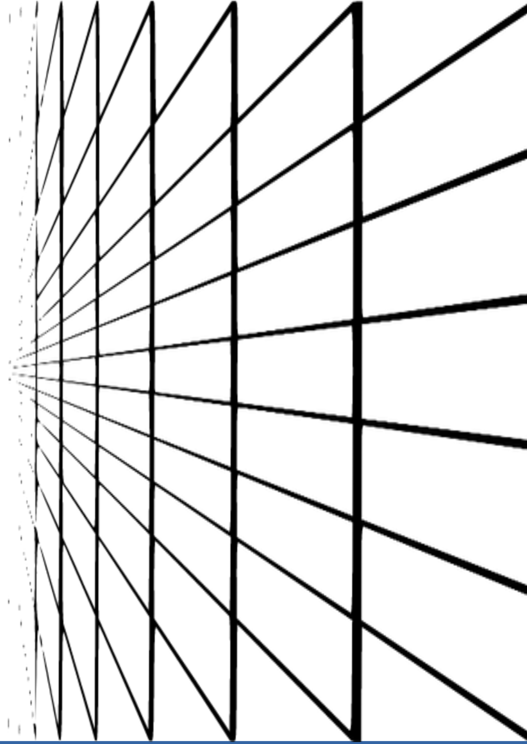
Computer Vision

Class 03



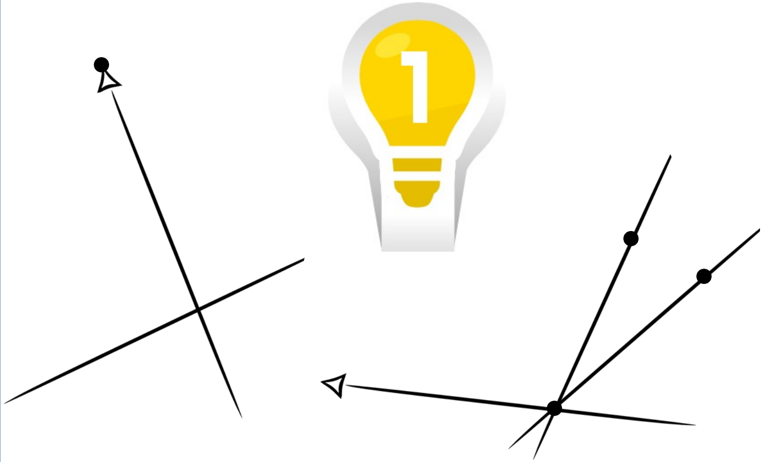
Raquel Frizera Vassallo

Projective 2D Geometry

Summary

- Points and Lines
- Ideal Points
- Lines at Infinity
- Projective Plane
- Projective Transformations

Points and Lines



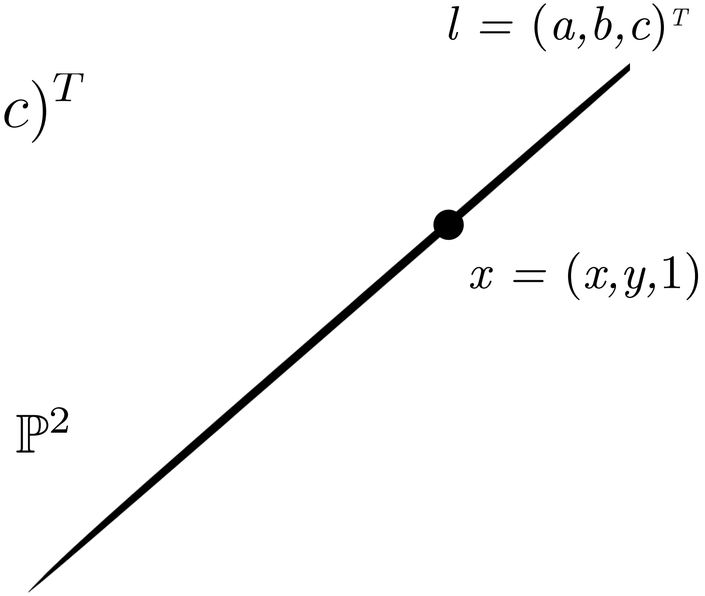
Points & Lines

- Homogeneous representation of lines

$$ax + by + c = 0 \rightarrow (a, b, c)^T$$

$$(ka)x + (kb)y + kc = 0, \forall k \neq 0 \rightarrow k(a, b, c)^T$$

$$(a, b, c)^T \sim k(a, b, c)^T$$



Equivalence class of vectors.

Set of all equivalence classes in $\mathbb{R}^3 - (0,0,0)^T$ forms $\mathbb{P}^2$

Points & Lines

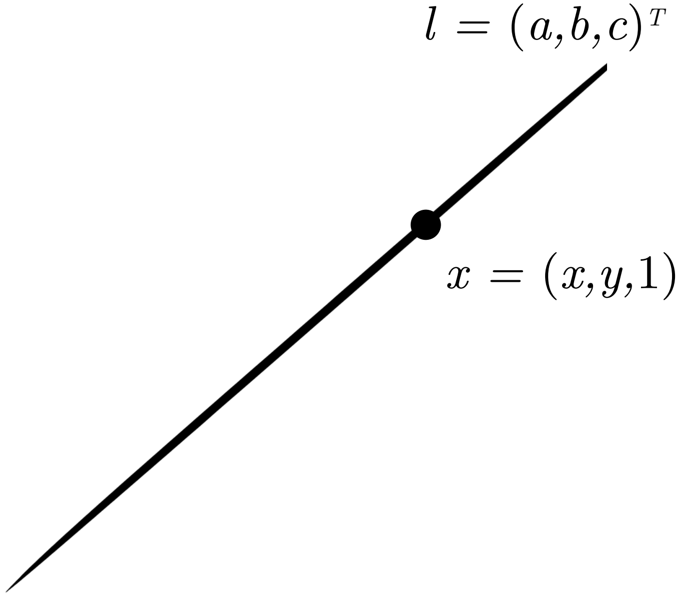
- Homogeneous representation of points

$$x = (x, y)^T \rightarrow \text{Cartesian Space}$$
$$x = (x, y, 1)^T \rightarrow \text{Homogeneous Space}$$

Point $x = (x, y, 1)^T$ is on the line $l = (a, b, c)^T$ if and only if $ax + by + c = 0$

Thus $\quad (x, y, 1)(a, b, c)^T = 0 \rightarrow x^T l = 0$

Also $\quad (x, y, 1)^T \sim k(x, y, 1)^T, \forall k \neq 0$



Points & Lines

- The point x lies on the line l if and only if

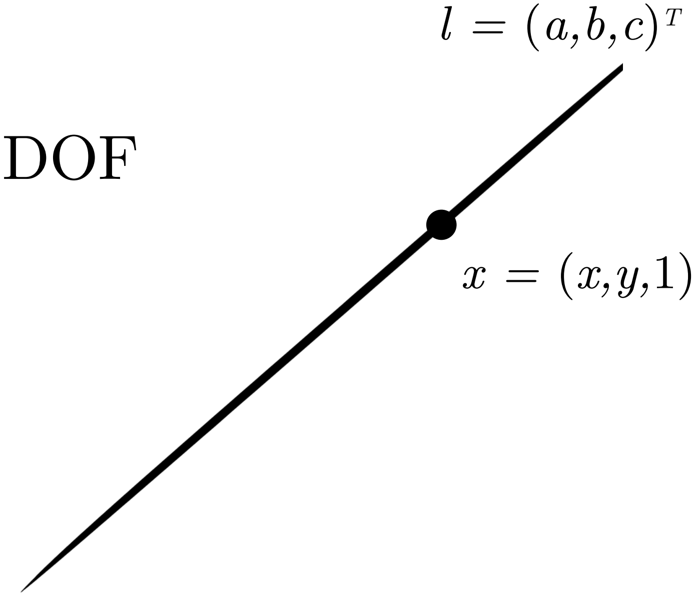
$$x^T l = l^T x = 0$$

Point homogeneous coordinates

$$x = (x_1, x_2, x_3)^T \text{ but only 2DOF}$$

Line representation

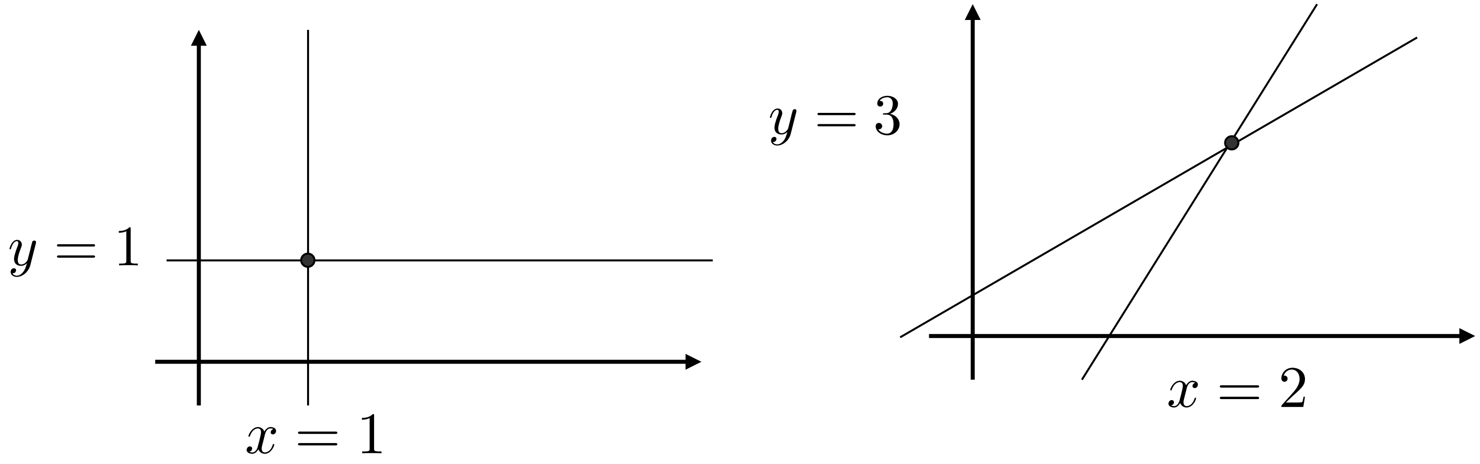
$$l = (a, b, c)^T$$



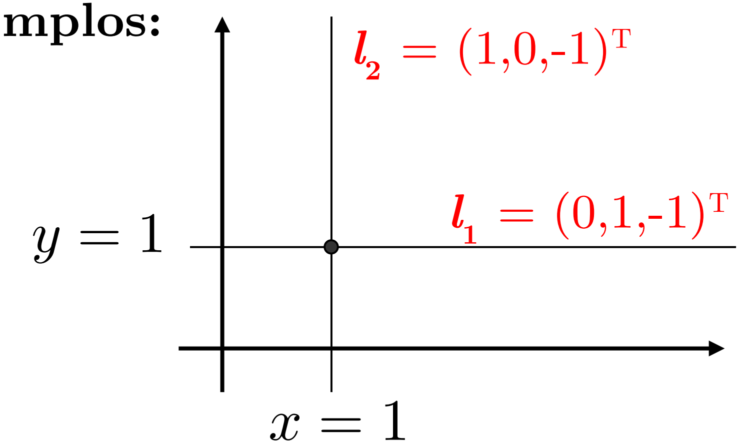
Points & Lines

- Intersections of lines $\rightarrow$ points

The intersection of two lines l and l' is $\boxed{p = l \times l'}$

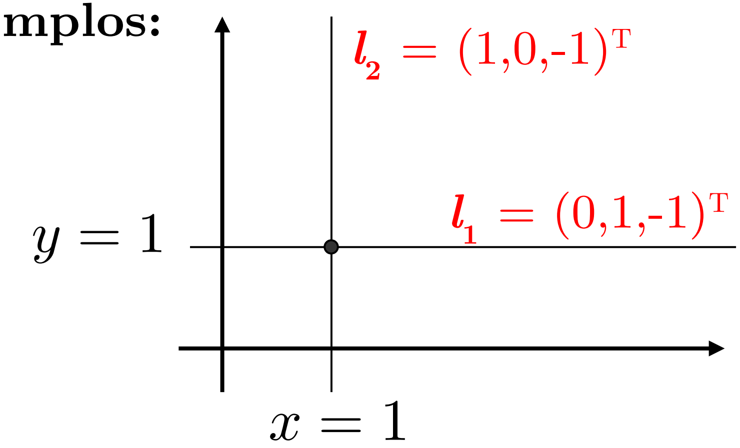


Exemplos:



$$p = l_1 \times l_2 = \hat{l_1} l_2 = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Exemplos:



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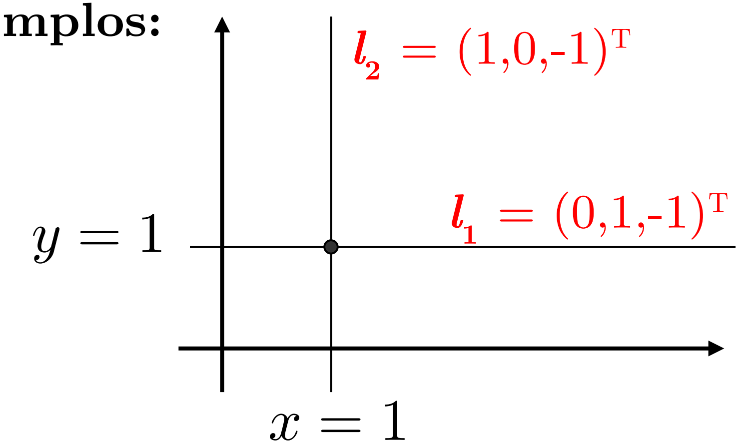
$$l_1 \times l_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \boxed{p = (1,1,1)^T}$$

Exemplos:



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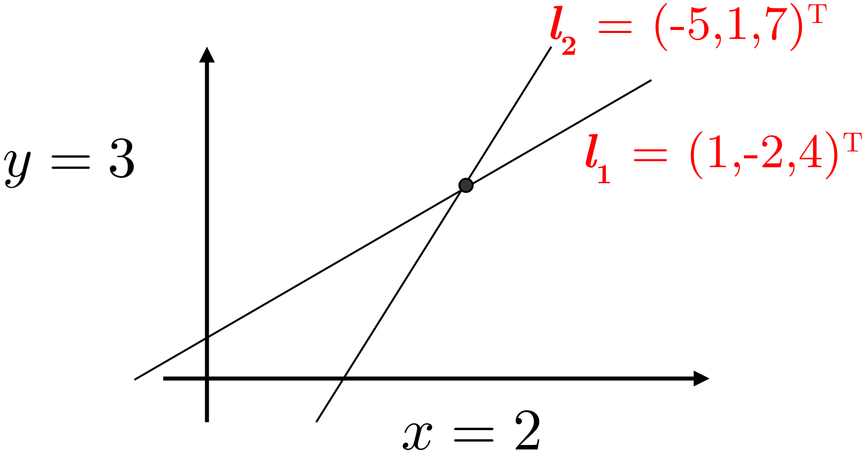
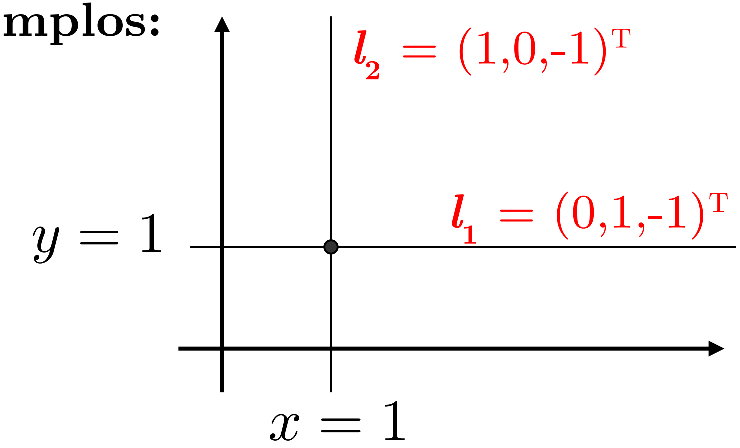


$$p = l_1 \times l_2 = \hat{l}_1 l_2 = \begin{bmatrix} 0 & -4 & -2 \\ 4 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \\ 7 \end{bmatrix}$$

Exemplos:



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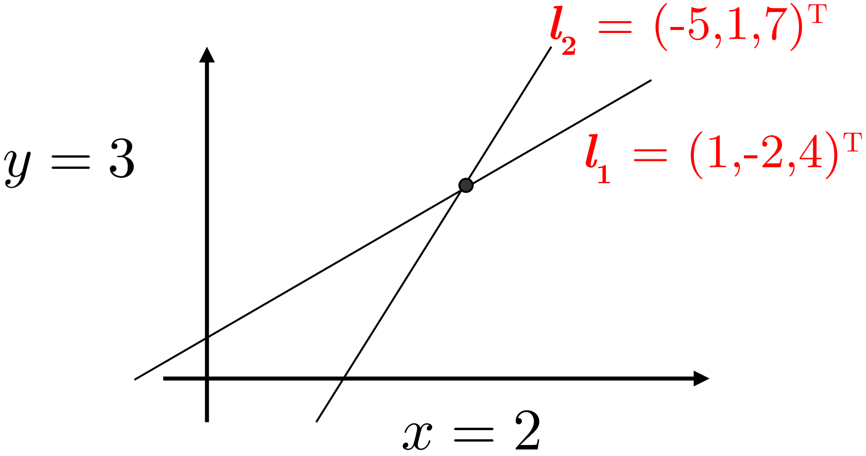
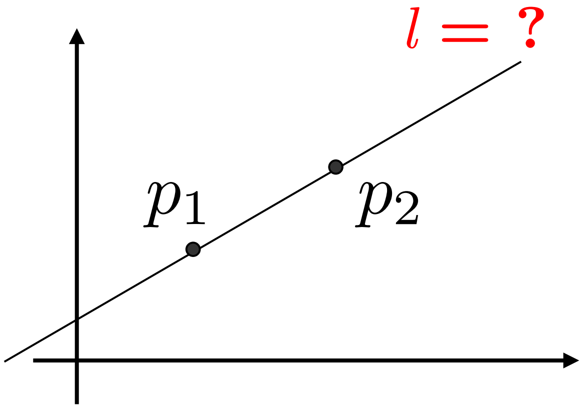


$$p = l_1 \times l_2 = \hat{l}_1 l_2 = \begin{bmatrix} 0 & -4 & -2 \\ 4 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \\ 7 \end{bmatrix}$$

$$l_1 \times l_2 = \begin{bmatrix} -18 \\ -27 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \Rightarrow \boxed{p = (2,3,1)^T}$$

Lines from points

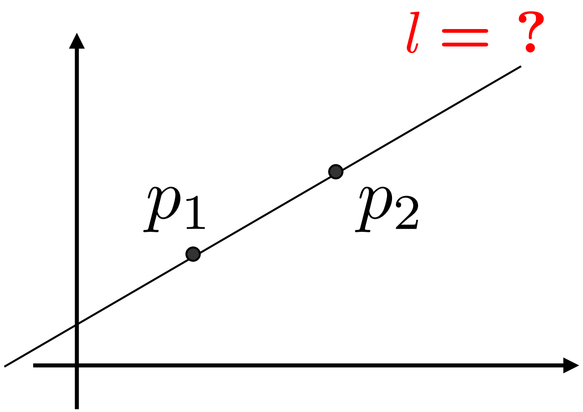
The line through points p_1 and p_2 is $\boxed{l = p_1 \times p_2}$



$p_1 = (1, 2, 1)^T \qquad p_2 = (3, 5, 1)^T \qquad l = p_1 \times p_2$

Lines from points

The line through points p_1 and p_2 is $\boxed{l = p_1 \times p_2}$

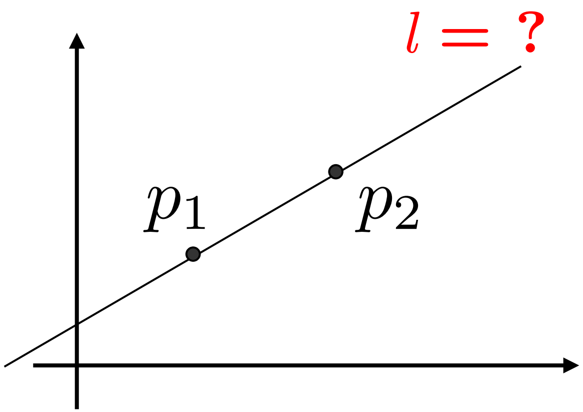


$p_1 = (1, 2, 1)^T \qquad p_2 = (3, 5, 1)^T \qquad l = p_1 \times p_2$

$$l = \hat{p}_1 p_2 = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Lines from points

The line through points p_1 and p_2 is $\boxed{l = p_1 \times p_2}$



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$$\boxed{l = (3,-2,1)^T} \Rightarrow \boxed{3x - 2y + 1 = 0}$$

Parallel lines

$l = (a, b, c)^T$ and $l' = (a, b, c')^T$

$l \times l' = (b, -a, 0)^T$

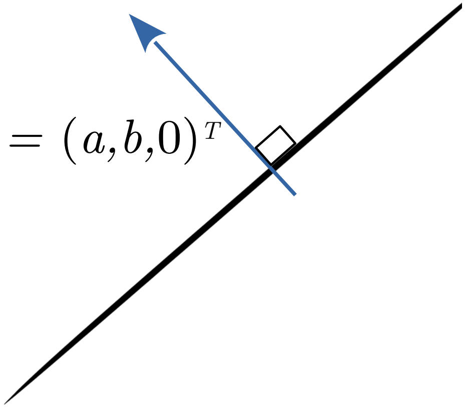
$$\begin{vmatrix} a & b & c \\ a & b & c' \\ i & j & k \end{vmatrix} = \text{(c'-c)(b,-a,0)}^T$$

$(b, -a)$ tangent vector

(a, b) normal direction

$n^T(b, -a) = 0$

$n = (a, b)$



Parallel lines

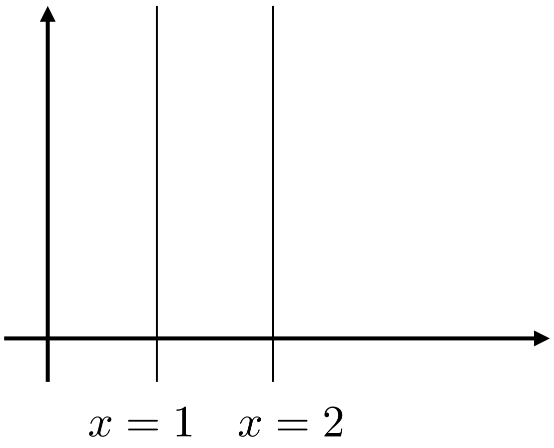
Line $x = 1 \rightarrow l_1 = (1, 0, -1)^T$

Line $x = 2 \rightarrow l_2 = (1, 0, -2)^T$

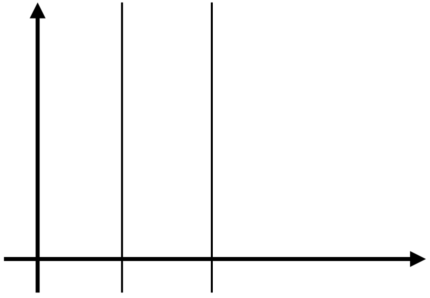
Tangent vector (0,-1,0) or (0,1,0)

Normal vector (1,0,0)

Example



Example



$$l_1 = (1, 0, -1)^T \qquad l_2 = (1, 0, -2)^T \qquad p_\infty = l_1 \times l_2$$

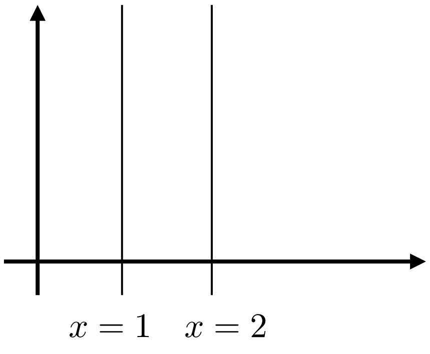
$$p_\infty = \hat{l}_1 l_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\boxed{p_\infty = (0, 1, 0)^T}$ ➡ Tangent vector

$n^T (0, 1, 0) = 0 \rightarrow \boxed{n = (1, 0, 0)^T}$ ➡ Normal direction

Example

$$l_1 = (1, 0, -1)^T \qquad l_2 = (1, 0, -2)^T \qquad p_\infty = l_1 \times l_2$$

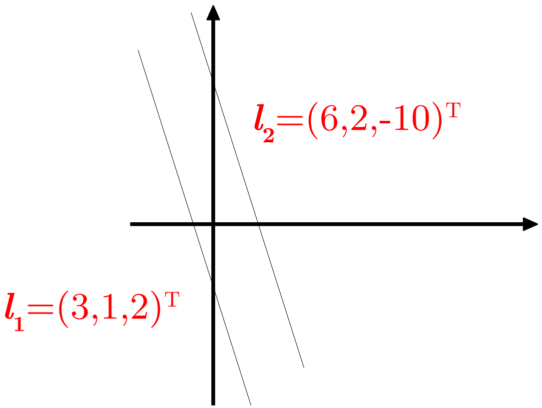


$$p_\infty = \hat{l}_1 l_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

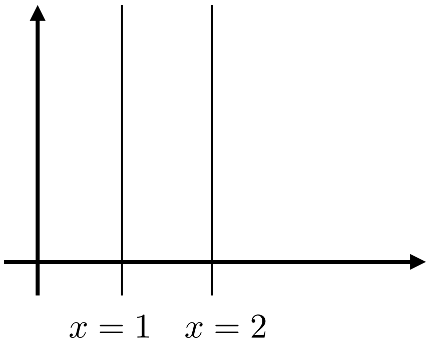
$$\boxed{p_\infty = (0, 1, 0)^T} \Rightarrow \text{Tangent vector}$$

$$n^T (0, 1, 0) = 0 \rightarrow \boxed{n = (1, 0, 0)^T} \Rightarrow \text{Normal direction}$$

$$l_1 = (3, 1, 2)^T \qquad l_2 = (6, 2, -10)^T \qquad p_\infty = l_1 \times l_2$$



Example

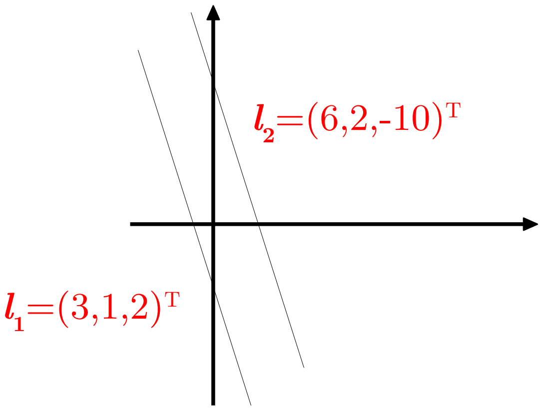


$$l_1 = (1, 0, -1)^T \qquad l_2 = (1, 0, -2)^T \qquad p_\infty = l_1 \times l_2$$

$$p_\infty = \hat{l}_1 l_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\boxed{p_\infty = (0, 1, 0)^T}$ ➡ Tangent vector

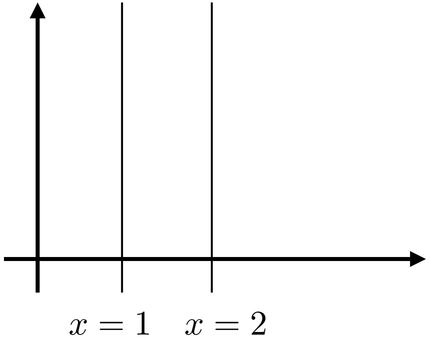
$n^T (0, 1, 0) = 0 \rightarrow \boxed{n = (1, 0, 0)^T}$ ➡ Normal direction



$$l_1 = (3, 1, 2)^T \qquad l_2 = (6, 2, -10)^T \qquad p_\infty = l_1 \times l_2$$

$$p_\infty = \hat{l}_1 l_2 = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ -10 \end{bmatrix} = \begin{bmatrix} -14 \\ 42 \\ 0 \end{bmatrix}$$ dividing by -14

Example

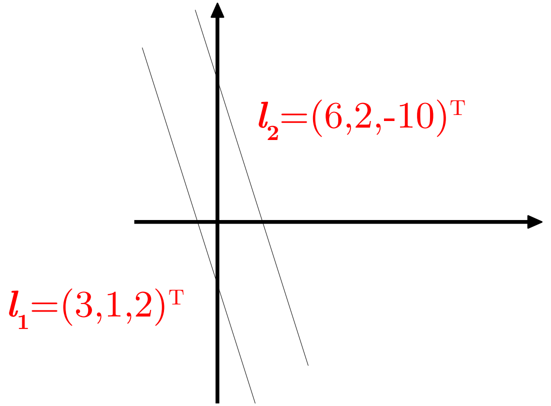


$$l_1 = (1,0,-1)^T \qquad l_2 = (1,0,-2)^T \qquad p_\infty = l_1 \times l_2$$

$$p_\infty = \hat{l}_1 l_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\boxed{p_\infty = (0,1,0)^T}$ ➡ Tangent vector

$n^T(0,1,0) = 0 \rightarrow \boxed{n = (1,0,0)^T}$ ➡ Normal direction



$$l_1 = (3,1,2)^T \qquad l_2 = (6,2,-10)^T \qquad p_\infty = l_1 \times l_2$$

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$\boxed{p_\infty = (1,-3,0)^T}$ ➡ Tangent vector

$n^T(1,-3,0) = 0 \rightarrow \boxed{n = (3,1,0)^T}$ ➡ Normal direction

Ideal points and lines at infinity

Ideal points $\rightarrow (x_1, x_2, 0)^T$

Line at infinity $\rightarrow l_\infty = (0, 0, 1)^T$

$$\mathbb{P}^2 = \mathbb{R}^2 \cup l_\infty$$

Note that in $\mathbb{P}^2$ there is no distinction between ideal points and others

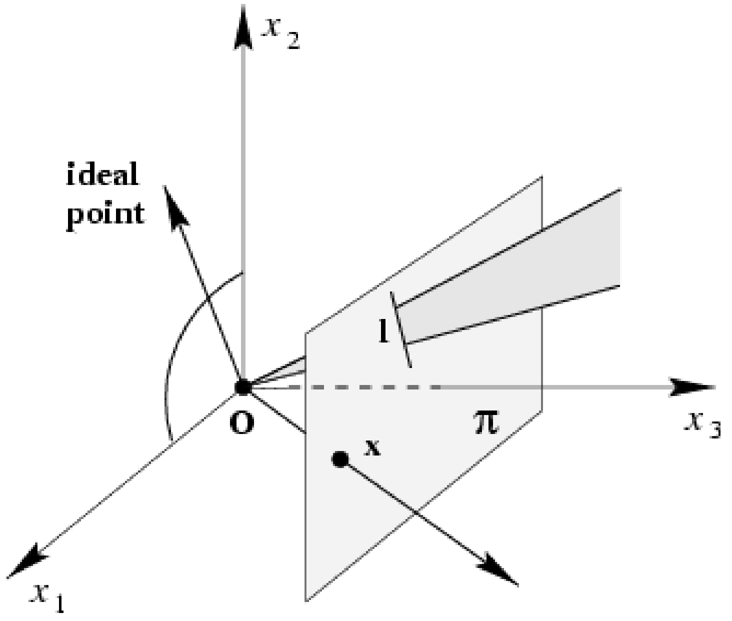
Duality

- Duality principle:
  - To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

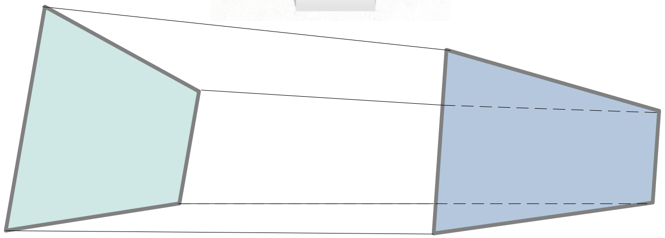
$$
\begin{array}{rcl}
x & \longleftrightarrow & l \\
x^T l = 0 & \longleftrightarrow & l^T x = 0 \\
x = l \times l' & \longleftrightarrow & l = x \times x'
\end{array}
$$

Model of the projective plane

- Projective Plane at $x_3 = 1$
- Rays through the origin $\rightarrow$ points
- Planes through the origin $\rightarrow$ lines
- Two non-identical rays lie on a plane $\rightarrow$ two points define a line
- Two planes intersect in one ray $\rightarrow$ two lines intersect in a point
- Rays representing ideal points $(x_1, x_2, 0)$ and the plane representing the line at infinity are parallel to the plane $x_3 = 1 \rightarrow x_1\, x_2$ plane



Projective Transformation



Projective Transformations

- A projectivity is an invertible mapping from points in $\mathbb{P}^2$ (homogeneous 3-vectors) to points in $\mathbb{P}^2$ that maps lines into lines.
- If three points x_1, x_2 and x_3 lie on a line, than $h(x_1)$, $h(x_2)$, and $h(x_3)$ also do.
- There is a non-singular 3 x 3 matrix H that: $h(x) = Hx$

x_1, x_2, x_3 lie on a line $l \rightarrow l^T x_i = 0$

Thus $l^T H^{-1} H x_i = 0$ and $h(x_i) = H x_i$

$\downarrow$

$h(x_1), h(x_2), h(x_3)$ lie on the line $H^{-T} l$

Point transformation

Line transformation

A hierarchy of transformations

Projective linear group $\rightarrow$ 8DOF

Affine group (last row (0,0,1)) $\rightarrow$ 6DOF

Similarity group (isotropic scaling) $\rightarrow$ 4DOF

Isometry group (upper left 2x2 orthogonal) $\rightarrow$ 3DOF

Class I: Isometries

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \varepsilon\cos\theta & -\sin\theta & t_x \\ \varepsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$\varepsilon = \pm 1$

orientation preserving: $\varepsilon = 1$
orientation reversing: $\varepsilon = -1$

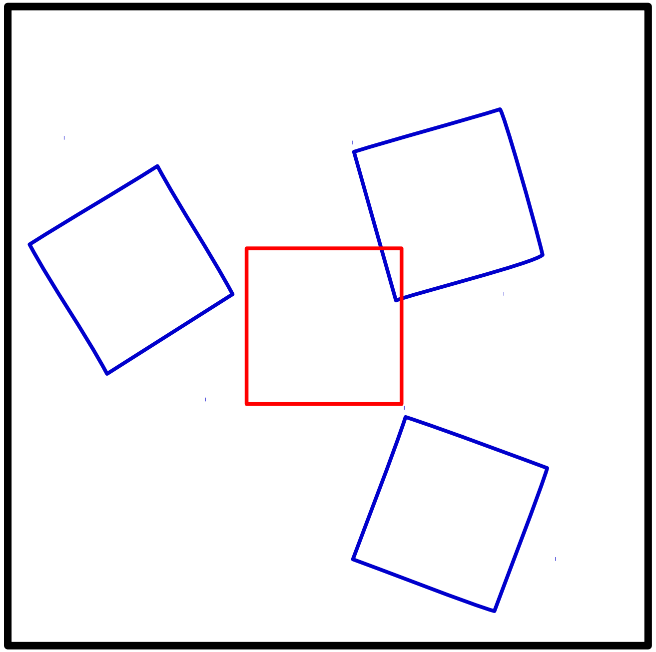
- (iso = same, metric = measure)
- 3DOF (1 rotation, 2 translation)
- Special cases: pure rotation, pure translation
- Orientation preserving $\rightarrow$ **Euclidean Transformation**
- **Invariants:** length, angle, area

$$x' = H_E x = \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix} x$$

$$R^T R = I$$

Class I: Isometries

- Example: Euclidean Transformation



Homogeneous:

$$\lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} w_{11} & w_{12} & \tau_x \\ w_{21} & w_{22} & \tau_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

Cartesian:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} \tau_x \\ \tau_y \end{bmatrix}$$

Class II: Similarities

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

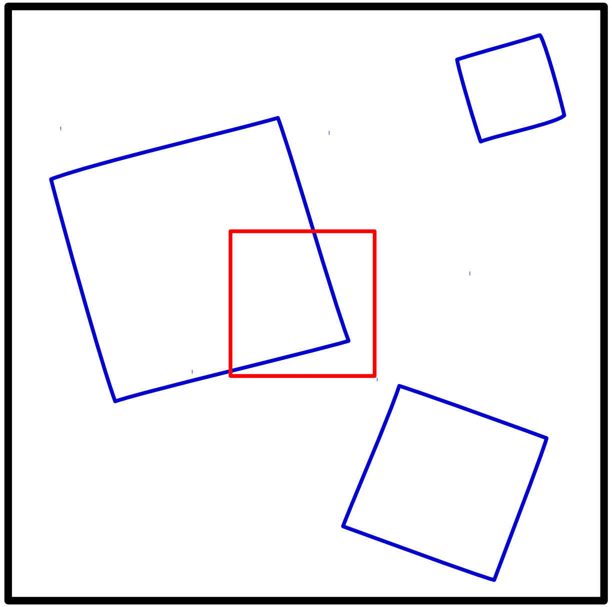
$$x' = H_S x = \begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix} x$$

$$R^T R = I$$

- (isometry + scale)
- 4DOF (1 scale, 1 rotation, 2 translation)
- Also know as equi-form (shape preserving)
- **Invariants:** ratios of length, angle, ratios of areas, parallel lines

Class II: Similarities

- Example



Homogeneous:

$$\lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \rho w_{11} & \rho w_{12} & \tau_x \\ \rho w_{21} & \rho w_{22} & \tau_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

Cartesian:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \rho w_{11} & \rho w_{12} \\ \rho w_{21} & \rho w_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} \tau_x \\ \tau_y \end{bmatrix}$$

Class III: Affine transformations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

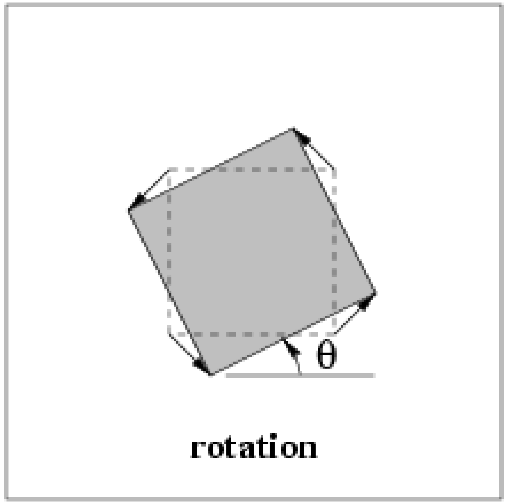
$$A = R(\theta)R(-\phi)DR(\phi) \qquad \mathrm{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

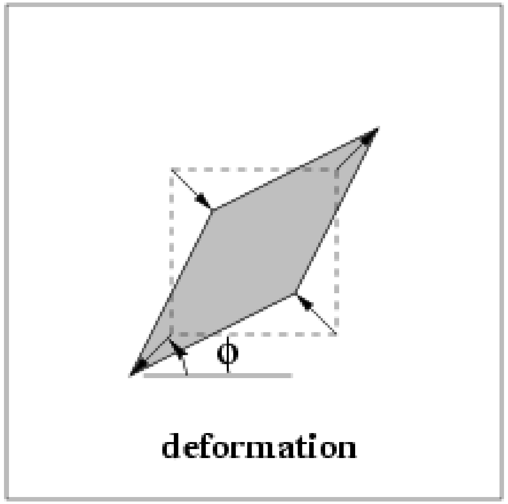
$$\mathrm{x}' = \mathrm{H}_A x = \begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix} x$$

- 6DOF (2 scale, 2 rotation, 2 translation)
- non-isotropic scaling! (2DOF: scale ratio and orientation)
- **Invariants:** parallel lines, ratios of parallel lengths, ratios of areas

Class III: Affine transformations

- Example





$$A = R(\theta)R(-\phi)DR(\phi) \qquad \mathrm{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\mathrm{x'} = \mathrm{H}_A x = \begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix} x$$

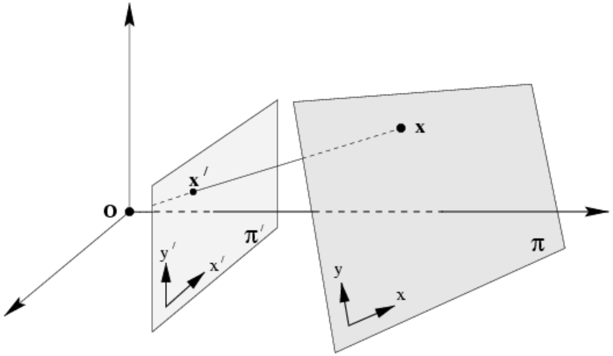
Class IV: Projective transformations

$$\text{x'} = \text{H}_P x = \begin{bmatrix} A & t \\ \nu^T & \nu \end{bmatrix} x \qquad \nu = (\nu_1, \nu_2)^T$$

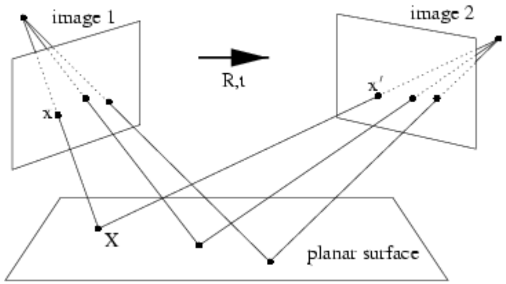
- 8DOF (2 scale, 2 rotation, 2 translation, 2 line at infinity)
- Action non-homogeneous over the plane
- **Invariants:** cross-ratio of four points on a line (ratio of ratio)
- **Perspectivity:** When two coordinate frames (on two planes) are both Euclidean, then the mapping defined by central projection is called perspectivity (more restricted than an arbitrary projectivity)

Class IV: Projective transformations

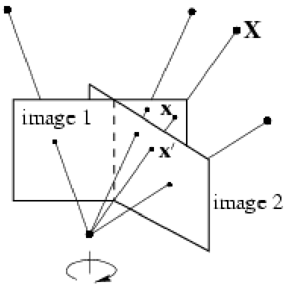
Mapping between planes



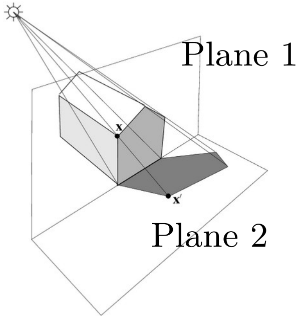
Transformation between two images induced by a world plane



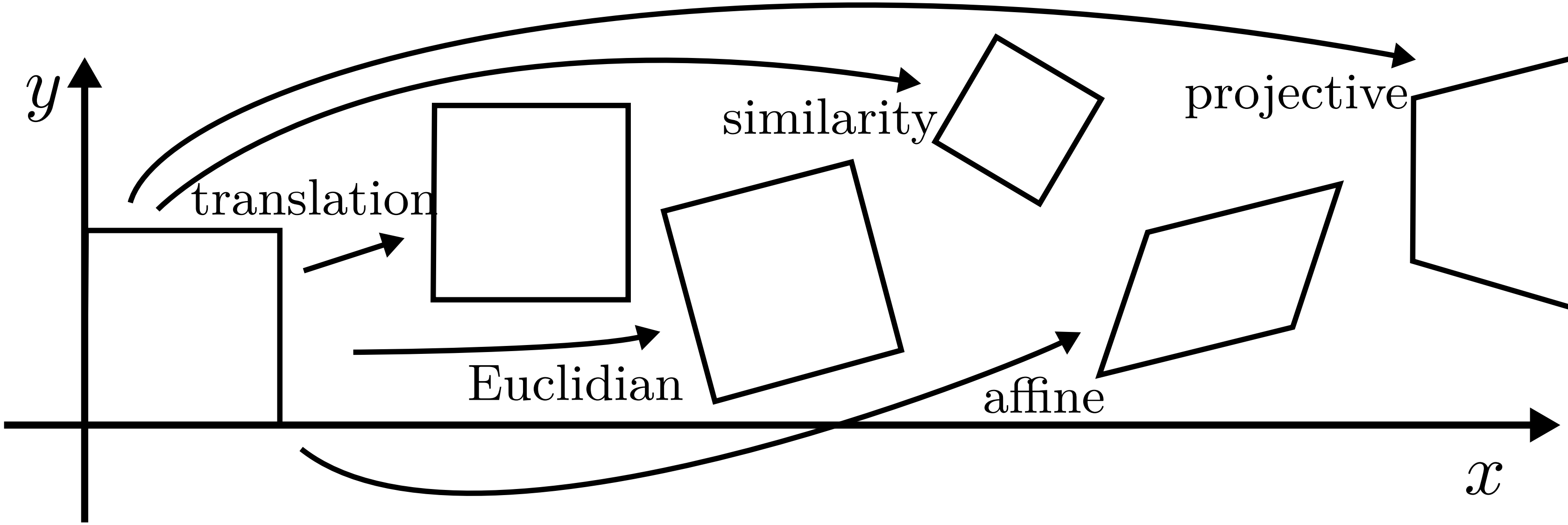
Transformation between two images with the same camera centre



Transformation between Planes 1 and 2: Perspectivity



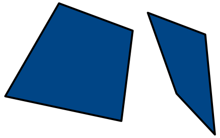
Overview of transformations



Overview of transformations

Projective
8DOF

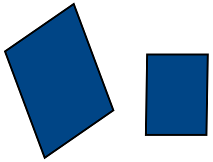
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio

Affine
6DOF

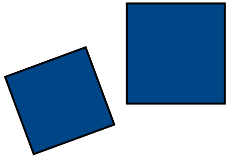
$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids).
The line at infinity $\boldsymbol{l_\infty}$

Similarity
4DOF

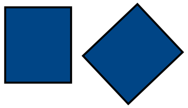
$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Ratios of lengths, angles.
The circular points I,J

Euclidean
3DOF

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Angles, lengths, areas, parallel lines.

Action of affinities and projectivities on points at infinity

- **Affinities:** Points at infinity stays at infinity, but move along line

$$\begin{bmatrix} A & t \\ 0^T & \nu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ 0 \end{bmatrix}$$

- **Projectivities:** Points at infinity becomes finite, allows to observe vanishing points, horizon

$$\begin{bmatrix} A & t \\ \nu^T & \nu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \nu_1 x_1 + \nu_2 x_2 \end{bmatrix}$$

Credits

- Richard Hartley and Andrew Zisserman.
  Multiple View Geometry in Computer Vision.
  Cambridge, ISBN 0521623049