CHAPTER 2

Differentiation

The world is not a static place; things are constantly changing. Galaxies are flying apart, nuclei are fusing in stellar interiors, planets are circling the sun, and you and I move about in our daily lives observing change all around us. The mathematical language of change is calculus. **Differential calculus** precisely defines how a function is changing and it derives techniques for calculating this change. **Integral calculus** presents techniques for determining the cumulative effect of a changing function. This chapter will focus on the **derivative** of a function, the quantity that will tell you about how the function is changing.

2.1 Functions

Suppose you were to take a walk along a straight path, stopping after you travel 100 m. You could record your displacement x at consecutive times t. For each time, there is a unique displacement and symbolically you could write

$$x = x(t)$$

Or suppose two charges are separated. Then for each value of separation distance r, there will be a unique force F that the charges exert on each other, and symbolically you could write

$$F = F(r)$$

A **function** is a relation that uniquely specifies the values of one quantity in terms of some other quantity. The quantity that is being determined (x or F) is called the **dependent variable** and the quantity (t or r) that fixes this variable is called the **independent variable.** The examples given are functions of one independent variable, but you know from physics that things can get much more complicated. For example, the magnitude of magnetic force experienced by a charge depends on the field strength B, charge q, speed v, and angle θ of velocity with respect to the field. In this case you could write

$$F = F(B, q, v, \theta)$$

a function of four variables. For now, let's keep to functions of a single variable. If you understand their properties, the generalization to multivariable functions will not be too difficult.

A function can be displayed visually in a graph. Accordingly, our examples of functions of displacement and force might appear as follows:

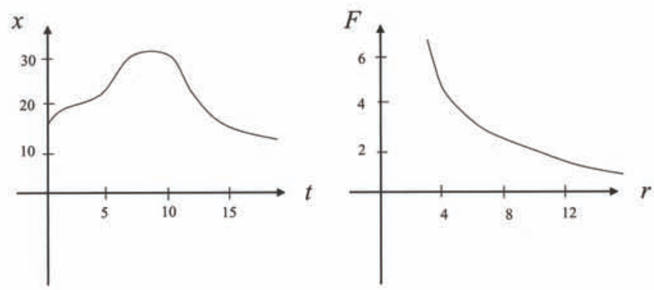

Figure 2.1

Each point on the graphs corresponds to the unique value of the dependent variable that is determined by the independent variable.

2.2 Linear Functions and Slope

As you might expect, the simplest of all functions shows the graph of a straight line. In such a function, the independent variable appears only to the first power, and you call the function **linear.** Following the notation of Algebra 1 textbooks throughout the world, let's denote y and x as dependent and independent variables, respectively. (In the numerical examples that follow, units of measure will be suppressed for mathematical and notational clarity.) In Algebra 1 you learned the slope-intercept form for the graph of a straight line:

$$y = y(x) = mx + b$$

Here m is the slope and b the y-intercept. The slope of the line can be determined if two points on the line are known.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

The slope of the line is what tells you how the function is changing. If the slope is 0, the graph is a horizontal line, and the value of y is not changing at all. If the slope is great, the graph is steep, and y is changing a lot for just a small change in x. For a straight line, the slope is constant, so it does not matter which two points you

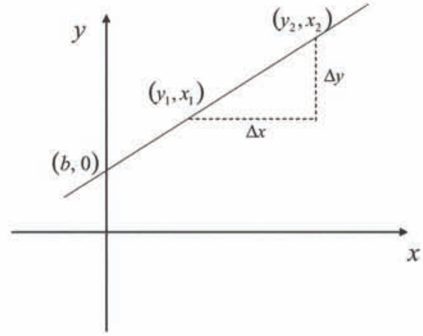

Figure 2.2

choose to calculate the slope. If we let point 2 of our preceding formula be arbitrary, we get the so-called **point-slope formula** for a line.

$$m = \frac{y - y_1}{x - x_1} \Rightarrow y = y_1 + m(x - x_1)$$

This last equation tells you that to find the value of y at some point x, just start with the value of y at x_1 and to it add the product of the slope and the difference between x and x_1. This material may all seem like the most basic to you, but in fact the definition of the derivative and its many uses depend on these concepts.

2.3 Nonlinear Functions and the Derivative

Unlike a linear or straight-line function, a **nonlinear function,** the graph of which shows curvature, does not have a well-defined slope. For the function graphed in Figure 2.3, the question "What is the slope of the curve?" is ambiguous.

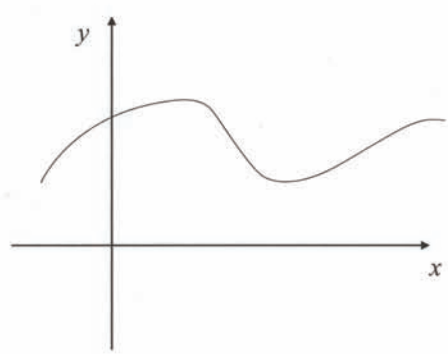

Figure 2.3

As you look at the graph, your intuition should tell you that the steepness is changing, but what is meant by *steepness* in this context? The answer lies in the observation that any curve will look like a straight line segment if you restrict yourself to a very small length of the curve. Imagine looking at a small region of the graph, centered on a particular point of your choosing, with a magnifying glass, as illustrated in Figure 2.4.

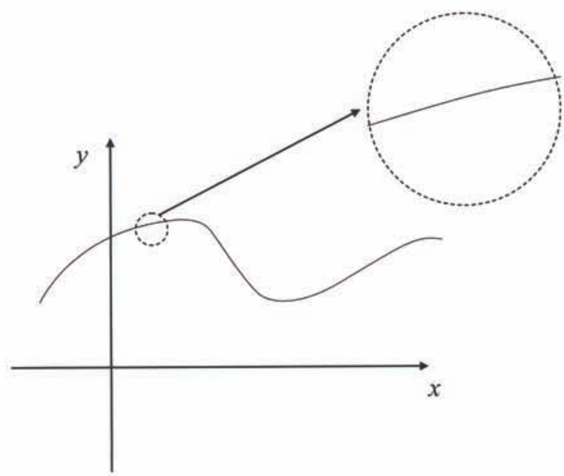

Figure 2.4 Magnifying a small portion of a curve produces a segment that approximates a straight line.

If the first magnification does not make the curve length look straight, then magnify a smaller region, again and again. In the *limit* of "infinite" magnification, certainly the graph will be straight. The concept of a limit is the new idea that calculus brings to the study of functions. So now we can answer the question posed above: The steepness of a function *at a particular point* is the slope of the line produced by infinite magnification of the function near that point. Thus the steepness depends on which point you choose, a result consistent with your intuition that the steepness is changing.

But how do you calculate this slope? The trick is to find the slope of the line that connects the point you have chosen with an arbitrary point that is close by. This line is called a **secant line** and is shown in Figure 2.5.

Figure 2.5 A secant line connects two points on a curve.

There are two things to note about the secant line.

- Its slope clearly is given by $\frac{\Delta y}{\Delta x}$; *and*
- Reducing the size of Δx is the equivalent of "magnifying." In the **limit** of very small Δx, the slope of the secant line will approach the slope of the curve as defined above through infinite magnification.

Now suppose you are given a function $y = y(x)$ or, more conventionally, $y = f(x)$. Then Δy is just the difference in the function at the two endpoints on the secant line

$$\Delta y = f(x + \Delta x) - f(x)$$

and the slope of the secant line is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Shrinking Δx to zero, you obtain the slope of a line that just touches the curve at the point chosen. This is called the **tangent line.**

$$\text{slope of tangent line} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The right-hand side of this equation is called the **derivative** of the function f, and is written symbolically as

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

To calculate the derivative, compute the slope of the secant line and take the limit $\Delta x \to 0$ after the computation.

Example 1 Find the derivative of the function $f(x) = 2x^3$ and find the slope of the line tangent to this function at $x = 1$.

Solution

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{2(x + \Delta x)^3 - 2x^3}{\Delta x} = \frac{2(x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3) - 2x^3}{\Delta x}$$
$$= 6x^2 + 6x\Delta x + 2\Delta x^2$$

Now if you take the limit $\Delta x \to 0$, only the first term survives.

$$\frac{df}{dx} = 6x^2$$

The slope of the tangent line at $x = 1$ is just the value of the derivative evaluated at this point.

$$\text{slope of tangent line} = 6(1)^2 = 6$$

You can generalize the result of Example 1 for any function that is a power of x other than 0.

$$f(x) = ax^n \qquad \frac{df}{dx} = nax^{n-1} \qquad n \neq 0$$

When $n = 0$, the function is just a constant with a graph that is a horizontal straight line with 0 slope, and since the tangent line of a line is just the line itself, it follows that the derivative is 0.

A polynomial is just a sum of such single-term functions and the derivative of a polynomial is just the sum of the derivatives of the terms.

Example 2 Find the derivative of the function $g(t) = 3t^4 - 2t^2 + 7$.

Solution You need to find $\frac{dg}{dt}$, and since this is just a sum of powers, the answer follows from the rule for the derivative of a power, remembering that the derivative of a constant is 0.

$$\frac{dg}{dt} = 12t^3 - 4t$$

To obtain the formulas for the derivatives of other common functions, you must use the definition of the derivative, but the limiting procedure for some functions can be a little complicated. Following are the formulas for the derivatives of some elementary functions. You may consult an introductory calculus book to see the details of their derivations, as well as the derivatives of many related functions.

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}\ln x = \frac{1}{x}$$

2.4 Linear Approximation to a Nonlinear Function

Recall the point-slope formula for a straight line:

$$y = y_1 + m(x - x_1)$$

This equation tells you that to find the value of y at some point x on a straight line, just start with the value of y at x_1 and add the product of the slope and the difference between x and x_1. This idea can also be applied to nonlinear functions $y = f(x)$. At some point (x_1, y_1) the tangent line itself will be a good approximation to the curve over some region (see Figure 2.6).

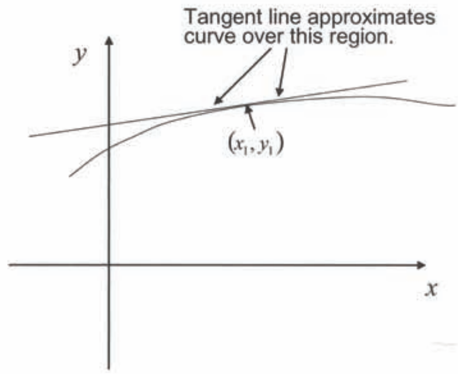

Figure 2.6

It is true that, if the function is changing rapidly, this region needs to be very small, but there will always be at least a small region where the **linear approximation** works very well. The slope of the tangent line at this point is

$$\text{slope} = \frac{df(x_1)}{dx}$$

The right-hand side of this equation means that you take the derivative with respect to x and then evaluate for x_1 using that derivative. Then, for points (x, y) nearby on the curve, we can use the point-slope formula to find the linear approximation to $f(x)$. This approximation, denoted as $f_1(x)$ because only x to the first power appears, is

$$f_1(x) = f(x_1) + \frac{df(x_1)}{dx}(x - x_1)$$

This equation is simply the point-slope formula for the tangent line drawn to the point (x_1, y_1), but you should now think of it as approximating the function itself, since the tangent line is so close to the function, over *at least some small region*. Linear approximations are used throughout physics and engineering for both practical and theoretical reasons.

Example 3 Find the linear approximation to the function $y = f(x) = x^2$ for values of x near 0.4. Compare the values of y that the linear approximation gives to the exact values.

Solution To proceed you need to find the equation of the line tangent to the function at $x = 0.4$. Since $\frac{df}{dx} = 2x$, our linear approximation is

$$f_1(x) = f(0.4) + \frac{df(0.4)}{dx}(x - 0.4) = 0.16 + 2(0.4)(x - 0.4)$$

$$f_1(x) = 0.8x - 0.16$$

A graph of the original function and the tangent line are shown in Figure 2.7.

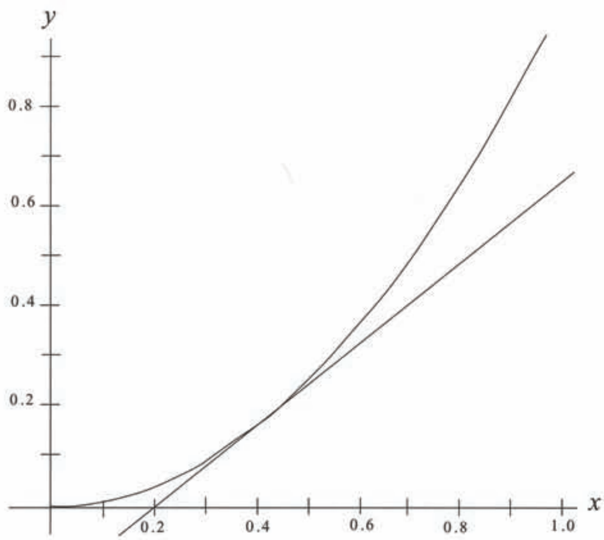

Figure 2.7

Notice how close the tangent line is to the function near $x = 0.4$. For $x = 0.42$, which is pretty close to 0.4, our linear approximation gives a value of $f_1(0.42) = 0.176$. The exact value obtained by plugging 0.42 directly into the original function x^2 is 0.1764, so our approximation gives less than 1% error. Looking at the graph, you can see the approximation will not work very well at $x = 0.6$. Here the linear approximation gives $f_1(0.6) = 0.32$ while the exact value is 0.36, an 11% error. Of course, you could get a better linear approximation near $x = 0.6$ by using a tangent line drawn closer to this point.

2.5 Rules for Differentiation

As a theoretical tool, the linear approximation can be used to derive some of the rules for finding derivatives. Understanding these derivations will give you more insight into the meaning of the rules and further solidify your understanding of the derivative. In this context you should imagine approximating the function over a small region of size Δx. Since you will be taking the limit $\Delta x \to 0$, the results obtained from such an approximation will be *exact*.

First, let's derive the rule for finding the derivative of products. Let $h(x) = f(x)g(x)$. Then

$$\frac{h(x + \Delta x) - h(x)}{\Delta x} = \frac{f(x + \Delta x)g(x + \Delta x) - f(x)g(x)}{\Delta x}$$

Now make the linear approximation to the first term in the numerator. The "1" subscript is suppressed since the approximation will be exact when the limit is taken.

$$f(x+\Delta x)g(x+\Delta x) = \left[f(x) + \Delta x\frac{df}{dx}\right]\left[g(x) + \Delta x\frac{dg}{dx}\right]$$
$$= f(x)g(x) + \Delta x\left[f(x)\frac{dg}{dx} + \frac{df}{dx}g(x)\right] + \Delta x^2\frac{df}{dx}\frac{dg}{dx}$$

Plugging this back into the first equation above gives

$$\frac{h(x+\Delta x) - h(x)}{\Delta x} = f(x)\frac{dg}{dx} + \frac{df}{dx}g(x) + \Delta x\frac{df}{dx}\frac{dg}{dx}$$

Now, taking the limit $\Delta x \to 0$, you get the **product rule** for the derivative.

$$h = fg \qquad \frac{dh}{dx} = f\frac{dg}{dx} + \frac{df}{dx}g$$

In this last step, the x dependence has been suppressed in a common notational convenience: you write f for $f(x)$, g for $g(x)$, etc.

Another important derivative formula is the **chain rule,** which tells you how to take the derivative of a function of a function. If you have two functions, $f(x)$ and $g(x)$, what is the derivative of the function $h(x) = f(g(x))$? The key to answering this question is to treat the function g as if it was an independent variable. Again, first use the linear approximation

$$g(x+\Delta x) = g(x) + \Delta x\frac{dg}{dx} = g + \Delta g$$

where $\Delta g \equiv \Delta x\frac{dg}{dx}$, in the last step, defines a small change in the function. Note that the derivative in this definition is evaluated at x. Next, let's look at

$$f(g(x+\Delta x)) = f(g+\Delta g)$$

Use the linear approximation on this expression for f.

$$f(g+\Delta g) = f(g(x)) + \Delta g\frac{df}{dg}$$

Note that in the last step, you differentiate with respect to g and evaluate everything at x. Now you are ready to put everything together.

$$\frac{dh}{dx} = \lim_{\Delta x\to 0}\frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(g+\Delta g) - f(g)}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(g) + \Delta g\frac{df}{dg} - f(g)}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{\Delta g}{\Delta x}\frac{df}{dg} = \frac{df}{dg}\frac{dg}{dx}$$

And so, when you have a function of a function of x, to find the overall derivative with respect to x, you first take the derivative with respect to the interior function and then multiply by the derivative of the interior function with respect to x.

Example 4 Find the derivative of the function $h(x) = \dfrac{1}{x^2 + 3}$.

Solution You can write this as $h(x) = (x^2 + 3)^{-1}$, and in the context of the previous discussion, $f(g) = g^{-1}$, $g(x) = x^2 + 3$ and $h(x) = f(g(x))$. Then, from the chain rule and your knowledge of the derivative of a power you get

$$\frac{dh}{dx} = \frac{df}{dg}\frac{dg}{dx} = (-1)g^{-2} \cdot 2x = -\frac{2x}{(x^2 + 3)^2}$$

You can use the chain rule with the product rule to derive a formula for the derivative of a quotient of two functions. Let $h(x) = \dfrac{r(x)}{s(x)} = rs^{-1}$. Then

$$\frac{dh}{dx} = \frac{dr}{dx}s^{-1} + r\frac{d(s^{-1})}{dx} = \frac{dr}{dx}s^{-1} - rs^{-2}\frac{ds}{dx} = \frac{1}{s}\frac{dr}{dx} - \frac{r}{s^2}\frac{ds}{dx}$$

Rather than memorize this messy expression as a separate formula, it is probably best to always treat quotients as products and just use the product rule and the chain rule.

2.6 Higher-Order Derivatives and Approximations

What has been referred to so far as the derivative is more precisely called the first derivative. The derivative is itself a function and you can take *its* derivative. This produces the derivative of a derivative, or the second derivative, which is written as

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right).$$

As you might guess, you can keep going to **higher-order derivatives,** the third derivative, fourth derivative, ...nth derivative, which are written as

$$\frac{d^3f}{dx^3}, \frac{d^4f}{dx^4}, \ldots \frac{d^nf}{dx^n}.$$

Example 5 Find the first and second derivatives of the function $f(x) = \dfrac{x}{x^3 + 1}$.

Solution Use the chain and product rules.

$$\frac{df}{dx} = \frac{1}{x^3 + 1} + x \cdot (-1)\frac{1}{(x^3 + 1)^2} \cdot (3x^2) = \left(\frac{1}{x^3 + 1}\right)\left(1 - \frac{3x^3}{x^3 + 1}\right)$$

$$\frac{d^2f}{dx^2} = \left(\frac{-3x^2}{(x^3 + 1)^2}\right)\left(1 - \frac{3x^3}{x^3 + 1}\right) + \left(\frac{1}{x^3 + 1}\right)\left\{\left(\frac{-9x^2}{x^3 + 1}\right) + \left(\frac{9x^5}{(x^3 + 1)^2}\right)\right\}$$

You have seen that the linear approximation, which involves the first derivative of a function, can be used to approximate a function over some region. Let's take another look at this approximation.

$$f_1(x) = f(x_1) + \frac{df(x_1)}{dx}(x - x_1)$$

Notice that when you evaluate this function at $x = x_1$ you get

$$f_1(x_1) = f(x_1)$$

and when you take the first derivative you get

$$\frac{df_1(x_1)}{dx} = \frac{df(x_1)}{dx}$$

The linear approximation agrees exactly with the actual function and its first derivative at the point where the approximation is applied. Of course, if the actual function has higher-order derivatives, the linear approximation tells you nothing about them. So you might ask: Can we extend the linear approximation so that the approximating function agrees with the actual function to higher-order derivatives? The answer is *yes* and you only need to generalize the form of the linear approximation. The function defined by

$$f_n(x) = f(x_1) + (x - x_1)\frac{df(x_1)}{dx} + \frac{1}{2!}(x - x_1)^2\frac{d^2f(x_1)}{dx^2} + \cdots + \frac{1}{n!}(x - x_1)^n\frac{d^nf(x_1)}{dx^n}$$

is identical to the exact function at $x = x_1$ up to the nth derivative

$$f_n(x_1) = f(x_1) \qquad \frac{df_n(x_1)}{dx} = \frac{df(x_1)}{dx} \qquad \cdots \frac{d^nf_n(x_1)}{dx^n} = \frac{d^nf(x_1)}{dx^n}$$

In using $f_n(x)$ you are approximating the exact function with a polynomial of degree n. Of course, the linear approximation uses a polynomial of degree 1.

Example 6 Find the second degree polynomial that best approximates the function $f(x) = \dfrac{1}{(x-2)^2 + 1}$ near the point $x = 1$.

Solution From the discussion above, it follows that the second order polynomial is

$$f_2(x) = f(1) + (x - 1)\frac{df(1)}{dx} + \frac{1}{2}(x - 1)^2\frac{d^2f(1)}{dx^2}$$

You need to calculate the first and second derivative of the function at $x = 1$.

$$\frac{df}{dx} = \frac{2(2-x)}{((x-2)^2+1)^2} \Rightarrow \qquad \frac{df(1)}{dx} = \frac{1}{2}$$
$$\frac{d^2f}{dx^2} = \frac{-2}{((x-2)^2+1)^2} + \frac{8(x-2)^2}{((x-2)^2+1)^3} \Rightarrow \frac{d^2f(1)}{dx^2} = \frac{1}{2}$$

Then

$$f_2(x) = \frac{1}{2} + \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2$$

Figure 2.8 shows the graph of the function in bold, as well as both the linear approximation $f_1(x)$ and the second order approximation $f_2(x)$. It is clear that the higher-order polynomial does a better job in approximating the function near $x = 1$.

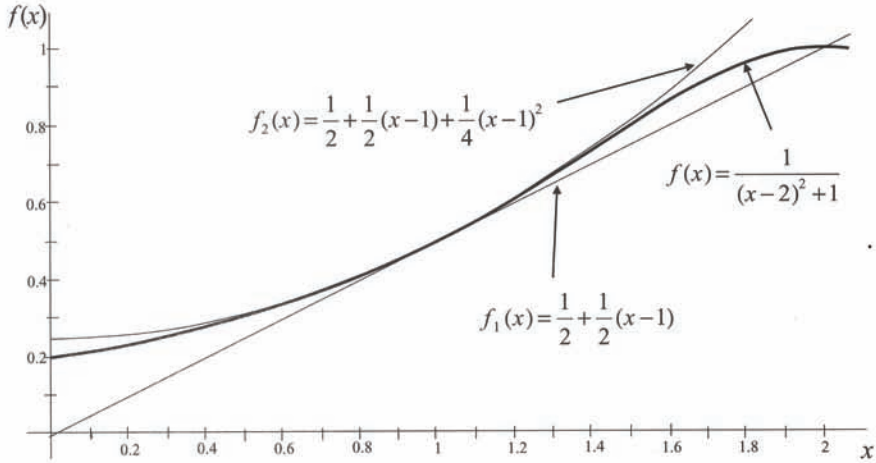

Figure 2.8

Suppose a function is well behaved at some point, so that both the function and its derivatives are defined at the point. Then it is possible to obtain better and better accuracy in approximating the function near that point by going to higher and higher orders in the polynomial. Theoretically you can represent the function *exactly* by keeping an infinite number of terms. Such a representation of a function is called a **Taylor series.**

$$f(x) = f(x_1) + (x - x_1)\frac{df(x_1)}{dx} + \cdots + \frac{1}{n!}(x - x_1)^n \frac{d^n f(x_1)}{dx^n} + \cdots$$

$$f(x) = f(x_1) + \sum_{n=1}^{\infty} \frac{1}{n!}(x - x_1)^n \frac{d^n f(x_1)}{dx^n}$$

If the point $x = 0$ is chosen to begin the expansion, the series is called a **Maclaurin series.**

$$f(x) = f(0) + \sum_{n=1}^{\infty} \frac{1}{n!} x^n \frac{d^n f(0)}{dx^n}$$

Because these series involve an infinite number of terms, there may be limitations on the values of x allowed. For values of x outside the allowed range, the series **diverges;** that is, as more terms are added, the computed value of the function grows without limit.

Example 7 The Maclaurin series for $\sin x$ **converges** for all values of x. Write the Maclaurin series for $f(x) = \sin x$.

Solution $f(0) = \sin 0 = 0$

$$\frac{d}{dx}\sin x = \cos x = 1 \text{ when } x = 0 \qquad \frac{d^2}{dx^2}\sin x = \frac{d}{dx}\cos x = -\sin x = 0 \text{ when } x = 0$$

Only the odd terms will contribute and it is not hard to see that

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-)^n \frac{x^{2n+1}}{(2n+1)!}$$

The Maclaurin series for $\cos x$ and e^x are

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots = \sum_{n=0}^{\infty} (-)^n \frac{x^{2n}}{(2n)!} \qquad \text{all real } x$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{all real } x$$

The Maclaurin series is very useful in approximating a function near $x = 0$. For small enough x, only the first couple of terms have to be kept.

Example 8 Write the first three terms of the Maclaurin series for $f(x) = (1 + x)^n$.

Solution

$$f(0) = 1 \qquad \frac{df}{dx} = n(1 + x)^{n-1} \Rightarrow \frac{df(0)}{dx} = n$$

$$\frac{d^2f}{dx^2} = n(n-1)(1 + x)^{n-2} \Rightarrow \frac{d^2f(0)}{dx^2} = n(n-1)$$

Then you can write

$$f(x) = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

When n is a positive integer so that f is a polynomial of order n, the series ends after n terms because the higher coefficients of x are all 0. The form for the series in this case looks like this:

$$f(x) = \sum_{m=0}^{\infty} \binom{n}{m} x^m \qquad \binom{n}{m} = \frac{n!}{m!(n-m)!}$$

The coefficients $\binom{n}{m}$ are called **binomial coefficients,** the familiar "n choose m" quantities that appear in statistics and probability.

Note: If n is not a positive integer, the series does not terminate, but the series will always converge to a finite result if $x < 1$. This Maclaurin series is called a **binomial expansion** because of the form of the coefficients.

2.7 Velocity and Acceleration

When an object moves in one dimension, its position x, as measured with respect to some origin, is a function of time t; that is, $x = x(t)$. The average velocity over some interval of time from t_1 to $t_1 + \Delta t$ is defined to be the change in position over the interval, or the displacement, divided by the size of the time interval.

$$v_{avg} = \frac{x(t_1 + \Delta t) - x(t_1)}{\Delta t} = \frac{\Delta x}{\Delta t}$$

If you were to make a graph of x against t, the average velocity would be the slope of the secant line connecting the two points on the graph that define the interval, as shown in Figure 2.9.

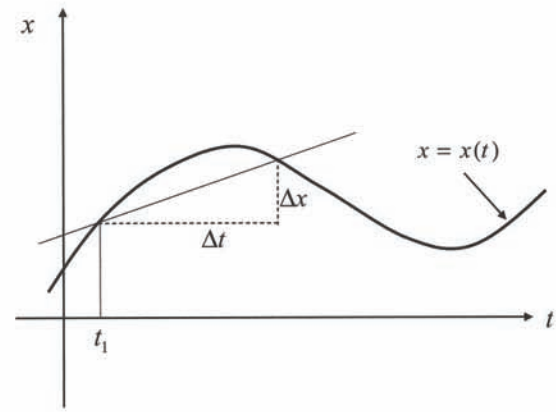

Figure 2.9

The instantaneous velocity is defined as the average velocity over an infinitesimal interval, and this is just the derivative of x with respect to t.

$$v(t_1) = \lim_{\Delta t \to 0} \frac{x(t_1 + \Delta t) - x(t_1)}{\Delta t} = \frac{dx(t_1)}{dt}$$

Of course, this is the slope of the line drawn tangent to the point x defining the beginning of the interval, as illustrated in Figure 2.10.

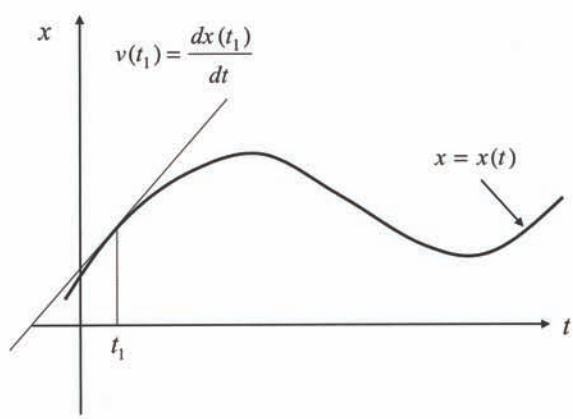

Figure 2.10

Example 9 A particle moves along the y-axis according to the equation $y(t) = 10t^2e^{-3t}$.

a. Find the instantaneous velocity of the particle at $t = 1$.

b. Find at what time the velocity is 0.

Solutions

$$v(t) = \frac{dy}{dt} = 20te^{-3t} - 30t^2e^{-3t} = 10te^{-3t}(2 - 3t)$$

So, for **a**: $v(1) = 10e^{-3}(-1) = -0.5$

for **b**: $v = 0 \Rightarrow 2 - 3t = 0 \qquad t = 0.67$

When an object has a nonconstant velocity, you say the object is **accelerating.** The average acceleration over some interval is defined as the change in the velocity during the interval, divided by the size of the time interval:

$$a_{avg} = \frac{v(t_1 + \Delta t) - v(t_1)}{\Delta t} = \frac{\Delta v}{\Delta t}$$

Note the similarity of this definition to that of average velocity. As you might suspect, instantaneous acceleration is obtained in the limit of infinitesimal Δt, yielding the derivative of velocity and the second derivative of position.

$$a(t) = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Example 10 A mass moves along the x-axis according to the relation $x(t) = 4 - 5t + 2t^3$. Find the acceleration of the mass at $t = 2$.

Solution Find the second derivative and evaluate at $t = 2$.

$$v = \frac{dx}{dt} = -5 + 6t^2 \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 12t \Rightarrow a(2) = 24$$

When an object moves in two or three dimensions, its position is described by the position vector. As time proceeds, the tip of this vector will map out a **trajectory,** or motion path, of the object. Figure 2.11 shows a two-dimensional example.

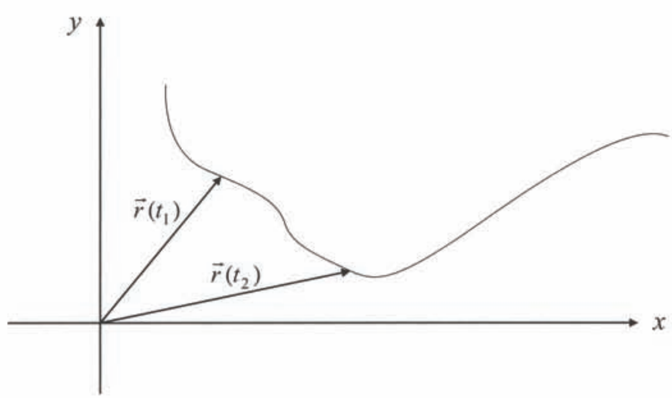

Figure 2.11

You define average velocity in two or more dimensions in a manner analogous to what is done in one dimension: the average velocity is the displacement, during the time interval, divided by the size of the interval:

$$\vec{v}_{avg} = \frac{\vec{r}(t_1 + \Delta t) - \vec{r}(t_1)}{\Delta t} = \frac{\Delta \vec{r}}{\Delta t}$$

The new element here is the fact that displacement and velocity are now vectors. Since dividing by Δt does not change the direction of a vector, the *direction* of the average velocity is the same as the direction $\Delta \vec{r}$, which Figure 2.12 shows as just a vector connecting the tips of the two displacement vectors.

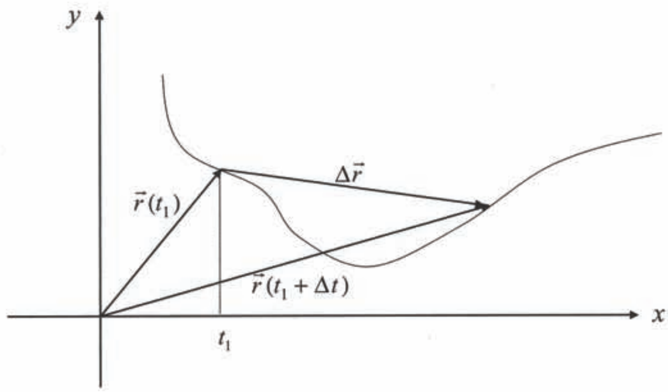

Figure 2.12

The instantaneous velocity is then obtained by taking the limit of infinitesimal Δt.

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt}$$

As the interval shrinks to 0, the difference vector $\Delta \vec{r}$ will become tangent to the trajectory, as seen in Figure 2.13, so that the instantaneous velocity is always tangent to the trajectory.

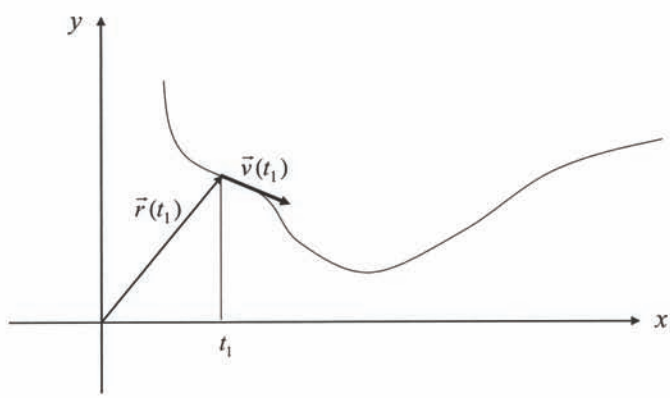

Figure 2.13

This gives you a graphical picture of the velocity, but how do you find the velocity analytically? Because the position vector is changing with time, each of its components is a function of time as well. Thus you might write

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

where the explicit dependence on t is shown. The unit vectors $\hat{i}$ and $\hat{j}$ do not vary: They are always length 1 pointing in either the x or y direction. The limiting process used in defining the derivative, therefore, implies that

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = v_x(t)\hat{i} + v_y(t)\hat{j}$$

Thus, each velocity component is obtained by taking the derivative of the respective position component.

Example 11 An object moves in two dimensions with position vector

$$\vec{r}(t) = (4 - e^{2t})\hat{i} + (3t^2 + 2t)\hat{j}$$

Find the velocity vector at $t = 2$.

Solution Take the derivative and then evaluate at $t = 2$.

$$\frac{d\vec{r}}{dt} = -2e^{2t}\hat{i} + (6t + 2)\hat{j} \qquad \vec{v}(2) = -109\hat{i} + 14\hat{j}$$

It is a straightforward extension of these ideas to define acceleration in two or more dimensions.

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

Example 12 Using the equation from the previous example, find the acceleration at $t = 1$.

Solution

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = -4e^{2t}\hat{i} + 6\hat{j} \qquad \vec{a}(1) = -29.56\hat{i} + 6\hat{j}$$

When an object is constrained to rotate about a fixed axis, every point on the object moves in a circle with radius determined by how far away from the axis it is.

Once you know the angle through which the object has rotated, you know the positions of every particle in the body because their relative positions do not change. Thus it is easiest to describe this kind of motion with the angle $\theta = \theta(t)$. Of course, this function is changing. The average angular velocity ω over a time interval is defined as the change in the angle, the angular displacement, divided by the magnitude of the time interval.

$$\omega_{avg} = \frac{\Delta\theta}{\Delta t}$$

As usual, you get the instantaneous value in the limit of infinitesimal time intervals, leading to the derivative.

$$\omega = \frac{d\theta}{dt}$$

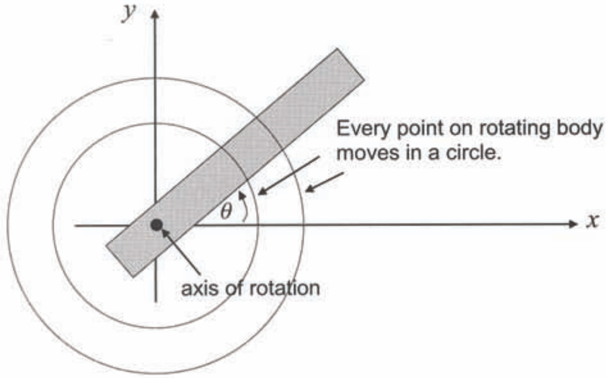

Figure 2.14

When you have a changing angular velocity, then angular acceleration is occurring. As you should expect by now, the instantaneous angular acceleration α is

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

2.8 Maxima, Minima, and Oscillations

If a function $f(x)$ is fairly complex, its graph may have "hills" and "valleys." The peak of a hill is called a **relative maximum** because the value of the function at this point is greater than at any nearby point. The bottom of a valley is called a **relative minimum** because the value of the function at this point is less than at any nearby point. The tangent line drawn to either a maximum or a minimum point will be horizontal with 0 slope. Thus the derivative $\frac{df}{dx}$, when evaluated at either a maximum or a minimum, as depicted in Figure 2.15, will be 0.

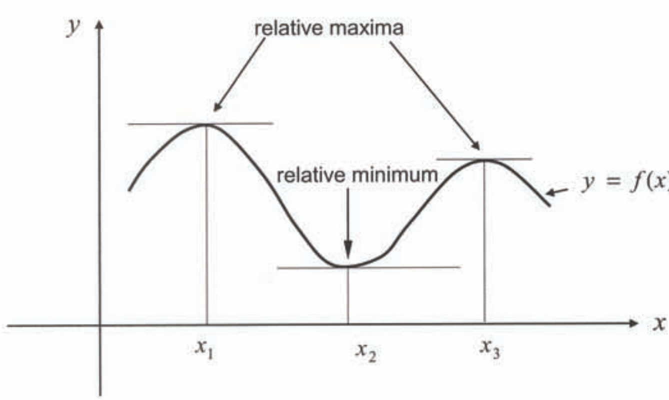

Figure 2.15 At maxima and minima, tangent lines have 0 slope.

$$\frac{df(x_1)}{dx} = \frac{df(x_2)}{dx} = \frac{df(x_3)}{dx} = 0$$

The fact that the first derivative must vanish at these relative extreme points gives you a condition to determine where these points are, provided you are given the function.

Example 13 Find the relative extreme points for the function

$$y = f(x) = x^3 - 6x^2 + 9x + 1$$

Solution The extreme points are determined by the condition $\frac{df}{dx} = 0$. Thus you have

$$\frac{df}{dx} = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0$$

This means that at $x = 1$ and at $x = 3$, the function reaches relative extreme points.

Of course, the next question is: How do you tell if the point is a maximum or a minimum? There are various ways to answer this question. You could type the equation into your graphing calculator and look at it. Another approach is to look at the asymptotic behavior, or limiting behavior, of the function. In the function from Example 13, as x gets very large as a negative, the cubic term dominates, and the function must be negative. As x gets very large as a positive, the cubic term dominates again, and the function must be positive. There are only two extreme points; therefore, these conditions can be satisfied only if the $x = 1$ point is a maximum and the $x = 3$ point is a minimum. For example, if $x = 3$ was a relative maximum and $x = 1$ was a relative minimum, there would have to be *another* maximum to the left of $x = 1$ to get the function into negative territory for large negative x. The function is graphed in Figure 2.16 to allow you to visualize the situation.

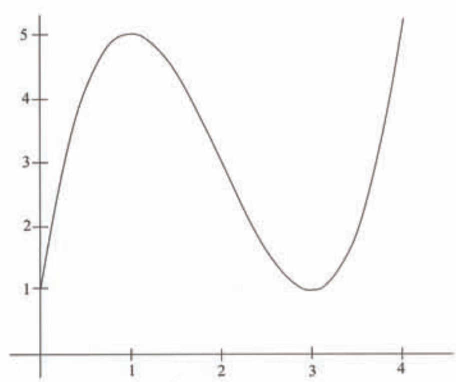

Figure 2.16

Another approach is to use physical intuition. If the particular problem has a physical interpretation, you can often infer from the physics whether the point is a "max" or a "min." There is a more systematic way of going about this task, which is useful in more difficult cases. Near a minimum, you can draw three tangent lines, one just to the left of the min, one at the min, and one just to the right of the min.

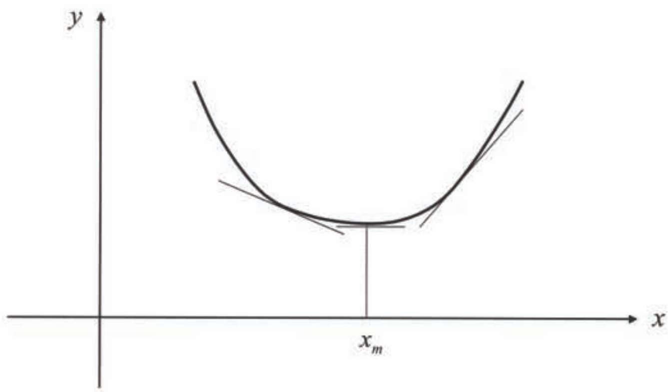

Figure 2.17 Near a minimum, the slope of the tangent line goes from negative to positive as x increases.

As the graph in Figure 2.17 indicates, the slope of the tangent line is increasing because it goes from negative to positive as x increases. This means that the first derivative $\frac{df}{dx}$ is an increasing function in the neighborhood of the minimum. Because it is an increasing function, its slope must be positive. But the slope of the first derivative is the second derivative, so it follows that for a minimum,

$$\frac{d^2f(x_m)}{dx^2} > 0 \qquad x_m \text{ a minimum point}$$

A similar argument can be made for a maximum point:

$$\frac{d^2f(x_m)}{dx^2} < 0 \qquad x_m \text{ a maximum point}$$

Let's go back to Example 13, where $y = f(x) = x^3 - 6x^2 + 9x + 1$. The extreme points were at $x = 1$ and $x = 3$. The second derivative is

$$\frac{d^2f}{dx^2} = 6x - 12$$

At $x = 1$ the second derivative is negative, so this point is a maximum. At $x = 3$ the second derivative is positive, so this point corresponds to a minimum.

An important physics application of relative minima is in the study of oscillations. Points of stable equilibrium are associated with the minima of **potential energy (PE).** If the system is displaced from the equilibrium position, it will oscillate about this point. The maximum excursions from the equilibrium point will be determined by how much energy the system has: When the total energy is all PE, the system has reached a turning point where the velocity is 0. In Figure 2.18, a one-dimensional potential energy function $U(x)$ is graphed. The turning points x_1 and x_2 are determined by the intersection of the PE graph with the total energy E.

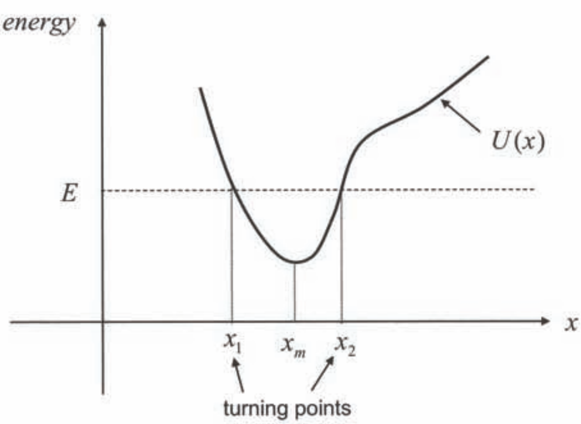

Figure 2.18

Suppose the system stays pretty close to the minimum; that is, the oscillations are small. Then it makes sense to approximate $U(x)$ with a polynomial near x_m.

$$U(x) = U(x_m) + (x - x_m)\frac{dU(x_m)}{dx} + \frac{1}{2!}(x - x_m)^2\frac{d^2U(x_m)}{dx^2} + \cdots$$

Since x_m is a minimum, the first derivative is 0. The constant $U(x_m)$ is not important since you can always redefine the zero point of energy by adding a constant. Thus, with such a redefinition of energy, the first two terms in the approximation vanish and you can write

$$U(x) = \frac{1}{2!}(x - x_m)^2\frac{d^2U(x_m)}{dx^2} + \cdots$$

If you define a new variable $x' = x - x_m$, the displacement from equilibrium, and if you ignore terms beyond second order because they are so very small for small oscillations, you can write

$$U(x') = \frac{1}{2}kx'^2 \qquad k \equiv \frac{d^2U(x_m)}{dx^2}$$

You should recognize this as the PE for a **simple harmonic oscillator**—for example, a spring-mass system. Thus, near a minimum of PE, you can always make the approximation of treating the system as a simple harmonic oscillator with a "k" value determined by the second derivative. From a graphic point of view, the PE is being approximated by a parabola. So we see the simple harmonic oscillator as such an important system to study not only because it can be easily solved, but also because of its wide applicability: An arbitrary PE function can always be approximated near a relative minimum with a simple harmonic oscillator PE function.

Example 14 Determine the potential energy function that can be used to approximate, to second order, the following function near its minimum at $x = 2$.

$$U(x) = -\frac{1}{(x-2)^2+1} + 1$$

Solution First, find the value of the second derivative at $x = 2$.

$$\frac{dU}{dx} = \frac{2x-4}{[(x-2)^2+1]^2} \qquad \frac{d^2U}{dx^2} = \frac{2}{[(x-2)^2+1]^2} - \frac{8(x-2)^2}{[(x-2)^2+1]^3}$$

$$\frac{d^2U(2)}{dx^2} = 2$$

Note that $\dfrac{dU(2)}{dx} = 0$, since this a minimum point. Thus you can write

$$U(x) = \frac{1}{2}(x-2)^2\frac{d^2U(2)}{dx^2} = \frac{1}{2}(2)(x-2)^2$$

A graph of the function and the approximating parabola are illustrated.

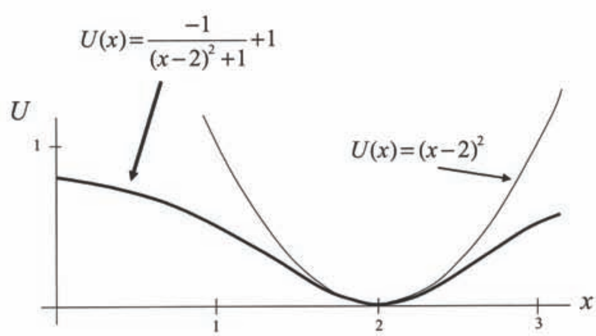

Figure 2.19

If you define $x' = x - 2$ as the displacement from the minimum point, then $U(x') = x'^2$. For small oscillations the mathematical description of this system is exactly the same as a spring-mass system with spring constant $k = 2$ units (of course the units will depend on the units used for x and U).

2.9 Vectors from Scalars

Suppose an object moves in one dimension, say along the x-axis, under the influence of a conservative force $\vec{F}$ that is directed along the x-axis. If the force is constant, then the work done by the force is $W_F = F\Delta x$ where Δx is the displacement. Because the force is conservative, the work done is the negative of the change in potential energy, so you can write

$$\Delta U = -F\Delta x \Rightarrow F = -\frac{\Delta U}{\Delta x}$$

If the force is not constant, but depends on x, then this equation will not be exact because the value of F changes over the displacement. However, in the limit of infinitesimal displacements, the equation will become exact and you have

$$F(x) = \lim_{\Delta x \to 0} \frac{-\Delta U}{\Delta x} = -\frac{dU}{dx}$$

In this one-dimensional example, the force will be along the x-axis, so differentiating the potential energy with respect to x yields the x component of the force.

These ideas extend in a natural way to applications in more than one dimension.

The work done by a force $\vec{F}$ over a displacement $\Delta\vec{r}$ is defined as

$$W_F = F\Delta r\cos\theta = \vec{F} \cdot \Delta\vec{r}$$

This formula assumes that the force is constant over the displacement. If the force doing the work is conservative, then the change in the potential energy associated with the force is just the negative of the work done by the force.

$$\Delta U = -F\Delta r\cos\theta = -\vec{F} \cdot \Delta\vec{r}$$

Suppose, as depicted in Figure 2.20, the object moves along some path.

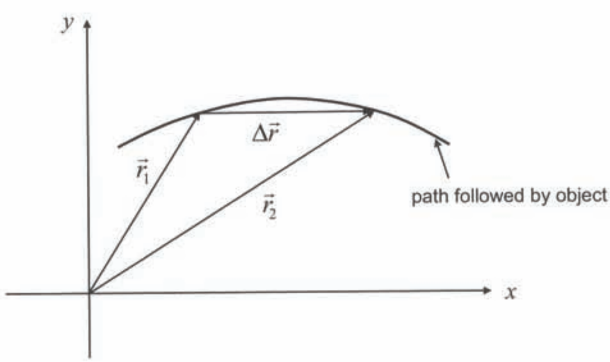

Figure 2.20

As long as the force is constant, then this equation is true for any $\Delta\vec{r}$. However, if the force is not constant but depends on position, the equation will still be almost true if the displacement is very small, because over a very small displacement, the force will not change very much. In the limit of infinitesimal displacements, the equation will be exact. Figure 2.21 illustrates the situation when the displacement is very small.

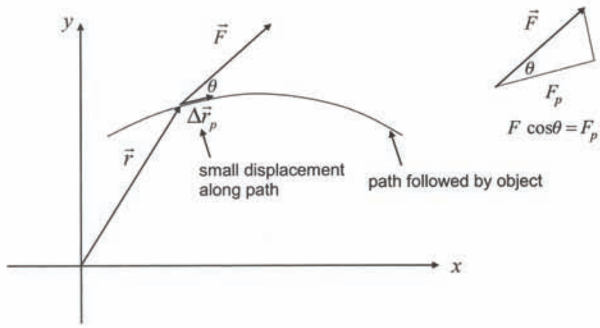

Figure 2.21

When the displacement is small, the displacement vector is nearly tangential to the path, and the size of the displacement is nearly equal to the actual distance traveled along the path, so let's denote it as Δr_p. The right side of the figure shows the component of $\vec{F}$ parallel to $\Delta\vec{r}_p$, denoted F_p. Then the small change in PE as the object moves this small distance along the path is given by

$$\Delta U = -F_p \Delta r_p \Rightarrow F_p = -\frac{\Delta U}{\Delta r_p}$$

In the limit of infinitesimal displacements, this becomes

$$F_p = -\frac{dU}{dr_p}$$

This equation tells you that you can get the components of the force by differentiating the potential energy function. To get the x component, you imagine that $\Delta r_p = \Delta x$ so that you have

$$F_x = -\frac{dU}{dx}$$

You get the y component by differentiating with respect to y, etc. Thus, by taking the appropriate derivatives, you can find the complete force vector $\vec{F}$ from the single scalar function U.

Example 15 Determine the force at an arbitrary point x for a particle moving in one dimension with potential energy given by

$$U(x) = \frac{1}{2}kx^2$$

Solution In one dimension, the force must be in the x direction, so just differentiate with respect to x.

$$F = F_x = -\frac{dU}{dx} = -kx$$

You may well recognize this as the **Hooke's law** force associated with springs and the PE associated with a stretched spring.

Gravity presents another simple one-dimensional example. The PE is given by $U(y) = mgy$ and the force, obtained from the negative derivative with respect to y, is just $F = -mg$. A slightly different example in one dimension, presented by Newtonian gravity, describes the potential energy of a mass m in the presence of another mass M.

$$U(r) = -G\frac{mM}{r}$$

The minus sign indicates that the force is attractive. For M at the origin, Figure 2.22 represents the situation in two dimensions.

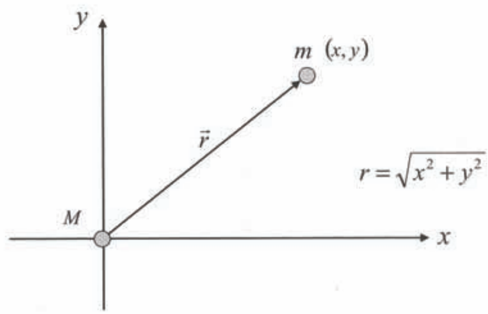

Figure 2.22

Only a change in r will change the value of the PE. By differentiating with respect to r, you can get the component of the force that points radially, along $\vec{r}$.

$$F_r = -\frac{dU}{dr} = -G\frac{mM}{r^2}$$

The minus sign indicates that the force points toward the origin. Since the PE does not change for displacements perpendicular to $\vec{r}$, there is no component of the force perpendicular to $\vec{r}$, so the magnitude of F_r is the magnitude of the total force. Of course, you could find the x and y components of the force directly from the PE function by expressing r in terms of these variables, and differentiating with respect to x and y, respectively. Given a function of two variables, to differentiate with respect to one variable, you assume the other one is a constant; for example,

$$F_x = -\frac{dU}{dx} = -\frac{d}{dx}\left(-G\frac{mM}{\sqrt{x^2 + y^2}}\right) = -\frac{GmM}{(x^2 + y^2)^{\frac{3}{2}}}x = -\frac{GmM}{r^3}x$$

where y was held constant while the x derivative was taken.

In electrostatics, if the electric field is constant and directed along the x-axis, then the electric potential difference between two points on the x-axis is given by the equation

$$\Delta V = -E_x\Delta x \Rightarrow E_x = -\frac{\Delta V}{\Delta x}$$

Even if the field is not constant, this equation will be true in the limit of infinitesimal displacements, so you have

$$E_x = -\frac{dV}{dx}$$

You can see that exactly the same mathematics describes the electric field-electric potential relation as describes the force-potential energy relation. It follows that in more than one dimension, you get the field components by differentiating with respect to each coordinate variable.

2.10 Differentials

In physics, the change in a quantity is represented with the Δ symbol: the change in x position is written Δx, the change in PE is written ΔU, and in general the change in any quantity f is written Δf. A **differential** is an infinitesimal change and it uses the letter d instead of the letter Δ. Thus you would write dx, dU and df for the infinitesimal changes in x, U, and f. If the quantity f depends on a variable x, then you can relate the differential of f to the differential of x through the chain rule.

$$df = \frac{df}{dx}dx$$

In words, you should think of this equation as saying, "The small change in f can be found by multiplying the rate at which f changes with respect to x times the small change in x."

Example 16 Find the differential of the function $U(x) = 2x^2 - 4\ln x$.

Solution Using the above definition, you have $dU = \left(4x - \frac{4}{x}\right)dx$

An understanding of differentials can help in numerical calculations, as the next example demonstrates.

Example 17 Suppose a string were wrapped tightly around Earth's equator. How much longer would the string have to be so that the string could lie 1 m above the ground all the way around the equator?

Solution A direct calculation of Earth's circumference would be difficult because most sources give the radius only to 3 or 4 significant figures, $R = 6.38 \times 10^6$ m. You cannot just add 1 to this value, since you have no knowledge of R to that level of precision; however, the answer is easy to get using differentials. The key is to treat the 1 m as a differential, certainly an excellent approximation. Then you have

$$C = 2\pi R \Rightarrow dC = 2\pi \cdot dR = 6.28 \cdot 1 = 6.28m$$

Thus the string must be 6.28 m longer to lie 1 m above the ground all the way around the equator—perhaps a surprising result.

Integration, which you will study in Chapter 3, often involves thinking in terms of differentials. For symmetric area and volume integrals, you will need to know how much the area of a circle or the volume of a sphere increases if you change the radius. Using differentials, these determinations are easy.

$$A(r) = \pi r^2 \Rightarrow dA = 2\pi r dr$$

$$V(r) = \frac{4}{3}\pi r^3 \Rightarrow dV = 4\pi r^2 dr$$

Example 18 Suppose a planet of uniform density in a primordial solar system was constantly bombarded by meteors of the same density, causing the radius of the planet to increase slowly. Assume the planet currently has a radius of 4×10^6 m and mass 2×10^{24} kg and can be approximated as a sphere. By how much will the radius of the planet increase if the mass increases by 1×10^{20} kg due to meteor strikes?

Solution First, you need to express the radius in terms of the mass, using the definition of volume and density. Since the density is constant, you can calculate it from the initial values.

$$\rho = \frac{M_0}{V_0} = \frac{2 \times 10^{24}}{\frac{4}{3}\pi(4 \times 10^6)^3} = 7.46 \times 10^3 \frac{kg}{m^3}$$

After the meteors have been absorbed, there will be a new mass and volume, but the density is the same.

$$M = \rho V = \rho\frac{4}{3}\pi r^3 \Rightarrow r = \left(\frac{3M}{4\pi\rho}\right)^{\frac{1}{3}}$$

Now find the differential increase in r due to the increase in M.

$$dr = \frac{1}{3}\left(\frac{3M}{4\pi\rho}\right)^{-\frac{2}{3}}\frac{3}{4\pi\rho}dM = \frac{1}{3}\left(\frac{3}{4\pi\rho}\right)^{\frac{1}{3}}\frac{dM}{M^{\frac{2}{3}}} = \frac{1}{3}(0.0317)\frac{1 \times 10^{20}}{(2 \times 10^{24})^{\frac{2}{3}}} = 66.56\ m$$

Thus the radius increase is about 1.67×10^{-3} %.

Practice Problems

1. Calculate the first derivative of the following functions.
   a. $f(x) = 3x^4 - 2x^2 + 7$ b. $g(t) = \dfrac{5}{3 + 2t^2}$
   c. $h(x) = 2x\sin 3x$ d. $f(y) = y^2 e^{-4y}$
   e. $q(t) = q_0\left(1 - e^{-\frac{t}{RC}}\right)$
   f. $V(t) = 4t^2 \ln(t^2 + 1)$
   g. $r(s) = \cos(\sin 5s)$ h. $F(x) = \dfrac{4\cos 2x}{(1 - x)^2}$
   i. $A(t) = 4\cos^2 3t$
   j. $g(x) = (2x^3 - 4)e^{-(1-x^2)}$

2. Find the slope of the tangent line to the following functions at the indicated point.
   a. $U(x) = 3x^3 - 4x + 2;\quad x = 2$
   b. $x(t) = 3\cos\left(\dfrac{2\pi}{3}t - \dfrac{\pi}{4}\right);\quad t = 6$
   c. $v(t) = 40\left(1 - e^{-\frac{t}{5}}\right);\quad t = 10$
   d. $f(v) = \dfrac{8}{\sqrt{v}}e^{-\frac{v^2}{10}};\quad v = 2$

3. Find the linear approximation to the following functions at the indicated point—that is, the equation of the straight line that best approximates the function near the point.
   a. $y(t) = 2t^3 - t^2 + 5t + 1;\quad t = 2$
   b. $U(x) = x^2 \ln 3x;\quad x = 1$
   c. $P(x) = e^{-\frac{(x-2)^2}{4}};\quad x = 3$
   d. $V(t) = 8\sin\dfrac{\pi}{10}t;\quad t = 4$

4. Find the second order polynomial that best approximates the following functions at the indicated point.
   a. $f(t) = t^4 - 3t^2 + 2;\quad t = 1$
   b. $I(t) = 5\cos\left(\dfrac{3\pi}{5}t + \dfrac{\pi}{5}\right);\quad t = 2$
   c. $E(r) = \dfrac{1}{(r + 2)^2};\quad r = 0$
   d. $J(x) = xe^{-\frac{x}{10}};\quad x = 5$

5. Two equal but opposite charges, separated by a distance $2a$, are called an **electric dipole,** as Figure 2.23 illustrates.

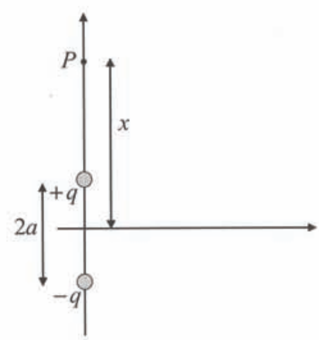

Figure 2.23

The form for the electric field magnitude due to a dipole, along the long axis at a distance x from an origin centered on the dipole, has the form from **Coulomb's law:**

$$E = k\frac{q}{(x-a)^2} - k\frac{q}{(x+a)^2} \qquad k \text{ is a constant}$$

Although this form is exact for all x, you are often just interested in the region where $x >> a$.

a. By factoring out $\frac{1}{x^2}$ in each term, show that you can write the above equation for the field strength as

$$E = \frac{kq}{x^2}\left\{\frac{1}{\left(1-\frac{a}{x}\right)^2} - \frac{1}{\left(1+\frac{a}{x}\right)^2}\right\}$$

b. Because $y = \frac{a}{x}$ is small if $x >> a$, it makes sense to use the binomial expansion to write each fraction as a polynomial in y. Show that to the lowest order in y, you can write the field strength as

$$E = 2k\frac{p}{x^3} \qquad p = 2qa$$

The quantity p is called the **dipole moment** of the charge configuration.

6. A wire loop of radius R carrying a current i produces a magnetic field B, along the axis at a distance x from the center of the loop, with a magnitude given by

$$B = k'\frac{iR^2}{(x^2+R^2)^{\frac{3}{2}}}$$

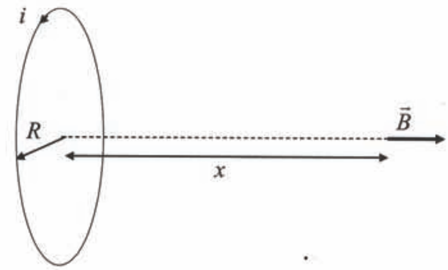

Figure 2.24

a. By factoring out R^3 from the denominator, show that you can write

$$B = \frac{k'}{R}\frac{i}{\left(1+\frac{x^2}{R^2}\right)^{\frac{3}{2}}}$$

b. For $x << R$, use the binomial expansion of the denominator to show that

$$B = \frac{k'i}{R}\left(1 - \frac{3}{2}\frac{x^2}{R^2}\right)$$

7. Prove Euler's identity:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad i = \sqrt{-1}$$

(*Hint:* Expand the exponential in its Maclaurin series.)

8. An object moves along the x-axis described by a position coordinate

$$x(t) = 3t^3 + t^2 - 2$$

a. Find the velocity of the object at $t = 1$.
b. Find the acceleration of the object at $t = 2$.

9. A one-dimensional oscillator is described by the coordinate

$$y(t) = 10\sin\left(\frac{\pi}{3}t + \frac{\pi}{6}\right)$$

a. Find the velocity of the oscillator at $t = 3$.

b. Find the acceleration of the oscillator at $t = 0$.

10. An object rotates about a fixed axis described by angle variable

$$\theta(t) = t\ln(1 + t)$$

a. Find the angular velocity at $t = 0$.

b. Find the angular acceleration at $t = 1$.

11. A projectile is described by the position vector

$$\vec{r}(t) = (3 - 2t)\hat{\imath} + (5 + 7t - 5t^2)\hat{\jmath}$$

a. Find the magnitude of the velocity at $t = 2$.

b. Find the x component of the position vector when the y component of the velocity is 0.

12. When an object moves in a circle at constant speed (**uniform circular motion**), the object is described by the position vector with origin at the circle center

$$\vec{r}(t) = r\cos\omega t\,\hat{\imath} + r\sin\omega t\,\hat{\jmath}$$

where r is the radius of the circle and ω is the constant angular velocity.

a. Show that the velocity is always perpendicular to the position vector. (*Hint:* Use dot product.)

b. Show that the acceleration is always directed toward the center of the circle.

13. Two objects are moving along the x-axis with position coordinates given by

$$x_1(t) = t^3 + 10 \qquad x_2(t) = 3t$$

How far apart are the two objects when they are closest together?

14. An object moves in two dimensions described by the position vector

$$\vec{r}(t) = (t^2e^{-t})\hat{\imath} + (\ln(1 + t^2))\hat{\jmath}$$

Find the y component of the velocity when the object reaches its maximum x-coordinate.

15. An object moves in one dimension subject to a potential energy

$$U(x) = \frac{2}{x^4} - \frac{1}{x}$$

a. This function has one relative extreme point for $x > 0$. Find it.

b. Verify with the second derivative test that this point is a minimum.

c. Expand the function about the minimum to second order. What is the effective "spring constant" for small oscillations about this point?

16. Suppose a particle moves in one dimension subject to a periodic potential energy of the form $U(x) = U_0 \sin ax$ where U_0 and a are constants.

a. Show that this potential energy has minima at $x = \dfrac{n\pi}{2a}$ $\quad n = 3, 7, 11 \ldots$

b. Expand the function to second order about the point $x = \dfrac{3\pi}{2a}$. Find the effective "spring constant" for small oscillations about this point.

17. The form for the potential energy of a system of two atoms is given by

$$U(r) = \frac{A}{r^{12}} - \frac{B}{r^6} \qquad A, B \text{ are constants}$$

Find the force between the atoms as a function of r, the separation between the two.

18. The electric potential along the axis of a ring of radius R, carrying a total charge Q uniformly distributed over the ring, is given by

$$V(x) = kQ\frac{1}{\sqrt{x^2 + R^2}} \qquad k \text{ is a constant}$$

Find the x component of the electric field along the axis of the ring.

19. The relation between momentum p and wavelength λ for an elementary particle is

$$p = \frac{h}{\lambda} \qquad h = \text{Planck's constant} = 6.63 \times 10^{-34}\ \text{J-s}$$

A proton with wavelength 5×10^{-11} m is accelerated so that its wavelength decreases by 2×10^{-13} m. By how much did the momentum increase? (*Hint:* Use differentials.)

20. The electric field along the axis of a disc of radius R, carrying a charge Q uniformly distributed over the disc, is given by

$$E = kQ\frac{1}{(x^2 + R^2)^{\frac{3}{2}}} \qquad k \text{ is a constant}$$

where x is the distance from the center of the disc. By how much will the field decrease at the point x if the radius of the disc is increased by a small amount ΔR, while the charge Q is kept constant? (*Hint:* Use differentials.)