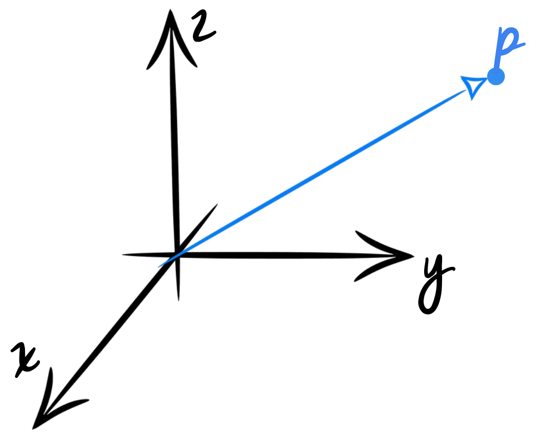
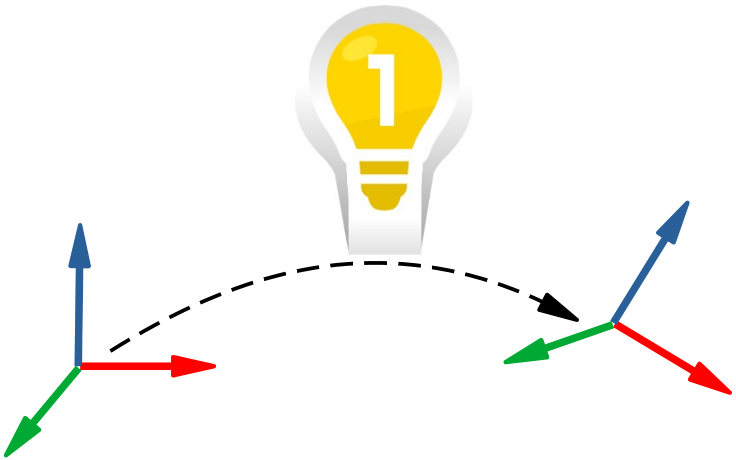
Computer Vision

Class 02



Raquel Frizera Vassallo

Tridimensional Representation and Rigid Body Motion



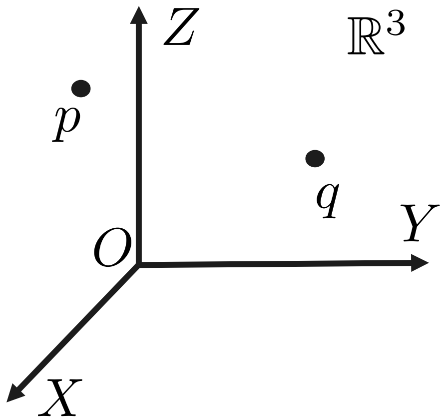
Summary

- **Euclidean 3D Space**
  - Cartesian Coordinates
  - Referential frames
- **Rigid-body Motion**
  - Homogeneous Coordinates
  - Translation and Rotation
  - Matricial Representation

Vectors

A vector can be defined by a pair of points (p, q) :

$$\mathbf{X_p} = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} \in \mathbb{R}^3, \; \mathbf{X_q} = \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} \in \mathbb{R}^3$$



Vector coordinates v :

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} X_2 - X_1 \\ Y_2 - Y_1 \\ Z_2 - Z_1 \end{bmatrix} \in \mathbb{R}^3$$

IMPORTANT:
Vector and Point are different!

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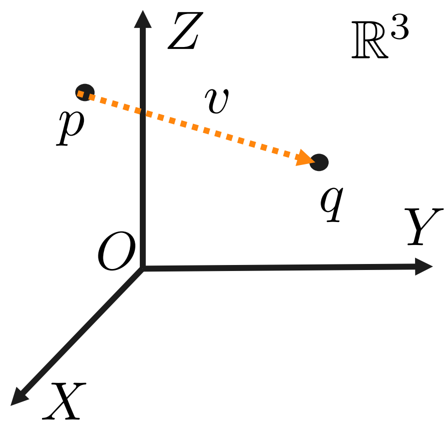

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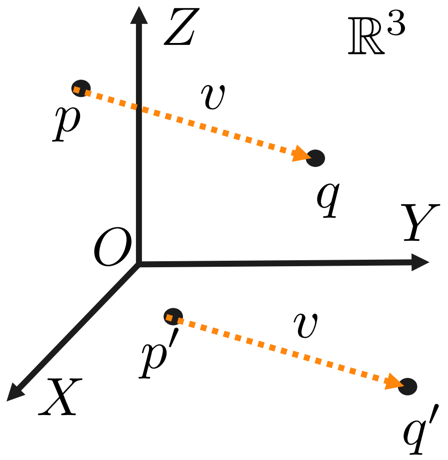

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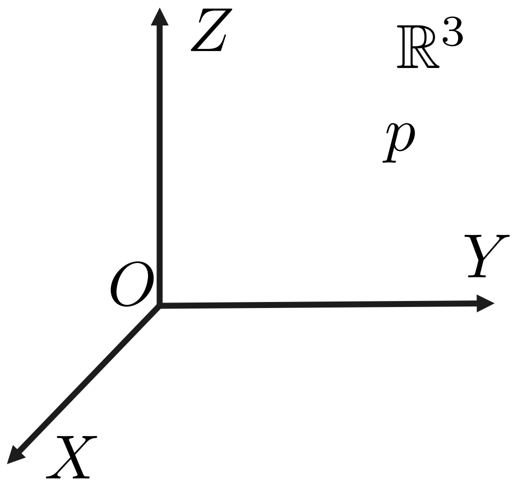
Euclidean Space

Base Vectors:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Coordinates of point p in 3D space:

$$\mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \in \mathbb{R}^3$$

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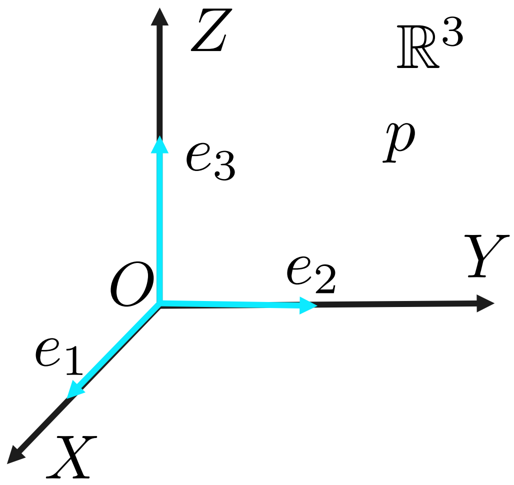

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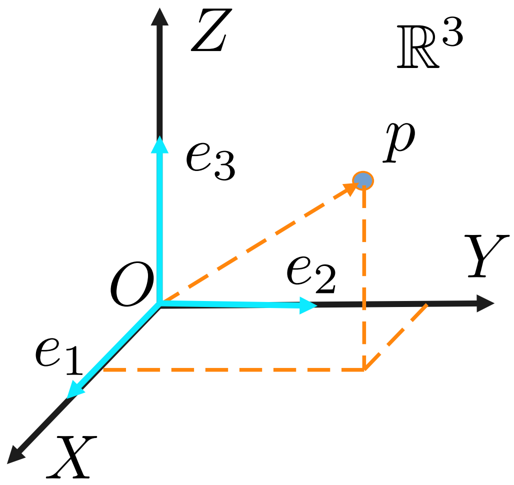

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Linear Space:

- Vectors can be multiplied by scalars and added with other vectors generating new vectors.
- Addition is commutative and associative.
- There is the identity zero
- Each element has an inverse, that if added to it results in zero.
- Addition and scalar multiplication accept the Distributive Law.

Linear combination of two vectors $v, u \in \mathbb{R}^3$

$$\alpha v + \beta u = [\alpha v_1 + \beta u_1, \alpha v_2 + \beta u_2, \alpha v_3 + \beta u_3]^T \in \mathbb{R}^3, \quad \forall \alpha, \beta \in \mathbb{R}.$$

Metric used for measuring distances and angles: **inner product**

Inner Product of two vectors:

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

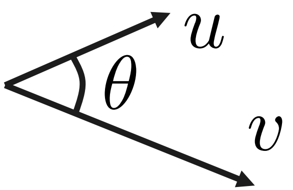
$$\langle u, v \rangle \doteq u^T v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Used to measure distances

$$\|u\| \doteq \sqrt{u^T u} = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Vector norm

$$\cos(\theta) = \frac{\langle u,v \rangle}{\|u\| \|v\|}$$



If the inner product of two vectors is zero, then the vectors are orthogonal to each other

Used to measure angles

Cross Product of two vectors:

$$u \times v \doteq \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} \in \mathbb{R}^3.$$

Attention:

$u \times (\alpha v + \beta w) = \alpha u \times v + \beta u \times w, \forall \alpha, \beta \in \mathbb{R}.$

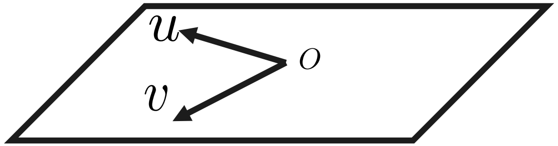
$\langle u \times v, u \rangle = \langle u \times v, v \rangle = 0$

$u \times v = -v \times u$

Resulting vector is perpendicular to the original ones.

$$u \times v \doteq \hat{u} v, \quad u, v \in \mathbb{R}^3$$

$$\hat{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$



$\hat{u}$ is a skew-symmetric matrix. Thus $\hat{u}$ is also square and $\hat{u}^T = -\hat{u}$

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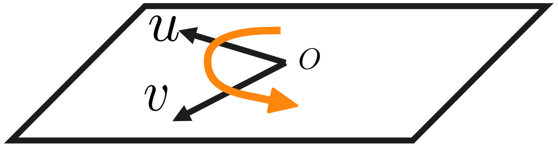

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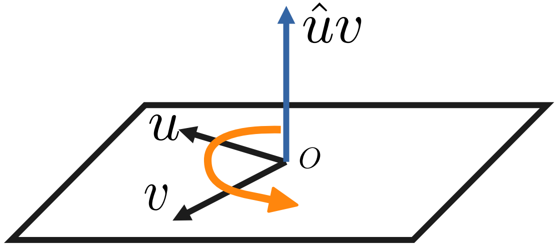

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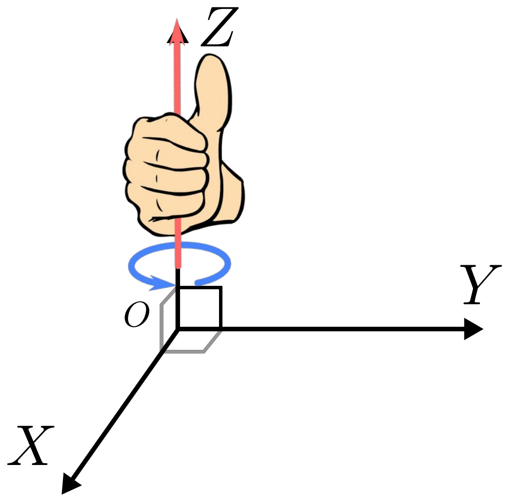
Coordinate Axis – Right-hand Rule

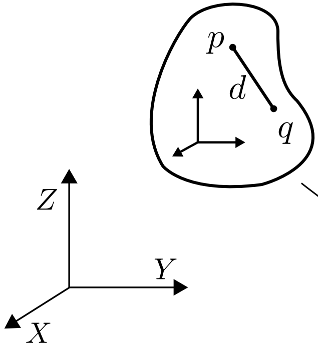
$$e_1 = [1, 0, 0]^T \qquad e_1 \times e_2 = e_3$$

$$e_2 = [0, 1, 0]^T \qquad e_2 \times e_3 = e_1$$

$$e_3 = [0, 0, 1]^T \qquad e_3 \times e_1 = e_2$$



Rigid-body Motion



It is sufficient to specify the motion of one point, and the motion of the coordinate frame attached to that point, since, for a rigid object, the distance between any two points on it does not change over time as the object moves.

Body Coordinate Frame

- Is a Special Euclidean Transformation.
- Preserves distances and orientations
- Preserves both the inner product and cross product.

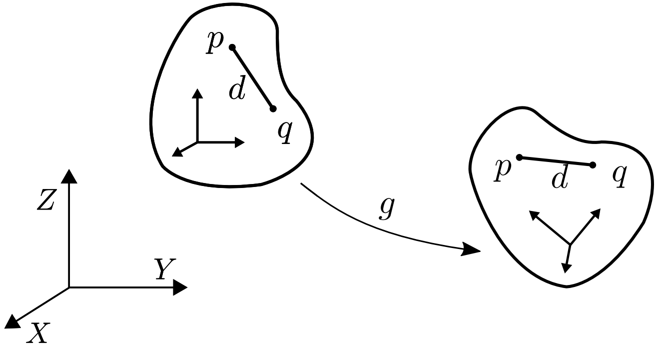
norm: $\|g_*(v)\| = \|v\|, \forall v \in \mathbb{R}^3,$

cross product: $g_*(u) \times g_*(v) = g_*(u \times v), \forall u, v \in \mathbb{R}^3,$

- Also preserves the triple product among three vectors, which corresponds to the volume of the parallelepiped spanned by the three vectors.

$$\langle g_*(u), g_*(v) \times g_*(w) \rangle = \langle u, v \times w \rangle$$

Rigid-body Motion



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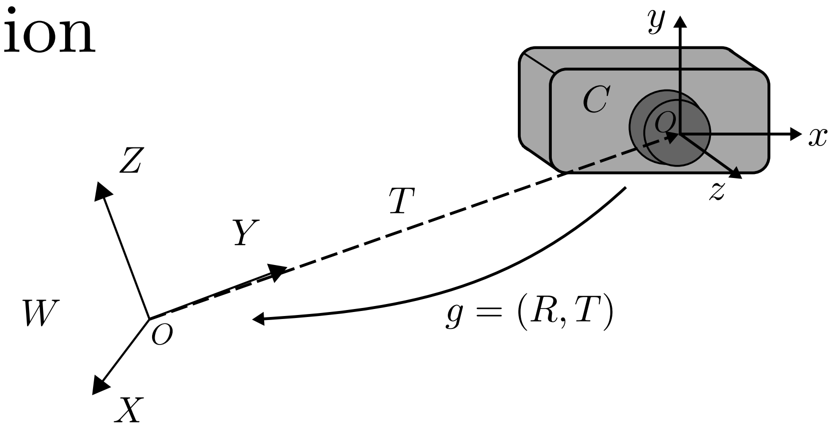
Rigid-body Motion



Consider a world frame W and a camera frame C:

- Translation corresponds to the vector that connects the origin of the frame W to that of the frame C;
- Rotation is the relative rotation between frames W and C.

PS: We can consider W as a rigid frame and C as a moving frame or vice-versa.

Rotation

Rotation matrix:

$$R \doteq [r_1, r_2, r_3] \in \mathbb{R}^{3\times 3}$$

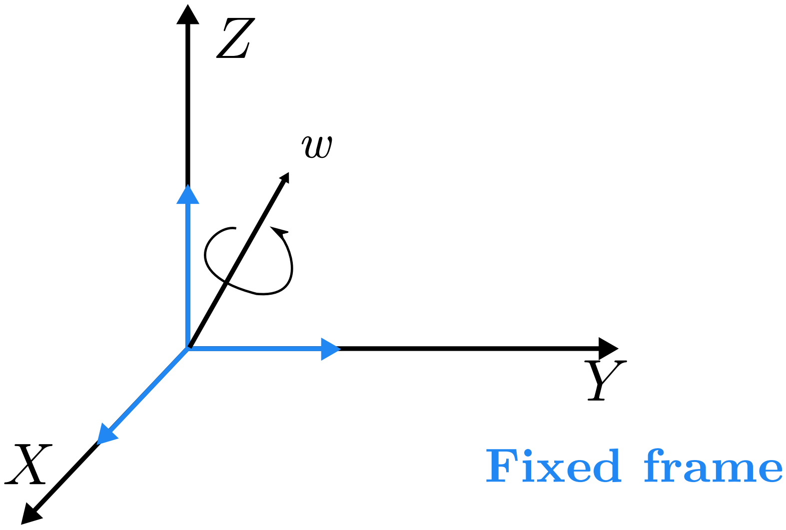
$$R^T R = I, det(R) = +1$$

Orthogonal Matrix $\rightarrow R^T = R^{-1}$

Det $= +1 \rightarrow$ Special Orthogonal

Preserves the inner product

Preserves body orientation



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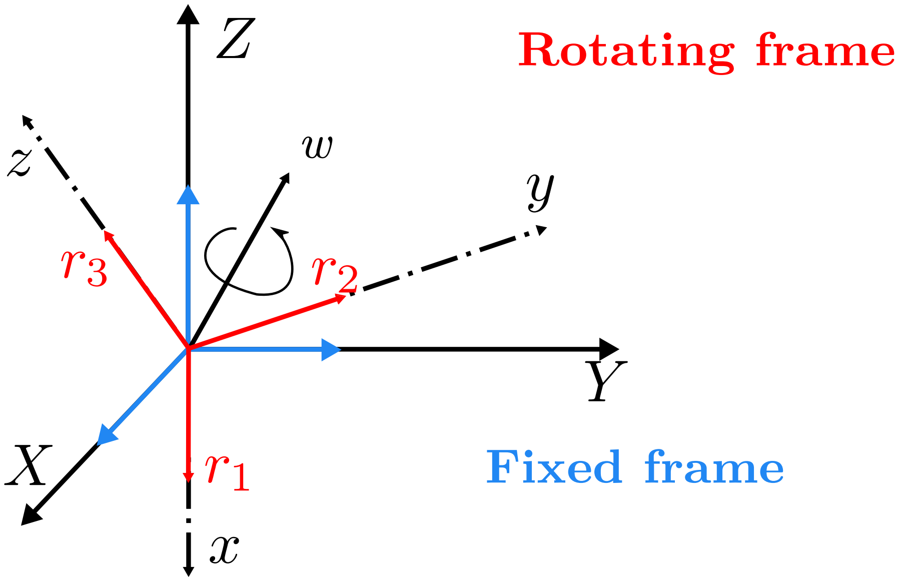


Rotation

$\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$ form an orthogonal frame.

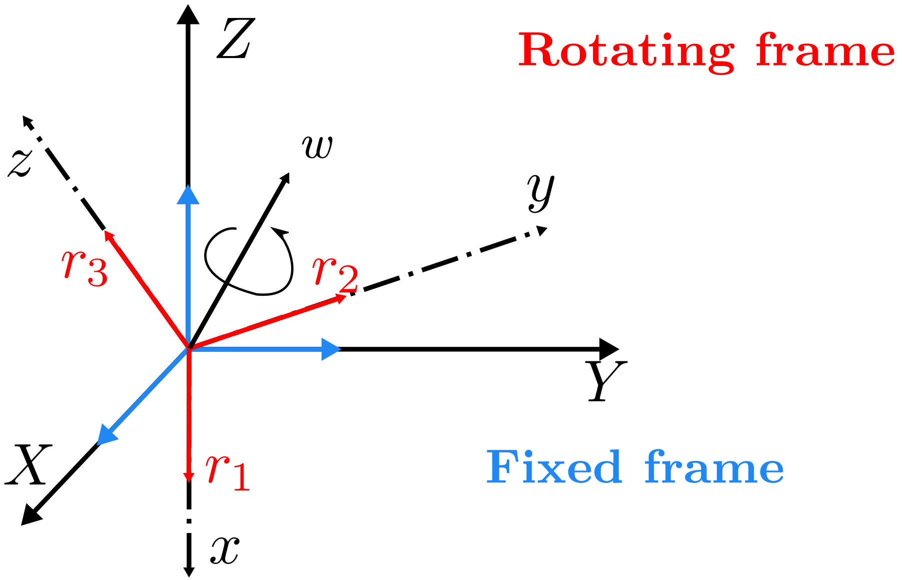
A point X_w in the fixed frame can be represented as a linear combination of $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ e $\boldsymbol{r}_3$ (Rotating referential frame).

$$\mathbf{X}_w = X_{1c}\boldsymbol{r}_1 + X_{2c}\boldsymbol{r}_2 + X_{3c}\boldsymbol{r}_3 = R_{wc}\mathbf{X}_c$$

Coordinates are related by:

$$\mathbf{X}_w(t) = R_{wc}(t)\mathbf{X}_c \qquad \mathbf{X}_c(t) = R_{cw}(t)\mathbf{X}_w$$

where $R_{cw} = R_{wc}^{-1} = R_{wc}^{T}$



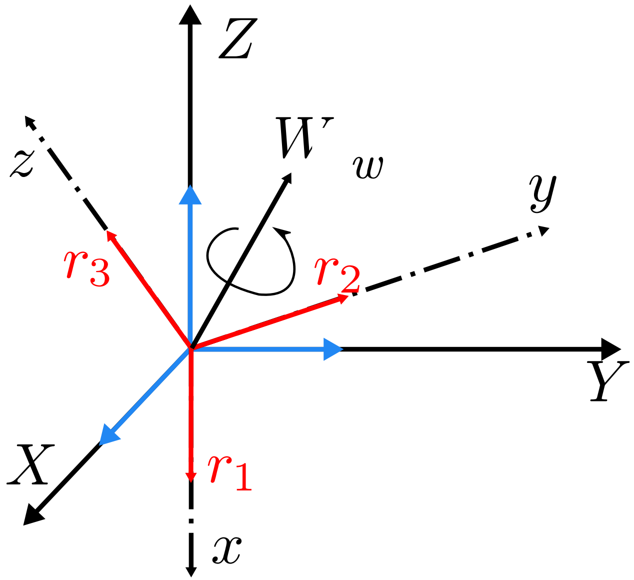
Rotation

Rodrigues' formula for a Rotation Matrix

w is the rotation axis and t is the rotation angle

$$\boxed{R(t) = e^{\hat{w}t}}$$

$$\boxed{e^{\hat{w}t} = I + \hat{w}\sin(t) + \hat{w}^2(1 - \cos(t))}$$



Consider w a rotation axis with unit norm, and t the rotation angle.

Consecutive rotations around the same axis are commutative

$$e^{\hat{w}t_1}e^{\hat{w}t_2} = e^{\hat{w}t_2}e^{\hat{w}t_1} = e^{\hat{w}(t_1+t_2)}, \quad \forall t_1, t_2 \in \mathbb{R}$$

Rotation

Given a rotation matrix $R \neq I$, the rotation axis and rotation angle can be recovered as follows.

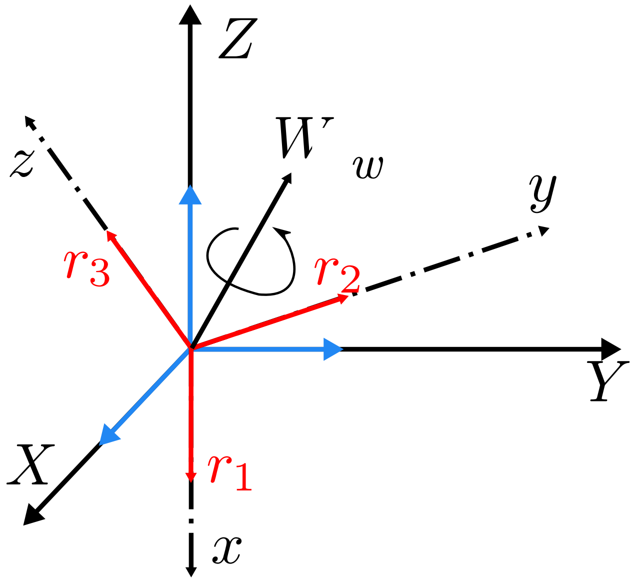
Theorem 2.1 (Surjectivity of the exponential map onto $SO(3)$). *For any $R \in SO(3)$, there exists a (not necessarily unique) $\omega \in \mathbb{R}^3, \|\omega\| = 1$ and $t \in \mathbb{R}$ such that $R = e^{\hat{\omega}t}$.*

Proof. The proof of this theorem is by construction: if the rotation matrix R is given as:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix},$$

the corresponding t and ω are given by:

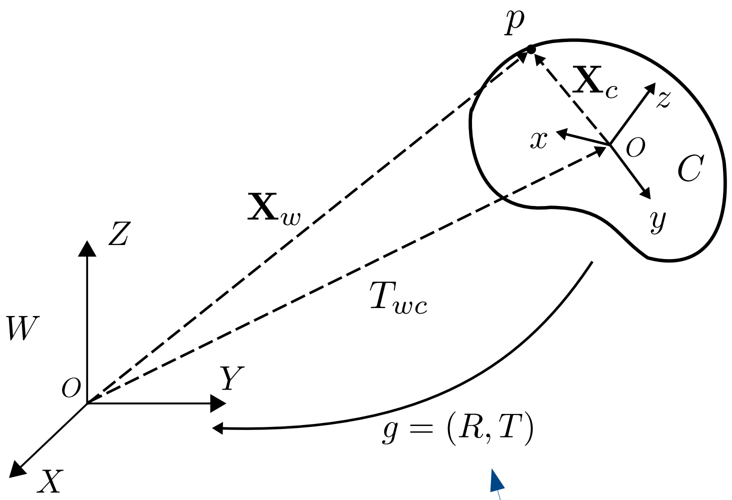
$$\boxed{t = \cos^{-1}\left(\frac{\operatorname{trace}(R)-1}{2}\right), \quad \omega = \frac{1}{2\sin(t)} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}.}$$



Rotation and Translation

Consider:

- Fixed world frame (W)
- Moving frame (C) attached to the object
- Point p on the body



The point coordinates $\boldsymbol{X}_w$ relative to the world frame:

$$\mathbf{X}_w = R_{wc}\mathbf{X}_c + T_{wc}$$

Representation of $\mathbf{X}_c$ according to the world frame orientation

Rigid-body motion

Homogenous Coordinates

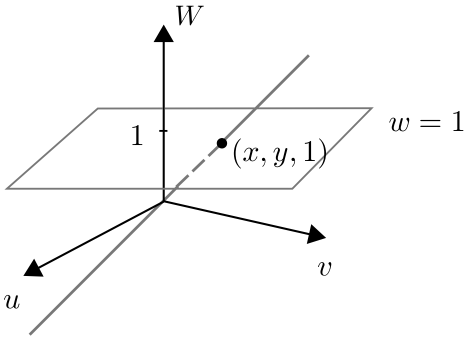
Homogeneous Coordinates

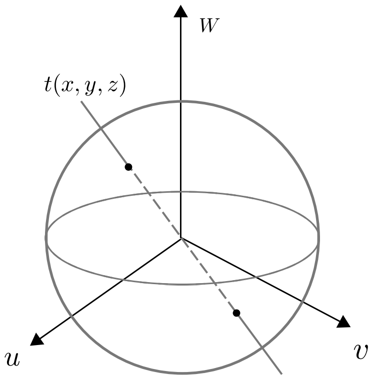
Homogeneous Coordinates for 2D Euclidean Space:

- (wx, wy, w) represent the same point for different w
- $(0,0,0)$ does not exist
- if the last coordinate is 0 (zero) $\rightarrow$ point at infinity or ideal point (direction)
- (wx, wy, w) divided by w gives $(x, y, 1)$ $\rightarrow$ euclidean coordinates (x, y)

Two interpretations:

Projective Plane $\rightarrow$ w=1





Unit Sphere

Spherical Model $\rightarrow$ norm (wx, wy, w) =1

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- (0,0,0,0) does not exist
- if the last coordinate is 0 (zero) $\rightarrow$ point at infinity or ideal point (direction)

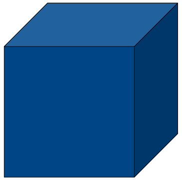
$$\bar{v} = \begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{X}(q) \\ 1 \end{bmatrix} - \begin{bmatrix} \mathbf{X}(p) \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix} \in \mathbb{R}^4$$

- (wx, wy, wz, w) divided by w gives $(x, y, z, 1) \rightarrow$ euclidean coordinates (x, y, z)

Good News:
Homogeneous coordinates allow performing point transformations as multiplication of matrices.

Translation

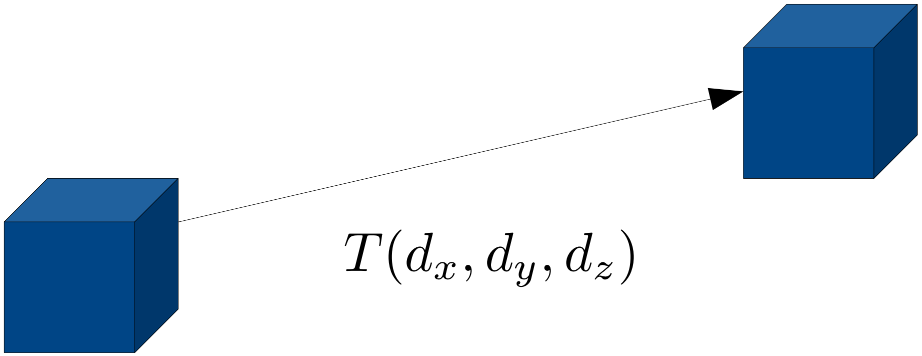
$$\mathbf{P}' = \mathbf{P} + T$$



Translation

$$\mathbf{P}' = \mathbf{P} + T$$

$$x' = x + d_x$$
$$y' = y + d_y$$
$$z' = z + d_z$$



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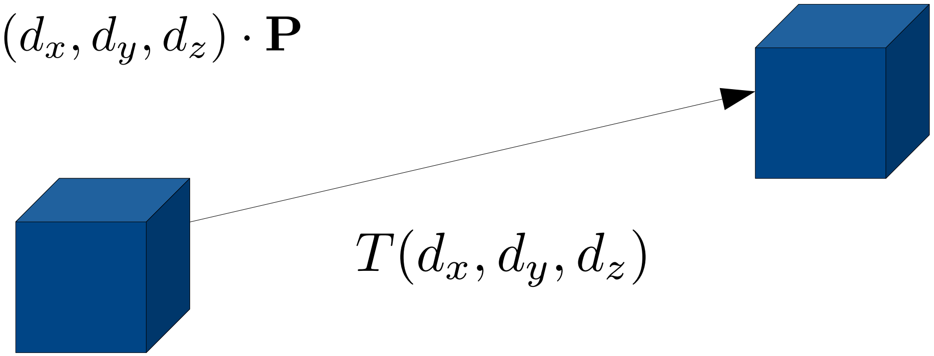
$$y' = y + d_y$$

$$z' = z + d_z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

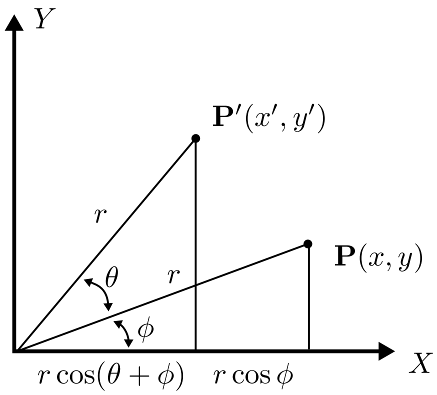
$$\mathbf{P}' = T(d_x, d_y, d_z) \cdot \mathbf{P}$$



Rotation

To facilitate understanding, consider a 2D rotation:

$$x_p = r\cos(\phi)$$
$$y_p = r\sin(\phi)$$



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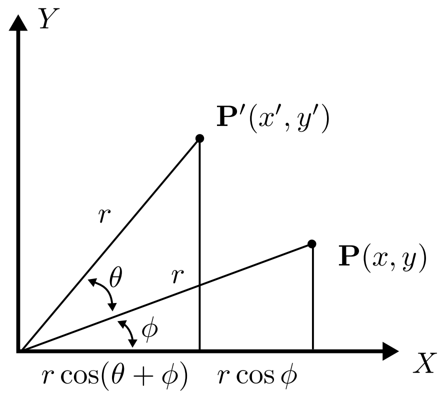

$$x'_p = r\cos(\phi+\theta) = \underbrace{r\cos(\phi)\cos(\theta) - r\sin(\phi)\sin(\theta)}_{x_p\cos(\theta) - y_p\sin(\theta)}$$

$$y'_p = r\sin(\phi+\theta) = \underbrace{r\sin(\phi)\cos(\theta) + r\sin(\theta)\cos(\phi)}_{x_p\sin(\theta) + y_p\cos(\theta)}$$

As a matrix operation: $\mathbf{P}' = R\mathbf{P}$ $\quad R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

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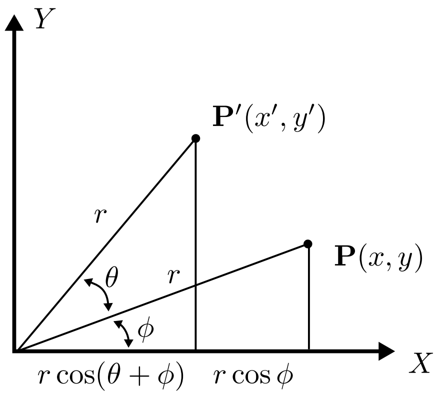


$$x'_p = r\cos(\phi + \theta) = \underline{r\cos(\phi)\cos(\theta) - r\sin(\phi)\sin(\theta)}$$

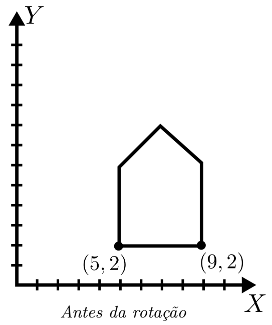
$$x_p\cos(\theta) - y_p\sin(\theta)$$

$$y'_p = r\sin(\phi + \theta) = \underline{r\sin(\phi)\cos(\theta) + r\sin(\theta)\cos(\phi)}$$

$$x_p\sin(\theta) + y_p\cos(\theta)$$

As a matrix operation: $\mathbf{P}' = R\mathbf{P}$ $\quad R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

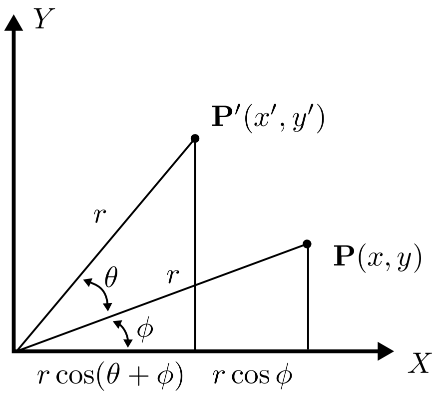
Important: Rotation happens with respect to the origin



Rotation

To facilitate understanding, consider a 2D rotation:

$$x_p = r\cos(\phi)$$
$$y_p = r\sin(\phi)$$

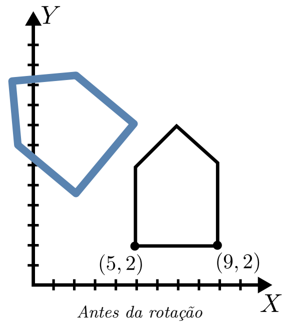


$$x'_p = r\cos(\phi+\theta) = \underbrace{r\cos(\phi)\cos(\theta) - r\sin(\phi)\sin(\theta)}_{x_p\cos(\theta) - y_p\sin(\theta)}$$

$$y'_p = r\sin(\phi+\theta) = \underbrace{r\sin(\phi)\cos(\theta) + r\sin(\theta)\cos(\phi)}_{x_p\sin(\theta) + y_p\cos(\theta)}$$

As a matrix operation: $\mathbf{P}' = R\mathbf{P}$ $\quad R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Important: Rotation happens with respect to the origin



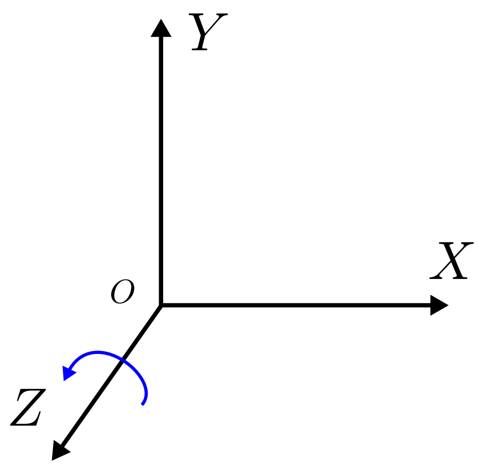
Rotation

Now in 3D:

Around Z-axis

$$\mathbf{P}' = R_z(\theta)\mathbf{P}$$

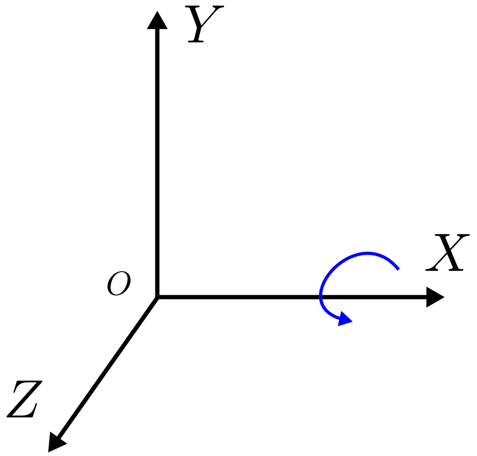
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Around X-axis

$$\mathbf{P}' = R_x(\theta)\mathbf{P}$$

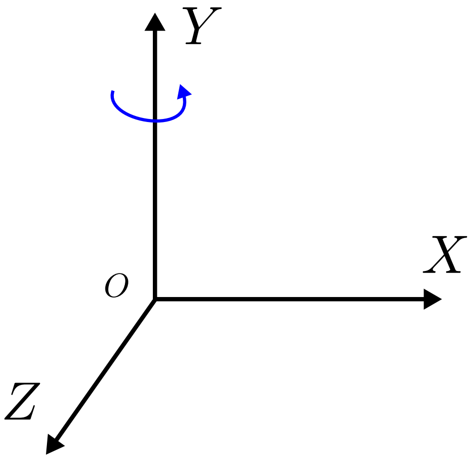
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Around Y-axis

$$\mathbf{P}' = R_y(\theta)\mathbf{P}$$

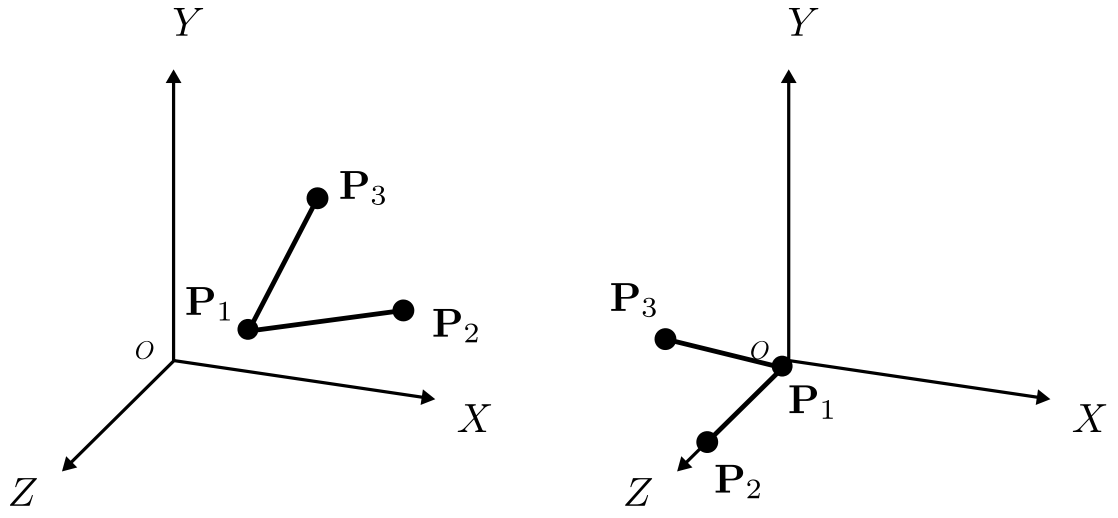
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Composition of transformations

Transform the segments $\boldsymbol{P}_1\boldsymbol{P}_2$ and $\boldsymbol{P}_1\boldsymbol{P}_3$ from the initial position (a) to the final position (b).

- $\boldsymbol{P}_1$ must stay at the origin
- $\boldsymbol{P}_1\boldsymbol{P}_2$ must lie on the Z-axis
- $\boldsymbol{P}_3$ must be on the positive YZ-plane



(a) Initial position

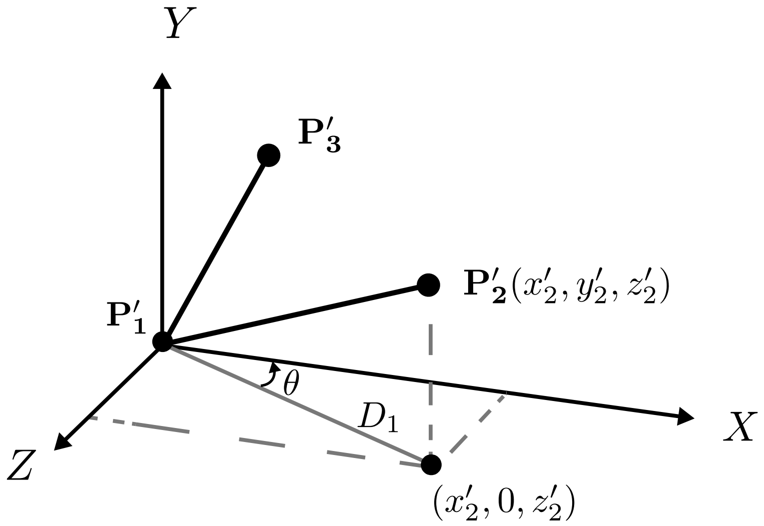
(b) Final position

First step: Translation of $\boldsymbol{P}_1 = (x_1, y_1, z_1)$ to the origin

$$T(-x_1, -y_1, -z_1) = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

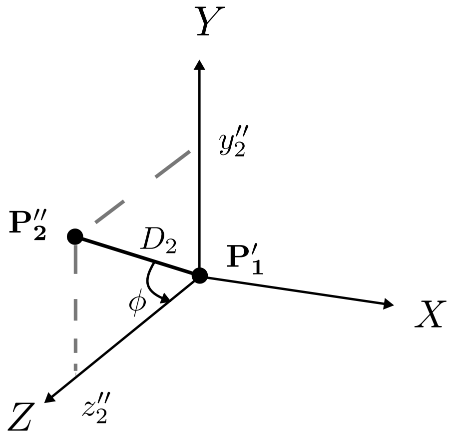
Second step: Rotation of $\boldsymbol{P}_1\boldsymbol{P}_2$ around the Y-axis, taking $\boldsymbol{P}_1\boldsymbol{P}_2$ to the YZ-plane

$$R_y(-(90 - \theta)) = R_y(\theta - 90)$$



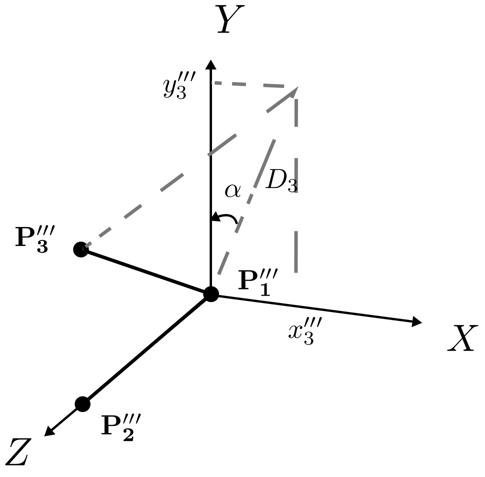
Third step: Rotation of $\boldsymbol{P}_1''\boldsymbol{P}_2''$ around the X-axis, placing $\boldsymbol{P}_1''\boldsymbol{P}_2''$ onto the X-axis.

$$R_x(\phi)$$



Fourth step: Rotation of $\boldsymbol{P}_1'''\boldsymbol{P}_3'''$ around the Z-axis, placing $\boldsymbol{P}_1'''\boldsymbol{P}_3'''$ onto the YZ-plane.

$$R_z(\alpha)$$



$$\boxed{\begin{gathered} M = R_z(\alpha) \cdot R_x(\phi) \cdot R_y(\theta - 90^\circ) \cdot T(-x_1, -y_1, -z_1) = R \cdot T \\ \mathbf{P_{final}} = M\mathbf{P_{initial}} \end{gathered}}$$

General form:

$$\bar{g} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$$

PS: Because vectors have the last homogeneous coordinate equal to zero, they are only influenced by rotation $\rightarrow v\ (x,y,z,0)$

$$\bar{g}_2\bar{g}_1 = \begin{bmatrix} R_2 & T_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_1 & T_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_2R_1 & R_2T_1 + T_2 \\ 0 & 1 \end{bmatrix}$$

$$\bar{g}^{-1} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T T \\ 0 & 1 \end{bmatrix}$$

Object Transformation x Changing Reference Frame

Transforming points of an object with respect to a frame **is different** from **changing point coordinates between two different frames.**

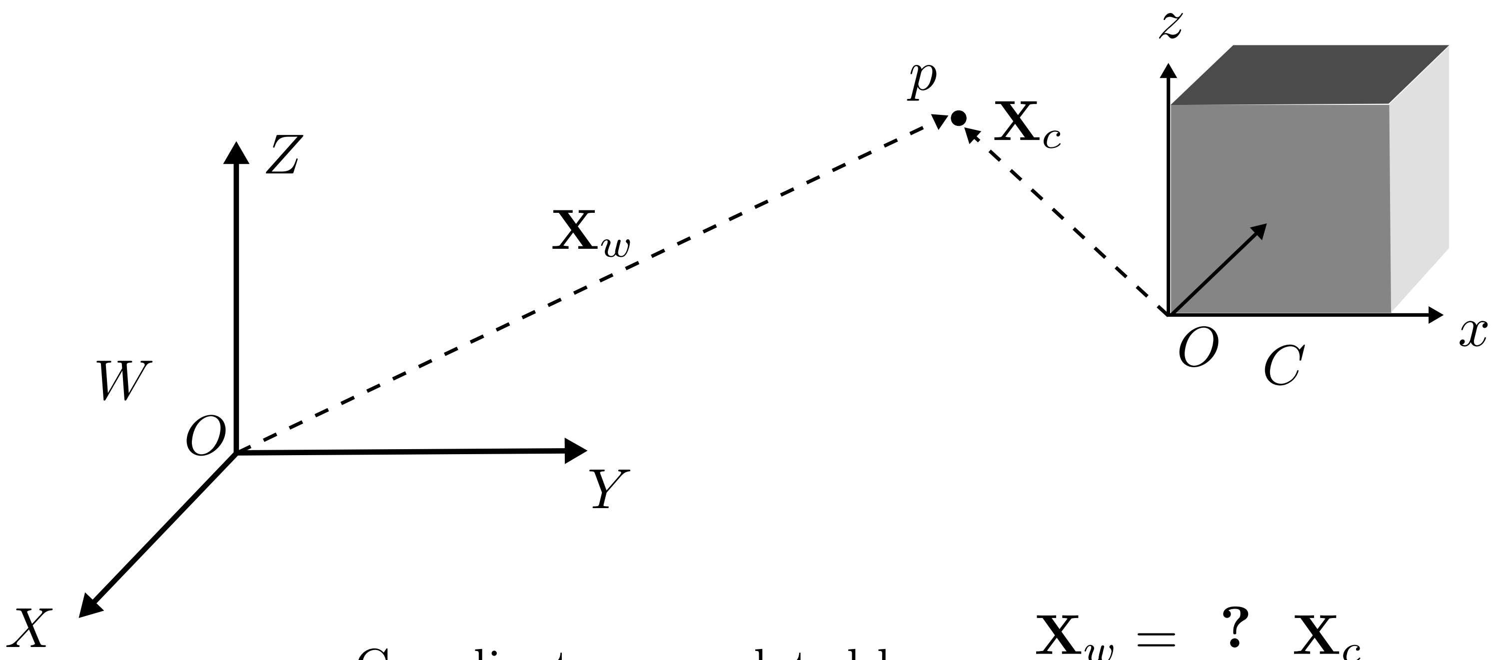
Transforming points with respect to a frame:

$\boldsymbol{P}_f = \overleftarrow{M_n \ldots\ldots M_3 M_2 M_1} \boldsymbol{P}_0$ where M_i are the successive transformations performed by the object.

Changing Reference Frame:

$\boldsymbol{P}_C = \overrightarrow{M_1 M_2 M_3 \ldots M_n} \boldsymbol{P}_W$ where M_i are the successive transformations performed by the frame C until it overlaps the frame W.

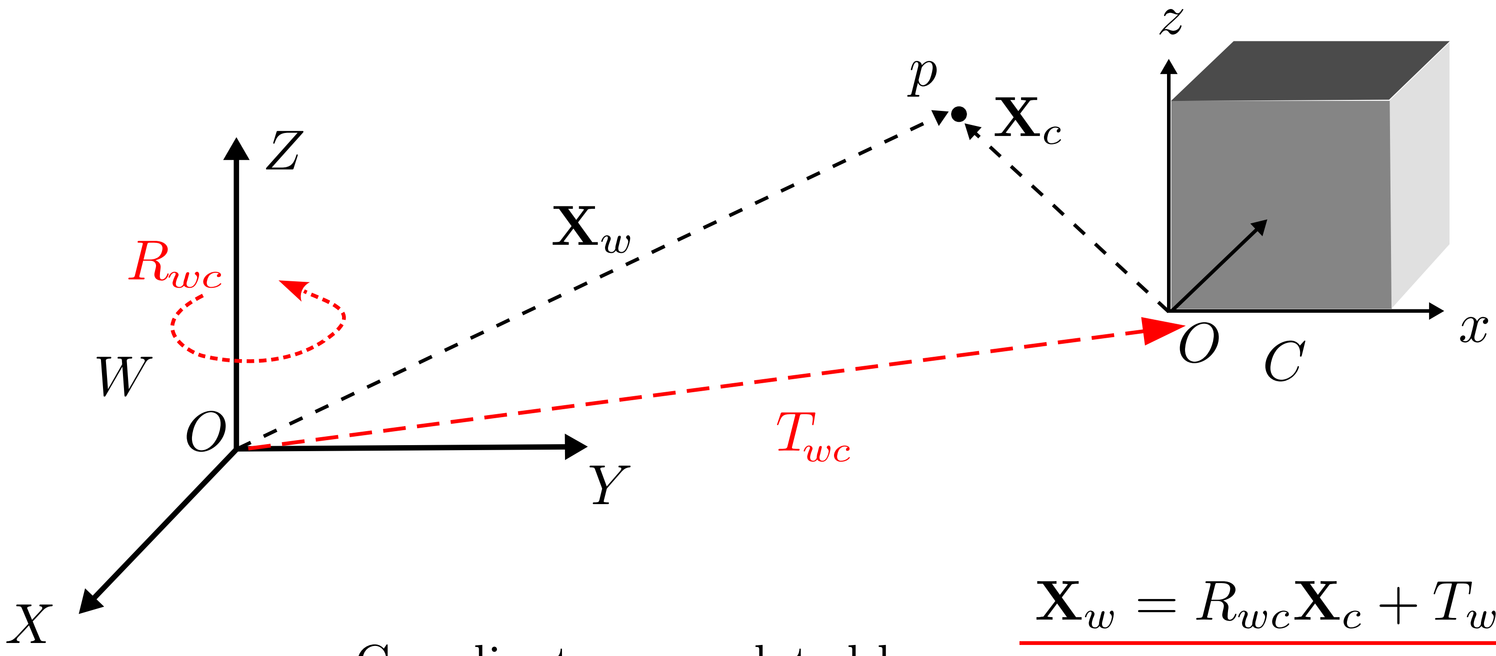
Rigid-body Motion



Coordinates are related by:

$$\mathbf{X}_w = \textbf{?} \; \mathbf{X}_c$$
$$\mathbf{X}_c = \textbf{?} \; \mathbf{X}_w$$

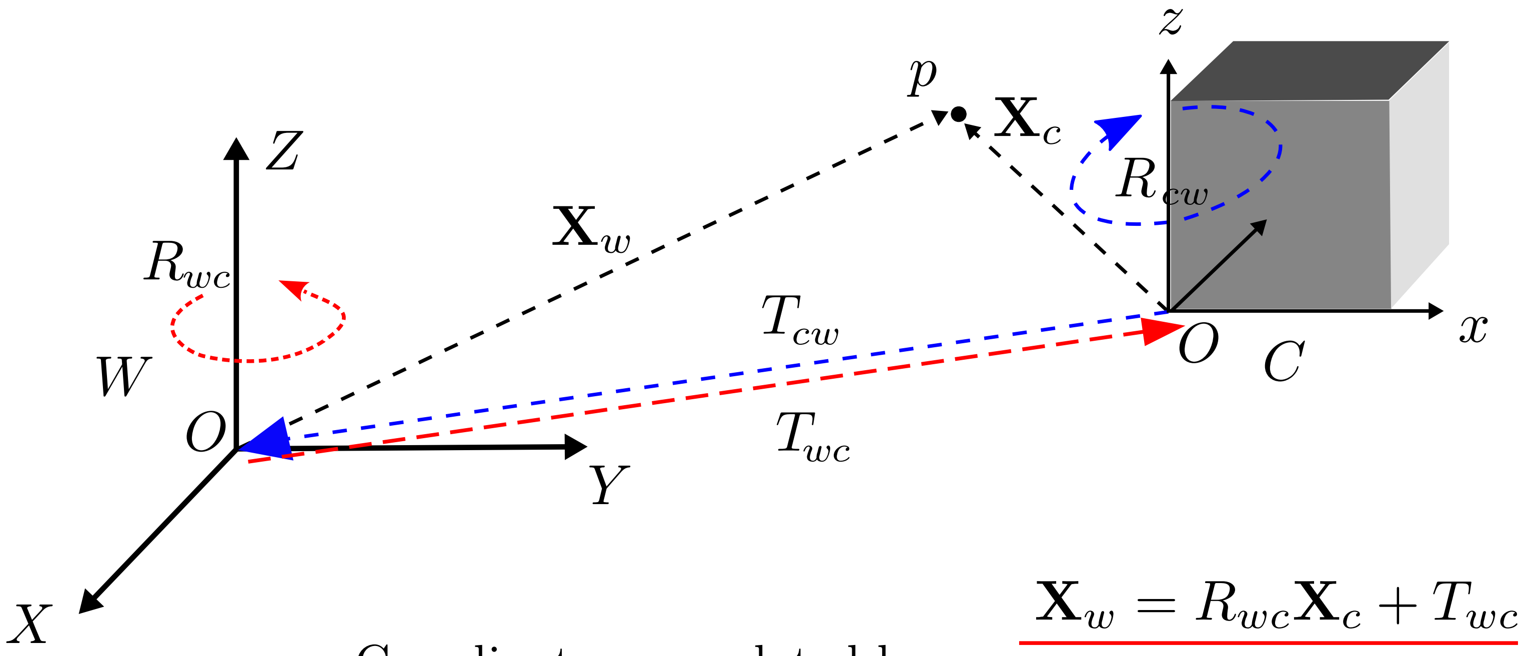
Rigid-body Motion



Coordinates are related by:

$$\mathbf{X}_w = R_{wc}\mathbf{X}_c + T_{wc}$$

Rigid-body Motion

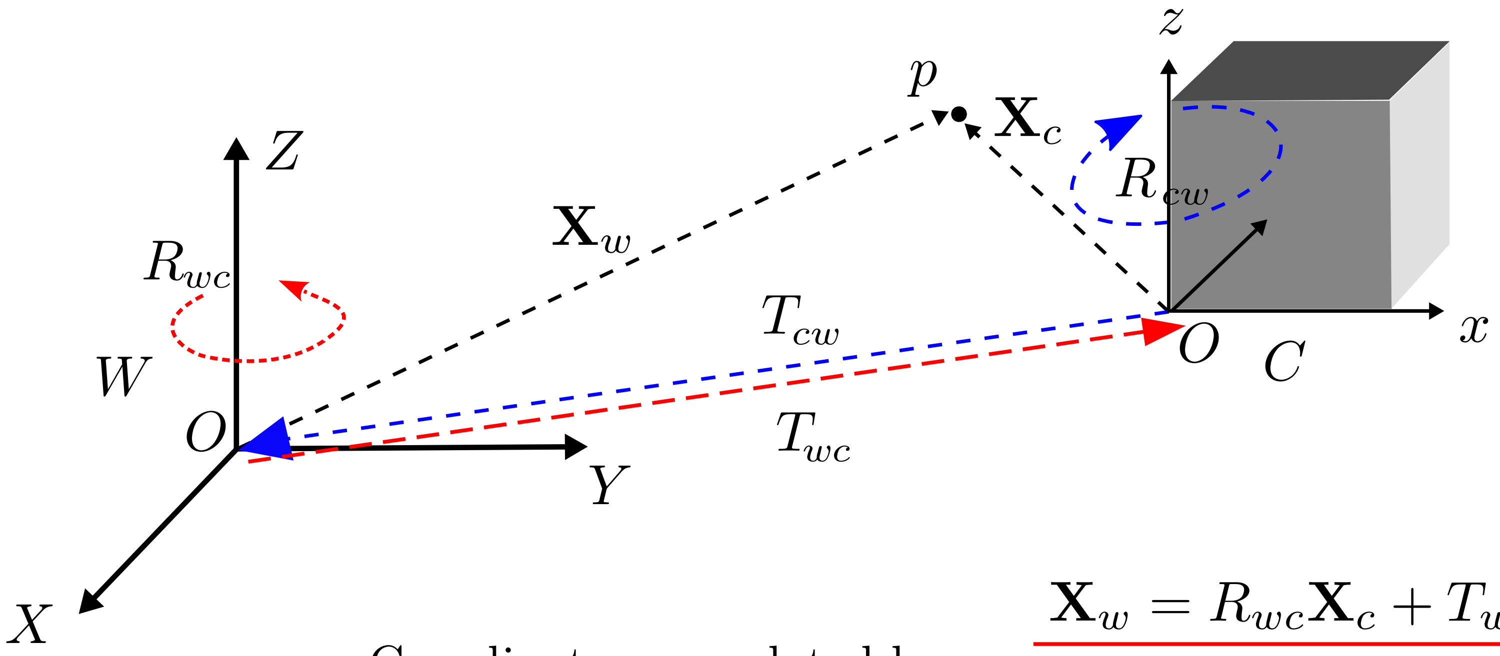


Coordinates are related by:

$$\mathbf{X}_w = R_{wc}\mathbf{X}_c + T_{wc}$$

$$\mathbf{X}_c = R_{cw}\mathbf{X}_w + T_{cw}$$

Rigid-body Motion



Coordinates are related by:

$$\mathbf{X}_w = R_{wc}\mathbf{X}_c + T_{wc}$$

$$\mathbf{X}_c = R_{cw}\mathbf{X}_w + T_{cw}$$

Velocities are related by:

$$\dot{\mathbf{X}}_c = \hat{\omega}\mathbf{X}_c + v$$

3D Coordinates: $\mathbf{X}_c = R\mathbf{X}_w + T$

Velocities: $\dot{\mathbf{X}}_c = \hat{w}\mathbf{X}_c + v$

Homogeneous Coordinates: $\mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \to \mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \in \mathbb{R}^4,$

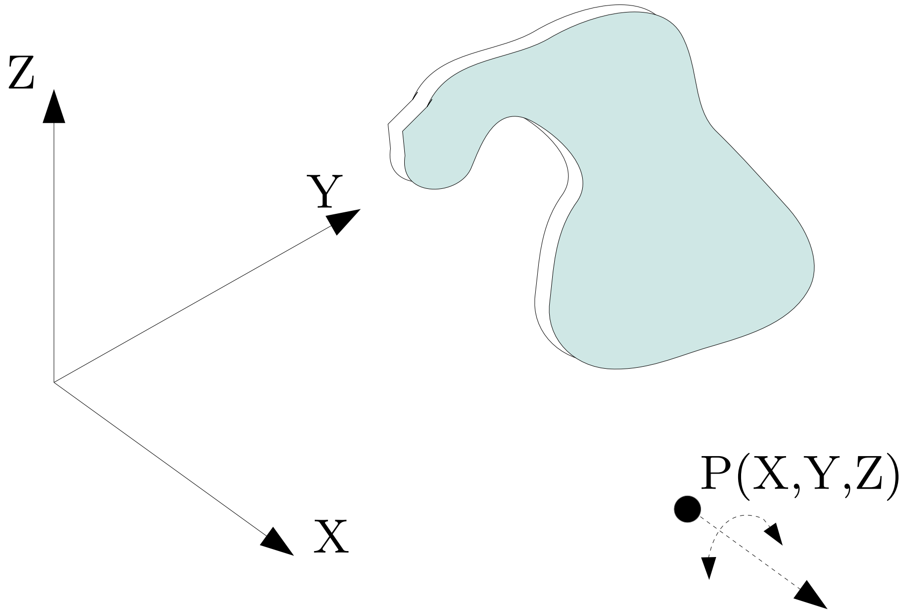
$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} \qquad \begin{bmatrix} \dot{X}_c \\ \dot{Y}_c \\ \dot{Z}_c \\ 1 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix}$$

$$\mathbf{X}_c(t) = g_{cw}(t)\mathbf{X}_w \qquad \dot{\mathbf{X}}_c(t) = \hat{V}^c_{cw}(t)\mathbf{X}_c(t)$$

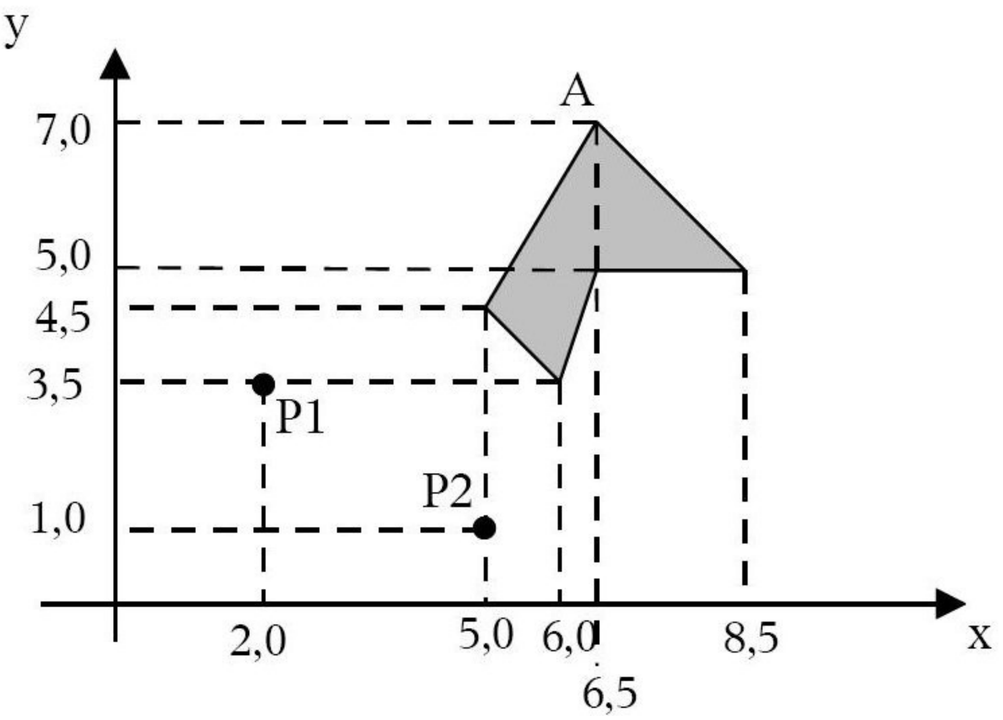
$\hat{V}^c_{cw}$ Is the world velocity with respect to the moving camera.

Exercises

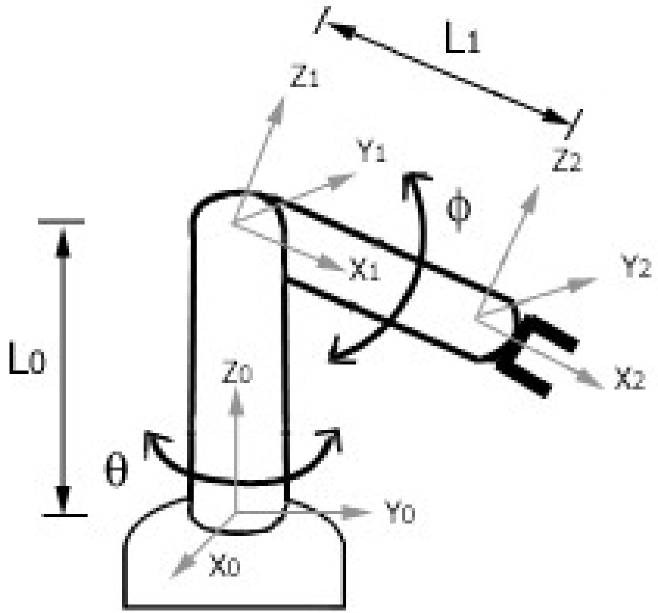
1- What should be done to rotate a 3D object around the X-axis from a point that is not the origin? Write down the transformation matrices in the order they should be applied.



2- What do you need to do to move the object so point A stays at point P1, and then rotates by 45 ° with respect to point P2? Define the transformation arrays and the order in which they should be applied.



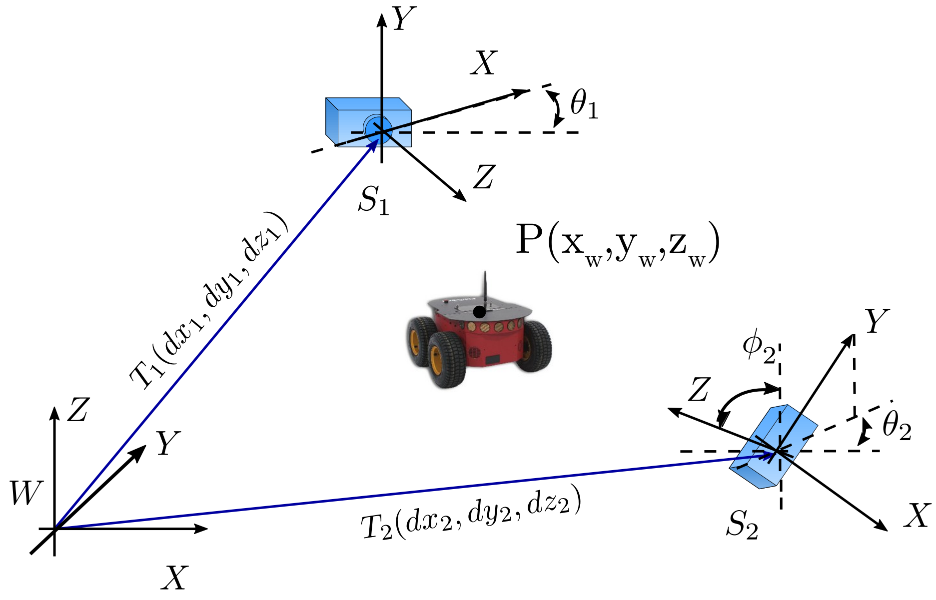
3- Homogeneous coordinates are commonly used in robotics to specify changes between the coordinate frames of each robot joint. This is easily achieved by composing transformations (multiplication of transformation matrices). The figure brings a typical case of manipulators. Specify the transformation matrices required to represent the coordinates of a point measured in frame S2 in the frame S0. (Hint: think how points in S2 are represented in S1 and how points in S1 are represented in S0).



4- A smart space has 2 cameras positioned at different points in a room. To control a robot in this space, it is necessary to obtain the robot coordinates in the world frame. Assuming that the reference point of the robot has coordinates P1 (x1, y1, z1), in the Camera1 frame (S1), and coordinates P2 (x2, y2, z2) in the Camera2 frame, determine how this point is represented in the world frame, considering the position and orientation shown in the figure. Determine the transformation from S1 to the world frame and also from S2 to the world frame.

Besides that, determine the transformation between references S1 and S2.

PS: T1 and T2 are measured in the world frame.



Credits

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  Springer, ISBN 0387008934
- James D. Foley, Andries van Dam, Steven K. Feiner, John F. Hughes.
  Computer Graphics: Principles and Practice. Second Edition in C.
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