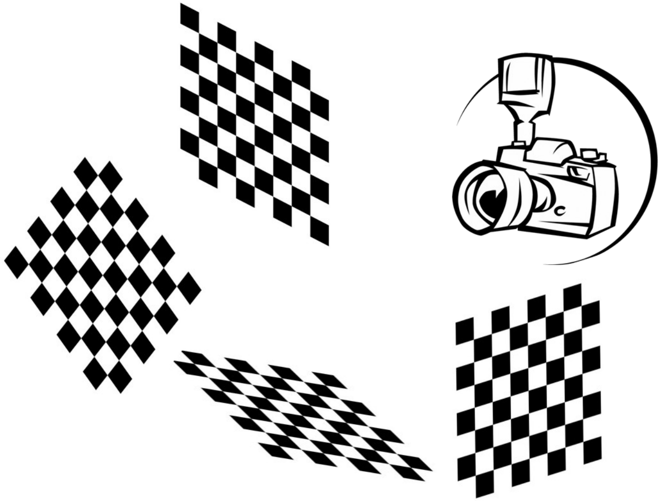
Computer Vision



Class 06

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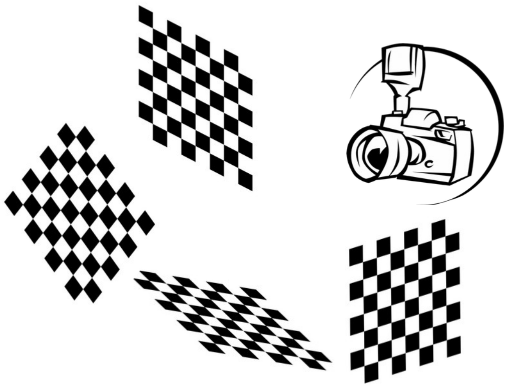
Camera Calibration

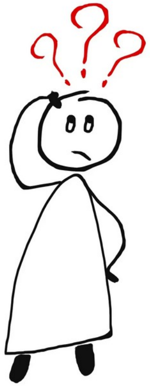
(Zhang's Algorithm)



Summary

- Camera Calibration
- Camera Model
- Homography between the model plane and its image
- Solving Camera Calibration
- Radial Distortion

What is Calibration?



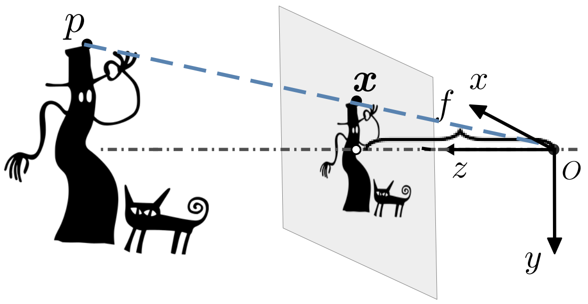
Camera Calibration

- Calibration tries to recover:
  - The intrinsic camera parameters, i.e., the inner transformations of the camera, including focal length, position of the principal point, sensor scale and skew.
  - The parameters of the non-linear lens distortion.
  - The extrinsic parameters (3D rotation and translation) for each of the given views of the reference pattern.

$$\lambda x = K\,[\mathrm{R},\mathrm{T}]\,X$$

Perspective Projection Model

- Pinhole Model
- Image plane is in front of the optical center, positioned at the distance $\boldsymbol{f}$ from the optical center $C = (0, 0, 0)^T$ and perpendicular to the optical axis
- C is the origin of the camera frame
- The optical axis aligns with the Z-axis and intersects the image plane at $(0, 0, f)^T$



Perspective Projection Model

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f s_x & f s_\theta & o_x \\ 0 & f s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

Consider the radial distortion before the image to sensor transformation (normalized image coordinates)

$$\boldsymbol{x} = f(r)(\boldsymbol{x}_d), \qquad r = \|\boldsymbol{x}_d\|$$

$$f(r) = 1 + a_1 r + a_2 r^2 + a_3 r^3 + a_4 r^4 + \cdots$$

Using the report notation

- Basic Equations:

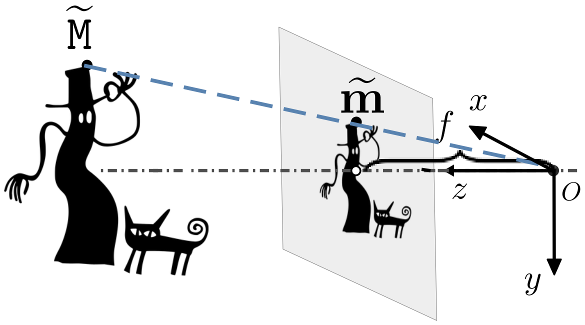
Image point: $\mathbf{m} = [u, v]^T$ In homogeneous coordinates: $\widetilde{\mathbf{m}} = [u, v, 1]^T$

3D point: $\mathtt{M} = [X, Y, Z]^T$ In homogeneous coordinates: $\widetilde{\mathtt{M}} = [X, Y, Z, 1]^T$

Projection onto the image plane: $s\widetilde{\mathbf{m}} = \mathbf{A}[\mathbf{R} \;\; \mathbf{t}]\widetilde{\mathtt{M}}$

- s is a scale factor
- $(\boldsymbol{R},\boldsymbol{t})$ are the extrinsic parameters
- $\boldsymbol{A}$ is the camera intrinsic matrix

$$\mathbf{A} = \begin{bmatrix} \alpha & \gamma & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

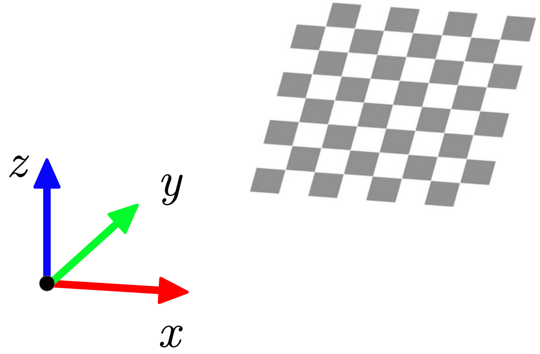


Homography between the model plane and its image

$$s\widetilde{\mathbf{m}} = \mathbf{A}[\mathbf{R} \quad \mathbf{t}]\widetilde{\mathtt{M}}$$

$$s\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{A}\begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{t} \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

where r_i is the i^{th} column of the rotation matrix $\boldsymbol{R}$



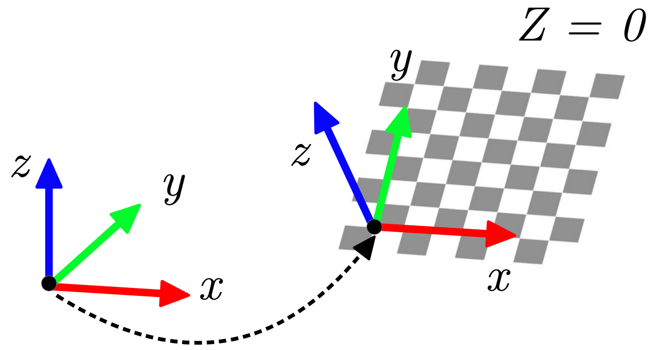
Homography between the model plane and its image

- Without loss of generality, we assume the model plane is on $Z = 0$ of the world coordinate system.

$$s\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{A}\begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{t} \end{bmatrix}\begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix}$$

$$= \mathbf{A}\begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{t} \end{bmatrix}\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

where r_i is the i th column of the rotation matrix $\boldsymbol{R}$



Homography between the model plane and its image

- By abuse of notation, we still use $\mathtt{M}$ to denote a point on the model plane, but $\mathtt{M} = [X, Y]^T$ since Z is always equal to 0. In turn, $\widetilde{\mathtt{M}} = [X, Y, 1]^T$. Therefore, a model point $\mathtt{M}$ and its image $\mathbf{m}$ is related by a homography $\mathbf{H}$:

$$s \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{t} \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \longrightarrow s\widetilde{\mathbf{m}} = \mathbf{H}\widetilde{\mathtt{M}} \quad \text{with} \quad \mathbf{H} = \mathbf{A} \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{t} \end{bmatrix}$$

As is clear, the 3×3 matrix $\mathbf{H}$ is defined up to a scale factor.

Given an image of the model plane, an homography can be estimated.

Estimation of the Homography

- Let $\mathtt{M}_i$ and $\mathbf{m}_i$ be the model and image points. Ideally, they satisfy the equation: $s\widetilde{\mathbf{m}} = \mathbf{H}\widetilde{\mathtt{M}}$
- In practice, they don't because of noise. Lets assume that $\mathbf{m}_i$ is corrupted by Gaussian noise with mean 0 and covariance matrix $\mathbf{\Lambda}_{\mathbf{m}_i}$.
- Then, the maximum likelihood estimation of $\mathbf{H}$ is obtained by minimizing the following functional and assuming $\mathbf{\Lambda}_{\mathbf{m}_i} = \sigma^2 I$ for all I

$$\sum_i (\mathbf{m}_i - \widehat{\mathbf{m}}_i)^T \mathbf{\Lambda}_{\mathbf{m}_i}^{-1} (\mathbf{m}_i - \widehat{\mathbf{m}}_i) \text{ where } \widehat{\mathbf{m}}_i = \frac{1}{\bar{\mathbf{h}}_3^T \mathtt{M}_i} \begin{bmatrix} \bar{\mathbf{h}}_1^T \mathtt{M}_i \\ \bar{\mathbf{h}}_2^T \mathtt{M}_i \end{bmatrix} \quad \text{with } \bar{\mathbf{h}}_i\text{, the } i^{th} \text{ row of } \mathbf{H}$$

- In this case, the above problem becomes a nonlinear least-squares one

$$\min_{\mathbf{H}} \sum_i \|\mathbf{m}_i - \widehat{\mathbf{m}}_i\|^2$$

Estimation of the Homography

- The nonlinear minimization is conducted with the Levenberg-Marquardt Algorithm.
- This requires an initial guess, which can be obtained as follows.

$$s\widetilde{\mathbf{m}} = \mathbf{H}\widetilde{\mathtt{M}} \xrightarrow{\widetilde{\mathbf{m}} \times \mathbf{H}\widetilde{\mathtt{M}} = 0} \begin{bmatrix} \widetilde{\mathtt{M}}^T & \mathbf{0}^T & -u\widetilde{\mathtt{M}}^T \\ \mathbf{0}^T & \widetilde{\mathtt{M}}^T & -v\widetilde{\mathtt{M}}^T \end{bmatrix} \mathbf{x} = \mathbf{0} \quad \text{with } \mathbf{x} = \left[\bar{\mathbf{h}}_1^T, \bar{\mathbf{h}}_2^T, \bar{\mathbf{h}}_3^T\right]^T$$

- For n points, we have n above equations, which can be written in matrix equation as $\mathbf{L}\mathbf{x} = \mathbf{0}$, where $\mathbf{L}$ is a $2n \times 9$ matrix.
- As $\mathbf{x}$ is defined up to a scale factor, the solution is well known to be the right singular vector of $\mathbf{L}$ associated with the smallest singular value (or equivalently, the eigenvector of $\mathbf{L}^T\mathbf{L}$ associated with the smallest eigenvalue).
- $\mathbf{x}$ will be the initial guess for the Levenberg-Marquardt Algorithm.
- Normalization must be done before estimating.

Estimation of the Homography

Another option is to use what we learned about DLT to estimate the Homography!

Constraints on the intrinsic parameters

- Let's denote the homography by $\mathbf{H} = [\mathbf{h}_1\ \mathbf{h}_2\ \mathbf{h}_3]$
- So:

$$\begin{bmatrix}\mathbf{h}_1 & \mathbf{h}_2 & \mathbf{h}_3\end{bmatrix} = \lambda \mathbf{A} \begin{bmatrix}\mathbf{r}_1 & \mathbf{r}_2 & \mathbf{t}\end{bmatrix} \text{ where } \lambda \text{ is an arbitrary scalar}$$

- Then $\mathbf{r}_1$ and $\mathbf{r}_2$ are can be written as :

$$\mathbf{h}_1 = \mathbf{A}\ \mathbf{r}_1 \longrightarrow \mathbf{r}_1 = \mathbf{A}^{-1}\ \mathbf{h}_1$$

$$\mathbf{h}_2 = \mathbf{A}\ \mathbf{r}_2 \longrightarrow \mathbf{r}_2 = \mathbf{A}^{-1}\ \mathbf{h}_2$$

Constraints on the intrinsic parameters

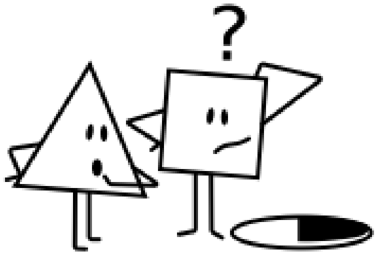
$$\mathbf{r}_1 = \mathbf{A}^{-1}\ \mathbf{h}_1 \qquad \mathbf{r}_2 = \mathbf{A}^{-1}\ \mathbf{h}_2$$

- Since $\mathbf{r}_1$ and $\mathbf{r}_2$ are orthonormal, we have:

$$\mathbf{r}_1^T \mathbf{r}_2 = 0 \longrightarrow \mathbf{h}_1^T \mathbf{A}^{-T} \mathbf{A}^{-1} \mathbf{h}_2 = 0$$

$$\mathbf{r}_1^T \mathbf{r}_1 = \mathbf{r}_2^T \mathbf{r}_2 \longrightarrow \mathbf{h}_1^T \mathbf{A}^{-T} \mathbf{A}^{-1} \mathbf{h}_1 = \mathbf{h}_2^T \mathbf{A}^{-T} \mathbf{A}^{-1} \mathbf{h}_2$$

Solving Camera Calibration



Closed-form solution

- Consider

$$\mathbf{B} = \mathbf{A}^{-T}\mathbf{A}^{-1} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\alpha^2} & -\frac{\gamma}{\alpha^2\beta} & \frac{v_0\gamma-u_0\beta}{\alpha^2\beta} \\ -\frac{\gamma}{\alpha^2\beta} & \frac{\gamma^2}{\alpha^2\beta^2}+\frac{1}{\beta^2} & -\frac{\gamma(v_0\gamma-u_0\beta)}{\alpha^2\beta^2}-\frac{v_0}{\beta^2} \\ \frac{v_0\gamma-u_0\beta}{\alpha^2\beta} & -\frac{\gamma(v_0\gamma-u_0\beta)}{\alpha^2\beta^2}-\frac{v_0}{\beta^2} & \frac{(v_0\gamma-u_0\beta)^2}{\alpha^2\beta^2}+\frac{v_0^2}{\beta^2}+1 \end{bmatrix}$$

- Note that $\mathbf{B}$ is symmetric, defined by a 6D vector:

$$\mathbf{b} = [B_{11}, B_{12}, B_{22}, B_{13}, B_{23}, B_{33}]^T$$

Closed-form solution

- Let the i^{th} column vector of $\mathbf{H}$ be: $\mathbf{h}_i = [h_{i1}, h_{i2}, h_{i3}]^T$

- Then, we have $\mathbf{h}_i^T \mathbf{B} \mathbf{h}_j = \mathbf{v}_{ij}^T \mathbf{b}$ with

$$\mathbf{v}_{ij} = [h_{i1}h_{j1}, h_{i1}h_{j2}+h_{i2}h_{j1}, h_{i2}h_{j2}, h_{i3}h_{j1}+h_{i1}h_{j3}, h_{i3}h_{j2}+h_{i2}h_{j3}, h_{i3}h_{j3}]^T$$

Closed-form solution

- Therefore, the two fundamental constraints, from a given homography, can be rewritten as 2 homogeneous equations in $\mathbf{b}$:

$$\mathbf{h}_1^T \mathbf{A}^{-T} \mathbf{A}^{-1} \mathbf{h}_2 = 0 \qquad \mathbf{h}_1^T \mathbf{B} \mathbf{h}_2 = \mathbf{v}_{12}^T \mathbf{b} \qquad \mathbf{v}_{12}^T \mathbf{b} = 0$$

$$\mathbf{h}_1^T \mathbf{A}^{-T} \mathbf{A}^{-1} \mathbf{h}_1 = \mathbf{h}_2^T \mathbf{A}^{-T} \mathbf{A}^{-1} \mathbf{h}_2 \qquad \mathbf{h}_1^T \mathbf{B} \mathbf{h}_1 = \mathbf{v}_{11}^T \mathbf{b} \qquad \mathbf{v}_{11}^T \mathbf{b} - \mathbf{v}_{22}^T \mathbf{b} = 0$$

$$\mathbf{h}_2^T \mathbf{B} \mathbf{h}_2 = \mathbf{v}_{22}^T \mathbf{b}$$

Closed-form solution

- Therefore, the two fundamental constraints, from a given homography, can be rewritten as 2 homogeneous equations in $\mathbf{b}$:

$$\mathbf{h}_1^T \mathbf{A}^{-T} \mathbf{A}^{-1} \mathbf{h}_2 = 0 \qquad \mathbf{h}_1^T \mathbf{B} \mathbf{h}_2 = \mathbf{v}_{12}^T \mathbf{b} \qquad \mathbf{v}_{12}^T \mathbf{b} = 0$$

$$\mathbf{h}_1^T \mathbf{A}^{-T} \mathbf{A}^{-1} \mathbf{h}_1 = \mathbf{h}_2^T \mathbf{A}^{-T} \mathbf{A}^{-1} \mathbf{h}_2 \qquad \mathbf{h}_1^T \mathbf{B} \mathbf{h}_1 = \mathbf{v}_{11}^T \mathbf{b} \qquad \mathbf{v}_{11}^T \mathbf{b} - \mathbf{v}_{22}^T \mathbf{b} = 0$$

$$\mathbf{h}_2^T \mathbf{B} \mathbf{h}_2 = \mathbf{v}_{22}^T \mathbf{b}$$

$$\begin{bmatrix} \mathbf{v}_{12}^T \\ (\mathbf{v}_{11} - \mathbf{v}_{22})^T \end{bmatrix} \mathbf{b} = \mathbf{0}$$

Closed-form solution

- If n images of the model plane are observed, by stacking n of such equations we have

$$\begin{bmatrix} \mathbf{v}_{12}^T \\ (\mathbf{v}_{11} - \mathbf{v}_{22})^T \end{bmatrix} \mathbf{b} = \mathbf{0} \quad \xrightarrow{\quad n \quad} \quad \mathbf{V}\mathbf{b} = \mathbf{0} \qquad \text{where } \mathbf{V} \text{ is a } 2n \times 6 \text{ matrix}$$

- If $n \geq 3$, we will have in general a unique solution $\mathbf{b}$ defined up to a scale factor.
- The solution is well known as the eigenvector of $\mathbf{V}^T\mathbf{V}$ associated with the smallest eigenvalue (equivalently, the right singular vector of $\mathbf{V}$ associated with the smallest singular value)

Closed-form solution

- Extraction of the Intrinsic Parameters from Matrix $\mathbf{B}$
  - Matrix $\mathbf{B}$ is estimated up to a scale factor, i.e., $\mathbf{B} = \lambda \mathbf{A}^{-T}\, \mathbf{A}$ with λ an arbitrary scale.
  - Extract the intrinsic parameters from matrix $\mathbf{B}$

$$
\begin{aligned}
v_0 &= (B_{12}B_{13} - B_{11}B_{23})/(B_{11}B_{22} - B_{12}^2) \\
\lambda &= B_{33} - \left[B_{13}^2 + v_0(B_{12}B_{13} - B_{11}B_{23})\right]/B_{11} \\
\alpha &= \sqrt{\lambda/B_{11}} \\
\beta &= \sqrt{\lambda B_{11}/(B_{11}B_{22} - B_{12}^2)} \\
\gamma &= -B_{12}\alpha^2\beta/\lambda \\
u_0 &= \gamma v_0/\beta - B_{13}\alpha^2/\lambda
\end{aligned}
$$

Closed-form solution

- Extraction of the Extrinsic Parameters from Matrix **H** and **A.**
- Once **A** is known, the extrinsic parameters for each image is readily computed.

$$s\widetilde{\mathbf{m}} = \mathbf{H}\widetilde{\mathtt{M}} \quad \text{with} \quad \mathbf{H} = \mathbf{A}\begin{bmatrix}\mathbf{r}_1 & \mathbf{r}_2 & \mathbf{t}\end{bmatrix} \longrightarrow \begin{aligned} \mathbf{r}_1 &= \lambda \mathbf{A}^{-1}\mathbf{h}_1 \\ \mathbf{r}_2 &= \lambda \mathbf{A}^{-1}\mathbf{h}_2 \\ \mathbf{r}_3 &= \mathbf{r}_1 \times \mathbf{r}_2 \\ \mathbf{t} &= \lambda \mathbf{A}^{-1}\mathbf{h}_3 \end{aligned}$$

$$\text{with } \lambda = 1/\|\mathbf{A}^{-1}\,\mathbf{h}_1\| = 1/\|\mathbf{A}^{-1}\,\mathbf{h}_2\|$$

- Because of noise in data, the so-computed matrix $\mathbf{R} = [\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3]$ does not in general satisfy the properties of a rotation matrix

Closed-form solution

- Approximating a 3×3 matrix by a Rotation Matrix
  - Consider $\mathbf{Q}$ as the solution obtained from the previous step: $\mathbf{Q} = [\ \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3]$
  - Solve the best rotation matrix $\mathbf{R}$ to approximate a given 3×3 matrix $\mathbf{Q}$
  - **"Best"** is in the sense of the smallest Frobenius norm of the difference $\mathbf{R} - \mathbf{Q}$

$$\min_{\mathbf{R}} \|\mathbf{R} - \mathbf{Q}\|_F^2 \quad \text{subject to } \mathbf{R}^T\mathbf{R} = \mathbf{I}$$

$$\begin{aligned} \|\mathbf{R} - \mathbf{Q}\|_F^2 &= \mathrm{trace}((\mathbf{R} - \mathbf{Q})^T(\mathbf{R} - \mathbf{Q})) \\ &= 3 + \mathrm{trace}(\mathbf{Q}^T\mathbf{Q}) - 2\mathrm{trace}(\mathbf{R}^T\mathbf{Q}) \end{aligned}$$

Closed-form solution

- Because of the definition of Frobenius norm, solving $\min_{\mathbf{R}} \|\mathbf{R}-\mathbf{Q}\|_F^2$ subject to $\mathbf{R}^T\mathbf{R}=\mathbf{I}$

  is equivalent to the one of maximizing trace($\mathbf{R}^T\mathbf{Q}$).

- Let the singular value decomposition of $\mathbf{Q}$ be $\mathbf{USV}^T$, where $\mathbf{S} = \text{diag}\,(\sigma_1, \sigma_2, \sigma_3)$
- If we define an orthogonal matrix $\mathbf{Z} = \mathbf{V}^T\mathbf{R}^T\mathbf{U}$, then

OBS: Trace is invariant under cyclic permutations

trace(ABC) = trace(BCA) = trace(CAB)

$$\begin{aligned}\text{trace}(\mathbf{R}^T\mathbf{Q}) &= \text{trace}(\mathbf{R}^T\mathbf{USV}^T) = \text{trace}(\mathbf{V}^T\mathbf{R}^T\mathbf{US}) \\ &= \text{trace}(\mathbf{ZS}) = \sum_{i=1}^{3} z_{ii}\sigma_i \leq \sum_{i=1}^{3}\sigma_i\end{aligned}$$

- The maximum is achieved by setting $\mathbf{R} = \mathbf{UV}^T$ because then $\mathbf{Z} = \mathbf{I}$.

Maximum likelihood estimation

- The previous solution (obtained through minimizing an algebraic distance). It can be refined through maximum likelihood inference.
- For n images of a model plane with m points on the model plane.
- Considering that the image points are corrupted by independent noise, the maximum likelihood estimate can be obtained by minimizing the following functional:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\|\mathbf{m}_{ij}-\hat{\mathbf{m}}(\mathbf{A},\mathbf{R}_i,\mathbf{t}_i,\mathtt{M}_j)\|^2$$

where $\hat{\mathbf{m}}(\mathbf{A}, \mathbf{R}_i , \mathbf{t}_i , \mathtt{M}_j)$ is the projection of point $_j$ in image i

- $\mathbf{R}$ is parameterized by a vector of 3 parameters, denoted by $\mathbf{r}$, which is parallel to the rotation axis and whose magnitude is the rotation angle. $\mathbf{R}$ and $\mathbf{r}$ are related by the Rodrigues formula.
- The minimization is solved with the Levenberg-Marquardt Algorithm.
- The initial guess for $\mathbf{A}$, $\{\mathbf{R}_i , \mathbf{t}_i \,|i = 1..n\}$ is obtained by the Close-form solution

Dealing with radial distortion

- Consider the first two terms of radial distortion
- Let (u, v) be the ideal (nonobservable distortion-free) pixel image coordinates, and $(\breve{u}, \breve{v})$ the corresponding real observed image coordinates.
- Similarly, (x, y) and $(\breve{x}, \breve{y})$ are the ideal (distortion-free) and real (distorted) normalized image coordinates

$$\breve{x} = x + x\left[k_1(x^2 + y^2) + k_2(x^2 + y^2)^2\right]$$

$$\breve{y} = y + y\left[k_1(x^2 + y^2) + k_2(x^2 + y^2)^2\right]$$

where k_1 and k_2 are the coefficients of the radial distortion.

Dealing with radial distortion

- Rewriting in image pixel coordinates

$$\breve{u} = u_0 + \alpha \breve{x} \qquad \breve{v} = v_0 + \beta \breve{y}$$

assuming $\gamma = 0$ (skew factor)

$$\breve{u} = u + (u - u_0)[k_1(x^2 + y^2) + k_2(x^2 + y^2)^2]$$

$$\breve{v} = v + (v - v_0)[k_1(x^2 + y^2) + k_2(x^2 + y^2)^2]$$

- The above equations can be rewritten as:

$$\begin{bmatrix} (u - u_0)(x^2 + y^2) & (u - u_0)(x^2 + y^2)^2 \\ (v - v_0)(x^2 + y^2) & (v - v_0)(x^2 + y^2)^2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} \breve{u} - u \\ \breve{v} - v \end{bmatrix}$$

- Given m points in n images, we can stack all equations together to obtain in total $2mn$ equations, or in matrix form as $\mathbf{Dk} = \mathbf{d}$, where $k = [\mathrm{k}_1, k_2]^T$.
- The linear least-squares solution is given by: $\mathbf{k} = (\mathbf{D}^T\mathbf{D})^{-1}\mathbf{D}^T\mathbf{d}$

Dealing with radial distortion

- Estimate the complete set of parameters by minimizing the following functional:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\|\mathbf{m}_{ij} - \breve{\mathbf{m}}(\mathbf{A}, k_1, k_2, \mathbf{R}_i, \mathbf{t}_i, \mathtt{M}_j)\|^2$$

  where $\breve{\mathbf{m}}(\mathbf{A}, k_1, k_2, \mathbf{R}_i, \mathbf{t}_i, \mathtt{M}_j)$ is the projection of point $\mathtt{M}_j$ in image i followed by radial distortion.

- Nonlinear minimization problem solved with the Levenberg-Marquardt Algorithm. The initial guess of $\mathbf{A}$ and $\{\mathbf{R}_i, \mathbf{t}_i | i = 1..n\}$ are done just like before.

- The initial guess of k_1 and k_2 can be obtained by the previous linear least-squares solution or can be set to 0 (zero).

Calibration Procedure

- The recommended calibration procedure is as follows:

  1. Print a pattern and attach it to a planar surface;

  2. Take a few images of the model plane under different orientations by moving either the plane or the camera;

  3. Detect the feature points in the images;

  4. Estimate the five intrinsic parameters and all the extrinsic parameters using the closed-form solution;

  5. Estimate the coefficients of the radial distortion by solving the linear least-squares;

  6. Refine all parameters by minimizing through Levenberg-Marquardt Algorithm.

Credits

- Zhengyou Zhang,
  A Flexible New Technique for Camera Calibration,
  Technical Report MSR-TR-98-71
- Zhengyou Zhang**, A Flexible New Technique for Camera Calibration**, IEEE Transactions on Pattern Analysis and Machine Intelligence, Vol. 22, No. 11, November 2000