T3

$$p(x) = \begin{cases} \dfrac{e^{-\frac{x}{\theta}}}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad \theta > 0$$

θ: 1) $\hat{\theta}_1 = \bar{x}$, 2) $\hat{\theta}_2 = x_{(1)}$

~~Проверим на несмещенность~~

$$M[\xi] = \int_{-\infty}^{\infty} x \cdot p(x)\, dx = \int_0^{\infty} \frac{x}{\theta} e^{-\frac{x}{\theta}} dx = \left\{ t = \frac{x}{\theta} \right\} =$$

$$= \theta \int_0^{\infty} t e^{-t} dt = \underbrace{-\theta t e^{-t} \Big|_0^{\infty}}_{=0} + \theta \int_0^{\infty} e^{-t} dt =$$

$$= -\theta e^{-t} \Big|_0^{\infty} = \theta$$

$$M[\xi^2] = \int_0^{\infty} \frac{x^2}{\theta} e^{-\frac{x}{\theta}} dx = \left\{ t = \frac{x}{\theta} \right\} = \theta^2 \int_0^{\infty} t^2 e^{-t} dt =$$

$$= \underbrace{-\theta^2 t^2 e^{-t} \Big|_0^{\infty}}_{=0} + 2\theta^2 \int_0^{\infty} t e^{-t} dt = 2\theta^2$$

$$D[\xi] = M[\xi^2] - M^2[\xi] = 2\theta^2 - \theta^2 = \theta^2$$

а) Проверим оценки на несмещенность!

1) $M[\hat{\theta}_1] = M[\frac{1}{n}\sum_{i=1}^{n} x_i] = \frac{1}{n} M[\sum_{i=1}^{n} x_i] =$

$= \frac{1}{n}\sum_{i=1}^{n} M[x_i] = M[\xi] = \theta$ — несмещенная

2) $F(x) = \frac{1}{\theta}\int_0^x e^{-\frac{t}{\theta}} dt = \{\frac{t}{\theta} = u\} = \int_0^{\frac{x}{\theta}} e^{-u} du =$

$= -e^{-u}\Big|_0^{\frac{x}{\theta}} = 1 - e^{-\frac{x}{\theta}}$

$\varphi(y) = n\frac{1}{\theta} e^{-\frac{y}{\theta}} C_{n-1}^{1} (1-(1-e^{-\frac{y}{\theta}}))^{n-2} (1-e^{-\frac{y}{\theta}}) =$

$= \frac{n(n-1)}{\theta} e^{-\frac{y}{\theta}} e^{-\frac{y}{\theta}(n-2)} (1-e^{-\frac{y}{\theta}}) =$

$= \frac{n(n-1)}{\theta} e^{-\frac{y}{\theta}n + 2\frac{y}{\theta} - \frac{y}{\theta}} (1-e^{-\frac{y}{\theta}}) = \frac{n(n-1)}{\theta} e^{-\frac{y}{\theta}(n-1)} (1-e^{-\frac{y}{\theta}})$

$= \frac{n(n-1)}{\theta} (e^{-\frac{y}{\theta}(n-1)} - e^{-\frac{y}{\theta}n})$

$M[\hat{\theta}_2] = \int_{-\infty}^{\infty} y\varphi(y) dy = \int_0^{\infty} \frac{n(n-1)y}{\theta} (e^{-\frac{y}{\theta}(n-1)} - e^{-\frac{y}{\theta}n}) dy =$

$= \frac{n(n-1)}{\theta}\int_0^{\infty} y e^{-\frac{(n-1)}{\theta}y} dy - \frac{n(n-1)}{\theta}\int_0^{\infty} y e^{-\frac{n}{\theta}y} dy =$

$=$

$= \frac{n\theta}{n-1}\int_0^{\infty} t e^{-t} dt - \frac{(n-1)\theta}{n}\int_0^{\infty} t e^{-t} dt =$

$= \underbrace{-\frac{n\theta}{n-1} t e^{-t}\Big|_0^{\infty}}_{=0} + \frac{n\theta}{n-1}\int_0^{\infty} e^{-t} dt + \underbrace{\frac{(n-1)\theta}{n} t e^{-t}\Big|_0^{\infty}}_{=0} -$

$$-\frac{(n-1)\theta}{n}\int_0^\infty e^{-t}dt = \frac{n\theta}{n-1} - \frac{(n-1)\theta}{n} = \frac{n^2\theta - \theta(n-1)^2}{n(n-1)} =$$

$$= \frac{n^2\theta - n^2\theta + 2n\theta - \theta}{n(n-1)} = \frac{2n-1}{n(n-1)}\theta$$ – смещенная

$$\tilde{\theta}_2' = \frac{n(n-1)}{2n-1} X_{(2)}$$ – несмещенная (исправленная)

б) Найдем эффективную

1) $$D[\tilde{\theta}_1] = D\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n^2} D\left[\sum_{i=1}^{n} x_i\right] =$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} D[x_i] = \frac{D(\xi)}{n} = \frac{\theta^2}{n}$$

2) $$M[\tilde{\theta}_2'] = \int_{-\infty}^{\infty} y^2 \varphi(y)\,dy = \int_0^\infty \frac{n(n-1)y^2}{\theta}\left(e^{-\frac{y}{\theta}(n-1)} - e^{-\frac{y}{\theta}n}\right)dy =$$

$$= \frac{n(n-1)}{\theta}\int_0^\infty y^2 e^{-\frac{(n-1)}{\theta}y}dy - \frac{n(n-1)}{\theta}\int_0^\infty y^2 e^{-\frac{n}{\theta}y}dy =$$

$$= -\frac{n\theta^2}{(n-1)^2} t^2 e^{-t}\Big|_0^\infty + 2\frac{n\theta^2}{(n-1)^2}\int_0^\infty t e^{-t}dt + \frac{(n-1)\theta^2}{n^2} t^2 e^{-t}\Big|_0^\infty =$$

$$-2\frac{(n-1)\theta^2}{n^2}\int_0^\infty t e^{-t}dt = -2\frac{n\theta^2}{(n-1)^2} t e^{-t}\Big|_0^\infty +$$

$$+ 2\frac{n\theta^2}{(n-1)^2}\int_0^\infty e^{-t}dt + 2\frac{(n-1)\theta^2}{n^2} t e^{-t}\Big|_0^\infty - 2\frac{(n-1)\theta^2}{n^2}\int_0^\infty e^{-t}dt =$$

$$= 2\frac{n\theta^2}{(n-1)^2} - 2\frac{(n-1)\theta^2}{n^2} = 2\theta^2\left(\frac{n}{(n-1)^2} - \frac{n-1}{n^2}\right) =$$

$$= 2\theta^2\left(\frac{n^3-(n-1)^3}{n^2(n-1)^2}\right)$$

2) $M[\tilde{\Theta}_2^2] = \int\limits_{-\infty}^{\infty} y^2 \varphi(y)\,dy = \int\limits_0^{\infty} \frac{n(n-1)y^2}{\theta}\left(e^{-\frac{y}{\theta}(n-1)} - e^{-\frac{y}{\theta}n}\right)dy$

$= \frac{n(n-1)}{\theta}\int\limits_0^{\infty} y^2 e^{-\frac{y}{\theta}(n-1)}\,dy - \frac{n(n-1)}{\theta}\int\limits_0^{\infty} y^2 e^{-\frac{y}{\theta}n}\,dy =$

$= \frac{n\theta^2}{(n-1)^2}\int\limits_0^{\infty} t^2 e^{-t}\,dt - \frac{(n-1)\theta^2}{n^2}\int\limits_0^{\infty} t^2 e^{-t}\,dt =$

$= \overbrace{-\frac{n\theta^2}{(n-1)^2} t^2 e^{-t}\Big|_0^{\infty}}^{=0} + 2\frac{n\theta^2}{(n-1)^2}\int\limits_0^{\infty} t e^{-t}\,dt + \overbrace{\frac{(n-1)\theta^2}{n} t^2 e^{-t}\Big|_0^{\infty}}^{=0} -$

$- 2\frac{(n-1)\theta^2}{n^2}\int\limits_0^{\infty} t e^{-t}\,dt = 2\frac{n\theta^2}{(n-1)^2} - 2\frac{(n-1)\theta^2}{n^2} =$

$= 2\theta^2\left(\frac{n}{(n-1)^2} - \frac{n-1}{n^2}\right) = 2\theta^2\left(\frac{n^3-(n-1)^3}{n^2(n-1)^2}\right) =$

$= 2\theta^2\left(\frac{\cancel{n^3} - \cancel{n^3} + 3n^2 - 3n + 1}{n^2(n-1)^2}\right) = 2\theta^2\left(\frac{3n^2-3n+1}{n^2(n-1)^2}\right)$

$D[\tilde{\Theta}_2] = M[\tilde{\Theta}_2^2] - M^2[\tilde{\Theta}_2] = 2\theta^2\left(\frac{3n^2-3n+1}{n^2(n-1)^2}\right) -$
$- \frac{(2n-1)^2}{n^2(n-1)^2}\theta^2 = \theta^2\frac{1}{n^2(n-1)^2}(6n^2-6n+2-4n^2+4n-1) =$

$= \theta^2\left(\frac{2n^2-2n+1}{n^2(n-1)^2}\right)$

$D[\tilde{\Theta}_2'] = \frac{n^2(n-1)^2}{(2n-1)^2} D[\tilde{\Theta}_2] = \theta^2\left(\frac{2n^2-2n+1}{(2n-1)^2}\right) =$

$= \theta^2\left(\frac{2n^2-2n+1}{4n^2-4n+1}\right)$

$\frac{1}{n} < \frac{2n^2-2n+1}{4n^2-4n+1} \quad \forall n > 1$

$\lim\limits_{n\to\infty}\frac{1}{n} = 0, \quad \lim\limits_{n\to\infty}\frac{2n^2-2n+1}{4n^2-4n+1} = \frac{1}{2}$

$n=3$:

$D[\tilde{\Theta}_1] = \frac{\Theta^2}{3}$, $D[\tilde{\Theta}_2'] = \frac{2\cdot 9 - 2\cdot 3 + 1}{4\cdot 9 - 4\cdot 3 + 1}\Theta^2 = \frac{13}{25}\Theta^2$

$\tilde{\Theta}_1$ – эффективная оценка.

c) Исследуем оценки на эфф-ность с помощью неравенства Крамера-Рао:

Проверим, регулярна ли модель:

1) $p(x,\theta)$ непр. дифф. по θ на $\Theta = (0, +\infty)$ ✓

2) $\int_0^\infty \frac{1}{\theta} e^{-\frac{x}{\theta}} dx = \frac{1}{\theta}\int_0^\infty e^{-\frac{x}{\theta}} dx = \int_0^\infty e^{-t} dt = 1$

$\frac{\partial}{\partial\theta}(1) = 0$

$\frac{\partial}{\partial\theta}\left(\frac{1}{\theta} e^{-\frac{x}{\theta}}\right) = -\frac{e^{-\frac{x}{\theta}}}{\theta^2} + \frac{x}{\theta^3} e^{-\frac{x}{\theta}}$

$\int_0^\infty \left(-\frac{1}{\theta^2} e^{-\frac{x}{\theta}} + \frac{x}{\theta^3} e^{-\frac{x}{\theta}}\right) dx = -\frac{1}{\theta^2}\int_0^\infty e^{-\frac{x}{\theta}} dx +$

$+ \frac{1}{\theta^3}\int_0^\infty x e^{-\frac{x}{\theta}} dx = -\frac{1}{\theta}\int_0^\infty e^{-t} dt + \frac{1}{\theta}\int_0^\infty t e^{-t} dt =$

$= -\frac{1}{\theta} - \frac{1}{\theta} \underbrace{t e^{-t}\Big|_0^\infty}_{=0} + \frac{1}{\theta}\int_0^\infty e^{-t} dt = -\frac{1}{\theta} + \frac{1}{\theta} = 0$ ✓

3) $\ln\left(\frac{e^{-\frac{x}{\theta}}}{\theta}\right) = -\frac{x}{\theta} - \ln\theta$

$$\frac{\partial \ln p(x,\theta)}{\partial \theta} = \frac{x}{\theta^2} - \frac{1}{\theta}$$

$$I(\theta) = \cancel{M\left(\frac{x}{\theta^2} - \frac{x}{\theta}\right)^2} M\left[\left(\frac{\partial \ln p(x,\theta)}{\partial \theta}\right)^2\right] =$$

$$= \int_0^\infty \left(\frac{x^2}{\theta^4} - 2\frac{x}{\theta^3} + \frac{1}{\theta^2}\right)\frac{1}{\theta}e^{-\frac{x}{\theta}}dx =$$

$$= \frac{1}{\theta^2}\left(\int_0^\infty t^2e^{-t}dt - 2\int_0^\infty te^{-t}dt + \int_0^\infty e^{-t}dt\right) =$$

$= \frac{1}{\theta^2}(2-2+1) = \frac{1}{\theta^2}$ – непр. на Θ и $I(\theta) > 0$ на Θ ✓

$\Rightarrow$ модель регулярна

Проверим, регулярна ли оценка:

1) $\tilde{\theta}_1$ – несмещ. оценка и $D[\tilde{\theta}_1] = \frac{\theta^2}{n}$ – огр. на $\forall$ компакте из Θ по $\theta \Rightarrow \tilde{\theta}_1$ – регулярна.

2) $\tilde{\theta}_2'$ – несмещ. оценка и $D[\tilde{\theta}_2'] = \theta^2\left(\frac{2n^2-2n+1}{4n^2-4n+1}\right)$ – огр. на $\forall$ компакте из Θ по $\theta \Rightarrow \tilde{\theta}_2'$ – регулярна

Применим неравенство Крамера-Рао:

1) $D[\tilde{\Theta}_1] \geq \frac{g'^2(\Theta)}{nI(\Theta)} = \frac{1}{3 \cdot \frac{1}{\Theta^2}} = \frac{\Theta^2}{3}$

$g(\Theta) = \Theta$, $g'(\Theta) = 1$

$D[\tilde{\Theta}_1] = \frac{\Theta^2}{3} \Rightarrow \tilde{\Theta}_1$ – эффективная

2) $D[\tilde{\Theta}_2'] = \frac{g'^2(\Theta)}{nI(\Theta)} = \frac{\Theta^2}{3}$

$D[\tilde{\Theta}_2'] = \Theta^2 \left(\frac{2n^2 - 2n + 1}{4n^2 - 4n + 1} \right) \Bigg|_{n=3} = \frac{13\Theta^2}{25} \neq \frac{\Theta^2}{3} \Rightarrow$

$\Rightarrow \tilde{\Theta}_2'$ – не эффективная оценка.