The problems of this homework are to be presented on **April 27, 2021**. Students should tick in TUWEL problems they have solved and are prepared to present their detailed solutions. The problems should be ticked and solution paths uploaded by **23:59 on April 26, 2021**.

(1) **Distribution of the maximum**
Let $X_1, X_2, \ldots$ be a sequence of i.i.d. with uniform $(0,1)$ distribution and let $X_{(n)} = \max_{1 \le i \le n} X_i$. Show that the sequence

$$Y_n = n(1 - X_{(n)}), \quad n \in \mathbb{N}$$

converges to an exponential $\exp(1)$ random variable as $n \to \infty$.

(2) **Coin throws**
An unfair coin is thrown 600 times. The probability of geting a tail in each throw is $\frac{1}{4}$.

(a) Use a Binomial distribution to compute the probability that the number of heads obtained does not differ more than 10 from 450.

(b) Use a Normal approximation without a continuity correction to calculate the probability in (a). How does the result change if the approimation is provided with a continuity correction?

(3) **Simulations**

(a) By applying the `R`-function `replicate()` generate a sample $X_1, \ldots, X_{10}$ of size 10 from an exponential distribution with a rate parameter 0.2 and sum up its elements. Do this sum 10 000 times and make a histogram of the simulation. Can you say something about the shape of distribution?

(b) Use `R` to simulate 50 tosses of a fair coin (0 and 1). We call a *run* a sequence of all 1's or all 0's. Estimate the average length of the longest run in 10000 trials and report the result.

Hint: Use the commands `rbinom` and `rle`. The command `rle`() stands for run length encoding. For example,

```
rle(rbinom(5, 1, 0.5))$lengths
```

is a vector of the lengths of all the different runs in trial of 5 flips of a fair coin.

(4) **Conditional variance**

(a) Show that for any two random variables X and Y the conditional variance identity holds

$$\mathbb{V}ar\, Y = \mathbb{E}\left(Var\left(Y|X\right)\right) + \mathbb{V}ar\left(\mathbb{E}\left(Y|X\right)\right),$$

provided that the expectations exist. The law of total expectation (the tower property) $\mathbb{E}X = \mathbb{E}(\mathbb{E}(X|Y))$ should be applied.

(b) Suppose that the distribution of Y conditional on $X = x$ is $\mathcal{N}(x, x^2)$ and that the marginal distribution of X is uniform on $(0, 1)$. Compute $\mathbb{E}Y$, $\mathbb{V}ar\, Y$ and $\mathbb{C}ov(X, Y)$.

(5) (a) **Delta method**

Let $X_1, \ldots, X_n$ be i.i.d. from normal distribution with unknown mean μ and known variance σ^2. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find the limiting distribution of $\sqrt{n}\left(\bar{X}^3 - c\right)$ for an appropriate constant c.

(b) **Logit transformation**

Let $X_n \sim bin(n, p)$. Consider the logit transformation, defined by

$$logit(y) = \ln \frac{y}{1 - y}, \qquad 0 < y < 1.$$

Determine the approximate distribution of $logit\left(\frac{X_n}{n}\right)$.