TIME SERIES

STATISTICAL TESTS AND
ANALYSIS IN THE FREQUENCY DOMAIN

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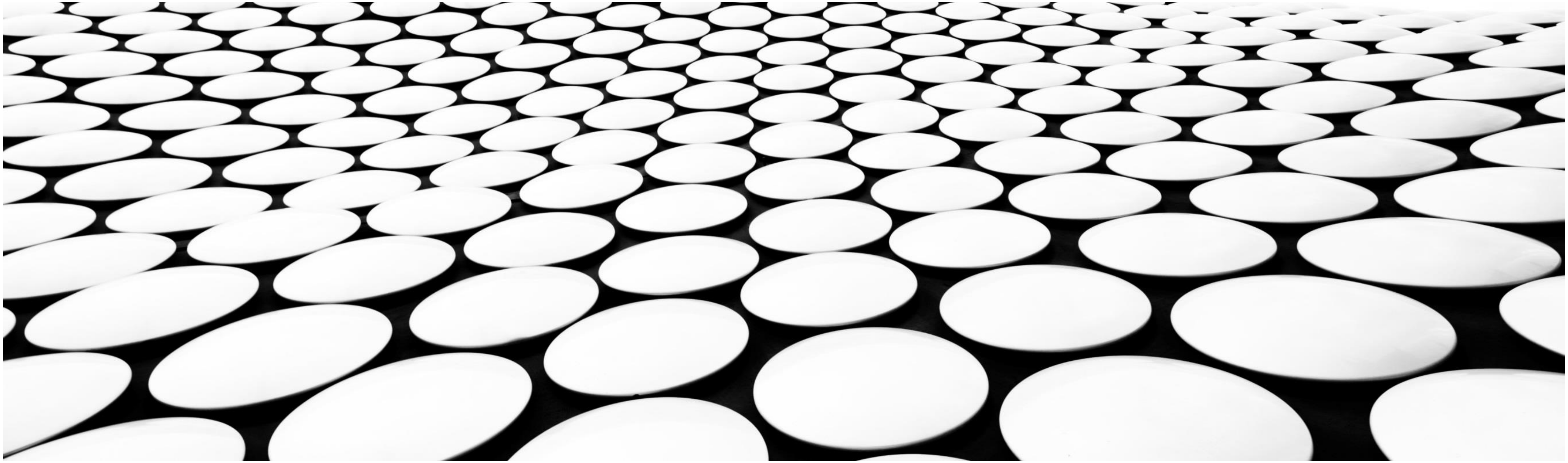

STATISTICAL TESTS

STATISTICAL TESTS

CONCEPT

Statistical tests are used in hypothesis testing that we can perform on data to infer some conditions of the data, such as stationarity, normality, independence, etc.

They assume a <u>null hypothesis</u> (h0) of occurrence of an event (for example: the time series is non-stationary). Statistical tests work by calculating a number that describes how likely the null hypothesis is true. This number is called <u>p-value</u> (probability value).

If the p-value is less than a chosen significance level α (commonly set at 0.05 or 0.01), we reject the null hypothesis in favor of an alternative hypothesis. Otherwise, the null hypothesis is retained.

Maintaining the null hypothesis does not necessarily mean that the null hypothesis is true, only that there is not <u>currently enough evidence to reject it</u>.

STATISTICAL TESTS

CONCEPT

For a statistical test to be valid, the sample size needs to be large enough to approximate the true distribution of the population being studied.

To determine which statistical test to use, we need to know whether your data meets certain assumptions.

Some statistical tests assume that the data have a certain distribution, such as a normal distribution. These tests are called <u>parametric tests</u>.

Other statistical tests are less restrictive, such that they work without any restriction on the distribution of the data. These tests are called <u>nonparametric tests</u>.

In addition to the distribution of data, statistical tests can make other assumptions that must be met for the correct evaluation of the test.

STATISTICAL TESTS

CONCEPT

Making a statistical decision always involves uncertainties, so the risks of making errors are unavoidable in hypothesis testing. There two types of errors:

Type I error (false positive): The test result says to reject the null hypothesis when it is actually true.

Type II error (false negative): The test result says to retain the null hypothesis when it is actually false. This is not quite the same as “accepting” the null hypothesis, because hypothesis testing can only tell us whether to reject the null hypothesis. The test may not have had enough statistical power to reject the null hypothesis.

The probability of making a Type I error is the significance level alpha (α), while the probability of making a Type II error is beta (β).

We can set a lower significance level to reduce the probability of committing a Type I error, but this increases the risk of a Type II error.

Type I and Type II Error

Null hypothesis is ...	True	False
Rejected	Type I error False positive Probability = α	Correct decision True positive Probability = $1 - \beta$
Not rejected	Correct decision True negative Probability = $1 - \alpha$	Type II error False negative Probability = β

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STATISTICAL TESTS

TESTS FOR NORMALITY

One of the most common assumptions for statistical tests is that the data used are normally distributed. The assumption of normal distribution is important for many methods and tests.

There are different tests available for this purpose. The most popular ones are:

Kolmogorov-Smirnov (KS) test: The null hypothesis is that your data are normally distributed. This test compares the ECDF (empirical cumulative distribution function) of your sample data with the distribution expected if the data were normal. If this observed difference is adequately large, the test will reject the null hypothesis.

Shapiro-Wilk (SW) test: The null hypothesis is that your data are normally distributed. This test assesses normality by calculating the correlation between the data and the normal counts of your data. If the correlation coefficient is close to 1, the population tends to be normal.

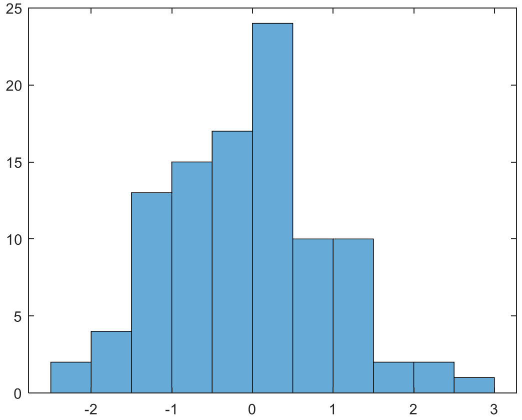
Shapiro-Francia (SF) test: The null hypothesis is that your data are normally distributed. Similar to Shapiro-Wilk test, but it is a simplification of the Shapiro-Wilk test.

The calculated p-value is affected by the size of the sample. Therefore, if you have a very small sample, your p-value may be much larger than 0.05, but if you have a very large sample, your p-value may be smaller than 0.05.

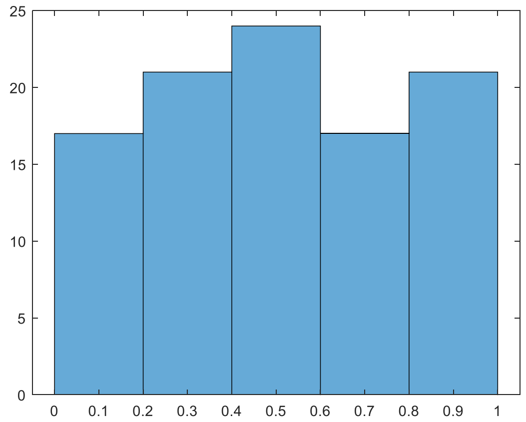
STATISTICAL TESTS

TESTS FOR NORMALITY

Example 1: Let X be a set of 100 values normally distributed with zero mean and unit variance.
Solution:
Kolmogorov-Smirnov test returned a p-value equals 0.4906 → retained the null hypothesis;
Shapiro-Wilk test returned a p-value equals 0.6584 → retained the null hypothesis;
Shapiro-Francia test returned a p-value equals 0.6044 → retained the null hypothesis;



Example 2: Let X be a set of 100 values uniformly distributed between 0 and 1.
Solution:
Kolmogorov-Smirnov test returned a p-value equals 0 → rejected the null hypothesis;
Shapiro-Wilk test returned a p-value equals 0.0022 → rejected the null hypothesis;
Shapiro-Francia test returned a p-value equals 0.0096 → retained the null hypothesis;



STATISTICAL TESTS

TESTS FOR HOMOSCEDASTICITY

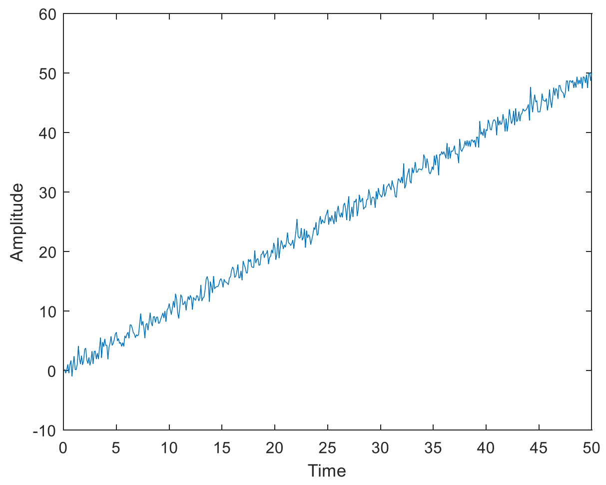
Test of homogeneity of covariances (or homoscedasticity) among several groups has many applications in statistical analysis.

Hawkins test: The null hypothesis is the homoscedasticity of the data. This test assumes the data are normally distributed.

Breusch-Pagan (BP) test: The null hypothesis is the homoscedasticity of the data. This test fits a linear regression model with the data, regresses the residuals on the fitted values, and checks whether they can explain any of the residual variance. It assumes that the residuals are normally distributed. A small p-value indicates that residual variance is non-constant (heteroscedastic).

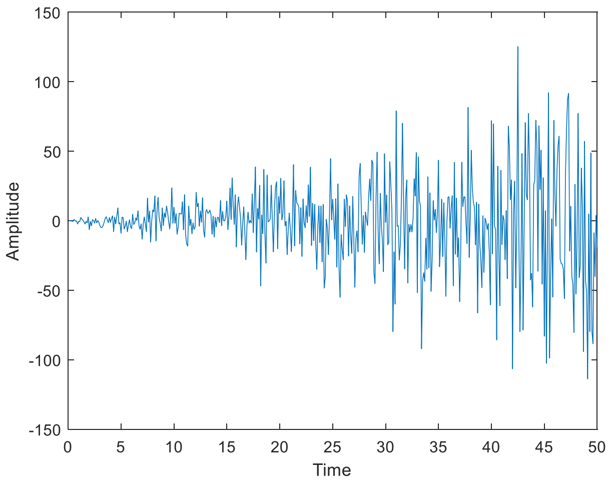
STATISTICAL TESTS
TESTS FOR HOMOSCEDASTICITY

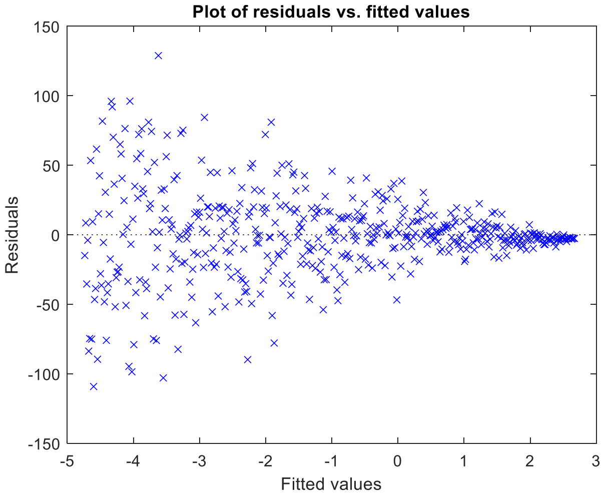
Example 3: Consider the data in the figures:



estimated values

Breusch-Pagan test: p-value = 0.9221
The null hypothesis is not rejected.





Breusch-Pagan test: p-value = 0
The null hypothesis is rejected.

STATISTICAL TESTS

TESTS FOR WEAK STATIONARITY

Weak stationarity is a property of time series which states that the mean and standard deviation of the variable does not change with time, i.e., variation in time does not serve as a factor that brings changes in these statistics.

There are different tests available for this purpose. The most popular ones are:

Augmented Dickey-Fuller (ADF) test: The null hypothesis of the test is that the time series is non-stationary (trend in the time series). This test assumes a homoscedastic time series. This test is based on an autoregressive model.

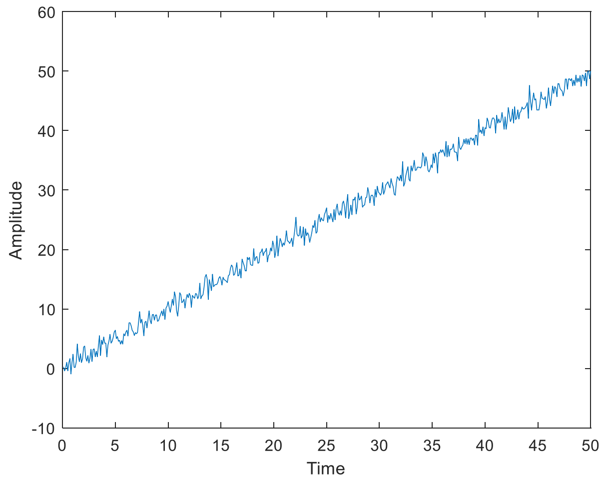
Philips-Perron (PP) test: The null hypothesis of the test is that the time series is non-stationary (trend in the time series). It is a non-parametric test and works well only in large samples.

Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test: The null hypothesis of the test is that an observable time series is stationary around a deterministic trend. This test considers the series is expressed as the sum of deterministic trend, random walk, and stationary error. The null hypothesis of trend stationarity corresponds to the hypothesis that the variance of the random walk equals zero. Under the additional assumptions that the random walk is normal and that the stationary error is normal white noise.

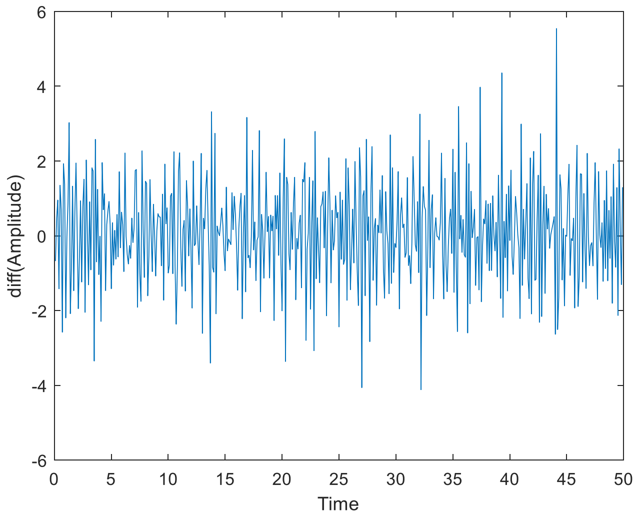
STATISTICAL TESTS

TESTS FOR WEAK STATIONARITY

Example 4: Consider the data in the figures:



KSSP test: p-value = 0.1 → retained the null hypothesis (the series is trend stationary).
ADF test: p-value = 0.8959 → retained the null hypothesis (time series is non-stationary).
PP test: p-value = 0.8959 → retained the null hypothesis (time series is non-stationary).



KSSP test: p-value = 0.1 → retained the null hypothesis (the series is trend stationary).
ADF test: p-value = 0.001 → rejected the null hypothesis.
PP test: p-value = 0.001 → rejected the null hypothesis.

STATISTICAL TESTS

TEST FOR CAUSALITY

Granger causality test

Identifying causality between variables is useful for determining whether one variable can be used to forecasting another.

Granger causality is a statistical concept of causality that is based on prediction. According to Granger causality, if a signal X "Granger-causes" (or "G-causes") a signal Y, then past values of X should contain information that helps predict Y above and beyond the information contained in past values of Y alone.

Its mathematical formulation is based on linear regression modeling. It does not capture instantaneous and non-linear causal relationships.

STATISTICAL TESTS

TEST FOR CAUSALITY

Granger causality test

To assess whether a variable Granger-causes another variable, we fit models, with n lags, to the series, so that we consider the relationship between variables given by:

$$X[t] = c_1 + \sum_{i=1}^{n} \alpha_{1,i}\, X[t-i] + \sum_{i=1}^{n} \beta_{1,i}\, Y[t-i] + \epsilon_{X_1}$$

$$Y[t] = c_2 + \sum_{i=1}^{n} \alpha_{2,i}\, X[t-i] + \sum_{i=1}^{n} \beta_{2,i}\, Y[t-i] + \epsilon_{X_2}$$

The null hypothesis is that a variable does not contain information about another variable. In other words:

- $Y[t-i]$ does not contain information about $X[t]$. This is equivalent to $\beta_{1,i} = 0$, for all i, and
- $X[t-i]$ does not contain information about $Y[t]$. This is equivalent to $\alpha_{2,i} = 0$, for all i.

The alternative hypothesis is: Past values of Y contain information about X whether at least one of $\beta_{1,i} \neq 0$, or Past values of X contain information about Y whether at least one of $\alpha_{2,i} \neq 0$.

STATISTICAL TESTS

TEST FOR CAUSALITY

Granger causality test

One way to estimate the coefficients α, β, and c is using ordinary least squares (OLS).
For example, suppose we want to estimate the coefficients of the equation:

$$X[t] = c + \alpha_{1,1}X[t-1] + \beta_{1,1}Y[t-1] + \epsilon$$

We can use the following equation to estimate the values of X:

$$\hat{T} = AW$$
$$W = \left(A^{\mathrm{T}}A\right)^{-1}A^{\mathrm{T}}T$$

The residual is the difference between the observed values (T) and the fitted values ($\hat{T}$) by the model.

where:

$$A = \begin{bmatrix} 1 & X[0] & Y[0] \\ 1 & X[1] & Y[1] \\ \vdots & \vdots & \vdots \\ 1 & X[t-1] & Y[t-1] \end{bmatrix} \qquad W = \begin{bmatrix} c \\ \alpha_{1,1} \\ \beta_{1,1} \end{bmatrix} \qquad T = \begin{bmatrix} X[1] \\ X[2] \\ \vdots \\ X[t] \end{bmatrix}$$

STATISTICAL TESTS

TEST FOR CAUSALITY

Example 5: Suppose the variables X and Y have the following values, where the last values are the most recent.
X = [0.84, 1.31, 1.84, 2.43, 2.63, 2.59]
Y = [0.10, 0.98, -1.25, -1.13, 0.85, 1.61]
We want to model the variable X by the equation:

$$X[t] = c + \alpha_{1,1}X[t-1] + \beta_{1,1}Y[t-1] + \epsilon$$

Solution:

$$A = \begin{bmatrix} 1 & X[0] & Y[0] \\ 1 & X[1] & Y[1] \\ 1 & X[2] & Y[2] \\ 1 & X[3] & Y[3] \\ 1 & X[4] & Y[4] \end{bmatrix} = \begin{bmatrix} 1 & 0.84 & 0.1 \\ 1 & 1.31 & 0.98 \\ 1 & 1.84 & -1.25 \\ 1 & 2.43 & -1.13 \\ 1 & 2.63 & 0.85 \end{bmatrix} \quad T = \begin{bmatrix} X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \end{bmatrix} = \begin{bmatrix} 1.31 \\ 1.84 \\ 2.43 \\ 2.63 \\ 2.59 \end{bmatrix}$$

$$W = \left(A^{\mathrm{T}}A\right)^{-1}A^{\mathrm{T}}T$$

$$W = \begin{bmatrix} c \\ \alpha_{1,1} \\ \beta_{1,1} \end{bmatrix} = \begin{bmatrix} 0.89 \\ 0.70 \\ -0.10 \end{bmatrix}$$

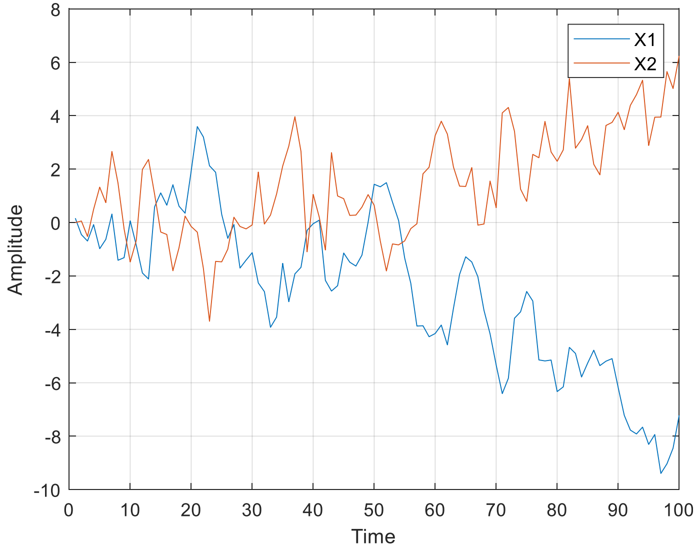
Residuals: $T - \hat{T} = T - AW$ = [-0.15, 0.14, 0.13, -0.07, -0.04]

Although the value of $\beta_{1,1}$ is different from zero, Y does not have any interference on X, but the model cannot set the value of $\beta_{1,1}$ to zero. The statistical test is used to verify whether the value can be considered zero or not.

STATISTICAL TESTS

TEST FOR CAUSALITY

Example 6: Let X_2 be a variable so that $X_2[t] = 0.1X_2[t-1] - 0.5X_1[t-1] + \epsilon$, where ϵ is a random Gaussian noise with zero mean and unit variance.



Solution: To assess whether each variable in a linear autoregressive model causes another variable, we fit models to the series with lags ranging from 1 to 4. We evaluate the models with the best fit to the data.

$$X_1[t] = c_1 + \sum_{i=1}^{4} \alpha_{1,i}\, X_1[t-i] + \sum_{i=1}^{4} \beta_{1,i}\, X_2[t-i] + \epsilon_{X_1}$$

$$X_2[t] = c_2 + \sum_{i=1}^{4} \alpha_{2,i}\, X_1[t-i] + \sum_{i=1}^{4} \beta_{2,i}\, X_2[t-i] + \epsilon_{X_2}$$

Conclusion: Excluding lagged X_2 in X_1 equation: Cannot reject null hypothesis.
Excluding lagged X_1 in X_2 equation: Reject null hypothesis.

MATHEMATICAL TRANSFORMATIONS IN TIME SERIES

MATHEMATICAL TRANSFORMATIONS IN TIME SERIES

BOX-COX

Box-Cox transforms nonnormally distributed data to a set of data that has approximately normal distribution. It is a power transformations and stabilize variance over time (it is useful for dealing with heteroscedasticity). The data is normalized and shifted to have a minimum of 1 so that negative powers and logarithms are always meaningful.

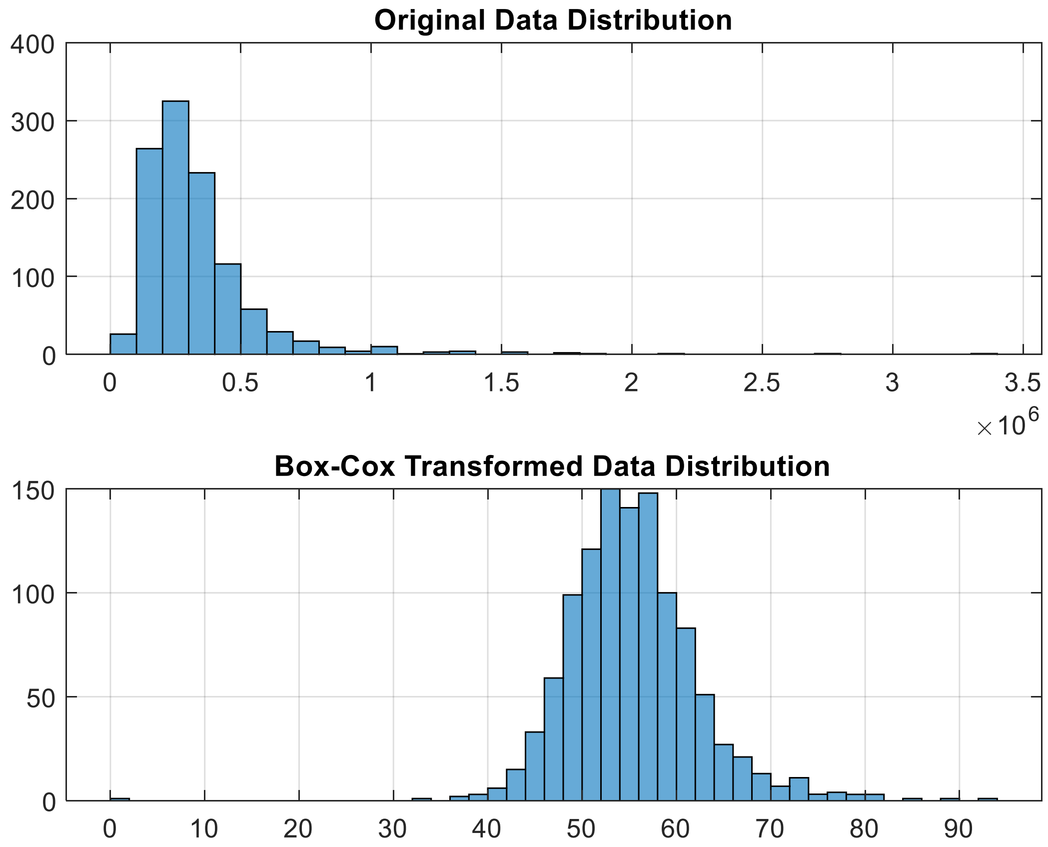
$$g(\lambda) = \begin{cases} \dfrac{data^{\lambda} - 1}{\lambda}, \text{if } \lambda \neq 0 \\ \ln(data), \text{if } \lambda = 0 \end{cases}$$

if

$\lambda = 1, g(\lambda)$ is the identity (data shifts down, but the shape of the data does not change).

$\lambda = -1, g(\lambda)$ is the inverse

$\lambda = 1/2, g(\lambda) = \sqrt{data}$



MATHEMATICAL TRANSFORMATIONS IN TIME SERIES

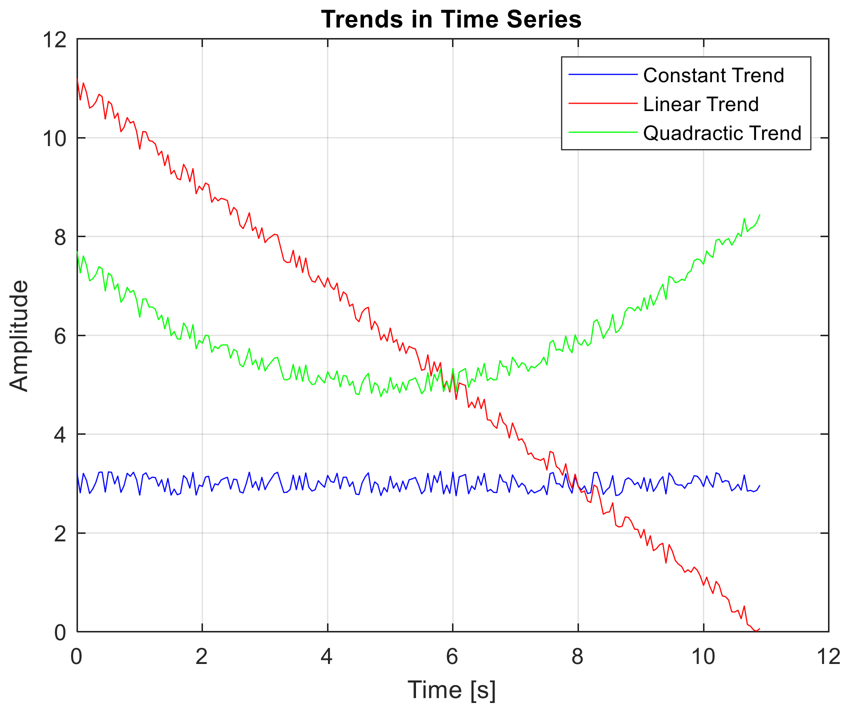
DIFFERENCE

<u>Difference</u> is an operation used to remove nonstationary level (first order) or slope (second order). It is the change between one observation and the next.

$$\Delta y[k] = y[k] - y[k-1]$$

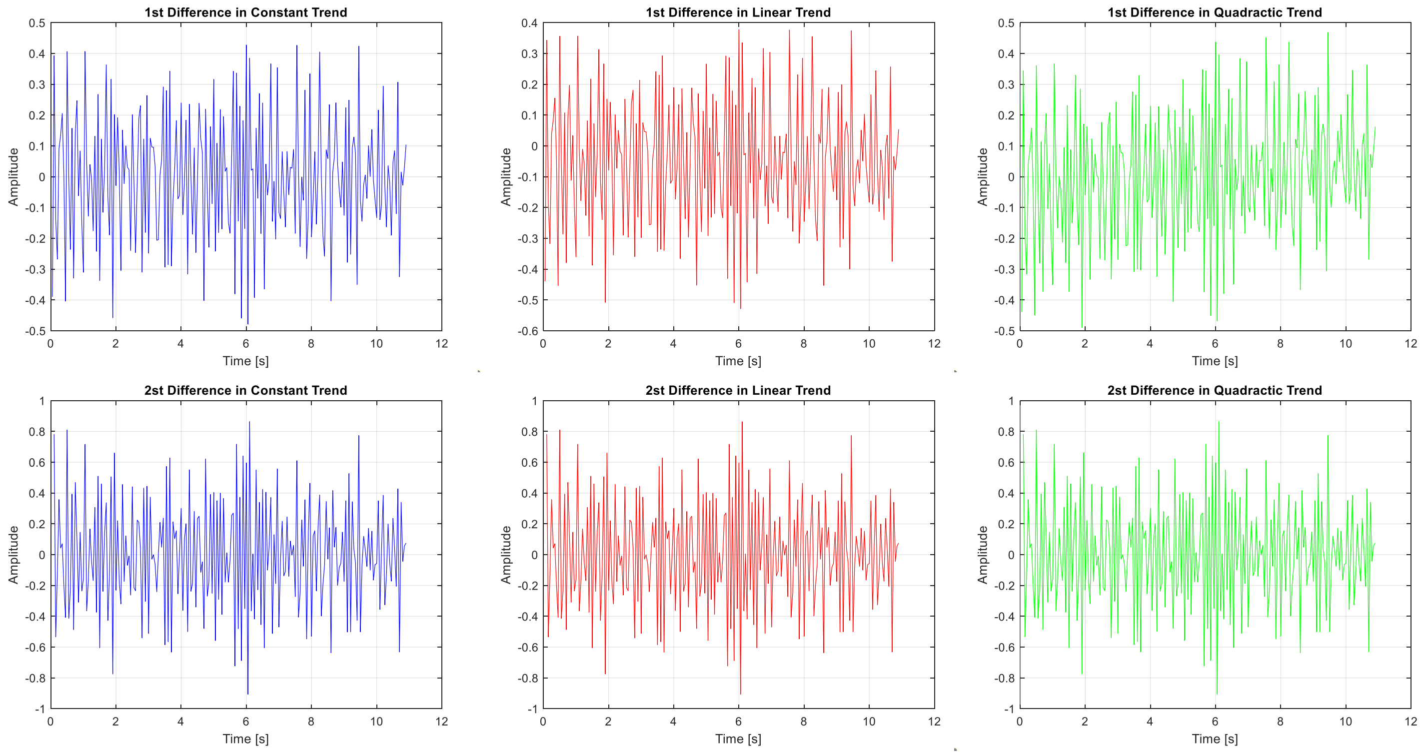
The n^{th} difference is:

$$\Delta^n y[k] = \Delta(\Delta^{n-1} y[k])$$



1st

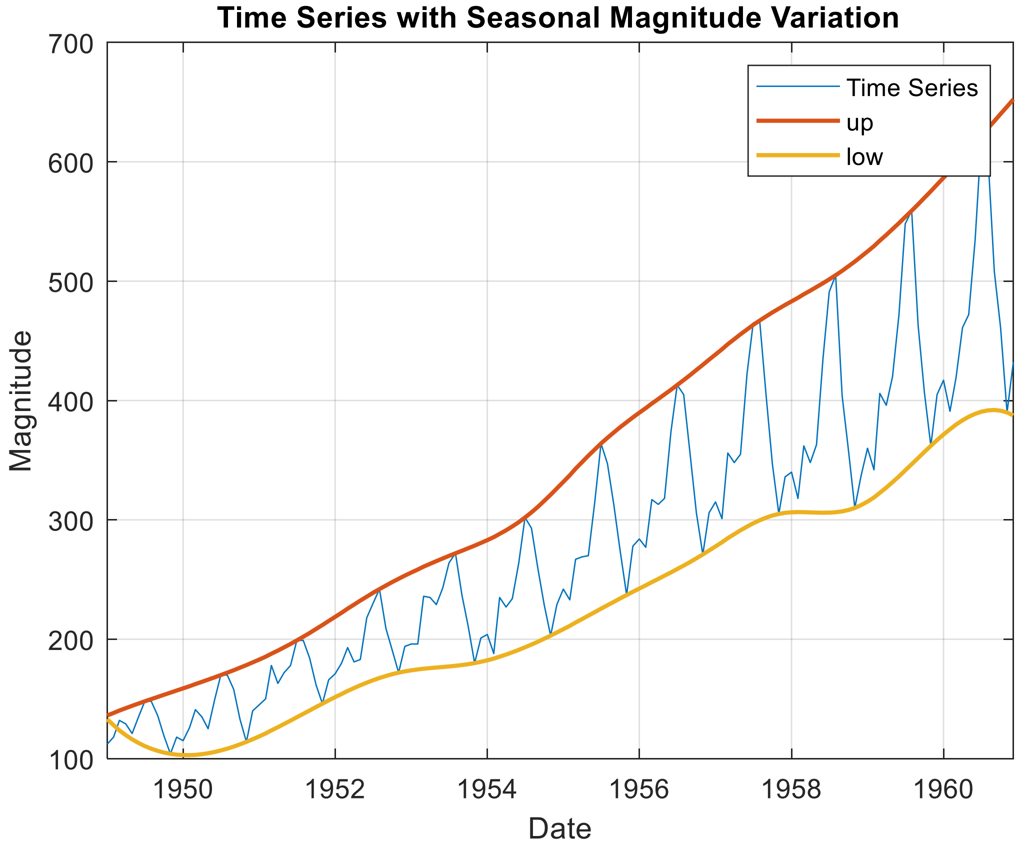
2st

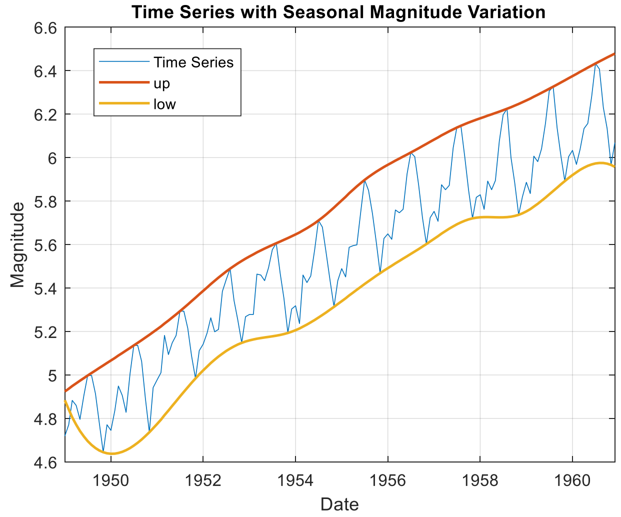


MATHEMATICAL TRANSFORMATIONS IN TIME SERIES

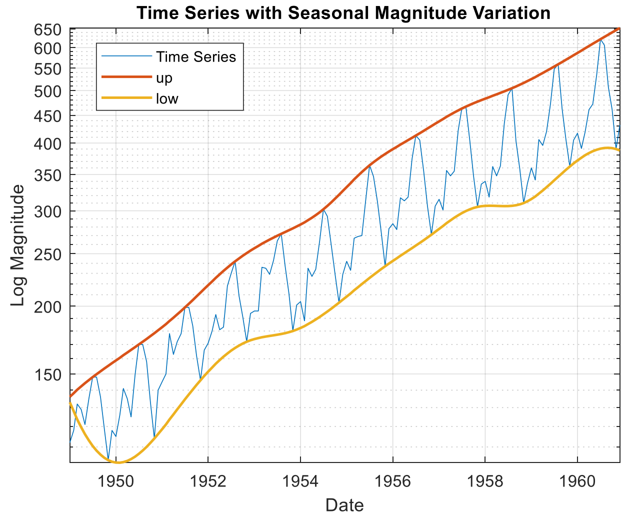
LOG FIT

Log Fit: The logarithmic transformation is used when the time series has an exponential growth trend.





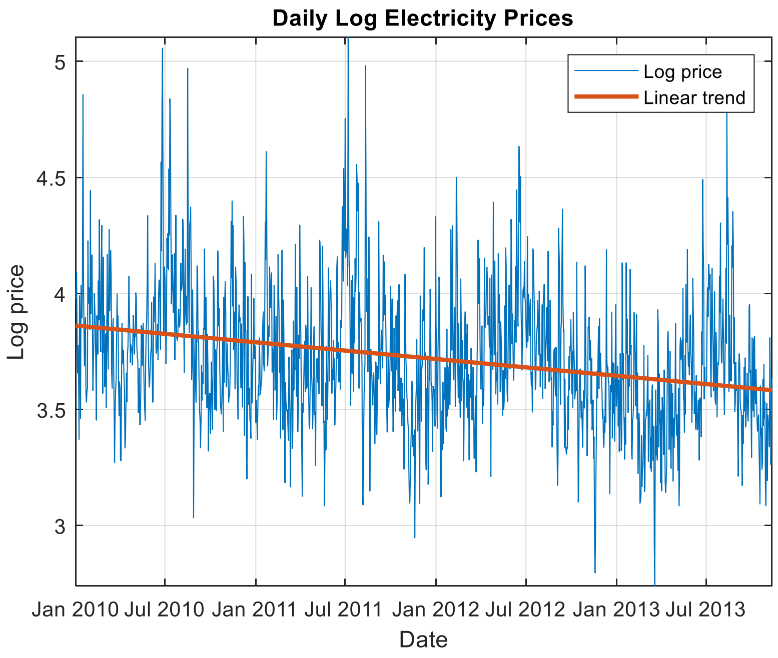
$$y = \log(y)$$



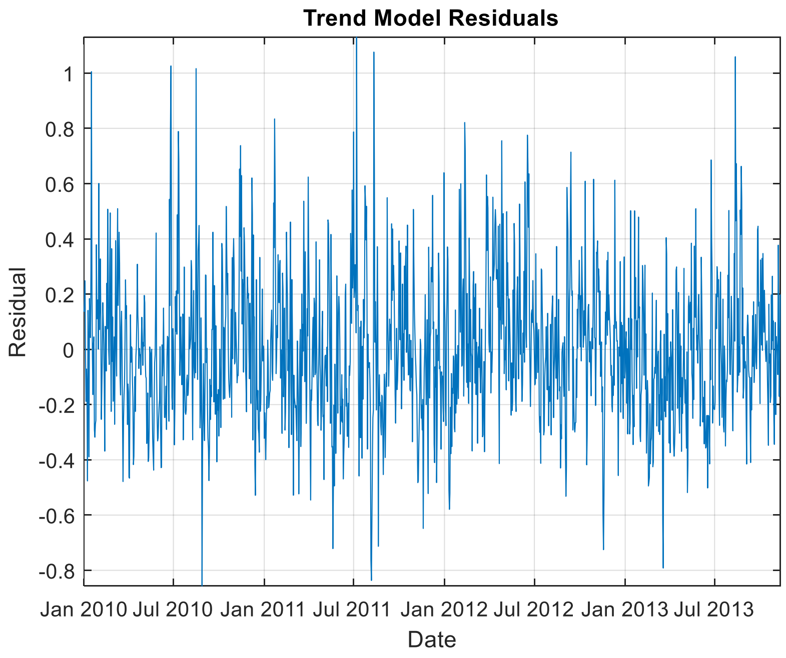
MATHEMATICAL TRANSFORMATIONS IN TIME SERIES

DETREND

Detrend: The trend of a series indicates its “long-term” behavior, that is, whether it increases, decreases, or remains stable, and how fast these changes. Transformations for detrend are implemented in several ways.



Form a detrended time series by removing the estimated long-term deterministic trend from the data. In other words, extract the residuals from the fitted linear regression model.

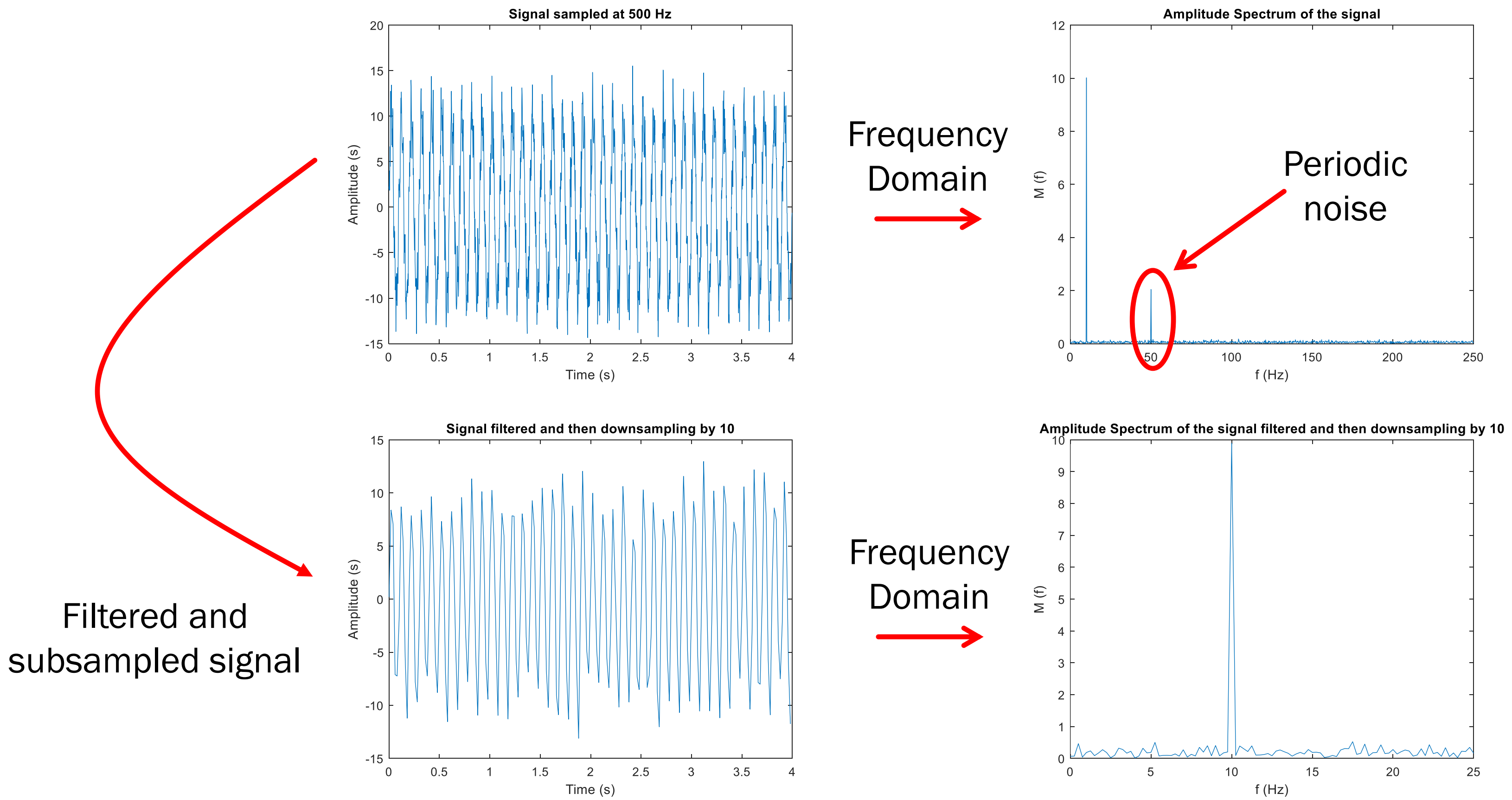


ANALYSIS IN THE FREQUENCY DOMAIN

ANALYSIS IN THE FREQUENCY DOMAIN

CONCEPT

So far we have analyzed time series in the time domain.
Analysis in the frequency domain allows us to identify which frequency components are relevant, which periodic noises can be filtered, and whether the signal can be subsampled to speed up its processing or improve storage, among other tasks. This analysis is most effective on stationary time series.



Sampling frequency (fs): 500 Hz.
Maximum signal frequency: 250 Hz

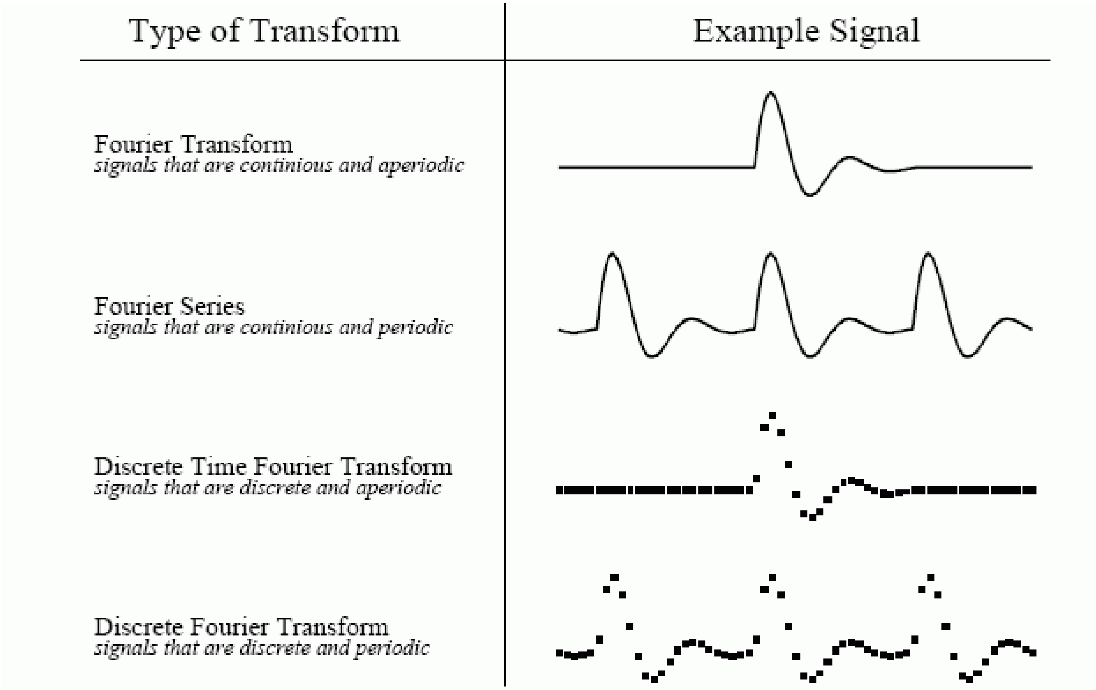
ANALYSIS IN THE FREQUENCY DOMAIN

CONCEPT

To analyze a time series in the frequency domain, we need to apply Fourier Transform. The Fourier transform is a mapping from time domain to the frequency domain.

There are four types of Fourier Transform, depending on the nature of the signal (all Fourier Transforms assume signals have infinite duration in time):

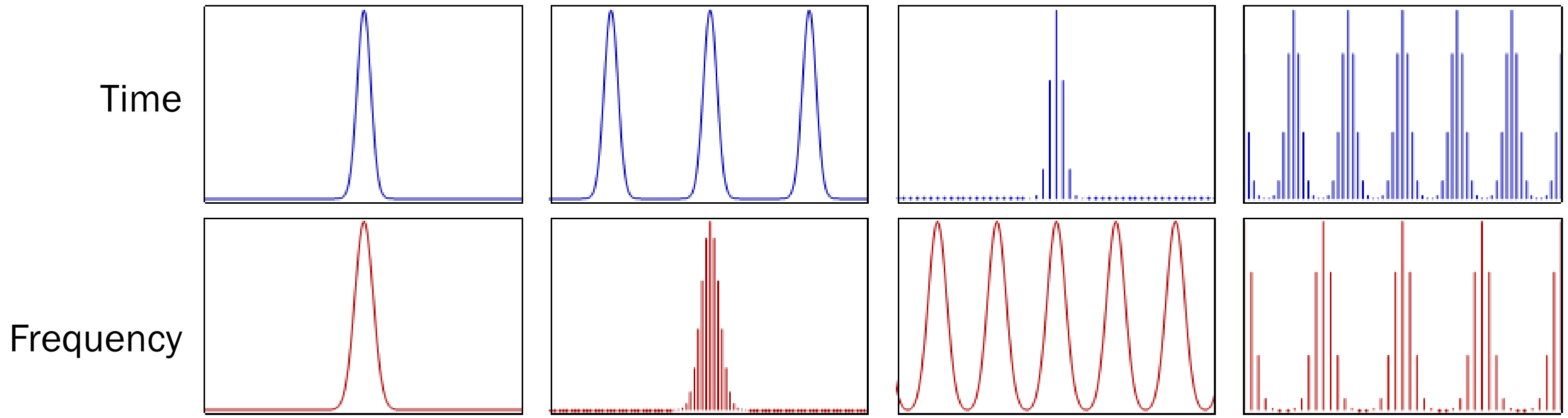
- Fourier Transform → for continuous and aperiodic signals;
- Fourier Series → for continuous and periodic signals;
- Discrete Time Fourier Transform (DTFT) → for discrete and aperiodic signals;
- Discrete Fourier Transform (DFT) → for discrete and periodic signals;



ANALYSIS IN THE FREQUENCY DOMAIN

CONCEPT

Relationship between time and frequency



Time	→	Frequency
Continuous in time	→	Aperiodic in frequency
Discrete in time	→	Periodic in frequency
Periodic in time	→	Discrete in frequency
Aperiodic in time	→	Continuous in frequency

impossible on the computer

impossible on the computer

Due to limitations, <u>we can only handle discrete signals</u> on computers. Therefore, we can only use a discrete and periodic signal in time when working with both domains (time and frequency).

ANALYSIS IN THE FREQUENCY DOMAIN

CONCEPT

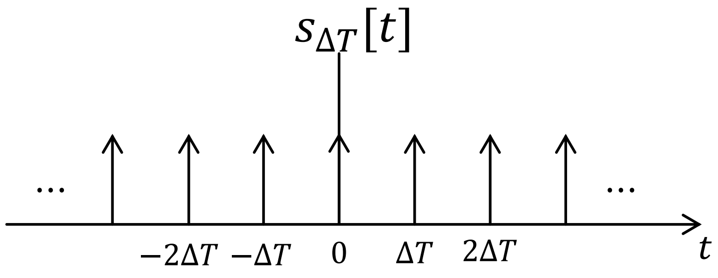
Due to the limitation of the type of data that we can process on the computer, we can only work with data that are discrete in both the time domain and the frequency domain.

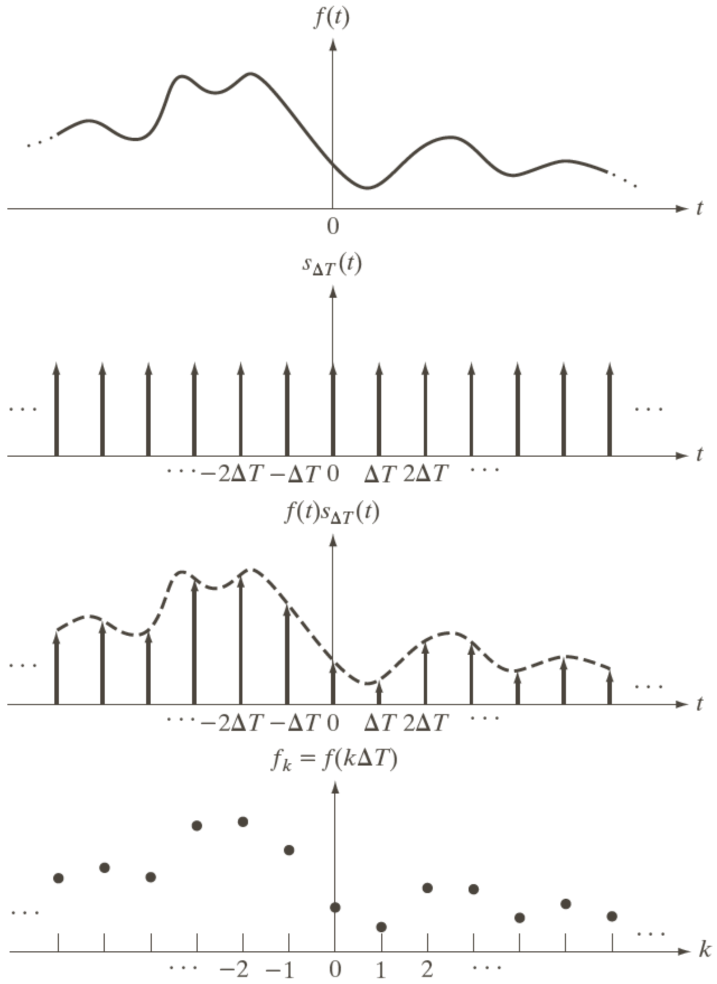
Therefore, before storing and processing any signal in a computer it must be discretized.

For this, it is required to perform a signal sampling. A signal is sampled by an impulse train.

Impulse train $s_{\Delta T}[t]$ is defined as the sum of infinitely many periodic impulses ΔT units apart ($f_s = 1/\Delta T$).

$$s_{\Delta T}[t] = \sum_{n=-\infty}^{\infty} \delta[t - n\Delta T]$$





ANALYSIS IN THE FREQUENCY DOMAIN

CONCEPT

Due to the limitation of the type of data that we can process on the computer, we can only work with data that are discrete in both the time domain and the frequency domain.

Time	→	Frequency
Continuous in time	→	Aperiodic in frequency
Discrete in time	→	Periodic in frequency
Periodic in time	→	Discrete in frequency
Aperiodic in time	→	Continuous in frequency

This limits us to using a signal that is discrete and periodic in time whose counterpart in the frequency domain is discrete and periodic.

The only Fourier Transform that we can use to process signals of this type is the Discrete Fourier Transform:

Discrete Fourier Transform (DFT): It treats periodic and discrete signals. It considers that the analysis interval is repeated periodically. The signal is represented by a finite number of senoids.

ANALYSIS IN THE FREQUENCY DOMAIN

DISCRETE FOURIER TRANSFORM (DFT)

Discrete Fourier Transform of a sampled function $f[x]$ returns an infinite and periodic sequence in the frequency domain. Therefore, all we need to characterize is only one period that is limited by the number of samples of sampled function $f[x]$.

Let M be the number of samples of $f[x]$, DFT of $f[x]$ is:

$$F[\mu] = \sum_{x=0}^{M-1} f[x] e^{-j2\pi\mu x/M} \quad \mu = 0,1,2,\dots,M-1$$

where $F[\mu]$ is the frequency component in the frequency μ.

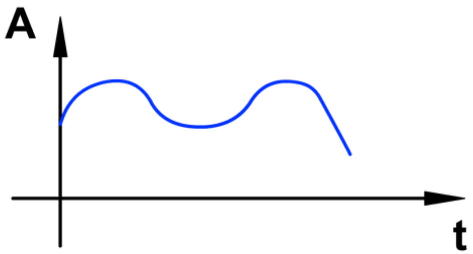
Inverse transform of DFT (IDFT) is:

$$f[x] = \frac{1}{M} \sum_{\mu=0}^{M-1} F[\mu] e^{j2\pi\mu x/M} \qquad x = 0,1,2,\dots,M-1$$

ANALYSIS IN THE FREQUENCY DOMAIN

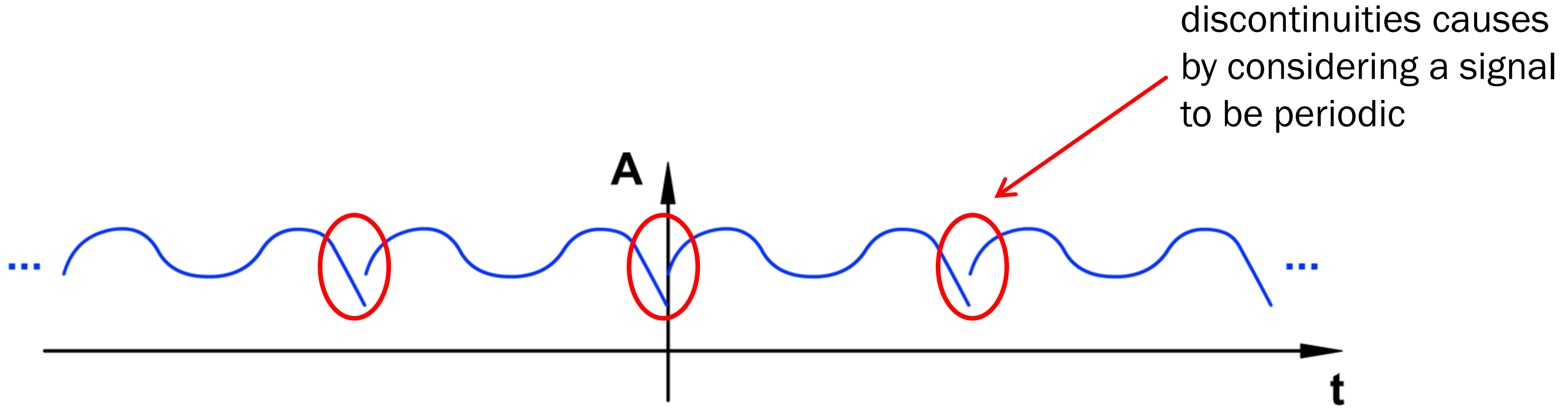
DISCRETE FOURIER TRANSFORM (DFT)

The Discrete Fourier Transform (DFT) assumes that the signal is infinitely periodic in time. However, this is not the real-world situation.



Discrete signal

In practice, the DFT assumes that there are infinite copies of the signal being processed.



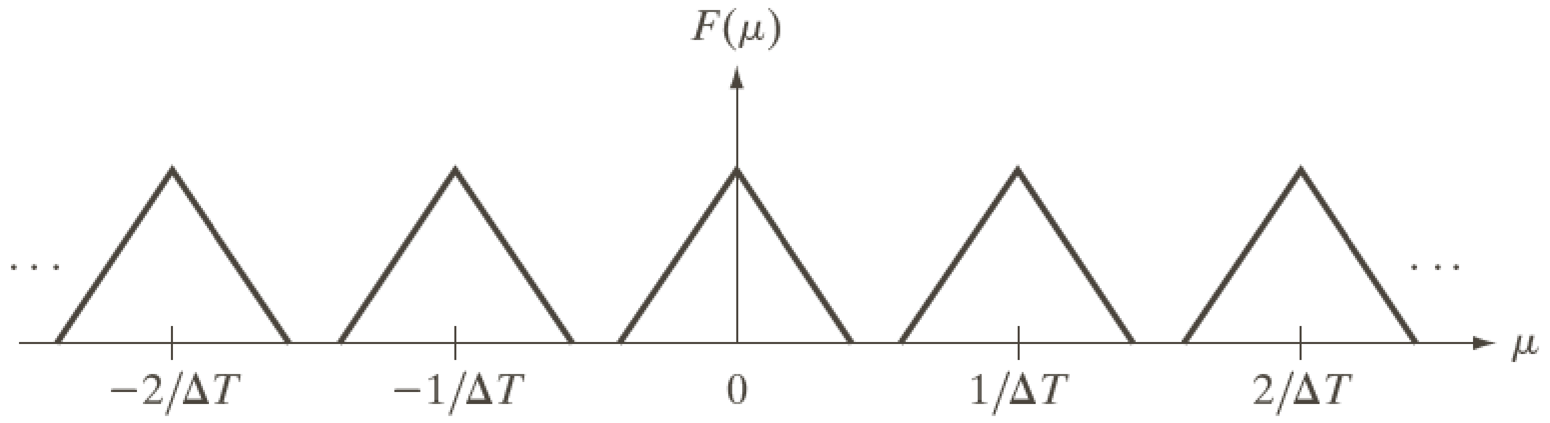
Discrete signal “seen” by DFT

ANALYSIS IN THE FREQUENCY DOMAIN

DISCRETE FOURIER TRANSFORM (DFT)

In the frequency domain, the DFT returns an infinite periodic sequence of the representation of the signal in the frequency domain (infinite copies). Therefore, to represent the signal in the frequency domain, we need only a single copy.

The separation between copies is determined by the value of $f_s = \frac{1}{\Delta T}$, where f_s is the sampling rate used to generate the sampled signal (discrete signal).



ANALYSIS IN THE FREQUENCY DOMAIN

DISCRETE FOURIER TRANSFORM (DFT)

Example 7: Compute DFT of four samples from a discrete signal obtained at intervals ΔT:

$$f[0] = 1 \;\; f[1] = 2 \;\; f[2] = 4 \;\; f[3] = 4$$

$$F[0] = \sum_{x=0}^{M-1} f[x] e^{-j2\pi 0x/M} = (1 + 2 + 4 + 4) \times 1 = 11$$

$$F[1] = \sum_{x=0}^{M-1} f[x] e^{-j2\pi 1x/M} = 1e^{0} + 2e^{-j\pi/2} + 4e^{-j\pi} + 4e^{-j3\pi/2} = -3 + 2j$$

$$F[2] = \sum_{x=0}^{M-1} f[x] e^{-j2\pi 2x/M} = -1$$

$$F[3] = \sum_{x=0}^{M-1} f[x] e^{-j2\pi 2x/M} = -3 - 2j$$

ANALYSIS IN THE FREQUENCY DOMAIN

DISCRETE FOURIER TRANSFORM (DFT)

A signal $f[t]$ in time is transformed to the frequency domain $F[\mu]$, where it usually has complex components.

A common representation of values in the frequency domain is:

$$F[\mu] = Re[\mu] + Im[\mu] = |F[\mu]|e^{j\theta[\mu]}, \text{ where } |F[\mu]| \geq 0 \text{ and } e^{j\theta[\mu]} = \cos(\theta[\mu]) + j\sin(\theta[\mu])$$

$$|F[\mu]| = \sqrt{Re^2[\mu] + Im^2[\mu]} \quad \phi[\mu] = \tan^{-1}\left(\frac{Im[\mu]}{Re[\mu]}\right)$$

We can represent a function $F[\mu]$ by a graph plotted between the Fourier coefficients and the frequency. This graph is known as Fourier spectrum.

The Fourier spectrum has two parts, both plotted as a function of frequency μ:

- Amplitude (magnitude) spectrum ($|F[\mu]|$);
- Phase spectrum (($\theta[\mu]$)).

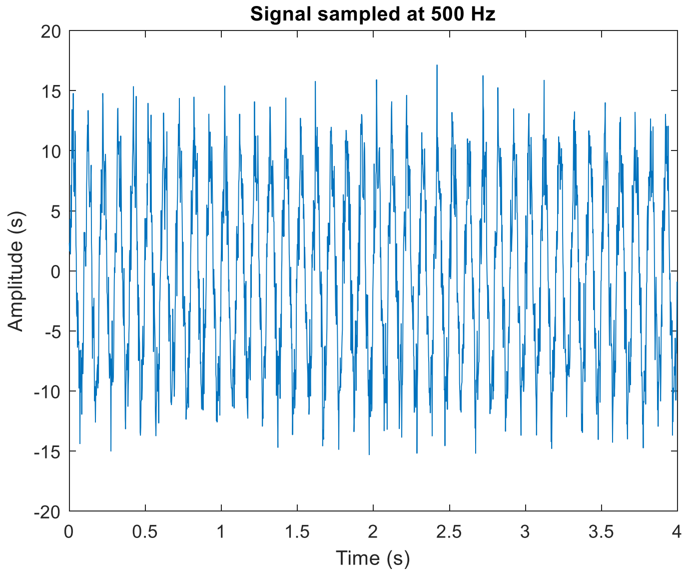
The representation by amplitude is easier to extract information intuitively.

Fourier Power Spectrum: $P[\mu] = |F[\mu]|^2$

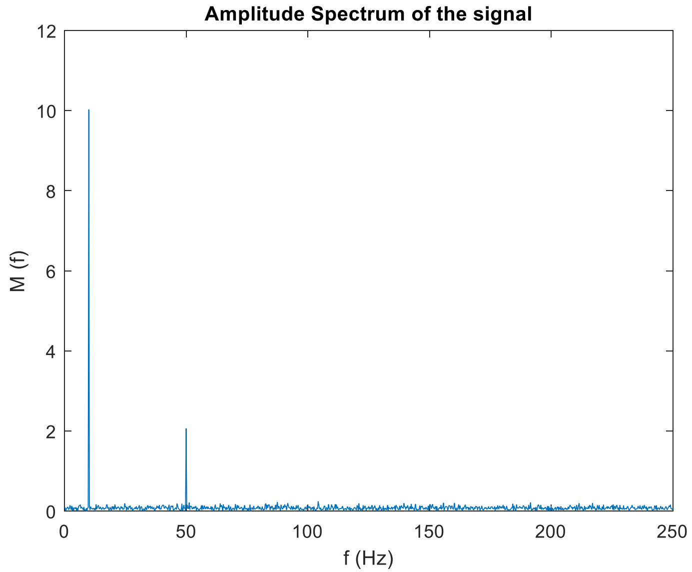
ANALYSIS IN THE FREQUENCY DOMAIN

DISCRETE FOURIER TRANSFORM (DFT)

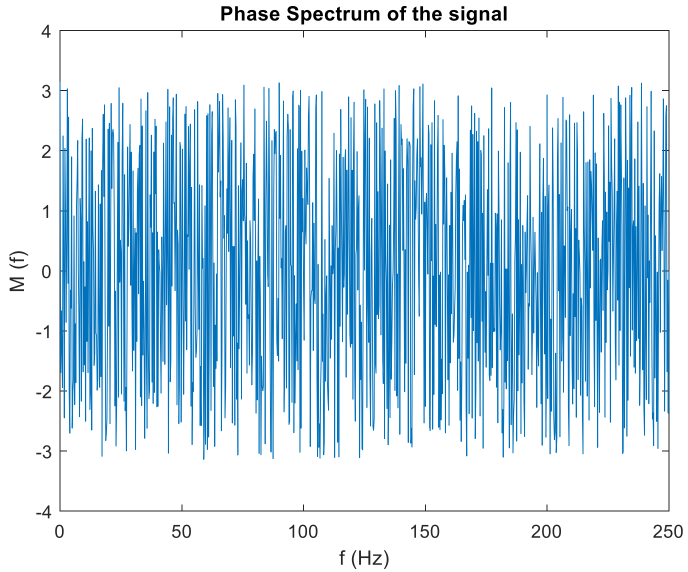
Graphic representation of Amplitude spectrum and Phase spectrum



Signal (in time)



Amplitude (in frequency)



Phase (in frequency)

ANALYSIS IN THE FREQUENCY DOMAIN

DISCRETE FOURIER TRANSFORM (DFT)

Nyquist rate and aliasing

If a signal is sampled at a rate that is not greater than twice its highest frequency (Nyquist rate) will happen a phenomenon known as frequency aliasing or simply as aliasing.

Aliasing is a process in which high frequency components of a continuous signal “masquerade” as lower frequencies in the sampled signal.

To avoid the aliasing problem, the continuous signal must pass by a lowpass filter to limit the band to a finite frequency range before the signal is sampled.

Unfortunately, except for some special cases, aliasing is always present in sampled signals because, even if the original sampled signal is band-limited[1], infinite frequency components are introduced the moment we limit the duration of the signal[2] (a signal that is band-limited must extend from $-\infty$ to ∞).

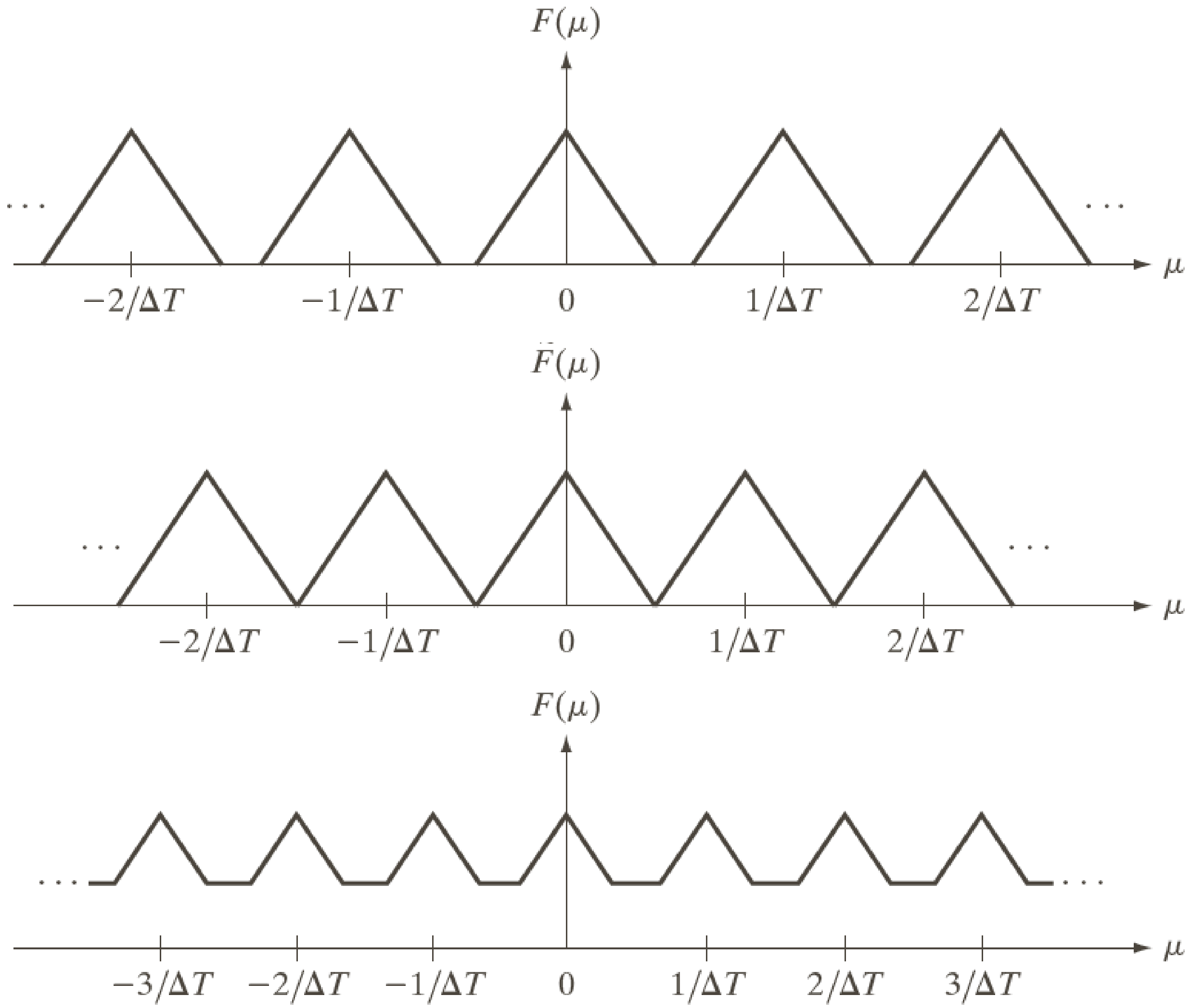
Aliasing cannot be undone after sampling.

[1] A function whose Fourier Transform is zero for values of frequencies outside a finite interval.
[2] See the discontinuities in the periodic signal of slide 29.

ANALYSIS IN THE FREQUENCY DOMAIN

DISCRETE FOURIER TRANSFORM (DFT)

Nyquist rate and aliasing



$f_s > 2f_{NR}$

$f_s = 2f_{NR}$

$f_s < 2f_{NR}$

ANALYSIS IN THE FREQUENCY DOMAIN

SOME PROPERTIES OF THE DFT

Periodicity

The Discrete Fourier Transform and its inverse are infinitely periodic, that is,

$$f[x] \Leftrightarrow f[x+kM]$$
$$F[\mu] \Leftrightarrow F[\mu+kM]$$

where k is integer and M is the signal size.

ANALYSIS IN THE FREQUENCY DOMAIN

SOME PROPERTIES OF THE DFT

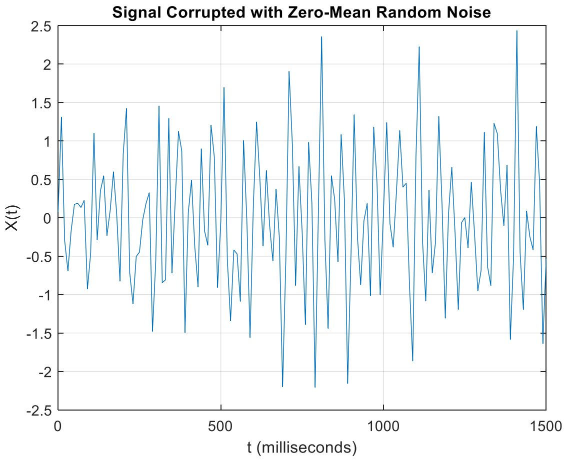
Symmetry

The DFT of a real function, $f[x]$, is conjugate symmetric, and the magnitude is mirrored in relation to the origin:

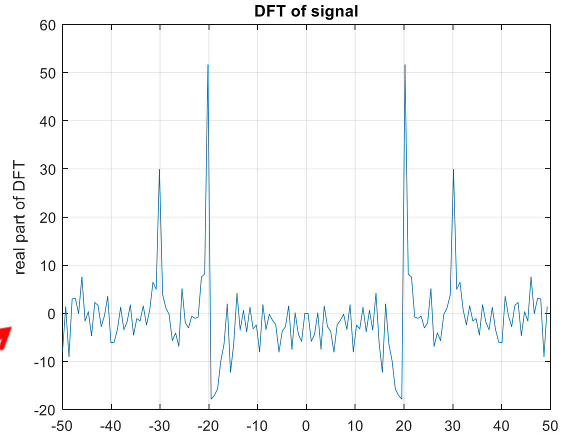
$$|F[\mu]| = |F[-\mu]|$$

$F[\mu]$ is the conjugate complex of $F[-\mu]$:

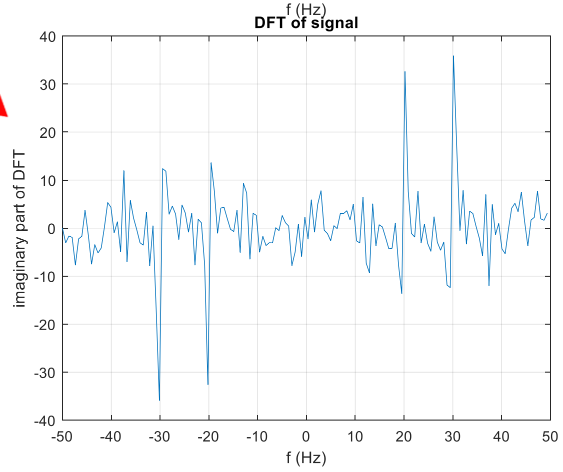
$$F[\mu] = F^*[-\mu]$$
$$F[\mu] = R[\mu] + jI[\mu]$$
$$F[-\mu] = R[\mu] - jI[\mu]$$



Signal



Real part



Imaginary part

ANALYSIS IN THE FREQUENCY DOMAIN

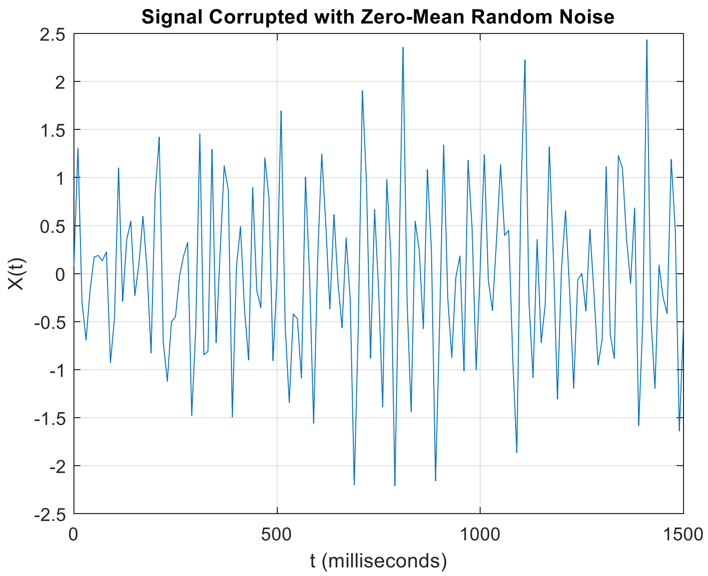
SOME PROPERTIES OF THE DFT

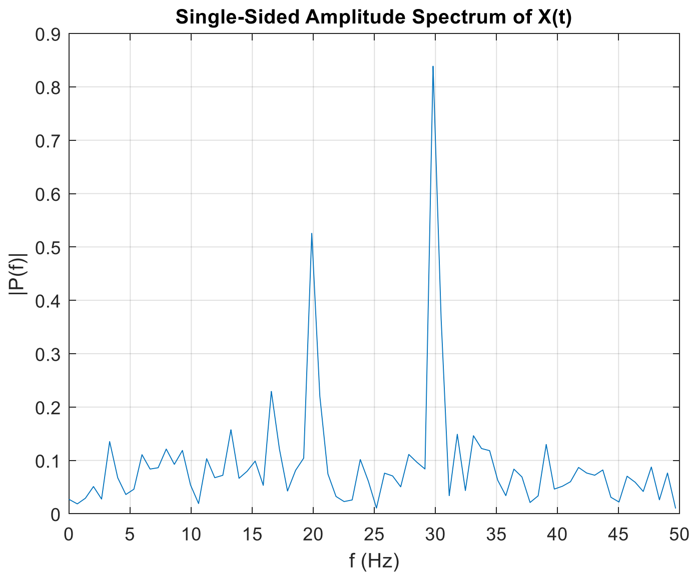
Symmetry

Due to symmetry, we usually only see one single-side of the Amplitude spectrum after DFT, so we calculate the amplitude divided by the signal size M and multiply by 2 the components with frequency greater than zero:

$F[\mu] = \Im(f[x])$, where $\Im$ is DFT

$$P[\mu] = \begin{cases} \dfrac{|F[\mu]|}{M}, \text{for } \mu = 0 \\ \dfrac{2|F[\mu]|}{M}, \text{for } 0 < \mu \leq \dfrac{M}{2} \end{cases}$$





Ex1Cap3

ANALYSIS IN THE FREQUENCY DOMAIN

SOME PROPERTIES OF THE DFT

Linearity

If a discrete function $f[x]$ is multiplied/divided by a constant, its transform $F[\mu]$ will be multiplied/divided by the same constant.

If a discrete function $f_1[x]$ is added to another function $f_2[x]$, the resulting transform will be the sum of its transforms.

$$af_1[x] + bf_2[x] \Leftrightarrow aF_1[\mu] + bF_2[\mu]$$

ANALYSIS IN THE FREQUENCY DOMAIN

SOME PROPERTIES OF THE DFT

Translation

$$f[x]e^{j2\pi(\mu_0 x/M)} \Leftrightarrow F[\mu - \mu_0]$$
$$f[x - x_0] \Leftrightarrow F[\mu]e^{-j2\pi(\mu x_0/M)}$$

Multiplying $f[x]$ by the exponential shown in the equation shifts the origin of the DFT to μ_0.

Conversely, multiplying $F[\mu]$ by the negative of that exponential shifts the origin of $f[x]$ to x_0.

Translation of $f[x]$ has no effect on the magnitude of $F[\mu]$, but affects the phase angle.

FREQUENCY DOMAIN FILTERING FUNDAMENTALS

FREQUENCY DOMAIN FILTERING FUNDAMENTALS

CHARACTERISTICS

The goal of the DFT is to decompose a sampled signal $f[x]$ into a set of DFT coefficients $F[\mu]$, which tell which frequency components are present in the signal as well as their relative intensity. Visualizing the Amplitude spectrum divided by M makes the visualization independent of the signal size.

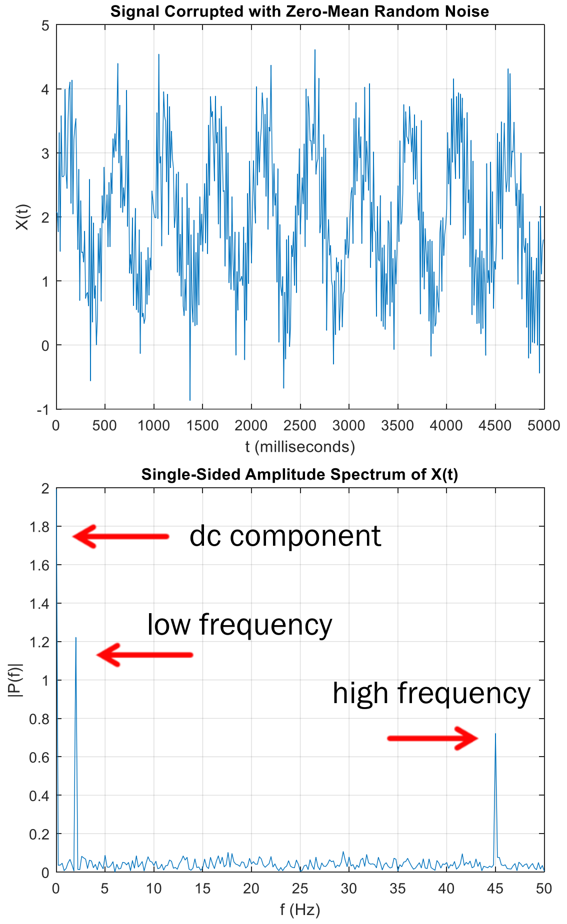
$$x[t] = 1.2\sin(2\pi 2t) + 0.7\sin(2\pi 45t) + 2 + \epsilon$$

The slowest varying frequency component ($\mu = 0$) of $F[\mu]$ is proportional to the average intensity of a signal ($\bar{f} = \frac{F[0]}{M}$).

As we move away from the origin of the transform, but close to it, the components correspond to low frequencies of $F[\mu]$, i.e., the slowly varying intensity components.

As we move further away from the origin, the higher frequencies begin to correspond to faster and faster intensity changes in the signal. These are the high frequencies of $F[\mu]$. These are the elements characterized by abrupt changes in intensity.

When we apply the DFT to the signal, it is not possible to identify at what instant in time a certain frequency component occurred.



FREQUENCY DOMAIN FILTERING FUNDAMENTALS

FILTERING IN THE FREQUENCY DOMAIN

The basic filtering equation has the form:

$$g[x] = \mathfrak{I}^{-1}(H[\mu]F[\mu])$$

where $\mathfrak{I}^{-1}$ is the IDFT, $F[\mu]$ is the DFT of the input signal, $H[\mu]$ is a filter function and $g[x]$ is the filtered (output) signal. All functions are of size M.

Note: If $H[\mu]$ is real and symmetric and $f[x]$ is real, then $g[x]$ will be real.

Due to some computational rounding, the inverse transform may return a residual imaginary part. In this case, we take only the real part:

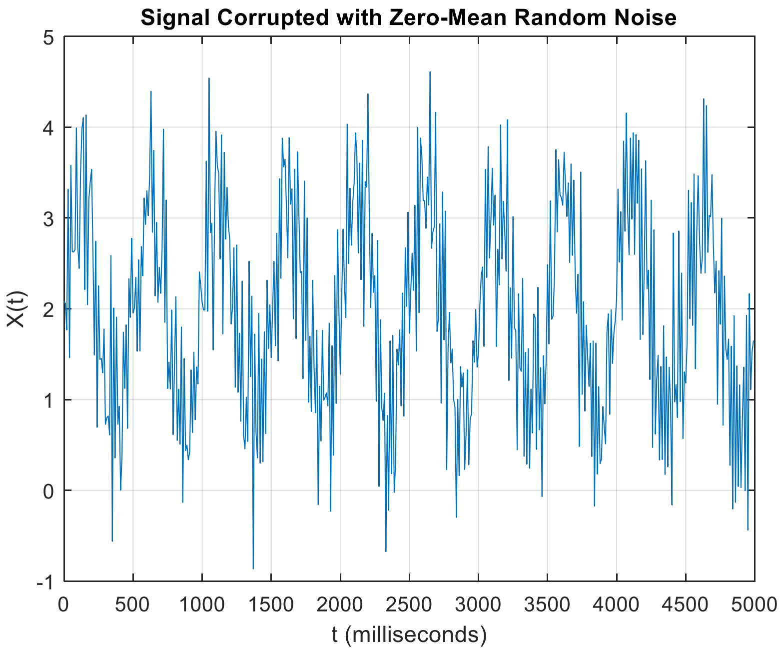
$$g[x] = \mathrm{real}(\mathfrak{I}^{-1}(H[\mu]F[\mu]))$$

FREQUENCY DOMAIN FILTERING FUNDAMENTALS

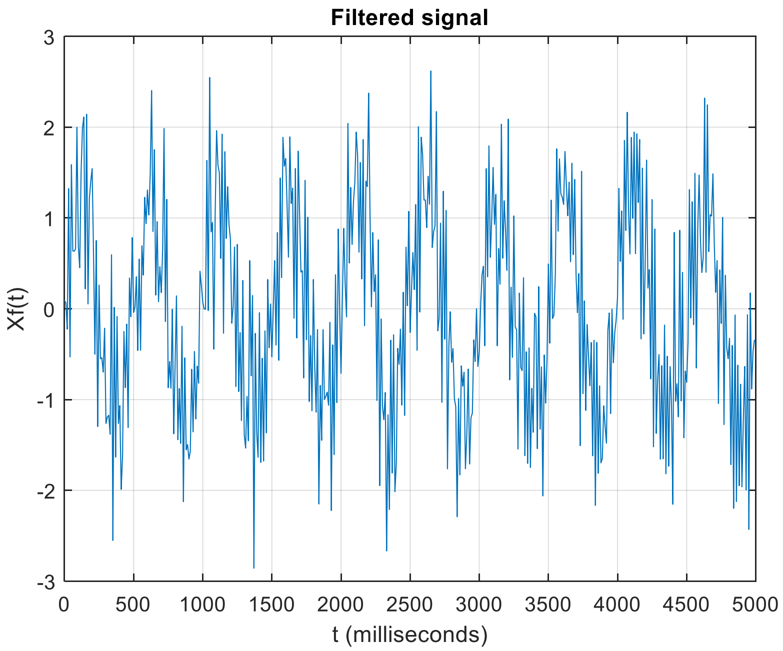
FILTERING IN THE FREQUENCY DOMAIN

Example: One of the simplest filters we can construct is a filter $H[\mu]$ that is 0 at $H[0]$ and 1 elsewhere. It rejects only the dc term (frequency zero).

$$H[\mu] = \begin{cases} 0; & \mu = 0 \\ 1; & \mu \neq 0 \end{cases}$$



$$\mathfrak{I}^{-1}\left(H[\mu]\mathfrak{I}(f[x])\right)$$



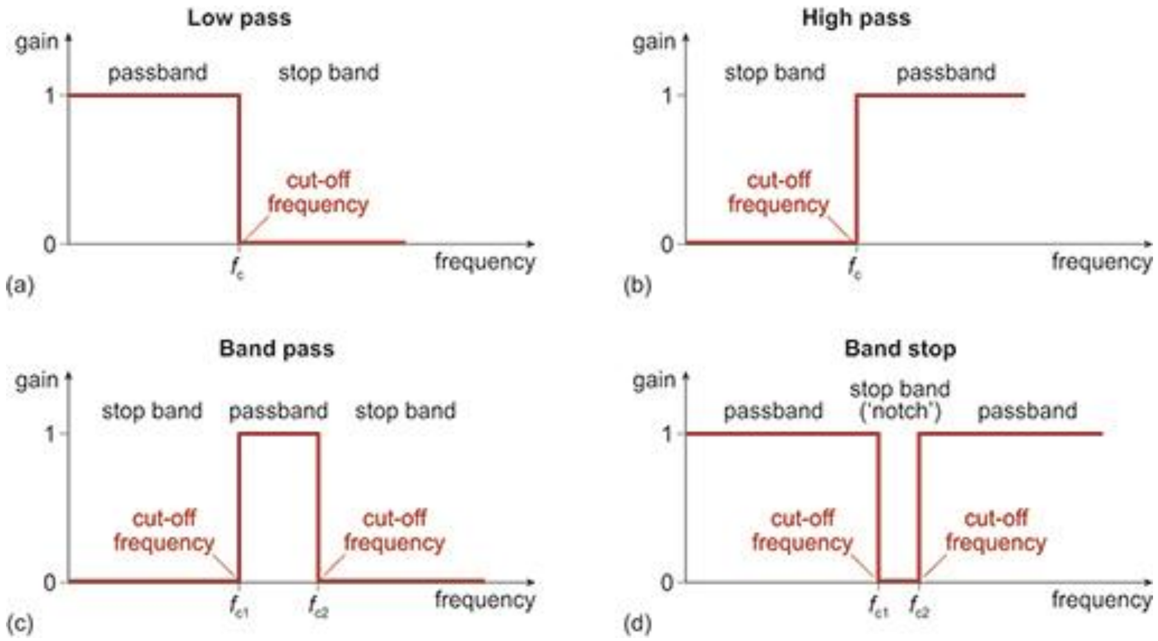
FREQUENCY DOMAIN FILTERING FUNDAMENTALS

FILTERS

In the design of frequency selective filters (which allow certain frequencies to pass and attenuate others), a unitary gain is generally adopted in the passband and zero in the stop-band, accepting some tolerances in the pass and rejection bands.

Typical filters are:

- Low-pass filters
- High-pass filters
- Band-pass filters
- Band-reject filters



Ideal filter: The cutoff frequency defines exactly which frequencies pass without attenuation and which are completely rejected. However, it adds artifacts.

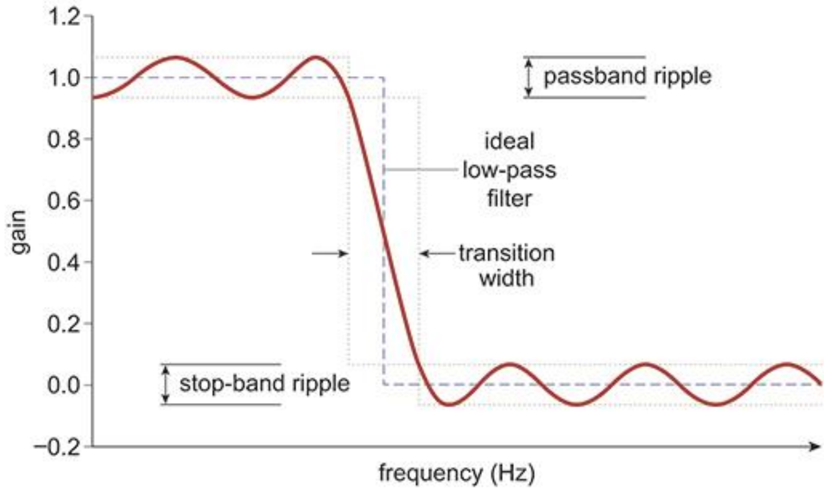
FREQUENCY DOMAIN FILTERING FUNDAMENTALS

FILTERS

Most filters are designed in the frequency domain. Input signals are characterized by their frequency spectrum and design filters to modify that spectrum by, for example, removing high-frequency noise with a low-pass filter.

The traditional method of designing many frequency selective filters is to first design a frequency-normalized prototype low-pass filter and then, using an algebraic transformation, derive the desired filter from the desired prototype low-pass filter.

Figure shows some of the characteristics of a typical low-pass filter. In comparison to the ideal shape (shown in dashed line), there is a transition region between the passband and stop-band sections, and also a ripple in both the passband and the stop band. These effects can be altered by changing various parameters in the design.



FREQUENCY DOMAIN FILTERING FUNDAMENTALS

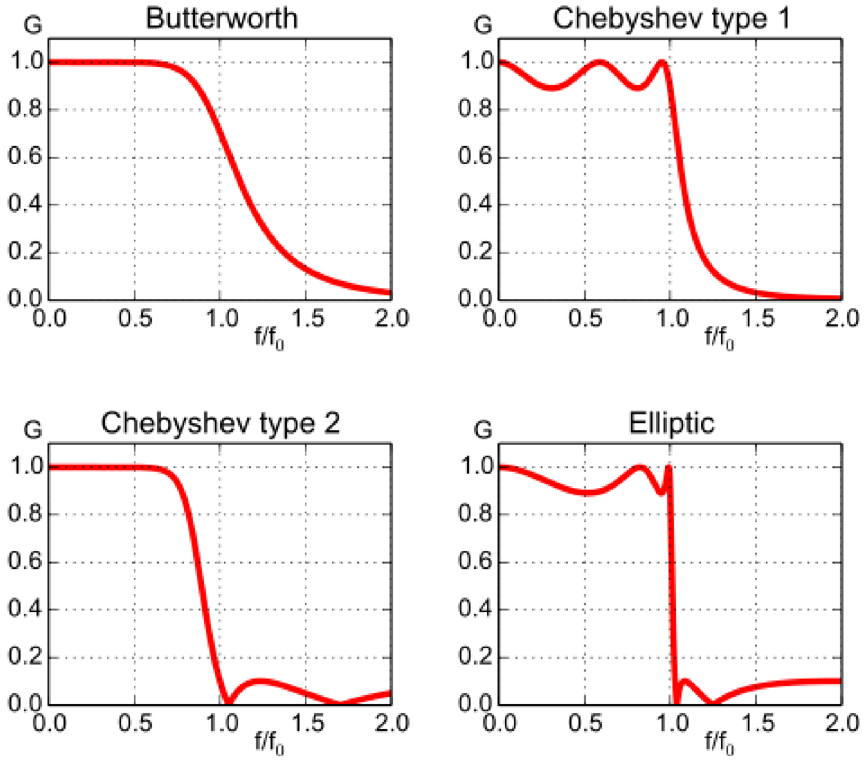
FILTERS

The most common filters in the frequency domain are:

- Butterworth: It is designed to have a frequency response that is as flat as possible in the passband. Its main characteristic is the smoothness of the transition between the passband and the stop-band. The disadvantage is that this transition can be slow, resulting in a filter that is not as sharp as other types.

- Chebyshev: Chebyshev filters come in two main types: Type I and Type II. Type I Chebyshev filters have ripples in the passband, while Type II filters have ripples in the stopband. The advantage of these filters is that they can transition more quickly between the passband and the stop-band than Butterworth filters, resulting in more efficient rejection of frequencies outside the passband.

- Elliptic: It is the most efficient in terms of rejecting unwanted frequencies, as it presents ripples in both the passband and the stop-band. This type of filter offers the fastest transition between the passband and the stop-band, and is ideal when very precise frequency discrimination is required. However, ripples can be a disadvantage depending on the application.

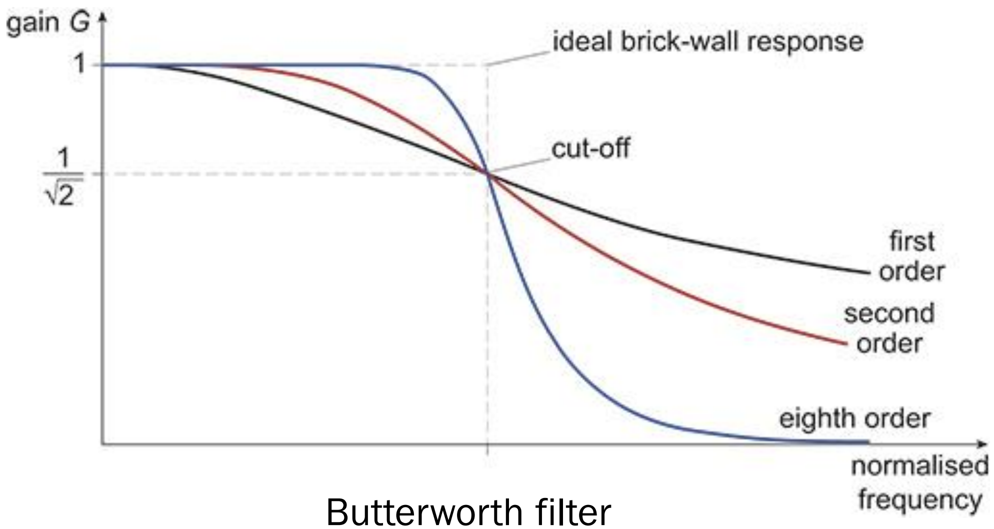


FREQUENCY DOMAIN FILTERING FUNDAMENTALS

FILTERS

Filter Specifications

- Filter type: low-pass, high-pass, band-pass, or band-band.
- Cutoff frequency: the frequency at which the filter response begins to attenuate. This is the frequency at which the amplitude attenuation is equal to $1/\sqrt{2}$, which is equivalent to attenuating the signal power by half.
- Stop-band attenuation: the amount of attenuation desired outside the passband.
- Passband and stop-band ripples: specifications that determine the type of filter (Butterworth, Chebyshev, Elliptical, etc.).



Butterworth filter

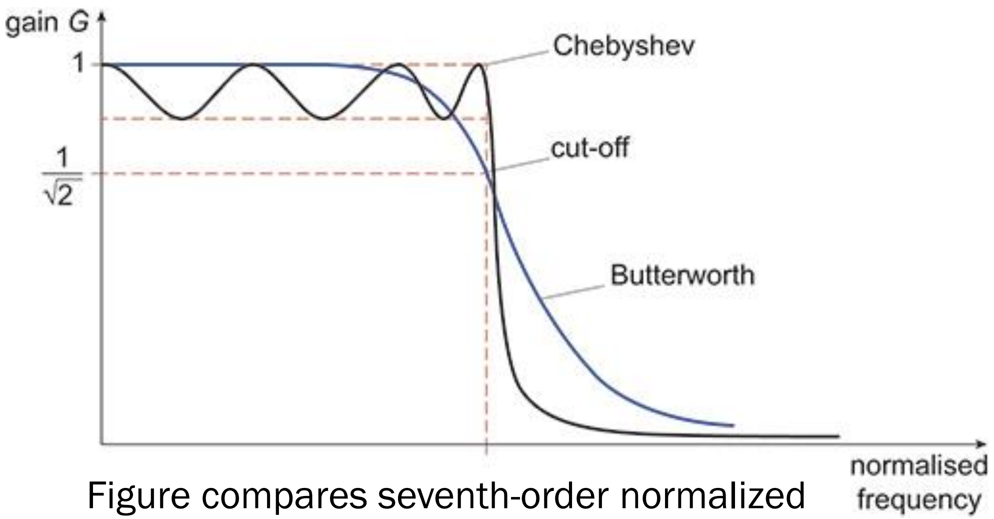


Figure compares seventh-order normalized Chebyshev and Butterworth filters

FREQUENCY DOMAIN FILTERING FUNDAMENTALS

FILTERS

Profile of filters in different orders

The attenuation of the filters can be given in decibels:

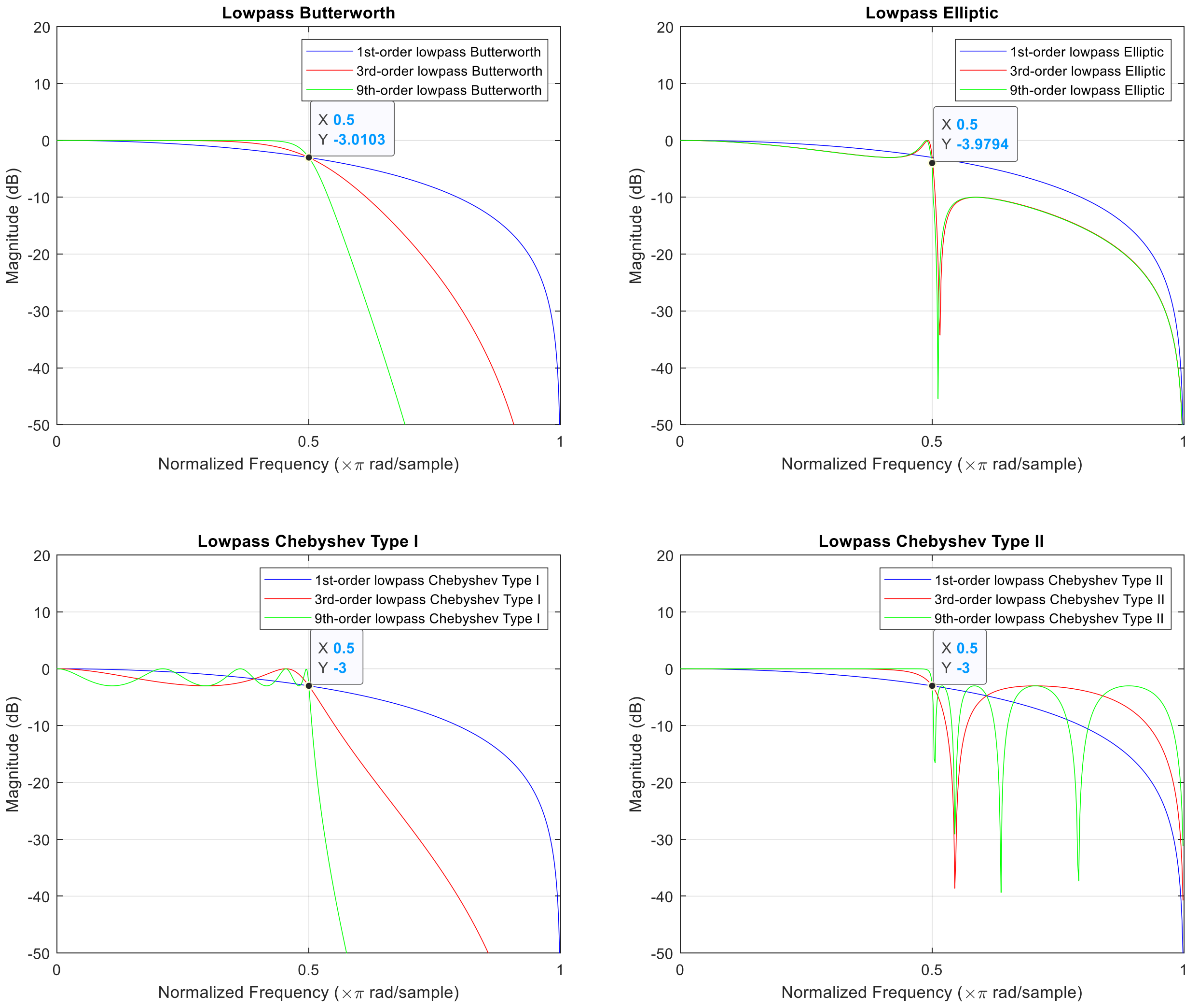
$$P_{dB}[\mu] = 10\log_{10}\left(\frac{P_{out}[\mu]}{P_{in}[\mu]}\right)$$

$$P_{dB}[\mu] = 10\log_{10}\left(\frac{\left|F_{filtered}[\mu]\right|^2}{|F[\mu]|^2}\right)$$

An attenuation of -3dB corresponds to a reduction of 50% in the original signal power:

$$P_{dB} = 10\log_{10}\left(\frac{0.5P_{in}[\mu]}{P_{in}[\mu]}\right) = -3.0103\text{ dB}$$

Cutoff frequency = 0.5/(fs/2)



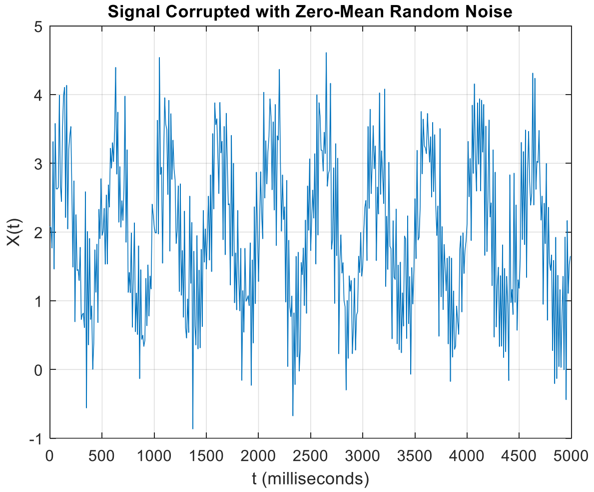
FREQUENCY DOMAIN FILTERING FUNDAMENTALS

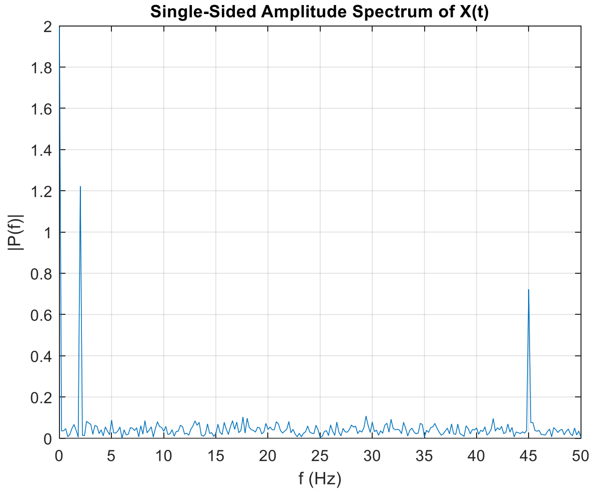
FILTERS

Example 8: Filter the signal $x[t] = 1.2\,sin(2\pi 2t) + 0.7\,sin(2\pi 45t) + 2 + \epsilon$ using a low-pass filter with a cutoff frequency of 30 Hz.

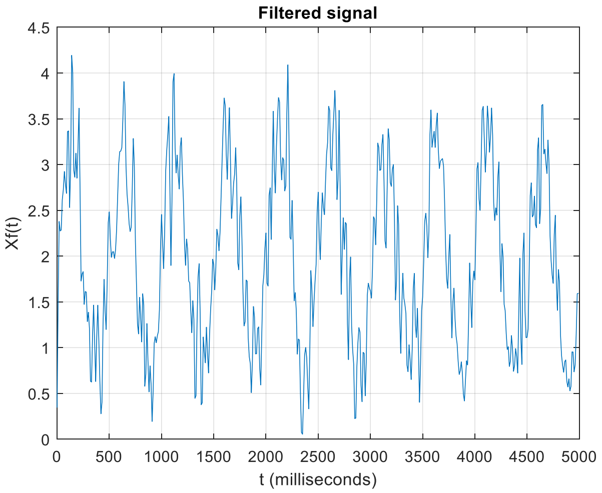
Solution:

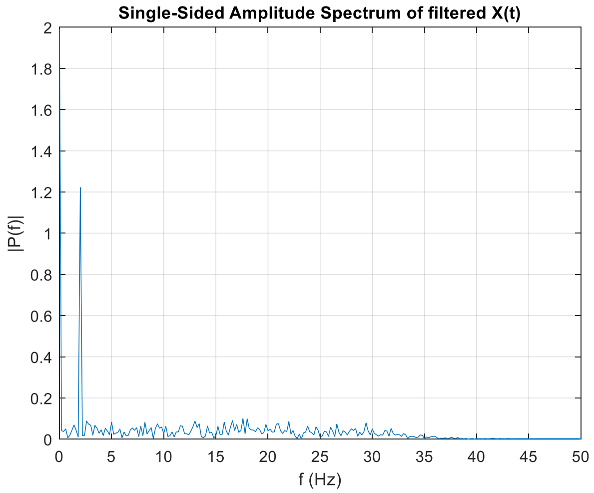
Before filtering





After filtering





FREQUENCY DOMAIN FILTERING FUNDAMENTALS

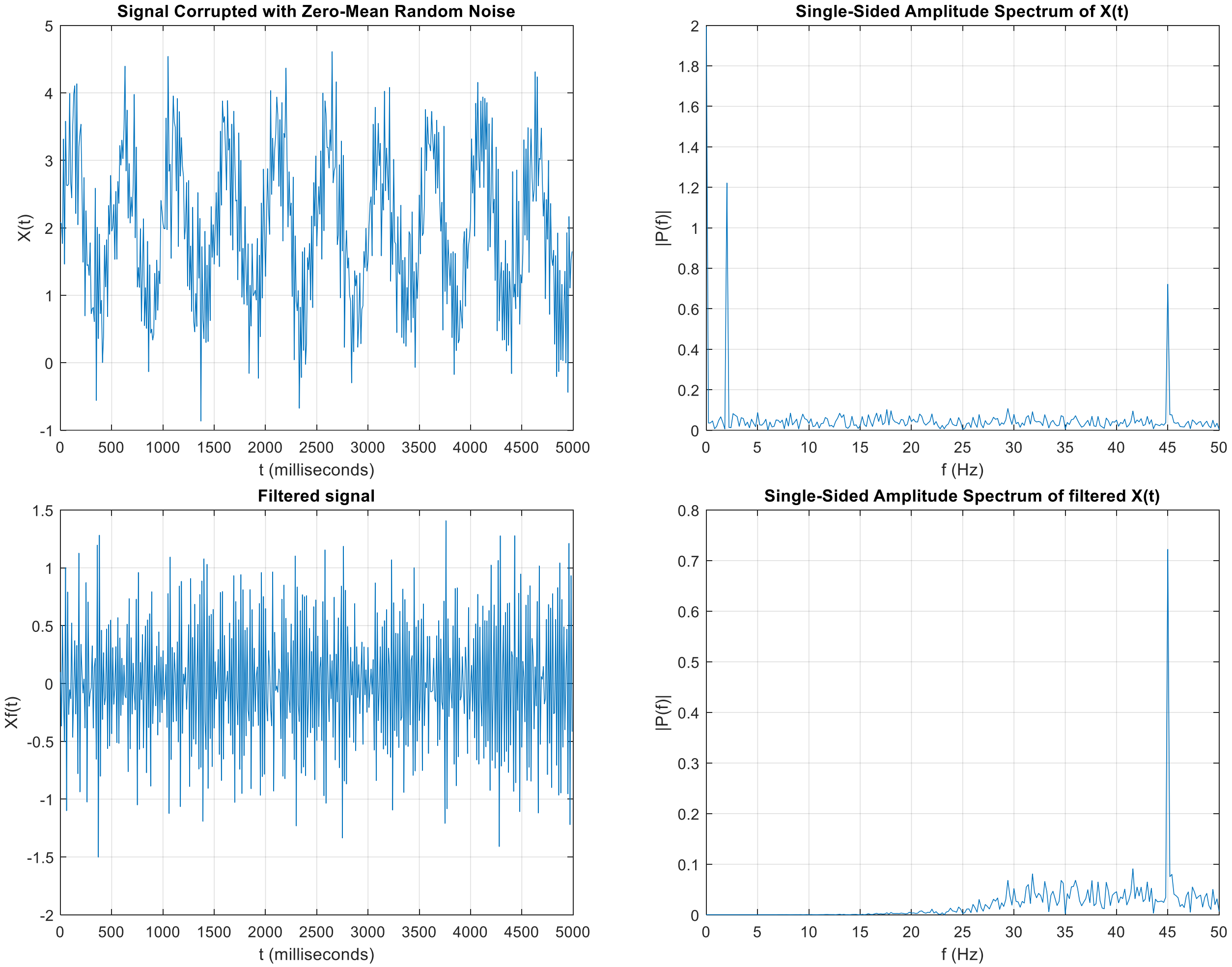
FILTERS

Example 9: Filter the signal $x[t] = 1.2\,sin(2\pi 2t) + 0.7\,sin(2\pi 45t) + 2 + \epsilon$ using a high-pass filter with a cutoff frequency of 30 Hz.

Solution:

Before filtering

After filtering



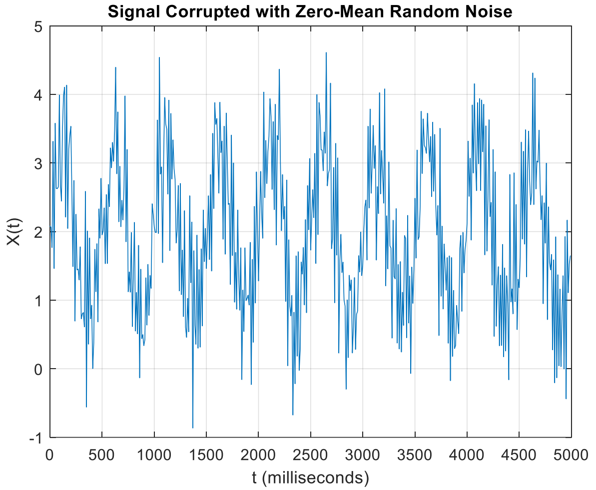
FREQUENCY DOMAIN FILTERING FUNDAMENTALS

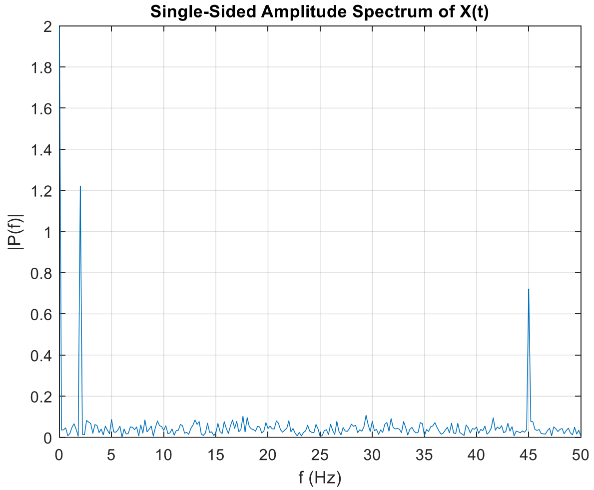
FILTERS

Example 10: Filter the signal $x[t] = 1.2\,sin(2\pi 2t) + 0.7\,sin(2\pi 45t) + 2 + \epsilon$ using a band-pass filter with cutoff frequencies of 0.5 Hz and 30 Hz.

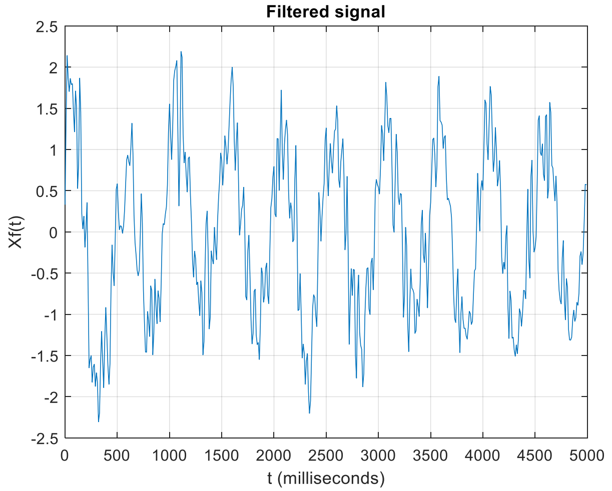
Solution:

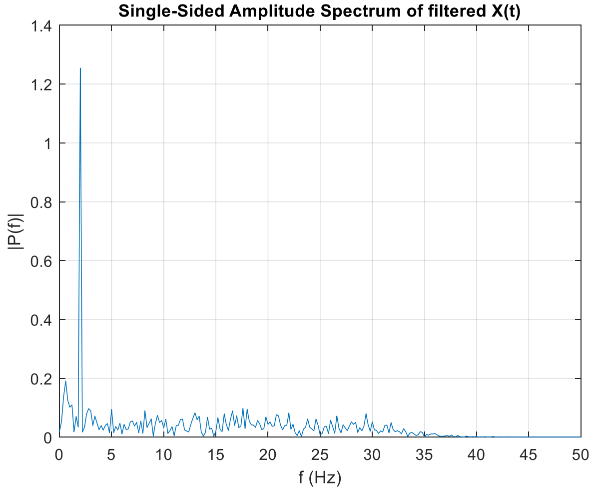
Before filtering





After filtering





FREQUENCY DOMAIN FILTERING FUNDAMENTALS

FILTERS

THE CONVOLUTION THEOREM

The product of the transforms in the frequency domain is equivalent to the convolution in the time domain. On the other hand, convolution of the transforms in frequency domain is equivalent to the product in the time domain.

$$F[\mu]H[\mu] = \Im\{f[x] \star h[x]\}$$
$$F[\mu] \star H[\mu] = \Im\{f[x]h[x]\}$$

where $\star$ is a convolution operator.

Let f be a signal with M samples:

$$g[x] = f[x] \star h[x] = \sum_{m=0}^{M-1} f[m]h[x-m]$$

FREQUENCY DOMAIN FILTERING FUNDAMENTALS

FILTERS

THE CONVOLUTION THEOREM

When convolution is calculated between f and h, it is necessary:

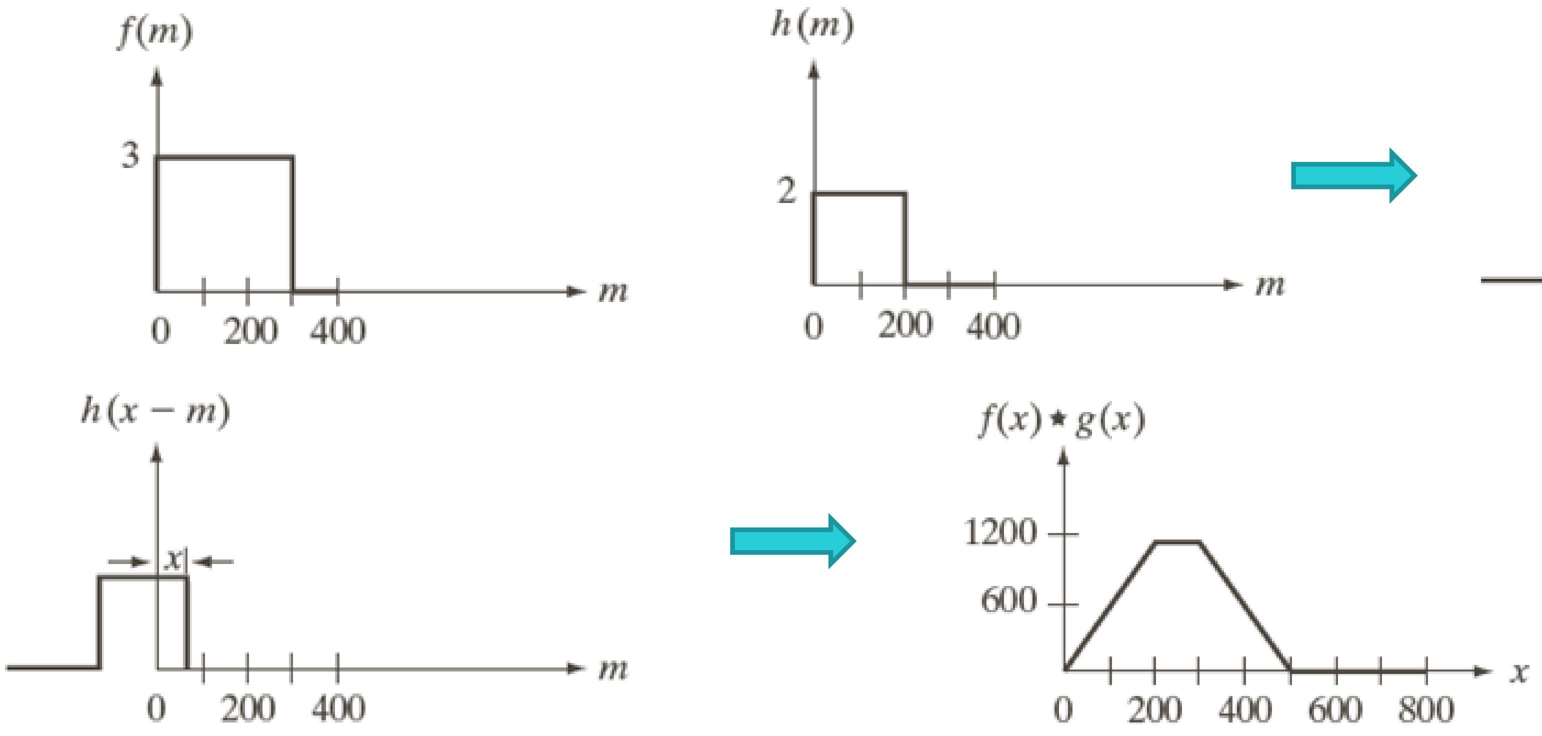
$$g[x] = f[x] \star h[x] = \sum_{m=0}^{M-1} f[m]h[x-m]$$

1) Rotating h by 180°;
2) Translating the mirrored filter h by an amount x;
3) Computing the entire sum of products between f and h for each value x of translation;
4) The displacement x must be large enough for the function h completely slides across f.

FREQUENCY DOMAIN FILTERING FUNDAMENTALS

FILTERS

THE CONVOLUTION THEOREM

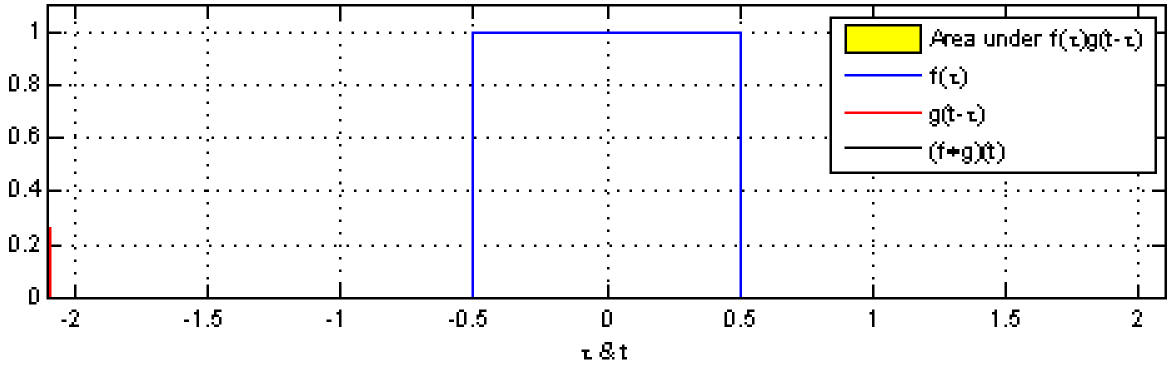


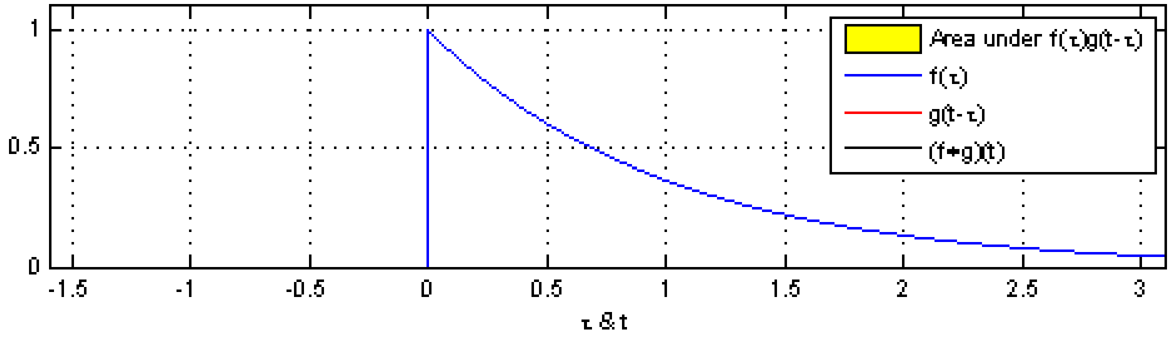
FREQUENCY DOMAIN FILTERING FUNDAMENTALS

FILTERS

THE CONVOLUTION THEOREM

The convolution can be described as the area under the function $f[\tau]$ weighted by the function $g[-\tau]$ shifted by amount t.





FREQUENCY DOMAIN FILTERING FUNDAMENTALS

FILTERS

THE CONVOLUTION THEOREM

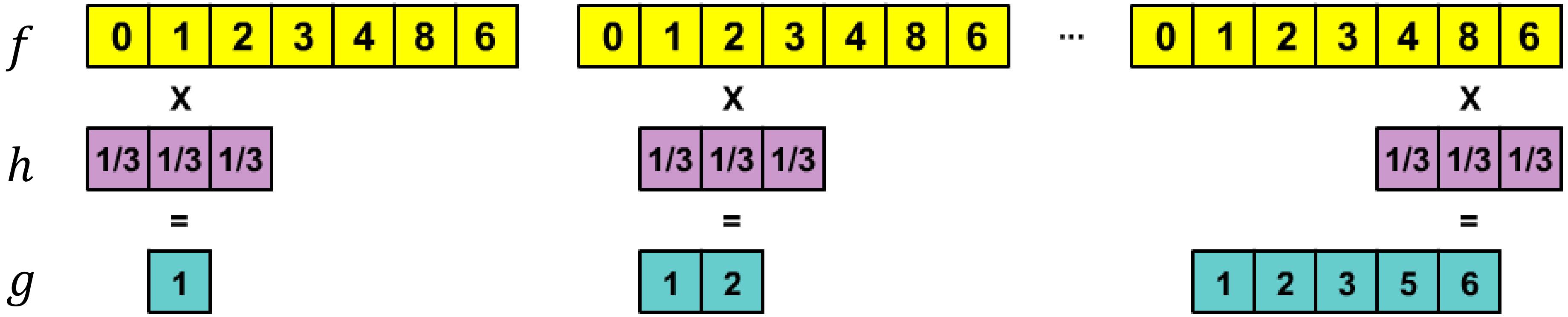
Example 11: Let f be a signal with values [0, 1, 2, 3, 4, 8, 6] and h a filter with values [1/3, 1/3, 1/3]. Compute the convolution between f and h.

Solution:

$$g[x] = f[x] \star h[x] = \sum_{m=0}^{M-1} f[m]h[x-m]$$

1) Rotating h by 180°: $h[m] = h[-m]$ (symmetric filter)
2) Translating the mirrored filter h by an amount x and computing the entire sum of products between f and h for each value x of translation:

The filter h must be entirely within f:



Note that g is not the same size as f

FREQUENCY DOMAIN FILTERING FUNDAMENTALS

FILTERS

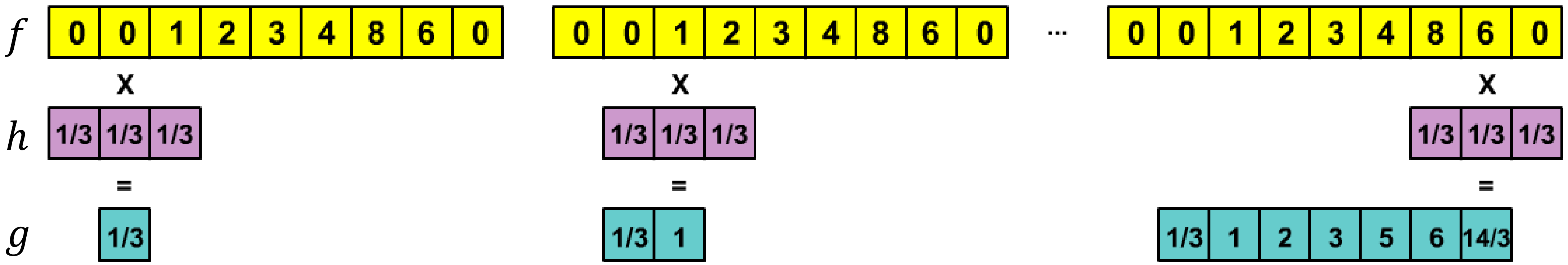
THE CONVOLUTION THEOREM

Example 12: Let f be a signal with values [0, 1, 2, 3, 4, 8, 6] and h a filter with values [1/3, 1/3, 1/3]. Compute the convolution between f and h, but the output signal must have the same size as the input signal.

Solution:

$$g[x] = f[x] \star h[x] = \sum_{m=0}^{M-1} f[m]h[x-m]$$

To make the output the same size as the input, we can apply zero padding at the beginning and end of the input signal. For a filter of size N, there must be $\lfloor N/2 \rfloor$ zeros on each side of the signal.



FREQUENCY DOMAIN FILTERING FUNDAMENTALS

FILTERS

THE CONVOLUTION THEOREM

Filters designed in the frequency domain, such as Butterworth, Chebyshev, and Elliptic, are often applied in the time domain through convolution. In the time domain, they are represented by:

$$h[z] = \frac{b_0 + \sum_{i=1}^{k} b_i z^{-i}}{1 + \sum_{i=1}^{q} a_i z^{-i}} \qquad \text{(causal filter)}$$

where a_i and b_i are the coefficients of the filter, z^{-i} is the ith delay in the signal, and the order of the filter is the greater of k or q.

For the case where f is the input signal and g is the output signal, we can write the equation as follows:

$$g[n] = b_0 f[n] + b_1 f[n-1] + \cdots + b_k f[n-k] - a_1 g[n-1] - \cdots - a_q g[n-q]$$

$$g[n] = \sum_{i=0}^{k} b_i f[n-i] - \sum_{i=1}^{q} a_i g[n-i]$$

These filters are called infinite impulse response (IIR) filters and are characterized by having feedback of the output signal. These filters have the potential to become unstable.

FREQUENCY DOMAIN FILTERING FUNDAMENTALS

FILTERS

THE CONVOLUTION THEOREM

Example 13: Let f be a signal with values [0, 1, 2, 3, 4, 8, 6] and $h[z] = 1/(1 - 0.2z^{-1})$. Compute the convolution between f and h, but the output signal must have the same size as the input signal.

Solution:

$$g[n] = f[n] + 0.2g[n-1]$$

$$
\begin{aligned}
&g[-1] = 0\\
&g[0] = 0 + 0.2 \times 0 = 0\\
&g[1] = 1 + 0.2 \times 0 = 1\\
&g[2] = 2 + 0.2 \times 1 = 2.2\\
&g[3] = 3 + 0.2 \times 2.2 = 3.44\\
&g[4] = 4 + 0.2 \times 3.44 = 4.688\\
&g[5] = 8 + 0.2 \times 4.688 = 8.9376\\
&g[6] = 6 + 0.2 \times 8{,}9376 = 7.7875
\end{aligned}
$$

$$g[n] = [0, 1, 2.2, 3.44, 4.688, 8.9376, 7.7875]$$

FREQUENCY DOMAIN FILTERING FUNDAMENTALS

FILTERS

THE CONVOLUTION THEOREM

In addition to IIR filters, there are finite impulse response (FIR) filters, given by the equation:

$$h[z] = b_0 + \sum_{i=1}^{k} b_i z^{-i} \qquad \text{(causal filter)}$$

where b_i are the coefficients of the filter, z^{-i} is the ith delay in the signal, and the order of the filter is k.

For the case where f is the input signal and g is the output signal, we can write the equation as follows:

$$g[n] = b_0 f[n] + b_1 f[n-1] + \cdots + b_k f[n-k]$$

$$g[n] = \sum_{i=0}^{k} b_i f[n-i]$$

These filters are characterized by not having a recursive part (feedback of the output signal). They are stable filters, but tend to have higher order than IIR filters.

FREQUENCY DOMAIN FILTERING FUNDAMENTALS

FILTERS

THE CONVOLUTION THEOREM

Some types of filters:

- **Simple moving average filter**: Used to smooth data. The size of the filter N determines the degree of smoothing that is applied to the data.

$$g[n] = \frac{1}{N}\sum_{i=0}^{N-1} f[n-i]$$

- **Exponential moving average filter**: It gives more weight to more recent values than the simple average. $\lambda = 1/N$ gives an idea of the value to choose for λ so that the to exponential filter has an efficiency similar to an average filter of order N.

$$g[n] = \lambda f[n] + (1-\lambda)g[n-1] \qquad \lambda \in \,]0,1]$$

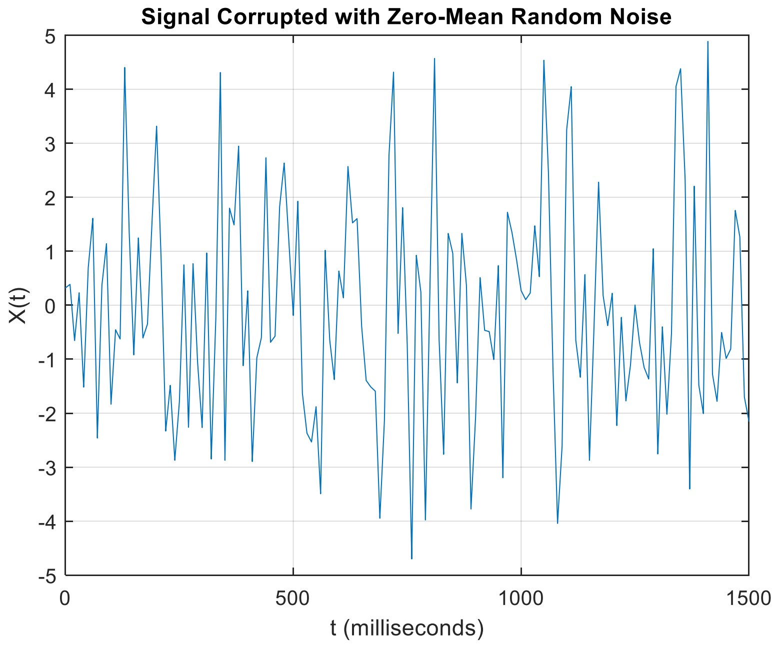
FREQUENCY DOMAIN FILTERING FUNDAMENTALS

FILTERS

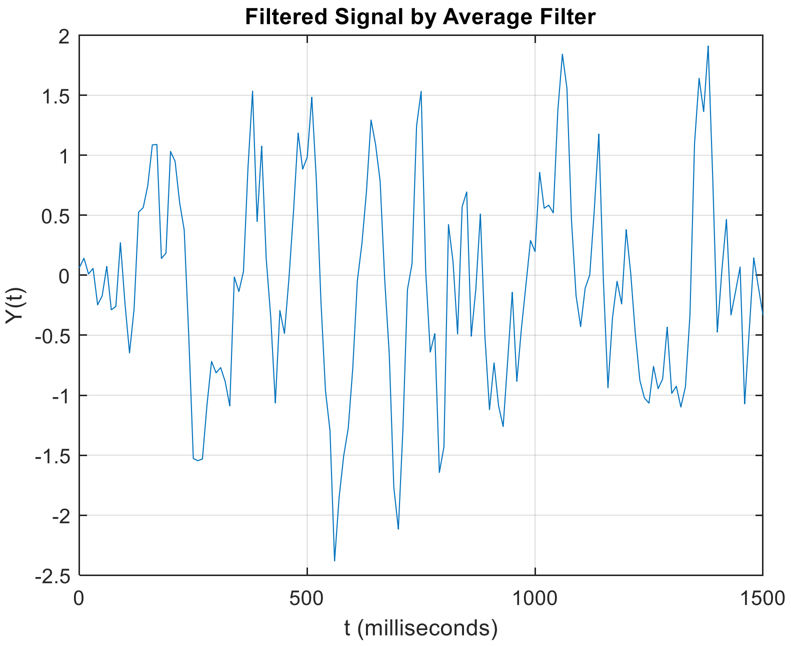
THE CONVOLUTION THEOREM

Example 14: Apply moving average and exponential filter to the time series.

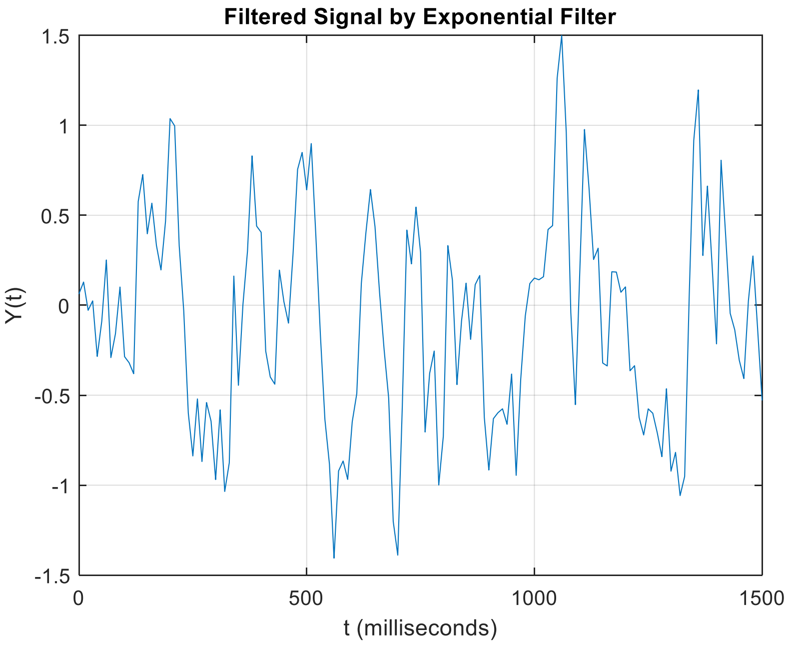
Solution:



Original data



Average filter: $N = 5$



Exponential filter: $\lambda = 0.2$

SHORT-TIME FOURIER TRANSFORM AND WAVELET TRANSFORM

SHORT-TIME FOURIER TRANSFORM (STFT)

CONCEPT

The Fourier transform is traditionally used to perform analysis of signals in the frequency domain.

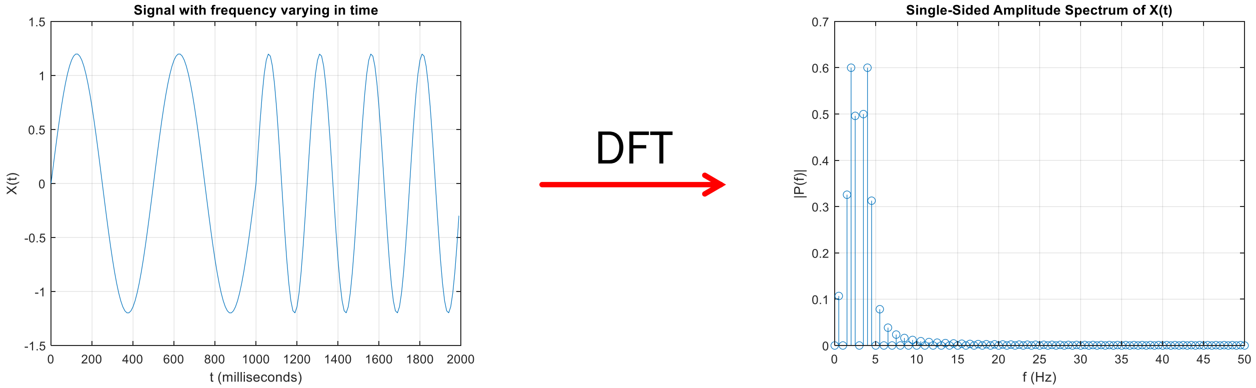
However, the Fourier transform rejects the notion of frequency varying with time, although such a notion is often useful. According to Fourier transform theory the notion of frequency varying with time is meaningless, because frequencies are always associated with an infinite time duration in signals.

In other words, it is not possible to identify at what instant in time a certain frequency component occurred, because the Fourier transform considers that the components occur over an infinite period of time.

SHORT-TIME FOURIER TRANSFORM (STFT)

CONCEPT

For example, if we apply the Fourier transform to the signal shown in figure, we will obtain an infinite number of frequencies in the frequency domain, which supposedly extend over the entire time domain, while in fact there are only two frequencies in the signal that change with time.



For these cases, it is desirable to find a time-frequency representation where the notion of frequency changing over time can exist.

For this purpose, techniques such as the windowed Fourier transform (STFT – Short-Time Fourier Transform) and a “generalized” form of it, known as the wavelet transform, have been developed.

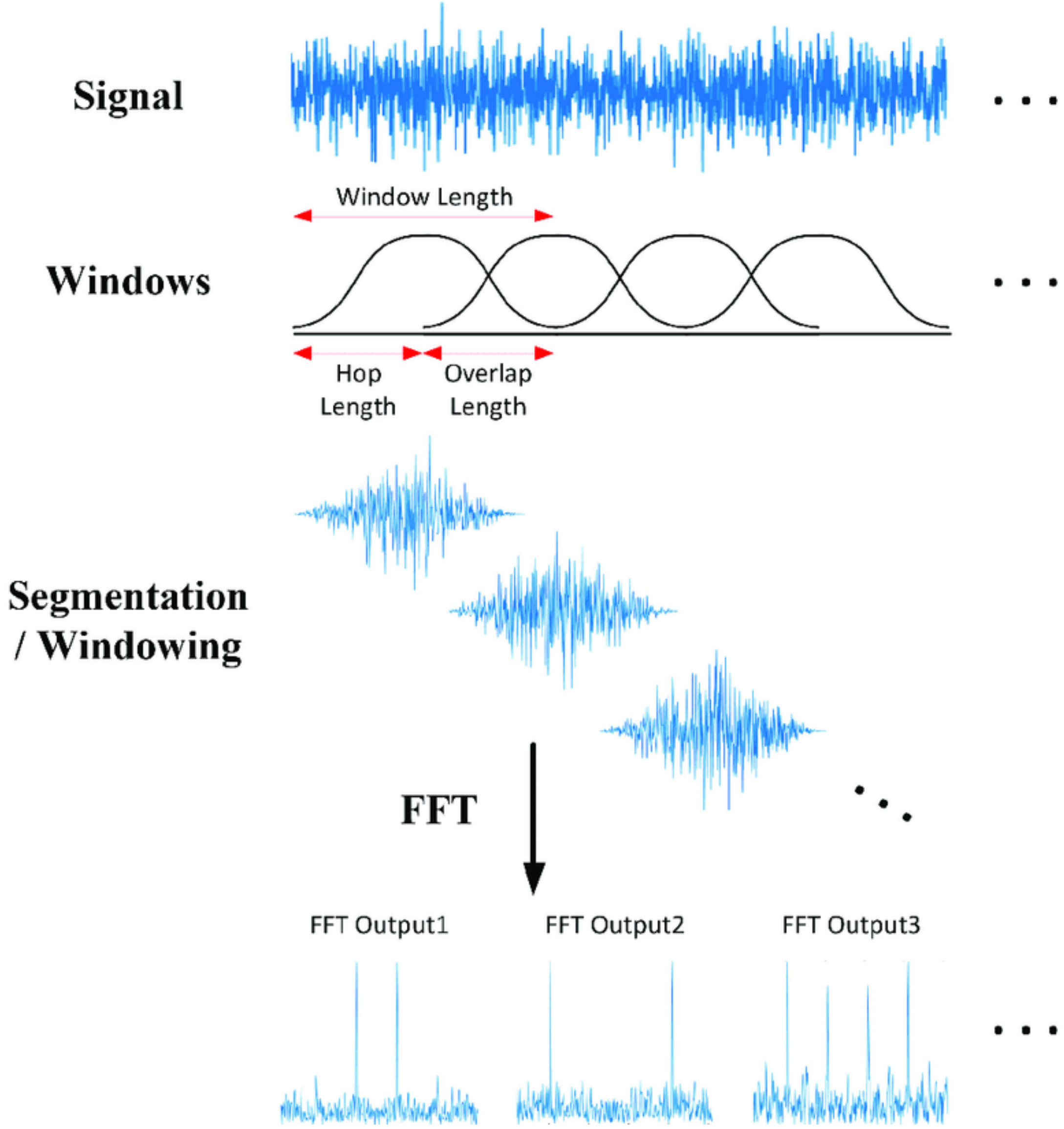
SHORT-TIME FOURIER TRANSFORM (STFT)

CONCEPT

Unlike the Fourier transform, which performs the signal analysis using probably all available samples of the signal, the **Short-Time Fourier Transform** (STFT) performs the Fourier transform for M successive samples of the signal and then repeats the operation for the next M samples.

Such an operation can be interpreted as a sliding window over a set of samples of the signal.

This operation results in a separate Fourier transform (DFT) for each location of the window.



SHORT-TIME FOURIER TRANSFORM (STFT)

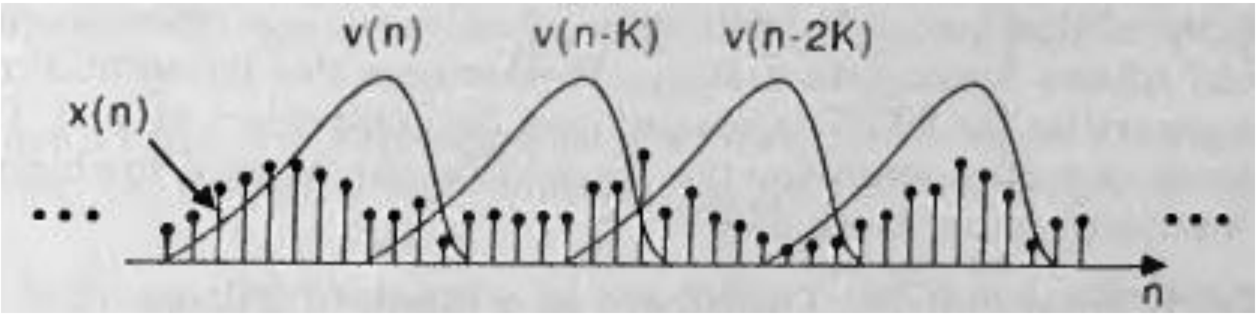
CONCEPT

In STFT, an input signal $x[n]$ is multiplied by a window $v[n]$ (typically of finite duration) and the Fourier transform is performed. Then the window is shifted in time and the Fourier transform is performed again, and this procedure is repeated and so on.

This operation results in a separate Fourier transform for each location m of the window center and thus we obtain:

$$X_{STFT\_m} = \Im(x[n]v[n-m])$$

where $\Im$ is the DFT. If is imposed $v[n] = 1$ for all n, then the above equation reduces to the Fourier transform for any chosen m.



The inverse transform of the STFT can be obtained based on the inverse transform of the Fourier transform:

$$x[n]v[n-m] = \Im^{-1}\left(X_{STFT\_m}\right)$$

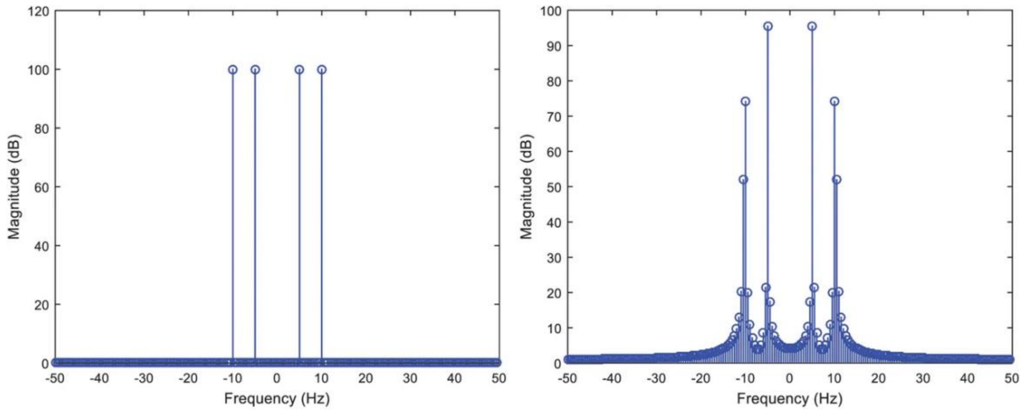
SHORT-TIME FOURIER TRANSFORM (STFT)

CONCEPT

Signal truncated by time window results in spectral leakage, which causes the signal levels to be reduced and redistributed over a broad frequency range. Any type of operation that creates frequency components that do not exist in the original signal may be referred to as spectral leakage in the broadest sense. Aliasing is a spectral leakage.

Since most measured signals will possess non-periodic signals and the signal truncated by the time window is not an integer multiple of the signal period length, the application of DFT will result in larger distortion and cause unpredictable spurious components and spectral leakage. The situation gets worse when using a window with few samples.

The figure shows the magnitude (in dB) of the DFT spectrum using a periodic and a non-periodic truncation, respectively.



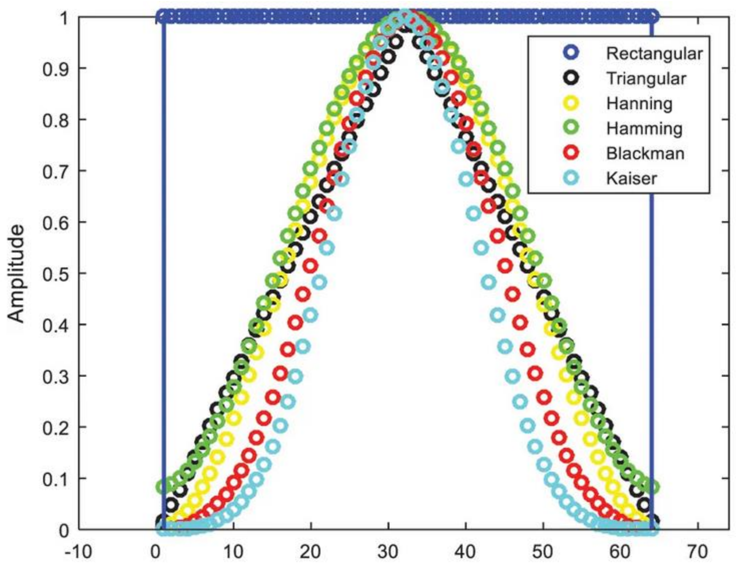
SHORT-TIME FOURIER TRANSFORM (STFT)

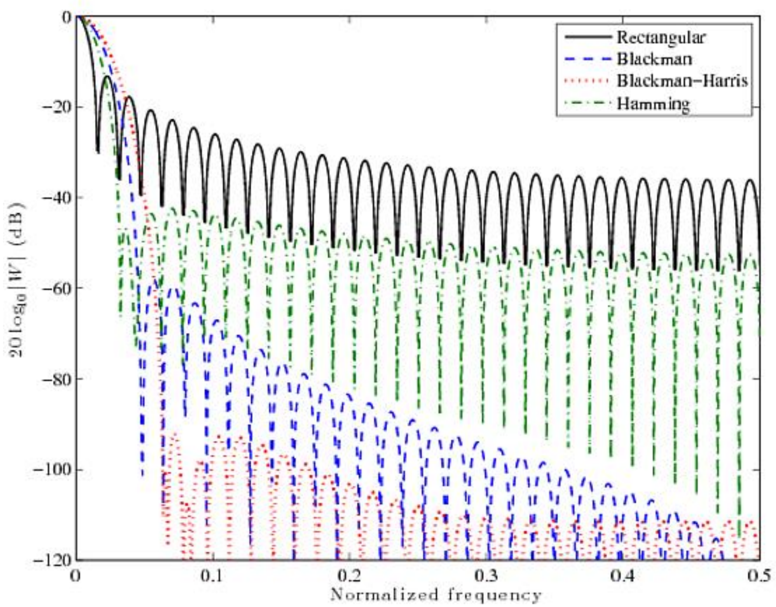
CONCEPT

Spectral leakage cannot be avoided in general, but we can minimize leakage using appropriated windows.

Some types of windows are:

- Rectangular Window: It is the simplest, but the one that returns the highest spectral leakage among the windows normally used;
- Triangular Window;
- Hanning Window;
- Hamming Window;
- Blackman Window;
- Blackman-Harris Window: It is a generalization of other windows;
- Kaiser Window.





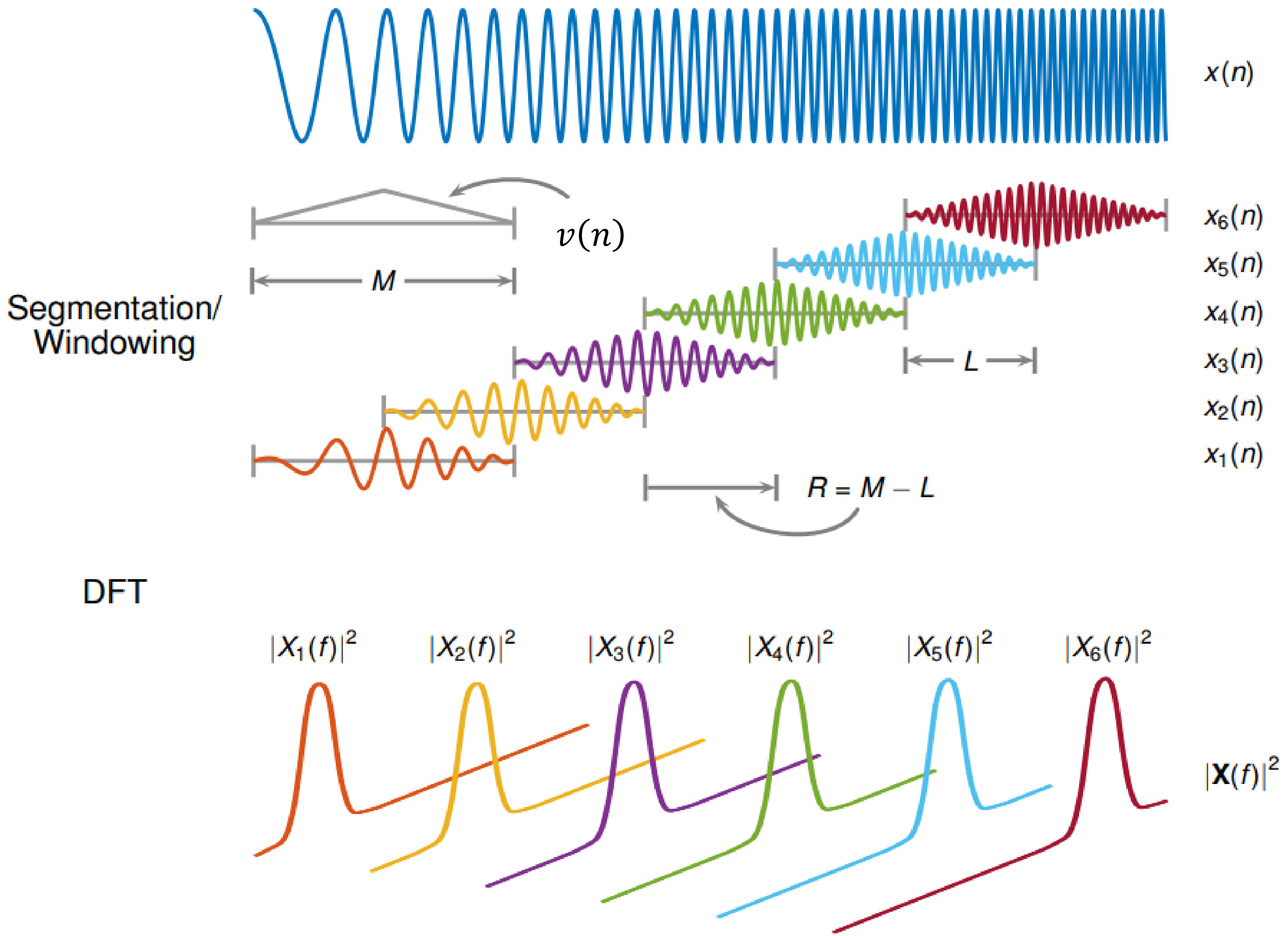
SHORT-TIME FOURIER TRANSFORM (STFT)

CONCEPT

Therefore, the STFT of a signal is computed by sliding an window $v[n]$ of length M over the signal and calculating the discrete Fourier transform (DFT) of each segment of windowed data.

The window hops over the original signal at intervals of R samples, equivalent to $L = M - R$ samples of overlap between adjoining segments.

The DFT of each windowed segment is added to a complex-valued matrix that contains the magnitude and phase for each point in time and frequency.



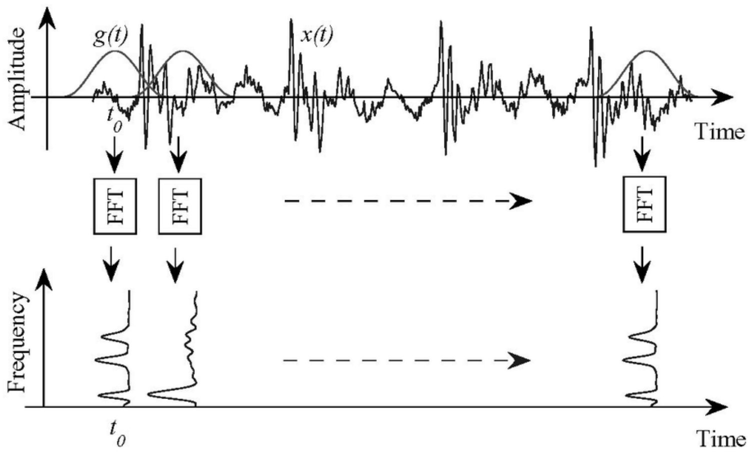
SHORT-TIME FOURIER TRANSFORM (STFT)

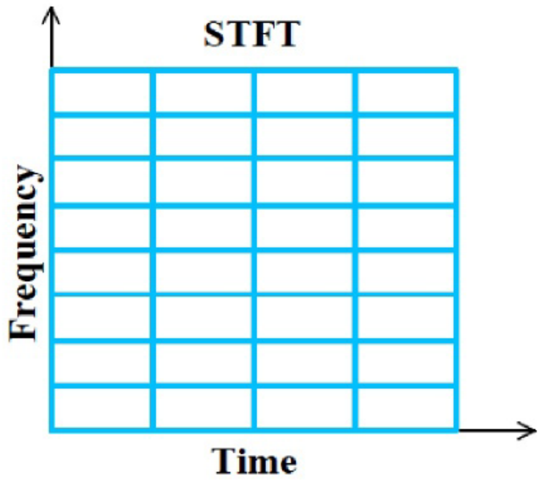
COMPROMISE BETWEEN FREQUENCY AND TIME

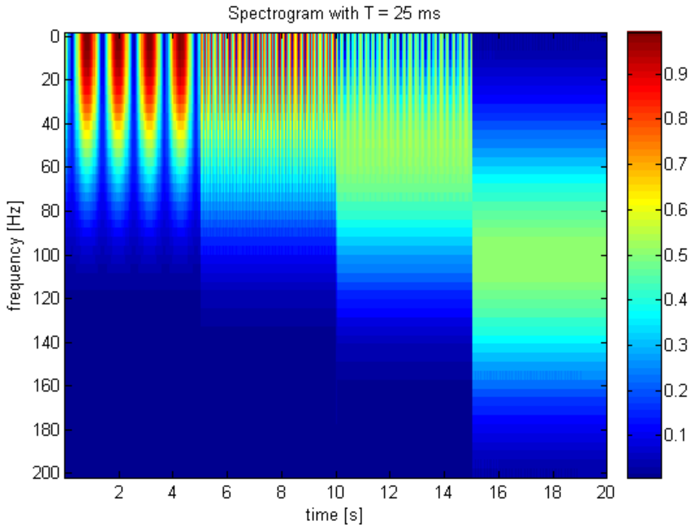
The matrix that contains the results of DFT can be represented by a graphic known as a spectrogram.

This graphic demonstrates a two-dimensional time-frequency grid for STFT. The horizontal lines represent the frequency components, and the vertical lines represent the center locations of the windows in time.

The intersections of the horizontal lines with a vertical line represent the location of a DFT of a windowed segment. This grid represents a uniform displacement about the frequency and time axes. The spacing between the time-domain samples corresponds to the step size for the window movement. The spacing between the frequencies is f_{max}/M, because there are M samples in each window, where f_{max} is half the sampling frequency.







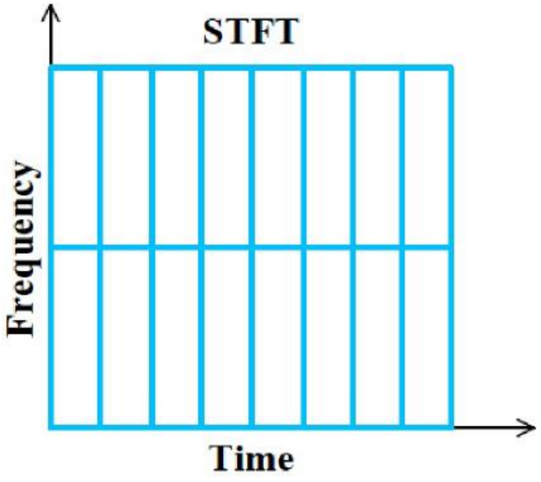
SHORT-TIME FOURIER TRANSFORM (STFT)

COMPROMISE BETWEEN FREQUENCY AND TIME

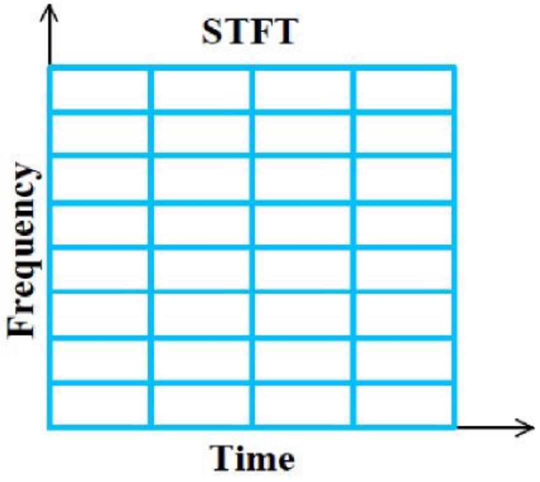
The choice of window size $v[n]$ determines the trade-off between time localization and frequency resolution.

If the window $v[n]$ is narrow, it is possible to localize information better in the time domain. However, the frequency resolution is reduced (the interval between one frequency component and another is large), being bad at analyzing low frequencies.

On the other hand, it is possible to increase the resolution in the frequency domain, but the window $v[n]$ becomes wide, compromising the location of information in the time domain.



narrow window in time, low resolution in frequency



wide window in time, high resolution frequency

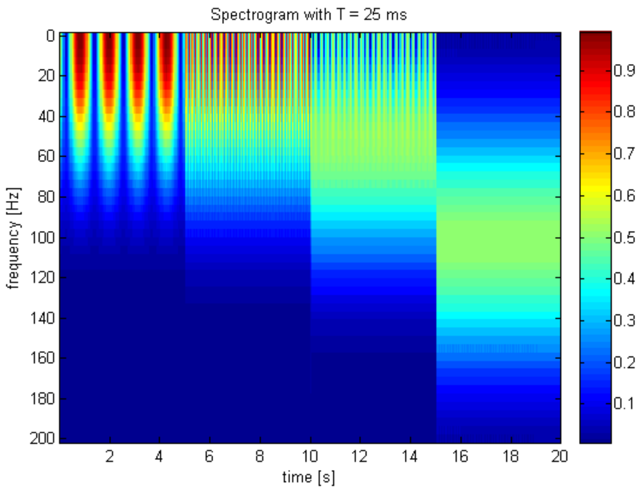
SHORT-TIME FOURIER TRANSFORM (STFT)

COMPROMISE BETWEEN FREQUENCY AND TIME

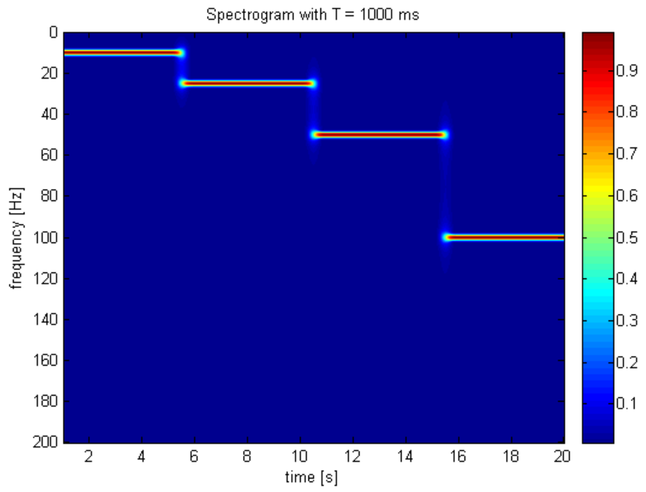
Example 11: Let x be a sampled signal at 400 Hz composed of a set of four sinusoidal waveforms joined together in sequence, compute its spectrogram with windows of 25 ms ($M = 10$) and 1000 ms ($M = 400$):

$$x[t] = \begin{cases} \cos(2\pi 10t), & 0\text{s} \leq t < 5\text{s} \\ \cos(2\pi 25t), & 5\text{s} \leq t < 10\text{s} \\ \cos(2\pi 50t), & 10\text{s} \leq t < 15\text{s} \\ \cos(2\pi 100t), & 15\text{s} \leq t < 20\text{s} \end{cases}$$

25 ms



1000 ms



The 25 ms window allows us to identify a precise time at which the signals change but the precise frequencies are difficult to identify. On the other hand, the 1000 ms window allows the frequencies to be precisely seen but the time between frequency changes is blurred.

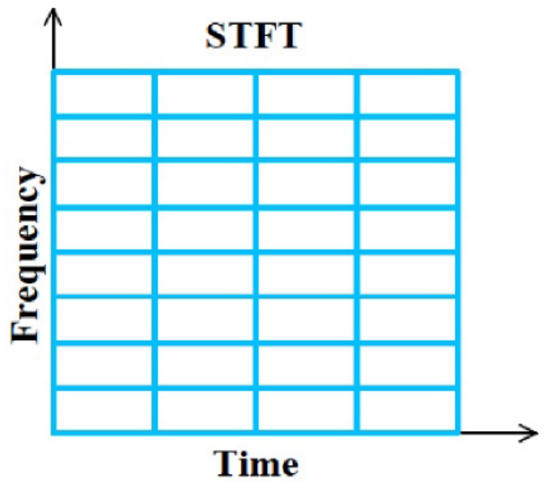
WAVELET TRANSFORM

CONCEPT

Although STFT allows the analysis of frequency changes over time (which is not possible with the traditional Fourier transform), it has some disadvantages.

Because the filters have the same bandwidth, it ends up imposing the same time-frequency resolution for the entire signal, which causes the loss of relevant information in the sections that require better definition in time and/or frequency.

For example, the accuracy of STFT is poor for low frequencies and improves as the frequency increases (due to the size of the window).

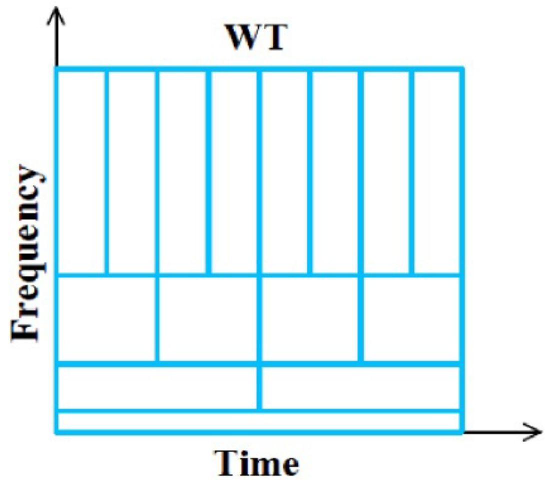


WAVELET TRANSFORM

CONCEPT

One way to overcome these disadvantages is to replace the window $v[t]$ (used so far) with a window that is a function of frequency and time, so that the window becomes wider in the time domain as the frequency decreases. By doing this, we can obtain approximately the same resolution regardless of the frequency range analyzed. Furthermore, as the window becomes wider it is desirable to have a larger step size to move the window.

These goals are achieved with the wavelet transform (WT). One of the advantages of WT is that its frequency components are localized in time, which allows the signal to be analyzed at different levels of resolution in time and frequency, making it suitable for detecting and locating transients with low and high frequency components.



WAVELET TRANSFORM

CONCEPT

There are basically two changes in the transition from STFT to wavelet transform (WT):

1) **Windows of different sizes**

In the WT, the window sizes in time domain are obtained by scaling the frequency of a prototype window (called the mother wavelet). There will be wider windows to capture lower frequencies with greater resolution and there will be smaller windows to locate better in time. Therefore, it is different from the case of STFT, where a single window of the same size is used.

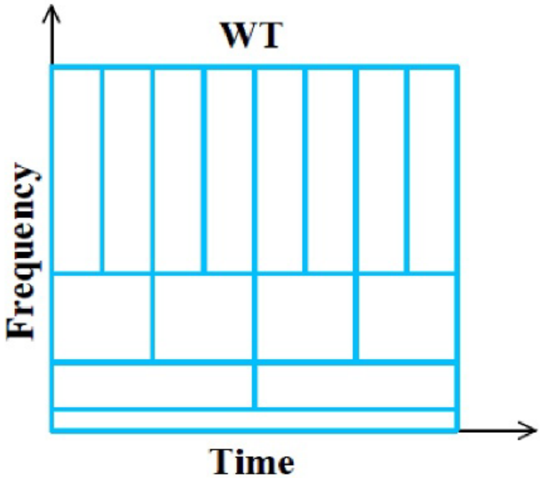
2) **Non-uniform displacement of windows**

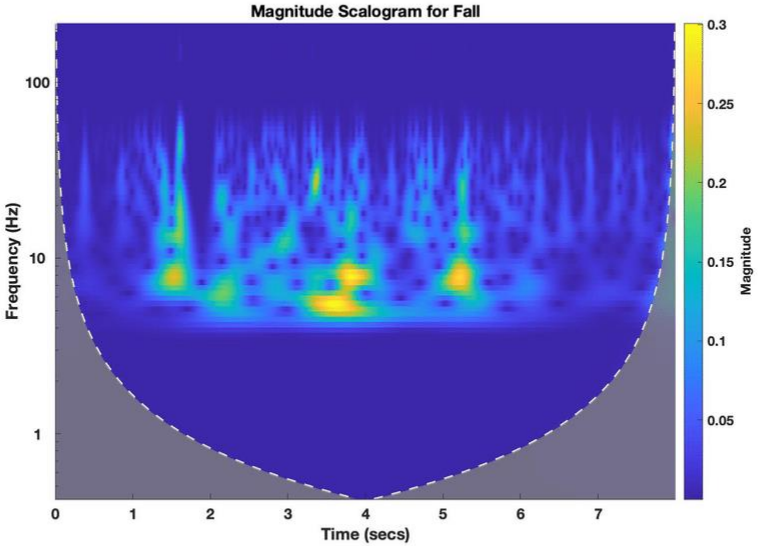
A larger time-domain window is shifted by a larger step size. Conversely, a narrow window is shifted by a smaller step size. Therefore, it is different from the case of STFT, where the window is shifted by the same step size.

WAVELET TRANSFORM

CONCEPT

In the wavelet transform, the time-frequency diagram has a spacing on the frequency axis that is smaller as the frequency is lower and the corresponding time spacing is larger, as shown in the figure.





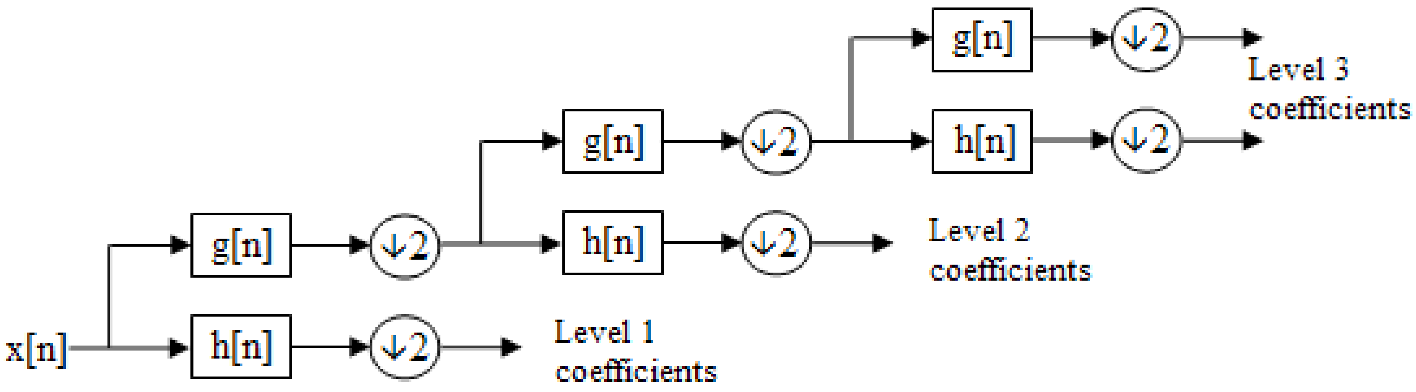
Boundary effects:
The cone of influence shows areas in the scalogram potentially affected by edge-effect artifacts.
They arise from areas where the stretched wavelets extend beyond the edges of the observation interval.

The wavelet transform is not implemented by a moving window, since there is not really just one window, but rather a family of windows.

WAVELET TRANSFORM

CONCEPT

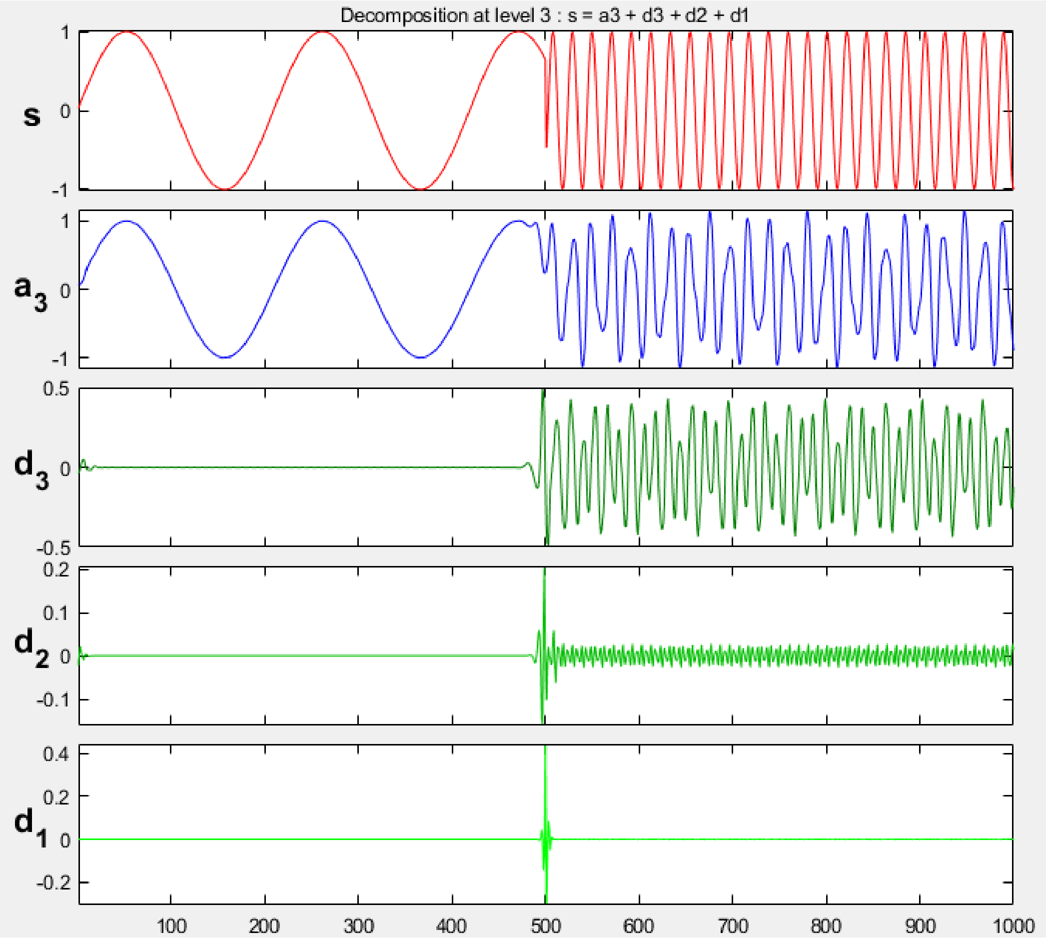
WT calculations can be interpreted as digital filtering processes followed by decimation. The figure shows the approximation and detail components of the WT at 3 levels of decomposition, where $h[n]$ are high-pass filters and $g[n]$are low-pass filters.

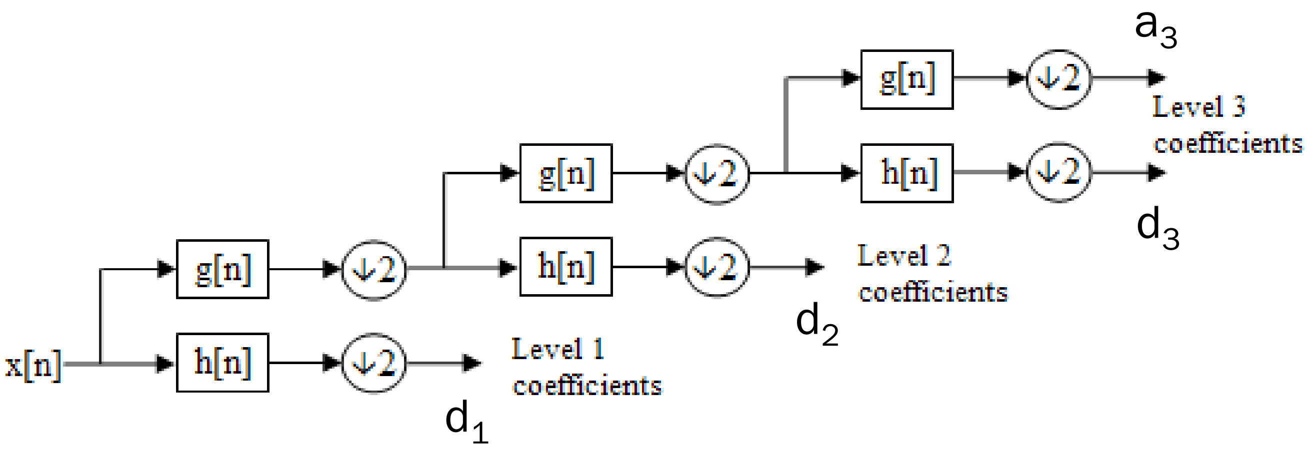


After each decomposition into approximation coefficients and detail coefficients, the time resolution is halved and the frequency resolution is doubled. Thus, different sub-bands of the signal are formed, which are the approximation coefficients and detail coefficients of the signal.

WAVELET TRANSFORM

CONCEPT





approximation coefficients: a_3
detail coefficients: d_1, d_2, and d_3

WAVELET TRANSFORM

CONCEPT

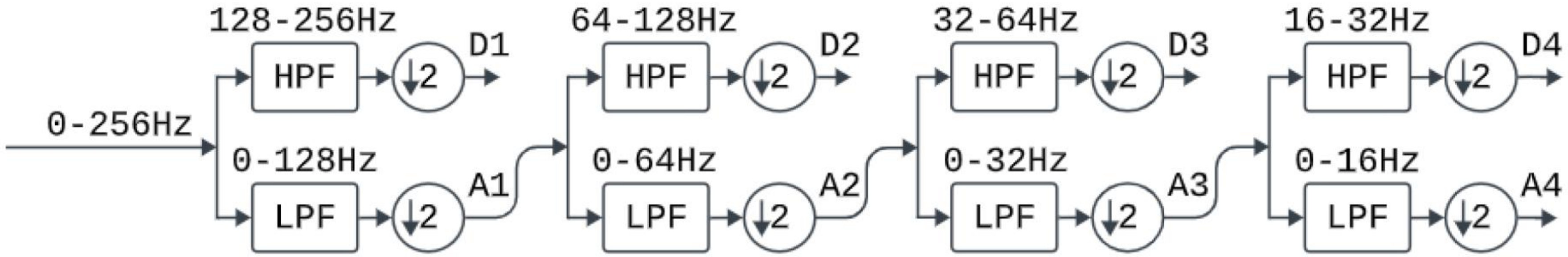
The equations below return the approximation and detail coefficients that are returned by TW:

$$a_j[m] = \sum_n a_{j-1}[n]g[2m-n]$$

$$d_j[m] = \sum_n a_{j-1}[n]h[2m-n]$$

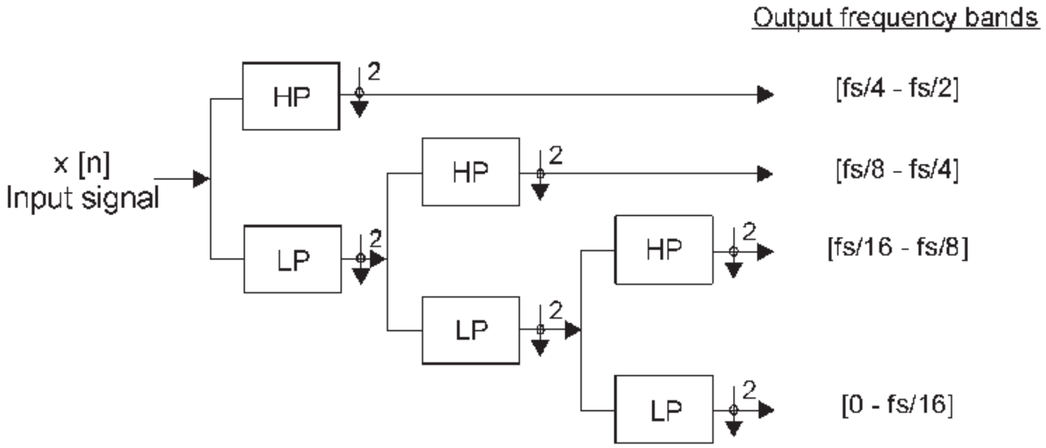
where $j = 1,2,\dots,nd$, where nd is the last level of decomposition, a_j and d_j are the approximation and detail coefficients of scale j, respectively, g and h are low-pass and high-pass filters, respectively, at $j = 1$, and a_0 is the original signal.

In this case, n is the nth coefficient at level $j-1$ and m is the mth coefficient at level j. The approximation coefficients a_j and detail coefficients d_j are obtained, respectively, by the convolution of the approximation coefficients a_{j-1} with the filters h and g, followed by a decimation of two.

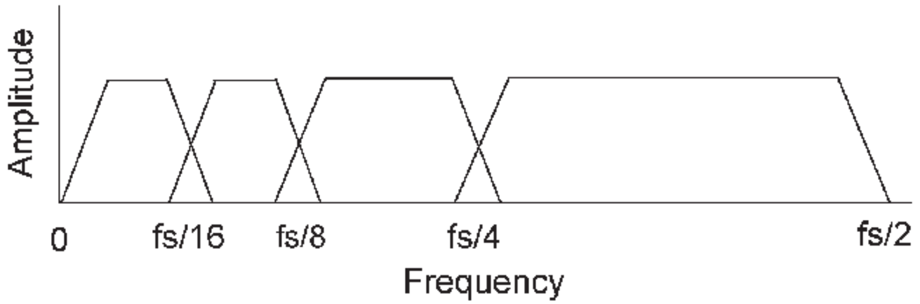


WAVELET TRANSFORM

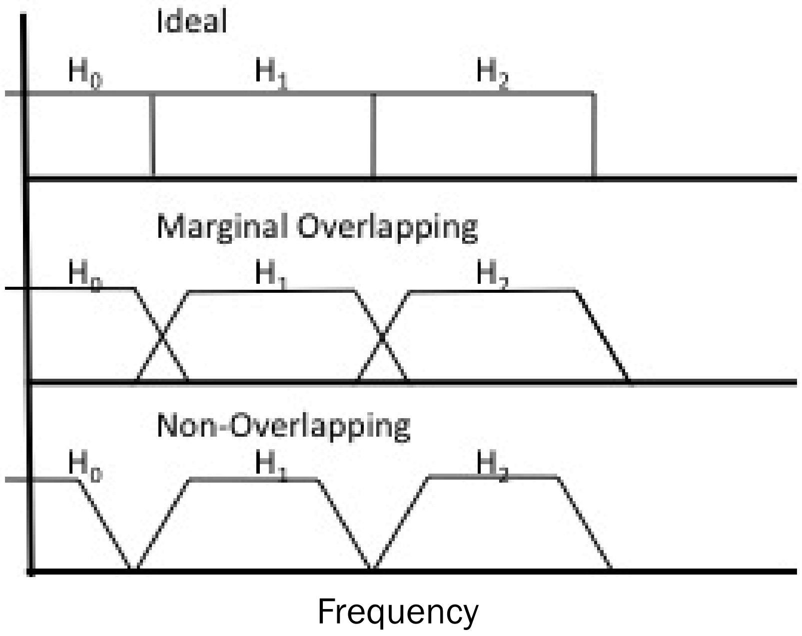
CONCEPT



(a)



(b)



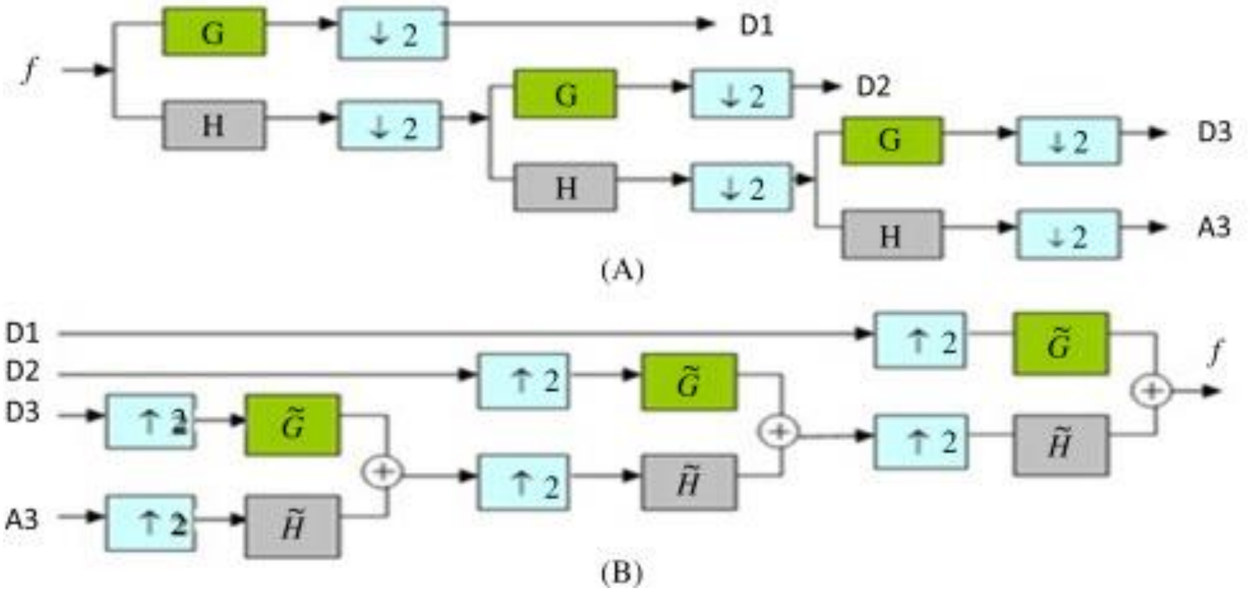
WAVELET TRANSFORM

CONCEPT

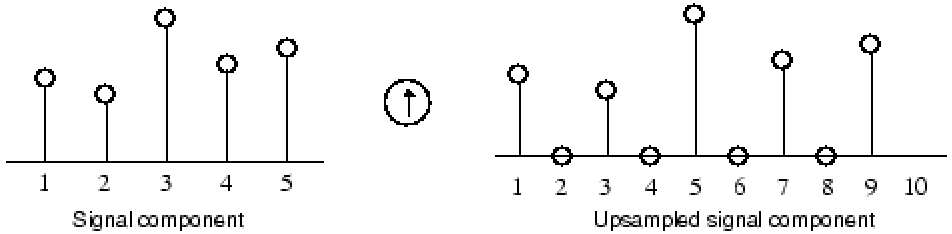
Inverse Wavelet Transform

WT

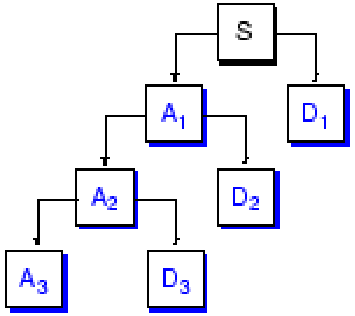
IWT



G is high-pass filter and H is low-pass filter



Reconstructed Signal Components



$$S = A_1 + D_1$$
$$= A_2 + D_2 + D_1$$
$$= A_3 + D_3 + D_2 + D_1$$

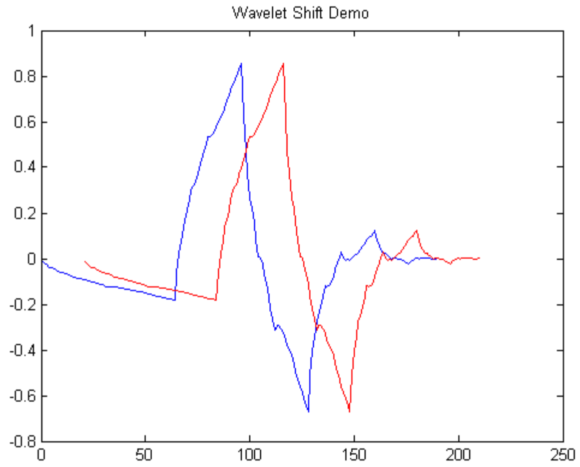
WAVELET TRANSFORM

CONCEPT

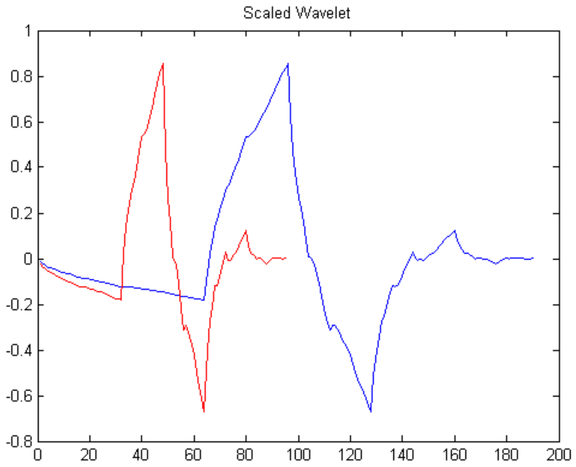
A mother wavelet is a mathematical function of a tiny wave that is used as a prototype filter that is translated and scaled to process the signal.

Some mother wavelets:

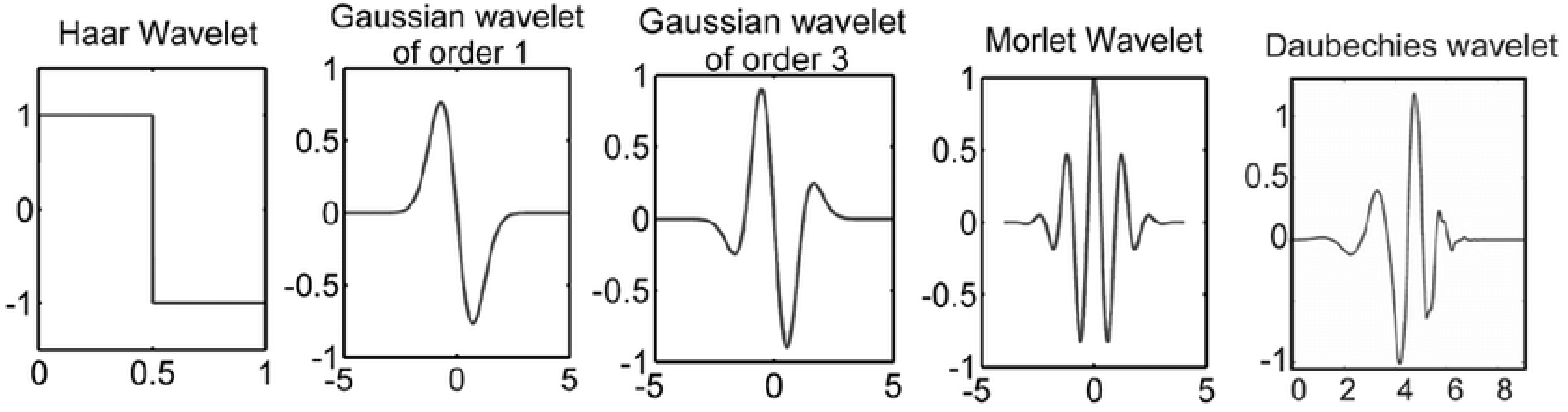
- Haar;
- Daubechies;
- Symlets;
- Coiflets;
- Gaussian;
- Biorthogonal;
- Morlet.



wavelet shift

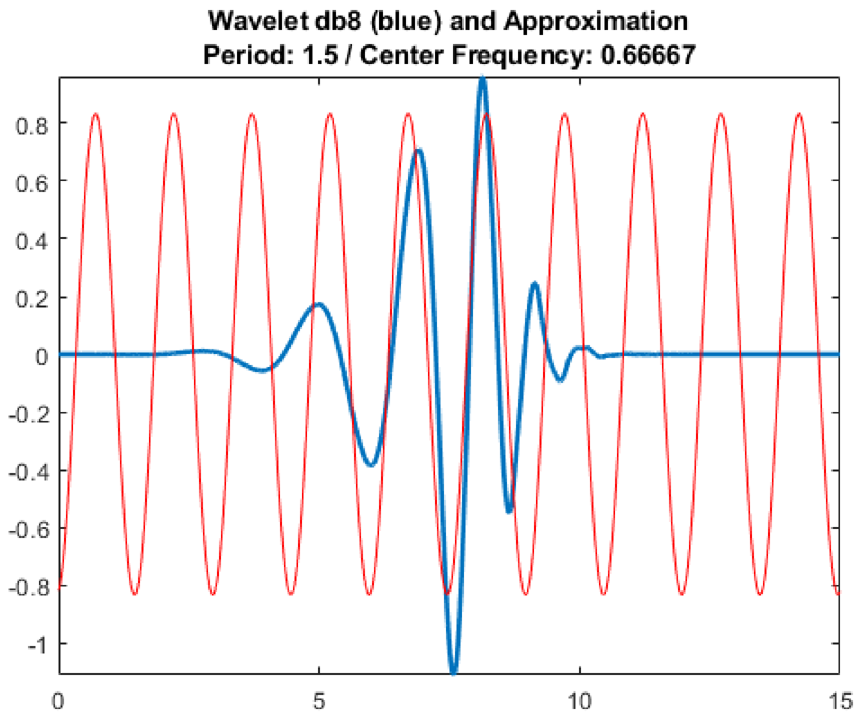


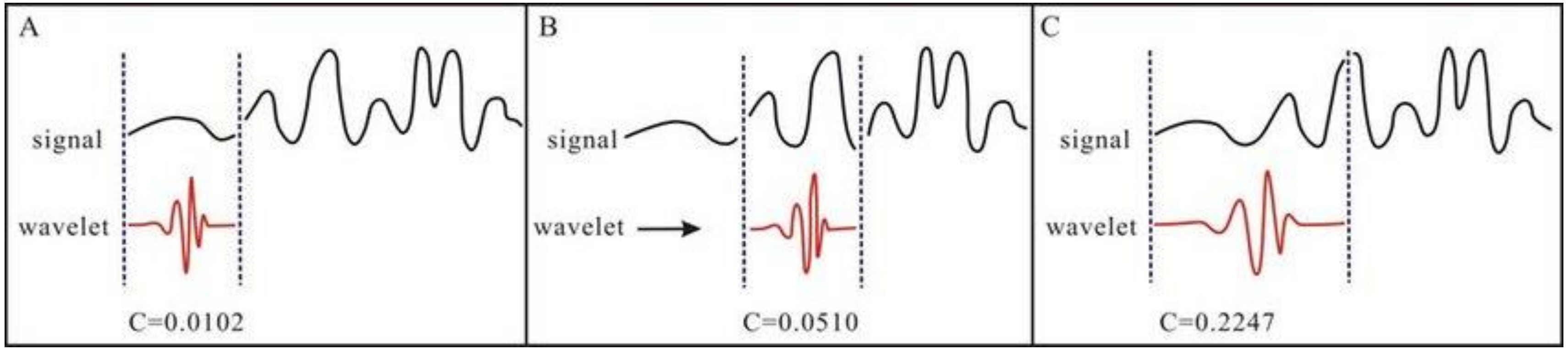
scaled wavelet



WAVELET TRANSFORM

CONCEPT



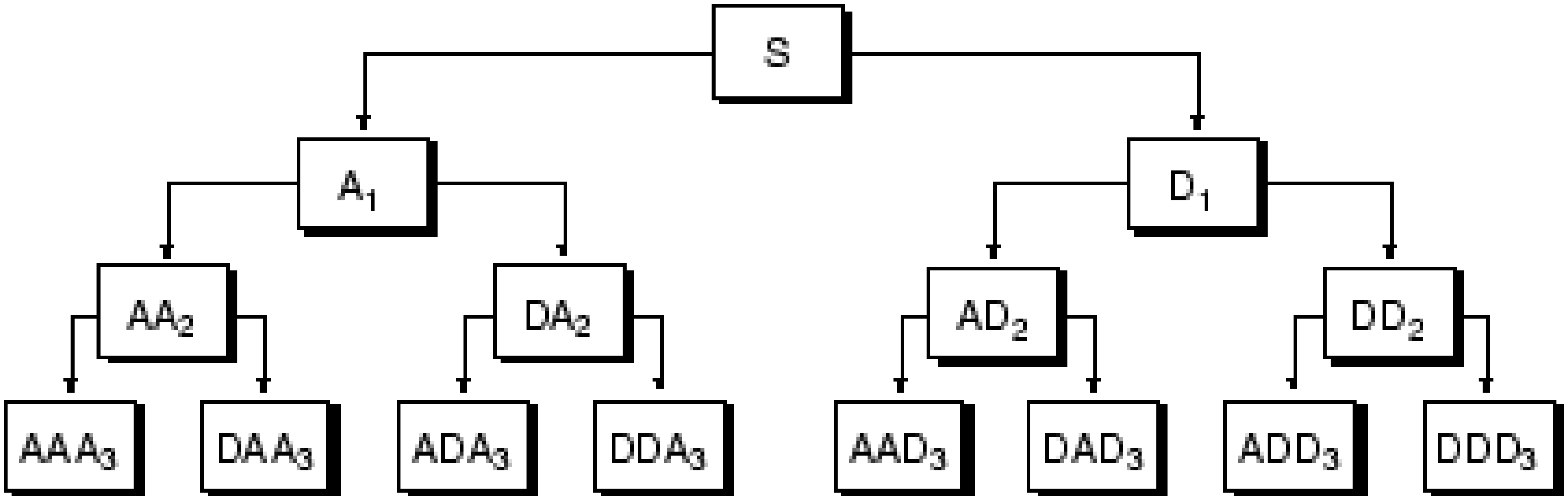


WAVELET TRANSFORM

CONCEPT

Wavelet Packet Transform

The wavelet packet method is a generalization of wavelet decomposition that offers a richer signal analysis.

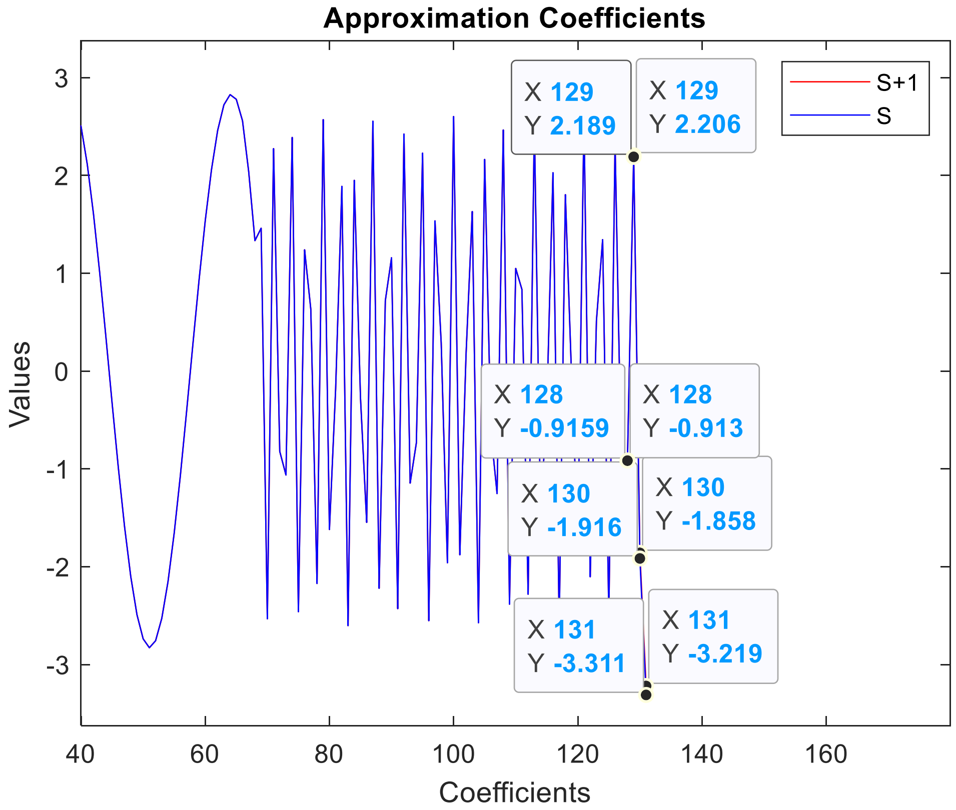


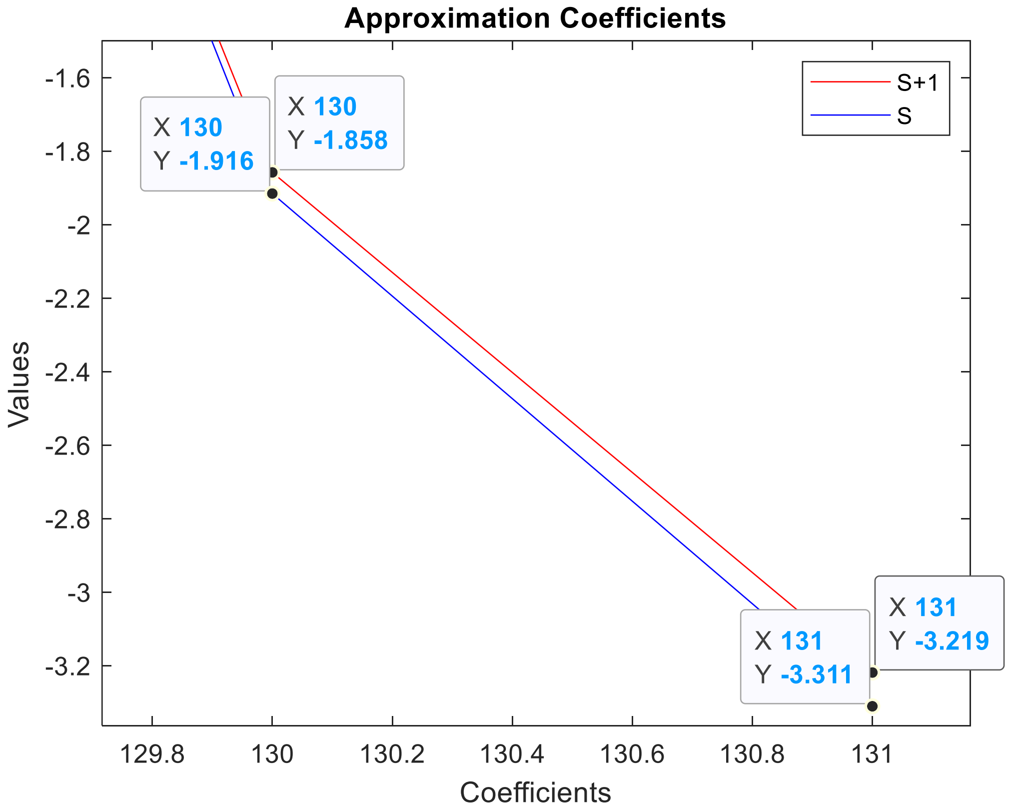
WAVELET TRANSFORM

CONCEPT

Maximal Overlapped Discrete Wavelet Transform

The WT presents drawbacks, such as time-variant transformation, which leads to problems in the real-time detection of non-stationary signals. This problem is due to border effects.



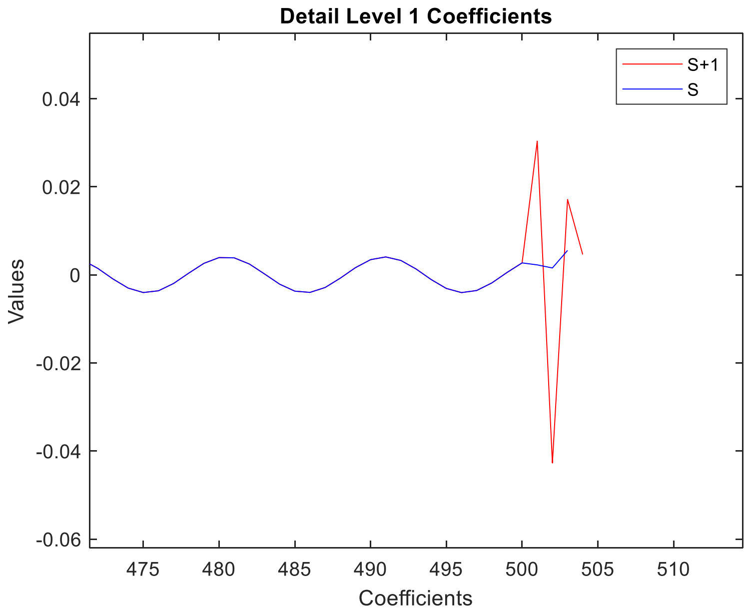


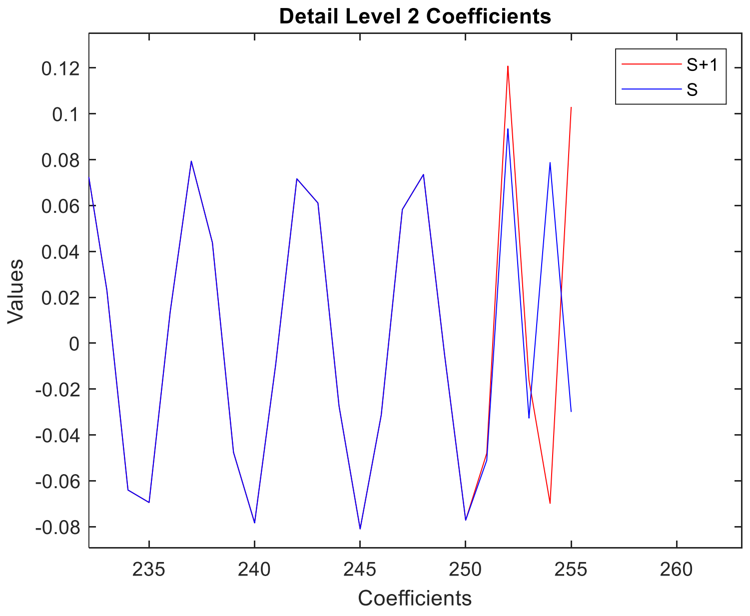
WAVELET TRANSFORM

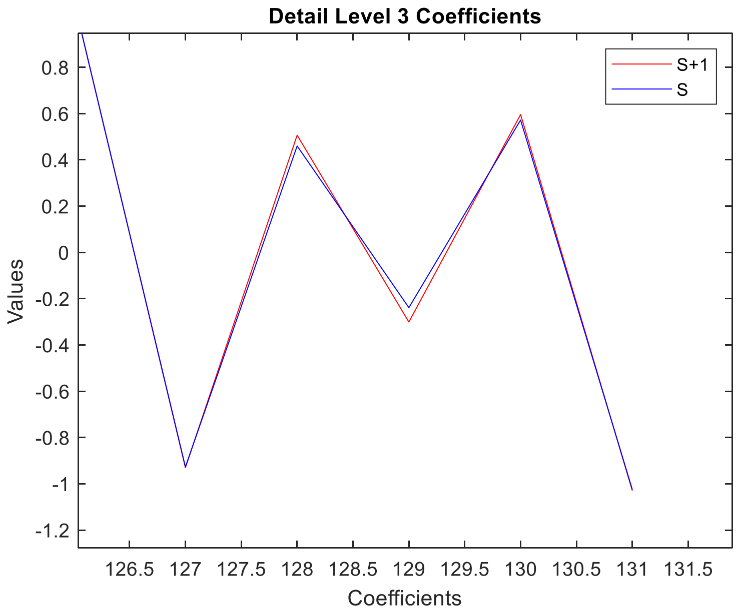
CONCEPT

Maximal Overlapped Discrete Wavelet Transform

The WT presents drawbacks, such as time-variant transformation, which leads to problems in the real-time detection of non-stationary signals. This problem is due to border effects.





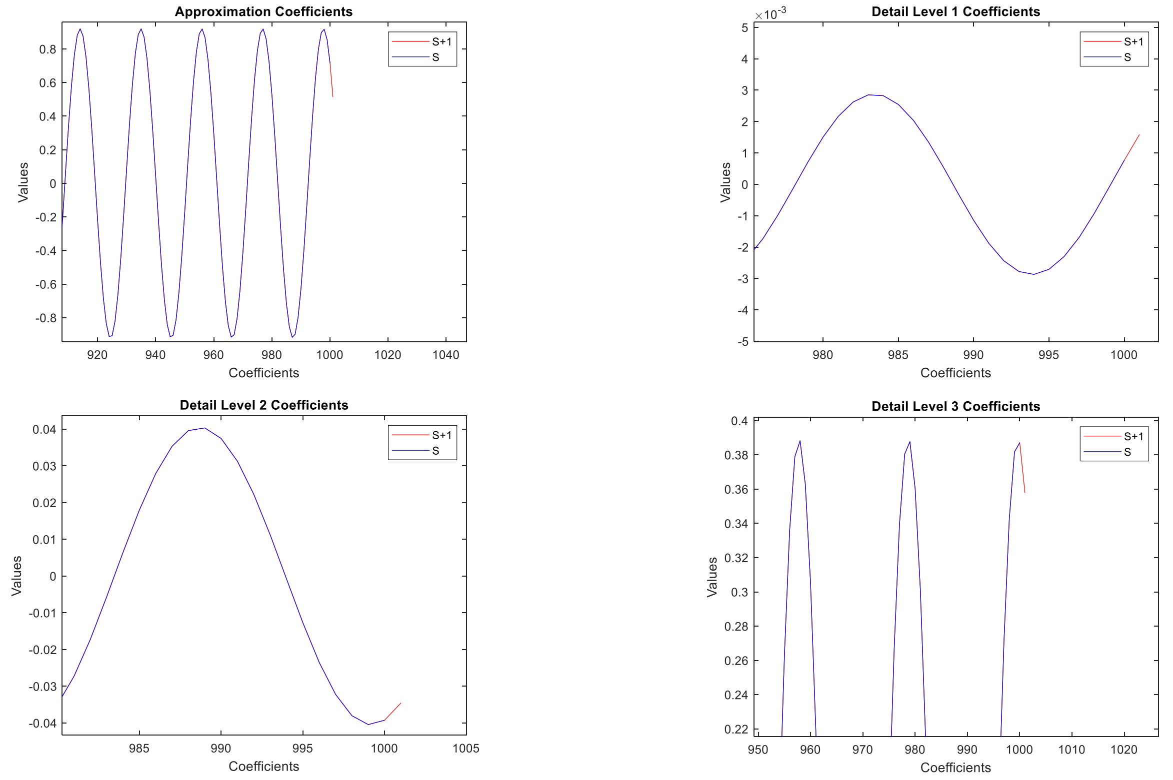


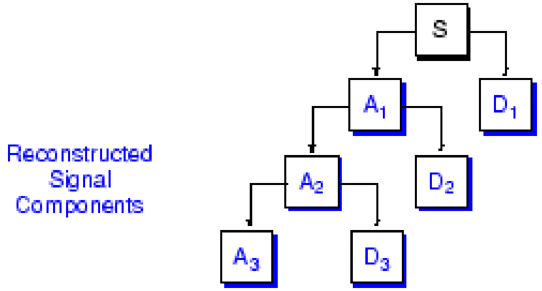
WAVELET TRANSFORM

CONCEPT

Maximal Overlapped Discrete Wavelet Transform

The maximal overlap DWT (MODWT) solves the issue of the time variance property, promoting no downsampling, i.e., all MODWT decomposition layers maintain the same time resolution without phase distortion, which is better for the real-time detection. The border effects are minimized, however, MODWT is highly redundant.





$$S = A_1 + D_1$$
$$= A_2 + D_2 + D_1$$
$$= A_3 + D_3 + D_2 + D_1$$