Harvey Mudd College Math Tutorial:

The Chain Rule

You probably remember the derivatives of $\sin(x)$, x^8, and e^x. But what about functions like $\sin(2x-1)$, $(3x^2-4x+1)^8$, or e^{-x^2}? How do we take the derivative of **compositions** of functions?

The **Chain Rule** allows us to use our knowledge of the derivatives of functions $f(x)$ and $g(x)$ to find the derivative of the composition $f(g(x))$:

Suppose a function $g(x)$ is differentiable at x and $f(x)$ is differentiable at $g(x)$. Then the composition $f(g(x))$ is differentiable at x.
Letting $y = f(g(x))$ and $u = g(x)$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Using alternative notation,

$$\begin{aligned} \frac{d}{dx}\left[f(g(x))\right] &= f'(g(x))g'(x), \\ \frac{d}{dx}\left[f(u)\right] &= f'(u)\frac{du}{dx}. \end{aligned}$$

Proof

The three formulations of the Chain Rule given here are identical in meaning. In words, the derivative of $f(g(x))$ is the derivative of f, evaluated at $g(x)$, multiplied by the derivative of $g(x)$.

Examples

- To differentiate $\sin(2x-1)$, we identify $u = 2x-1$. Then

$$\begin{aligned} \frac{d}{dx}\left[\sin(2x-1)\right] &= \frac{d}{dx}\left[\sin(u)\right] \cdot \frac{d}{dx}\left[2x-1\right] \\ &= \cos(u) \cdot 2 \\ &= 2\cos(2x-1). \end{aligned}$$

$$\begin{aligned} f(x) &= \sin(x) \\ g(x) &= 2x-1 \\ f(g(x)) &= \sin(2x-1) \end{aligned}$$

- To differentiate $(3x^2-4x+1)^8$, we identify $u = 3x^2-4x+1$. Then

$$\begin{aligned}\frac{d}{dx}\left[\left(3x^2-4x+1\right)^8\right] &= \frac{d}{du}\left[u^8\right]\cdot\frac{d}{dx}\left[3x^2-4x+1\right]\\ &= 8u^7\cdot(6x-4)\\ &= 8(6x-4)\left(3x^2-4x+1\right)^7.\end{aligned}$$

$$\boxed{\begin{aligned}f(x) &= x^8\\ g(x) &= 3x^2-4x+1\\ f(g(x)) &= \left(3x^2-4x+1\right)^8\end{aligned}}$$

- To differentiate e^{-x^2}, we identify $u=-x^2$. Then

$$\begin{aligned}\frac{d}{dx}\left[e^{-x^2}\right] &= \frac{d}{du}\left[e^u\right]\cdot\frac{d}{dx}\left[-x^2\right]\\ &= e^u\cdot(-2x)\\ &= -2xe^{-x^2}.\end{aligned}$$

$$\boxed{\begin{aligned}f(x) &= e^x\\ g(x) &= -x^2\\ f(g(x)) &= e^{-x^2}\end{aligned}}$$

Sometimes you will need to apply the Chain Rule several times in order to differentiate a function.

Example

We will differentiate $\sqrt{\sin^2(3x)+x}$.

$$\begin{aligned}\tfrac{d}{dx}\left[\sqrt{\sin^2(3x)+x}\right] &= \frac{1}{2\sqrt{\sin^2(3x)+x}}\cdot\tfrac{d}{dx}\left[\sin^2(3x)+x\right] && f(u)=\sqrt{u}\\ &= \frac{1}{2\sqrt{\sin^2(3x)+x}}\cdot\left(2\sin(3x)\tfrac{d}{dx}\left[\sin(3x)\right]+1\right) && \begin{aligned}f(u)&=u^2\\ \tfrac{d}{dx}[x]&=1\end{aligned}\\ &= \frac{1}{2\sqrt{\sin^2(3x)+x}}\cdot\left(2\sin(3x)\cos(3x)\tfrac{d}{dx}[3x]+1\right) && f(u)=\sin(u)\\ &= \frac{1}{2\sqrt{\sin^2(3x)+x}}\cdot(2\sin(3x)\cos(3x)\cdot 3+1)\\ &= \frac{6\sin(3x)\cos(3x)+1}{2\sqrt{\sin^2(3x)+x}}\end{aligned}$$

Key Concepts

Let $g(x)$ be differentiable at x and $f(x)$ be differentiable at $f(g(x))$. Then, if $y=f(g(x))$ and $u=g(x)$,

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

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Harvey Mudd College Math Tutorial:

Product Rule for Derivatives

In Calculus and its applications we often encounter functions that are expressed as the product of two other functions, like the following examples:

- $h(x) = xe^x = (x)(e^x)$,
- $h(x) = x^2 \sin x = (x^2)(\sin x)$,
- $h(x) = e^{-x^2} \cos 2x = (e^{-x^2})(\cos 2x)$.

In each of these examples, the values of the function h can be written in the form

$$h(x) = f(x)g(x)$$

for functions $f(x)$ and $g(x)$. If we know the derivative of $f(x)$ and $g(x)$, the **Product Rule** provides a formula for the derivative of $h(x) = f(x)g(x)$:

$$\boxed{h'(x) = [f(x)g(x)]' = f'(x)g(x) + f(x)g'(x).}$$

Proof

We illustrate this rule with the following examples.

- If $h(x) = xe^x$ then

$$\begin{aligned} h'(x) &= (x)'e^x + x(e^x)' \\ &= e^x + xe^x. \end{aligned}$$

- If $h(x) = x^2 \sin x$ then

$$\begin{aligned} h'(x) &= (x^2)' \sin x + (x^2)(\sin x)' \\ &= 2x \sin x + x^2 \cos x. \end{aligned}$$

- If $h(x) = e^{-x^2} \cos 2x$ then

$$\begin{aligned} h'(x) &= (e^{-x^2})' \cos 2x + e^{-x^2}(\cos 2x)' \\ &= -2xe^{-x^2} \cos 2x - 2e^{-x^2} \sin 2x. \end{aligned}$$

Key Concepts

Product Rule

Let $f(x)$ and $g(x)$ be differentiable at x. Then $h(x) = f(x)g(x)$ is differentiable at x and $h'(x) = f'(x)g(x) + f(x)g'(x)$.

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Harvey Mudd College Math Tutorial:

Quotient Rule for Derivatives

Suppose we are working with a function $h(x)$ that is a ratio of two functions $f(x)$ and $g(x)$.

How is the derivative of $h(x)$ related to $f(x)$, $g(x)$, and their derivatives?

Quotient Rule

Let f and g be differentiable at x with $g(x) \neq 0$. Then f/g is differentiable at x and

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

Proof

Examples

- If $f(x) = \dfrac{2x+1}{x-3}$, then

$$\begin{aligned} f'(x) &= \frac{(x-3)\frac{d}{dx}[2x+1] - (2x+1)\frac{d}{dx}[x-3]}{[x-3]^2} \\ &= \frac{(x-3)(2) - (2x+1)(1)}{(x-3)^2} \\ &= -\frac{7}{(x-3)^2}. \end{aligned}$$

- If $f(x) = \tan x = \dfrac{\sin x}{\cos x}$, then

$$\begin{aligned} f'(x) &= \frac{\cos(x)\frac{d}{dx}[\sin(x)] - \sin(x)\frac{d}{dx}[\cos x]}{[\cos x]^2} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} \\ &= \sec^2(x), \end{aligned}$$

  verifying the familiar differentiation formula for $\tan(x)$.

- If $f(x) = \dfrac{1}{g(x)}$, then

$$\begin{aligned} f'(x) = \left[\frac{1}{g(x)}\right]' &= \frac{g(x)\frac{d}{dx}[1] - (1)g'(x)}{[g(x)]^2} \\ &= \frac{g(x)(0) - (1)g'(x)}{[g(x)]^2} \\ &= -\frac{g'(x)}{[g(x)]^2}. \end{aligned}$$

For example, $\dfrac{d}{dx}[x^{-4}] = \dfrac{d}{dx}\left[\dfrac{1}{x^4}\right] = -\dfrac{\frac{d}{dx}[x^4]}{[x^4]^2} = -\dfrac{4x^3}{x^8} = -\dfrac{4}{x^5} = -4x^{-5}$.

Key Concepts

Quotient Rule

Let f and g be differentiable at x with $g(x) \neq 0$. Then f/g is differentiable at x and

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

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