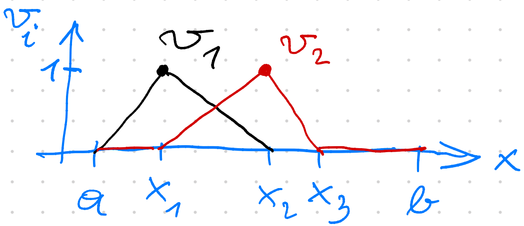
§5. Simulationsbeispiele - elliptische Probleme

schwache Formulierung:

ges: $u \in H$, so dass $a(u,v) = F(v) \quad \forall v \in H$

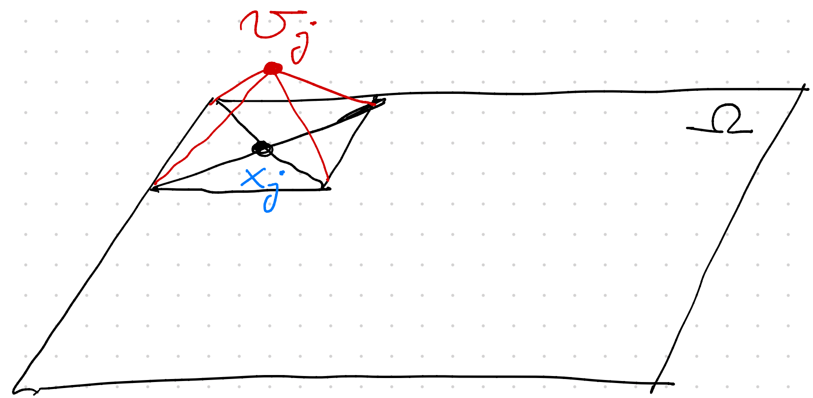
- **1D Finite Elemente Methode (FEM):**



$H := \operatorname{span}\{v_i\} \subset H_0^1(a,b)$

$\dim H < \infty$

2D FEM:



- Methode 2. Ordnung, d.h.: $\|\text{Fehler}\|_{L^\infty(\Omega)} \leq C\, h_{max}^2$

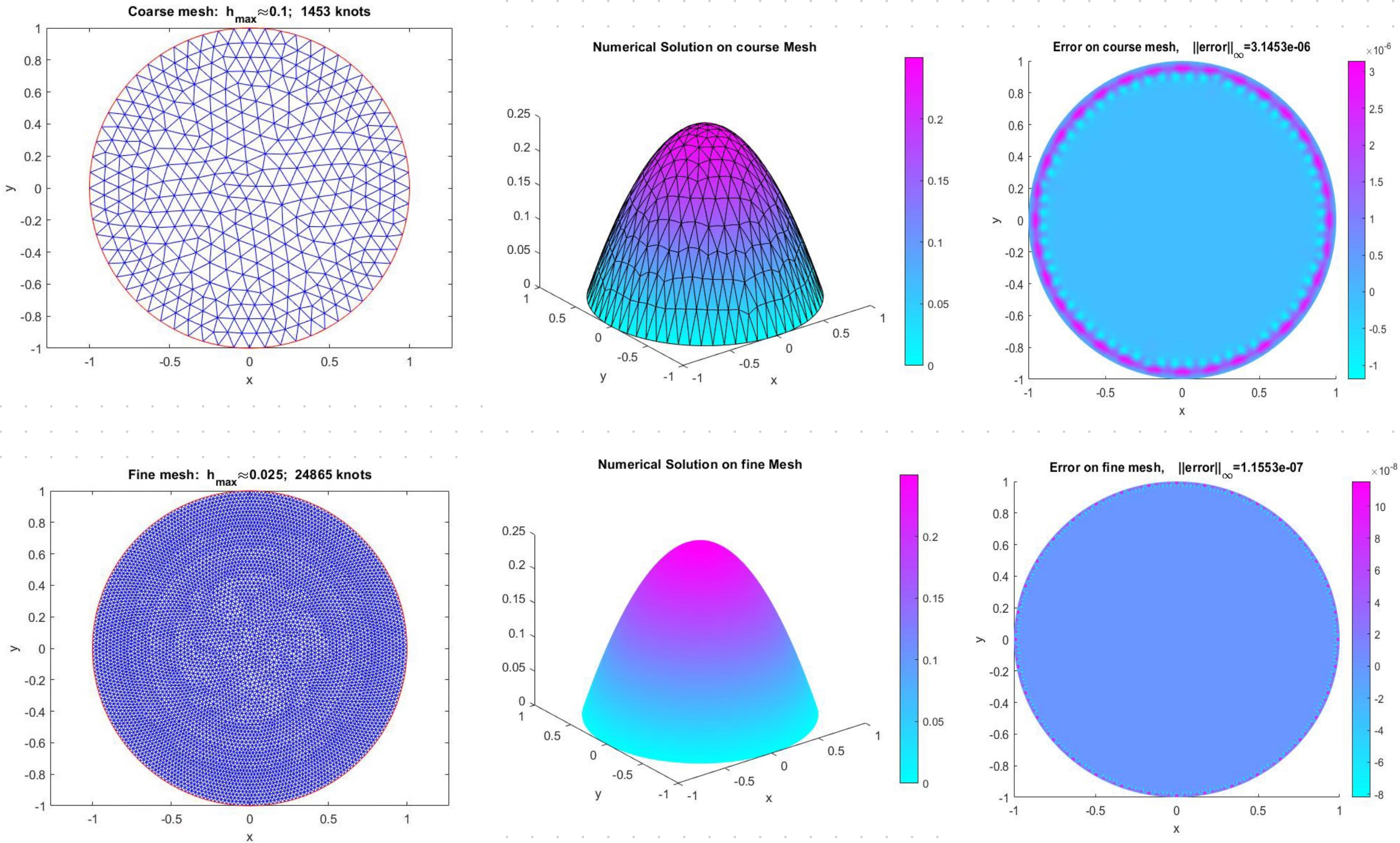
Bsp 1:

$$\begin{cases} -\Delta u = 1, & \Omega = K_1(0) \subset \mathbb{R}^2 \\ u = 0, & \partial\Omega \end{cases}$$

- $-\Delta u \geq 0 \Rightarrow$ Min Prinzip:

  Min. von u am Rand.

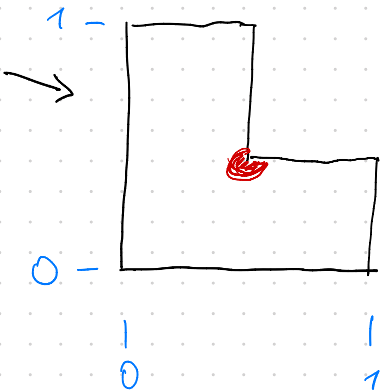
- explizite Lösung: $u(x,y) = \dfrac{1-x^2-y^2}{4}$

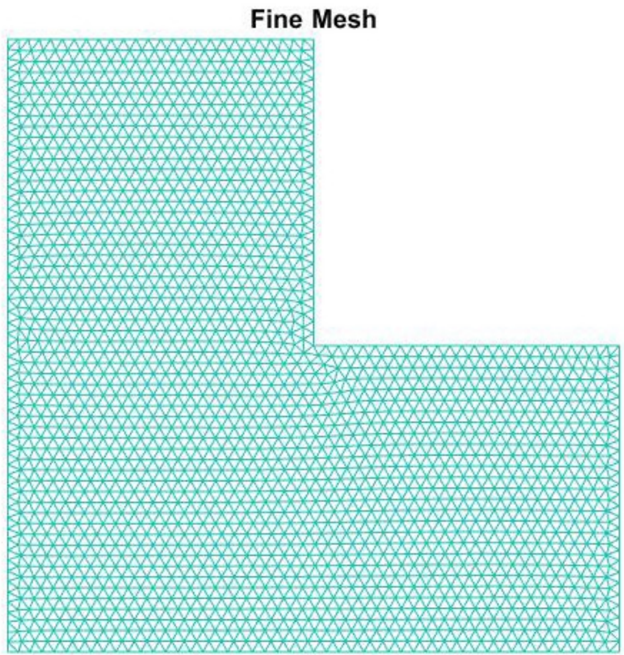


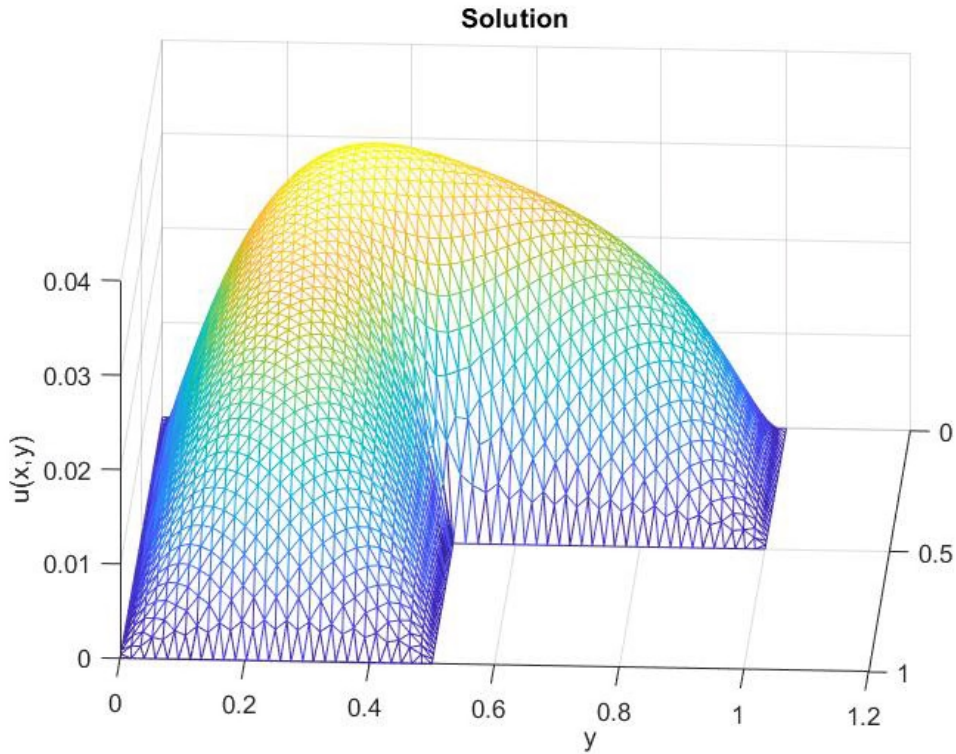
Coarse mesh: h_max≈0.1; 1453 knots
Numerical Solution on course Mesh
Error on course mesh, ||error||_∞=3.1453e-06
×10^-6
Fine mesh: h_max≈0.025; 24865 knots
Numerical Solution on fine Mesh
Error on fine mesh, ||error||_∞=1.1553e-07
×10^-8
x
y

Bsp 2:

$$\begin{cases} -\Delta u = 1, & \Omega \\ u = 0, & \partial\Omega \end{cases} \rightarrow$$

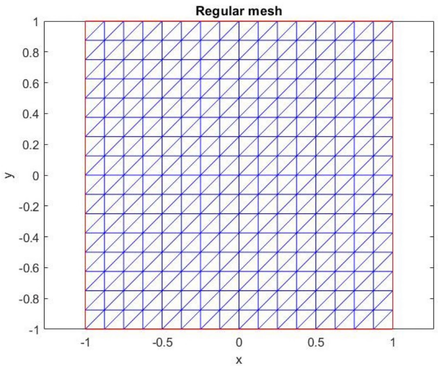


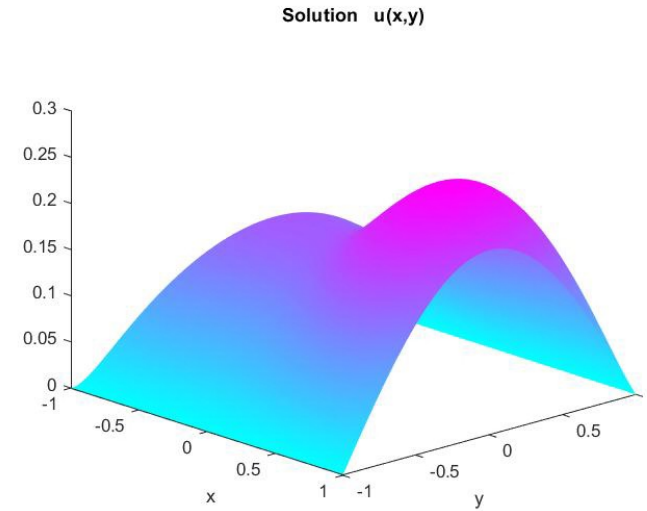

Fine Mesh

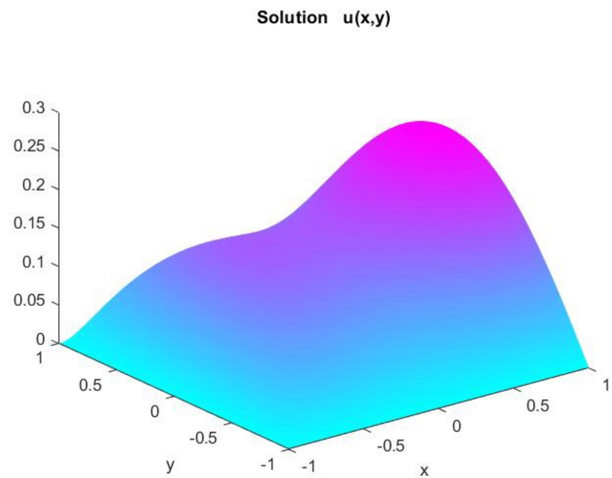

Solution
u(x,y)
0.04
0.03
0.02
0.01
0
0
0.2
0.4
0.6
0.8
1
1.2
y
0
0.5
1

Bsp 3:

$$\begin{cases} -\Delta u = 3x^2, \quad \Omega = (-1,1)^2 \\ \text{RB: } u(1,y) = 0.2\,(1-y^2) \\ \qquad u|_{\partial\Omega} = 0, \quad \text{sonst} \end{cases}$$


Regular mesh
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
y
-1
-0.5
0
0.5
1
x

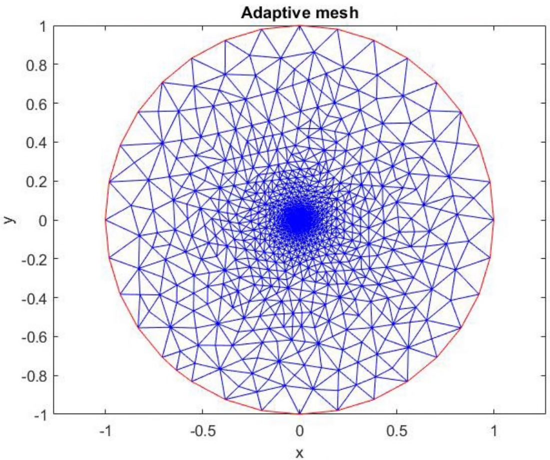

Solution u(x,y)
0.3
0.25
0.2
0.15
0.1
0.05
0
-1
-0.5
0
0.5
1
x
-1
-0.5
0
0.5
1
y

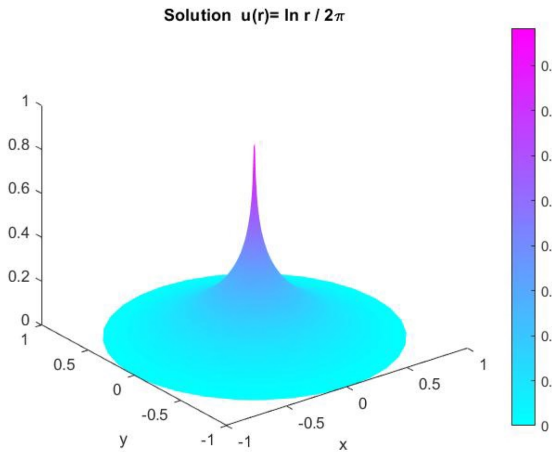

Solution u(x,y)
0.3
0.25
0.2
0.15
0.1
0.05
0
1
0.5
0
-0.5
-1
y
-1
-0.5
0
0.5
1
x

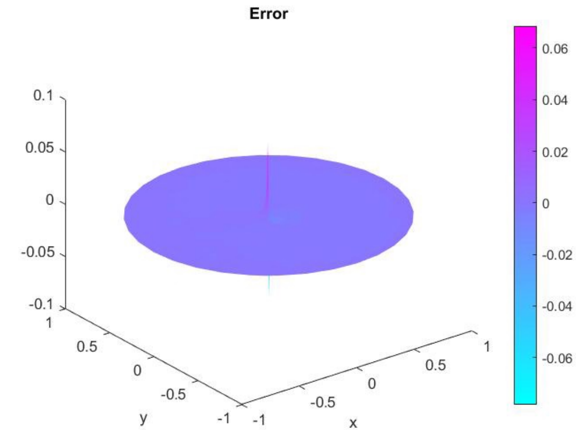
Bsp 4 (Greensche Funktion):

$$\begin{cases} -\Delta u = \delta \,, & \Omega = K_1(0) \subset \mathbb{R}^2 \\ u = 0 \,, & \partial\Omega \end{cases}$$

$$\Rightarrow u(x,y) = -\frac{1}{2\pi} \ln r \,, \quad r = \sqrt{x^2+y^2}$$


Adaptive mesh
x
y


Solution u(r)= ln r / 2π
x
y


Error
x
y