2

Projective Geometry and Transformations of 2D

This chapter introduces the main geometric ideas and notation that are required to understand the material covered in this book. Some of these ideas are relatively familiar, such as vanishing point formation or representing conics, whilst others are more esoteric, such as using circular points to remove perspective distortion from an image. These ideas can be understood more easily in the planar (2D) case because they are more easily visualized here. The geometry of 3-space, which is the subject of the later parts of this book, is only a simple generalization of this planar case.

In particular, the chapter covers the geometry of projective transformations of the plane. These transformations model the geometric distortion which arises when a plane is imaged by a perspective camera. Under perspective imaging certain geometric properties are preserved, such as collinearity (a straight line is imaged as a straight line), whilst others are not, for example parallel lines are not imaged as parallel lines in general. Projective geometry models this imaging and also provides a mathematical representation appropriate for computations.

We begin by describing the representation of points, lines and conics in homogeneous notation, and how these entities map under projective transformations. The line at infinity and the circular points are introduced, and it is shown that these capture the affine and metric properties of the plane. Algorithms for rectifying planes are then given which enable affine and metric properties to be computed from images. We end with a description of fixed points under projective transformations.

2.1 Planar geometry

The basic concepts of planar geometry are familiar to anyone who has studied mathematics even at an elementary level. In fact, they are so much a part of our everyday experience that we take them for granted. At an elementary level, geometry is the study of points and lines and their relationships.

To the purist, the study of geometry ought properly to be carried out from a "geometric" or coordinate-free viewpoint. In this approach, theorems are stated and proved in terms of geometric primitives only, without the use of algebra. The classical approach of Euclid is an example of this method. Since Descartes, however, it has been seen that geometry may be algebraicized, and indeed the theory of geometry may be developed from an algebraic viewpoint. Our approach in this book will be a hybrid approach, sometimes using geometric, and sometimes algebraic methods. In the algebraic approach, geometric entities are described in terms of coordinates and algebraic entities. Thus, for instance a point is identified with a vector in terms of some coordinate basis. A line is also identified with a vector, and a conic section (more briefly, a conic) is represented by a symmetric matrix. In fact, we often carry this identification so far as to consider that the vector actually *is* a point, or the symmetric matrix *is* a conic, at least for convenience of language. A significant advantage of the algebraic approach to geometry is that results derived in this way may more easily be used to derive algorithms and practical computational methods. Computation and algorithms are a major concern in this book, which justifies the use of the algebraic method.

2.2 The 2D projective plane

As we all know, a point in the plane may be represented by the pair of coordinates (x, y) in $\mathbb{R}^2$. Thus, it is common to identify the plane with $\mathbb{R}^2$. Considering $\mathbb{R}^2$ as a vector space, the coordinate pair (x, y) is a vector – a point is identified as a vector. In this section we introduce the *homogeneous* notation for points and lines on a plane.

Row and column vectors. Later on, we will want to consider linear mappings between vector spaces, and represent such mappings as matrices. In the usual manner, the product of a matrix and a vector is another vector, the image under the mapping. This brings up the distinction between "column" and "row" vectors, since a matrix may be multiplied on the right by a column and on the left by a row vector. Geometric entities will by default be represented by column vectors. A bold-face symbol such as $\mathbf{x}$ always represents a column vector, and its transpose is the row vector $\mathbf{x}^\mathsf{T}$. In accordance with this convention, a point in the plane will be represented by the column vector $(x, y)^\mathsf{T}$, rather than its transpose, the row vector (x, y). We write $\mathbf{x} = (x, y)^\mathsf{T}$, both sides of this equation representing column vectors.

2.2.1 Points and lines

Homogeneous representation of lines. A line in the plane is represented by an equation such as $ax + by + c = 0$, different choices of a, b and c giving rise to different lines. Thus, a line may naturally be represented by the vector $(a, b, c)^\mathsf{T}$. The correspondence between lines and vectors $(a, b, c)^\mathsf{T}$ is not one-to-one, since the lines $ax + by + c = 0$ and $(ka)x + (kb)y + (kc) = 0$ are the same, for any non-zero constant k. Thus, the vectors $(a, b, c)^\mathsf{T}$ and $k(a, b, c)^\mathsf{T}$ represent the same line, for any non-zero k. In fact, two such vectors related by an overall scaling are considered as being equivalent. An equivalence class of vectors under this equivalence relationship is known as a *homogeneous* vector. Any particular vector $(a, b, c)^\mathsf{T}$ is a representative of the equivalence class. The set of equivalence classes of vectors in $\mathbb{R}^3 - (0, 0, 0)^\mathsf{T}$ forms the *projective space* $\mathbb{P}^2$. The notation $-(0, 0, 0)^\mathsf{T}$ indicates that the vector $(0, 0, 0)^\mathsf{T}$, which does not correspond to any line, is excluded.

Homogeneous representation of points. A point $\mathbf{x} = (x, y)^\mathsf{T}$ lies on the line $\mathbf{l} = (a, b, c)^\mathsf{T}$ if and only if $ax + by + c = 0$. This may be written in terms of an inner product of vectors representing the point as $(x, y, 1)(a, b, c)^\mathsf{T} = (x, y, 1)\mathbf{l} = 0$; that is the point $(x, y)^\mathsf{T}$ in $\mathbb{R}^2$ is represented as a 3-vector by adding a final coordinate of 1. Note that for any non-zero constant k and line $\mathbf{l}$ the equation $(kx, ky, k)\mathbf{l} = 0$ if and only if $(x, y, 1)\mathbf{l} = 0$. It is natural, therefore, to consider the set of vectors $(kx, ky, k)^\mathsf{T}$ for varying values of k to be a representation of the point $(x, y)^\mathsf{T}$ in $\mathbb{R}^2$. Thus, just as with lines, points are represented by homogeneous vectors. An arbitrary homogeneous vector representative of a point is of the form $\mathbf{x} = (x_1, x_2, x_3)^\mathsf{T}$, representing the point $(x_1/x_3, x_2/x_3)^\mathsf{T}$ in $\mathbb{R}^2$. Points, then, as homogeneous vectors are also elements of $\mathbb{P}^2$.

One has a simple equation to determine when a point lies on a line, namely

Result 2.1. *The point* $\mathbf{x}$ *lies on the line* $\mathbf{l}$ *if and only if* $\mathbf{x}^\mathsf{T}\mathbf{l} = 0$.

Note that the expression $\mathbf{x}^\mathsf{T}\mathbf{l}$ is just the inner or scalar product of the two vectors $\mathbf{l}$ and $\mathbf{x}$. The scalar product $\mathbf{x}^\mathsf{T}\mathbf{l} = \mathbf{l}^\mathsf{T}\mathbf{x} = \mathbf{x}.\mathbf{l}$. In general, the transpose notation $\mathbf{l}^\mathsf{T}\mathbf{x}$ will be preferred, but occasionally, we will use a . to denote the inner product. We distinguish between the *homogeneous coordinates* $\mathbf{x} = (x_1, x_2, x_3)^\mathsf{T}$ of a point, which is a 3-vector, and the *inhomogeneous* coordinates $(x, y)^\mathsf{T}$, which is a 2-vector.

Degrees of freedom (dof). It is clear that in order to specify a point two values must be provided, namely its x- and y-coordinates. In a similar manner a line is specified by two parameters (the two independent ratios $\{a : b : c\}$) and so has two degrees of freedom. For example, in an inhomogeneous representation, these two parameters could be chosen as the gradient and y intercept of the line.

Intersection of lines. Given two lines $\mathbf{l} = (a, b, c)^\mathsf{T}$ and $\mathbf{l}' = (a', b', c')^\mathsf{T}$, we wish to find their intersection. Define the vector $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$, where $\times$ represents the vector or cross product. From the triple scalar product identity $\mathbf{l}.(\mathbf{l} \times \mathbf{l}') = \mathbf{l}'.(\mathbf{l} \times \mathbf{l}') = 0$, we see that $\mathbf{l}^\mathsf{T}\mathbf{x} = \mathbf{l}'^\mathsf{T}\mathbf{x} = 0$. Thus, if $\mathbf{x}$ is thought of as representing a point, then $\mathbf{x}$ lies on both lines $\mathbf{l}$ and $\mathbf{l}'$, and hence is the intersection of the two lines. This shows:

Result 2.2. *The intersection of two lines* $\mathbf{l}$ *and* $\mathbf{l}'$ *is the point* $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$.

Note that the simplicity of this expression for the intersection of the two lines is a direct consequence of the use of homogeneous vector representations of lines and points.

Example 2.3. Consider the simple problem of determining the intersection of the lines $x = 1$ and $y = 1$. The line $x = 1$ is equivalent to $-1x + 1 = 0$, and thus has homogeneous representation $\mathbf{l} = (-1, 0, 1)^\mathsf{T}$. The line $y = 1$ is equivalent to $-1y + 1 = 0$, and thus has homogeneous representation $\mathbf{l}' = (0, -1, 1)^\mathsf{T}$. From result 2.2 the intersection point is

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

which is the inhomogeneous point $(1, 1)^\mathsf{T}$ as required. △

Line joining points. An expression for the line passing through two points $\mathbf{x}$ and $\mathbf{x}'$ may be derived by an entirely analogous argument. Defining a line $\mathbf{l}$ by $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$, it may be verified that both points $\mathbf{x}$ and $\mathbf{x}'$ lie on $\mathbf{l}$. Thus

Result 2.4. *The line through two points* $\mathbf{x}$ *and* $\mathbf{x}'$ *is* $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$.

2.2.2 Ideal points and the line at infinity

Intersection of parallel lines. Consider two lines $ax+by+c = 0$ and $ax+by+c' = 0$. These are represented by vectors $\mathbf{l} = (a, b, c)^\mathsf{T}$ and $\mathbf{l}' = (a, b, c')^\mathsf{T}$ for which the first two coordinates are the same. Computing the intersection of these lines gives no difficulty, using result 2.2. The intersection is $\mathbf{l} \times \mathbf{l}' = (c' - c)(b, -a, 0)^\mathsf{T}$, and ignoring the scale factor $(c' - c)$, this is the point $(b, -a, 0)^\mathsf{T}$.

Now if we attempt to find the inhomogeneous representation of this point, we obtain $(b/0, -a/0)^\mathsf{T}$, which makes no sense, except to suggest that the point of intersection has infinitely large coordinates. In general, points with homogeneous coordinates $(x, y, 0)^\mathsf{T}$ do not correspond to any finite point in $\mathbb{R}^2$. This observation agrees with the usual idea that parallel lines meet at infinity.

Example 2.5. Consider the two lines $x = 1$ and $x = 2$. Here the two lines are parallel, and consequently intersect "at infinity". In homogeneous notation the lines are $\mathbf{l} = (-1, 0, 1)^\mathsf{T}$, $\mathbf{l}' = (-1, 0, 2)^\mathsf{T}$, and from result 2.2 their intersection point is

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ -1 & 0 & 2 \end{vmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

which is the point at infinity in the direction of the y-axis. $\triangle$

Ideal points and the line at infinity. Homogeneous vectors $\mathbf{x} = (x_1, x_2, x_3)^\mathsf{T}$ such that $x_3 \neq 0$ correspond to finite points in $\mathbb{R}^2$. One may augment $\mathbb{R}^2$ by adding points with last coordinate $x_3 = 0$. The resulting space is the set of all homogeneous 3-vectors, namely the projective space $\mathbb{P}^2$. The points with last coordinate $x_3 = 0$ are known as *ideal* points, or points at infinity. The set of all ideal points may be written $(x_1, x_2, 0)^\mathsf{T}$, with a particular point specified by the ratio $x_1 : x_2$. Note that this set lies on a single line, the *line at infinity*, denoted by the vector $\mathbf{l}_\infty = (0, 0, 1)^\mathsf{T}$. Indeed, one verifies that $(0, 0, 1)(x_1, x_2, 0)^\mathsf{T} = 0$.

Using result 2.2 one finds that a line $\mathbf{l} = (a, b, c)^\mathsf{T}$ intersects $\mathbf{l}_\infty$ in the ideal point $(b, -a, 0)^\mathsf{T}$ (since $(b, -a, 0)\mathbf{l} = 0$). A line $\mathbf{l}' = (a, b, c')^\mathsf{T}$ *parallel* to $\mathbf{l}$ intersects $\mathbf{l}_\infty$ in the same ideal point $(b, -a, 0)^\mathsf{T}$ irrespective of the value of c'. In inhomogeneous notation $(b, -a)^\mathsf{T}$ is a vector tangent to the line, and orthogonal to the line normal (a, b), and so represents the line's *direction*. As the line's direction varies the ideal point $(b, -a, 0)^\mathsf{T}$ varies over $\mathbf{l}_\infty$. For these reasons the line at infinity can be thought of as the set of directions of lines in the plane.

Note how the introduction of the concept of points at infinity serves to simplify the intersection properties of points and lines. In the projective plane $\mathbb{P}^2$, one may state without qualification that two distinct lines meet in a single point and two distinct

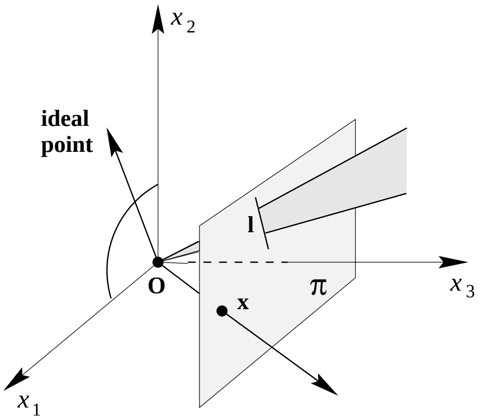


Fig. 2.1. **A model of the projective plane.** *Points and lines of* $\mathbb{P}^2$ *are represented by rays and planes, respectively, through the origin in* $\mathbb{R}^3$*. Lines lying in the* x_1x_2*-plane represent ideal points, and the* x_1x_2*-plane represents* $\mathbf{l}_\infty$.

points lie on a single line. This is not true in the standard Euclidean geometry of $\mathbb{R}^2$, in which parallel lines form a special case.

The study of the geometry of $\mathbb{P}^2$ is known as projective geometry. In a coordinate-free purely geometric study of projective geometry, one does not make any distinction between points at infinity (ideal points) and ordinary points. It will, however, serve our purposes in this book sometimes to distinguish between ideal points and non-ideal points. Thus, the line at infinity will at times be considered as a special line in projective space.

A model for the projective plane. A fruitful way of thinking of $\mathbb{P}^2$ is as a set of rays in $\mathbb{R}^3$. The set of all vectors $k(x_1, x_2, x_3)^\mathsf{T}$ as k varies forms a ray through the origin. Such a ray may be thought of as representing a single point in $\mathbb{P}^2$. In this model, the lines in $\mathbb{P}^2$ are planes passing through the origin. One verifies that two non-identical rays lie on exactly one plane, and any two planes intersect in one ray. This is the analogue of two distinct points uniquely defining a line, and two lines always intersecting in a point.

Points and lines may be obtained by intersecting this set of rays and planes by the plane $x_3 = 1$. As illustrated in figure 2.1 the rays representing ideal points and the plane representing $\mathbf{l}_\infty$ are parallel to the plane $x_3 = 1$.

Duality. The reader has probably noticed how the role of points and lines may be interchanged in statements concerning the properties of lines and points. In particular, the basic incidence equation $\mathbf{l}^\mathsf{T}\mathbf{x} = 0$ for line and point is symmetric, since $\mathbf{l}^\mathsf{T}\mathbf{x} = 0$ implies $\mathbf{x}^\mathsf{T}\mathbf{l} = 0$, in which the positions of line and point are swapped. Similarly, result 2.2 and result 2.4 giving the intersection of two lines and the line through two points are essentially the same, with the roles of points and lines swapped. One may enunciate a general principle, the *duality principle* as follows:

Result 2.6. Duality principle. *To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the roles of points and lines in the original theorem.*

In applying this principle, concepts of incidence must be appropriately translated as well. For instance, the line through two points is dual to the point through (that is the point of intersection of) two lines.

Note that is it not necessary to prove the dual of a given theorem once the original theorem has been proved. The proof of the dual theorem will be the dual of the proof of the original theorem.

2.2.3 Conics and dual conics

A conic is a curve described by a second-degree equation in the plane. In Euclidean geometry conics are of three main types: hyperbola, ellipse, and parabola (apart from so-called degenerate conics, to be defined later). Classically these three types of conic arise as conic sections generated by planes of differing orientation (the degenerate conics arise from planes which contain the cone vertex). However, it will be seen that in 2D projective geometry all non-degenerate conics are equivalent under projective transformations.

The equation of a conic in inhomogeneous coordinates is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

i.e. a polynomial of degree 2. "Homogenizing" this by the replacements:
$x \mapsto x_1/x_3,\ y \mapsto x_2/x_3$ gives

$$a{x_1}^2 + bx_1x_2 + c{x_2}^2 + dx_1x_3 + ex_2x_3 + f{x_3}^2 = 0 \tag{2.1}$$

or in matrix form

$$\mathbf{x}^{\mathsf{T}}\mathtt{C}\mathbf{x} = 0 \tag{2.2}$$

where the conic coefficient matrix $\mathtt{C}$ is given by

$$\mathtt{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}. \tag{2.3}$$

Note that the conic coefficient matrix is symmetric. As in the case of the homogeneous representation of points and lines, only the ratios of the matrix elements are important, since multiplying $\mathtt{C}$ by a non-zero scalar does not affect the above equations. Thus $\mathtt{C}$ is a homogeneous representation of a conic. The conic has five degrees of freedom which can be thought of as the ratios $\{a : b : c : d : e : f\}$ or equivalently the six elements of a symmetric matrix less one for scale.

Five points define a conic. Suppose we wish to compute the conic which passes through a set of points, $\mathbf{x}_i$. How many points are we free to specify before the conic is determined uniquely? The question can be answered constructively by providing an algorithm to determine the conic. From (2.1) each point $\mathbf{x}_i$ places one constraint on the conic coefficients, since if the conic passes through (x_i, y_i) then

$$a{x_i}^2 + bx_iy_i + c{y_i}^2 + dx_i + ey_i + f = 0.$$

This constraint can be written as

$$\begin{pmatrix} x_i^2 & x_iy_i & y_i^2 & x_i & y_i & 1 \end{pmatrix} \mathbf{c} = 0$$

where $\mathbf{c} = (a, b, c, d, e, f)^\mathsf{T}$ is the conic $\mathtt{C}$ represented as a 6-vector.

Stacking the constraints from five points we obtain

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = \mathbf{0} \tag{2.4}$$

and the conic is the null vector of this 5×6 matrix. This shows that a conic is determined uniquely (up to scale) by five points in general position. The method of fitting a geometric entity (or relation) by determining a null space will be used frequently in the computation chapters throughout this book.

Tangent lines to conics. The line $\mathbf{l}$ tangent to a conic at a point $\mathbf{x}$ has a particularly simple form in homogeneous coordinates:

Result 2.7. *The line* $\mathbf{l}$ *tangent to* $\mathtt{C}$ *at a point* $\mathbf{x}$ *on* $\mathtt{C}$ *is given by* $\mathbf{l} = \mathtt{C}\mathbf{x}$.

Proof. The line $\mathbf{l} = \mathtt{C}\mathbf{x}$ passes through $\mathbf{x}$, since $\mathbf{l}^\mathsf{T}\mathbf{x} = \mathbf{x}^\mathsf{T}\mathtt{C}\mathbf{x} = 0$. If $\mathbf{l}$ has one-point contact with the conic, then it is a tangent, and we are done. Otherwise suppose that $\mathbf{l}$ meets the conic in another point $\mathbf{y}$. Then $\mathbf{y}^\mathsf{T}\mathtt{C}\mathbf{y} = 0$ and $\mathbf{x}^\mathsf{T}\mathtt{C}\mathbf{y} = \mathbf{l}^\mathsf{T}\mathbf{y} = 0$. From this it follows that $(\mathbf{x} + \alpha\mathbf{y})^\mathsf{T}\mathtt{C}(\mathbf{x} + \alpha\mathbf{y}) = 0$ for all α, which means that the whole line $\mathbf{l} = \mathtt{C}\mathbf{x}$ joining $\mathbf{x}$ and $\mathbf{y}$ lies on the conic $\mathtt{C}$, which is therefore degenerate (see below). $\square$

Dual conics. The conic $\mathtt{C}$ defined above is more properly termed a *point* conic, as it defines an equation on points. Given the duality result 2.6 of $\mathbb{P}^2$ it is not surprising that there is also a conic which defines an equation on lines. This *dual* (or line) conic is also represented by a 3×3 matrix, which we denote as $\mathtt{C}^*$. A line $\mathbf{l}$ *tangent* to the conic $\mathtt{C}$ satisfies $\mathbf{l}^\mathsf{T}\mathtt{C}^*\mathbf{l} = 0$. The notation $\mathtt{C}^*$ indicates that $\mathtt{C}^*$ is the adjoint matrix of $\mathtt{C}$ (the adjoint is defined in section A4.2(*p*580) of appendix 4(*p*578)). For a non-singular symmetric matrix $\mathtt{C}^* = \mathtt{C}^{-1}$ (up to scale).

The equation for a dual conic is straightforward to derive in the case that $\mathtt{C}$ has full rank: From result 2.7, at a point $\mathbf{x}$ on $\mathtt{C}$ the tangent is $\mathbf{l} = \mathtt{C}\mathbf{x}$. Inverting, we find the point $\mathbf{x}$ at which the line $\mathbf{l}$ is tangent to $\mathtt{C}$ is $\mathbf{x} = \mathtt{C}^{-1}\mathbf{l}$. Since $\mathbf{x}$ satisfies $\mathbf{x}^\mathsf{T}\mathtt{C}\mathbf{x} = 0$ we obtain $(\mathtt{C}^{-1}\mathbf{l})^\mathsf{T}\mathtt{C}(\mathtt{C}^{-1}\mathbf{l}) = \mathbf{l}^\mathsf{T}\mathtt{C}^{-1}\mathbf{l} = 0$, the last step following from $\mathtt{C}^{-\mathsf{T}} = \mathtt{C}^{-1}$ because $\mathtt{C}$ is symmetric.

Dual conics are also known as conic envelopes, and the reason for this is illustrated

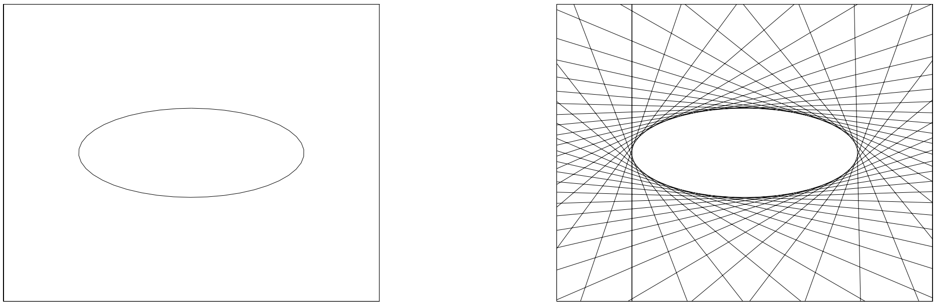


Fig. 2.2. *(a) Points $\mathbf{x}$ satisfying $\mathbf{x}^\mathsf{T}\mathtt{C}\mathbf{x} = 0$ lie on a point conic. (b) Lines $\mathbf{l}$ satisfying $\mathbf{l}^\mathsf{T}\mathtt{C}^*\mathbf{l} = 0$ are tangent to the point conic $\mathtt{C}$. The conic $\mathtt{C}$ is the envelope of the lines $\mathbf{l}$.*

in figure 2.2. A dual conic has five degrees of freedom. In a similar manner to points defining a point conic, it follows that five lines in general position define a dual conic.

Degenerate conics. If the matrix $\mathtt{C}$ is not of full rank, then the conic is termed degenerate. Degenerate point conics include two lines (rank 2), and a repeated line (rank 1).

Example 2.8. The conic

$$\mathtt{C} = \mathbf{l}\,\mathbf{m}^\mathsf{T} + \mathbf{m}\,\mathbf{l}^\mathsf{T}$$

is composed of two lines $\mathbf{l}$ and $\mathbf{m}$. Points on $\mathbf{l}$ satisfy $\mathbf{l}^\mathsf{T}\mathbf{x} = 0$, and are on the conic since $\mathbf{x}^\mathsf{T}\mathtt{C}\mathbf{x} = (\mathbf{x}^\mathsf{T}\mathbf{l})(\mathbf{m}^\mathsf{T}\mathbf{x}) + (\mathbf{x}^\mathsf{T}\mathbf{m})(\mathbf{l}^\mathsf{T}\mathbf{x}) = 0$. Similarly, points satisfying $\mathbf{m}^\mathsf{T}\mathbf{x} = 0$ also satisfy $\mathbf{x}^\mathsf{T}\mathtt{C}\mathbf{x} = 0$. The matrix $\mathtt{C}$ is symmetric and has rank 2. The null vector is $\mathbf{x} = \mathbf{l} \times \mathbf{m}$ which is the intersection point of $\mathbf{l}$ and $\mathbf{m}$. $\triangle$

Degenerate *line* conics include two points (rank 2), and a repeated point (rank 1). For example, the line conic $\mathtt{C}^* = \mathbf{x}\mathbf{y}^\mathsf{T} + \mathbf{y}\mathbf{x}^\mathsf{T}$ has rank 2 and consists of lines passing through either of the two points $\mathbf{x}$ and $\mathbf{y}$. Note that for matrices that are not invertible $(\mathtt{C}^*)^* \neq \mathtt{C}$.

2.3 Projective transformations

In the view of geometry set forth by Felix Klein in his famous "Erlangen Program", [Klein-39], geometry is the study of properties invariant under groups of transformations. From this point of view, 2D projective geometry is the study of properties of the projective plane $\mathbb{P}^2$ that are invariant under a group of transformations known as *projectivities*.

A projectivity is an invertible mapping from points in $\mathbb{P}^2$ (that is homogeneous 3-vectors) to points in $\mathbb{P}^2$ that maps lines to lines. More precisely,

Definition 2.9. A *projectivity* is an invertible mapping h from $\mathbb{P}^2$ to itself such that three points $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ lie on the same line if and only if $h(\mathbf{x}_1)$, $h(\mathbf{x}_2)$ and $h(\mathbf{x}_3)$ do.

Projectivities form a group since the inverse of a projectivity is also a projectivity, and so is the composition of two projectivities. A projectivity is also called a *collineation* (a helpful name), a *projective transformation* or a *homography*: the terms are synonymous.

In definition 2.9, a projectivity is defined in terms of a coordinate-free geometric concept of point line incidence. An equivalent algebraic definition of a projectivity is possible, based on the following result.

Theorem 2.10. *A mapping $h : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is a projectivity if and only if there exists a non-singular 3×3 matrix $\mathtt{H}$ such that for any point in $\mathbb{P}^2$ represented by a vector $\mathbf{x}$ it is true that $h(\mathbf{x}) = \mathtt{H}\mathbf{x}$.*

To interpret this theorem, any point in $\mathbb{P}^2$ is represented as a homogeneous 3-vector, $\mathbf{x}$, and $\mathtt{H}\mathbf{x}$ is a linear mapping of homogeneous coordinates. The theorem asserts that any projectivity arises as such a linear transformation in homogeneous coordinates, and that conversely any such mapping is a projectivity. The theorem will not be proved in full here. It will only be shown that any invertible linear transformation of homogeneous coordinates is a projectivity.

Proof. Let $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ lie on a line $\mathbf{l}$. Thus $\mathbf{l}^\mathsf{T}\mathbf{x}_i = 0$ for $i = 1, \ldots, 3$. Let $\mathtt{H}$ be a non-singular 3×3 matrix. One verifies that $\mathbf{l}^\mathsf{T}\mathtt{H}^{-1}\mathtt{H}\mathbf{x}_i = 0$. Thus, the points $\mathtt{H}\mathbf{x}_i$ all lie on the line $\mathtt{H}^{-\mathsf{T}}\mathbf{l}$, and collinearity is preserved by the transformation.

The converse is considerably harder to prove, namely that each projectivity arises in this way. □

As a result of this theorem, one may give an alternative definition of a projective transformation (or collineation) as follows.

Definition 2.11. Projective transformation. A planar projective transformation is a linear transformation on homogeneous 3-vectors represented by a non-singular 3×3 matrix:

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{2.5}$$

or more briefly, $\mathbf{x}' = \mathtt{H}\mathbf{x}$.

Note that the matrix $\mathtt{H}$ occurring in this equation may be changed by multiplication by an arbitrary non-zero scale factor without altering the projective transformation. Consequently we say that $\mathtt{H}$ is a *homogeneous* matrix, since as in the homogeneous representation of a point, only the ratio of the matrix elements is significant. There are eight independent ratios amongst the nine elements of $\mathtt{H}$, and it follows that a projective transformation has eight degrees of freedom.

A projective transformation projects every figure into a projectively equivalent figure, leaving all its projective properties invariant. In the ray model of figure 2.1 a projective transformation is simply a linear transformation of $\mathbb{R}^3$.

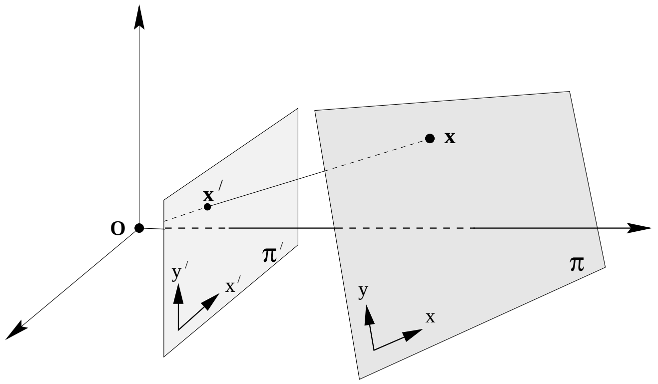


Fig. 2.3. **Central projection maps points on one plane to points on another plane.** *The projection also maps lines to lines as may be seen by considering a plane through the projection centre which intersects with the two planes* π *and* π'*. Since lines are mapped to lines, central projection is a projectivity and may be represented by a linear mapping of homogeneous coordinates* $\mathbf{x}' = \mathtt{H}\mathbf{x}$.

Mappings between planes. As an example of how theorem 2.10 may be applied, consider figure 2.3. Projection along rays through a common point (the centre of projection) defines a mapping from one plane to another. It is evident that this point-to-point mapping preserves lines in that a line in one plane is mapped to a line in the other. If a coordinate system is defined in each plane and points are represented in homogeneous coordinates, then the *central projection* mapping may be expressed by $\mathbf{x}' = \mathtt{H}\mathbf{x}$ where $\mathtt{H}$ is a non-singular 3×3 matrix. Actually, if the two coordinate systems defined in the two planes are both Euclidean (rectilinear) coordinate systems then the mapping defined by central projection is more restricted than an arbitrary projective transformation. It is called a *perspectivity* rather than a full projectivity, and may be represented by a transformation with six degrees of freedom. We return to perspectivities in section A7.4(*p*632).

Example 2.12. Removing the projective distortion from a perspective image of a plane.

Shape is distorted under perspective imaging. For instance, in figure 2.4a the windows are not rectangular in the image, although the originals are. In general parallel lines on a scene plane are not parallel in the image but instead converge to a finite point. We have seen that a central projection image of a plane (or section of a plane) is related to the original plane via a projective transformation, and so the image is a projective distortion of the original. It is possible to "undo" this projective transformation by computing the inverse transformation and applying it to the image. The result will be a new synthesized image in which the objects in the plane are shown with their correct geometric shape. This will be illustrated here for the front of the building of figure 2.4a. Note that since the ground and the front are not in the same plane, the projective transformation that must be applied to rectify the front is not the same as the one used for the ground.

Computation of a projective transformation from point-to-point correspondences will be considered in great detail in chapter 4. For now, a method for computing the trans-

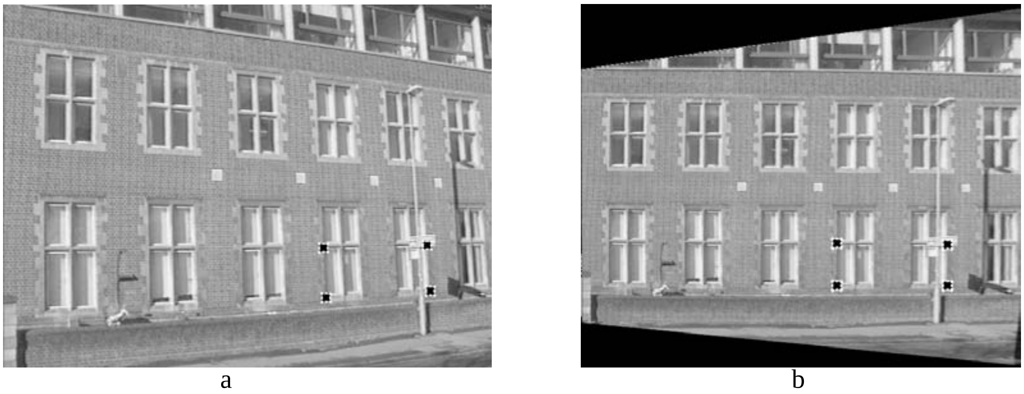


Fig. 2.4. **Removing perspective distortion.** *(a) The original image with perspective distortion – the lines of the windows clearly converge at a finite point. (b) Synthesized frontal orthogonal view of the front wall. The image (a) of the wall is related via a projective transformation to the true geometry of the wall. The inverse transformation is computed by mapping the four imaged window corners to corners of an appropriately sized rectangle. The four point correspondences determine the transformation. The transformation is then applied to the whole image. Note that sections of the image of the ground are subject to a further projective distortion. This can also be removed by a projective transformation.*

formation is briefly indicated. One begins by selecting a section of the image corresponding to a planar section of the world. Local 2D image and world coordinates are selected as shown in figure 2.3. Let the inhomogeneous coordinates of a pair of matching points $\mathbf{x}$ and $\mathbf{x}'$ in the world and image plane be (x, y) and (x', y') respectively. We use inhomogeneous coordinates here instead of the homogeneous coordinates of the points, because it is these inhomogeneous coordinates that are measured directly from the image and from the world plane. The projective transformation of (2.5) can be written in inhomogeneous form as

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}, \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}.$$

Each point correspondence generates two equations for the elements of $\mathtt{H}$, which after multiplying out are

$$\begin{aligned} x'(h_{31}x + h_{32}y + h_{33}) &= h_{11}x + h_{12}y + h_{13} \\ y'(h_{31}x + h_{32}y + h_{33}) &= h_{21}x + h_{22}y + h_{23}. \end{aligned}$$

These equations are *linear* in the elements of $\mathtt{H}$. Four point correspondences lead to eight such linear equations in the entries of $\mathtt{H}$, which are sufficient to solve for $\mathtt{H}$ up to an insignificant multiplicative factor. The only restriction is that the four points must be in "general position", which means that no three points are collinear. The inverse of the transformation $\mathtt{H}$ computed in this way is then applied to the whole image to undo the effect of perspective distortion on the selected plane. The results are shown in figure 2.4b. △

Three remarks concerning this example are appropriate: first, the computation of the rectifying transformation $\mathtt{H}$ in this way does not require knowledge of *any* of the camera's parameters or the pose of the plane; second, it is not always necessary to

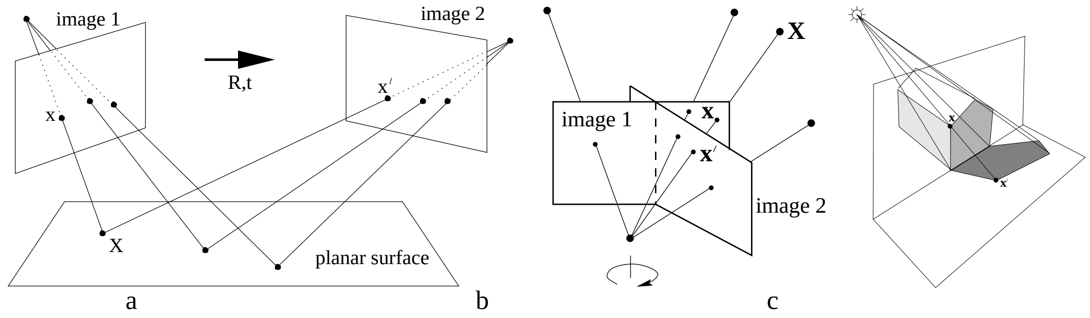


Fig. 2.5. **Examples of a projective transformation, $\mathbf{x}' = \mathtt{H}\mathbf{x}$, arising in perspective images.** *(a) The projective transformation between two images induced by a world plane (the concatenation of two projective transformations is a projective transformation); (b) The projective transformation between two images with the same camera centre (e.g. a camera rotating about its centre or a camera varying its focal length); (c) The projective transformation between the image of a plane (the end of the building) and the image of its shadow onto another plane (the ground plane). Figure (c) courtesy of Luc Van Gool.*

know coordinates for four points in order to remove projective distortion: alternative approaches, which are described in section 2.7, require less, and different types of, information; third, superior (and preferred) methods for computing projective transformations are described in chapter 4.

Projective transformations are important mappings representing many more situations than the perspective imaging of a world plane. A number of other examples are illustrated in figure 2.5. Each of these situations is covered in more detail later in the book.

2.3.1 Transformations of lines and conics

Transformation of lines. It was shown in the proof of theorem 2.10 that if points $\mathbf{x}_i$ lie on a line $\mathbf{l}$, then the transformed points $\mathbf{x}'_i = \mathtt{H}\mathbf{x}_i$ under a projective transformation lie on the line $\mathbf{l}' = \mathtt{H}^{-\mathsf{T}}\mathbf{l}$. In this way, incidence of points on lines is preserved, since $\mathbf{l}'^{\mathsf{T}}\mathbf{x}'_i = \mathbf{l}^{\mathsf{T}}\mathtt{H}^{-1}\mathtt{H}\mathbf{x}_i = 0$. This gives the transformation rule for lines:
Under the point transformation $\mathbf{x}' = \mathtt{H}\mathbf{x}$, a line transforms as

$$\mathbf{l}' = \mathtt{H}^{-\mathsf{T}}\mathbf{l}. \tag{2.6}$$

One may alternatively write $\mathbf{l}'^{\mathsf{T}} = \mathbf{l}^{\mathsf{T}}\mathtt{H}^{-1}$. Note the fundamentally different way in which lines and points transform. Points transform according to $\mathtt{H}$, whereas lines (as rows) transform according to $\mathtt{H}^{-1}$. This may be explained in terms of "covariant" or "contravariant" behaviour. One says that points transform *contravariantly* and lines transform *covariantly*. This distinction will be taken up again, when we discuss tensors in chapter 15 and is fully explained in appendix 1(*p*562).

Transformation of conics. Under a point transformation $\mathbf{x}' = \mathtt{H}\mathbf{x}$, (2.2) becomes

$$\begin{aligned} \mathbf{x}^{\mathsf{T}}\mathtt{C}\mathbf{x} &= \mathbf{x}'^{\mathsf{T}}[\mathtt{H}^{-1}]^{\mathsf{T}}\mathtt{C}\mathtt{H}^{-1}\mathbf{x}' \\ &= \mathbf{x}'^{\mathsf{T}}\mathtt{H}^{-\mathsf{T}}\mathtt{C}\mathtt{H}^{-1}\mathbf{x}' \end{aligned}$$

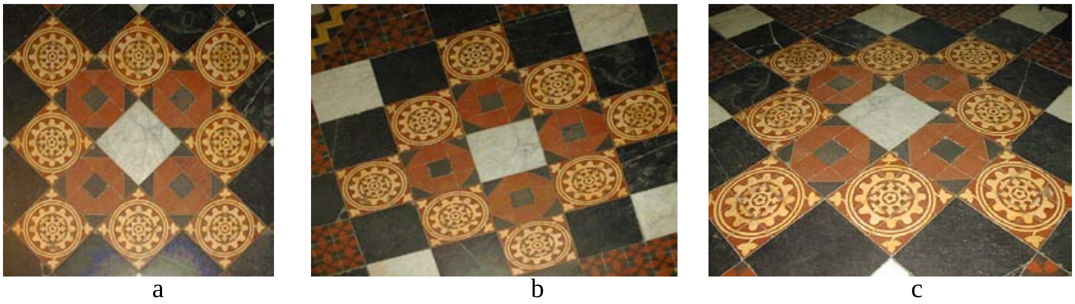


Fig. 2.6. **Distortions arising under central projection.** *Images of a tiled floor. (a)* **Similarity***: the circular pattern is imaged as a circle. A square tile is imaged as a square. Lines which are parallel or perpendicular have the same relative orientation in the image. (b)* **Affine***: The circle is imaged as an ellipse. Orthogonal world lines are not imaged as orthogonal lines. However, the sides of the square tiles, which are parallel in the world are parallel in the image. (c)* **Projective***: Parallel world lines are imaged as converging lines. Tiles closer to the camera have a larger image than those further away.*

which is a quadratic form $\mathbf{x}'^{\mathsf{T}}\mathtt{C}'\mathbf{x}'$ with $\mathtt{C}' = \mathtt{H}^{-\mathsf{T}}\mathtt{C}\mathtt{H}^{-1}$. This gives the transformation rule for a conic:

Result 2.13. *Under a point transformation* $\mathbf{x}' = \mathtt{H}\mathbf{x}$, *a conic* $\mathtt{C}$ *transforms to* $\mathtt{C}' = \mathtt{H}^{-\mathsf{T}}\mathtt{C}\mathtt{H}^{-1}$.

The presence of $\mathtt{H}^{-1}$ in this equation may be expressed by saying that a conic transforms *covariantly*. The transformation rule for a dual conic is derived in a similar manner. This gives:

Result 2.14. *Under a point transformation* $\mathbf{x}' = \mathtt{H}\mathbf{x}$, *a dual conic* $\mathtt{C}^*$ *transforms to* $\mathtt{C}^{*\prime} = \mathtt{H}\mathtt{C}^*\mathtt{H}^{\mathsf{T}}$.

2.4 A hierarchy of transformations

In this section we describe the important specializations of a projective transformation and their geometric properties. It was shown in section 2.3 that projective transformations form a group. This group is called the *projective linear group*, and it will be seen that these specializations are *subgroups* of this group.

The group of invertible $n \times n$ matrices with real elements is the (real) general linear group on n dimensions, or $GL(n)$. To obtain the projective linear group the matrices related by a scalar multiplier are identified, giving $PL(n)$ (this is a quotient group of $GL(n)$). In the case of projective transformations of the plane $n = 3$.

The important subgroups of $PL(3)$ include the *affine group*, which is the subgroup of $PL(3)$ consisting of matrices for which the last row is $(0, 0, 1)$, and the *Euclidean group*, which is a subgroup of the affine group for which in addition the upper left hand 2×2 matrix is orthogonal. One may also identify the *oriented Euclidean group* in which the upper left hand 2×2 matrix has determinant 1.

We will introduce these transformations starting from the most specialized, the isometries, and progressively generalizing until projective transformations are reached.

This defines a *hierarchy* of transformations. The distortion effects of various transformations in this hierarchy are shown in figure 2.6.

Some transformations of interest are not groups, for example, perspectivities (because the composition of two perspectivities is a projectivity, not a perspectivity). This point is covered in section A7.4(*p*632).

Invariants. An alternative to describing the transformation *algebraically*, i.e. as a matrix acting on coordinates of a point or curve, is to describe the transformation in terms of those elements or quantities that are preserved or *invariant*. A (scalar) invariant of a geometric configuration is a function of the configuration whose value is unchanged by a particular transformation. For example, the separation of two points is unchanged by a Euclidean transformation (translation and rotation), but not by a similarity (e.g. translation, rotation and isotropic scaling). Distance is thus a Euclidean, but not similarity invariant. The angle between two lines is both a Euclidean and a similarity invariant.

2.4.1 Class I: Isometries

Isometries are transformations of the plane $\mathbb{R}^2$ that preserve Euclidean distance (from *iso* = same, *metric* = measure). An isometry is represented as

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \epsilon\cos\theta & -\sin\theta & t_x \\ \epsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

where $\epsilon = \pm 1$. If $\epsilon = 1$ then the isometry is *orientation-preserving* and is a *Euclidean* transformation (a composition of a translation and rotation). If $\epsilon = -1$ then the isometry reverses orientation. An example is the composition of a reflection, represented by the matrix $\text{diag}(-1, 1, 1)$, with a Euclidean transformation.

Euclidean transformations model the motion of a rigid object. They are by far the most important isometries in practice, and we will concentrate on these. However, the orientation reversing isometries often arise as ambiguities in structure recovery.

A planar Euclidean transformation can be written more concisely in block form as

$$\mathbf{x}' = \mathtt{H}_{\mathrm{E}}\mathbf{x} = \begin{bmatrix} \mathtt{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \mathbf{x} \tag{2.7}$$

where $\mathtt{R}$ is a 2×2 rotation matrix (an orthogonal matrix such that $\mathtt{R}^{\mathsf{T}}\mathtt{R} = \mathtt{R}\mathtt{R}^{\mathsf{T}} = \mathtt{I}$), $\mathbf{t}$ a translation 2-vector, and $\mathbf{0}$ a null 2-vector. Special cases are a pure rotation (when $\mathbf{t} = \mathbf{0}$) and a pure translation (when $\mathtt{R} = \mathtt{I}$). A Euclidean transformation is also known as a *displacement*.

A planar Euclidean transformation has three degrees of freedom, one for the rotation and two for the translation. Thus three parameters must be specified in order to define the transformation. The transformation can be computed from two point correspondences.

Invariants. The invariants are very familiar, for instance: length (the distance between two points), angle (the angle between two lines), and area.

Groups and orientation. An isometry is orientation-preserving if the upper left hand 2×2 matrix has determinant 1. Orientation-*preserving* isometries form a group, orientation-*reversing* ones do not. This distinction applies also in the case of similarity and affine transformations which now follow.

2.4.2 Class II: Similarity transformations

A similarity transformation (or more simply a *similarity*) is an isometry composed with an isotropic scaling. In the case of a Euclidean transformation composed with a scaling (i.e. no reflection) the similarity has matrix representation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}. \tag{2.8}$$

This can be written more concisely in block form as

$$\mathbf{x}' = \mathtt{H}_{\mathrm{S}}\mathbf{x} = \begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \mathbf{x} \tag{2.9}$$

where the scalar s represents the isotropic scaling. A similarity transformation is also known as an *equi-form* transformation, because it preserves “shape” (form). A planar similarity transformation has four degrees of freedom, the scaling accounting for one more degree of freedom than a Euclidean transformation. A similarity can be computed from two point correspondences.

Invariants. The invariants can be constructed from Euclidean invariants with suitable provision being made for the additional scaling degree of freedom. Angles between lines are not affected by rotation, translation or isotropic scaling, and so are similarity invariants. In particular parallel lines are mapped to parallel lines. The length between two points is not a similarity invariant, but the *ratio* of two lengths is an invariant, because the scaling of the lengths cancels out. Similarly a ratio of areas is an invariant because the scaling (squared) cancels out.

Metric structure. A term that will be used frequently in the discussion on reconstruction (chapter 10) is *metric*. The description *metric structure* implies that the structure is defined up to a similarity.

2.4.3 Class III: Affine transformations

An affine transformation (or more simply an *affinity*) is a non-singular linear transformation followed by a translation. It has the matrix representation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{2.10}$$

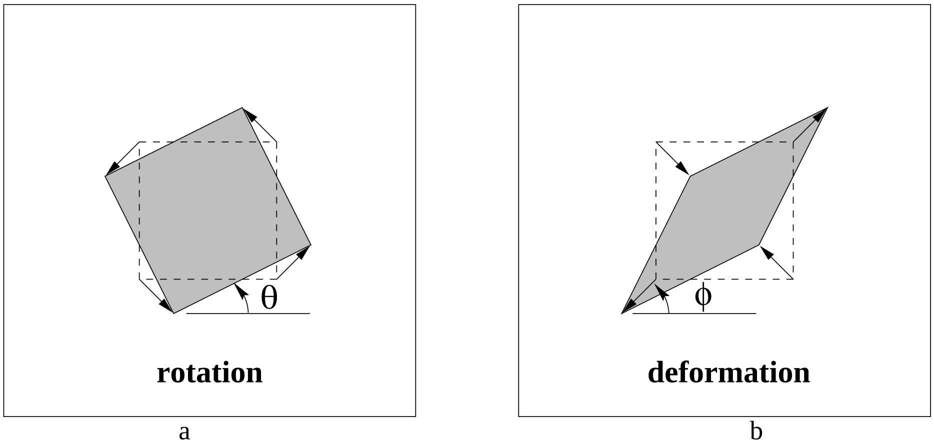


Fig. 2.7. **Distortions arising from a planar affine transformation.** *(a) Rotation by* $\mathtt{R}(\theta)$. *(b) A deformation* $\mathtt{R}(-\phi)\,\mathtt{D}\,\mathtt{R}(\phi)$. *Note, the scaling directions in the deformation are orthogonal.*

or in block form

$$\mathbf{x}' = \mathtt{H}_{\mathtt{A}}\mathbf{x} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \mathbf{x} \tag{2.11}$$

with $\mathtt{A}$ a 2×2 non-singular matrix. A planar affine transformation has six degrees of freedom corresponding to the six matrix elements. The transformation can be computed from three point correspondences.

A helpful way to understand the geometric effects of the linear component $\mathtt{A}$ of an affine transformation is as the composition of two fundamental transformations, namely rotations and non-isotropic scalings. The affine matrix $\mathtt{A}$ can always be decomposed as

$$\mathtt{A} \quad = \quad \mathtt{R}(\theta)\,\mathtt{R}(-\phi)\,\mathtt{D}\,\mathtt{R}(\phi) \tag{2.12}$$

where $\mathtt{R}(\theta)$ and $\mathtt{R}(\phi)$ are rotations by θ and ϕ respectively, and $\mathtt{D}$ is a diagonal matrix:

$$\mathtt{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

This decomposition follows directly from the SVD (section A4.4(*p*585)): writing $\mathtt{A} = \mathtt{UDV}^{\mathsf{T}} = (\mathtt{UV}^{\mathsf{T}})(\mathtt{VDV}^{\mathsf{T}}) = \mathtt{R}(\theta)\,(\mathtt{R}(-\phi)\,\mathtt{D}\,\mathtt{R}(\phi))$, since $\mathtt{U}$ and $\mathtt{V}$ are orthogonal matrices.

The affine matrix $\mathtt{A}$ is hence seen to be the concatenation of a rotation (by ϕ); a scaling by λ_1 and λ_2 respectively in the (rotated) x and y directions; a rotation back (by $-\phi$); and finally another rotation (by θ). The only "new" geometry, compared to a similarity, is the non-isotropic scaling. This accounts for the two extra degrees of freedom possessed by an affinity over a similarity. They are the angle ϕ specifying the scaling direction, and the ratio of the scaling parameters $\lambda_1 : \lambda_2$. The essence of an affinity is this scaling in orthogonal directions, oriented at a particular angle. Schematic examples are given in figure 2.7.

Invariants. Because an affine transformation includes non-isotropic scaling, the similarity invariants of length ratios and angles between lines are not preserved under an affinity. Three important invariants are:

(i) **Parallel lines.** Consider two parallel lines. These intersect at a point $(x_1, x_2, 0)^\mathsf{T}$ at infinity. Under an affine transformation this point is mapped to another point at infinity. Consequently, the parallel lines are mapped to lines which still intersect at infinity, and so are parallel after the transformation.

(ii) **Ratio of lengths of parallel line segments.** The length scaling of a line segment depends only on the angle between the line direction and scaling directions. Suppose the line is at angle α to the x-axis of the orthogonal scaling direction, then the scaling magnitude is $\sqrt{\lambda_1^2 \cos^2 \alpha + \lambda_2^2 \sin^2 \alpha}$. This scaling is common to all lines with the same direction, and so cancels out in a ratio of parallel segment lengths.

(iii) **Ratio of areas.** This invariance can be deduced directly from the decomposition (2.12). Rotations and translations do not affect area, so only the scalings by λ_1 and λ_2 matter here. The effect is that area is scaled by $\lambda_1\lambda_2$ which is equal to $\det \mathtt{A}$. Thus the area of any shape is scaled by $\det \mathtt{A}$, and so the scaling cancels out for a ratio of areas. It will be seen that this does not hold for a projective transformation.

An affinity is orientation-preserving or -reversing according to whether $\det \mathtt{A}$ is positive or negative respectively. Since $\det \mathtt{A} = \lambda_1\lambda_2$ the property depends only on the sign of the scalings.

2.4.4 Class IV: Projective transformations

A projective transformation was defined in (2.5). It is a general non-singular linear transformation of *homogeneous* coordinates. This generalizes an affine transformation, which is the composition of a general non-singular linear transformation of *inhomogeneous* coordinates and a translation. We have earlier seen the action of a projective transformation (in section 2.3). Here we examine its block form

$$\mathbf{x}' = \mathtt{H}_\mathrm{P}\mathbf{x} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix} \mathbf{x} \tag{2.13}$$

where the vector $\mathbf{v} = (v_1, v_2)^\mathsf{T}$. The matrix has nine elements with only their ratio significant, so the transformation is specified by eight parameters. Note, it is not always possible to scale the matrix such that v is unity since v might be zero. A projective transformation between two planes can be computed from four point correspondences, with no three collinear on either plane. See figure 2.4.

Unlike the case of affinities, it is not possible to distinguish between orientation preserving and orientation reversing projectivities in $\mathbb{P}^2$. We will return to this point in section 2.6.

Invariants. The most fundamental projective invariant is the cross ratio of four collinear points: a ratio of lengths on a line is invariant under affinities, but not under projectivities. However, a ratio of ratios or *cross ratio* of lengths on a line is a projective invariant. We return to properties of this invariant in section 2.5.

2.4.5 Summary and comparison

Affinities (6 dof) occupy the middle ground between similarities (4 dof) and projectivities (8 dof). They generalize similarities in that angles are not preserved, so that shapes are skewed under the transformation. On the other hand their action is homogeneous over the plane: for a given affinity the $\det \mathtt{A}$ scaling in area of an object (e.g. a square) is the same anywhere on the plane; and the orientation of a transformed line depends only on its initial orientation, not on its position on the plane. In contrast, for a given projective transformation, area scaling varies with position (e.g. under perspective a more distant square on the plane has a smaller image than one that is nearer, as in figure 2.6); and the orientation of a transformed line depends on both the orientation and position of the source line (however, it will be seen later in section 8.6(*p*213) that a line's vanishing point depends only on line orientation, not position).

The key difference between a projective and affine transformation is that the vector $\mathbf{v}$ is not null for a projectivity. This is responsible for the non-linear effects of the projectivity. Compare the mapping of an ideal point $(x_1, x_2, 0)^\mathsf{T}$ under an affinity and projectivity: First the affine transformation

$$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathtt{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}. \tag{2.14}$$

Second the projective transformation

$$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathtt{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}. \tag{2.15}$$

In the first case the ideal point remains ideal (i.e. at infinity). In the second it is mapped to a finite point. It is this ability which allows a projective transformation to model vanishing points.

2.4.6 Decomposition of a projective transformation

A projective transformation can be decomposed into a chain of transformations, where each matrix in the chain represents a transformation higher in the hierarchy than the previous one.

$$\mathtt{H} = \mathtt{H}_\mathrm{S}\, \mathtt{H}_\mathrm{A}\, \mathtt{H}_\mathrm{P} = \begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathtt{K} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathtt{I} & \mathbf{0} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix} \tag{2.16}$$

with $\mathtt{A}$ a non-singular matrix given by $\mathtt{A} = s\mathtt{R}\mathtt{K} + \mathbf{t}\mathbf{v}^\mathsf{T}$, and $\mathtt{K}$ an upper-triangular matrix normalized as $\det \mathtt{K} = 1$. This decomposition is valid provided $v \neq 0$, and is unique if s is chosen positive.

Each of the matrices $\mathtt{H}_S, \mathtt{H}_A, \mathtt{H}_P$ is the "essence" of a transformation of that type (as indicated by the subscripts S, A, P). Consider the process of rectifying the perspective image of a plane as in example 2.12: $\mathtt{H}_P$ (2 dof) moves the line at infinity; $\mathtt{H}_A$ (2 dof) affects the affine properties, but does not move the line at infinity; and finally, $\mathtt{H}_S$ is a general similarity transformation (4 dof) which does not affect the affine or projective properties. The transformation $\mathtt{H}_P$ is an *elation,* described in section A7.3(*p*631).

Example 2.15. The projective transformation

$$\mathtt{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

may be decomposed as

$$\mathtt{H} = \begin{bmatrix} 2\cos 45^\circ & -2\sin 45^\circ & 1 \\ 2\sin 45^\circ & 2\cos 45^\circ & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}.$$

△

This decomposition can be employed when the objective is to only partially determine the transformation. For example, if one wants to measure length ratios from the perspective image of a plane, then it is only necessary to determine (rectify) the transformation up to a similarity. We return to this approach in section 2.7.

Taking the inverse of $\mathtt{H}$ in (2.16) gives $\mathtt{H}^{-1} = \mathtt{H}_P^{-1}\,\mathtt{H}_A^{-1}\,\mathtt{H}_S^{-1}$. Since $\mathtt{H}_P^{-1}, \mathtt{H}_A^{-1}$ and $\mathtt{H}_S^{-1}$ are still projective, affine and similarity transformations respectively, a general projective transformation may also be decomposed in the form

$$\mathtt{H} = \mathtt{H}_P\,\mathtt{H}_A\,\mathtt{H}_S = \begin{bmatrix} \mathtt{I} & \mathbf{0} \\ \mathbf{v}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathtt{K} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \tag{2.17}$$

Note that the actual values of $\mathtt{K}$, $\mathtt{R}$, $\mathbf{t}$ and $\mathbf{v}$ will be different from those of (2.16).

2.4.7 The number of invariants

The question naturally arises as to how many invariants there are for a given geometric configuration under a particular transformation. First the term "number" needs to be made more precise, for if a quantity is invariant, such as length under Euclidean transformations, then any function of that quantity is invariant. Consequently, we seek a counting argument for the number of functionally independent invariants. By considering the number of transformation parameters that must be eliminated in order to form an invariant, it can be seen that:

Result 2.16. *The number of functionally independent invariants is equal to, or greater than, the number of degrees of freedom of the configuration less the number of degrees of freedom of the transformation.*

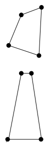
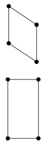
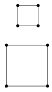
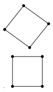
Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

Table 2.1. **Geometric properties invariant to commonly occurring *planar* transformations.** *The matrix* $\mathtt{A} = [a_{ij}]$ *is an invertible* 2×2 *matrix,* $\mathtt{R} = [r_{ij}]$ *is a 2D rotation matrix, and* (t_x, t_y) *a 2D translation. The distortion column shows typical effects of the transformations on a square. Transformations higher in the table can produce all the actions of the ones below. These range from Euclidean, where only translations and rotations occur, to projective where the square can be transformed to any arbitrary quadrilateral (provided no three points are collinear).*

For example, a configuration of four points in general position has 8 degrees of freedom (2 for each point), and so 4 similarity, 2 affinity and zero projective invariants since these transformations have respectively 4, 6 and 8 degrees of freedom.

Table 2.1 summarizes the 2D transformation groups and their invariant properties. Transformations lower in the table are specializations of those above. A transformation lower in the table inherits the invariants of those above.

2.5 The projective geometry of 1D

The development of the projective geometry of a line, $\mathbb{P}^1$, proceeds in much the same way as that of the plane. A point x on the line is represented by homogeneous coordinates $(x_1, x_2)^\mathsf{T}$, and a point for which $x_2 = 0$ is an ideal point of the line. We will use the notation $\bar{\mathbf{x}}$ to represent the 2-vector $(x_1, x_2)^\mathsf{T}$. A projective transformation of a line is represented by a 2×2 homogeneous matrix,

$$\bar{\mathbf{x}}' = \mathtt{H}_{2\times 2}\bar{\mathbf{x}}$$

and has 3 degrees of freedom corresponding to the four elements of the matrix less one for overall scaling. A projective transformation of a line may be determined from three corresponding points.

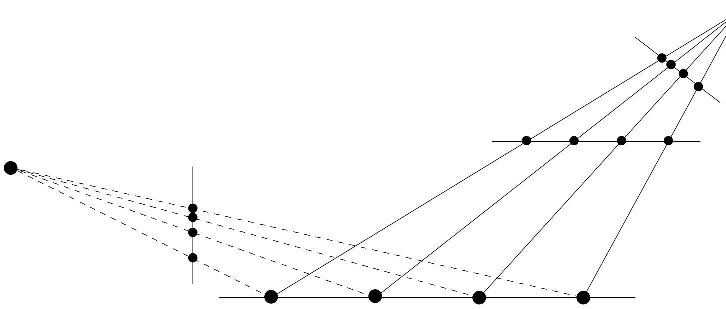


Fig. 2.8. **Projective transformations between lines.** *There are four sets of four collinear points in this figure. Each set is related to the others by a line-to-line projectivity. Since the cross ratio is an invariant under a projectivity, the cross ratio has the same value for all the sets shown.*

The cross ratio. The cross ratio is the basic projective invariant of $\mathbb{P}^1$. Given 4 points $\bar{\mathbf{x}}_i$ the *cross ratio* is defined as

$$\text{Cross}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4) = \frac{|\bar{\mathbf{x}}_1\bar{\mathbf{x}}_2||\bar{\mathbf{x}}_3\bar{\mathbf{x}}_4|}{|\bar{\mathbf{x}}_1\bar{\mathbf{x}}_3||\bar{\mathbf{x}}_2\bar{\mathbf{x}}_4|}$$

where

$$|\bar{\mathbf{x}}_i\bar{\mathbf{x}}_j| = \det \begin{bmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{bmatrix} .$$

A few comments on the cross ratio:

(i) The value of the cross ratio is not dependent on which particular homogeneous representative of a point $\bar{\mathbf{x}}_i$ is used, since the scale cancels between numerator and denominator.

(ii) If each point $\bar{\mathbf{x}}_i$ is a finite point and the homogeneous representative is chosen such that $x_2 = 1$, then $|\bar{\mathbf{x}}_i\bar{\mathbf{x}}_j|$ represents the signed distance from $\bar{\mathbf{x}}_i$ to $\bar{\mathbf{x}}_j$.

(iii) The definition of the cross ratio is also valid if one of the points $\bar{\mathbf{x}}_i$ is an ideal point.

(iv) The value of the cross ratio is invariant under any projective transformation of the line: if $\bar{\mathbf{x}}' = \mathtt{H}_{2\times 2}\bar{\mathbf{x}}$ then

$$\text{Cross}(\bar{\mathbf{x}}'_1, \bar{\mathbf{x}}'_2, \bar{\mathbf{x}}'_3, \bar{\mathbf{x}}'_4) = \text{Cross}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4). \tag{2.18}$$

The proof is left as an exercise. Equivalently stated, the cross ratio is invariant to the projective coordinate frame chosen for the line.

Figure 2.8 illustrates a number of projective transformations between lines with equivalent cross ratios.

Under a projective transformation of the plane, a 1D projective transformation is induced on any line in the plane.

Concurrent lines. A configuration of concurrent lines is dual to collinear points on a line. This means that concurrent lines on a plane also have the geometry $\mathbb{P}^1$. In particular four concurrent lines have a cross ratio as illustrated in figure 2.9a.

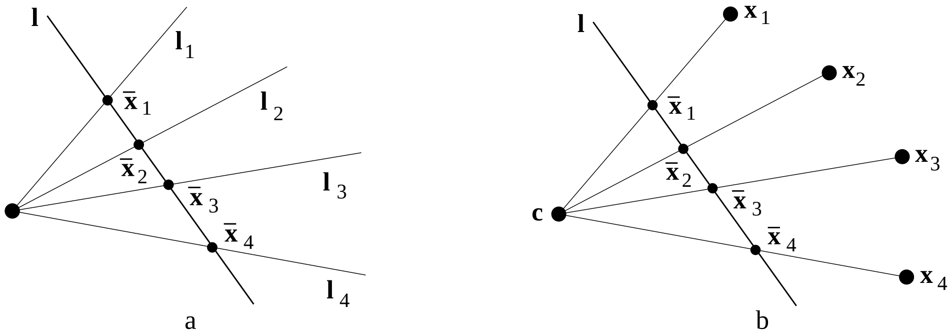


Fig. 2.9. **Concurrent lines.** *(a) Four concurrent lines* $\mathbf{l}_i$ *intersect the line* $\mathbf{l}$ *in the four points* $\bar{\mathbf{x}}_i$*. The cross ratio of these lines is an invariant to projective transformations of the plane. Its value is given by the cross ratio of the points, Cross*$(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4)$*. (b) Coplanar points* $\mathbf{x}_i$ *are imaged onto a line* $\mathbf{l}$ *(also in the plane) by a projection with centre* $\mathbf{c}$*. The cross ratio of the image points* $\bar{\mathbf{x}}_i$ *is invariant to the position of the image line* $\mathbf{l}$*.*

Note how figure 2.9b may be thought of as representing projection of points in $\mathbb{P}^2$ into a 1-dimensional image. In particular, if $\mathbf{c}$ represents a camera centre, and the line $\mathbf{l}$ represents an image line (1D analogue of the image plane), then the points $\bar{\mathbf{x}}_i$ are the projections of points $\mathbf{x}_i$ into the image. The cross ratio of the points $\bar{\mathbf{x}}_i$ characterizes the projective configuration of the four image points. Note that the actual position of the image line is irrelevant as far as the projective configuration of the four image points is concerned – different choices of image line give rise to projectively equivalent configurations of image points.

The projective geometry of concurrent lines is important to the understanding of the projective geometry of epipolar lines in chapter 9.

2.6 Topology of the projective plane

We make brief mention of the topology of $\mathbb{P}^2$*. Understanding of this section is not required for following the rest of the book.*

We have seen that the projective plane $\mathbb{P}^2$ may be thought of as the set of all homogeneous 3-vectors. A vector of this type $\mathbf{x} = (x_1, x_2, x_3)^\mathsf{T}$ may be normalized by multiplication by a non-zero factor so that $x_1^2 + x_2^2 + x_3^2 = 1$. Such a point lies on the unit sphere in $\mathbb{R}^3$. However, any vector $\mathbf{x}$ and $-\mathbf{x}$ represent the same point in $\mathbb{P}^2$, since they differ by a multiplicative factor, -1. Thus, there is a two-to-one correspondence between the unit sphere S^2 in $\mathbb{R}^3$ and the projective plane $\mathbb{P}^2$. The projective plane may be pictured as the unit sphere with opposite points identified. In this representation, a line in $\mathbb{P}^2$ is modelled as a great circle on the unit sphere (as ever, with opposite points identified). One may verify that any two distinct (non-antipodal) points on the sphere lie on exactly one great circle, and any two great circles intersect in one point (since antipodal points are identified).

In the language of topology, the sphere S^2 is a 2-sheeted covering space of $\mathbb{P}^2$. This implies that $\mathbb{P}^2$ is not *simply-connected*, which means that there are loops in $\mathbb{P}^2$ which cannot be contracted to a point inside $\mathbb{P}^2$. To be technical, the fundamental group of $\mathbb{P}^2$ is the cyclic group of order 2.

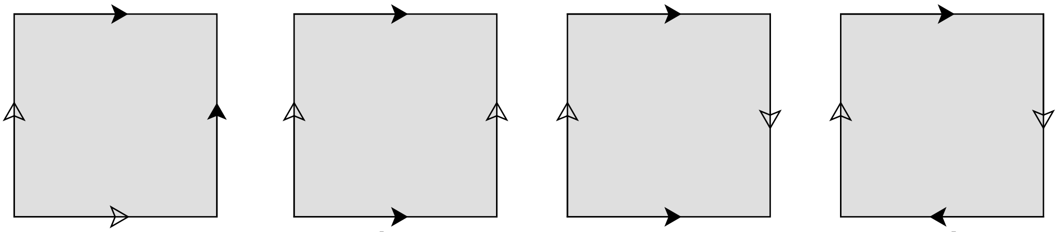


Fig. 2.10. **Topology of surfaces.** *Common surfaces may be constructed from a paper square (topologically a disk) with edges glued together. In each case, the matching arrow edges of the square are to be glued together in such a way that the directions of the arrows match. One obtains (a) a sphere, (b) a torus, (c) a Klein bottle and (d) a projective plane. Only the sphere and torus are actually realizable with a real sheet of paper. The sphere and torus are orientable but the projective plane and Klein bottle are not.*

In the model for the projective plane as a sphere with opposite points identified one may dispense with the lower hemisphere of S^2, since points in this hemisphere are the same as the opposite points in the upper hemisphere. In this case, $\mathbb{P}^2$ may be constructed from the upper hemisphere by identifying opposite points on the equator. Since the upper hemisphere of S^2 is topologically the same as a disk, $\mathbb{P}^2$ is simply a disk with opposite points on its boundary identified, or glued together. This is not physically possible. Constructing topological spaces by gluing the boundary of a disk is a common method in topology, and in fact any 2-manifold may be constructed in this way. This is illustrated in figure 2.10.

A notable feature of the projective plane $\mathbb{P}^2$ is that it is non-orientable. This means that it is impossible to define a local orientation (represented for instance by a pair of oriented coordinate axes) that is consistent over the whole surface. This is illustrated in figure 2.11 in which it is shown that the projective plane contains an orientation-reversing path.

The topology of $\mathbb{P}^1$. In a similar manner, the 1-dimensional projective line may be identified as a 1-sphere S^1 (that is, a circle) with opposite points identified. If we omit the lower half of the circle, as being duplicated by the top half, then the top half of a circle is topologically equivalent to a line segment. Thus $\mathbb{P}^1$ is topologically equivalent to a line segment with the two endpoints identified – namely a circle, S^1.

2.7 Recovery of affine and metric properties from images

We return to the example of projective rectification of example 2.12(*p*34) where the aim was to remove the projective distortion in the perspective image of a plane to the extent that similarity properties (angles, ratios of lengths) could be measured on the original plane. In that example the projective distortion was completely removed by specifying the position of four reference points on the plane (a total of 8 degrees of freedom), and explicitly computing the transformation mapping the reference points to their images. In fact this overspecifies the geometry – a projective transformation has only 4 degrees of freedom more than a similarity, so it is only necessary to specify 4

a b

Fig. 2.11. **Orientation of surfaces.** *A coordinate frame (represented by an L in the diagram) may be transported along a path in the surface eventually coming back to the point where it started. (a) represents a projective plane. In the path shown, the coordinate frame (represented by a pair of axes) is reversed when it returns to the same point, since the identification at the boundary of the square swaps the direction of one of the axes. Such a path is called an orientation-reversing path, and a surface that contains such a path is called non-orientable. (b) shows the well known example of a Möbius strip obtained by joining two opposite edges of a rectangle (M.C. Escher's "Moebius Strip II [Red Ants]", 1963. ©2000 Cordon Art B.V. – Baarn-Holland. All rights reserved). As can be verified, a path once around the strip is orientation-reversing.*

degrees of freedom (not 8) in order to determine metric properties. In projective geometry these 4 degrees of freedom are given "physical substance" by being associated with geometric objects: the line at infinity $\mathbf{l}_\infty$ (2 dof), and the two *circular points* (2 dof) on $\mathbf{l}_\infty$. This association is often a more intuitive way of reasoning about the problem than the equivalent description in terms of specifying matrices in the decomposition chain (2.16).

In the following it is shown that the projective distortion may be removed once the image of $\mathbf{l}_\infty$ is specified, and the affine distortion removed once the image of the circular points is specified. Then the only remaining distortion is a similarity.

2.7.1 The line at infinity

Under a projective transformation ideal points may be mapped to finite points (2.15), and consequently $\mathbf{l}_\infty$ is mapped to a finite line. However, if the transformation is an affinity, then $\mathbf{l}_\infty$ is not mapped to a finite line, but remains at infinity. This is evident directly from the line transformation (2.6–*p*36):

$$\mathbf{l}'_\infty = \mathtt{H}_\mathtt{A}^{-\mathsf{T}}\mathbf{l}_\infty = \begin{bmatrix} \mathtt{A}^{-\mathsf{T}} & \mathbf{0} \\ -\mathbf{t}^\mathsf{T}\mathtt{A}^{-\mathsf{T}} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{l}_\infty.$$

The converse is also true, i.e. an affine transformation is the most general linear transformation that fixes $\mathbf{l}_\infty$, and may be seen as follows. We require that a point at infinity, say $\mathbf{x} = (1, 0, 0)^\mathsf{T}$, be mapped to a point at infinity. This requires that $h_{31} = 0$. Similarly, $h_{32} = 0$, so the transformation is an affinity. To summarize,

Result 2.17. *The line at infinity, $\mathbf{l}_\infty$, is a fixed line under the projective transformation* $\mathtt{H}$ *if and only if* $\mathtt{H}$ *is an affinity.*

However, $\mathbf{l}_\infty$ is not fixed pointwise under an affine transformation: (2.14) showed that under an affinity a point on $\mathbf{l}_\infty$ (an ideal point) is mapped to a point on $\mathbf{l}_\infty$, but

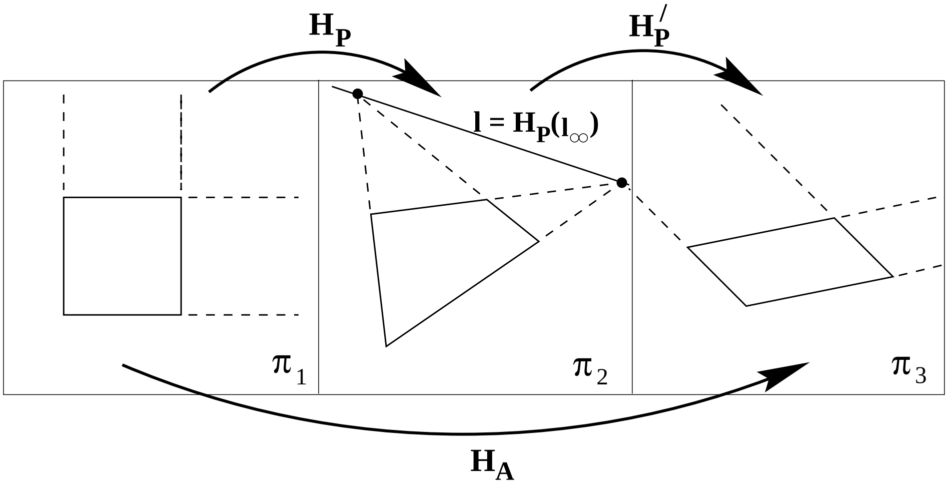


Fig. 2.12. **Affine rectification.** *A projective transformation maps* $\mathbf{l}_\infty$ *from* $(0,0,1)^\mathsf{T}$ *on a Euclidean plane* $\boldsymbol{\pi}_1$ *to a finite line* $\mathbf{l}$ *on the plane* $\boldsymbol{\pi}_2$. *If a projective transformation is constructed such that* $\mathbf{l}$ *is mapped back to* $(0,0,1)^\mathsf{T}$ *then from result 2.17 the transformation between the first and third planes* must be an affine transformation *since the canonical position of* $\mathbf{l}_\infty$ *is preserved. This means that affine properties of the first plane can be measured from the third, i.e. the third plane is within an affinity of the first.*

it is not the same point unless $\mathtt{A}(x_1, x_2)^\mathsf{T} = k(x_1, x_2)^\mathsf{T}$. It will now be shown that identifying $\mathbf{l}_\infty$ allows the recovery of affine properties (parallelism, ratio of areas).

2.7.2 Recovery of affine properties from images

Once the imaged line at infinity is identified in an image of a plane, it is then possible to make affine measurements on the original plane. For example, lines may be identified as parallel on the original plane if the imaged lines intersect on the imaged $\mathbf{l}_\infty$. This follows because parallel lines on the Euclidean plane intersect on $\mathbf{l}_\infty$, and after a projective transformation the lines still intersect on the imaged $\mathbf{l}_\infty$ since intersections are preserved by projectivities. Similarly, once $\mathbf{l}_\infty$ is identified a length ratio on a line may be computed from the cross ratio of the three points specifying the lengths together with the intersection of the line with $\mathbf{l}_\infty$ (which provides the fourth point for the cross ratio), and so forth.

However, a less tortuous path which is better suited to computational algorithms is simply to transform the identified $\mathbf{l}_\infty$ to its canonical position of $\mathbf{l}_\infty = (0,0,1)^\mathsf{T}$. The (projective) matrix which achieves this transformation can be applied to every point in the image in order to affinely rectify the image, i.e. after the transformation, affine measurements can be made directly from the rectified image. The key idea here is illustrated in figure 2.12.

If the imaged line at infinity is the line $\mathbf{l} = (l_1, l_2, l_3)^\mathsf{T}$, then provided $l_3 \neq 0$ a suitable projective point transformation which will map $\mathbf{l}$ back to $\mathbf{l}_\infty = (0,0,1)^\mathsf{T}$ is

$$\mathtt{H} = \mathtt{H}_\mathrm{A} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix} \tag{2.19}$$

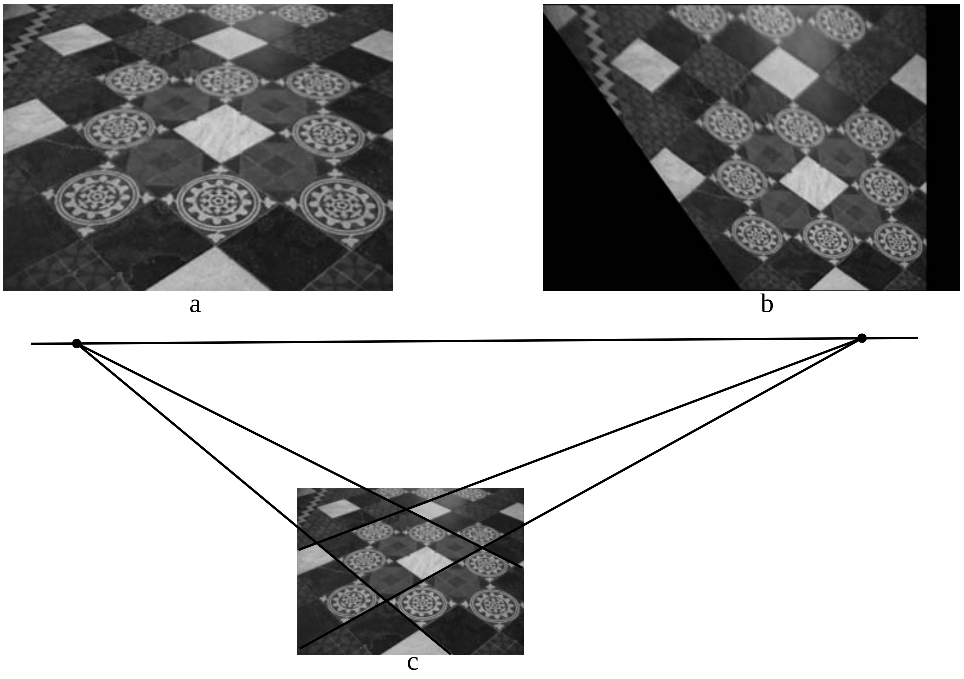


Fig. 2.13. **Affine rectification via the vanishing line.** *The vanishing line of the plane imaged in (a) is computed (c) from the intersection of two sets of imaged parallel lines. The image is then projectively warped to produce the affinely rectified image (b). In the affinely rectified image parallel lines are now parallel. However, angles do not have their veridical world value since they are affinely distorted. See also figure 2.17.*

where $\mathtt{H}_\mathtt{A}$ is any affine transformation (the last row of $\mathtt{H}$ is $\mathbf{l}^\mathsf{T}$). One can verify that under the line transformation (2.6–*p*36) $\mathtt{H}^{-\mathsf{T}}(l_1, l_2, l_3)^\mathsf{T} = (0, 0, 1)^\mathsf{T} = \mathbf{l}_\infty$.

Example 2.18. Affine rectification
In a perspective image of a plane, the line at infinity on the world plane is imaged as the vanishing line of the plane. This is discussed in more detail in chapter 8. As illustrated in figure 2.13 the vanishing line $\mathbf{l}$ may be computed by intersecting imaged parallel lines. The image is then rectified by applying a projective warping (2.19) such that $\mathbf{l}$ is mapped to its canonical position $\mathbf{l}_\infty = (0, 0, 1)^\mathsf{T}$. △

This example shows that affine properties may be recovered by simply specifying a line (2 dof). It is equivalent to specifying only the projective component of the transformation decomposition chain (2.16). Conversely if affine properties are known, these may be used to determine points and the line at infinity. This is illustrated in the following example.

Example 2.19. Computing a vanishing point from a length ratio. Given two intervals on a line with a known length ratio, the point at infinity on the line may be determined. A typical case is where three points $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ are identified on a line in an image. Suppose $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the corresponding collinear points on the world line, and the length ratio $d(\mathbf{a}, \mathbf{b}) : d(\mathbf{b}, \mathbf{c}) = a : b$ is known (where $d(\mathbf{x}, \mathbf{y})$ is the Euclidean

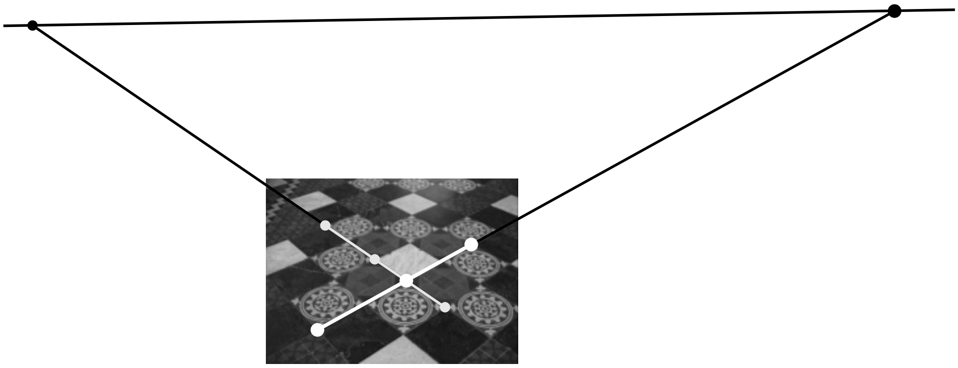


Fig. 2.14. *Two examples of using equal length ratios on a line to determine the point at infinity. The line intervals used are shown as the thin and thick white lines delineated by points. This construction determines the vanishing line of the plane. Compare with figure 2.13c.*

distance between the points $\mathbf{x}$ and $\mathbf{y}$). It is possible to find the vanishing point using the cross ratio. Equivalently, one may proceed as follows:

(i) Measure the distance ratio in the image, $d(\mathbf{a}', \mathbf{b}') : d(\mathbf{b}', \mathbf{c}') = a' : b'$.

(ii) Points $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ may be represented as coordinates 0, a and $a+b$ in a coordinate frame on the line $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle$. For computational purposes, these points are represented by homogeneous 2-vectors $(0, 1)^\mathsf{T}$, $(a, 1)^\mathsf{T}$ and $(a + b, 1)^\mathsf{T}$. Similarly, $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ have coordinates 0, a' and $a' + b'$, which may also be expressed as homogeneous vectors.

(iii) Relative to these coordinate frames, compute the 1D projective transformation $\mathtt{H}_{2\times 2}$ mapping $\mathbf{a} \mapsto \mathbf{a}'$, $\mathbf{b} \mapsto \mathbf{b}'$ and $\mathbf{c} \mapsto \mathbf{c}'$.

(iv) The image of the point at infinity (with coordinates $(1, 0)^\mathsf{T}$) under $\mathtt{H}_{2\times 2}$ is the vanishing point on the line $\langle \mathbf{a}', \mathbf{b}', \mathbf{c}' \rangle$.

An example of vanishing points computed in this manner is shown in figure 2.14. △

Example 2.20. Geometric construction of vanishing points from a length ratio. The vanishing points shown in figure 2.14 may also be computed by a purely geometric construction consisting of the following steps:

(i) Given: three collinear points, $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$, in an image corresponding to collinear world points with interval ratio $a : b$.

(ii) Draw any line $\mathbf{l}$ through $\mathbf{a}'$ (not coincident with the line $\mathbf{a}'\mathbf{c}'$), and mark off points $\mathbf{a} = \mathbf{a}'$, $\mathbf{b}$ and $\mathbf{c}$ such that the line segments $\langle \mathbf{ab} \rangle$, $\langle \mathbf{bc} \rangle$ have length ratio $a : b$.

(iii) Join $\mathbf{bb}'$ and $\mathbf{cc}'$ and intersect in $\mathbf{o}$.

(iv) The line through $\mathbf{o}$ parallel to $\mathbf{l}$ meets the line $\mathbf{a}'\mathbf{c}'$ in the vanishing point $\mathbf{v}'$.

This construction is illustrated in figure 2.15. △

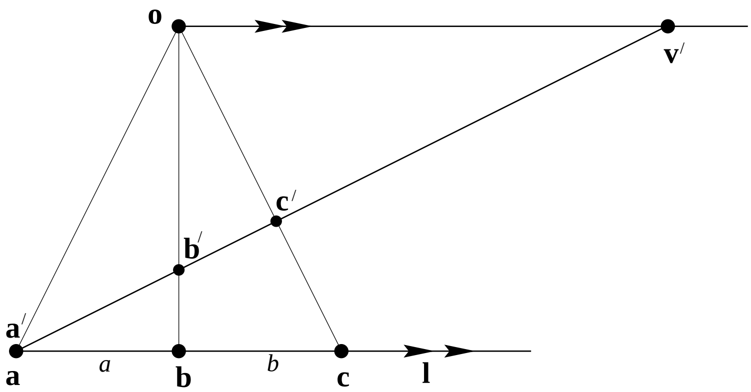


Fig. 2.15. *A geometric construction to determine the image of the point at infinity on a line given a known length ratio. The details are given in the text.*

2.7.3 The circular points and their dual

Under any similarity transformation there are two points on $\mathbf{l}_\infty$ which are fixed. These are the *circular points* (also called the *absolute points*) $\mathbf{I}, \mathbf{J}$, with canonical coordinates

$$\mathbf{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad \mathbf{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}.$$

The circular points are a pair of complex conjugate ideal points. To see that they are fixed under an orientation-preserving similarity:

$$\begin{aligned} \mathbf{I}' &= \mathtt{H}_{\mathtt{S}}\mathbf{I} \\ &= \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \\ &= se^{-i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathbf{I} \end{aligned}$$

with an analogous proof for $\mathbf{J}$. A reflection swaps $\mathbf{I}$ and $\mathbf{J}$. The converse is also true, i.e. if the circular points are fixed then the linear transformation is a similarity. The proof is left as an exercise. To summarize,

Result 2.21. *The circular points, $\mathbf{I}, \mathbf{J}$, are fixed points under the projective transformation $\mathtt{H}$ if and only if $\mathtt{H}$ is a similarity.*

The name "circular points" arises because every circle intersects $\mathbf{l}_\infty$ at the circular points. To see this, start from equation (2.1–*p*30) for a conic. In the case that the conic is a circle: $a = c$ and $b = 0$. Then

$$x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

where a has been set to unity. This conic intersects $\mathbf{l}_\infty$ in the (ideal) points for which $x_3 = 0$, namely

$$x_1^2 + x_2^2 = 0$$

with solution $\mathbf{I} = (1, i, 0)^\mathsf{T}$, $\mathbf{J} = (1, -i, 0)^\mathsf{T}$, i.e. any circle intersects $\mathbf{l}_\infty$ in the circular points. In Euclidean geometry it is well known that a circle is specified by three points. The circular points enable an alternative computation. A circle can be computed using the general formula for a conic defined by five points (2.4–*p*31), where the five points are the three points augmented with the two circular points.

In section 2.7.5 it will be shown that identifying the circular points (or equivalently their dual, see below) allows the recovery of similarity properties (angles, ratios of lengths). Algebraically, the circular points are the orthogonal directions of Euclidean geometry, $(1, 0, 0)^\mathsf{T}$ and $(0, 1, 0)^\mathsf{T}$, packaged into a single complex conjugate entity, e.g.

$$\mathbf{I} = (1, 0, 0)^\mathsf{T} + i(0, 1, 0)^\mathsf{T}.$$

Consequently, it is not so surprising that once the circular points are identified, orthogonality, and other metric properties, are then determined.

The conic dual to the circular points. The conic

$$\mathtt{C}^*_\infty = \mathbf{IJ}^\mathsf{T} + \mathbf{JI}^\mathsf{T} \tag{2.20}$$

is dual to the circular points. The conic $\mathtt{C}^*_\infty$ is a degenerate (rank 2) line conic (see section 2.2.3), which consists of the two circular points. In a Euclidean coordinate system it is given by

$$\mathtt{C}^*_\infty = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -i & 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \begin{pmatrix} 1 & i & 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The conic $\mathtt{C}^*_\infty$ is fixed under similarity transformations in an analogous fashion to the fixed properties of circular points. A conic is fixed if the same matrix results (up to scale) under the transformation rule. Since $\mathtt{C}^*_\infty$ is a dual conic it transforms according to result 2.14(*p*37) ($\mathtt{C}^{*\prime} = \mathtt{HC}^*\mathtt{H}^\mathsf{T}$), and one can verify that under the point transformation $\mathbf{x}' = \mathtt{H}_\mathrm{S}\mathbf{x}$,

$$\mathtt{C}^{*\,\prime}_\infty = \mathtt{H}_\mathrm{S}\mathtt{C}^*_\infty\mathtt{H}_\mathrm{S}^\mathsf{T} = \mathtt{C}^*_\infty.$$

The converse is also true, and we have

Result 2.22. *The dual conic $\mathtt{C}^*_\infty$ is fixed under the projective transformation $\mathtt{H}$ if and only if $\mathtt{H}$ is a similarity.*

Some properties of $\mathtt{C}^*_\infty$ in any projective frame:

(i) $\mathtt{C}^*_\infty$ has 4 degrees of freedom: a 3×3 homogeneous symmetric matrix has 5 degrees of freedom, but the constraint $\det \mathtt{C}^*_\infty = 0$ reduces the degrees of freedom by 1.

(ii) $\mathbf{l}_\infty$ is the null vector of $\mathtt{C}^*_\infty$. This is clear from the definition: the circular points lie on $\mathbf{l}_\infty$, so that $\mathbf{I}^\mathsf{T}\mathbf{l}_\infty = \mathbf{J}^\mathsf{T}\mathbf{l}_\infty = 0$; then

$$\mathtt{C}^*_\infty \mathbf{l}_\infty = (\mathbf{I}\mathbf{J}^\mathsf{T} + \mathbf{J}\mathbf{I}^\mathsf{T})\mathbf{l}_\infty = \mathbf{I}(\mathbf{J}^\mathsf{T}\mathbf{l}_\infty) + \mathbf{J}(\mathbf{I}^\mathsf{T}\mathbf{l}_\infty) = \mathbf{0}.$$

2.7.4 Angles on the projective plane

In Euclidean geometry the angle between two lines is computed from the dot product of their normals. For the lines $\mathbf{l} = (l_1, l_2, l_3)^\mathsf{T}$ and $\mathbf{m} = (m_1, m_2, m_3)^\mathsf{T}$ with normals parallel to $(l_1, l_2)^\mathsf{T}, (m_1, m_2)^\mathsf{T}$ respectively, the angle is

$$\cos\theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}. \tag{2.21}$$

The problem with this expression is that the first two components of $\mathbf{l}$ and $\mathbf{m}$ do not have well defined transformation properties under projective transformations (they are not tensors), and so (2.21) cannot be applied after an affine or projective transformation of the plane. However, an analogous expression to (2.21) which is invariant to projective transformations is

$$\cos\theta = \frac{\mathbf{l}^\mathsf{T}\mathtt{C}^*_\infty\mathbf{m}}{\sqrt{(\mathbf{l}^\mathsf{T}\mathtt{C}^*_\infty\mathbf{l})(\mathbf{m}^\mathsf{T}\mathtt{C}^*_\infty\mathbf{m})}} \tag{2.22}$$

where $\mathtt{C}^*_\infty$ is the conic dual to the circular points. It is clear that in a Euclidean coordinate system (2.22) reduces to (2.21). It may be verified that (2.22) is invariant to projective transformations by using the transformation rules for lines (2.6–*p*36) ($\mathbf{l}' = \mathtt{H}^{-\mathsf{T}}\mathbf{l}$) and dual conics (result 2.14(*p*37)) ($\mathtt{C}^{*\prime} = \mathtt{H}\mathtt{C}^*\mathtt{H}^\mathsf{T}$) under the point transformation $\mathbf{x}' = \mathtt{H}\mathbf{x}$. For example, the numerator transforms as

$$\mathbf{l}^\mathsf{T}\mathtt{C}^*_\infty\mathbf{m} \mapsto \mathbf{l}^\mathsf{T}\mathtt{H}^{-1}\mathtt{H}\mathtt{C}^*_\infty\mathtt{H}^\mathsf{T}\mathtt{H}^{-\mathsf{T}}\mathbf{m} = \mathbf{l}^\mathsf{T}\mathtt{C}^*_\infty\mathbf{m}.$$

It may also be verified that the scale of the homogeneous objects cancels between the numerator and denominator. Thus (2.22) is indeed invariant to the projective frame. To summarize, we have shown

Result 2.23. *Once the conic $\mathtt{C}^*_\infty$ is identified on the projective plane then Euclidean angles may be measured by (2.22).*

Note, as a corollary,

Result 2.24. *Lines $\mathbf{l}$ and $\mathbf{m}$ are orthogonal if $\mathbf{l}^\mathsf{T}\mathtt{C}^*_\infty\mathbf{m} = 0$.*

Geometrically, if $\mathbf{l}$ and $\mathbf{m}$ satisfy $\mathbf{l}^\mathsf{T}\mathtt{C}^*_\infty\mathbf{m} = 0$, then the lines are conjugate (see section 2.8.1) with respect to the conic $\mathtt{C}^*_\infty$.

Length ratios may also be measured once $\mathtt{C}^*_\infty$ is identified. Consider the triangle shown in figure 2.16 with vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}$. From the standard trigonometric sine rule the ratio of lengths $d(\mathbf{b}, \mathbf{c}) : d(\mathbf{a}, \mathbf{c}) = \sin\alpha : \sin\beta$, where $d(\mathbf{x}, \mathbf{y})$ denotes the Euclidean distance between the points $\mathbf{x}$ and $\mathbf{y}$. Using (2.22), both $\cos\alpha$ and $\cos\beta$ may be computed from the lines $\mathbf{l}' = \mathbf{a}' \times \mathbf{b}'$, $\mathbf{m}' = \mathbf{c}' \times \mathbf{a}'$ and $\mathbf{n}' = \mathbf{b}' \times \mathbf{c}'$ for any

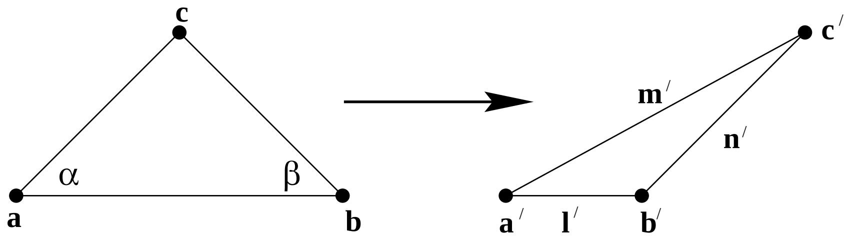


Fig. 2.16. **Length ratios.** *Once* C^*_∞ *is identified the Euclidean length ratio* $d(\mathbf{b}, \mathbf{c}) : d(\mathbf{a}, \mathbf{c})$ *may be measured from the projectively distorted figure. See text for details.*

projective frame in which C^*_∞ is specified. Consequently both $\sin\alpha$, $\sin\beta$, and thence the ratio $d(\mathbf{a}, \mathbf{b}) : d(\mathbf{c}, \mathbf{a})$, may be determined from the projectively mapped points.

2.7.5 Recovery of metric properties from images

A completely analogous approach to that of section 2.7.2 and figure 2.12, where affine properties are recovered by specifying $\mathbf{l}_\infty$, enables metric properties to be recovered from an image of a plane by transforming the circular points to their canonical position. Suppose the circular points are identified in an image, and the image is then rectified by a projective transformation H that maps the imaged circular points to their canonical position (at $(1, \pm i, 0)^\mathsf{T}$) on $\mathbf{l}_\infty$. From result 2.21 the transformation between the world plane and the rectified image is then a similarity since it is projective and the circular points are fixed.

Metric rectification using C^*_∞. The dual conic C^*_∞ neatly packages all the information required for a metric rectification. It enables both the projective and affine components of a projective transformation to be determined, leaving only similarity distortions. This is evident from its transformation under a projectivity. If the point transformation is $\mathbf{x}' = H\mathbf{x}$, where the $\mathbf{x}$-coordinate frame is Euclidean and $\mathbf{x}'$ projective, C^*_∞ transforms according to result 2.14(*p*37) ($C^{*\prime} = HC^*H^\mathsf{T}$). Using the decomposition chain (2.17–*p*43) for H

$$\begin{aligned} {C^*_\infty}' &= (H_P\, H_A\, H_S)\, C^*_\infty\, (H_P\, H_A\, H_S)^\mathsf{T} = (H_P\, H_A)\left(H_S\, C^*_\infty H_S^\mathsf{T}\right)\left(H_A^\mathsf{T}\, H_P^\mathsf{T}\right) \\ &= (H_P\, H_A)\, C^*_\infty \left(H_A^\mathsf{T}\, H_P^\mathsf{T}\right) \\ &= \begin{bmatrix} KK^\mathsf{T} & KK^\mathsf{T}\mathbf{v} \\ \mathbf{v}^\mathsf{T}KK^\mathsf{T} & \mathbf{v}^\mathsf{T}KK^\mathsf{T}\mathbf{v} \end{bmatrix}. \end{aligned} \tag{2.23}$$

It is clear that the projective ($\mathbf{v}$) and affine (K) components are determined directly from the image of C^*_∞, but (since C^*_∞ is invariant to similarity transformation by result 2.22) the similarity component is undetermined. Consequently,

Result 2.25. *Once the conic* C^*_∞ *is identified on the projective plane then projective distortion may be rectified up to a similarity.*

Actually, a suitable rectifying homography may be obtained directly from the identified ${C^*_\infty}'$ in an image using the SVD (section A4.4(*p*585)): writing the SVD of ${C^*_\infty}'$

as

$$\mathtt{C}^{*\prime}_{\infty} = \mathtt{U}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathtt{U}^{\mathsf{T}}$$

then by inspection from (2.23) the rectifying projectivity is $\mathtt{H} = \mathtt{U}$ up to a similarity.

The following two examples show typical situations where $\mathtt{C}^*_\infty$ may be identified in an image, and thence a metric rectification obtained.

Example 2.26. Metric rectification I

Suppose an image has been affinely rectified (as in example 2.18 above), then we require two constraints to specify the 2 degrees of freedom of the circular points in order to determine a metric rectification. These two constraints may be obtained from two imaged right angles on the world plane.

Suppose the lines $\mathbf{l}', \mathbf{m}'$ in the affinely rectified image correspond to an orthogonal line pair $\mathbf{l}, \mathbf{m}$ on the world plane. From result 2.24 $\mathbf{l}'^{\mathsf{T}}\mathtt{C}^{*\prime}_{\infty}\mathbf{m}' = 0$, and using (2.23) with $\mathbf{v} = \mathbf{0}$

$$\begin{pmatrix} l'_1 & l'_2 & l'_3 \end{pmatrix}\begin{bmatrix} \mathtt{KK}^{\mathsf{T}} & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 0 \end{bmatrix}\begin{pmatrix} m'_1 \\ m'_2 \\ m'_3 \end{pmatrix} = 0$$

which is a *linear* constraint on the 2×2 matrix $\mathtt{S} = \mathtt{KK}^{\mathsf{T}}$. The matrix $\mathtt{S} = \mathtt{KK}^{\mathsf{T}}$ is symmetric with three independent elements, and thus 2 degrees of freedom (as the overall scaling is unimportant). The orthogonality condition reduces to the equation $(l'_1, l'_2)\mathtt{S}(m'_1, m'_2)^{\mathsf{T}} = 0$ which may be written as

$$(l'_1 m'_1, l'_1 m'_2 + l'_2 m'_1, l'_2 m'_2)\,\mathbf{s} = 0,$$

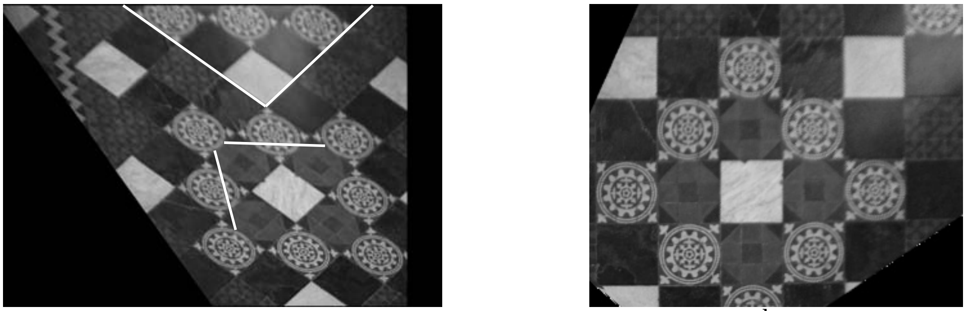
where $\mathbf{s} = (s_{11}, s_{12}, s_{22})^{\mathsf{T}}$ is $\mathtt{S}$ written as a 3-vector. Two such orthogonal line pairs provide two constraints which may be stacked to give a 2×3 matrix with $\mathbf{s}$ determined as the null vector. Thus $\mathtt{S}$, and hence $\mathtt{K}$, is obtained up to scale (by Cholesky decomposition, section A4.2.1(*p*582)). Figure 2.17 shows an example of two orthogonal line pairs being used to metrically rectify the affinely rectified image computed in figure 2.13. △

Alternatively, the two constraints required for metric rectification may be obtained from an imaged circle or two known length ratios. In the case of a circle, the image conic is an ellipse in the affinely rectified image, and the intersection of this ellipse with the (known) $\mathbf{l}_\infty$ directly determines the imaged circular points.

The conic $\mathtt{C}^*_\infty$ can alternatively be identified directly in a perspective image, without first identifying $\mathbf{l}_\infty$, as is illustrated in the following example.

Example 2.27. Metric rectification II

We start here from the original perspective image of the plane (not the affinely rectified image of example 2.26). Suppose lines $\mathbf{l}$ and $\mathbf{m}$ are images of orthogonal lines on the world plane; then from result 2.24 $\mathbf{l}^{\mathsf{T}}\mathtt{C}^*_\infty\mathbf{m} = 0$, and in a similar manner to constraining



a b

Fig. 2.17. **Metric rectification via orthogonal lines I.** *The affine transformation required to metrically rectify an affine image may be computed from imaged orthogonal lines. (a) Two (non-parallel) line pairs identified on the affinely rectified image (figure 2.13) correspond to orthogonal lines on the world plane. (b) The metrically rectified image. Note that in the metrically rectified image all lines orthogonal in the world are orthogonal, world squares have unit aspect ratio, and world circles are circular.*

a b

Fig. 2.18. **Metric rectification via orthogonal lines II.** *(a) The conic* $\mathtt{C}^*_\infty$ *is determined on the perspectively imaged plane (the front wall of the building) using the five orthogonal line pairs shown. The conic* $\mathtt{C}^*_\infty$ *determines the circular points, and equivalently the projective transformation necessary to metrically rectify the image (b). The image shown in (a) is the same perspective image as that of figure 2.4(*p*35), where the perspective distortion was removed by specifying the world position of four image points.*

a conic to contain a point (2.4–*p*31), this provides a linear constraint on the elements of $\mathtt{C}^*_\infty$, namely

$$(l_1m_1, (l_1m_2 + l_2m_1)/2, l_2m_2, (l_1m_3 + l_3m_1)/2, (l_2m_3 + l_3m_2)/2, l_3m_3)\,\mathbf{c} = 0$$

where $\mathbf{c} = (a, b, c, d, e, f)^\mathsf{T}$ is the conic matrix (2.3–*p*30) of $\mathtt{C}^*_\infty$ written as a 6-vector. Five such constraints can be stacked to form a 5×6 matrix, and $\mathbf{c}$, and hence $\mathtt{C}^*_\infty$, is obtained as the null vector. This shows that $\mathtt{C}^*_\infty$ can be determined linearly from the images of five line pairs which are orthogonal on the world plane. An example of metric rectification using such line pair constraints is shown in figure 2.18. △

Stratification. Note, in example 2.27 the affine and projective distortions are determined in one step by specifying $\mathtt{C}^*_\infty$. In the previous example 2.26 first the projective and subsequently the affine distortions were removed. This two-step approach is termed *stratified*. Analogous approaches apply in 3D, and are employed in chapter 10

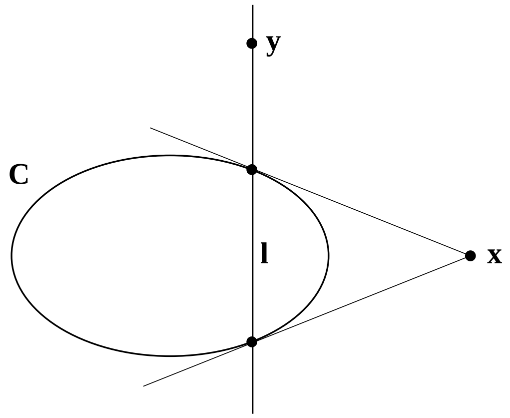


Fig. 2.19. **The pole–polar relationship.** *The line* $\mathbf{l} = \mathtt{C}\mathbf{x}$ *is the* polar *of the point* $\mathbf{x}$ *with respect to the conic* $\mathtt{C}$*, and the point* $\mathbf{x} = \mathtt{C}^{-1}\mathbf{l}$ *is the* pole *of* $\mathbf{l}$ *with respect to* $\mathtt{C}$*. The polar of* $\mathbf{x}$ *intersects the conic at the points of tangency of lines from* $\mathbf{x}$*. If* $\mathbf{y}$ *is on* $\mathbf{l}$ *then* $\mathbf{y}^{\mathsf{T}}\mathbf{l} = \mathbf{y}^{\mathsf{T}}\mathtt{C}\mathbf{x} = 0$*. Points* $\mathbf{x}$ *and* $\mathbf{y}$ *which satisfy* $\mathbf{y}^{\mathsf{T}}\mathtt{C}\mathbf{x} = 0$ *are* conjugate.

on 3D reconstruction and chapter 19 on auto-calibration, when obtaining a metric from a 3D projective reconstruction.

2.8 More properties of conics

We now introduce an important geometric relation between a point, line and conic, which is termed *polarity*. Applications of this relation (to the representation of orthogonality) are given in chapter 8.

2.8.1 The pole–polar relationship

A point $\mathbf{x}$ and conic $\mathtt{C}$ define a line $\mathbf{l} = \mathtt{C}\mathbf{x}$. The line $\mathbf{l}$ is called the *polar* of $\mathbf{x}$ with respect to $\mathtt{C}$, and the point $\mathbf{x}$ is the *pole* of $\mathbf{l}$ with respect to $\mathtt{C}$.

- *The polar line* $\mathbf{l} = \mathtt{C}\mathbf{x}$ *of the point* $\mathbf{x}$ *with respect to a conic* $\mathtt{C}$ *intersects the conic in two points. The two lines tangent to* $\mathtt{C}$ *at these points intersect at* $\mathbf{x}$*.*

This relationship is illustrated in figure 2.19.

Proof. Consider a point $\mathbf{y}$ on $\mathtt{C}$. The tangent line at $\mathbf{y}$ is $\mathtt{C}\mathbf{y}$, and this line contains $\mathbf{x}$ if $\mathbf{x}^{\mathsf{T}}\mathtt{C}\mathbf{y} = 0$. Using the symmetry of $\mathtt{C}$, the condition $\mathbf{x}^{\mathsf{T}}\mathtt{C}\mathbf{y} = (\mathtt{C}\mathbf{x})^{\mathsf{T}}\mathbf{y} = 0$ is that the point $\mathbf{y}$ lies on the line $\mathtt{C}\mathbf{x}$. Thus the polar line $\mathtt{C}\mathbf{x}$ intersects the conic in the point $\mathbf{y}$ at which the tangent line contains $\mathbf{x}$. $\square$

As the point $\mathbf{x}$ approaches the conic the tangent lines become closer to collinear, and their contact points on the conic also become closer. In the limit that $\mathbf{x}$ lies on $\mathtt{C}$, the polar line has two-point contact at $\mathbf{x}$, and we have:

- *If the point* $\mathbf{x}$ *is on* $\mathtt{C}$ *then the polar is the tangent line to the conic at* $\mathbf{x}$*.*

See result 2.7(*p*31).

Example 2.28. A circle of radius r centred on the x-axis at $x = a$ has the equation $(x-a)^2 + y^2 = r^2$, and is represented by the conic matrix

$$\mathtt{C} = \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & 0 \\ -a & 0 & a^2 - r^2 \end{bmatrix}.$$

The polar line of the origin is given by $\mathbf{l} = \mathtt{C}(0,0,1)^{\mathsf{T}} = (-a, 0, a^2 - r^2)^{\mathsf{T}}$. This is a vertical line at $x = (a^2 - r^2)/a$. If $r = a$ the origin lies on the circle. In this case the polar line is the y-axis and is tangent to the circle. △

It is evident that the conic induces a map between points and lines of $\mathbb{P}^2$. This map is a projective construction since it involves only intersections and tangency, both properties that are preserved under projective transformations. A projective map between points and lines is termed a *correlation* (an unfortunate name, given its more common usage).

Definition 2.29. A *correlation* is an invertible mapping from points of $\mathbb{P}^2$ to lines of $\mathbb{P}^2$. It is represented by a 3×3 non-singular matrix $\mathtt{A}$ as $\mathbf{l} = \mathtt{A}\mathbf{x}$.

A correlation provides a systematic way to dualize relations involving points and lines. It need not be represented by a symmetric matrix, but we will only consider symmetric correlations here, because of the association with conics.

- **Conjugate points.** *If the point* $\mathbf{y}$ *is on the line* $\mathbf{l} = \mathtt{C}\mathbf{x}$ *then* $\mathbf{y}^{\mathsf{T}}\mathbf{l} = \mathbf{y}^{\mathsf{T}}\mathtt{C}\mathbf{x} = 0$. *Any two points* $\mathbf{x}, \mathbf{y}$ *satisfying* $\mathbf{y}^{\mathsf{T}}\mathtt{C}\mathbf{x} = 0$ *are* conjugate *with respect to the conic* $\mathtt{C}$.

  The conjugacy relation is symmetric:

- *If* $\mathbf{x}$ *is on the polar of* $\mathbf{y}$ *then* $\mathbf{y}$ *is on the polar of* $\mathbf{x}$.

This follows simply because of the symmetry of the conic matrix – the point $\mathbf{x}$ is on the polar of $\mathbf{y}$ if $\mathbf{x}^{\mathsf{T}}\mathtt{C}\mathbf{y} = 0$, and the point $\mathbf{y}$ is on the polar of $\mathbf{x}$ if $\mathbf{y}^{\mathsf{T}}\mathtt{C}\mathbf{x} = 0$. Since $\mathbf{x}^{\mathsf{T}}\mathtt{C}\mathbf{y} = \mathbf{y}^{\mathsf{T}}\mathtt{C}\mathbf{x}$, if one form is zero, then so is the other. There is a dual conjugacy relationship for lines: two lines $\mathbf{l}$ and $\mathbf{m}$ are conjugate if $\mathbf{l}^{\mathsf{T}}\mathtt{C}^*\mathbf{m} = 0$.

2.8.2 Classification of conics

This section describes the projective and affine classification of conics.

Projective normal form for a conic. Since $\mathtt{C}$ is a symmetric matrix it has real eigenvalues, and may be decomposed as a product $\mathtt{C} = \mathtt{U}^{\mathsf{T}}\mathtt{D}\mathtt{U}$ (see section A4.2(*p*580)), where $\mathtt{U}$ is an orthogonal matrix, and $\mathtt{D}$ is diagonal. Applying the projective transformation represented by $\mathtt{U}$, conic $\mathtt{C}$ is transformed to another conic $\mathtt{C}' = \mathtt{U}^{-\mathsf{T}}\mathtt{C}\mathtt{U}^{-1} = \mathtt{U}^{-\mathsf{T}}\mathtt{U}^{\mathsf{T}}\mathtt{D}\mathtt{U}\mathtt{U}^{-1} = \mathtt{D}$. This shows that any conic is equivalent under projective transformation to one with a diagonal matrix. Let $\mathtt{D} = \mathrm{diag}(\epsilon_1 d_1, \epsilon_2 d_2, \epsilon_3 d_3)$ where $\epsilon_i = \pm 1$ or 0 and each $d_i > 0$. Thus, $\mathtt{D}$ may be written in the form

$$\mathtt{D} = \mathrm{diag}(s_1, s_2, s_3)^{\mathsf{T}}\mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_3)\mathrm{diag}(s_1, s_2, s_3)$$

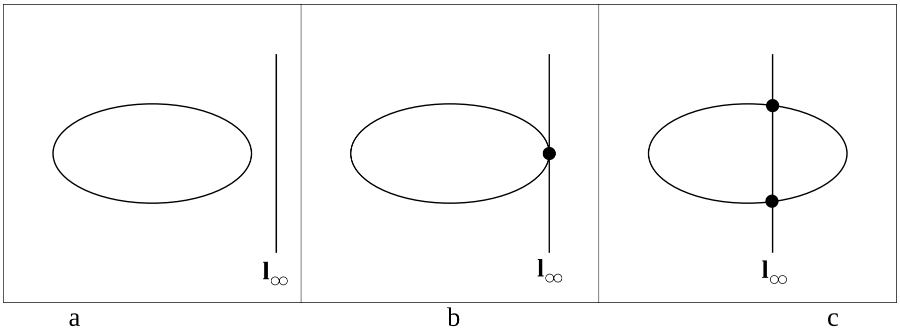


Fig. 2.20. **Affine classification of point conics.** *A conic is an (a) ellipse, (b) parabola, or (c) hyperbola; according to whether it (a) has no real intersection, (b) is tangent to (2-point contact), or (c) has 2 real intersections with $\mathbf{l}_\infty$. Under an affine transformation $\mathbf{l}_\infty$ is a fixed line, and intersections are preserved. Thus this classification is unaltered by an affinity.*

where $s_i^2 = d_i$. Note that $\text{diag}(s_1, s_2, s_3)^\mathsf{T} = \text{diag}(s_1, s_2, s_3)$. Now, transforming once more by the transformation $\text{diag}(s_1, s_2, s_3)$, the conic $\mathtt{D}$ is transformed to a conic with matrix $\text{diag}(\epsilon_1, \epsilon_2, \epsilon_3)$, with each $\epsilon_i = \pm 1$ or 0. Further transformation by permutation matrices may be carried out to ensure that values $\epsilon_i = 1$ occur before values $\epsilon_i = -1$ which in turn precede values $\epsilon_i = 0$. Finally, by multiplying by -1 if necessary, one may ensure that there are at least as many $+1$ entries as -1. The various types of conics may now be enumerated, and are shown in table 2.2.

Diagonal	Equation	Conic type
$(1, 1, 1)$	$x^2 + y^2 + w^2 = 0$	Improper conic – no real points.
$(1, 1, -1)$	$x^2 + y^2 - w^2 = 0$	Circle
$(1, 1, 0)$	$x^2 + y^2 = 0$	Single real point $(0, 0, 1)^\mathsf{T}$
$(1, -1, 0)$	$x^2 - y^2 = 0$	Two lines $x = \pm y$
$(1, 0, 0)$	$x^2 = 0$	Single line $x = 0$ counted twice.

Table 2.2. **Projective classification of point conics.** *Any plane conic is projectively equivalent to one of the types shown in this table. Those conics for which $\epsilon_i = 0$ for some i are known as degenerate conics, and are represented by a matrix of rank less than 3. The conic type column only describes the real points of the conics – for example as a complex conic $x^2 + y^2 = 0$ consists of the line pair $x = \pm iy$.*

Affine classification of conics. The classification of (non-degenerate, proper) conics in Euclidean geometry into hyperbola, ellipse and parabola is well known. As shown above in projective geometry these three types of conic are projectively equivalent to a circle. However, in affine geometry the Euclidean classification is still valid because it depends only on the relation of $\mathbf{l}_\infty$ to the conic. The relation for the three types of conic is illustrated in figure 2.20.

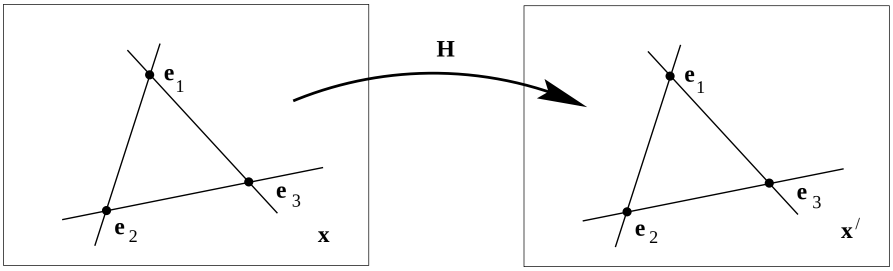


Fig. 2.21. **Fixed points and lines of a plane projective transformation.** *There are three fixed points, and three fixed lines through these points. The fixed lines and points may be complex. Algebraically, the fixed points are the eigenvectors, $\mathbf{e}_i$, of the point transformation ($\mathbf{x}' = \mathtt{H}\mathbf{x}$), and the fixed lines eigenvectors of the line transformation ($\mathbf{l}' = \mathtt{H}^{-\mathsf{T}}\mathbf{l}$). Note, the fixed line is* not *fixed pointwise: under the transformation, points on the line are mapped to other points on the line; only the fixed points are mapped to themselves.*

2.9 Fixed points and lines

We have seen, by the examples of $\mathbf{l}_\infty$ and the circular points, that points and lines may be fixed under a projective transformation. In this section the idea is investigated more thoroughly.

Here, the source and destination planes are identified (the same) so that the transformation maps points $\mathbf{x}$ to points $\mathbf{x}'$ in the same coordinate system. The key idea is that an *eigenvector* corresponds to a *fixed point* of the transformation, since for an eigenvector $\mathbf{e}$ with eigenvalue λ,

$$\mathtt{H}\mathbf{e} = \lambda\mathbf{e}$$

and $\mathbf{e}$ and $\lambda\mathbf{e}$ represent the same point. Often the eigenvector and eigenvalue have physical or geometric significance in computer vision applications.

A 3×3 matrix has three eigenvalues and consequently a plane projective transformation has up to three fixed points, if the eigenvalues are distinct. Since the characteristic equation is a cubic in this case, one or three of the eigenvalues, and corresponding eigenvectors, is real. A similar development can be given for *fixed lines*, which, since lines transform as (2.6–*p*36) $\mathbf{l}' = \mathtt{H}^{-\mathsf{T}}\mathbf{l}$, correspond to the eigenvectors of $\mathtt{H}^{\mathsf{T}}$.

The relationship between the fixed points and fixed lines is shown in figure 2.21. Note the lines are fixed as a set, not fixed pointwise, i.e. a point on the line is mapped to another point on the line, but in general the source and destination points will differ. There is nothing mysterious here: The projective transformation of the plane induces a 1D projective transformation on the line. A 1D projective transformation is represented by a 2×2 homogeneous matrix (section 2.5). This 1D projectivity has two fixed points corresponding to the two eigenvectors of the 2×2 matrix. These fixed points are those of the 2D projective transformation.

A further specialization concerns repeated eigenvalues. Suppose two of the eigenvalues (λ_2, λ_3 say) are identical, and that there are two distinct eigenvectors ($\mathbf{e}_2, \mathbf{e}_3$), corresponding to $\lambda_2 = \lambda_3$. Then the line containing the eigenvectors $\mathbf{e}_2, \mathbf{e}_3$ will be fixed pointwise, i.e. it is a line of fixed points. For suppose $\mathbf{x} = \alpha\mathbf{e}_2 + \beta\mathbf{e}_3$; then

$$\mathtt{H}\mathbf{x} = \lambda_2\alpha\mathbf{e}_2 + \lambda_2\beta\mathbf{e}_3 = \lambda_2\mathbf{x}$$

i.e. a point on the line through two degenerate eigenvectors is mapped to itself (only differing by scale). Another possibility is that $\lambda_2 = \lambda_3$, but that there is only one corresponding eigenvector. In this case, the eigenvector has *algebraic dimension* equal to two, but *geometric dimension* equal to one. Then there is one fewer fixed point (2 instead of 3). Various cases of repeated eigenvalues are discussed further in appendix 7(*p*628).

We now examine the fixed points and lines of the hierarchy of projective transformation subgroups of section 2.4. Affine transformations, and the more specialized forms, have two eigenvectors which are ideal points ($x_3 = 0$), and which correspond to the eigenvectors of the upper left 2×2 matrix. The third eigenvector is finite in general.

A Euclidean matrix. The two ideal fixed points are the complex conjugate pair of circular points $\mathbf{I}, \mathbf{J}$, with corresponding eigenvalues $\{e^{i\theta}, e^{-i\theta}\}$, where θ is the rotation angle. The third eigenvector, which has unit eigenvalue, is called the *pole*. The Euclidean transformation is equal to a pure rotation by θ about this point with no translation.

A special case is that of a pure translation (i.e. where $\theta = 0$). Here the eigenvalues are triply degenerate. The line at infinity is fixed pointwise, and there is a pencil of fixed lines through the point $(t_x, t_y, 0)^\mathsf{T}$ which corresponds to the translation direction. Consequently lines parallel to $\mathbf{t}$ are fixed. This is an example of an elation (see section A7.3(*p*631)).

A similarity matrix. The two ideal fixed points are again the circular points. The eigenvalues are $\{1, se^{i\theta}, se^{-i\theta}\}$. The action can be understood as a rotation and isotropic scaling by s about the finite fixed point. Note that the eigenvalues of the circular points again encode the angle of rotation.

An affine matrix. The two ideal fixed points can be real or complex conjugates, but the fixed line $\mathbf{l}_\infty = (0, 0, 1)^\mathsf{T}$ through these points is real in either case.

2.10 Closure

2.10.1 The literature

A gentle introduction to plane projective geometry, written for computer vision researchers, is given in the appendix of Mundy and Zisserman [Mundy-92]. A more formal approach is that of Semple and Kneebone [Semple-79], but [Springer-64] is more readable.

On the recovery of affine and metric scene properties for an imaged plane, Collins and Beveridge [Collins-93] use the vanishing line to recover affine properties from satellite images, and Liebowitz and Zisserman [Liebowitz-98] use metric information on the plane, such as right angles, to recover the metric geometry.

2.10.2 Notes and exercises

(i) **Affine transformations.**

(a) Show that an affine transformation can map a circle to an ellipse, but cannot map an ellipse to a hyperbola or parabola.

(b) Prove that under an affine transformation the ratio of lengths on parallel line segments is an invariant, but that the ratio of two lengths that are not parallel is not.

(ii) **Projective transformations.** Show that there is a three-parameter family of projective transformations which fix (as a set) a unit circle at the origin, i.e. a unit circle at the origin is mapped to a unit circle at the origin (hint, use result 2.13(*p*37) to compute the transformation). What is the geometric interpretation of this family?

(iii) **Isotropies.** Show that two lines have an invariant under a similarity transformation; and that two lines and two points have an invariant under a projective transformation. In both cases the equality case of the counting argument (result 2.16(*p*43)) is violated. Show that for these two cases the respective transformation cannot be fully determined, although it is partially determined.

(iv) **Invariants.** Using the transformation rules for points, lines and conics show:

(a) Two lines, $\mathbf{l}_1$, $\mathbf{l}_2$, and two points, $\mathbf{x}_1$, $\mathbf{x}_2$, not lying on the lines have the invariant

$$I = \frac{(\mathbf{l}_1^{\mathsf{T}}\mathbf{x}_1)(\mathbf{l}_2^{\mathsf{T}}\mathbf{x}_2)}{(\mathbf{l}_1^{\mathsf{T}}\mathbf{x}_2)(\mathbf{l}_2^{\mathsf{T}}\mathbf{x}_1)}$$

(see the previous question).

(b) A conic $\mathtt{C}$ and two points, $\mathbf{x}_1$ and $\mathbf{x}_2$, in general position have the invariant

$$I = \frac{(\mathbf{x}_1^{\mathsf{T}}\mathtt{C}\mathbf{x}_2)^2}{(\mathbf{x}_1^{\mathsf{T}}\mathtt{C}\mathbf{x}_1)(\mathbf{x}_2^{\mathsf{T}}\mathtt{C}\mathbf{x}_2)}.$$

(c) Show that the projectively invariant expression for measuring angles (2.22) is equivalent to Laguerre's projectively invariant expression involving a cross ratio with the circular points (see [Springer-64]).

(v) **The cross ratio.** Prove the invariance of the cross ratio of four collinear points under projective transformations of the line (2.18–*p*45). Hint, start with the transformation of two points on the line written as $\bar{\mathbf{x}}'_i = \lambda_i \mathtt{H}_{2\times2}\bar{\mathbf{x}}_i$ and $\bar{\mathbf{x}}'_j = \lambda_j \mathtt{H}_{2\times2}\bar{\mathbf{x}}_j$, where equality is *not* up to scale, then from the properties of determinants show that $|\bar{\mathbf{x}}'_i\bar{\mathbf{x}}'_j| = \lambda_i\lambda_j \det \mathtt{H}_{2\times2}|\bar{\mathbf{x}}_i\bar{\mathbf{x}}_j|$ and continue from here. An alternative derivation method is given in [Semple-79].

(vi) **Polarity.** Figure 2.19 shows the geometric construction of the polar line for a point $\mathbf{x}$ *outside* an ellipse. Give a geometric construction for the polar when the point is inside. Hint, start by choosing any line through $\mathbf{x}$. The pole of this line is a point on the polar of $\mathbf{x}$.

(vii) **Conics.** If the sign of the conic matrix $\mathtt{C}$ is chosen such that two eigenvalues are positive and one negative, then internal and external points may be distinguished according to the sign of $\mathbf{x}^{\mathsf{T}}\mathtt{C}\mathbf{x}$: the point $\mathbf{x}$ is inside/on/outside the conic

$\mathtt{C}$ if $\mathbf{x}^{\mathsf{T}}\mathtt{C}\mathbf{x}$ is negative/zero/positive respectively. This can seen by example from a circle $\mathtt{C} = \text{diag}(1, 1, -1)$. Under projective transformations internality is invariant, though its interpretation requires care in the case of an ellipse being transformed to a hyperbola (see figure 2.20).

(viii) **Dual conics.** Show that the matrix $[\mathbf{l}]_\times \mathtt{C} [\mathbf{l}]_\times$ represents a rank 2 dual conic which consists of the two points at which the line $\mathbf{l}$ intersects the (point) conic $\mathtt{C}$ (the notation $[\mathbf{l}]_\times$ is defined in (A4.5–*p*581)).

(ix) **Special projective transformations.** Suppose points on a scene plane are related by reflection in a line: for example, a plane object with bilateral symmetry. Show that in a perspective image of the plane the points are related by a projectivity $\mathtt{H}$ satisfying $\mathtt{H}^2 = \mathtt{I}$. Furthermore, show that under $\mathtt{H}$ there is a line of fixed points corresponding to the imaged reflection line, and that $\mathtt{H}$ has an eigenvector, not lying on this line, which is the vanishing point of the reflection direction ($\mathtt{H}$ is a planar harmonic homology, see section A7.2(*p*629)).

Now suppose that the points are related by a finite rotational symmetry: for example, points on a hexagonal bolt head. Show in this case that $\mathtt{H}^n = \mathtt{I}$, where n is the order of rotational symmetry (6 for a hexagonal symmetry), that the eigenvalues of $\mathtt{H}$ determine the rotation angle, and that the eigenvector corresponding to the real eigenvalue is the image of the centre of the rotational symmetry.