4. *(Galerkin method for the Poisson equation)* Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with smooth boundary. For $f \in L^2(\Omega)$ construct a solution of the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1}$$

using a Galerkin method. To do this, let $\{\phi_k\}$ for $k \in \mathbb{N}$ denote the eigenfunctions of the Laplacian with homogeneous Dirichlet boundary data on Ω. Then prove that for any $m \in \mathbb{N}$ there exists

$$u_m = \sum_{k=1}^{m} \mathbf{d}_m^k \phi_k, \quad \text{for } \mathbf{d}_m^k \in \mathbb{R}$$

that satisfies

$$\int_\Omega \nabla u_m \cdot \nabla \phi_k \, dx = \int_\Omega f \phi_k, \quad \text{for } k = 1, \dots, m.$$

To finish, show that the sequence $\{u_m\}_{m\in\mathbb{N}}$ converges weakly in $H_0^1(\Omega)$ to a weak solution of (1).

$\forall k \in \mathbb{N}: \; -\Delta \phi_k = \lambda_k \phi_k, \; T(\phi_k) = 0 \text{ (auf } \partial\Omega)$, $\quad \lambda_k \in \mathbb{C}$, sogar $\lambda_k \in \mathbb{R}$, weil $-\Delta: H_0^1 \to H_0^1$ symmetrisch ? $\quad \phi_k \in L^2(\partial\Omega)$?

$$\int_\Omega \nabla u_m \cdot \nabla \phi_k \, dx \overset{!}{=} \int_\Omega f \phi_k \, dx \le \|f\|_{L^2(\Omega)} \|\phi_k\|_{L^2(\Omega)} < \infty$$

$\| \text{ Gauß ?}$

$$-\int_\Omega \Delta u_m \, \phi_k \, dx + \int_{\partial\Omega} \underbrace{\phi_k}_{=0, \text{ weil homogene DB}} (\nabla u_m \cdot \nu) \, ds = \sum_{j=1}^m d_m^j \int_\Omega -\Delta \phi_j \, \phi_k \, dx$$

$$= \sum_{j=1}^m d_m^j \lambda_j \overbrace{\int_\Omega \phi_j \phi_k \, dx}^{\text{ONS} \,\overset{?}{=}\, \Delta \text{ kompakt}} = d_m^k \lambda_k \underbrace{\|\phi_k\|_{L^2(\Omega)}}_{\substack{=1\\ \text{o.B.d.A.}}} = d_m^k \lambda_k \Rightarrow d_m^k := \frac{1}{\lambda_k} \int_\Omega f \phi_k \, dx \quad \lambda_k \neq 0, \text{ weil } -\Delta \text{ injektiv}$$

z.z.: $\exists v \in H_0^1: \forall w \in H_0^1: \lim_{m\to\infty} \langle u_m, w \rangle = \langle v, w \rangle$

$$|\langle u_m, v\rangle_{H_0^1} - \langle u_n, w\rangle_{H_0^1}| = |\langle u_m - u_n, w\rangle_{H_0^1}| = |\langle u_m - u_n, w\rangle_{L^2} + \int_\Omega \nabla(u_m - u_n) \cdot \nabla w \, dx| =$$

Sei v die schwache Lsg.

$$|\langle v - u_m, w\rangle_{H_0^1}| = |\langle v - \sum_{k=1}^m d_m^k \phi_k, w\rangle_{H_0^1}| = |\langle v - \sum_{k=1}^m \frac{1}{\lambda_k} \langle f, \phi_k\rangle \phi_k, w\rangle_{H_0^1}| =$$

$$= |\langle v, w\rangle - \langle u_m, w\rangle|$$

$$\langle v, w\rangle_{H_0^1} = \langle v, w\rangle_{L^2(\Omega)} + \langle \nabla v, \nabla w\rangle_{L^2(\Omega)} = \langle v, w\rangle_{L^2(\Omega)} + \langle f, w\rangle_{L^2(\Omega)} = \langle v, w\rangle + \sum_{j\in\mathbb{N}} \langle w, \phi_j\rangle \langle f, \phi_j\rangle$$

$$\langle u_m, w\rangle_{H_0^1} = \langle u_m, w\rangle_{L^2(\Omega)} + \langle \nabla u_m, \nabla w\rangle_{L^2(\Omega)}$$

$$= \langle u_m, w\rangle_{L^2(\Omega)} + \langle \nabla u_m, \nabla \sum_{j\in\mathbb{N}} \langle w, \phi_j\rangle \phi_j\rangle_{L^2(\Omega)}$$

$$\overset{?}{=} \langle u_m, w\rangle_{L^2(\Omega)} + \sum_{j\in\mathbb{N}} \langle w, \phi_j\rangle \langle \nabla u_m, \nabla \phi_j\rangle_{L^2(\Omega)}$$

$$= \langle u_m, w\rangle_{L^2(\Omega)} + \sum_{j=m+1}^{\infty} \langle w, \phi_j\rangle \langle \nabla u_m, \nabla \phi_j\rangle_{L^2(\Omega)} + \sum_{j=1}^{m} \langle w, \phi_j\rangle \langle f, \phi_j\rangle \, .$$

$$= \langle u_m, w\rangle_{L^2(\Omega)} + \sum_{j=m+1}^{\infty} \langle w, \phi_j\rangle \langle \nabla u_m, \nabla \phi_j\rangle_{L^2(\Omega)} + \langle f, \sum_{j=1}^{m} \langle w, \phi_j\rangle, \phi_j\rangle$$

$\overset{?}{}$ also $\lim_{m\to\infty} \langle \nabla u_m, \nabla w\rangle_{L^2} \to \langle \nabla v, \nabla w\rangle$

$$\left| \langle v - u_m, w \rangle_{H_0^1} \right| = \left| \langle v - u_m, w \rangle_{L^2(\Omega)} + \sum_{j=m+1}^{\infty} \langle w, \phi_j \rangle_{L^2} \left(\langle f, \phi_j \rangle_{L^2} - \langle \nabla u_m, \nabla \phi_j \rangle_{L^2} \right) \right| \leq$$

$$\leq \left| \langle v - u_m, w \rangle_{L^2(\Omega)} \right| + \left| \sum_{j=m+1}^{\infty} \langle w, \phi_j \rangle_{L^2} \left(\langle f, \phi_j \rangle_{L^2} - \langle \nabla u_m, \nabla \phi_j \rangle_{L^2} \right) \right|$$