CHAPTER 4

Differential Equations

In physics, often the fundamental laws governing the behavior of a system are not directly expressed just in terms of the basic parameters used to describe the system. Rather, the laws involve these parameters but also their rates of change—that is, their derivatives—as well. An equation that relates a quantity and its derivatives is called a **differential equation.** The topic of differential equations is vast, with many books devoted to the subject. To begin this chapter, you will learn a bit about how differential equations are categorized. In AP Physics, you will need to be familiar with a few basic differential equations that involve derivatives with respect to time. It is on these that the chapter will mainly focus.

4.1 Nomenclature

Suppose a quantity $f = f(x, y)$ depends on two independent variables x and y. An equation that relates this function and its partial derivatives with respect to x and y is called a **partial differential equation.** A partial derivative of a function involves taking the derivative with respect to one variable while holding other variables fixed. You write partial derivatives like this:

$$\frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y \partial x}, \text{ etc.}$$

In AP Physics you will have to be familiar only with functions of a single variable. An equation that relates such a function $g = g(x)$ and its derivatives is called an **ordinary differential equation.** Here is an example of an ordinary differential equation.

$$\frac{d^2 g}{dx^2} + 5x\frac{dg}{dx} - g^2 = x^3 + 2$$

This equation is said to be a second-order differential equation because the highest order derivative is the second derivative. A differential equation is called **linear** if it is linear in the function and its derivatives; that is, no powers of the function or its derivatives higher that 1 appear. The equation above is **nonlinear** because of the g^2 term. An example of a linear equation would be

$$\frac{d^2g}{dx^2} + 5x\frac{dg}{dx} - g = x^3 + 2$$

The standard form for writing a differential equation is to put all the terms involving the function and its derivatives on the left side of the equation. Anything left over is put on the right-hand side. If there is nothing left over, so that the right-hand side is 0, the differential equation is said to be **homogeneous.** The examples shown above are **nonhomogeneous** because of the $x^3 + 2$ term on the right-hand side. An example of a homogeneous, linear, second-order differential equation would be

$$\frac{d^2g}{dx^2} + 5x\frac{dg}{dx} - g = 0$$

In AP Physics, the independent variable in the equations will always be the time, t. The quantities for which you might have to solve are position, velocity, electric charge, electric current, and others, but these quantities will all be functions of time only. The differential equations will come about as a result of analyzing the system in terms of **Newton's second law,** if it is a mechanical system, or **Kirchhoff's laws,** if it is an electric circuit. In the following sections you will see all the common differential equations that appear in AP Physics. You will become familiar with their solutions and you will also learn some of the techniques needed to solve differential equations.

4.2 First Order Examples from AP Physics C

Suppose an object is sliding across a frictionless horizontal surface subject to a retarding force $\vec{F} = -k\vec{v}$ that is proportional to the velocity of the object (k is a constant).

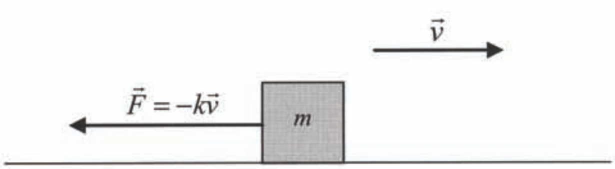


Figure 4.1

When you apply Newton's second law to this system, remembering that $\vec{a} = \frac{d\vec{v}}{dt}$, you get

$$m\vec{a} = m\frac{d\vec{v}}{dt} = -k\vec{v}$$

Rearranging and dropping the vector notation, because all motions are in the horizontal direction, you get

$$\frac{dv}{dt} + \frac{k}{m}v = 0$$

You should recognize this as a first-order, linear, homogeneous equation. The solution to this equation will be a function $v = v(t)$ that satisfies the equation. In simple terms, this particular equation says that the function v is such that if you take its derivative and add it back to a simple multiple of the function, you always get 0. In other words, the derivative of the function is proportional to the function itself. You should ask yourself if you know of any functions that have this property. Of course you do! The exponential function e^t works this way. Thus you expect your solution to involve an exponential function. Let's see how this works. First, rewrite the equation to get the "v stuff" on one side and the "t stuff" on the other.

$$\frac{dv}{v} = -\frac{k}{m}dt$$

This equation relates the infinitesimal changes of v and t. If you add up all these changes on each side of the equation over the same physical interval, the sums will still be equal. Thus you can integrate both sides of this equation over the same physical interval, during which time varies from 0 to t and velocity varies from v_0 to v.

$$\int_{v_0}^{v} \frac{dv}{v} = \int_0^t -\frac{k}{m}dt \qquad \Rightarrow \qquad \ln\frac{v}{v_0} = -\frac{k}{m}t$$

You can now exponentiate both sides of the last equation using e. Then, remembering that $e^{\ln x} = x$, you get

$$v(t) = v_0 e^{-\frac{k}{m}t}$$

Your expectations have been realized; an exponential was involved, in this case an exponential with a negative argument, a decaying exponential. Functions like $v(t)$ are said to exhibit **exponential decay.**

Notice that, in the process of solving the differential equation, it was necessary to introduce the initial velocity of the mass. The original differential equation, derived from the second law, did not contain this parameter. In fact, you could have started the mass with any initial velocity, and the same differential equation would have described it. Put another way, there are an infinite number of solutions to the original differential equation. They all have the form

$$v(t) = Ce^{-\frac{k}{m}t} \qquad \text{C is a constant}$$

It is only when you specify the initial velocity that you pick out the unique solution to the problem. This is one of the many aspects of differential equations that make them different from algebraic equations. When you solve an algebraic system of equations—one without derivatives or integrals—the solution only involves the parameters initially in the problem. A first-order differential equation always has one undetermined constant in its general solution. This constant is usually determined by the particular initial conditions of the system that obeys the equation. A second-order differential equation will have two undetermined constants in its general solution. These constants will also be determined by the initial conditions of the system under study.

The solution for $v(t)$ above tells you that, as time increases, the mass goes slower and slower, as Figure 4.2 shows in the graph of this function.

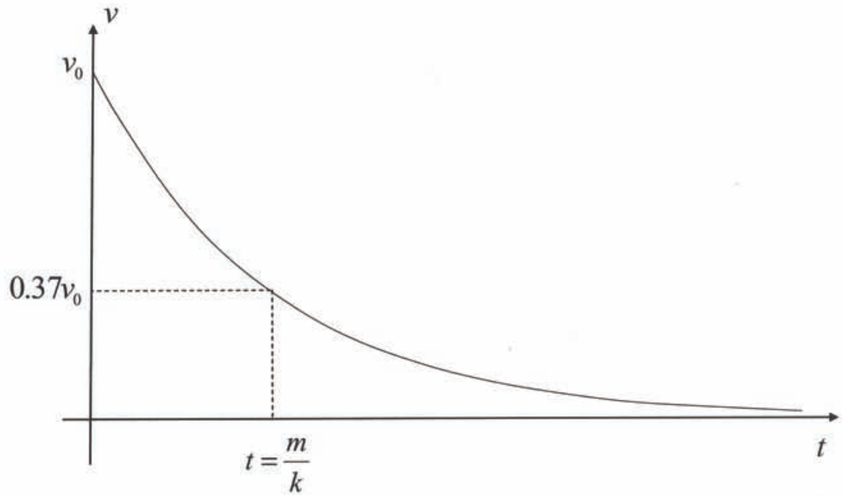


Figure 4.2

At a time $t = \frac{m}{k}$, the velocity of the object has been reduced to $v = v_0 e^{-1} = 0.37 v_0$. The quantity $\frac{m}{k}$ is sometimes called the **decay constant.** After a time of one decay constant, the velocity has dropped to 37% of its initial value.

Even though the formula for $v(t)$ theoretically says the object never completely stops, the total displacement of the mass is finite. To prove this statement, recall that $v = \frac{dx}{dt}$. Then you can write

$$dx = vdt = v_0 e^{-\frac{k}{m}t} dt$$

You may think of dx as the little displacement in the little time dt. To find the total displacement after a time T, add up (integrate) all these little displacements.

$$\Delta x = \int dx = \int_0^t v_0 e^{-\frac{k}{m}t} dt = -\frac{mv_0}{k}\left[e^{-\frac{k}{m}t}\right]_0^t = \frac{mv_0}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

Figure 4.3 shows a graph of this function.

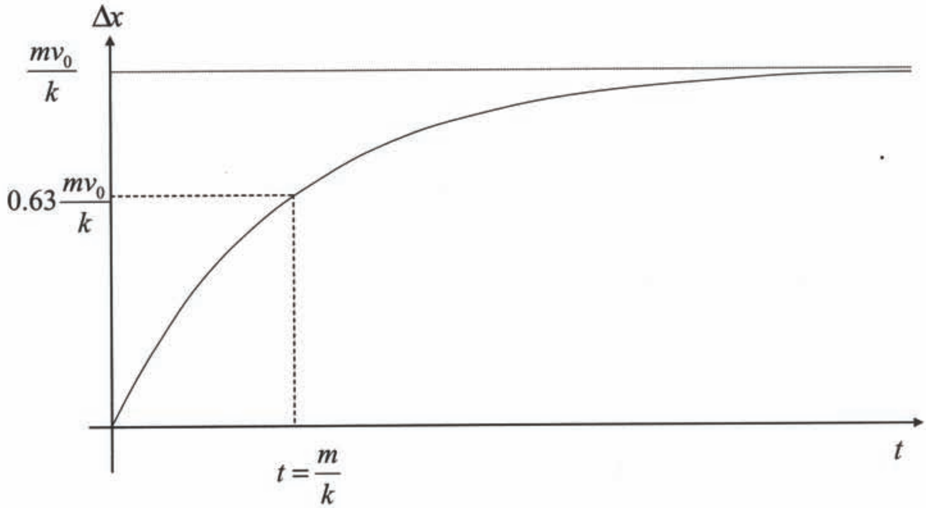


Figure 4.3

As $t \to \infty$, you can see that the total displacement approaches $\frac{mv_0}{k}$. After a time $t = \frac{m}{k}$, the displacement is

$$\Delta x = \frac{mv_0}{k}(1 - e^{-1}) = 0.63\frac{mv_0}{k}$$

Instead of moving horizontally, let's suppose the object is falling vertically but still subject to a retarding force proportional to its velocity.

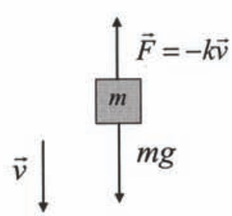


Figure 4.4

The new wrinkle here is, of course, the presence of gravity. Application of the second law in the vertical direction gives

$$ma = m\frac{dv}{dt} = mg - kv \qquad \Rightarrow \qquad \frac{dv}{dt} + \frac{k}{m}v = g$$

You can see that the second law has now produced a nonhomogeneous, linear, first-order equation. There are several ways to get to the solution of this equation. You could "brute force" the solution, following the technique used to solve the homogeneous case. That approach will be left as an exercise for you (see Practice Problem 2).

A second approach relies on a theorem from the theory of differential equations saying that, if you know the most general solution to the homogeneous equation, you can get the most general solution to the nonhomogeneous equation by finding *any* particular solution to the nonhomogeneous equation and adding it to the homogeneous solution. This possibility makes guessing a very effective way of solving a nonhomogeneous equation. You need to come up with *just one* special solution and add it to the general homogeneous solution to get the most general nonhomogeneous solution. In this case, you already know the most general homogeneous solution

$$v(t) = Ce^{-\frac{k}{m}t} \qquad \text{C is a constant}$$

Can you guess a solution for the nonhomogeneous case? Well, the simplest guess would be $v = \alpha$, a constant. If you substitute this into the differential equation, remembering that the derivative of a constant is 0, you get

$$0 + \frac{k}{m}\alpha = g \Rightarrow \alpha = \frac{mg}{k}$$

Thus the simple constant function $v(t) = \frac{mg}{k}$ is a particular solution to the nonhomogeneous equation. It follows from the theorem cited above, then, that the most general solution is

$$v(t) = Ce^{-\frac{k}{m}t} + \frac{mg}{k}$$

You can see from the general form for $v(t)$ that, as time increases, the velocity approaches a limiting value $v = \frac{mg}{k}$. This is called the **terminal speed** of the object. The constant C will still be determined by initial conditions. For example, if the object starts from rest at $t = 0$, you can write

$$0 = C + \frac{mg}{k} \Rightarrow C = -\frac{mg}{k}$$

Then the form for v is

$$v(t) = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

The next illustration, Figure 4.5, shows a graph of this function.

Note the similarity to Figure 4.3. In this case, the velocity approaches a limiting value, whereas in the former case it was the displacement that approached a limiting value. For this reason the original nonhomogeneous differential equation is said to define **exponential approach.**

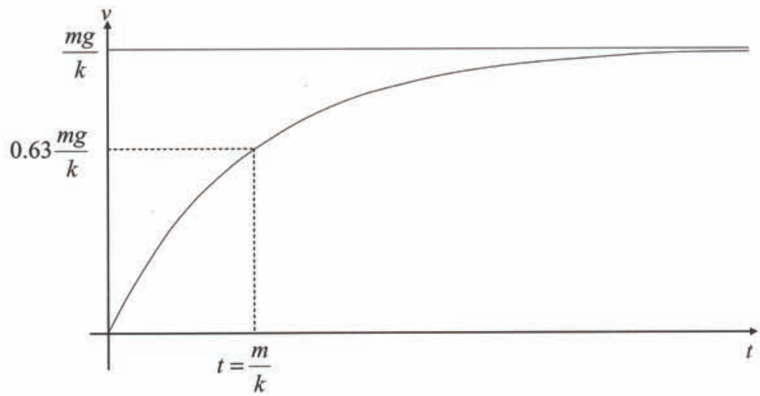


Figure 4.5

A third approach to solving the equation

$$\frac{dv}{dt} + \frac{k}{m}v = g$$

involves transforming it into a homogeneous equation. First rewrite it as

$$\frac{dv}{dt} + \frac{k}{m}\left(v - \frac{mg}{k}\right) = 0$$

Then define a new velocity variable $v' = v - \frac{mg}{k}$. Note that v' is just the difference between the actual velocity and the terminal velocity. Then since v' differs from v by just a constant, it obeys the homogeneous equation

$$\frac{dv'}{dt} + \frac{k}{m}v' = 0$$

You already know the most general solution to this equation.

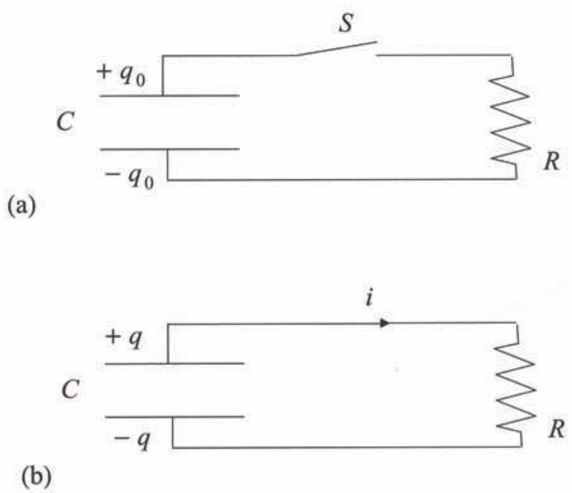
$$v'(t) = Ce^{-\frac{k}{m}t} \qquad C \text{ is a constant}$$

Thus you can write

$$v'(t) = Ce^{-\frac{k}{m}t} = v(t) - \frac{mg}{k}$$

and you can solve for $v(t)$. Of course, as before, C is determined by initial conditions.

One of the fascinating things occuring in physics is that the same mathematical description can apply to widely different phenomena. You have just seen the mathematical description of an object moving under the influence of a retarding force proportional to its velocity. There are many other systems that obey the same differential equation and will thus behave exactly the same from a mathematical point of view. However, the variables describing the other systems need not be velocity and position. In AP Physics, circuits that contain a resistor and a capacitor provide an example of this analogous behavior. Figure 4.6a shows a resistor with resistance R connected to a capacitor with capacitance C through a switch S that is initially open. The capacitor carries initial charge q_0. Figure 4.6b shows the situation after the switch has been closed. The capacitor begins to discharge and a current i will flow through the resistor.



Figures 4.6a and 4.6b

The drops in voltage across the capacitor and the resistor are $\frac{q}{C}$ and $-iR$, respectively, for a charge moving clockwise. Kirchhoff's second law tells you that the sum of the drops in voltage around a closed loop must be 0. This means that

$$-iR + \frac{q}{C} = 0$$

The current is related to the charge on the capacitor through the equation

$$i = -\frac{dq}{dt} \text{ (as } i \text{ increases, } q \text{ decreases, hence the minus sign)}$$

Thus you have a differential equation for the charge on the capacitor.

$$\frac{dq}{dt} + \frac{q}{RC} = 0$$

This is the same basic equation you saw with the horizontally moving object studied previously. The charge q has replaced the velocity v, and the constant $\frac{1}{RC}$

has replaced $\frac{k}{m}$. However, the math leading to the solution will be exactly the same and you get

$$q(t) = q_0 e^{-\frac{t}{RC}}$$

The next figure, 4.7, is a graph of this equation.

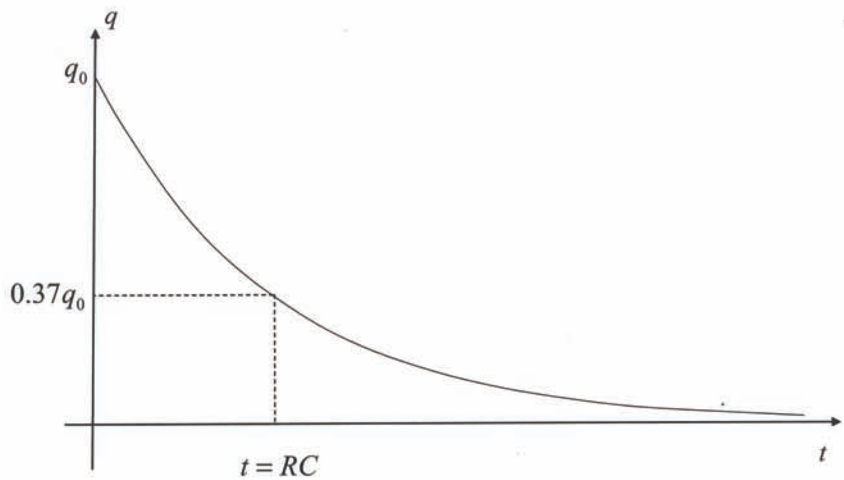


Figure 4.7

The quantity $\tau = RC$ has the dimensions of time and is called the ***RC* time constant** of the circuit. After one time constant, the charge on the capacitor has been reduced to 37% of its initial value.

If the capacitor is initially uncharged and placed in series with a resistor and an ideal battery having voltage V_B, as shown in Figure 4.8, a current i will flow.

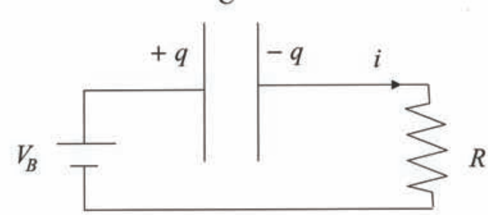


Figure 4.8

Once again, Kirchhoff's second law will yield an equation for the sum of the voltage drops.

$$+V_B - \frac{q}{C} - iR = 0$$

Because in this case $i = \frac{dq}{dt}$, upon rearrangement, you get the differential equation

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{V_B}{R}$$

This equation is analogous to the falling object subject to a retarding force proportional to its velocity. The charge q has replaced the velocity v, the constant $\frac{1}{RC}$ has replaced $\frac{k}{m}$, and the constant $\frac{V_B}{R}$ has replaced g. Now you can immediately write down the solution

$$q(t) = CV_B\left(1 - e^{-\frac{t}{RC}}\right).$$

Figure 4.9 shows a graph of this function.

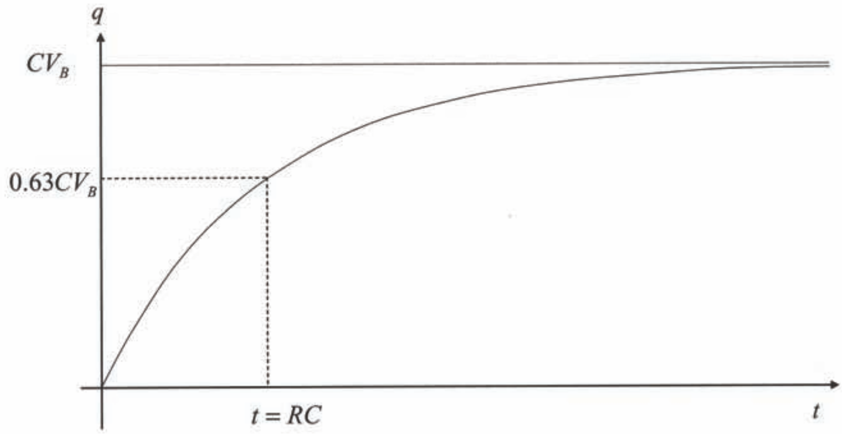


Figure 4.9

The charge on the capacitor approaches the limiting value of CV_B, and after one time constant it has reached 63% of the limiting value.

4.3 The Harmonic Oscillator

In his well-known book on quantum mechanics, R. Shankar begins Chapter 7 with the sentence, "In this section I will put the harmonic oscillator in its place—on a pedestal."* The **harmonic oscillator** is a system with an extremely high degree of physical relevance. It forms the basis for describing oscillations in a wide range of systems at both the classical and quantum level. What is more, at both the classical and quantum level it is an exactly solvable model. No approximations have to be introduced to arrive at its solutions. That such an important model is exactly solvable is certainly fortunate and a firm grasp of the harmonic oscillator is a requirement of every serious physics student. Your study of this model begins in AP Physics.

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*Shankar, R., *Principles of Quantum Physics*, 2nd edition, Kluwer Academic/Plenum Publishers, New York, NY, 1980

The first place you meet this model is in the description of a spring-mass system undergoing horizontal oscillations on a frictionless surface.

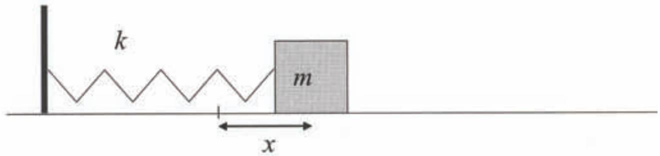


Figure 4.10

Hooke's law states that, if the mass is displaced from its equilibrium position, it will experience a restoring force F that tends to bring it back to the equilibrium position. This force is proportional to the displacement x from the equilibrium point.

$$F = -kx \qquad \text{Hooke's law}$$

The constant k is called the **spring constant** and indicates how stiff the spring is. Applying Newton's second law, you get

$$ma = m\frac{d^2x}{dt^2} = -kx$$

You can write this equation in the following way

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

This is called the **harmonic oscillator equation.** You will note that it is a second-order, linear, homogeneous equation. The task at hand is to find the most general function $x(t)$ that will solve the equation. Describing the equation in words, you could say that if you take the second derivative of the function and add it to the function itself, multiplied by a constant, you always get 0, no matter what the time. In other words, the second derivative must be directly proportional to the function itself. Do you know of any functions that give you back the function after taking two derivatives? Sure you do! The sine, cosine, and exponential functions will all do this. Let's make a guess at the form of the solution and try

$$x(t) = B\sin\omega t$$

As of now, the constants B and ω are not known. It is hoped that, by substituting into the differential equation, they can be determined, so let's substitute.

$$\frac{d^2}{dt^2}(B\cos\omega t) + \frac{k}{m}B\cos\omega t = 0$$

$$-\omega^2 B\sin\omega t + \frac{k}{m}B\sin\omega t = 0 \Rightarrow \left(-\omega^2 + \frac{k}{m}\right)B\sin\omega t = 0$$

Because this equation must be true for all time, it follows that

$$\omega^2 = \frac{k}{m}$$

The quantity ω is called the **angular frequency** of the oscillator. The constant B is not determined by the substitution, so it remains arbitrary. Thus you have a solution $x(t) = B\sin\left(\sqrt{\frac{k}{m}}t\right)$, but is it the most general solution? You should suspect not, because there was nothing singular about substituting sine rather than cosine. You might just as well have tried

$$x(t) = B'\cos\omega' t$$

Here B' and ω' are different constants. When you substitute this form into the oscillator equation you get

$$\left(-\omega'^2 + \frac{k}{m}\right)B'\cos\omega t = 0$$

Thus $\omega'^2 = \omega^2 = \frac{k}{m}$, but B' is arbitrary. So now you have two solutions. Are there more? Yes! Because the oscillator equation is linear, if you have two solutions $x_1(t)$ and $x_2(t)$, then the combination

$$x(t) = a_1x_1 + a_2x_2 \qquad a_1 \text{ and } a_2 \text{ are constants}$$

is also a solution. It is left as an exercise to verify this. Thus you are assured that

$$x(t) = B\sin\omega t + B'\cos\omega t \qquad \omega = \sqrt{\frac{k}{m}}$$

is a solution. In fact, this is the most general solution to the oscillator equation. It contains two undetermined constants, as expected for a second-order equation. These two constants can be determined from the initial conditions, that is, from the initial position and velocity of the oscillator. (You will see how this works shortly.) But this form for the solution is not the most useful, so let's derive a slightly different form. You can define two new constants, A and ϕ, from B and B' as follows.

$$B = -A\sin\phi \qquad B' = A\cos\phi$$

Because B and B' are arbitrary, A and ϕ are as well. Now substitute into the general form for the solution.

$$x(t) = -A\sin\phi\,\sin\omega t + A\cos\phi\,\cos\omega t = A(\cos\phi\,\cos\omega t - \sin\phi\,\sin\omega t)$$
$$x(t) = A\cos(\omega t + \phi)$$

To arrive at the last step, a trigonometric identity was used. You now have the most commonly seen form for the general solution to the oscillator equation. Because neither sine nor cosine ever get bigger than $+1$ or smaller than -1, it follows that the largest magnitude of x is A. This is why A is called the **amplitude** of the oscillation. The argument of the cosine, $\omega t + \phi$, is called the **phase** of the oscillator. Nomenclature for the constant ϕ is less consistent; it is often simply called the **phase angle.** You might wonder how the solution ended up involving cosine and not sine. Actually, sine and cosine simply differ by $\frac{\pi}{2}$ in the phase angle.

$$\cos\left(x - \frac{\pi}{2}\right) = \sin x$$

This means that by simply redefining the phase angle (it is arbitrary anyway), you can write the most general solution as

$$x(t) = A\sin(\omega t + \phi')$$

Thus you can use either the sine or the cosine to write the most general solution.

Both the sine and cosine are periodic with period 2π. This means that the value of x will repeat with a period T that satisfies

$$\omega t + \phi + 2\pi = \omega(t + T) + \phi \Rightarrow T = \frac{2\pi}{\omega}$$

Example 1 A spring-mass system with angular frequency ω is displaced an amount x_0 from equilibrium and released with 0 initial velocity. Obtain an expression for its position at any later time.

Solution The initial conditions are given

$$x(0) = x_0 \qquad v(0) = \frac{dx(0)}{dt} = 0$$

From the general form for the solution you have

$$x(0) = A\cos(\phi) \qquad \frac{dx}{dt} = -\omega A\sin(\omega t + \phi) \Rightarrow \frac{dx(0)}{dt} = -\omega A\sin(\phi)$$

Equating the respective quantities gives

$$x_0 = A\cos\phi \qquad 0 = -\omega A\sin\phi$$

The right equation tells you that $\phi = 0$ and thus $x_0 = A$. The solution in this case is then

$$x(t) = x_0\cos\omega t$$

When a spring hangs vertically, the gravitational force must be accounted for in applying Newton's second law. Figure 4.11 depicts a spring, initially unstretched, to which a mass is attached and then set in vibration. A free body diagram is shown on the right.

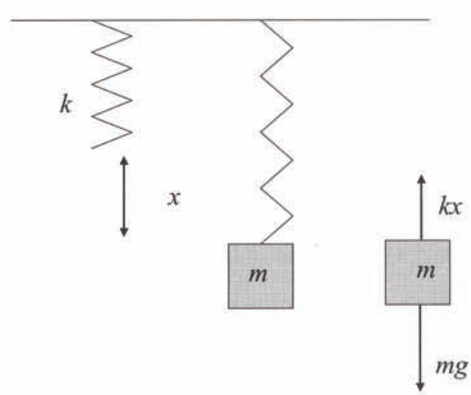


Figure 4.11

Applying the second law gives you

$$ma = m\frac{d^2x}{dt^2} = mg - kx \qquad \Rightarrow \qquad \frac{d^2x}{dt^2} + \frac{k}{m}x = g$$

This differs from the harmonic oscillator equation by the nonhomogeneous constant g on the right-hand side. Could it be that the motion is now more complex? No, not really. A quick way to see this is to rewrite the last equation.

$$\frac{d^2x}{dt^2} + \frac{k}{m}\left(x - \frac{mg}{k}\right) = 0$$

Now define a new displacement variable $x' = x - \frac{mg}{k}$. Because this differs from x by a constant, the derivatives of x' are the same as those of x, so the equation becomes

$$\frac{d^2x'}{dt^2} + \frac{k}{m}x' = 0$$

Thus you see that x' obeys the harmonic oscillator equation with the same angular frequency and period as the corresponding horizontal oscillator. The physical interpretation of x' is easy to establish. If the mass is gently added to the spring and allowed to come to rest, the spring will be stretched an amount $x = \frac{mg}{k}$. The variable x' measures the displacement from this new equilibrium point. Vertical springs behave as horizontal springs do, except that the center of oscillation is displaced by a constant amount $x = \frac{mg}{k}$. Save for this one aspect, gravity can be ignored in the analysis.

Perhaps the first oscillating system you ever encountered was a pendulum. When you sat on a swing and someone gave you a push, you and the swing formed a pendulum system. A simple pendulum, abstracted from this model, consists of a point mass m attached to a light string of length L. The mass moves only under the influence of the string and gravity. Because the mass is constrained to move along an arc, the displacement from equilibrium is given by $x = L\theta$, with the angle measured in radian units. The restoring force on the mass is given by $F = -mg\sin\theta$, as shown in Figure 4.12. The minus sign indicates that as the angle increases, the velocity decreases, since $F = m\frac{dv}{dt}$.

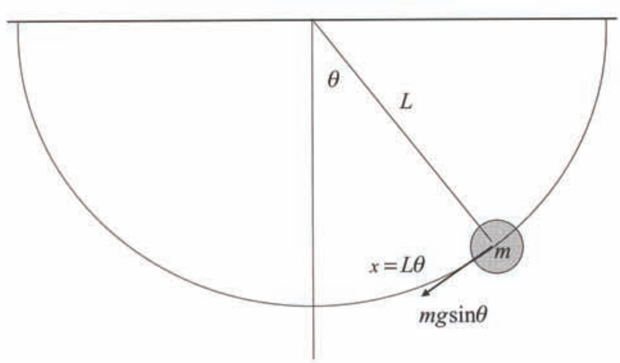


Figure 4.12

By applying the second law, you get

$$m\frac{d^2x}{dt^2} = mL\frac{d^2\theta}{dt^2} = -mg\sin\theta \qquad \Rightarrow \qquad \frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

This is a second-order, homogeneous equation, but it is nonlinear. Because you can write $\sin\theta$ as a series of powers of θ, this equation is actually very complicated. However, for values of θ that are not too big, a good approximation is

$$\sin\theta \cong \theta \qquad \text{radian measure used}$$

For example, even at the fairly large angle of 30° or $\frac{\pi}{6}$ rad,

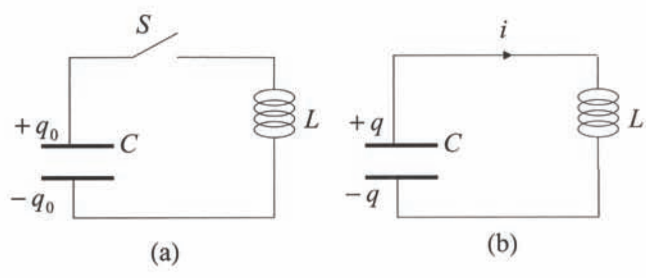
$$\frac{\theta}{\sin\theta} = 1.047$$

so there is less than a 5% error when using this approximation for angles this large. When you were on the swing, small angles were probably not too interesting for you, but the simple pendulum is only solvable in terms of elementary functions in this **small angle approximation.** Under these conditions, the second law equation then gives

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

You will recognize this as a harmonic oscillator equation with $\omega^2 = \sqrt{\frac{g}{L}}$. So you can say that a simple pendulum acts like a harmonic oscillator in the small angle approximation.

In AP Physics, oscillations also appear in circuit analysis. Figure 4.13a shows a capacitor with capacitance C and an inductor with self-inductance L connected in series with the capacitor initially charged to q_0. When the switch S is closed, the capacitor can discharge and a current will flow through the inductor as shown in Figure 4.13b.



Figures 4.13a and 4.13b

The voltage drops across the respective devices are given by

$$V_C = \frac{q}{C} \qquad V_L = -L\frac{di}{dt}$$

Applying Kirchhoff's second law, that the sum of the voltages around a closed loop must equal 0, you get

$$\frac{q}{C} - L\frac{di}{dt} = 0$$

Because $i = -\frac{dq}{dt}$ (increasing i means decreasing q, hence the minus sign) you can substitute and rearrange to get

$$\frac{d^2q}{dt^2} + \frac{q}{LC} = 0$$

Once again you have encountered the harmonic oscillator equation. The charge on the capacitor will oscillate with angular frequency and period

$$\omega = \frac{1}{\sqrt{LC}} \qquad T = 2\pi\sqrt{LC}$$

4.4 Key Formulas

Exponential decay equation	$\frac{dv}{dt} + kv = 0$	
Exponential decay solution	$v(t) = v_0 e^{-kt}$	k is a constant
Exponential approach equation	$\frac{dv}{dt} + kv = k'$	
Exponential approach solution	$v(t) = Ae^{-kt} + \frac{k'}{k}$	k', k, A constant
	$v(t) = \frac{k'}{k}(1 - e^{-kt})$	if $v(0) = 0$
Harmonic oscillator equation	$\frac{d^2x}{dt^2} + \omega^2 x = 0$	
Harmonic oscillator solution	$x(t) = A\cos(\omega t + \phi)$	A, ϕ constant

Practice Problems

1. A mass m falls vertically, subject to a retarding force that is proportional to the square of its velocity

$$F_{retard} = -kv^2$$

   a. Write the differential equation you get by analyzing this system with Newton's second law.
   b. Identify the type of differential equation you get.
   c. Find the terminal speed of the mass. (*Hint:* At terminal speed, the acceleration is 0.)

2. The generic exponential approach differential equation has the form

$$\frac{dv}{dt} + kv = k'$$

   Use "brute force" direct integration to find the solution to this equation. Assume $v(0) = v_0$.

3. How many RC time constants will it take for the capacitor in Figure 4.6 to decay to 1% of its initial charge?

4. Figure 4.14 shows a movable conducting rail of mass m and length L supported without friction by fixed conductors such that the movable rail and the conductors form a closed loop. Although the conductors have no electrical resistance, a resistor of resistance R is connected as shown. A magnetic field of strength B is directed into the page as indicated by the x symbols.

   When the rail has a velocity $\vec{v}$ directed to the right, a current is induced in the loop that flows counterclockwise as shown. The field then exerts a force $\vec{F}$ on the rail

$$\vec{F} = -\frac{B^2L^2}{R}\vec{v}$$

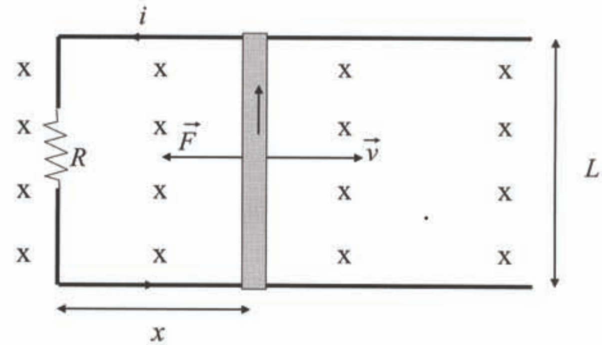


Figure 4.14

   a. Apply Newton's second law to this system to obtain a differential equation.
   b. Assuming the initial velocity of the rail is $\vec{v}_0$ directed to the right, obtain the full solution to the differential equation.
   c. Calculate the total displacement of the rail.

5. A linear, ordinary differential equation does not contain products of the function $f(x)$ or its derivatives. The most general form for such an equation is

$$\frac{d^nf}{dx^n} + a_{n-1}\frac{d^{n-1}f}{dx^{n-1}} + \cdots + a_0f = g(x)$$

   In this equation the coefficients a_m are constants and the function $g(x)$ is an arbitrary function of x. For example, if the equation is second-order, it would have the form

$$\frac{d^2f}{dx^2} + a_1\frac{df}{dx} + a_0f = g(x)$$

   When $g(x) = 0$, the equation is homogeneous. Suppose you have two solutions to the homogeneous equation, $f_1(x)$ and $f_2(x)$. Show that the linear combination

$$f_L(x) = c_1f_1(x) + c_2f_2(x)$$

   is also a solution to the homogeneous equation for any constants c_1 and c_2. The fact that you can add solutions to get new solutions is what makes linear homogeneous equations relatively simple.

6. A spring-mass system with mass 2 kg is described by the equation

$$x(t) = 5\cos\left(\frac{2\pi}{5}t + \frac{\pi}{4}\right)$$

Assume x is measured in meters. Calculate the following quantities:

a. period

b. maximum speed

c. spring constant

7. Within Chapter 4 you learned that the general solution to the harmonic oscillator equation could be expressed either in terms of a sine or a cosine function. However, the differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

will also allow exponential functions as solutions since the second derivative of an exponential is proportional to the original function. This problem is designed to let you investigate the nature of these exponential solutions.

a. Assume a solution for x with the form

$$x(t) = Ae^{\alpha t} \qquad \alpha, A \text{ are constants}$$

Show that this form will solve the equation for any A if $\alpha = \pm i\omega$. Here $i = \sqrt{-1}$. Thus the exponential solutions involve imaginary numbers in the argument of the exponential.

b. From the results of the previous problem it follows that the linear combination

$$x(t) = A_+e^{i\omega t} + A_-e^{-i\omega t} \qquad A_+, A_- \text{ are constants}$$

is also a solution. Show that if

$$A_+ = A_- = \frac{A}{2}$$

then $x(t) = A\cos\omega t$

Show also that if $A_+ = -A_- = \frac{A}{2i}$, then

$$x(t) = A\sin\omega t$$

(*Hint:* Use the results of Practice Problem 7, Chapter 2.)

These results basically tell you that the exponential solutions are contained in the trigonometric solutions already studied so they are not really independent solutions. However, in more advanced work you will find that working with exponential functions with imaginary arguments is easier than working with trigonometric functions.

8. A simple pendulum of mass m and length L is observed to be passing through the lowest point of its path with a velocity v_0 at $t = 0$ in the direction of increasing angle θ. Obtain an expression for its velocity at an arbitrary later time t.