AP Physics

Important Calculus Rules

In the following discussions, $f(x)$ and $g(x)$ are arbitrary functions of x. a is a constant.

Differentiation

The derivative of a constant times a function

$$\frac{d}{dx}(a\,f(x)) = a\frac{d}{dx}f(x) \tag{1.1}$$

Example:

$$\frac{d}{dx}(6x^2) = 6\frac{d}{dx}(x^2) = 6\times 2x = 12x \tag{1.2}$$

The derivative of the sum of two functions

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x)+\frac{d}{dx}g(x) \tag{1.3}$$

Example:

$$\frac{d}{dx}(6x^2+x) = \frac{d}{dx}(6x^2)+\frac{d}{dx}(x) = 12x+1 \tag{1.4}$$

The derivative of the product of two functions

$$\frac{dy}{dx}(f(x)g(x)) = g(x)\frac{d}{dx}f(x)+f(x)\frac{d}{dx}g(x) \tag{1.5}$$

Example:

$$\frac{d}{dx}(x^2\sin x) = \sin x\frac{d}{dx}(x^2)+x^2\frac{d}{dx}(\sin x) = 2x\sin x + x^2\cos x \tag{1.6}$$

The derivative of the quotient of two functions

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{g(x)^2} \tag{1.7}$$

Example:

$$\frac{d}{dx}\left(\frac{2x+1}{x-3}\right) = \frac{(x-3)\frac{d}{dx}(2x+1)-(2x+1)\frac{d}{dx}(x-3)}{(x-3)^2} = \frac{2x-6-2x-1}{(x-3)^2} \tag{1.8}$$

$$\frac{d}{dx}\left(\frac{2x+1}{x-3}\right) = \frac{-7}{(x-3)^2} \tag{1.9}$$

The Chain Rule

If $f(x)$ and $g(x)$ are functions of x and $h(x) = f(g(x))$ then

$$\frac{d}{dx}h(x) = \frac{d\,f(x)}{d\,g(x)}\frac{d\,g(x)}{dx} \tag{1.10}$$

Example 1:

$$h(x) = \frac{1}{x^3 + 3} \tag{1.11}$$

In this example, $g(x) = x^3 + 3$ and $f(x) = (g(x))^{-1}$

Therefore,

$$\frac{d\,f(x)}{d\,g(x)} = \frac{d}{d\,g(x)}g(x)^{-1} = -(g(x))^{-2} \tag{1.12}$$

and

$$\frac{d}{dx}g(x) = \frac{d}{dx}(x^2 + 3) = 2x \tag{1.13}$$

Applying the rule from 1.10 yields

$$\frac{d\,f(x)}{d\,g(x)}\frac{d\,g(x)}{dx} = -\frac{2x}{(x^2 + 3)^2} \tag{1.14}$$

Example 2:

$$h(x) = (4x + 3)^2 \tag{1.15}$$

In this example, $g(x) = 4x + 3$ and $f(x) = (g(x))^2$

Therefore,

$$\frac{d\,f(x)}{d\,g(x)} = \frac{d}{d\,g(x)}g(x)^2 = 2g(x) = 2(4x + 3) \tag{1.16}$$

and

$$\frac{d}{dx}g(x) = \frac{d}{dx}(4x + 3) = 4 \tag{1.17}$$

Applying the rule from1.10 yields

$$\frac{d\,f(x)}{d\,g(x)}\frac{d\,g(x)}{dx} = 2(4x + 3)4 = 8(4x + 3) = 32x + 24 \tag{1.18}$$

Example 3:

$$h(x) = \sin(3x^2 + 2x + 5) \tag{1.19}$$

Did you see that $g(x) = 3x^2 + 2x + 5$ and $f(x) = \sin g(x)$?

Therefore

$$\frac{d}{dx}\sin(3x^2 + 2x + 5) = (6x + 2)\cos(3x^2 + 2x + 5) \tag{1.20}$$

Integration

Integral of a constant times a function

$$\int a\, f(x)dx = a\int f(x)dx \tag{2.1}$$

Example:

$$\int_a^b 5x^2 dx = 5\int_a^b x^2 dx = 5\left(\frac{x^3}{3}\right)\Bigg|_a^b = \frac{5}{3}\left(b^3 - a^3\right) \tag{2.2}$$

Integral of a sum of two functions

$$\int \left[f(x) + g(x)\right]dx = \int f(x)dx + \int g(x)dx \tag{2.3}$$

Example:

$$\int_a^b \left(5x^2 + x\right)dx = \int_a^b 5x^2\, dx + \int_a^b x\, dx = \left(\frac{5}{3}x^3\right)\Bigg|_a^b + \left(\frac{x^2}{2}\right)\Bigg|_a^b \tag{2.4}$$

$$\int_a^b \left(5x^2 + x\right)dx = \frac{5}{3}(b^3 - a^3) + \frac{1}{2}(b^2 - a^2) \tag{2.5}$$

Change of limits of integration

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx \tag{2.6}$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx \tag{2.7}$$

Example Problem 1

The acceleration of a mass pushed back and forth by a spring is $a(t) = B\cos\omega t$ where B and ω are constants. Find the position as a function of time. Assume that $v = 0$ m/s and $x = 0$ at $t = 0$.

This is a two-step process. First, integrate acceleration to yield velocity, and then integrate velocity to yield position. Here, $t_f = t$ and $t_0 = 0$.

$$v(t_f) - v(t_0) = \int_{t_0}^{t_f} B\cos\omega t\, dt = B\frac{1}{\omega}\sin\omega t\Bigg|_0^t = \frac{B}{\omega}\sin\omega t \tag{3.1}$$

Next, integrate $v(t)$ to yield position.

$$x(t) = \int_0^t v(t)dt = \int_0^t \frac{B}{\omega}\sin\omega t\, dt = \frac{B}{\omega}\left(-\frac{1}{\omega}\cos\omega t\right)\Bigg|_0^t \tag{3.2}$$

$$x(t) = -\frac{B}{\omega^2}\cos\omega t + \frac{B}{\omega^2} \tag{3.3}$$

Example Problem 2

The instantaneous velocity of a projectile traveling through air is the following function of time:

$$v(t) = 3.26t^2 - 61.14t + 655.9 \tag{4.1}$$

where $v(t)$ is measured in meters per second and t is measured in seconds. Assuming that $x = 0$ at $t = 0$, what is the position as a function of time? What is the position at $t = 3.0$ s?

$$x(t_f) - x(t_0) = \int_{t_0}^{t_f} 3.26t^2 - 61.14t + 655.9 \, dt \tag{4.2}$$

$$x(t) = 3.26\int_0^t t^2 \, dt - 61.14\int_0^t t \, dt + 655.9\int_0^t dt \tag{4.3}$$

$$x(t) = 3.26\left(\frac{t^3}{3}\right)\Bigg|_0^t - 61.14\left(\frac{t^2}{2}\right)\Bigg|_0^t + 655.9t\Big|_0^t \tag{4.4}$$

$$x(t) = 3.26\left(\frac{t^3}{3}\right) - 61.14\left(\frac{t^2}{2}\right) + 655.9t \tag{4.5}$$

$$x(t) = 1.087t^3 - 30.57t^2 + 655.9t \tag{4.6}$$

At $t = 3.0$ s, this yields

$$x(3.0) = 1.087 \times (3.0)^3 - 30.7 \times (3.0)^2 + 655.9 \times (3.0) \tag{4.7}$$

$$x(3.0) = 1721.92 \tag{4.8}$$

Example Problem 3

A particle undergoes an acceleration given by $a(t) = -2.5t + 4$ where acceleration is in m/s^2 and time is in seconds. At $t = 0$, the particle has an initial velocity of +15 m/s. Determine the velocity of the particle at $t = 7.0$ s.

$$v(t_f) - v(t_0) = \int_{t_0}^{t_f} -2.5t + 4 \, dt \tag{5.1}$$

$$v(t) - 15.0 = \int_0^t -2.5t \, dt + \int_0^t 4 \, dt = -2.5\left(\frac{t^2}{2}\right)\Bigg|_0^t + 4t\Big|_0^t \tag{5.2}$$

$$v(7.0) - 15.0 = -2.5\left(\frac{(7.0)^2}{2}\right) + 4(7.0) \tag{5.3}$$

$$v(7.0) = -79 \text{ m/s} \tag{5.4}$$