Convert a general linear program forms into standard form

  $f(x_1, x_2) = 3x_1 + 2x_2 \rightarrow \min$

  $$2x_1 + x_2 \le 7$$

  $$x_1 + 2x_2 = 8$$

  $$x_1 - x_2 \ge 2$$

  $$x_1, x_2 \in R, x_2 \ge 0$$

- Example: Convert a general linear program forms into standard form
  - Substitution: $x_1 = x_1^+ - x_1^-$

  $f(x_1^+, x_1^-, x_2) = -3x_1^+ + 3x_1^- - 2x_2 \rightarrow \max$

  $2x_1^+ - 2x_1^- + x_2 \leq 7$

  $x_1^+ - x_1^- + 2x_2 \leq 8$

  $-x_1^+ + x_1^- - 2x_2 \leq -8$

  $-x_1^+ + x_1^- + x_2 \leq -2$

  $x_1^+, x_1^-, x_2 \in R,\ x_1^+, x_1^-, x_2 \geq 0$

GEOMETRIC APPROACH

- Constraints (inequalities) form a feasible region
- Optimal points will be one of the corners of the feasible region

GEOMETRIC APPROACH

- Example

$$f(x_1, x_2) = 3x_1 + 2x_2 \rightarrow \max$$

$$2x_1 + x_2 \leq 7$$

$$x_1 + 2x_2 \leq 8$$

$$x_1 - x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

- Example

$$f(x_1, x_2) = 3x_1 + 2x_2 \rightarrow \max$$

$$2x_1 + x_2 \leq 7$$

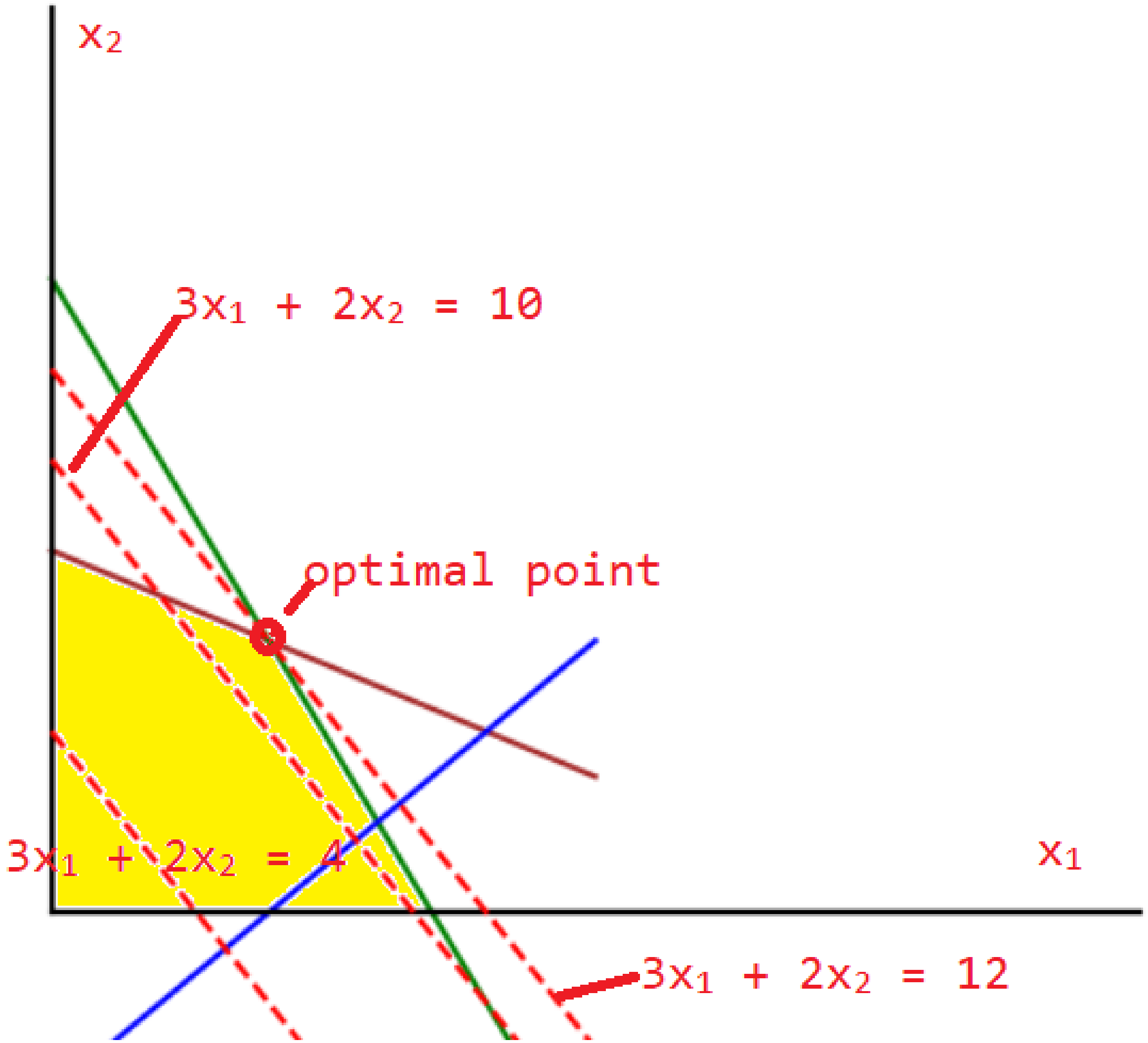
$$x_1 + 2x_2 \leq 8$$

$$x_1 - x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

- Example

$$
\begin{aligned}
f(x_1, x_2) \quad = \quad & 3x_1 + 2x_2 \rightarrow \max \\
& 2x_1 + x_2 \le 7 \\
& x_1 + 2x_2 \le 8 \\
& x_1 - x_2 \le 2 \\
& x_1, x_2 \ge 0
\end{aligned}
$$



GEOMETRIC APPROACH

- Special cases
  - Problem does not have optimal solutions
  - Problem does not have feasible solutions

$$
\begin{aligned}
f(x_1, x_2) = 3x_1 + 2x_2 &\rightarrow \max \\
-2x_1 - x_2 &\leq -7 \\
x_1 - x_2 &\leq 2 \\
x_1, x_2 \in R,\ &x_1, x_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
f(x_1, x_2) = 3x_1 + 2x_2 &\rightarrow \max \\
2x_1 + x_2 &\leq 7 \\
-4x_1 - 2x_2 &\leq -16 \\
x_1, x_2 \in R,\ &x_1, x_2 \geq 0
\end{aligned}
$$

GEOMETRIC APPROACH

- Exercise
  - A company must decide to make a plan to produce 2 products P1, P2.
    - The revenue received when selling 1 unit of P1 and P2 are respectively 5$ and 7$
    - The manufacturing cost when producing P1 and P2 are respectively 5$ and 3$
    - The storage cost in warehouses for 1 unit of P1 and P2 are respectively 2$ and 3$
  - Compute the production plan so that
    - Total manufacturing cost is less than or equal to 200$
    - Total storage cost is less than or equal to 150$
    - Total revenue is maximal

BASIC FEASIBLE SOLUTION

- Standard form to standard equational form by adding slack variables $y_1, y_2, \ldots, y_m$

Standard form

$$f(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \max$$

$$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n \le b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n \le b_2$$

$$\ldots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n \le b_m$$

$$x_1, x_2, \ldots x_n \in R,\ x_1, x_2, \ldots x_n \ge 0$$

Standard equality form

$$f(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \max$$

$$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n + y_1 = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n + y_2 = b_2$$

$$\ldots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n + y_m = b_m$$

$$x_1, x_2, \ldots x_n \in R,\ x_1, x_2, \ldots x_n, y_1, y_2, \ldots, y_m \ge 0$$

BASIC FEASIBLE SOLUTION

- Consider a Linear Program (LP) under a standard equational form

Standard equational form

$f(x) = c_1x_1 + c_2x_2 + \dots + c_nx_n \rightarrow$ max

$a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n = b_1$

$a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n = b_2$

$\dots$

$a_{m,1}x_1 + a_{m,2}x_2 + \dots + a_{m,n}x_n = b_m$

$x_1, x_2, \dots x_n \in R, x_1, x_2, \dots x_n \geq 0$

Standard equality form

$f(x) = c^Tx \rightarrow$ max

$Ax = b$

$x \geq 0$

BASIC FEASIBLE SOLUTION

- Consider a Linear Program (LP) under a standard equational form
- Suppose rank(A) = m
- Let *B* be the matrix of *m* linearly independent columns (indexed $j_1, j_2, \ldots, j_m$) of *A*: $B = (A(j_1), A(j_2), \ldots, A(j_m))$
  - Solution *x* is called a **basic solution** if :
    - $x_j = 0$ for $j \in \{1, 2, \ldots, n\} \setminus \{j_1, j_2, \ldots, j_m\}$
    - Remain variables are found by solving this equation:

$$\begin{pmatrix} a_{1,j_1} & a_{1,j_2} & \ldots & a_{1,j_m} \\ a_{2,j_1} & a_{2,j_2} & \ldots & a_{2\ j_m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m,j_1} & a_{m,j_2} & \ldots & a_{m,j_m} \end{pmatrix} \begin{pmatrix} x_{j_1} \\ x_{j_2} \\ \ldots \\ x_{j_m} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{pmatrix}$$

$f(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \max$

$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n = b_1$

$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n = b_2$

$\ldots$

$a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n = b_m$

$x_1, x_2, \ldots x_n \in R,\ x_1, x_2, \ldots x_n \geq 0$

BASIC FEASIBLE SOLUTION

- Let B be the matrix of m linearly independent columns (indexed $j_1, j_2, \ldots, j_m$) of A: $B = (A(j_1), A(j_2), \ldots, A(j_m))$
  - Solution x is called a basic solution if :
    - $x_j = 0$ for $j \in \{1, 2, \ldots, n\} \setminus \{j_1, j_2, \ldots, j_m\}$
    - Remain variables are found by solving this equation:

$$\begin{pmatrix} a_{1,j_1} & a_{1,j_2} & \ldots & a_{1,j_m} \\ a_{2,j_1} & a_{2,j_2} & \ldots & a_{2\ j_m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m,j_1} & a_{m,j_2} & \ldots & a_{m,j_n} \end{pmatrix} \begin{pmatrix} x_{j_1} \\ x_{j_2} \\ \ldots \\ x_{j_m} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{pmatrix}$$

  - B is called a basis
  - $j_1, j_2, \ldots, j_m$: basic indices, $j \in \{1, 2, \ldots, n\} \setminus \{j_1, j_2, \ldots, j_m\}$ is called non-basic index
  - $x_{j_1}, x_{j_2}, \ldots, x_{j_m}$: basic variables and x_j ($j \in \{1, 2, \ldots, n\} \setminus \{j_1, j_2, \ldots, j_m\}$) is called non-basic variable
  - A **basic solution** x with $x \geq 0$ is called a **basic feasible solution**

- Consider a linear program under a standard form

$$
\begin{aligned}
& z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \max \\
& \quad a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n \leq b_1 \\
& \quad a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n \leq b_2 \\
& \quad \ldots \\
& \quad a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n \leq b_m \\
& \quad b_1, b_2, \ldots b_m \geq 0 \\
& \quad x_1, x_2, \ldots x_n \in R, x_1, x_2, \ldots x_n \geq 0
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
& z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \max \\
& \quad a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n + y_1 = b_1 \\
& \quad a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n + y_2 = b_2 \\
& \quad \ldots \\
& \quad a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n + \quad + y_m = b_m \\
& \quad b_1, b_2, \ldots b_m \geq 0 \\
& \quad x_1, x_2, \ldots x_n \in R, x_1, x_2, \ldots x_n, y_1, y_2, \ldots, y_m \geq 0
\end{aligned}
$$

	1	2	…	n	n+1	n+2	…	n+m		
0	x_1	x_2	…	x_n	y_1	y_2	…	y_m	z	RHS
1	$a_{1,1}$	$a_{1,2}$	…	$a_{1,n}$	1	0	…	0	0	b_1
2	$a_{2,1}$	$a_{2,2}$	…	$a_{2,n}$	0	1	…	0	0	b_2
…	…	…	…	…	…		…	…	..	…
m	$a_{m,1}$	$a_{m,2}$	…	$a_{m,n}$	0	0	…	1	0	b_m
m+1	$-c_1$	$-c_2$	…	$-c_n$	0	0	..	0	1	0

SIMPLEX METHOD: Tabular form

	1	2	...	n	n+1	n+2	...	n+m	n+m+1	
0	x_1	x_2	…	x_n	x_{n+1}	x_{n+2}	…	x_{n+m}	z	RHS
1	$\alpha_{1,1}$	$\alpha_{1,2}$	…	$\alpha_{1,n}$	$\alpha_{1,n+1}$	$\alpha_{1,n+2}$	…	$\alpha_{1,n+m}$	$\alpha_{1,n+m+1}$	β_1
2	$\alpha_{2,1}$	$\alpha_{2,2}$	…	$\alpha_{2,n}$	$\alpha_{2,n+1}$	$\alpha_{2,n+2}$	…	$\alpha_{2,n+m}$	$\alpha_{2,n+m+1}$	β_2
...	…	…	…	…	…		…	…	..	…
m	$\alpha_{m,1}$	$\alpha_{m,2}$	…	$\alpha_{m,n}$	$\alpha_{m,n+1}$	$\alpha_{m,n+2}$	…	$\alpha_{m,n+m}$	$\alpha_{m,n+m+1}$	β_m
m+1	$\alpha_{m+1,1}$	$\alpha_{m+1,2}$	…	$\alpha_{m+1,n}$	$\alpha_{m+1,n+1}$	$\alpha_{m+1,n+2}$	…	$\alpha_{m+1,n+m}$	$\alpha_{m+1,n+m+1}$	β_{m+1}

- $J = \{1, 2, \ldots, n, n+1, \ldots, n+m\}$
- Maintain linear constraints on each row k (k = 1, 2, ..., m+1):

$$\alpha_{k,1} x_1 + \alpha_{k,2} x_2 + \ldots + \alpha_{k,n} x_n + \alpha_{k,n+1} x_{n+1} + \ldots + \alpha_{k,n+m} x_{n+m} + \alpha_{k,n+m+1} z = \beta_k \qquad (*)$$

- Let R_k be a vector containing elements on row k of the table (k = 1, 2, ..., m+1)
- Perform linear transformation below, constraint (*) is still satisfied:
  - Replace $R_k = R_k + \delta^* R_i$ (k, i = 1, 2, ..., m+1), with some constant δ

SIMPLEX METHOD: Tabular form

- **Optimality**

	1	2	...	m	$m+1$	$m+2$	...	$n+m$	$n+m+1$	
0	x_1	x_2	…	x_m	x_{m+1}	x_{m+2}	…	x_{n+m}	z	RHS
1	1	0	…	0	$\alpha_{1,m+1}$	$\alpha_{1,m+2}$	…	$\alpha_{1,n+m}$	0	β_1
2	0	1	…	0	$\alpha_{2,m+1}$	$\alpha_{2,m+2}$	…	$\alpha_{2,n+m}$	0	β_2
...	…	…	…	…	…		…	…	..	…
m	0	0	…	1	$\alpha_{m,m+1}$	$\alpha_{m,m+2}$	…	$\alpha_{m,n+m}$	0	β_m
$m+1$	0	0	…	0	$\alpha_{m+1,m+1}$	$\alpha_{m+1,m+2}$	…	$\alpha_{m+1,n+m}$	1	β_{m+1}

- With $\beta_1, \beta_2, \ldots, \beta_m \geq 0$, $\exists\, J_B = \{j_1, j_2, \ldots, j_m\}$ such that $\alpha_{m+1,j} = 0, \forall j \in J_B$, $\alpha_{m+1,j} \geq 0\ \forall j \in J \backslash J_B$, columns $j_1, j_2, \ldots, j_m$ forms a unit matrix
- Without loss of generality, suppose that $J_B = \{1, 2, \ldots, n\}$, coefficients $\alpha_{m+1,m+1}, \alpha_{m+1,m+2}, \ldots, \alpha_{m+1,n+m} \geq 0$, columns 1, ..., m forms a unit matrix: $\alpha_{1,1}, \alpha_{2,2}, \ldots, \alpha_{m,m} = 1$
- Constraint (*) is still satisfied. We have $\alpha_{m+1,m+1}x_{m+1} + \alpha_{m+1,m+2}x_{m+2} + \ldots, \alpha_{m+1,n+m}x_{n+m} + z = \beta_{m+1}$
- $z = \beta_{m+1} - (\alpha_{m+1,m+1}x_{m+1} + \alpha_{m+1,m+2}x_{m+2} + \ldots, \alpha_{m+1,n+m}x_{n+m}) \leq \beta_{m+1}$ (because $\alpha_{m+1,m+1}, \alpha_{m+1,m+2}, \ldots, \alpha_{m+1,n+m} \geq 0$ and $x_{m+1}, \ldots, x_{n+m} \geq 0$).

SIMPLEX METHOD: Tabular form

- **Optimality**

	1	2	…	m	$m+1$	$m+2$	…	$n+m$	$n+m+1$	
0	x_1	x_2	…	x_m	x_{m+1}	x_{m+2}	…	x_{n+m}	z	RHS
1	1	0	…	0	$\alpha_{1,m+1}$	$\alpha_{1,m+2}$	…	$\alpha_{1,n+m}$	0	β_1
2	0	1	…	0	$\alpha_{2,m+1}$	$\alpha_{2,m+2}$	…	$\alpha_{2,n+m}$	0	β_2
…	…	…	…	…	…		…	…	..	…
m	0	0	…	1	$\alpha_{m,m+1}$	$\alpha_{m,m+2}$	…	$\alpha_{m,n+m}$	0	β_m
$m+1$	0	0	…	0	$\alpha_{m+1,m+1}$	$\alpha_{m+1,m+2}$	…	$\alpha_{m+1,n+m}$	1	β_{m+1}

- Moreover, there exists a solution (nonnegative values for variables x_1, x_2, ..., x_{n+m}) described below:
  - $x_1 = \beta_1$, $x_2 = \beta_2$, . . ., $x_m = \beta_m$
  - $x_{m+1} = x_{m+2} = . . . = x_{n+m} = 0$

  Satisfying given constraints. Also, the objective value at this solution is equal to the upper bound β_{m+1}. It means that this solution is an optimal solution to the given problem.

SIMPLEX METHOD: Tabular form

- **Simplex step**

	1	2	...	m	m+1		i	...	n+m			
0	x_1	x_2	…	x_m	x_{m+1}	…	x_i	…	x_{n+m}	z	RHS	E
1	1	0	…	0	$\alpha_{1,m+1}$	…	$\alpha_{1,i}$	…	$\alpha_{1,n+m}$	0	β_1	E_1
2	…	…	…	…	…	…	…	…	…	…	…	…
...	0	1	…	0	$\alpha_{k,m+1}$	…	$\alpha_{k,i}$	…	$\alpha_{k,n+m}$	0	β_k	E_k
m	0	0	…	1	$\alpha_{m,m+1}$	…	$\alpha_{m,i}$	…	$\alpha_{m,n+m}$	0	β_m	E_m
m+1	0	0	…	0	$\alpha_{m+1,m+1}$	…	$\alpha_{m+1,i}$	…	$\alpha_{m+1,n+m}$	1	β_{m+1}	

- Select column i such that the element on row m+1 (which is $\alpha_{m+1,i}$) is negative minimal
- Compute evaluations (column E): $E_j = +\infty$, if $\alpha_{j,i} \leq 0$, and $E_j = \dfrac{\beta_j}{\alpha_{j,i}}$, if $\alpha_{j,i} > 0$, $j = 1, 2, \ldots, m$
- Select the row k such that E_k is minimal: if $E_k = +\infty$, then the problem is **unbounded**, otherwise
  - Update:
    - Row $R_k = R_k / \alpha_{k,i}$
    - Row $R_j = R_j - \alpha_{j,i} * R_k$, $j = \{1, 2, \ldots, m+1\} \setminus \{k\}$

- **Example**

$$\begin{aligned} z = \ & 3x_1 + 2x_2 \rightarrow \max \\ & 2x_1 + x_2 \le 7 \\ & x_1 + 2x_2 \le 8 \\ & x_1 - x_2 \le 2 \\ & x_1, x_2 \in R,\ x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} z = \ & 3x_1 + 2x_2 \rightarrow \max \\ & 2x_1 + x_2 + x_3 = 7 \\ & x_1 + 2x_2 + x_4 = 8 \\ & x_1 - x_2 + x_5 = 2 \\ & x_1, x_2, x_3, x_4, x_5 \in R,\ x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

x_1	x_2	x_3	x_4	x_5	z	RHS	E
2	1	1	0	0	0	7	
1	2	0	1	0	0	8	
1	-1	0	0	1	0	2	
-3	-2	0	0	0	1	0	

SIMPLEX METHOD: Tabular form

- **Example**

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	2	1	1	0	0	0	7	
2	1	2	0	1	0	0	8	
3	1	-1	0	0	1	0	2	
4	-3	-2	0	0	0	1	0	

SIMPLEX METHOD: Tabular form

- **Example**

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	2	1	1	0	0	0	7	
2	1	2	0	1	0	0	8	
3	1	-1	0	0	1	0	2	
4	-3	-2	0	0	0	1	0	

- Select column 1: the corresponding element at the last row is minimal

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	2	1	1	0	0	0	7	
2	1	2	0	1	0	0	8	
3	1	-1	0	0	1	0	2	
4	-3	-2	0	0	0	1	0	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	2	1	1	0	0	0	7	
2	1	2	0	1	0	0	8	
3	1	-1	0	0	1	0	2	
4	-3	-2	0	0	0	1	0	

- Select column 1: the corresponding element at the last row is minimal
- Compute evaluation (column E)

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	2	1	1	0	0	0	7	7/2
2	1	2	0	1	0	0	8	8/1
3	1	-1	0	0	1	0	2	2/1
4	-3	-2	0	0	0	1	0	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	2	1	1	0	0	0	7	
2	1	2	0	1	0	0	8	
3	1	-1	0	0	1	0	2	
4	-3	-2	0	0	0	1	0	

- Select column 1: the corresponding element at the last row is minimal
- Compute evaluation (column E)
- Select row R3: evaluation is minimal
- Update R3 = R3/1

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	2	1	1	0	0	0	7	7/2
2	1	2	0	1	0	0	8	8/1
3	1	-1	0	0	1	0	2	2/1
4	-3	-2	0	0	0	1	0	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	2	1	1	0	0	0	7	
2	1	2	0	1	0	0	8	
3	1	-1	0	0	1	0	2	
4	-3	-2	0	0	0	1	0	

- Select column 1: the corresponding element at the last row is minimal
- Compute evaluation (column E)
- Select row R3: evaluation is minimal
- Update R3 = R3/1
- R1 = R1 – 2R3; R2 = R2 – R3; R4 = R4 + 3R3;

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	
2	0	3	0	1	-1	0	6	
3	1	-1	0	0	1	0	2	
4	0	-5	0	0	3	1	6	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	
2	0	3	0	1	-1	0	6	
3	1	-1	0	0	1	0	2	
4	0	-5	0	0	3	1	6	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	
2	0	3	0	1	-1	0	6	
3	1	-1	0	0	1	0	2	
4	0	-5	0	0	3	1	6	

- Select column 2: the corresponding element at the last row is minimal

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	
2	0	3	0	1	-1	0	6	
3	1	-1	0	0	1	0	2	
4	0	-5	0	0	3	1	6	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	
2	0	3	0	1	-1	0	6	
3	1	-1	0	0	1	0	2	
4	0	-5	0	0	3	1	6	

- Select column 2: the corresponding element at the last row is minimal
- Compute evaluations: column E

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	3/3
2	0	3	0	1	-1	0	6	6/3
3	1	-1	0	0	1	0	2	$+\infty$
4	0	-5	0	0	3	1	6	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	
2	0	3	0	1	-1	0	6	
3	1	-1	0	0	1	0	2	
4	0	-5	0	0	3	1	6	

- Select column 2: the corresponding element at the last row is minimal
- Compute evaluations: column E
- Select row R1: minimum evaluation

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	3/3
2	0	3	0	1	-1	0	6	6/3
3	1	-1	0	0	1	0	2	$+\infty$
4	0	-5	0	0	3	1	6	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	
2	0	3	0	1	-1	0	6	
3	1	-1	0	0	1	0	2	
4	0	-5	0	0	3	1	6	

- Select column 2: the corresponding element at the last row is minimal
- Compute evaluations: column E
- Select row R1: minimum evaluation
- Update R1 = R1/3

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	3	0	1	-1	0	6	
3	1	-1	0	0	1	0	2	
4	0	-5	0	0	3	1	6	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	
2	0	3	0	1	-1	0	6	
3	1	-1	0	0	1	0	2	
4	0	-5	0	0	3	1	6	

- Select column 2: the corresponding element at the last row is minimal
- Compute evaluations: column E
- Select row R1: minimum evaluation
- Update R1 = R1/3
- R2 = R2 – 3R1; R3 = R3 + R1; R4 = R4 + 5R1

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

- Select column 5: the corresponding element at the last row is minimal

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

- Select column 5: the corresponding element at the last row is minimal
- Compute evaluations: column E

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	$+\infty$
2	0	0	-1	1	1	0	3	3
3	1	0	1/3	0	1/3	0	3	9
4	0	0	5/3	0	-1/3	1	11	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

- Select column 5: the corresponding element at the last row is minimal
- Compute evaluations: column E
- Select row 2: minimum evaluation

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	$+\infty$
2	0	0	-1	1	1	0	3	3
3	1	0	1/3	0	1/3	0	3	9
4	0	0	5/3	0	-1/3	1	11	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

- Select column 5: the corresponding element at the last row is minimal
- Compute evaluations: column E
- Select row 2: minimum evaluation
- Update: R2 = R2/1

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

- Select column 5: the corresponding element at the last row is minimal
- Compute evaluations: column E
- Select row 2: minimum evaluation
- Update: R2 = R2/1
- R1 = R1 + (2/3)R2; R3 = R3 – (1/3)R2; R4 = R4+(1/3)R2

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	-1/3	2/3	0	0	3	
2	0	0	-1	1	1	0	3	
3	1	0	2/3	-1/3	0	0	2	
4	0	0	4/3	1/3	0	1	12	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

- Select column 5: the corresponding element at the last row is minimal
- Compute evaluations: column E
- Select row 2: minimum evaluation
- Update: R2 = R2/1
- R1 = R1 + (2/3)R2; R3 = R3 – (1/3)R2; R4 = R4+(1/3)R2

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	-1/3	2/3	0	0	3	
2	0	0	-1	1	1	0	3	
3	1	0	2/3	-1/3	0	0	2	
4	0	0	4/3	1/3	0	1	12	

- Phương án tối ưu của bài toán là $x_1 = 2$ và $x_2 = 3$, $x_3 = 0$, $x_4 = 0$, $x_5 = 3$.
- Giá trị hàm mục tiêu tối ưu bằng 12

TWO-PHASE SIMPLEX METHOD

- **Consider a linear program under a standard equational form**

(LP) $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \max$

$$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n = b_2$$

$$\ldots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n = b_m$$

Coefficients $b_1, b_2, \ldots b_m \geq 0$

$x_1, x_2, \ldots x_n \in R$, $x_1, x_2, \ldots x_n \geq 0$

TWO-PHASE SIMPLEX METHOD

- **Consider a linear program under a standard equational form**

(LP) $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \max$

$$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n = b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n = b_2$$
$$\ldots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n = b_m$$

Coefficients $b_1, b_2, \ldots b_m \geq 0$

$x_1, x_2, \ldots x_n \in R, x_1, x_2, \ldots x_n \geq 0$

Introduce an auxiliary linear program (ALP) with m artificial variables $y_1, y_2, \ldots, y_m$

(ALP) $g = -y_1 - y_2 - \ldots - y_m \rightarrow \max$

$$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n + y_1 = b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n + y_2 = b_2$$
$$\ldots \qquad \ldots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n + \quad +y_m = b_m$$

$b_1, b_2, \ldots b_m \geq 0$

$x_1, x_2, \ldots x_n \in R, x_1, x_2, \ldots x_n, y_1, y_2, \ldots, y_m \geq 0$

TWO-PHASE SIMPLEX METHOD

$$
\begin{aligned}
\text{(ALP)}\quad & g = -y_1 - y_2 - \ldots - y_m \rightarrow \max \\
& a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n + y_1 = b_1 \\
& a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n + y_2 = b_2 \\
& \ldots \\
& a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n + {} + y_m = b_m \\
& b_1, b_2, \ldots b_m \geq 0 \\
& x_1, x_2, \ldots x_n \in R,\ x_1, x_2, \ldots x_n, y_1, y_2, \ldots, y_m \geq 0
\end{aligned}
$$

- Solve the (ALP) by the Simplex Method under the Tabular form: Basis is the column vectors corresponding to artificial variables → obtain an optimal solution (x^*, y^*) and basic indices set is $J_B{}^*$
- Proposition: The original (LP) problem has feasible solutions iff the corresponding (ALP) has an optimal solution (x^*, y^*) in which $y^* = 0$ (**proof ?**)
- If $y^* \neq 0$, then the original (LP) problem does not have feasible solutions
- We consider the case that $y^* = 0$

TWO-PHASE SIMPLEX METHOD

	1	2	...	n	n+1	n+2	...	n+m	n+m+1	
0	x_1	x_2	...	x_n	y_1	y_2	...	Y_m	z	RHS
1	$\alpha_{1,1}$	$\alpha_{1,2}$	...	$\alpha_{1,n}$	$\alpha_{1,n+1}$	$\alpha_{1,n+2}$	...	$\alpha_{1,n+m}$	0	β_1
2	$\alpha_{2,1}$	$\alpha_{2,2}$	...	$\alpha_{2,n}$	$\alpha_{2,n+1}$	$\alpha_{2,n+2}$	...	$\alpha_{2,n+m}$	0	β_2
...	...	...	...	...	...	. . .	...	...	..	...
m	$\alpha_{m,1}$	$\alpha_{m,2}$	...	$\alpha_{m,n}$	$\alpha_{m,n+1}$	$\alpha_{m,n+2}$	...	$\alpha_{m,n+m}$	0	β_m
m+1	$\alpha_{m+1,1}$	$\alpha_{m+1,2}$	...	$\alpha_{m+1,n}$	$\alpha_{m+1,n+1}$	$\alpha_{m+1,n+2}$	...	$\alpha_{m+1,n+m}$	1	β_{m+1}

- Case 1: $J_B{}^*$ does not contain indices of artificial variables
    - Move to the second phase, solve the original (LP) problem
        - Remove columns corresponding to artificial variables: n+1, . . ., n+m
        - Recompute elements on row m+1 (based on the original objective function)

TWO-PHASE SIMPLEX METHOD

	1	2	...	n	$n+1$	$n+2$	...	$n+m$	$n+m+1$	
0	x_1	x_2	…	x_n	Y_1	y_2	…	Y_m	z	RHS
1	$\alpha_{1,1}$	$\alpha_{1,2}$	…	$\alpha_{1,n}$	$\alpha_{1,n+1}$	$\alpha_{1,n+2}$	…	$\alpha_{1,n+m}$	0	β_1
2	$\alpha_{2,1}$	$\alpha_{2,2}$	…	$\alpha_{2,n}$	$\alpha_{2,n+1}$	$\alpha_{2,n+2}$	…	$\alpha_{2,n+m}$	0	β_2
…	…	…	…	…	…	. . .	…	…	..	…
m	$\alpha_{m,1}$	$\alpha_{m,2}$	…	$\alpha_{m,n}$	$\alpha_{m,n+1}$	$\alpha_{m,n+2}$	…	$\alpha_{m,n+m}$	0	β_m
$m+1$	$\alpha_{m+1,1}$	$\alpha_{m+1,2}$	…	$\alpha_{m+1,n}$	$\alpha_{m+1,n+1}$	$\alpha_{m+1,n+2}$	…	$\alpha_{m+1,n+m}$	1	β_{m+1}

- Case 2: $J_B{}^*$ contains some indices of artificial variables
  - Suppose $J_B{}^*$ contains index $n+j$ of the artificial variable (y_j), perform the linear transformation to remove index $n+j$ from $J_B{}^*$ as follows:
    - Consider row k such that the element in row k and column $n+j$ is 1 (column vector corresponding to column $n+j$ is a unit vector)
    - Case 2.1: If all elements on row k, from column 1 to column n are equal to 0 ($\alpha_{k,1} = \ldots = \alpha_{k,n} = 0$): it means, the constraint of row k is linear dependent on other constraints → we can remove this row k and column $n+j$ from the table

TWO-PHASE SIMPLEX METHOD

	1	2	…	n	$n+1$	$n+2$	…	$n+m$	$n+m+1$	
0	x_1	x_2	…	x_n	y_1	y_2	…	Y_m	z	RHS
1	$\alpha_{1,1}$	$\alpha_{1,2}$	…	$\alpha_{1,n}$	$\alpha_{1,n+1}$	$\alpha_{1,n+2}$	…	$\alpha_{1,n+m}$	0	β_1
2	$\alpha_{2,1}$	$\alpha_{2,2}$	…	$\alpha_{2,n}$	$\alpha_{2,n+1}$	$\alpha_{2,n+2}$	…	$\alpha_{2,n+m}$	0	β_2
…	…	…	…	…	…	. . .	…	…	..	…
m	$\alpha_{m,1}$	$\alpha_{m,2}$	…	$\alpha_{m,n}$	$\alpha_{m,n+1}$	$\alpha_{m,n+2}$	…	$\alpha_{m,n+m}$	0	β_m
$m+1$	$\alpha_{m+1,1}$	$\alpha_{m+1,2}$	…	$\alpha_{m+1,n}$	$\alpha_{m+1,n+1}$	$\alpha_{m+1,n+2}$	…	$\alpha_{m+1,n+m}$	1	β_{m+1}

- Case 2: $J_B{}^*$ contains some indices of artificial variables
  - Case 2.2: There exists a column i such that $\alpha_{k,i} \neq 0$
  - In this optimal table, all artificial variables are equal to 0, so β_k is equal to 0
  - Perform the rotation with the pivot $\alpha_{k,i}$. With this rotation, column RHS is unchanged due to the fact that β_k is equal to 0. Hence β_{m+1} is always 0. It means that the new table corresponds to another optimal solution in which one artificial variable is replaced by an original variable x_i
  - The above procedure is repeated until all artificial variables are removed from the basic
  - We now process the computation as the case 1 (above)

- **Example 1 (exercise in class)**

(LP) $z = x_1 + 2x_2 - x_3 + x_4 \rightarrow \max$

$$\begin{aligned} x_1 + x_2 - x_3 - x_4 &= 4 \\ x_1 + x_3 + x_4 &= 7 \\ 2x_1 + x_2 &= 2 \\ x_1, x_2, x_3, x_4 \in R,\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

- **Example 2 (exercise in class)**

(LP) $z = x_1 + 2x_2 - x_3 + x_4 \rightarrow \max$

$$x_1 + x_2 - x_3 - x_4 = 4$$

$$x_1 + x_3 + x_4 = 7$$

$$x_1 - x_2 - x_3 = 2$$

$$x_1, x_2, x_3, x_4 \in R, x_1, x_2, x_3, x_4 \geq 0$$

- **Example 3 (exercise in class)**

$$
\begin{aligned}
\text{(LP)}\quad z = \; & 40x_1 + 10x_2 + 7x_5 + 14x_6 \rightarrow \max \\
& x_1 - x_2 + 2x_5 = 0 \\
& -2x_1 + x_2 - 2x_5 = 0 \\
& x_1 + x_3 + x_5 - x_6 = 3 \\
& x_2 + x_3 + x_4 + 2x_5 + x_6 = 4 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \in R,\; x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}
$$

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THANK YOU !