Solutions to Practice Problems

Chapter 1

1.

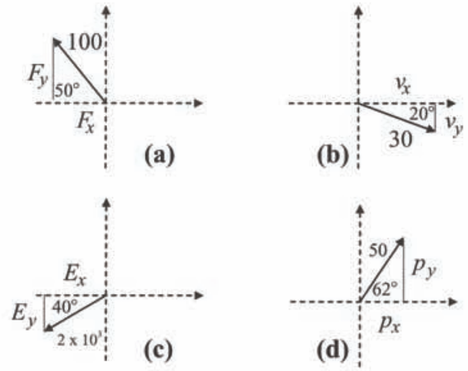

Figure 5.1

Diagrams for the four situations are shown.

a. $F_x = -100\cos 50 = -64.27$
$F_y = 100\sin 50 = 76.60$

b. $v_x = 30\cos 20 = 28.19$
$v_y = -30\sin 20 = -10.26$

c. $E_x = -2 \times 10^3 \cos 40 = -1.53 \times 10^3$
$E_y = -2 \times 10^3 \sin 40 = -1.29 \times 10^3$

d. $p_x = 50\cos 62 = 23.47$
$p_y = 50\sin 62 = 44.15$

2.

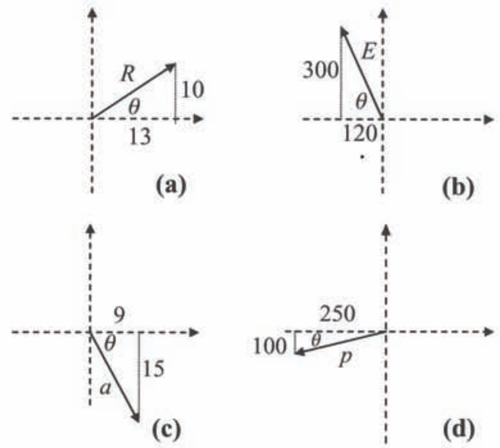

Figure 5.2

Diagrams for the four situations are shown in Figure 5.2. Note that the depicted angle θ is the answer for the angle only in **2a.** In the other cases the actual angle with the $+x$-axis can be calculated from θ by simply adding to or subtracting from 90° or 180°.

a. $R = \sqrt{(10)^2 + (13^2)} = 16.40$

$$\theta = \tan^{-1}\frac{10}{13} = 37.57°$$

b. $E = \sqrt{(120)^2 + (300^2)} = 323$

$$\theta = \tan^{-1}\frac{300}{120} = 68.20°$$

c. $a = \sqrt{(15)^2 + (9^2)} = 17.49$

$$\theta = \tan^{-1}\frac{15}{9} = 59.04°$$

d. $p = \sqrt{(100)^2 + (250^2)} = 269$

$$\theta = \tan^{-1}\frac{100}{250} = 21.80°$$

3. The coordinates of a point are the components of the position vector.

$$x = 20\cos 40 = 15.32$$
$$y = -20\sin 40 = -12.86$$

4. To make a unit vector out of any vector, multiply the vector by the inverse of the magnitude of the vector.

a. $B = \sqrt{(5)^2 + (-8)^2} = 9.43$

$\hat{B} = \dfrac{1}{9.43}(5\hat{i} - 8\hat{j}) = 0.53\hat{i} - 0.85\hat{j}$

b. $r = \sqrt{(-4)^2 + (-8)^2 + (6)^2} = 10.77$

$\hat{r} = \dfrac{1}{10.77}(-4\hat{i} - 8\hat{j} + 6\hat{k})$

$= -0.37\hat{i} - 0.74\hat{j} + 0.56\hat{k}$

c. $V = \sqrt{(14)^2 + (-9)^2} = 16.64$

$\hat{V} = \dfrac{1}{16.64}(14\hat{j} - 9\hat{k}) = 0.84\hat{j} - 0.54\hat{k}$

d. $F = \sqrt{(9)^2 + (12)^2 + (-3)^2} = 15.30$

$\hat{F} = \dfrac{1}{15.30}(9\hat{i} + 12\hat{j} - 3\hat{k})$

$= 0.59\hat{i} + 0.78\hat{j} - 0.20\hat{k}$

5. **a.** $8\hat{i} + 2\hat{j} - \hat{k}$ **b.** $5\hat{i} + 6\hat{j} - 4\hat{k}$
c. $2\hat{i} - 2\hat{j} - 4\hat{k}$ **d.** $3\hat{i} - 6\hat{j} - 14\hat{k}$
e. $16\hat{i} + 18\hat{j} - 23\hat{k}$

6. You need to find the components of the forces and add them to find the components of the net force. Then use these components to find the magnitude and direction of the net force.

$$F_{1x} = 30\cos 40 = 22.98$$
$$F_{1y} = 30\sin 40 = 19.28$$
$$F_{2x} = -60\cos 25 = -54.38$$
$$F_{2y} = 60\sin 25 = 25.36$$
$$netF_x = 22.98 - 54.38 = -31.40$$
$$netF_y = 19.28 + 25.36 = 44.64$$

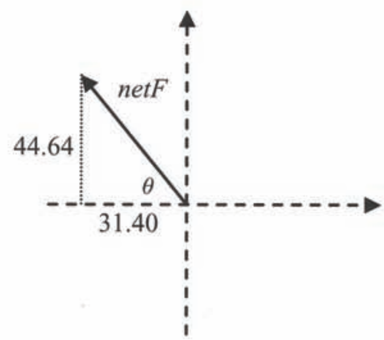

Figure 5.3

$$netF = \sqrt{(44.64)^2 + (31.40)^2} = 54.58$$
$$\theta = \tan^{-1}\frac{44.64}{31.40} = 54.88°$$

The angle with the $+x$-axis is then $180 - 54.88 = 125.12°$.

7. **a.** The position vector points from the origin to the object, so in this coordinate system the length of the position vector is the length of the radius. Then find the components when the angle is $\theta = \omega t$.

$$\vec{r} = R\cos\omega t\,\hat{i} + R\sin\omega t\,\hat{j}$$

b. With P as the origin, the position vector is just the sum of the constant vector $R\,\hat{j}$ and the old position vector, as can be seen from Figure 5.4.

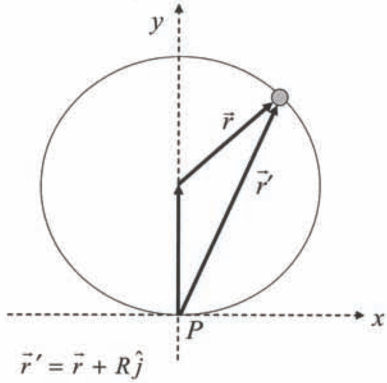

Figure 5.4

Thus you can write

$$\vec{r}\,' = R\cos\omega t\;\hat{i} + R(1 + \sin\omega t)\hat{j}$$

8. Since $\vec{S} = \vec{T} - \vec{R}$ and you are given the components of $\vec{R}$, just find the components of $\vec{T}$ and perform the subtraction.

$$T_x = 15\cos 60 = 7.50$$
$$T_y = 15\sin 60 = 12.99$$

Then

$$S_x = 7.50 - 4 = 3.4 \quad S_y = 12.99 - 10 = 2.99$$

9. The equilibrium condition tells you that $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0$ or $\vec{F}_3 = -(\vec{F}_1 + \vec{F}_2)$. Then you have

$$F_{3x} = -8 + 15 = 7 \qquad F_{3y} = 12 + 10 = 22$$

10. a. $\vec{F}\cdot\vec{v} = -3\cdot 6 + 5\cdot 14 = 52 \qquad F = \sqrt{34}$

$$v = \sqrt{232}$$
$$\vec{F}\cdot\vec{v} = Fv\cos\theta$$
$$52 = \sqrt{34\cdot 232}\cos\theta \Rightarrow \theta = 54.16°$$

b. $\vec{B}\cdot\vec{l} = 6\cdot 2 + 2\cdot(-6) = 0$ These two vectors are perpendicular.

c. $\vec{f}\cdot\Delta\vec{r} = (-10)\cdot 6 + (-8)\cdot(-4) + 3\cdot(-2)$
$= -34$

$$f = \sqrt{173} \qquad \Delta r = \sqrt{56}$$
$$\vec{f}\cdot\Delta\vec{r} = f\Delta r\cos\theta$$
$$-34 = \sqrt{173\cdot 56}\cos\theta \Rightarrow \theta = 110.20°$$

11. Find the displacement vector and take the dot product with the force.

$$\Delta\vec{r} = \vec{r}(2) - \vec{r}(1) = \{1.64\hat{i} + 10\hat{j}\} - \{0.90\hat{i} + 2\hat{j}\}$$
$$= 0.74\hat{i} + 8\hat{j}$$
$$\vec{F}\cdot\Delta\vec{r} = 10\cdot 0.74 + (-4)\cdot 8 = -24.6$$

12. Find the velocity at $t = 1$, then take the dot product with the force.

$$\vec{v}(1) = 5\hat{i} + 0.33\hat{j}$$
$$\vec{F}\cdot\vec{v} = 20\cdot 5 + 35\cdot 0.33 = 111.55$$

13. In each case you can use the distributive law for multiplication and the basic rules for the unit vectors that follow from the definition of the cross product.

$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0 \qquad \hat{i}\times\hat{j} = \hat{k}$$
$$\hat{j}\times\hat{k} = \hat{i} \qquad \hat{k}\times\hat{i} = \hat{j}$$

Alternatively, you could use the formula quoted in the text. You can get the angle between the vectors from the formula that relates the magnitudes and the angle. Use the Pythagorean theorem to calculate the magnitudes of the vectors.

$$C = AB\sin\theta$$

a. $(3\hat{i}-2\hat{j})\times(6\hat{i}+\hat{k}) = 18\hat{i}\times\hat{i}+3\hat{i}\times\hat{k}$
$- 12\hat{j}\times\hat{i}-2\hat{j}\times\hat{k} = 0-3\hat{j}+12\hat{k}-2\hat{i}$
$\sqrt{157} = \sqrt{13\cdot 37}\sin\theta \Rightarrow \theta = 34.84°$

b. $(10\hat{i}+7\hat{k})\times(-4\hat{i}-8\hat{j}) = -40\hat{i}\times\hat{i}$
$-80\hat{i}\times\hat{j}-28\hat{k}\times\hat{i}-56\hat{k}\times\hat{j}$
$= 0-80\hat{k}-28\hat{j}+56\hat{i}$
$\sqrt{10320} = \sqrt{149\cdot 80}\sin\theta \Rightarrow \theta = 68.50°$

c. $\vec{E}\times\vec{B} = 17\hat{i}-10\hat{j}+36\hat{k}$
$\sqrt{1685} = \sqrt{49\cdot 41}\sin\theta \Rightarrow \theta = 66.32°$

14. a. $(4\hat{i}+6\hat{j})\times(10\hat{i}+10\hat{j}) = 40\hat{i}\times\hat{i}+40\hat{i}\times\hat{j}$
$+60\hat{j}\times\hat{i}+60\hat{j}\times\hat{j}$
$= 0+40\hat{k}-60\hat{k}+0 = -20\hat{k}$

b. The vector that locates the point of application of the force in the new coordinate system is

$$\vec{r}_2 = 2\hat{i}-6\hat{j}$$

The unit vectors are the same in both coordinate systems since the new x- and y-axes are parallel to the old axes.

$(2\hat{i}-6\hat{j})\times(10\hat{i}+10\hat{j}) = 20\hat{i}\times\hat{i}+$
$20\hat{i}\times\hat{j}-60\hat{j}\times\hat{i}-60\hat{j}\times\hat{j}$
$= 0+20\hat{k}+60\hat{k}+0 = 80\hat{k}$

Thus $\vec{\tau}_2$ is in the opposite direction to $\vec{\tau}_1$ and has 4 times the magnitude.

c. If the origin is placed at the point of application of the force, the magnitude of the position vector will be 0, so the torque will also be 0.

Chapter 2

1. a. $\dfrac{df}{dx} = 12x^3-4x$

b. $\dfrac{dg}{dt} = \dfrac{-20t}{(3+2t^2)^2}$

c. $\dfrac{dh}{dx} = 2\sin 3x + 6x\cos 3x$

d. $\dfrac{df}{dy} = (2y-4y^2)e^{-4y}$

e. $\dfrac{dq}{dt} = -\dfrac{q_0}{RC}e^{-\frac{t}{RC}}$

f. $\dfrac{dV}{dt} = 8t\ln(t^2+1)+\dfrac{8t^3}{t^2+1}$

g. $\dfrac{dr}{ds} = -5\sin(\sin 5s)\cos 5s$

h. $\dfrac{dF}{dx} = \dfrac{-8\sin 2x}{(1-x)^2}+\dfrac{8\cos 2x}{(1-x)^3}$

i. $\dfrac{dA}{dt} = -24\cos 3t\sin 3t$

j. $\dfrac{dg}{dx} = (6x^2+4x^4-8x)e^{-(1-x^2)}$

2. In each case, find the equation of the tangent line from the first derivative, then evaluate at the given point.

a. $\dfrac{dU}{dx} = 9x^2-4; \dfrac{dU(2)}{dx} = 32$

b. $\dfrac{dx}{dt} = -2\pi\sin\left(\dfrac{2\pi}{3}t-\dfrac{\pi}{4}\right); \dfrac{dx(6)}{dt} = -1.23$

c. $\dfrac{dv}{dt} = 8e^{-\frac{t}{5}}; \dfrac{dv(10)}{dt} = 1.08$

d. $\dfrac{df}{dv} = \left(-\dfrac{4}{v^{\frac{3}{2}}}-\dfrac{8v}{5}\right)e^{-\frac{v^2}{10}}; \dfrac{df(2)}{dv} = -3.09$

3. In each case find the value of the function at the given point. Then find the slope of the tangent line at the given point by evaluating the first derivative. Then use the point-slope formula for a straight line to find the equation of the line with this slope that goes through the given point.

a. $y(2) = 23; \dfrac{dy}{dt} = 6t^2-2t+5; \dfrac{dy(2)}{dt} = 25;$
$y_1(t) = 23+25(t-2)$

b. $U(1) = 1.10; \dfrac{dU}{dx} = 2x\ln 3x + x; \dfrac{dU(1)}{dx} = 3.20;$
$U_1(x) = 1.10+3.20(x-1)$

c. $P(3) = 0.78; \dfrac{dP}{dx} = -\dfrac{x-2}{2}e^{-\frac{(x-2)^2}{4}};$
$\dfrac{dP(3)}{dx} = -0.39;$
$P_1(x) = 0.78-0.39(x-3)$

d. $V(4) = 7.61; \dfrac{dV}{dt} = 2.51\cos\dfrac{\pi}{10}t; \dfrac{dV}{dt} = 0.78;$
$V_1(t) = 7.61 + 0.78(t-4)$

4. In each case, use the Taylor series up to the second-order term.

a. $\dfrac{df}{dt} = 4t^3 - 6t; \dfrac{d^2f}{dt^2} = 12t^2 - 6$

$f(1) = 0 \quad \dfrac{df(1)}{dt} = -2 \quad \dfrac{d^2f(1)}{dt^2} = 6 \Rightarrow$

$f_2(t) = -2(t-1) + 3(t-1)^2$

b. $\dfrac{dI}{dt} = -3\pi\sin\left(\dfrac{3\pi}{5}t + \dfrac{\pi}{5}\right);$

$\dfrac{d^2I}{dt^2} = -\dfrac{9\pi^2}{5}\cos\left(\dfrac{3\pi}{5}t + \dfrac{\pi}{5}\right)$

$I(2) = -1.55 \quad \dfrac{dI(2)}{dt} = 8.96 \quad \dfrac{d^2I(2)}{dt^2} = 5.49 \Rightarrow$

$I_2(t) = -1.55 + 8.96(t-2) + 2.74(t-2)^2$

c. $\dfrac{dE}{dr} = -\dfrac{2}{(r+2)^3}; \dfrac{d^2E}{dr^2} = \dfrac{6}{(r+2)^4}$

$E(0) = 0.25 \quad \dfrac{dE(0)}{dr} = -0.25 \quad \dfrac{d^2E(0)}{dr^2} = 0.38 \Rightarrow$

$E_2(r) = 0.25 - 0.25r + 0.19r^2$

d. $\dfrac{dJ}{dx} = \left(1 - \dfrac{x}{10}\right)e^{-\frac{x}{10}};$

$\dfrac{d^2J}{dx^2} = -\dfrac{1}{10}\left(2 - \dfrac{x}{10}\right)e^{-\frac{x}{10}};$

$J(5) = 3.03 \quad \dfrac{dJ(5)}{dx} = 0.30 \quad \dfrac{d^2J(5)}{dx^2} = -0.091 \Rightarrow$

$J_2(x) = 3.03 + 0.30(x-5) - 0.045(x-5)^2$

5. a. $$E = kq\left[\frac{1}{(x-a)^2} - \frac{1}{(x+a)^2}\right]$$
$$= kq\left[\frac{1}{x^2\left(1-\frac{a}{x}\right)^2} - \frac{1}{x^2\left(1+\frac{a}{x}\right)^2}\right]$$
$$= \frac{kq}{x^2}\left[\frac{1}{\left(1-\frac{a}{x}\right)^2} - \frac{1}{\left(1+\frac{a}{x}\right)^2}\right]$$

b. Using the form for the answer to **5a**, expand each fraction using the binomial expansion, and keep only the first two terms.

$$\left(1 - \frac{a}{x}\right)^{-2} = 1 + 2\frac{a}{x}$$
$$\left(1 + \frac{a}{x}\right)^{-2} = 1 - 2\frac{a}{x}$$

Then

$$E = \frac{kq}{x^2}\left(1 + 2\frac{a}{x} - 1 + 2\frac{a}{x}\right) = 2k\frac{2qa}{x^3}$$

6. a. $$B = k'\frac{iR^2}{(x^2+R^2)^{\frac{3}{2}}} = k'R^2\frac{i}{R^3\left(\frac{x^2}{R^2}+1\right)^{\frac{3}{2}}}$$
$$= \frac{k'}{R}\frac{i}{\left(1+\frac{x^2}{R^2}\right)^{\frac{3}{2}}}$$

b. Since $\dfrac{x^2}{R^2}$ is small, expand the fraction, keeping only the first two terms in the binomial expansion.

$$\left(1 + \frac{x^2}{R^2}\right)^{-\frac{3}{2}} = 1 - \frac{3}{2}\frac{x^2}{R^2}$$

7. Look at the first four terms in the Maclaurin series for $e^{i\theta}$. Then compare that series to the series for the right-hand side of the identity.

$$e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \cdots$$
$$= \left(1 - \frac{\theta^2}{2!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \cdots\right)$$
$$= \cos\theta + i\sin\theta$$

8. The velocity and acceleration will be given by the first and second derivatives, respectively, evaluated at the given time.

a. $v(t) = 9t^2 + 2t \qquad v(1) = 11$
b. $a(t) = 18t + 2 \qquad a(2) = 38$

9. a. $v(t) = \dfrac{dy}{dt} = \dfrac{10\pi}{3}\cos\left(\dfrac{\pi}{3}t + \dfrac{\pi}{6}\right)$
$v(3) = -9.07$

b. $a(t) = \dfrac{d^2y}{dt^2} = -\dfrac{10\pi^2}{9}\sin\left(\dfrac{\pi}{3}t + \dfrac{\pi}{6}\right)$
$a(0) = -5.48$

10. a. $\omega(t) = \dfrac{d\theta}{dt} = \ln(1+t) + \dfrac{t}{1+t} \quad \omega(0) = 0$

b. $\alpha(t) = \dfrac{d^2\theta}{dt^2} = \dfrac{2}{1+t} - \dfrac{t}{(1+t)^2}$
$\alpha(1) = \dfrac{3}{4}$

11. a. The velocity vector is given by the first derivative of the position vector. The magnitude of the velocity is then found from the Pythagorean theorem using the components.

$\vec{v}(t) = -2\hat{i} + (7 - 10t)\hat{j} \quad \vec{v}(2) = -2\hat{i} - 13\hat{j}$
$v = \sqrt{4 + 169} = 13.15$

b. $v_y(t) = 7 - 10t = 0 \Rightarrow t = 0.7$
$x(0.7) = 3 - 2(0.7) = 1.6$

12. a. $\vec{v}(t) = -\omega r \sin\omega t\ \hat{i} + \omega r \cos\omega t\ \hat{j}$

$\vec{r}\cdot\vec{v} =$
$-\omega r^2 \cos\omega t \sin\omega t + \omega r^2 \sin\omega t \cos\omega t = 0$

Since the dot product is 0 for all times, the two vectors are always perpendicular.

b. $\vec{a}(t) = -\omega^2 r\cos\omega t\ \hat{i} - \omega^2 r \sin\omega t\ \hat{j} =$
$-\omega^2(r\cos\omega t\ \hat{i} + r\sin\omega t\ \hat{j})$
$\vec{a}(t) = -\omega^2\vec{r}(t)$

The minus sign indicates that the acceleration is in the opposite direction from the position vector; i.e., directed toward the circle center.

13. The distance between the two objects is the difference between their positions. This needs to be minimized.

$$\Delta x(t) = x_1 - x_2 = t^3 - 3t + 10$$
$$\frac{d}{dt}\Delta x = 3t^2 - 3 = 0 \Rightarrow t = 1$$
$$\Delta x(1) = 8$$

14. You need to maximize the x-coordinate and then use this value of the time to find the y velocity component.

$$\frac{dx}{dt} = (2t - t^2)e^{-t} = 0 \Rightarrow t = 0, 2$$

Direct evaluation into $x(t)$ tells you that $t = 2$ is a relative maximum. If you use the 2nd derivative test, you find

$$\frac{d^2x}{dt^2} = (t^2 - 4t + 2)e^{-t}$$
$$\frac{d^2x(2)}{dt^2} = (t^2 - 4t + 2)e^{-t}$$
$$= -0.27 < 0$$

Then you have

$$v_y(t) = \frac{2t}{1+t^2} \qquad v_y(2) = 0.8$$

15. a. $\dfrac{dU}{dx} = \dfrac{-8}{x^5} + \dfrac{1}{x^2} = 0 \Rightarrow x = 2$

b. $\dfrac{d^2U}{dx^2} = \dfrac{40}{x^6} - \dfrac{2}{x^3}$
$\dfrac{d^2U(2)}{dx^2} = \dfrac{40}{2^6} - \dfrac{2}{2^3} = 0.375 > 0$

c. $U(2) = -0.375$
$U_2(x) = -0.375 + 0.188(x-2)^2$

If you add 0.375 to all energies and define the displacement from the minimum point as $x' = x - 2$, then you have

$$U_2(x') = 0.188x'^2 = \frac{1}{2}(0.375)x'^2$$

Thus, the effective spring constant is 0.375.

16. a. $\frac{dU}{dx} = aU_0 \cos ax = 0 \Rightarrow ax = n\frac{\pi}{2}$
$n = 1, 3, 5\ldots$

It is easy to see by direct substitution into U that $n = 1, 5, 9\ldots$ correspond to maxima, and that $n = 3, 7, 11\ldots$ correspond to minima.

b. $\frac{d^2U}{dx^2} = -a^2U_0 \sin ax. \quad \frac{d^2U\left(\frac{3\pi}{2a}\right)}{dx^2} = a^2U_0$

$$U_2(x) = -U_0 + \frac{1}{2}a^2U_0\left(x - \frac{3\pi}{2a}\right)^2$$

Inspection of the quadratic term tells you that the effective spring constant is a^2U_0.

17. The force is the negative derivative with respect to r.

$$F(r) = -\frac{dU}{dr} = \frac{12A}{r^{13}} - \frac{6B}{r^7}$$

18. The x component of the field is the negative derivative of the potential with respect to x.

$$E_x = -\frac{dV}{dx} = kQ\frac{x}{(x^2 + R^2)^{\frac{3}{2}}}$$

19. Taking the differential of p, you have

$$dp = -\frac{h}{\lambda^2}d\lambda$$

Now treat the actual changes as if they were infinitesimal. Because the wavelength decreases, the change is negative.

$$\Delta p \cong -\frac{6.63 \times 10^{-34}}{(5 \times 10^{-11})^2}(-2 \times 10^{-13})$$
$$= 5.30 \times 10^{-26}\frac{J \cdot s}{m}$$

20. Find the differential change in the field due to a small change in R, and then treat the small change as an infinitesimal.

$$dE = kQ\frac{-3RdR}{(x^2 + R^2)^{\frac{5}{2}}} \qquad \Delta E \cong kQ\frac{-3R\Delta R}{(x^2 + R^2)^{\frac{5}{2}}}$$

Chapter 3

1. a. $\int_{-1}^{2}(4t^3 - t)dt = \left[t^4 - \frac{1}{2}t^2\right]_{-1}^{2}$
$= (16 - 2) - \left(1 - \frac{1}{2}\right) = 13.5$

b. $\int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} 3\cos 2x\, dx = \left[\frac{3}{2}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{3\pi}{2}}$
$= \frac{3}{2}\left(\sin 3\pi - \sin\frac{\pi}{2}\right) = -\frac{3}{2}$

c. $\int_{0}^{2} 6e^{-\frac{y}{4}}dy = \left[-24e^{-\frac{y}{4}}\right]_0^2 = -24\left(e^{-\frac{1}{2}} - 1\right)$
$= 9.44$

d. $\int_{2}^{10}\left(\sin \pi x - \frac{3}{x}\right)dx = \left[-\frac{1}{\pi}\cos \pi x - 3\ln x\right]_2^{10}$
$= \left(-\frac{1}{\pi}\cos 10\pi - 3\ln 10\right)$
$- \left(-\frac{1}{\pi}\cos 2\pi - 3\ln 2\right)$
$= 4.83$

2. To find the area, you calculate the definite integral with limits defined by the interval.

a. $\int_{0}^{1}(x^2 - 4x + 3)dx = \left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^1$
$= \left(\frac{1}{3} - 2 + 3\right) - 0$
$= 1.33$

b. $\int_1^3 (x^2 - 4x + 3)dx = \left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3$

$= (9 - 18 + 9) - \left(\frac{1}{3} - 2 + 3\right) = -1.33$

The negative result in **2b** indicates that the area lies below the x-axis.

3. Over a very small (infinitesimal) interval of time, the velocity will not change, so the displacement over this interval will be

$$dx = v(t)dt = 60\left(1 - e^{-\frac{t}{10}}\right)dt$$

Add up (integrate) all the little displacements to get the total.

$$\Delta x = \int dx = \int_0^t 60\left(1 - e^{-\frac{t}{10}}\right)dt$$

$$= \left[60\left(t + 10e^{-\frac{t}{10}}\right)\right]_0^t$$

$$= 60\left(t + 10\left(e^{-\frac{t}{10}} - 1\right)\right)$$

4. **a.** Over a very small (infinitesimal) interval of time, the acceleration will not change, so the change in velocity over this interval will be

$$dv = a(t)dt = 10\cos\left(\frac{\pi}{6}t\right)dt$$

Integrate to find the total change in the velocity.

$$\Delta v = \int dv = \int_0^3 10\cos\left(\frac{\pi}{6}t\right)dt$$

$$= \left[\frac{60}{\pi}\sin\frac{\pi}{6}t\right]_0^3 = \frac{60}{\pi} = 19.1\,\frac{\text{m}}{\text{s}}$$

b. Since the change in the velocity is just the end value minus the start value, $\Delta v = v - v_0$, you have

$$\Delta v = 19.1 = v - 2 \Rightarrow v = 21.1\,\frac{\text{m}}{\text{s}}$$

c. Over an infinitesimal interval, the velocity will not change and you can write

$$dx = v(t)dt$$

You can obtain the general expression for the change in velocity after a time t by replacing the upper limit in the integral of **4a** with t. Thus you have

$$\Delta v(t) = \frac{60}{\pi}\sin\frac{\pi}{6}t$$

Then, since the initial value is 2 m/s, you have for the velocity

$$v(t) = 2 + \Delta v = 2 + \frac{60}{\pi}\sin\frac{\pi}{6}t$$

Now integrate to find the total displacement.

$$\Delta x = \int dv = \int_0^3 v(t)dt = \int_0^3 \left(2 + \frac{60}{\pi}\sin\frac{\pi}{6}t\right)dt$$

$$= \left[2t - \frac{360}{\pi^2}\cos\frac{\pi}{6}t\right]_0^3$$

$$\Delta x = 6 + \frac{360}{\pi^2}\cos = 42.48\text{ m}$$

5. **a.** Break up the stick into little pieces, as shown in Figure 5.5, where the element of distance from the axis is given as x. Because the mass is distributed uniformly, the mass per length of stick is just $\lambda = \frac{M}{L}$.

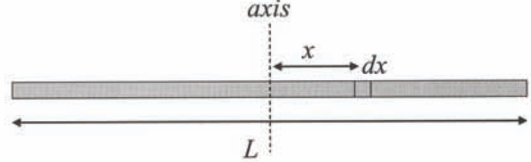

Figure 5.5

Each little piece will have a little mass $dm = \lambda dx = \frac{M}{L}dx$. This means that each little piece will contribute a little bit to the moment of inertia

$$dI = x^2 dm = x^2 \frac{M}{L} dx$$

To find the total moment of inertia, you integrate, recognizing that x varies from $-L/2$ to $+L/2$.

$$I_{center} = \int dI = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{M}{L} x^2 dx = \left[\frac{M}{3L} x^3 \right]_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= \frac{1}{12} ML^2$$

b. When the axis shifts to the end, all that changes in the analysis is the limits on the integral. Now x will vary from $0 \to L$ and you have

$$I_{end} = \int dI = \int_0^L \frac{M}{L} x^2 dx = \left[\frac{M}{3L} x^3 \right]_0^L = \frac{1}{3} ML^2$$

6. Set up your integration as shown in Figure 3.6. The little piece dx will contribute a little bit of mass

$$dm = \lambda dx = \lambda_0 \frac{x}{L} dx$$

To find the total mass m, add (integrate) all of these contributions.

$$m = \int dm = \int_0^L \lambda_0 \frac{x}{L} dx = \left[\lambda_0 \frac{x^2}{2L} \right]_0^L = \frac{1}{2} \lambda_0 L$$

To find the center of mass, you have to integrate the quantity xdm over the length of the stick.

$$X_{CM} = \frac{\oint_{stick} xdm}{m} = \frac{\int_0^L \lambda_0 \frac{x^2}{L} dx}{\frac{1}{2}\lambda_0 L} = \frac{\left[\lambda_0 \frac{x^3}{3L} \right]_0^L}{\frac{1}{2}\lambda_0 L} = \frac{2}{3} L$$

7. Following the hint given in Problem 7, set up your integral as suggested in this illustration.

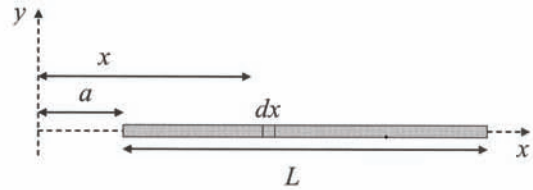

Figure 5.6

Since the charge is spread uniformly over the rod, the charge per length is given by

$$\lambda = \frac{Q}{L}$$

Then the little bit of charge dQ in the length dx is given by

$$dQ = \lambda dx = \frac{Q}{L} dx$$

From the formula for the potential of a point charge, the contribution to the potential at the origin due to this little piece will be

$$dV = k\frac{dQ}{x} = k\frac{Q}{L}\frac{dx}{x}$$

The total potential at the origin then is obtained by integrating these contributions, recognizing that x varies from a to $a + L$.

$$V = \int dV = \int_a^{a+L} k\frac{Q}{L}\frac{dx}{x} = \left[k\frac{Q}{L} \ln x \right]_a^{a+L}$$

$$= k\frac{Q}{L} \ln \frac{a+L}{a}$$

8. If you break the disc up into tiny rings centered about the center of the disc, every point on each ring will be the same distance r from the point on the axis where you want to calculate the potential. This means that each ring will contribute the amount $dV = k\frac{dQ}{r}$ to the total potential, where dQ is the charge of one of the ring elements.

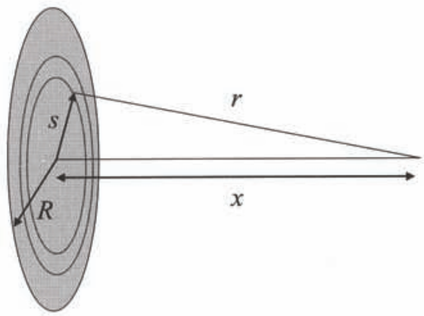

Figure 5.7

Since the charge is spread uniformly over the disc, the charge per area is given by $\sigma = \frac{Q}{\pi R^2}$. Then the little bit of charge in a ring is given by

$$dQ = \sigma dA = \sigma(2\pi s ds)$$

Here s is used as the radius variable of the ring. Putting this together, you can write

$$V = \int dV = \int_0^R k\sigma \frac{2sds}{\sqrt{s^2 + x^2}}$$

This integral can be done with a simple substitution. Let $u = s^2 + x^2$. Then $du = 2sds$, which is the numerator of the fraction in the integral. Now substitute these new quantities into the integral. Note that for $s = 0$, $u = x^2$, and for $s = R$, $u = R^2 + x^2$. Thus you have

$$V = \int_0^R k\sigma \frac{2sds}{\sqrt{s^2 + x^2}} = \int_{x^2}^{R^2+x^2} k\sigma \frac{du}{\sqrt{u}}$$
$$= 2k\sigma\left[\sqrt{u}\right]_{x^2}^{R^2+x^2} = 2k\sigma\left(\sqrt{R^2 + x^2} - x\right)$$

9. Because the charge density only depends on r, there is spherical symmetry. Take advantage of the spherical symmetry and break the shell into little shells for the integration. The little bit of charge in each little shell will be

$$dq = \rho dV = \rho_0 \frac{ab}{r^2} e^{-\alpha r}(4\pi r^2 dr)$$
$$= 4\pi\rho_0 abe^{-\alpha r}dr$$

Now integrate to find the total charge. Note that r varies from a to b.

$$q = \int dq = \int_a^b 4\pi\rho_0 abe^{-\alpha r}dr$$
$$= -\frac{4\pi\rho_0 ab}{\alpha}\left[e^{-\alpha r}\right]_a^b = \frac{4\pi\rho_0 ab}{\alpha}(e^{-\alpha a} - e^{-ab})$$

10. The $d\vec{l}$ path element is always parallel to the field $\vec{B}$ along the circular loop (compare Figure 5.8 with the figure in the statement of Problem 10).

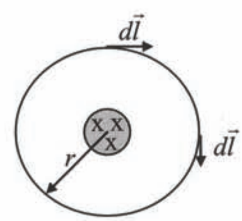

Figure 5.8

Thus $\vec{B} \cdot d\vec{l} = Bdl$. Because the field only depends on r, its magnitude does not change along the loop, and it will factor out of the integral. Thus you can write

$$\oint_{loop} \vec{B} \cdot d\vec{l} = \oint_{loop} Bdl = B \oint_{loop} dl = B(2\pi r) = 2\pi k' i$$

11. The line integral around the rectangular loop breaks up into four parts.

$$\oint_{loop} \vec{B} \cdot d\vec{l} = \oint_{top} \vec{B} \cdot d\vec{l} + \oint_{\substack{right\\side}} \vec{B} \cdot d\vec{l} + \oint_{bottom} \vec{B} \cdot d\vec{l} + \oint_{left\ side} \vec{B} \cdot d\vec{l}$$

On the two sides $\vec{B} \cdot d\vec{l} = 0$, since the field is perpendicular to the line element. On the top and bottom $\vec{B} \cdot d\vec{l} = Bdl$ since along these segments the field and line element are parallel. Note that on the bottom the $d\vec{l}$ vector points to the left, in the same direction as the field. Since the magnitude of the field is constant along the path, you can write

$$\oint_{loop} \vec{B} \cdot d\vec{l} = \oint_{top} Bdl + \oint_{bottom} Bdl = B \oint_{top} dl + B \oint_{bottom} dl = 2BL$$

12. a. Path 1 has two parts. Along the x-axis, because $y = 0$, you have

$$\vec{F} = 2x\hat{i} + \frac{x^2}{2}\hat{j} \qquad d\vec{r} = dx\hat{i}$$

$$\vec{F} \cdot d\vec{r} = 2xdx$$

Along the second part of the path, parallel to the y-axis, x is fixed at $x = 3$. Thus you can write

$$\vec{F} = 3(y + 2)\hat{i} + \frac{9}{2}\hat{j} \qquad d\vec{r} = dy\hat{j}$$

$$\vec{F} \cdot d\vec{r} = \frac{9}{2}dy$$

Then, adding the two contributions and putting in the correct limits, you get

$$\oint_{Path1} \vec{F} \cdot d\vec{r} = \int_0^3 2xdx + \int_0^2 \frac{9}{2}dy = \left[x^2\right]_0^3 + \frac{9}{2}\left[y\right]_0^2 = 9 + 9 = 18$$

Path 2 also has two parts. Along the y-axis you have, since $x = 0$,

$$\vec{F} = 0 \qquad d\vec{r} = dy\hat{j} \qquad \vec{F} \cdot d\vec{r} = 0$$

Along the second part of the path, parallel to the x-axis, y is fixed at $y = 2$. Thus you can write

$$\vec{F} = 4x\hat{i} + \frac{x^2}{2}\hat{j} \quad d\vec{r} = dx\hat{i} \quad \vec{F} \cdot d\vec{r} = 4xdx$$

Then, adding the two contributions and putting in the correct limits, you get

$$\oint_{Path2} \vec{F} \cdot d\vec{r} = 0 + \int_0^3 4xdx = \left[2x^2\right]_0^3 = 18$$

the same result.

b. The straight line that connects the points (0, 0) and (3, 2) satisfies the equation $y = \frac{2}{3}x$, ensuring that along this path $dy = \frac{2}{3}dx$. To calculate the line integral along this path, express the integrand in terms of just one variable and then integrate over the limits of that variable. If you choose to use x as the variable to keep, then along the path you have

$$\vec{F} = x\left(\frac{2x}{3} + 2\right)\hat{i} + \frac{x^2}{2}\hat{j}$$

$$d\vec{r} = dx\hat{i} + \frac{2}{3}dx\hat{j}$$

$$\vec{F} \cdot d\vec{r} = \left(\frac{2}{3}x^2 + 2x + \frac{1}{3}x^2\right)dx = (x^2 + 2x)dx$$

$$\oint_{path} \vec{F} \cdot d\vec{r} = \int_0^3 (x^2 + 2x)dx = \left[\frac{1}{3}x^3 + x^2\right]_0^3 = 18$$

This again gives the same result.

13. Because the field is uniform and the area is planar, the flux is given by

$$\Phi = \vec{B} \cdot \vec{A} = \frac{B_0}{\sqrt{30}}(2\hat{i} + 5\hat{j} - k) \cdot \frac{ab}{\sqrt{2}}(\hat{i} - \hat{j})$$
$$= \frac{B_0 ab}{\sqrt{60}}(2 - 5) = -3\frac{B_0 ab}{\sqrt{60}}$$

14. The cube has three pairs of faces, each face having area a^2. Because there is no component of the field in the z direction, there is no flux through the faces parallel to the x-y plane. The y component of the field is constant. This means that the flux through the two faces parallel to the x-z plane at $y = 0$ and $y = a$ will cancel because the outward normal to these two faces is in the opposite direction. Specifically, at the $y = 0$ face, the outward normal unit vector is $-\hat{j}$, and the flux through this face is

$$\int_{y=0} \vec{E} \cdot d\vec{A} = \int_{y=0} (x^2\hat{i} + 3\hat{j}) \cdot (-dA\hat{j})$$
$$= -3 \int_{y=0} d\vec{A} = -3a^2$$

At the $y = a$ face, the outward normal unit vector is $+\hat{j}$. The flux through this face is then $+3a^2$ and the total flux through these two faces is 0. This leaves only the two faces parallel to the y-z plane, one with $x = 0$ and the other with $x = a$.

$$\Phi = \oint_{cube} \vec{E} \cdot d\vec{A} = \oint_{x=0} \vec{E} \cdot d\vec{A} + \oint_{x=a} \vec{E} \cdot d\vec{A}$$

Along the $x = 0$ face

$$d\vec{A} = -dA\hat{i} \qquad \vec{E} = 3\hat{j} \qquad \vec{E} \cdot d\vec{A} = 0$$

Along the $x = a$ face

$$d\vec{A} = +dA\hat{i} \qquad \vec{E} = a^2\hat{i} + 3\hat{j} \qquad \vec{E} \cdot d\vec{A} = a^2 dA$$

Putting this together you get

$$\Phi = \oint_{cube} \vec{E} \cdot d\vec{A} = \oint_{x=0} \vec{E} \cdot d\vec{A} + \oint_{x=a} \vec{E} \cdot d\vec{A}$$
$$= 0 + a^2 \oint_{x=a} dA = a^4$$

15. Because the field is cylindrically symmetric, use little rings with area $dA = 2\pi r dr$ as your area element, with r varying from 0 to a. Choosing the direction of the area vector to be into the page, the little contribution to the flux for one of these area elements is

$$d\Phi = BdA = B_0\frac{r^2}{R^2}(2\pi r dr)$$

Integrating over the limits of r gives

$$\Phi = \int d\Phi = \int_0^a B_0\frac{r^2}{R^2}(2\pi r dr) = B_0\frac{\pi a^4}{2R^2}$$

16. Break the loop up into strips of width a and thickness dx. Over a strip, the field is constant, so the little bit of flux contributed by a strip is given by

$$d\Phi = BdA = C\left(\frac{1}{x} - \frac{1}{d - x}\right)adx$$

The variable x ranges from L to $L + b$. Integration then gives

$$\Phi = \int_{area} BdA = \int_L^{L+b} C\left(\frac{1}{x} - \frac{1}{d - x}\right)adx$$
$$= Ca\Big[\ln x + \ln(x - d)\Big]_L^{L+b}$$
$$\Phi = Ca\Big[\ln x(x - d)\Big]_L^{L+b}$$
$$= Ca\ln\frac{(L + b)(L + b - d)}{L(L - d)}$$

Chapter 4

1. a. Gravity acts downward and the retarding force acts upward, so the second law gives

$$ma = m\frac{dv}{dt} = mg - kv^2 \Rightarrow$$
$$\frac{dv}{dt} + \frac{k}{m}v^2 = g$$

b. Written in terms of v, this equation is first-order, nonlinear (because of the v^2 term) and nonhomogeneous (because of the g

on the right-hand side). It is an ordinary differential equation, because it depends only on the one variable t.

c. When the terminal speed v_t is reached, $\frac{dv}{dt} = 0$. Thus you get

$$0 + \frac{k}{m}v_t^2 = g \qquad \Rightarrow \qquad v_t = \sqrt{\frac{mg}{k}}$$

2. First, reorganize the differential equation.

$$\frac{dv}{dt} = k' - kv = -k\left(v - \frac{k'}{k}\right)$$

Then get the "v stuff" on one side and the "t stuff" on the other.

$$\frac{dv}{v - \frac{k'}{k}} = -kdt$$

Now integrate. As t varies from 0 to some final value t, the velocity will vary from v_0 to some final v.

$$\int_{v_0}^{v} \frac{dv}{v - \frac{k'}{k}} = -\int_0^t kdt \Rightarrow \ln\left(\frac{v - \frac{k'}{k}}{v_0 - \frac{k'}{k}}\right) = -kt$$

Exponentiate both sides, using $e^{\ln x} = x$.

$$\frac{v - \frac{k'}{k}}{v_0 - \frac{k'}{k}} = e^{-kt} \Rightarrow v(t) = \frac{k'}{k} + \left(v_0 - \frac{k'}{k}\right)e^{-kt}$$

Since you are given that $v_0 = 0$, it follows that

$$v(t) = \frac{k'}{k}(1 - e^{-kt})$$

3. You need the value of the charge to reach $q(t) = 0.01q_0$. Now use the solution for exponential decay and let $t = nRC$ where n is to be determined.

$$q(t) = q_0 e^{-\frac{t}{RC}} \qquad q(nRC) = 0.01q_0 = q_0 e^{-n}$$

The q_0 cancels. Take the natural log of both sides, using $\ln e^x = x$.

$$n = -\ln 0.01 = 4.61$$

4. a. Since the velocity and force have only horizontal components, you can drop the vector notation.

$$ma = m\frac{dv}{dt} = -\frac{B^2L^2}{R}v \Rightarrow$$
$$\frac{dv}{dt} + \frac{B^2L^2}{mR}v = 0$$

Note that this is an exponential decay. The rail will slow to a stop.

b. The decay constant in this case will be $\frac{mR}{B^2L^2}$ and the solution will have the form $v(t) = v_0 e^{-\frac{B^2L^2}{mR}t}$.

c. In a small time interval dt, the rail will undergo a displacement

$$dx = vdt = v_0 e^{-\frac{B^2L^2}{mR}t} dt$$

Integrate over all time from 0 to ∞ to find the total displacement.

$$\Delta x = \int dx = \int_0^\infty v_0 e^{-\frac{B^2L^2}{mR}t} dt$$
$$= -\frac{mRv_0}{B^2L^2}\left[e^{-\frac{B^2L^2}{mR}t}\right]_0^\infty = \frac{mRv_0}{B^2L^2}$$

5. By assumption, both f_1 and f_2 solve the equation, so you can write

$$\frac{d^2f_1}{dx^2} + a_1\frac{df_1}{dx} + a_0 f_1 = 0$$
$$\frac{d^2f_2}{dx^2} + a_1\frac{df_2}{dx} + a_0 f_2 = 0$$

Multiply the left equation by c_1 and the right by c_2 and then add the two equations.

$$\frac{d^2(c_1f_1 + c_2f_2)}{dx^2} + a_1\frac{d(c_1f_1 + c_2f_2)}{dx} + a_0(c_1f_1 + c_2f_2) = 0$$

Since $f_L = (c_1 f_1 + c_2 f_2)$, you have

$$\frac{d^2 f_L}{dx^2} + a_1 \frac{df_L}{dx} + a_0 f_L = 0$$

This completes the proof.

6. a. By inspection $\omega = \frac{2\pi}{5}$. Since $T = \frac{2\pi}{\omega}$, it follows that $T = 5$ s.

b. Differentiate to find the velocity at any time.

$$v(t) = \frac{dx}{dt} = -2\pi \sin\left(\frac{2\pi}{5}t + \frac{\pi}{4}\right)$$

Since sine stays between ± 1, the maximum speed is $2\pi = 6.28\frac{\text{m}}{\text{s}}$

c. $\omega^2 = \frac{k}{m} \Rightarrow k = m\omega^2 = 2 \cdot \left(\frac{2\pi}{5}\right)^2 = 3.16\frac{\text{N}}{\text{m}}$

7. a. If $x(t) = Ae^{\alpha t}$, then $\frac{d^2x}{dt^2} = \alpha^2 Ae^{\alpha t}$.

Substitute into the differential equation.

$$\alpha^2 Ae^{\alpha t} + \omega^2 Ae^{\alpha t} = 0 \quad \Rightarrow \quad (\alpha^2 + \omega^2)Ae^{\alpha t} = 0$$

Since the last equation must be true for all t, it follows that

$$\alpha^2 = -\omega^2 \quad \Rightarrow \quad \alpha = \pm\sqrt{-\omega^2} = \pm\sqrt{-1}\omega = \pm i\omega$$

b. Euler's formula, derived in Practice Problem 7 of Chapter 2, tells you

$$e^{ix} = \cos x + i \sin x$$

Replace x by $-x$ and you get

$$e^{-ix} = \cos x - i \sin x$$

You can then use these two equations to solve for either the sine or the cosine in terms of the complex exponentials. The results obtained from simple algebra are

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Thus

$$x(t) = \frac{A}{2}e^{i\omega t} + \frac{A}{2}e^{-i\omega t} = A\frac{e^{i\omega t} + e^{-i\omega t}}{2} = A\cos\omega t$$

$$x(t) = \frac{A}{2i}e^{i\omega t} - \frac{A}{2i}e^{-i\omega t} = A\frac{e^{i\omega t} - e^{-i\omega t}}{2i} = A\sin\omega t$$

8. The general form for the displacement is $x(t) = A\cos(\omega t + \phi)$ where $\omega^2 = \frac{g}{L}$. The initial conditions given are

$$x(0) = 0 \qquad v(0) = v_0$$

From the general form this becomes

$$0 = A\cos\phi \qquad v_0 = -\omega A \sin\phi$$

The left equation tells you that $\phi = \pm\frac{\pi}{2}$. Because the initial velocity was designated as positive, the right equation tells you to choose negative value for ϕ (ω, A are positive). Thus $A = \frac{v_0}{\omega}$ and the solution is

$$x(t) = \frac{v_0}{\omega}\cos\left(\omega t - \frac{\pi}{2}\right) = \frac{v_0}{\omega}\sin\omega t$$