(3) **Chi squared distribution**

Let X and Y be independent and identically distributed (i.i.d.) $\mathcal{N}(0,1)$ random variables. Define $Z = \min\{X, Y\}$. Show that $Z^2 \sim \chi_1^2$, i.e. show that the pdf of Z^2 is given by

$$f_{Z^2}(z) = \frac{1}{\sqrt{2\pi}} \cdot z^{-\frac{1}{2}} \cdot e^{-\frac{z}{2}} \cdot \mathbf{1}_{\{z>0\}},$$

$$\mathbb{P}(Z<z) = 1 - \mathbb{P}(Z \geq z) = 1 - \mathbb{P}(\min\{X,Y\} \geq z) = 1 - \mathbb{P}(X \geq z \wedge Y \geq z)$$

$$= 1 - \mathbb{P}(X \geq z)\,\mathbb{P}(Y \geq z) = 1 - \frac{1}{2\pi}\left(\int_z^\infty e^{-s^2/2}\, ds\right)^2$$

$$f_Z(z) = -\frac{1}{\pi} \int_z^\infty e^{-\frac{s^2}{2}} ds \left(-e^{-z^2/2}\right) = \frac{1}{\pi} e^{-z^2/2} \int_z^\infty e^{-\frac{s^2}{2}} ds$$

$g: \mathbb{R}\setminus\{0\} \to \mathbb{R}^+ : z \mapsto z^2$ has got two right inverses, namely

$h_-: (0,\infty) \to (-\infty, 0) : u \mapsto -\sqrt{u}$ and $h_+ : (0,\infty) \to (0,\infty) : u \mapsto \sqrt{u}$

$h_-'(u) = -\frac{1}{2} u^{-1/2}$ and $h_+(u) = \frac{1}{2} u^{-\frac{1}{2}}$, hence for all $u \in \mathbb{R}^+$, we obtain

$$f_{Z^2}(u) = f_Z(\sqrt{u})\frac{1}{2\sqrt{u}} + f_Z(-\sqrt{u})\frac{1}{2\sqrt{u}}$$

$$= \frac{1}{\pi} e^{-u/2} \int_{\sqrt{u}}^\infty e^{-\frac{s^2}{2}} ds \frac{1}{2\sqrt{u}} + \frac{1}{\pi} e^{-u/2} \int_{-\sqrt{u}}^\infty e^{-\frac{s^2}{2}} ds$$

$$= \frac{1}{\sqrt{2\pi}} u^{1/2} e^{-u/2} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds = \frac{1}{\sqrt{2\pi}} u^{1/2} e^{-\frac{u}{2}}$$

Thus, $f_{Z^2}(u) = \frac{1}{\sqrt{2\pi}} u^{\frac{1}{2}} e^{-\frac{u}{2}} \mathbb{1}_{\mathbb{R}^+}(u)$