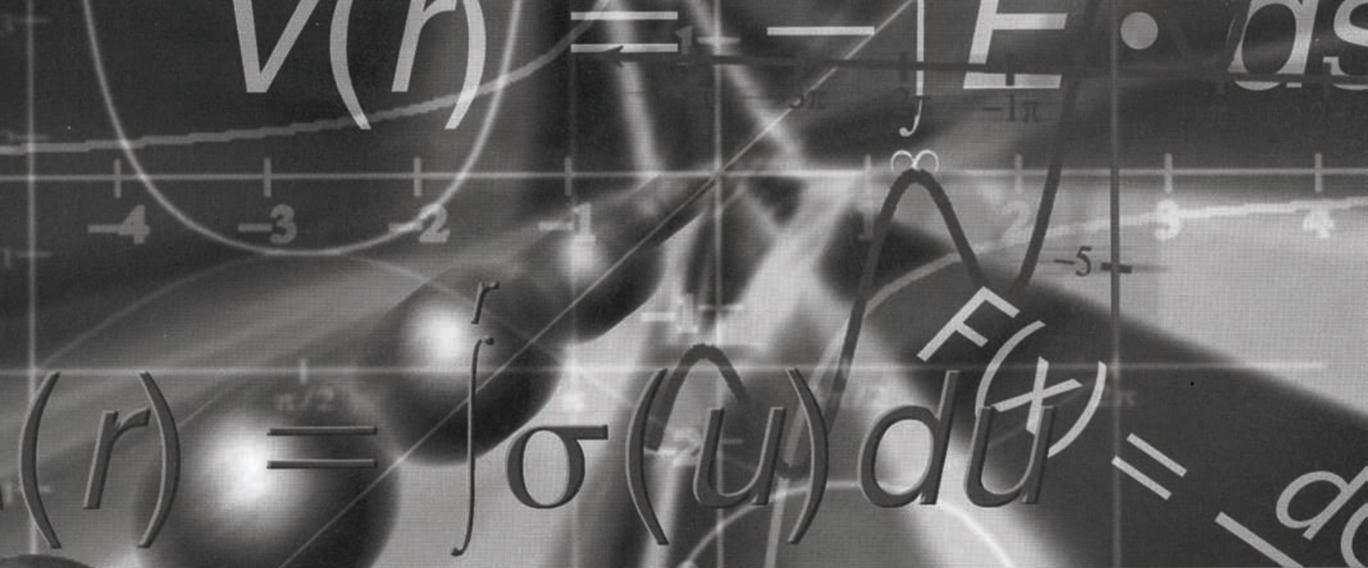


CHAPTER 3

Integration

In physics, it is often necessary to calculate the effects of a quantity when that quantity is not a constant. For example, the total mass of an object depends on its density and volume. But if the density is not uniform throughout the volume, its overall contributions to the mass will differ for the different regions of the volume. How do you express this changing effect in mathematical language?

Consider another example: The principle of superposition tells you that the total electric potential at a point in space is the sum of the contributions from all the charges in the region. The charge could be spread continuously throughout a nonconductor in such a way that the contribution of each part of the nonconductor is different. How do you calculate the total electric potential in such a case? The answer to both questions is the same: You integrate! In this chapter you will learn exactly what this process means.

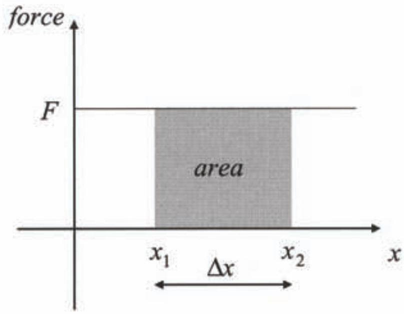
3.1 A Summation Limit

Let's use a specific example from physics to explore the meaning of integration. An object moves along the x-axis from a point x_1 to another point x_2. A constant force of magnitude F acts in the direction of the displacement. This means that the **work** done by the force is

$$W = F \cdot (x_2 - x_1) = F\Delta x$$

Figure 3.1 shows a graph of this force vs. position.

Notice that the work done—in this case of a constant force—is just the area between the x-axis and the force line. If the area was below the x-axis because the force was negative, then the area is considered to be negative. Similarly, if the object moved from x_2 to x_1, so that the displacement was negative, the area is said to be negative also.



$W = F\Delta x = area$

Figure 3.1

Now suppose that the force is not constant, but varies with position, $F = F(x)$. This means that the graph of the force is no longer a simple horizontal line. As the object moves from x_1 to x_2, the force will be constantly changing, as seen in Figure 3.2.

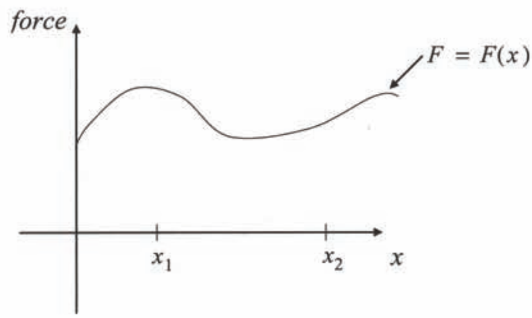


Figure 3.2

You can no longer say $W = F\Delta x$ because there is no single value of F that you could use. However, if you break the overall displacement into many very small displacements, the value of the force will be nearly constant over each of these small intervals. Over the small interval Δx, the force does change a little, but you can always make the Δx as small as you like to minimize this change. Figure 3.3 shows an example of this technique.

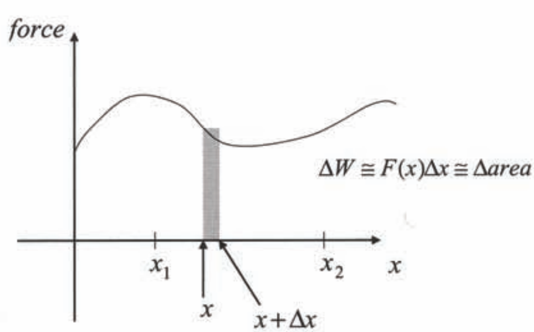


Figure 3.3

Eventually the limit $\Delta x \to 0$ will be taken and the effects of the force changing will go to zero also. There will be a little bit of work ΔW done by the force over this interval. This little bit of work is nearly equal to the value of F at the beginning of the interval multiplied by the displacement.

$$\Delta W \cong F(x)\Delta x$$

As the interval gets smaller, this equation becomes more exact. Notice also from Figure 3.3 that ΔW is nearly equal to the area ($\Delta area$) under the force curve over the small interval. It is not exact, owing to the little piece of rectangle that sticks out above the curve. But as the intervals get smaller, this extra piece gets smaller as well.

Now imagine looking at the work done over many of the small intervals, as depicted in Figure 3.4. You then get many of these rectangles.

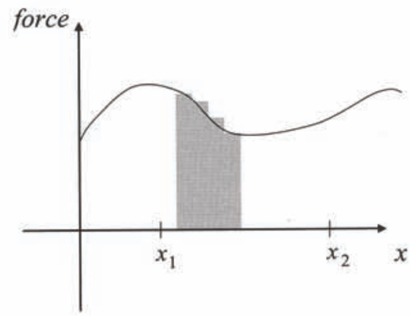


Figure 3.4

The total work done will be the sum of the little contributions of each little interval.

$$W \cong \sum_{\text{intervals}} \Delta W \cong \sum_{\text{intervals}} F(x)\Delta x \cong \sum_{\text{intervals}} \Delta area \cong area$$

The relations are not yet exact because the intervals, while quite small, are not infinitely small. But note that when you add together all the little areas ($\Delta area$) of each interval, you get the total area under the force curve from x_1 to x_2, except for some little pieces that will shrink to 0 when the limit is taken. So now let's explicitly write in the limit.

$$W = \lim_{\Delta x \to 0} \sum_{\text{intervals}} \Delta W = \lim_{\Delta x \to 0} \sum_{\text{intervals}} F(x)\Delta x = \lim_{\Delta x \to 0} \sum_{\text{intervals}} \Delta area = area$$

At this point, the equal signs have replaced the "nearly equal" signs. An important point to grasp is that the work done is still the area under the curve, just as it was in the case of the constant force. But our new procedure defines a strategy for exactly calculating the area: Break up the interval into many little pieces, find the area of each little piece, and add them up as you take the limit $\Delta x \to 0$.

An important point about notation: the quantity

$$\lim_{\Delta x \to 0} \sum_{\text{intervals}} F(x)\Delta x$$

is cumbersome to write, so it is always written as

$$\int_{x_1}^{x_2} F(x)dx$$

You should think of the integral symbol, "$\int$," as a fancy S for *sum*. The limits over which the summation is to take place are put at the bottom and top of the symbol. Because there are definite limits on the integral, it is said to be a **definite integral.** You replace Δx with dx to indicate the infinitesimally small intervals you are summing over. Of course, writing it in a different way does not help us calculate the result, and it is not yet clear how to perform this limiting process. But the notation is compact and contains the meaning of the more cumbersome expression. Now you can express the result for the work as

$$W = \int_{x_1}^{x_2} F(x)dx = area$$

In the next section, you will learn specifically how to calculate a definite integral.

3.2 The Fundamental Theorem

If you were good with spreadsheets or programming on a computer, you could imagine setting up a numerical calculation of the quantity

$$\lim_{\Delta x \to 0} \sum_{\text{intervals}} F(x)\Delta x = \int_{x_1}^{x_2} F(x)dx$$

by breaking the interval into several hundred (or thousand) little intervals, computing each little contribution and then adding all of them up. Indeed, handheld calculators can do this type of approximation, which can be very precise but never exact. You might ask if there is an exact answer for the definite integral. The answer is yes, and it is supplied by the so-called **Fundamental Theorem of Calculus:**

> **Suppose the function $G(x)$ is such that its derivative gives $F(x)$**
>
> $$F(x) = \frac{dG}{dx}$$
>
> **Then**
>
> $$\int_{x_1}^{x_2} F(x)dx = G(x_2) - G(x_1)$$

The meaning is this: If you can figure out which function has $F(x)$ as its derivative, you just evaluate this function at the upper and lower limit and subtract to calculate the definite integral. This calculation also gives you the area under $F(x)$ over the limits of the interval. It is because of this theorem that you may say that integration and differentiation are the inverse of each other, because to calculate an integral of a function you need to know what function it is the derivative of.

The integrated function, like $G(x)$ in the theorem, can be expressed as an indefinite integral, without limits on it.

$$\int F(x)dx = G(x) + C \qquad C \text{ is an arbitrary constant}$$

This equation is a shorthand way of saying that $F(x) = \dfrac{dG}{dx}$. Since the derivative of a constant is 0, you can add any constant to $G(x)$ and its derivative will still give you $F(x)$. The important indefinite integrals you need to know for AP Physics are:

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C \quad n \neq -1 \qquad \int \frac{1}{x}dx = \ln x + C$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax + C \qquad \int \cos ax dx = \frac{1}{a}\sin ax + C$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax} + C$$

At this point you might think that the problem of integration is solved, and it is, in principle. However, though differentiation and integration might be the inverse of each other, there is a big difference in the complexity of their operations. For any function you can write down, you can take the derivative—just follow the rules for differentiation. Although the final result may be complex and messy, there is a clear path to the final answer.

Integration is not so simple. You have to come up with the right integrated function such that its derivative will give you the function you are integrating. In a calculus course, you will learn certain techniques that will work in some situations, but the fact is that many functions do not have indefinite integrals that can be expressed in terms of a finite number of terms. In such cases, and these are quite common in practice, you will have to resort to the numerical techniques mentioned at the beginning of this section. However, you can rest assured that you will not meet such complex integrals in AP Physics.

Example 1

A mass moves in one dimension under the influence of a force that depends on position as

$$F(x) = 3x^2 + 5x - \frac{4}{x}$$

Find the work done by this force as the mass moves from $x = 2$ to $x = 3$.

Solution

You need to find the definite integral of the force evaluated at the endpoints of the displacement.

$$W = \int_2^3 \left(3x^2 + 5x - \frac{4}{x}\right)dx = \left[x^3 + \frac{5}{2}x^2 - 4\ln x\right]_2^3$$

$$W = \left(27 + \frac{45}{2} - 4\ln 3\right) - \left(8 + \frac{20}{2} - 4\ln 2\right) = 29.88$$

If position is measured in meters and force in newtons, the work units will be joules.

A common notation was introduced in Example 1. In the step after the integral, the integrated function is shown inside square brackets with the limits on the end bracket. This notation means to evaluate the quantity in brackets at the two limits and subtract the lower limit value from the upper limit value, as shown in the last line of computation.

3.3 Finding the Elementary Contribution

Recall that the definite integral $\int_{x_1}^{x_2} F(x)dx$ is really a shorthand notation for the complex process described by $\lim_{\Delta x \to 0} \sum_{\text{intervals}} F(x)\Delta x$. Now that you have an idea of what is involved in the process, you need not repeat all the steps every time you are trying to set up an integral. In Example 1, you were trying to find the work done by a nonconstant force over an interval. Another way to get to the answer is to first identify an arbitrary point along the path of the displacement and find the contribution to the work due to an infinitesimal displacement from that point along the path. Symbolically, you write this as

$$dW = F(x)dx$$

The contribution from the infinitesimal displacement dx will itself be an infinitesimal, so we call it dW. (It is common notation to use Δ for a finite change of a quantity, and d for an infinitesimal change.) To find the total work, you add up all the infinitesimal contributions. Adding up an infinite number of infinitesimals means integrating, so you have

$$W = \int dW = \int_{x_1}^{x_2} F(x)dx$$

Example 2 An object moves in one dimension with a velocity that depends on time, as follows:

$$v(t) = 5e^{-\frac{t}{2}}$$

Find the displacement of the object from $t = 0$ to $t = 4$.

Solution Choose an arbitrary time t and imagine an infinitesimal increase dt in the time. During this interval, the velocity will not change, so the infinitesimal displacement will be

$$dx = v(t)dt = 5e^{-\frac{t}{2}}dt$$

Now add up—that is, integrate—all the little displacements.

$$\Delta x = \int dx = \int_0^4 5e^{-\frac{t}{2}}dt = \left[-10e^{-\frac{t}{2}}\right]_0^4 = -10(e^{-2} - 1) = 8.65$$

To describe this process in somewhat loose but descriptive language, you find the contribution from an arbitrary little piece and then add up all the pieces. You treat infinitesimals such as dx and dt as if they were ordinary real valued quantities, just very small, so that functions like $v(t)$ or $F(x)$ do not change for such small changes in their arguments.

Example 3 Consider a rod of length L with a nonuniform **mass per unit length λ**

$$\lambda(x) = \lambda_0 \frac{x}{L} \qquad \lambda_0 \text{ is a constant}$$

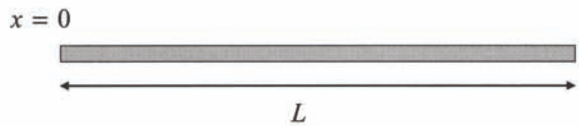


Figure 3.5

Find the total mass of the rod.

Solution First identify an arbitrary piece of the rod and label it with appropriate parameters.

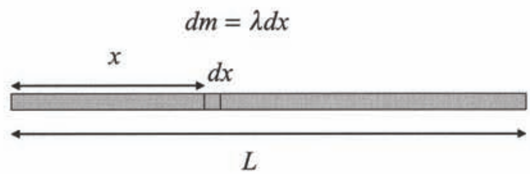


Figure 3.6

The little piece dx will have a little mass $dm = \lambda dx = \lambda_0 \frac{x}{L} dx$.

Now add up all the pieces; that is, integrate. Note that x varies from 0 to L.

$$m = \int dm = \int_0^L \lambda_0 \frac{x}{L} dx = \left[\frac{\lambda_0}{2L} x^2 \right]_0^L = \frac{\lambda_0 L}{2}$$

The previous two examples both involved integration in just one dimension. Integration in more than one dimension is a staple of analysis in physics. In AP Physics you will have to be familiar with certain types of two- and three-dimensional integrations, but there will always be enough symmetry in the situation to effectively reduce it to a one-dimensional integral. The key, once again, is to find the contribution of a little piece, but be sure to choose your piece in such a way as to maintain the symmetry in your integration setup.

Let's look at a specific example to see how integration in these cases can work. Consider a thin disc of radius R that carries a nonuniform **charge per unit area** σ that depends on r, the distance from the center, only.

$$\sigma(r) = \sigma_0 \frac{r^2}{R^2} \qquad \sigma_0 \text{ is a constant}$$

Suppose you want to determine the total charge on the disc. From the center of the disc, all directions are equivalent; that is, there is radial symmetry about the center. You want to choose your little piece to respect this symmetry. You do so by choosing little rings of thickness dr for your elementary contribution, as Figure 3.7 illustrates.

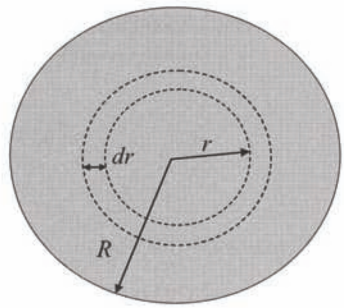


Figure 3.7

These little rings have an infinitesimal area obtained by increasing the area of the inner circle of radius r by the amount dr. Using differentials, you can easily obtain the expression for this little area.

$$A(r) = \pi r^2 \qquad dA = 2\pi r dr$$

You can think of the little area as the product of the inner circumference and the thickness of the ring. Notice that the area element depends on r so that the outer rings will contribute more. This is a property of higher dimension integrations not seen in one dimension. Since all points on the ring are the same distance from the center, the contribution to the total charge from this little piece will be

$$dq = \sigma dA = \sigma_0 \frac{r^2}{R^2} 2\pi r dr$$

Now add up (integrate) the little contributions, noting that r varies from 0 to R.

$$q = \int dq = \int_0^R \frac{2\pi\sigma_0}{R^2} r^3 dr = \left[\frac{2\pi\sigma_0}{4R^2} r^4 \right]_0^R = \frac{\sigma_0 \pi R^2}{2}$$

Because of the radial symmetry in the problem, you were able to choose a special kind of elementary contribution. This choice then only required a single integration to get the answer. Without this symmetry, you would have to do two integrations, moving through the disc radially for a fixed angle, and then allowing the angle to vary.

Here is another example of a 2-dimensional integration that can be reduced to a single integration because of the symmetry present. Suppose a thin sheet has width a

and length b and the mass density per area of the sheet depends only on the distance from one end of the sheet, according to the equation.

$$\sigma(x) = \sigma_0 \frac{b}{a + x}$$

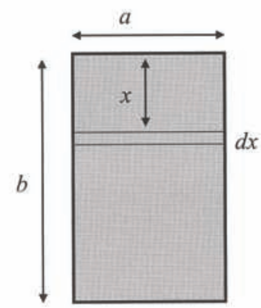


Figure 3.8

Since the density does not vary across the width of the strip, you can choose the width of your elementary contribution to be the entire width of the strip. These little strips have length a and width dx. They have an area given by

$$dA = adx$$

The little contribution to the total mass due to an arbitrary strip is

$$dm = \sigma dA = \sigma_0 \frac{b}{a + x} adx$$

Sum up (integrate) to find the total mass, noting that x varies from 0 to b.

$$m = \int dm = \int_0^b \sigma_0 \frac{b}{a + x} adx = \Big[\sigma_0 ab \ln(a + x)\Big]_0^b = \sigma_0 ab \ln \frac{a + b}{a}$$

Once again, because of the symmetry of the mass distribution, you could choose a special elementary contribution to avoid doing a double integral. A more complex mass distribution would not let you lump the entire width into a single elementary contribution.

In AP Physics, you need to know how to do certain three-dimensional integrals also. Once again, there will always be enough symmetry present to reduce the three-dimensional integration to a single integration. The two important cases involve cylindrical symmetry and spherical symmetry. In the case of cylindrical symmetry, you will want to use cylindrical shells as your little pieces to sum up. Since the volume of a cylinder of length L and radius r is $V = \pi r^2 L$, it follows that the infinitesimal volume of such a shell with thickness dr is given by the differential

$$dV = 2\pi r L dr$$

which you can think of as the inner sheath area multiplied by the thickness. Figure 3.9 gives a representation of this situation.

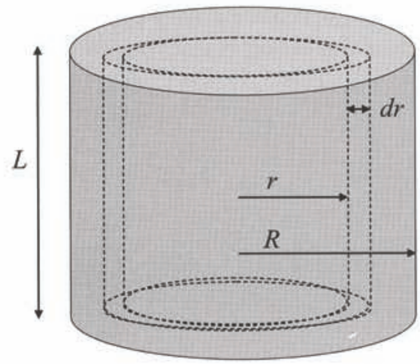


Figure 3.9

For spherical symmetry, you will want to use spherical shells as your little pieces to sum up. Since the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$, it follows that the infinitesimal volume of such a shell with thickness dr is given by the differential

$$dV = 4\pi r^2 dr$$

which you can think of as the inner spherical surface area multiplied by the shell thickness. Figure 3.10 represents this situation.

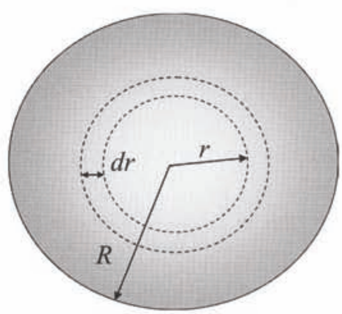


Figure 3.10

Example 4 Charge is distributed through a nonconducting sphere of radius R with a **charge density** given by

$$\rho = \rho_0 \frac{r^2}{R^2} \qquad \rho_0 \text{ is a constant}$$

Find the total charge in the sphere.

Solution Use the spherical shells depicted as having thickness dr in the preceding figure. The little bit of charge contained in each shell is given by

$$dq = \rho dV = \left(\rho_0 \frac{r^2}{R^2}\right)(4\pi r^2 dr) = \frac{4\pi\rho_0}{R^2} r^4 dr$$

Then integrate over the entire sphere so that r varies from $0 \to R$.

$$q = \int_0^R \frac{4\pi\rho_0}{R^2} r^4 dr = \frac{4}{5}\pi\rho_0 R^3$$

Example 5 The **moment of inertia** of a distribution of mass about an axis is defined as

$$I = \int_{body} r^2 dm$$

This equation tells you to take the mass of each little piece, multiply it by the square of its distance from the axis, and then add up all the little contributions. Suppose you have a cylinder of radius R and length L with a mass density given by

$$\rho = \rho_0 \frac{r}{R} \qquad \rho_0 \text{ is a constant}$$

Calculate the moment of inertia of the cylinder about its central axis.

Solution Use the cylindrical shells shown in Figure 3.9 above. All the mass on a shell has the same distance r from the axis. The little bit of mass of each shell is

$$dm = \rho dV = \left(\rho_0 \frac{r}{R}\right)(2\pi r L dr) = \frac{2\pi L \rho_0}{R} r^2 dr$$

The little contribution of this shell to the moment of inertia is given by

$$dI = r^2 dm = \frac{2\pi L \rho_0}{R} r^4 dr$$

Integrating from $r = 0$ to $r = R$ then gives

$$I = \int_{body} r^2 dm = \int_0^R \frac{2\pi L \rho_0}{R} r^4 dr = \frac{2}{5}\pi L \rho_0 R^4$$

3.4 Line Integrals

A **line integral** involves the integration of a vector in which only the component of the vector along a particular path is included in the integration. Perhaps the most intuitive appearance of a line integral is in the definition of work. Figure 3.11 depicts an object moving in one dimension, directly from A to B, under the influence of a constant force, $\vec{F}$.

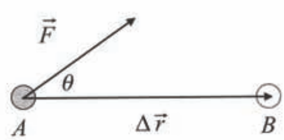


Figure 3.11

The work done by the force in this case is defined as the component of the force parallel to the displacement, multiplied by the magnitude of the displacement.

$$W = (F\cos\theta)\Delta r = \vec{F} \cdot \Delta\vec{r}$$

Using the dot product is a concise way of representing the fact that only the component parallel to $\Delta\vec{r}$ is included in the calculation.

If the object is allowed to move along a more complex path in two or three dimensions, you need to think of the work done over each little displacement along the path. The path itself is described by a position vector that varies along the path. Figure 3.12 shows an example in two dimensions.

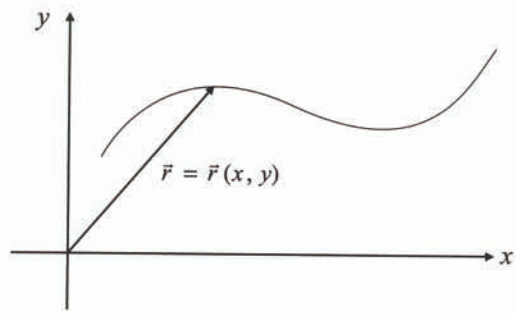


Figure 3.12 A path is described by a changing position vector.

The vector $\Delta\vec{r}$ that connects two points on the path is shown in the next illustration.

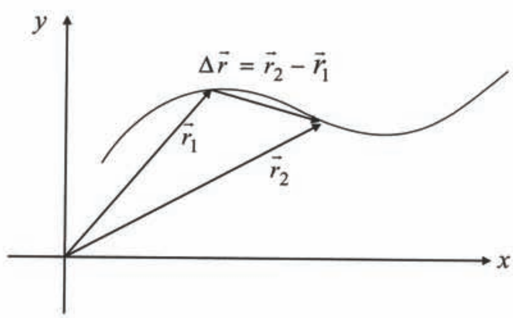
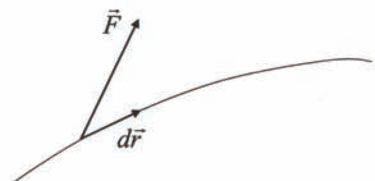


Figure 3.13

If the two points are infinitesimally close to each other along the path, you replace $\Delta\vec{r}$ with $d\vec{r}$ and it should be clear that, in this limit, $d\vec{r}$ is tangential to the path. Figure 3.14 shows a magnification of the path with the force included.



$$dW = \vec{F} \cdot d\vec{r}$$

Figure 3.14

The little bit of work done by the force over this little displacement will be

$$dW = \vec{F} \cdot d\vec{r}$$

To find the total work done by the force as the object moves from point A to point B along the path, you sum up (integrate) all the little bits of work.

$$W = \int dW = \int_{\vec{r}_A}^{\vec{r}_B} \vec{F} \cdot d\vec{r}$$

Figure 3.15 illustrates this process.

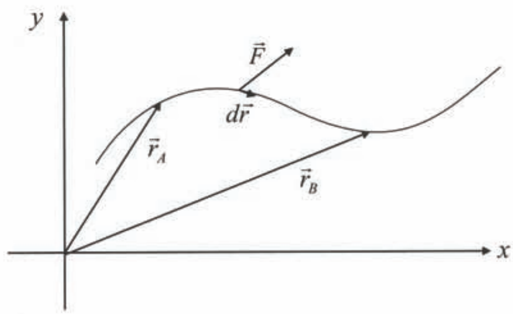


Figure 3.15

If the force is constant, you can factor it out of the integral to obtain

$$W = \vec{F} \cdot \int_{\vec{r}_A}^{\vec{r}_B} d\vec{r} = \vec{F} \cdot (\vec{r}_B - \vec{r}_A) = \vec{F} \cdot \Delta\vec{r}$$

For a constant force, you get the same formula for the work as in the one-dimensional example; it only depends on the displacement $\Delta\vec{r}$ and not on the details of the path connecting the two end points of the integration. However, if the force is not constant, it does not factor out of the integral and you must compute the contribution from each piece of the path. The most general form for the work done by a force is thus

$$W = \int_{\vec{r}_A}^{\vec{r}_B} \vec{F} \cdot d\vec{r}$$

This is a line integral. Sometimes you will see it written

$$W = \oint_{path} \vec{F} \cdot d\vec{r}$$

In general the value of the integral will depend on the particular path used to connect the two points. Line integrals will appear in AP Physics in your study of electricity and magnetism. Line integrals of the electric field and magnetic field

$$\oint_{path} \vec{E} \cdot d\vec{r} \qquad \oint_{path} \vec{B} \cdot d\vec{r}$$

appear in the formulation of the laws of electricity and magnetism.

Example 6 In a certain region of space, a position-dependent force acts on an object. The force depends only on x and y according to the formula

$$\vec{F} = x(y + 1)\hat{i} - y^2\hat{j}$$

Find the work done by this force on the object as it moves from the origin, first to the point (3, 0) on the x-axis, then from there to the point (3, 2) moving parallel to the y-axis.

Solution The path is made up of two pieces, as shown in Figure 3.16.

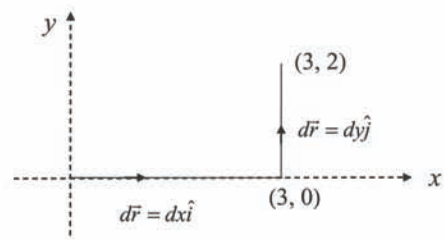


Figure 3.16

To calculate the total line integral, find the contribution from each piece of the path and add. For the horizontal path, since $y = 0$ along this section, you have

$$\vec{F} = x\hat{i} \qquad d\vec{r} = dx\hat{i} \qquad \vec{F} \cdot d\vec{r} = xdx$$

Because x varies from $0 \to 3$ along this path, you get (suppressing units)

$$\int_{\substack{\text{horiz}\\\text{path}}} \vec{F} \cdot d\vec{r} = \int_0^3 xdx = \left[\frac{x^2}{2}\right]_0^3 = \frac{9}{2}$$

For the vertical path, x is fixed at 3, so along this section you have

$$\vec{F} = 3(y + 1)\hat{i} - y^2\hat{j} \qquad d\vec{r} = dy\hat{j} \qquad \vec{F} \cdot d\vec{r} = -y^2dy$$

Over this path y varies from $0 \to 2$, so you get

$$\int_{\substack{\text{vert}\\\text{path}}} \vec{F} \cdot d\vec{r} = \int_0^2 -y^2dy = \left[-\frac{y^3}{3}\right]_0^2 = -\frac{8}{3}$$

Thus the total work done by the force along the overall path is

$$W = \int_{\substack{\text{horiz}\\\text{path}}} \vec{F} \cdot d\vec{r} + \int_{\substack{\text{vert}\\\text{path}}} \vec{F} \cdot d\vec{r} = \frac{9}{2} - \frac{8}{3} = \frac{11}{6}$$

3.5 Conservative Forces

You saw in the last section that, for a constant force, the work done did not depend on the particular path that connected the two end points of the integration. A force that satisfies this condition is said to be **conservative.** You might well ask if all forces are conservative; the answer is no. Let's look again at Example 6, and now calculate the work done in getting to the same point but by a different path. This time, first displace vertically and then horizontally, as shown in Figure 3.17.

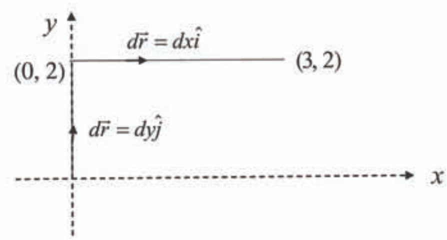


Figure 3.17

Along the vertical path $x = 0$, so you get

$$\vec{F} = -y^2\hat{j} \qquad d\vec{r} = dy\hat{j} \qquad \vec{F} \cdot d\vec{r} = -y^2 dy$$

Along this path, y varies from $0 \to 2$, so you get

$$\int_{\substack{\text{vert}\\\text{path}}} \vec{F} \cdot d\vec{r} = \int_0^2 -y^2 dy = \left[-\frac{y^3}{3}\right]_0^2 = -\frac{8}{3}$$

This value is the same as for the vertical path in the previous example. Along the horizontal path, $y = 2$, so you get

$$\vec{F} = 3x\hat{i} - 4\hat{j} \qquad d\vec{r} = dx\hat{i} \qquad \vec{F} \cdot d\vec{r} = 3xdx$$

Along this path x varies from $0 \to 3$, so you get

$$\int_{\substack{\text{horiz}\\\text{path}}} \vec{F} \cdot d\vec{r} = \int_0^3 3xdx = \left[\frac{3x^2}{2}\right]_0^3 = \frac{27}{2}$$

Thus the total work done by this overall path is

$$W = \int_{\substack{\text{horiz}\\\text{path}}} \vec{F} \cdot d\vec{r} + \int_{\substack{\text{vert}\\\text{path}}} \vec{F} \cdot d\vec{r} = \frac{27}{2} - \frac{8}{3} = +\frac{65}{6}$$

Since the work done is different along the two paths that end at the same point, the force is not conservative. Now you might ask if there are any conservative forces other than simple constant forces. The answer to this question is yes. For a force to be conservative outside of one dimension, its components must obey certain relationships. For a two-dimensional force, such as the one in this example, the force components must satisfy

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

The fancy $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x}$ symbols mean that you take the derivative with respect to just that variable, treating other variables as constants. For example, the force

$$\vec{F} = x(y + 2)\hat{\imath} + \frac{x^2}{2}\hat{\jmath}$$

satisfies this condition (see Practice Problem 12). In a full three-dimensional situation, the conditions are even more complex. In one-dimensional models, every force that depends on position only is conservative. This follows from the Fundamental Theorem

$$\int_{x_1}^{x_2} F(x)dx = G(x_2) - G(x_1)$$

which clearly depends only on x_1 and x_2 and not the path taken between them.

Conservative forces are important in physics because it is only for them that you can define potential energy in a meaningful way. Because potential energy is defined in terms of the work done against a force to move from one reference point to some other point, the value will not be unique unless the force is conservative.

3.6 Flux of a Vector Field

A line integral involves the integration of a vector along a one-dimensional path. There are also important applications in physics where a vector is integrated over an area. The **flux** Φ of a vector field through an area is a measure of the degree to which the vector field cuts through the area. In AP Physics, you will have to calculate the flux of both the electric and magnetic fields through different areas. For a uniform vector field $\vec{V}$, the flux through a planar area is defined as the product of the component of the field perpendicular to the area and the magnitude of the area. Figure 3.18 depicts this situation.

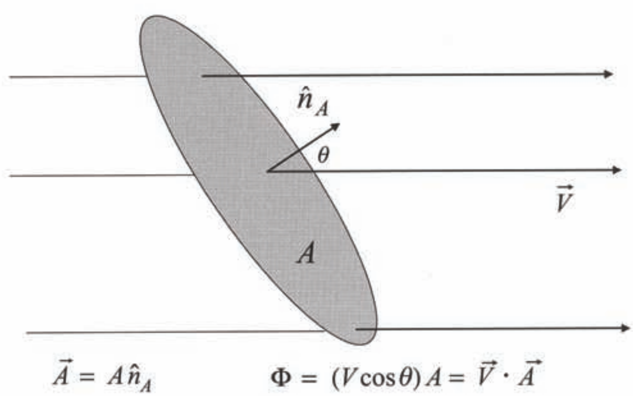


Figure 3.18

The component of the field perpendicular to the area is $V\cos\theta$, so the flux in this case is

$$\Phi = (V\cos\theta)A$$

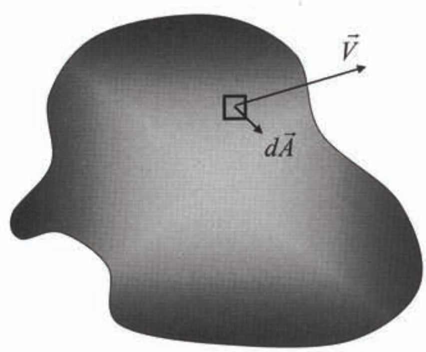
You can introduce an **area vector** with magnitude equal to the size of the area and direction defined as perpendicular to the area. If you define a unit vector $\hat{n}_A$ perpendicular to the area, then the area vector can be defined as $A\hat{n}_A$. Note that there is some ambiguity here since the vector $-\hat{n}_A$ is also perpendicular to the area. In actual physics applications there will be other conditions that uniquely define the perpendicular direction. With these definitions, you can use the dot product to represent the flux in a very elegant way.

$$\Phi = \vec{V} \cdot \vec{A}$$

If the field is not uniform or if the area is more contorted, you have to be a little more careful in defining the flux, but the basic concept is still the same. Imagine breaking the area up into infinitesimal pieces of area dA. For each piece, define an area vector $d\vec{A}$ to be perpendicular to the piece with magnitude equal to the size of the area. Then the little bit of flux through this little bit of area is given by

$$d\Phi = \vec{V} \cdot d\vec{A}$$

as Figure 3.19 illustrates.



$$d\Phi = \vec{V} \cdot d\vec{A}$$

Figure 3.19

To find the total flux through the entire surface, you add up (integrate) all the little contributions.

$$\Phi = \int_{\text{surface}} d\Phi = \int_{\text{surface}} \vec{V} \cdot d\vec{A}$$

Sometimes the flux integral is also written like this:

$$\Phi = \oint_{\text{surface}} \vec{V} \cdot d\vec{A}$$

As you might suspect, calculating a flux integral can be quite complicated. However, in applications relevant to AP Physics, you will find that there is always enough symmetry present so that the two-dimensional integral can be reduced to just a single integration or, even more simply, that the field will not vary over the surface and the integration will reduce to a trivial summation of areas.

Example 7 The magnetic field produced by a long straight wire has a magnitude

$$B(r) = k'\frac{i}{r}$$

where i is the current, r is the distance from the wire, and k' is a constant. Find the flux of the magnetic field through a rectangular loop that is arranged, as in Figure 3.20, so that the field is perpendicular to the loop, directed into the page.

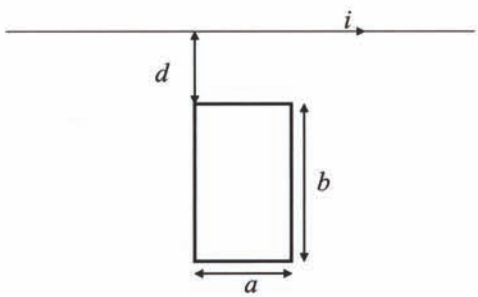


Figure 3.20

Solution Break the area up into strips so that each strip is the same distance from the wire. This ensures that the value of the field is the same over the strip.

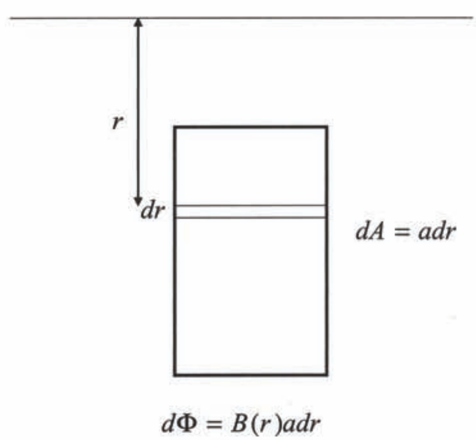


Figure 3.21

Choosing the area element direction to be into the page also, the contribution to the flux from a strip that is a distance r from the wire is

$$d\Phi = B(r)adr = k'\frac{ia}{r}dr$$

The total flux will be the sum of all these contributions as you let r vary over the length of the strip. This means that r varies from $d \to d + b$, so this determines the limits on the integral.

$$\Phi = \int d\Phi = \int_{\text{loop}} B(r) a dr = \int_{d}^{d+b} k' \frac{ia}{r} dr$$

$$\Phi = k' ia \ln\left(\frac{d + b}{d}\right)$$

Note that in Example 7, you could opt to use an area element that extended across the entire width of the loop. This was because the field depended only on r, and across the width of the loop r did not change. Because of the symmetry of the problem, a two-dimensional area integration was reduced to a simple one-dimensional integral.

There are cases where the symmetry is so high that the entire surface integral can be carried out almost immediately. In AP Physics, these situations occur in the application of **Gauss's law** to spherical and cylindrical charge distributions. Gauss's law applies to closed surfaces. For such a surface, the normal vector to the surface is defined to point *out of* the surface.

Example 8 Find the flux of the electric field through a sphere of radius R that is concentric to a spherically symmetric positive charge distribution of radius R_0.

Solution Since the charge distribution is spherically symmetric, the electric field will point radially outward. This will make the field parallel to the area vectors on the sphere.

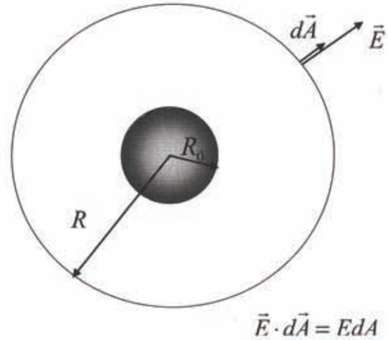


Figure 3.22

Everywhere along the surface, the angle between the field and area element is 0. Thus, you can write

$$\vec{E} \cdot d\vec{A} = EdA$$

To find the total flux, add (integrate) these.

$$\Phi = \int_{\text{sphere}} \vec{E} \cdot d\vec{A} = \int_{\text{sphere}} E dA$$

Because of the spherical symmetry of the charge distribution, the magnitude of the field will be the same over the entire sphere of radius R. Thus it factors out of the integral and you get

$$\Phi = \int_{\text{sphere}} \vec{E} \cdot d\vec{A} = \int_{\text{sphere}} E dA = E \int_{\text{sphere}} dA = E(4\pi R^2)$$

Although the diagram shows a sphere with $R > R_0$, this result is also valid if $R < R_0$. All that is required is that the charge distribution be spherically symmetric, so that $\vec{E}$ is radial with a magnitude that is constant at a fixed distance from the center of the charge distribution.

Example 9 Find the flux of the electric field through a cylinder of radius R and length L that is outside a cylindrically symmetric positive charge distribution of radius R_0.

Solution Since the charge distribution is cylindrically symmetric, the electric field will point radially outward. This makes the field parallel to the area vectors on the sheath of the cylinder and perpendicular to the area vectors on the caps of the cylinder. Thus there is no flux through the caps and only the sheath contributes to the overall flux.

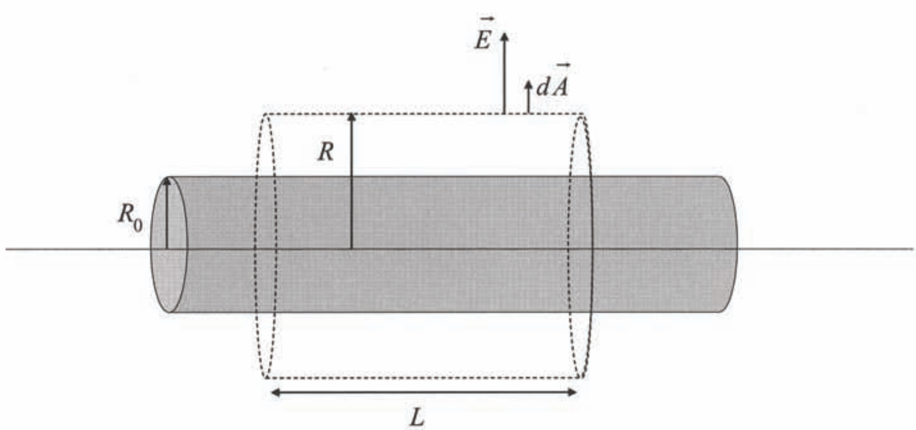


Figure 3.23

Everywhere along the sheath, the angle between the field and area element is 0. Thus, you can write

$$\vec{E} \cdot d\vec{A} = EdA$$

To find the total flux, add (integrate) these.

$$\Phi = \int_{\text{cylinder}} \vec{E} \cdot d\vec{A} = \int_{\text{sheath}} E dA$$

Because of the cylindrical symmetry of the charge distribution, the magnitude of the field will be the same over the entire sheath of radius R. Thus, it factors out of the integral and you get

$$\Phi = \int_{\text{cylinder}} \vec{E} \cdot d\vec{A} = \int_{\text{sheath}} E dA = E \int_{\text{sheath}} dA = E(2\pi RL)$$

Although the diagram shows a cylinder with $R > R_0$, this result is also valid if $R < R_0$. All that is required is that the charge distribution be cylindrically symmetric so that $\vec{E}$ is radial with a magnitude that is constant at a fixed distance from the axis of the charge distribution.

Practice Problems

1. Evaluate the following definite integrals.

   a. $\int_{-1}^{2} (4t^3 - t)dt$ b. $\int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} 3\cos 2x \, dx$

   c. $\int_{0}^{2} 6e^{-\frac{y}{4}}dy$ d. $\int_{2}^{10} \left(\sin \pi x - \frac{3}{x}\right)dx$

2. Find the area under the graph defined by

$$y = x^2 - 4x + 3$$

   a. over the interval from $0 \rightarrow 1$.

   b. over the interval from $1 \rightarrow 3$.

3. An object falls from rest from a great height. Because of the combined effects of gravity and air resistance, its velocity satisfies the equation

$$v(t) = 60\left(1 - e^{-\frac{t}{10}}\right)$$

   Find the expression for the displacement of the object as a function of time.

4. An object moves in one dimension with a time-dependent acceleration given by

$$a(t) = 10\cos\frac{\pi}{6}t$$

   a. Find the total change in velocity, expressed in m/s, of the object over the interval $t = 0$ to $t = 3$ s.

   b. Find the velocity of the object at $t = 3$ s if the initial velocity was 2 m/s.

   c. Find the displacement of the object over the same time interval, assuming the initial velocity given in **4b.**

5. The moment of inertia of a mass distribution about an axis was defined in this chapter as

$$I = \int_{\text{body}} r^2 dm$$

   which means you take each little bit of mass, multiply by the square of the element of distance from the axis, and sum up all of these contributions. Find the moment of inertia of a uniform stick of mass M and length L about an axis

   a. through its center and perpendicular to the stick.

   b. fixed at one end of the stick and perpendicular to the stick.

6. The **center of mass** of a one-dimensional object of total mass m is defined as

$$X_{CM} = \frac{\int_{\text{body}} x dm}{m}$$

   which means you take each little bit of mass, multiply by its coordinate, add up all of these contributions, and finally divide by the total mass. Suppose you have a nonuniform stick, like that in Example 3, with a mass per unit length given by

$$\lambda(x) = \lambda_0 \frac{x}{L} \qquad \lambda_0 \text{ is a constant}$$

   The position x is measured from the left end of the stick, as shown in Figure 3.5 of that example. Find the center of mass of the stick.

7. The **electric potential** at a distance r from a single point charge q is given by the formula

$$V(r) = k\frac{q}{r} \qquad k \text{ is a constant}$$

Find the electric potential at the origin due to a length L of charged nonconductor carrying total charge Q spread uniformly over the length, as shown in Figure 3.24. (*Hint:* Break up the length into little pieces dQ that can each be treated as a point charge, and then integrate.)

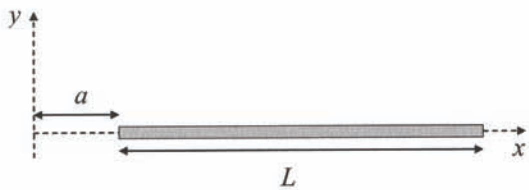


Figure 3.24

8. A thin disc of nonconducting material has a radius R and a total charge Q spread uniformly over the disc. Find the value of the electric potential at a distance x from the center of the disc and along the axis as shown in Figure 3.25. (See Problem 7 and the hint there. Take advantage of symmetry in your choice of dQ.)

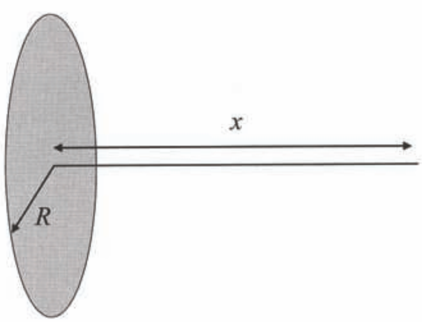


Figure 3.25

9. A spherical shell of nonconducting material has an inner radius a and outer radius b, as given in Figure 3.26. Charge is spread nonuniformly within this shell with a charge density

$$\rho(r) = \rho_0 \frac{ab}{r^2} e^{-\alpha r} \qquad \rho_0, \alpha \text{ are constants}$$

Find the total charge in the shell.

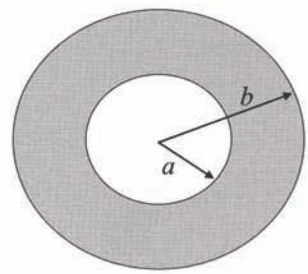


Figure 3.26

10. A long wire carrying a constant electric current i creates a magnetic field $\vec{B}$ in the region around the wire. The magnitude of the field depends only on the current and the distance r from the wire through the formula

$$B = k'\frac{i}{r} \qquad k' \text{ is a constant}$$

The direction of the field is tangential to any circle centered on the wire, as shown in Figure 3.27, where the x's indicate current flowing into the page.

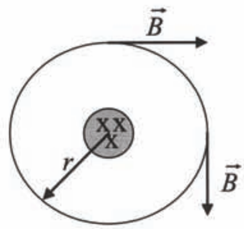


Figure 3.27 Magnetic field of a long wire carrying current into the page.

Calculate the line integral $\oint_{\text{loop}} \vec{B} \cdot d\vec{l}$ for the path consisting of the closed loop shown in the figure. The vector $d\vec{l}$ is an infinitesimal element of the path pointing tangential to the path in the clockwise direction.

11. An infinite sheet of conductor lies in the x- plane and carries a uniform current densit directed into the page. This distribution c current produces a magnetic field that ha constant magnitude outside of the sheet and i directed in the $+x$ direction above the shee and in the $-x$ direction below the sheet. portion of the sheet is shown in Figure 3.28 Calculate the line integral $\oint_{\text{loop}} \vec{B} \cdot d\vec{l}$

for the rectangular loop shown in the fig ure. The vector $d\vec{l}$ is an infinitesimal elemen of the path pointing tangential to the path i the clockwise direction.

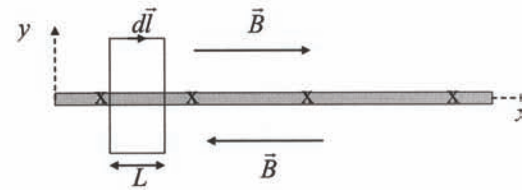


Figure 3.28

12. a. Show that the line integral of the forc $\vec{F} = x(y + 2)\hat{i} + \frac{x^2}{2}\hat{j}$ is the same for the tw paths described:

<u>Path 1</u>: (0, 0) to (3, 0) along the x-axis, the (3,0) to (3, 2) parallel to the y-axis.

<u>Path 2</u>: (0, 0) to (0, 2) along the y-axis, the (0, 2) to (3, 2) parallel to the x-axis.

b. Show that you get the same result for th line integral calculated along the path tha follows the straight line from (0, 0) to (3, 2

13. A rectangular area of length a and width b is oriented in space so that a unit vector perpendicular to the area is

$$\hat{n} = \frac{1}{\sqrt{2}}(\hat{i} - \hat{j})$$

A uniform magnetic field

$$\vec{B} = \frac{B_0}{\sqrt{30}}(2\hat{i} + 5\hat{j} - \hat{k})$$

exists in the region. Find the flux of the field through the area.

14. A cube of side a sits with one vertex at the origin such that the coordinates of any point in the cube are positive. An electric field exists in the region given by the formula

$$\vec{E} = x^2\hat{i} + 3\hat{j}$$

Find the flux of the electric field through the cube.

15. A cylindrically symmetric magnetic field exists in a certain region of space. It is directed into the page (x's signify into page), as shown in Figure 3.29, and has a magnitude

$$B = B_0 \frac{r^2}{R^2} \qquad B_0 \text{ is a constant}$$

The quantity r is the distance from the origin. Calculate the flux of the magnetic field through a circular area of radius a centered on the origin.

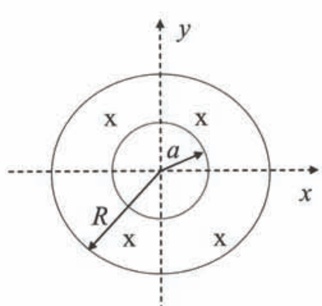


Figure 3.29

16. The magnetic field in the region between two long parallel wires, separated by a distance d, has a magnitude given by

$$B = C\left(\frac{1}{x} + \frac{1}{d-x}\right) \qquad C \text{ is a constant}$$

In the formula, x is the distance from one of the wires, as shown in Figure 3.30.

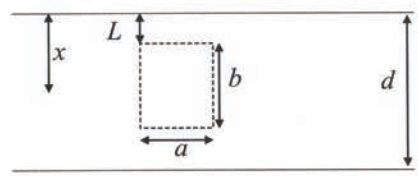


Figure 3.30

Find the flux of the magnetic field through the rectangular loop illustrated. You may assume that the field is directed into the page.