A Calculus Primmer – Part 1: Derivatives
The Reader's Digest® Version as Applied to Physics

Introduction

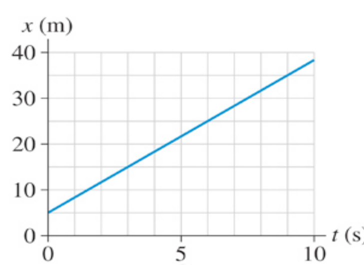
The notion of calculus being used in physics goes back to the days of Isaac Newton and Gottfried Leibnitz, the inventors of the calculus. Newton is given the credit, although both he and Leibnitz came up with the fundamental ideas about the same time.

So why calculus? We've already seen how we can solve position, velocity, and acceleration problems with conventional algebra. Why do we need to create more difficulty for ourselves? If you recall, the problems we dealt with in the past all assumed that something was *constant* – constant v, constant a, constant whatever. What if things aren't constant? What if, for example, a force acting on an object depends on the velocity of that object? This happens when we consider air resistance, a factor we conveniently swept under the rug in our previous classes.

This article is a brief explanation of how calculus is used in physics. It is not meant to replace the rigorous investigation you will be undertaking in your mathematics classes. It is just something we can put in place so we can get started looking at natural phenomena in ways that more closely represent real situations.

Differentiation

If we consider the motion of a particle as represented by the graph to the right, we understand that the particle is constantly moving farther and farther from the origin. Because the graph is linear, we can infer that the speed at which the particle is moving is constant. We know from our previous classes that the slope of the x/t graph is the speed of the particle. If we take the slope of this graph, we find that it is a constant, supporting our earlier inference.



In the case above, the speed of the particle is the same anywhere in time. The instantaneous speed is about 3.4 m/s and is the same for any value of t. We know that because the change in position (the y-axis) divided by the change in time (the x-axis) is the same anywhere we look.

When considering the equation for a straight line, the familiar formula is

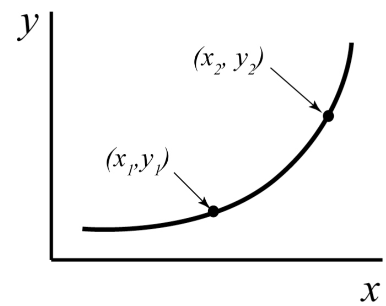
$$y = mx + b$$

where m is the slope of the line, or

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Notice that it doesn't matter where we choose to pick x_1, x_2, y_1, or y_2 simply because the line has the same slope everywhere.

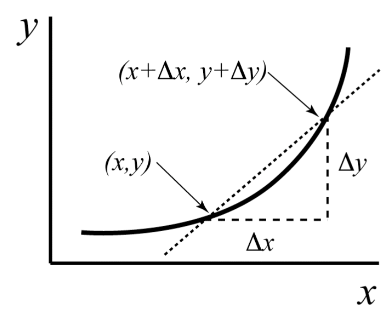
Now, what if we have a particle with a graph that looks more like the curve in the illustration to the right? Clearly, the slope of that line is not constant. For any points (x_1, y_1) and (x_2, y_2), the slope of the line, which is just the tangent to the line at that point, will be different. If this is a displacement/time graph where the slope of the line represents the speed of the particle, the *instantaneous* speed of this particle will not be constant.



So, what is the instantaneous slope at some point (x,y)?

Visualize this: pick a point (x,y) and a neighboring point close by at (x+Δx, y+Δy), and compute the slope of the line through those points, as illustrated to the right. The slope is given by

$$m = \frac{(y + \Delta y) - y}{(x + \Delta x) - x} = \frac{\Delta y}{\Delta x}$$



Now, let's let Δx get smaller. So small, in fact, that it approaches zero. This is called taking the limit of Δx and is designated mathematically as

$$\lim_{\Delta x \to 0} \Delta x = dx$$

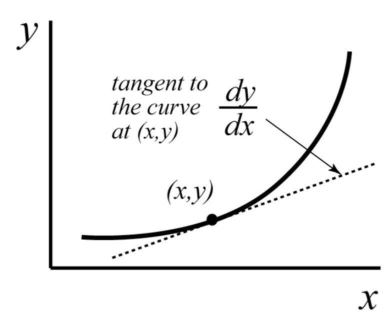
where *dx* represents the limit operation as applied to variable x.

Because y is a function of x, that is, *y=f(x),* y at some point x+Δx is just *y=f(x+Δx).* The slope of the tangent of the line at point (x,y) is

$$\text{slope of the tangent} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

This can be rewritten symbolically as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$



The expression $\frac{dy}{dx}$ is called the ***derivative*** of the function of y.

Let's try an example. Consider a particle that is moving in the y-plane with respect to time with the relationship

$$y = 6t^2$$

and we want to find the velocity of the particle with respect to time. We know the slope of the position/time curve will give us velocity.

At some time Δt after t, the position of the particle will be

$$y = 6(t + \Delta t)^2$$

Expanding this expression we get

$$y = 6t^2 + 12t\Delta t + 6\Delta t^2$$

The difference in the y-position from t to t+Δt will be

$$\Delta y = y_2 - y_1 = (6t^2 + 12t\Delta t + 6\Delta t^2) - 6t^2$$

or

$$\Delta y = y_2 - y_1 = 12t\Delta t + 6\Delta t^2$$

dividing by Δt will give us the average velocity of the particle in the interval of Δt.

$$\frac{\Delta y}{\Delta t} = \frac{12t\Delta t + 6\Delta t^2}{\Delta t} = 12t + 6\Delta t$$

Taking the above expression to the limit as Δt approaches zero results in

$$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \lim_{\Delta t \to 0} 12t + 6\Delta t$$

Notice that the 6Δt disappears as Δt goes to zero. Therefore, we can state

$$v = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt} = 12t$$

We have just taken the derivative of $y = 6t^2$ and found it to be $12t$. So let's consider the particle at some specific time, say t=3 s. The position of the particle is given by $y = 6t^2$, so at t= 3 s, the particle is at y=54 m; its instantaneous velocity is given by $v = 12t$, so v =36 m/s.

In the parlance of calculus, we have found that the derivative of the position/time graph yields the velocity/time graph. We also found that if we have a function for the velocity, we can find the instantaneous velocity at any time.

Although this discussion centered on position and velocity, the same consideration can be given to velocity and acceleration since acceleration is the slope of the velocity/time curve. Therefore, $a = \frac{dv}{dt}$.

Although the procedure described in the example above can be used to find the derivative of other functions, it is handy, and a huge time saver, to note that there are patterns that can be applied to general forms. What follows are a few common derivative relationships as found in the appendix of Halliday, Resnick and Walker. In these forms, the letters *u* and *v* stand for functions of *x*, and *a* and *m* are constants.

1. $\frac{dx}{dx} = 1$

2. $\frac{d}{dx}(au) = a\frac{du}{dx}$

3. $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$

4. $\frac{d}{dx}x^m = mx^{m-1}$

5. $\frac{d}{dx}\ln x = \frac{1}{x}$

6. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

7. $\frac{d}{dx}e^x = e^x$

8. $\frac{d}{dx}\sin x = \cos x$

9. $\frac{d}{dx}\cos x = -\sin x$

10. $\frac{d}{dx}\tan x = \sec^2 x$

11. $\frac{d}{dx}\cot x = -\csc^2 x$

12. $\frac{d}{dx}\sec x = \tan x \sec x$

13. $\frac{d}{dx}\csc x = -\cot x \csc x$

14. $\frac{d}{dx}e^u = e^u\frac{du}{dx}$

15. $\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$

16. $\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$

Here are a few more interesting ones you may see in the second semester.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{1}{v}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx} \qquad \frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dx} \qquad \frac{d}{dx}(e^u) = e^u\frac{du}{dx}$$