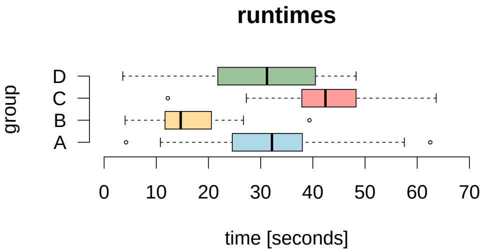
Descriptive Statistics

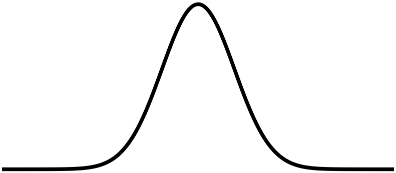


For comments please email: M. Messer, michael.messer@tuwien.ac.at

Overview

We differentiate:

Probability theory
(Stochastics)
$=$
Theory of randomness



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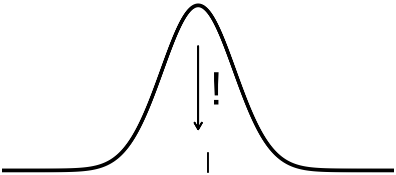

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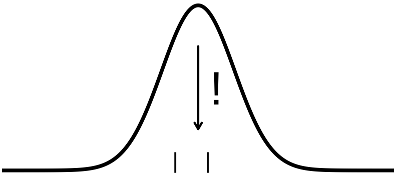

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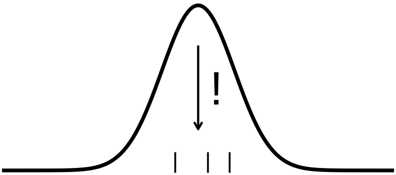

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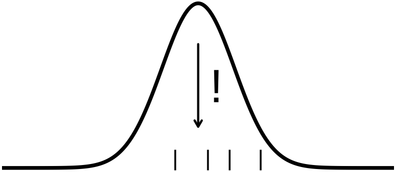

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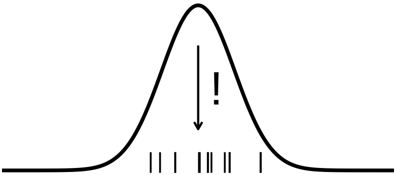

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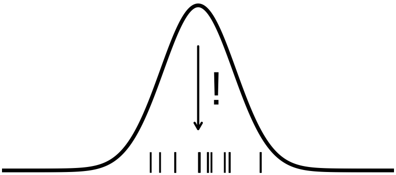

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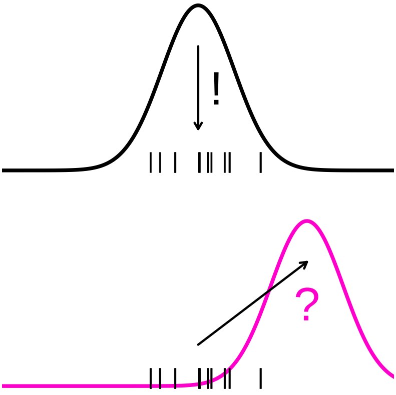

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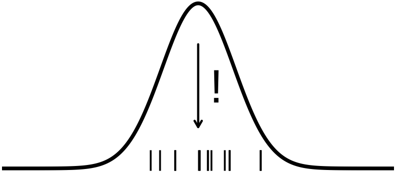

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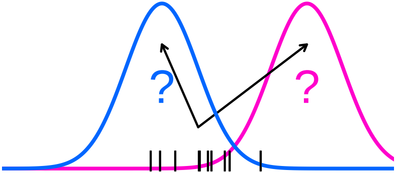

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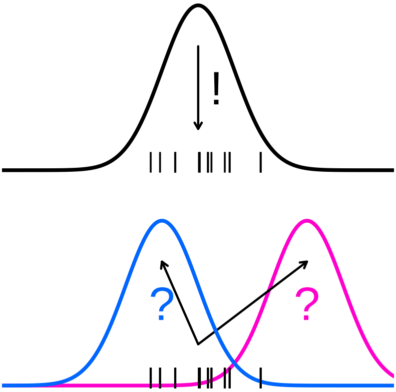

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How do data look like? How can they be summarized?

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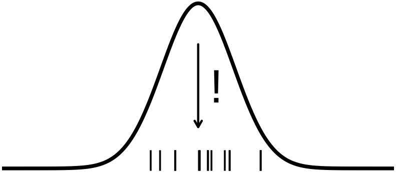

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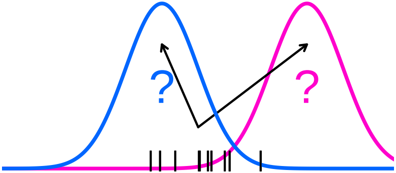

Today: <u>Short excursion to descriptive Statistics</u>
How do data look like? How can they be summarized?

From then on: <u>inferential Statistics</u> (Modelling)
How did the data occur?

Scales

We differentiate scales

- Categorial data (nominal scale, no ordering)
  - Do you drink coffee? yes or no (two categories)
  - What is the color of your hair? blond, brown, black, red, neither (five categories)

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  - How large are you? size in cm
  - How long is the runtime of an algorithm that you implemented? time in seconds

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(Today we stick to metric data)

Data collection

How long is the runtime of an algorithm that you implemented?

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Results (in seconds):

24.6, 24, 31.4, 29.9, 37.8, 19.9, 46.1, 32.8, 30.3, 29, 47.1, 27.8, 33.8, 30.1, 53.3, 23.8, 32.1, 4.2, 42.8, 25.2, 52.3, 35, 30.1, 43.2, 25.4, 62.5, 35.4, 25.2, 37.6, 37.1, 22.9, 29.5, 44.5, 34.8, 33.3, 21.9, 37.2, 24, 37, 34, 24.1, 10.8, 24.9, 37.2, 52, 30.8, 22, 18.6, 22, 26.8, 52.3, 27, 23.6, 33.5, 30.8, 20.9, 35.6, 37.2, 57.5, 46.2, 36.1, 19.8, 38.1, 36.9, 26.5, 23.6, 30.3, 49.9, 39, 50.2, 35.7, 11.4, 24.1, 27.5, 36.4, 29.8, 49, 42.6, 22.5, 32.7, 34.3, 21.4, 34.7, 47.3, 20.3, 35.4, 41.8, 24.9, 15.2, 42.2, 29.1, 25.1, 22.7, 41, 28.2, 30.3, 25.6, 41.8, 16.6, 38, 43.1, 29.5, 40.3, 20.5, 39.9, 24.5, 33.7, 14.6, 23.3, 36.7, 34.7, 34.9, 39.1, 32.2, 43, 12.1, 19.8, 27.4, 39.3, 35, 46.3

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We see: n data: $x_1 = 24.6, x_2 = 24.0, \ldots, x_n = 46.3$

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We understand: nothing?

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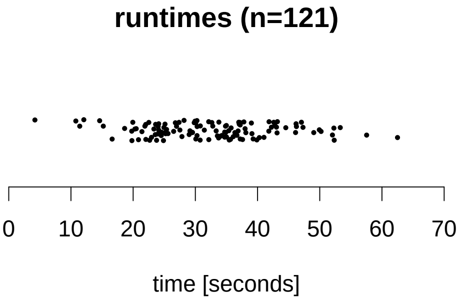
We see: n data: $x_1 = 24.6, x_2 = 24.0, \ldots, x_n = 46.3$

We understand: nothing?

Thus: descriptive Statistics $\rightarrow$ graphical representation and summary of data

Stripchart



At first sight we understand how the n data distribute:

- Many data lie close to 30 (typical runtime)

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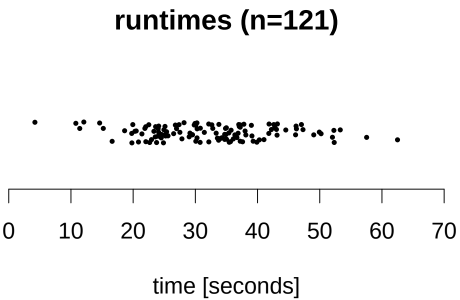

At first sight we understand how the n data distribute:

- Many data lie close to 30 (typical runtime)
- The minimum is about 5 (fastest runtime),
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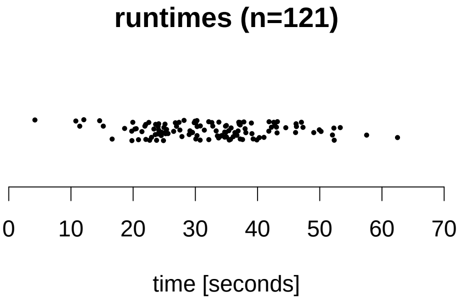
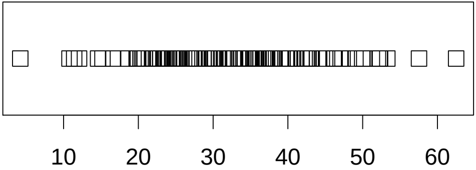


At first sight we understand how the n data distribute:

- Many data lie close to 30 (typical runtime)
- The minimum is about 5 (fastest runtime),
  the maximum is about 65 (slowest runtime)
- Remark.: the y-value has no meaning. The data are 'jittered' along the y-direction for a better overview.

Stripchart in R

```
#Enter data
x <- c(24.6, 24.0, 31.4, 29.9,...,39.3, 35.0, 46.3)
#Create stripchart
stripchart(x)
```

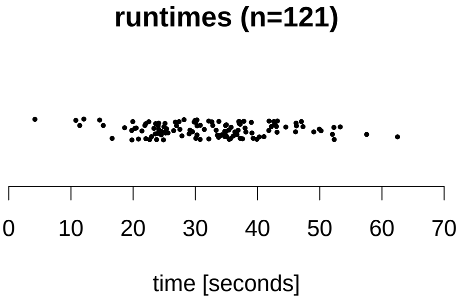


We don't understand too much - points superposed, axes annotations are missing, title is missing etc.

→ customize graphic using additional arguments or lowlevel graphics

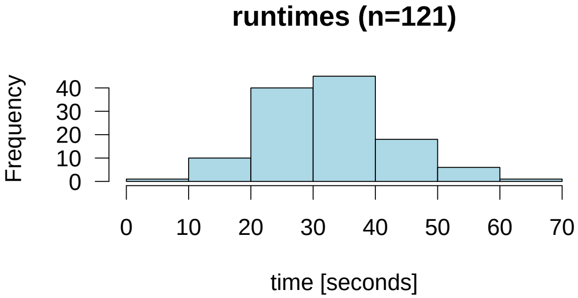
Stripchart in R

```
#Enter data
x <- c(24.6, 24.0, 31.4, 29.9,...,39.3, 35.0, 46.3)
#Create stripchart with additional arguments
stripchart(x,method="jitter",pch=19,cex=0.4,axes=FALSE,
   xlim=c(0,70),main="runtimes␣(n=121)",xlab="time␣[seconds]")
#add x-axis (lowlevelgraphic)
axis(1,at=seq(0,70,10))
```



Much more informative!

Histogram



- Description of the distribution of data
  Here: approximately *bell-shaped*, i.e., unimodal and symmetric

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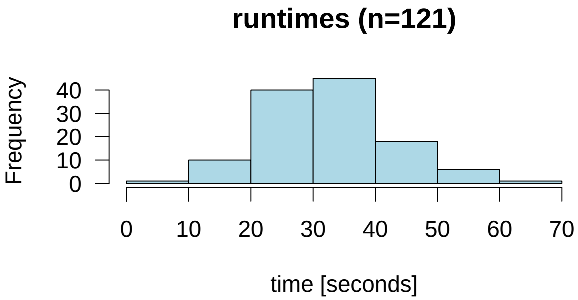

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- Absolute frequencies in the intervals $\{(10k, 10(k+1)] : k = 0, 1, \ldots, 6\}$ given through the height of the bars

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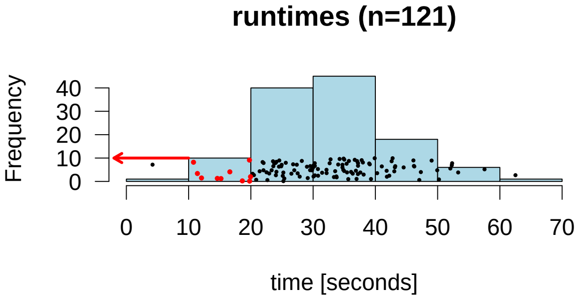

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  e.g.: 10 data are > 10 and $\leqslant 20$, for short $\sum_{i=1}^{n} \mathbb{1}_{(10,20]}(x_i) = 10$

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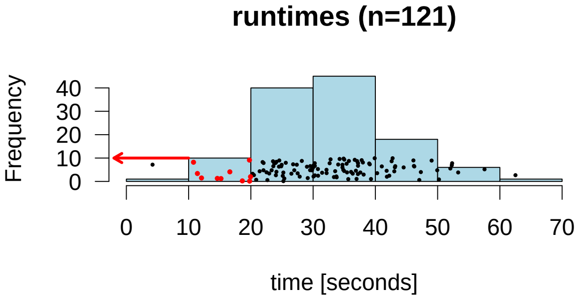

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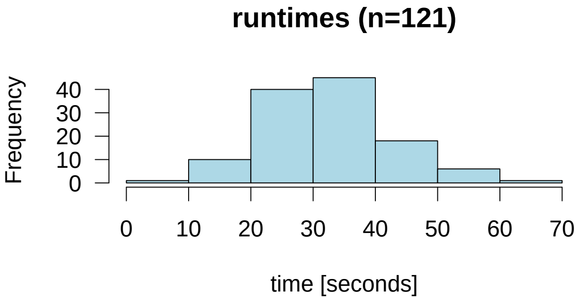
  e.g.: 10 data are > 10 and $\leqslant 20$, for short $\sum_{i=1}^{n} \mathbb{1}_{(10,20]}(x_i) = 10$
  Consequence: The sum of the bar heights is $n = 121$

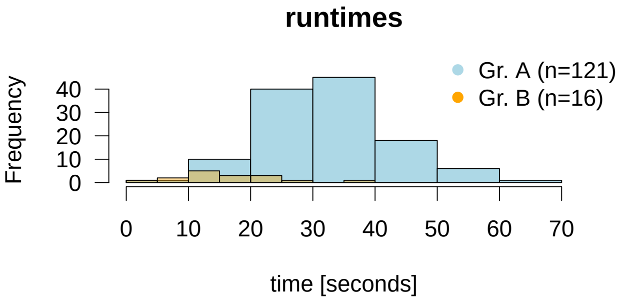
Histogram in R

```r
# Histogram with additional arguments
hist(x,las=1,xlab="time␣[seconds]",ylab="Frequency",
main="runtimes␣(n=121)",col="lightblue")
```



Histogram

The same algorithm was implemented by 16 other students after they attended a certain programming course (group B)



- Comparison of group A ($n_A = 121$) and group B ($n_B = 16$) inappropriate, because the sizes of the groups differ tremendously.

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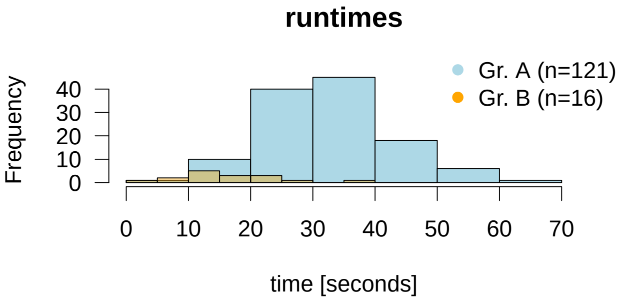

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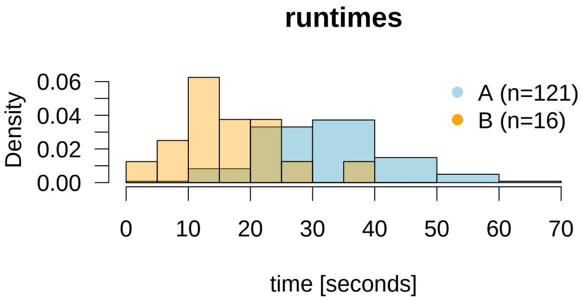
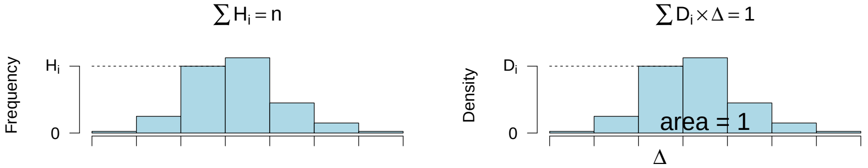


- Comparison of group A ($n_A = 121$) and group B ($n_B = 16$) inappropriate, because the sizes of the groups differ tremendously.
- Idea: Norm the areas $\rightarrow$ total area of 1 each
  The distributions are now nicely visible:
  shifted against each other and about bell-shaped each.

Histogram

What happens when norming?



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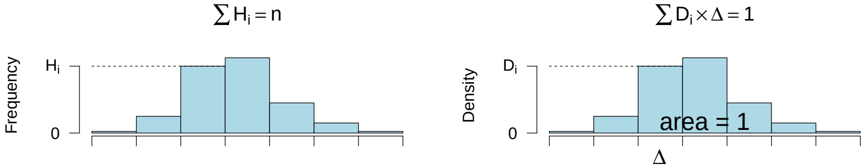

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  Search D_i such that total area $\sum D_i \cdot \Delta \overset{!}{=} 1$

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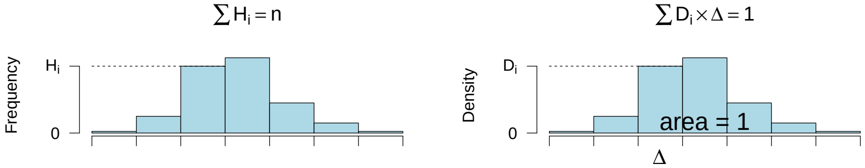

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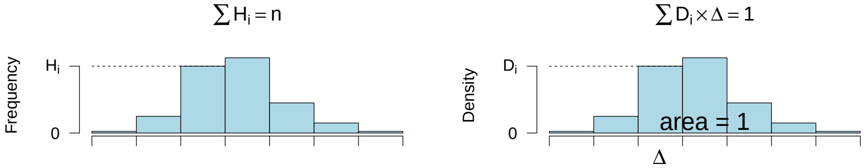

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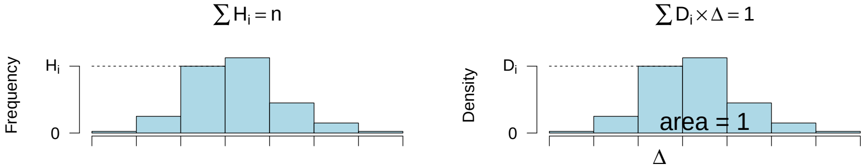

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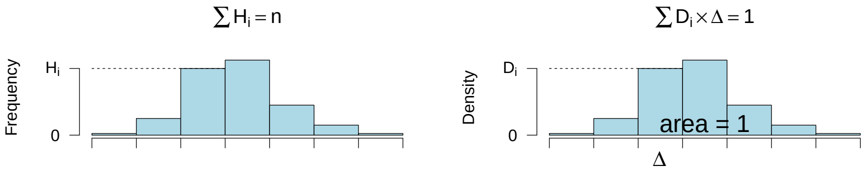

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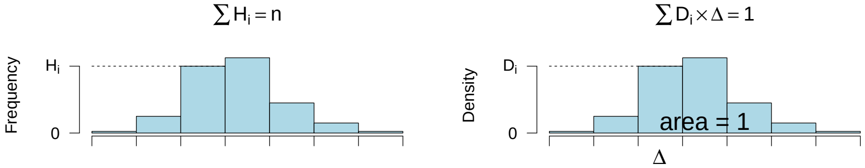
  Search D_i such that total area $\sum D_i \cdot \Delta \stackrel{!}{=} 1$

  $\sum H_i = n \Leftrightarrow 1 = \sum \frac{H_i}{n} = \sum \frac{H_i}{n \cdot \Delta} \cdot \Delta$, hence $D_i = \frac{H_i}{n \cdot \Delta}$

Histogram

What happens when norming?



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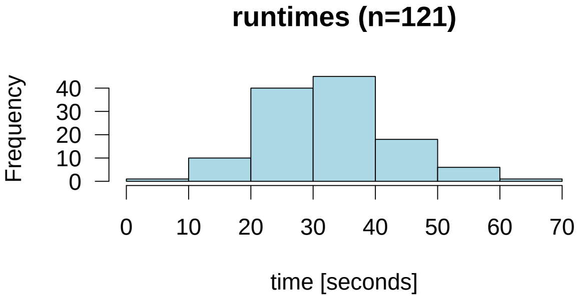
- R normes automatically via `hist(...,prob=TRUE)`

Mean and empirical standard deviation



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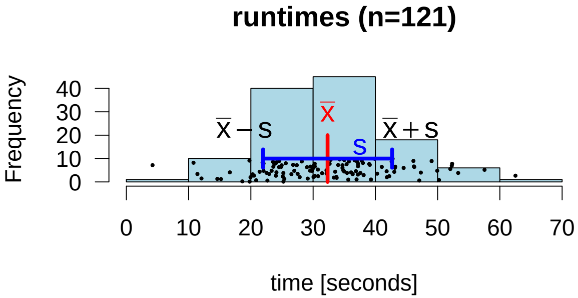

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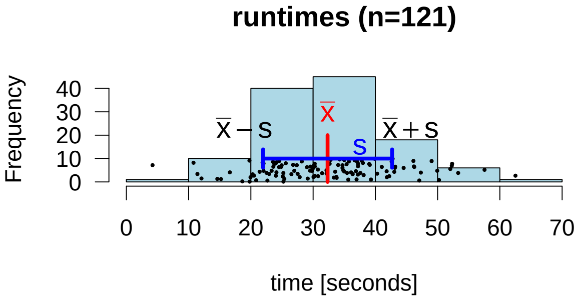

If the data distribute approximately bell-shaped, then they can be summarized nicely by two prominent *statistics*, i.e., functions of the data:

- 1. the mean $\bar{x}$ $\rightarrow$ where? (location)
- 2. the (empirical) standard deviation s $\rightarrow$ how variable? (dispersion)

Mean and empirical standard deviation

Data $x_1, x_2, \ldots, x_n$

Mean and empirical standard deviation

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$$\boxed{\bar{x} := \frac{1}{n}\sum_{i=1}^{n} x_i}$$

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- The (empirical) standard deviation is

$$s = \sqrt{s^2}$$

  ’the square root of the variance’

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Remark:

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Random variable X (here discrete)

$$\mathbb{E}[X] := \sum x \cdot \mathbb{P}(X = x) \qquad \mathbb{V}ar(X) := \mathbb{E}[(X - \mathbb{E}[X])^2] \qquad \sigma_X := \sqrt{\mathbb{V}ar(X)}$$

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- Analogy to the 'universe of randomness': mean $\leftrightarrow$ expectation

Mean and empirical standard deviation

Excursion: Analogy to the 'universe of randomness'. Reminder

Lemma: Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbb{E}[|X_1|^4] < \infty$.
Set $\mu := \mathbb{E}[X_1]$, $\sigma^2 := \mathbb{V}ar(X_1)$ and $\nu^2 := \mathbb{E}[(X_1 - \mu)^4] - \sigma^4$.
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$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \text{and} \qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

it holds

unbiasedness:

$$\mathbb{E}[\bar{X}] = \mu \qquad \mathbb{E}[S^2] = \sigma^2 \quad (\forall n \geqslant 2) \tag{1}$$

Ideas for proofs: (1) linearity of the expectation (correction $n-1$ yields unbiasedness of S^2),

Mean and empirical standard deviation

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(strong) consistency:

$$\bar{X} \xrightarrow{a.s.} \mu \qquad S^2 \xrightarrow{a.s.} \sigma^2 \qquad (n \to \infty) \qquad (2)$$

'$\xrightarrow{a.s.}$' denotes convergence with probability 1, i.e., 'almost surely'. Throughout the course we implicitly consider all random variables to derive from a single probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

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asymptotic normality:

$$\sqrt{n}[\bar{X} - \mu] \xrightarrow{d} N(0, \sigma^2) \qquad \sqrt{n}[S^2 - \sigma^2] \xrightarrow{d} N(0, \nu^2) \qquad (n \to \infty) \tag{3}$$

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'$\xrightarrow{d}$' denotes convergence in distribution

<u>Ideas for proofs:</u> (1) linearity of the expectation (correction $n-1$ yields unbiasedness of S^2), (2) Strong law of large numbers, (3) Central limit theorem / delta method

Notation

<u>Convention:</u>
We use *capital letters* for random variables, e.g.,

$$X_1, X_2, \ldots, X_n \qquad\qquad (\text{'random'})$$

and *lowercase letters* for data or realizations of the random variables

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Outlook:

The main idea of statistical modelling:

Treat data $x_1, x_2, \ldots, x_n$ ('real world')

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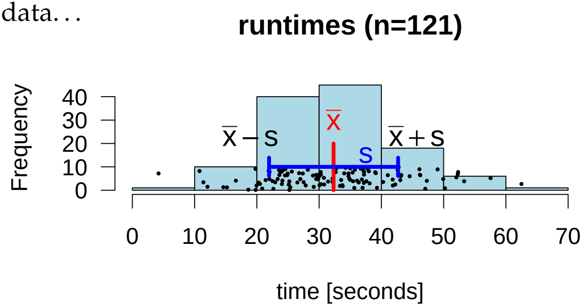
Note that we evaluate *statistics* either on data, e.g., $\bar{x} = (1/n) \sum^n x_i$ ($\rightarrow$ non-random), or on random variables $\overline{X} = (1/n) \sum^n X_i$ ($\rightarrow$ random)

more on modeling in the following sessions

Mean and empirical standard deviation

Back to the data…



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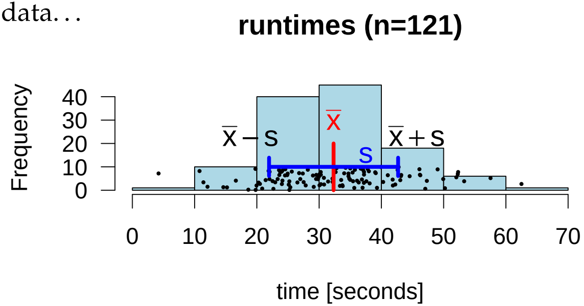

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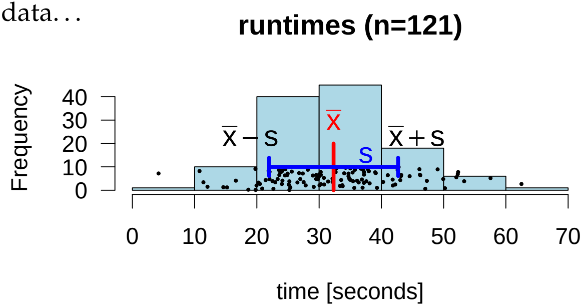

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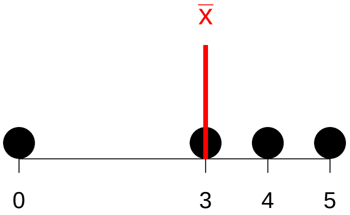
in R via

`mean(x)` `var(x)` `sd(x)`

Mean and empirical standard deviation

Geometrical interpretation of the mean $\bar{x}$



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Mean and empirical standard deviation

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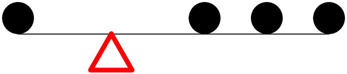

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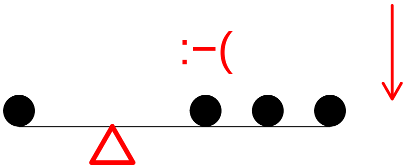

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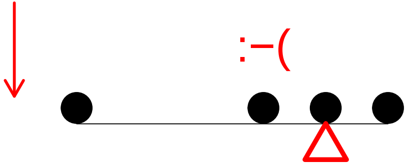

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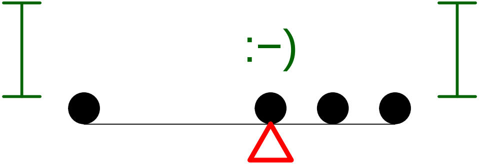

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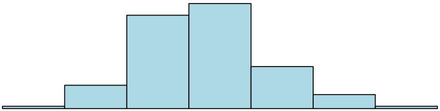

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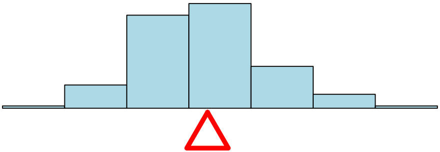

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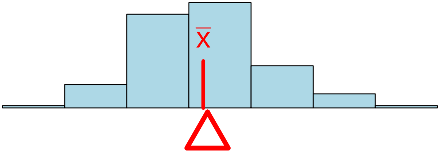

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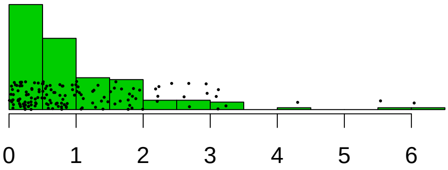

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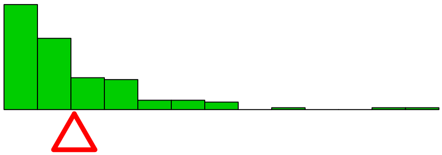

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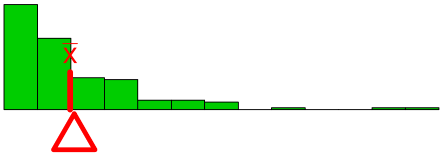

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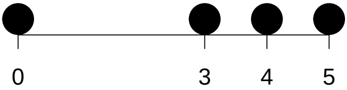

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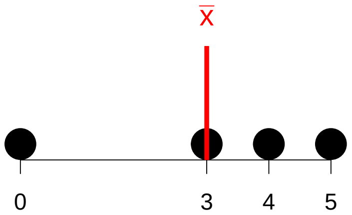

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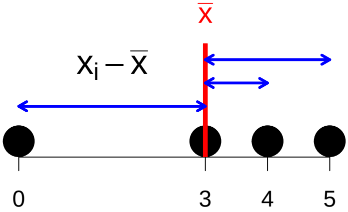

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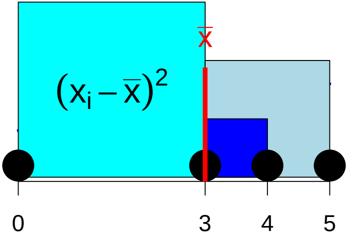

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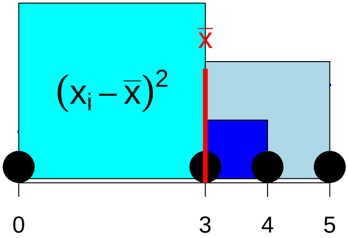

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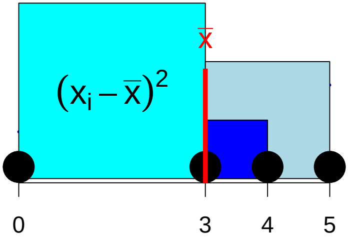

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Mean and empirical standard deviation

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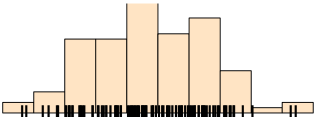

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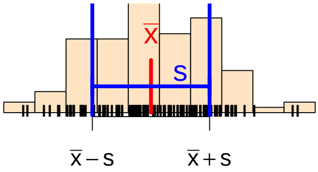

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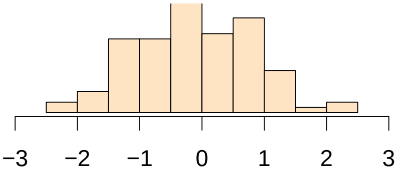

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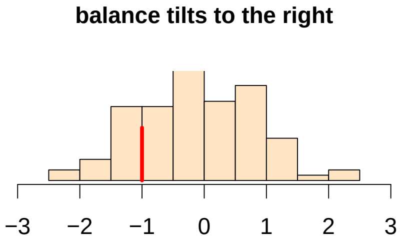

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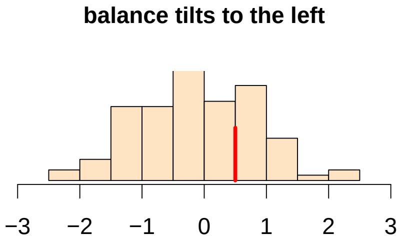

- Fact: About 2/3 of the data lie in the s-neighborhood of $\bar{x}$
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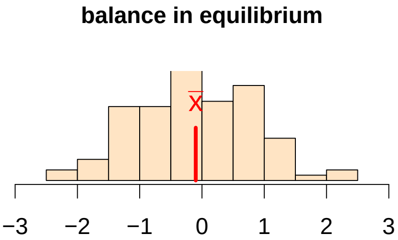

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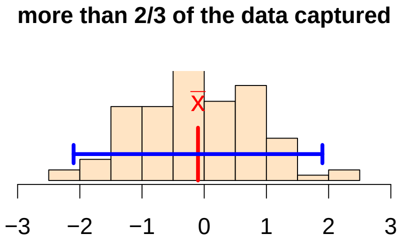

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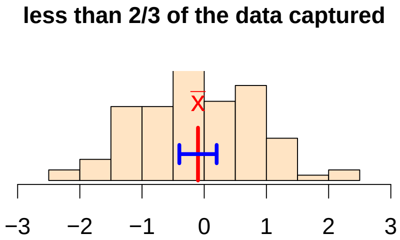

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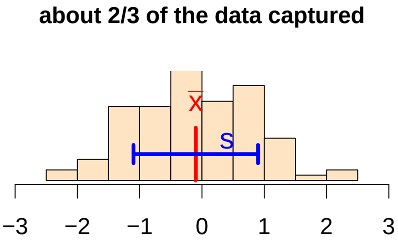

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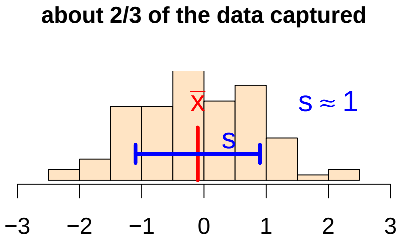

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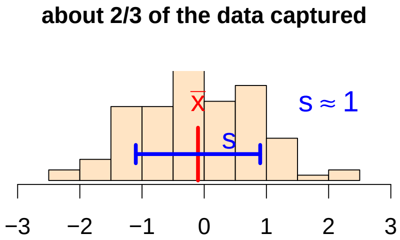

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- Numerically: $\bar{x} \approx -0.1$ and $s \approx 0.94$

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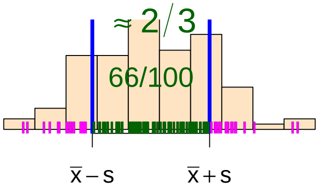

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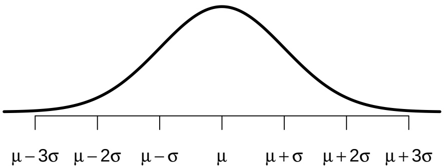

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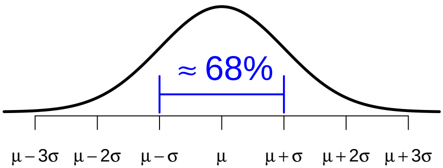

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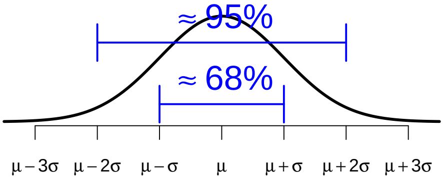

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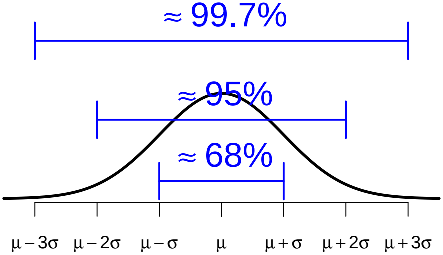

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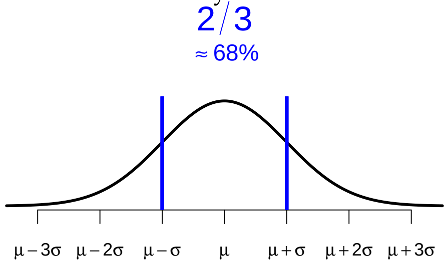

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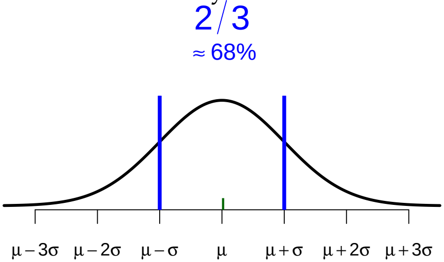

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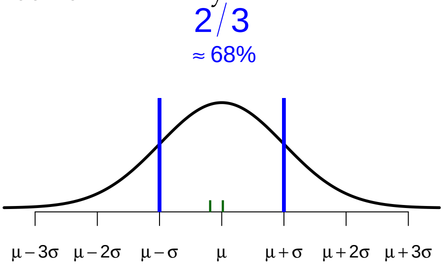

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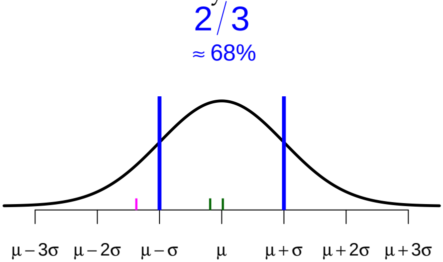

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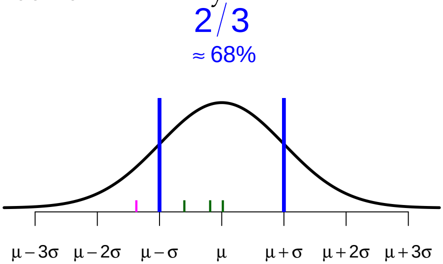

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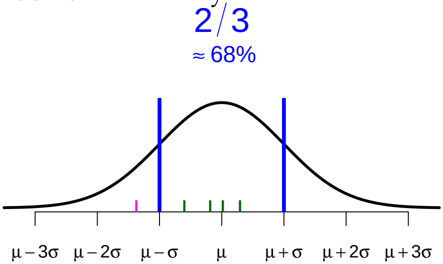

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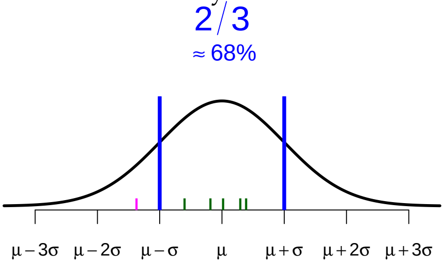

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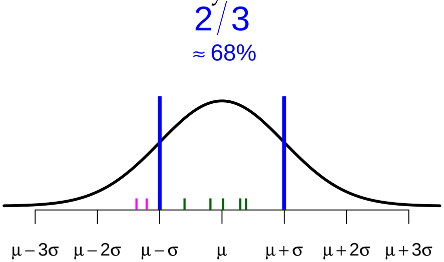

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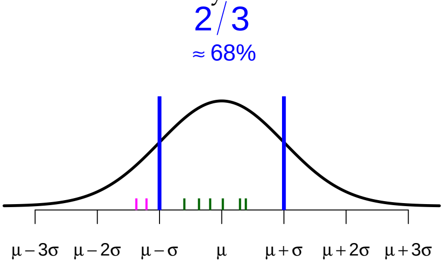

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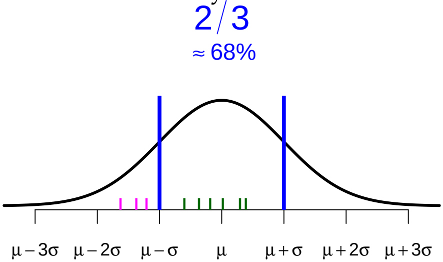

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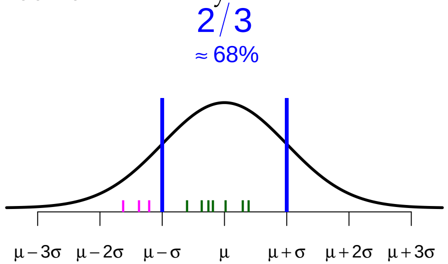

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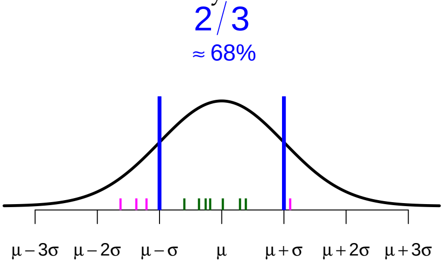

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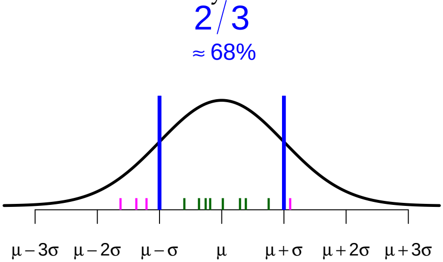

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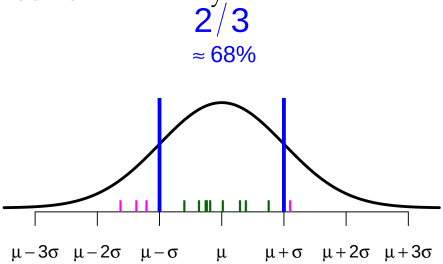

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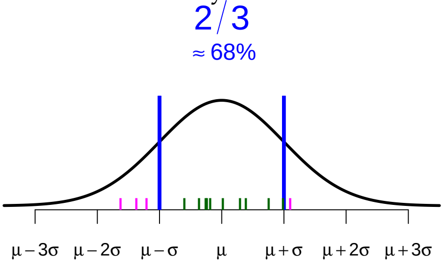

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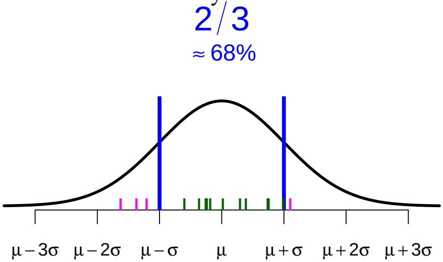

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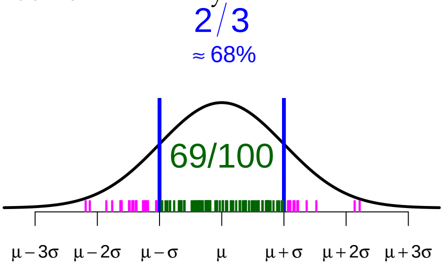

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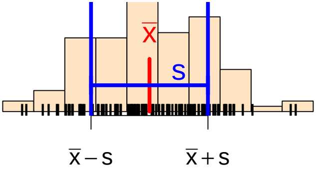

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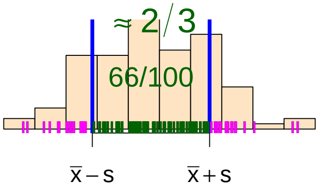

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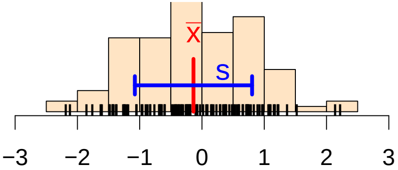
    $\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{[\mu \pm \sigma]}(X_i) \xrightarrow{\text{a.s.}} \mathbb{E}[\mathbb{1}_{[\mu \pm \sigma]}(X_1)] = \mathbb{P}(X_1 \in [\mu \pm \sigma]) \approx 2/3$, as $n \to \infty$
  - $\bar{X}$ and S consistently *estimate* μ and σ ($\rightarrow$ Law of large numbers)
    $\bar{X} \xrightarrow{\text{a.s.}} \mu$ and $S \xrightarrow{\text{a.s.}} \sigma$, as $n \to \infty$
  - Also the *proportion* within $\bar{X} \pm S$ is close to 2/3
    $\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{[\bar{X} \pm S]}(X_i) \xrightarrow{\text{a.s.}} \mathbb{E}[\mathbb{1}_{[\mu \pm \sigma]}(X_1)] \approx 2/3$, as $n \to \infty$ (note that the CDF of X is continuous)

Mean and empirical standard deviation

Interpretation (only for bell-shaped distributions of data)



- $\bar{x}$ is interpreted as a *typical observation*

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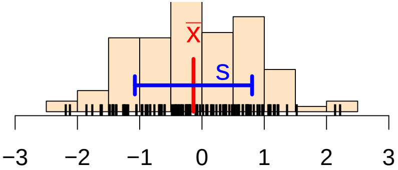

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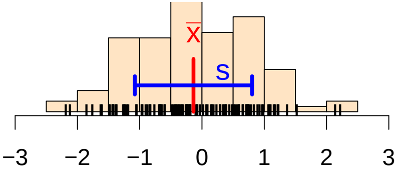
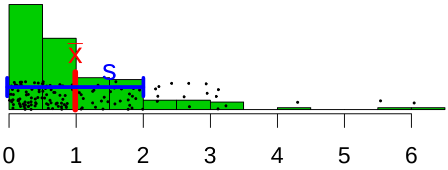


- $\bar{x}$ is interpreted as a *typical observation*
- s is interpreted as the *typical deviation* of an observation (from the mean)
- These two statistics (only two!) suitably summarize the whole set of data (many!)

Mean and empirical standard deviation



- If the data are not distributed approximately bell-shaped, then this interpretation is not useful

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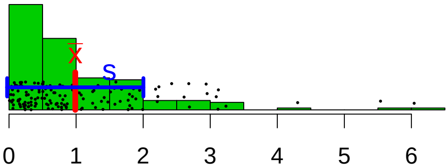

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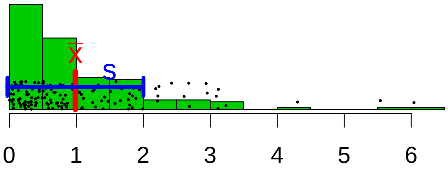

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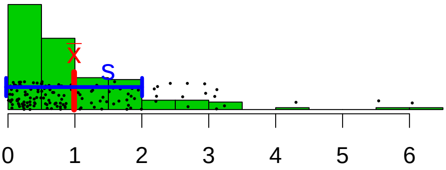

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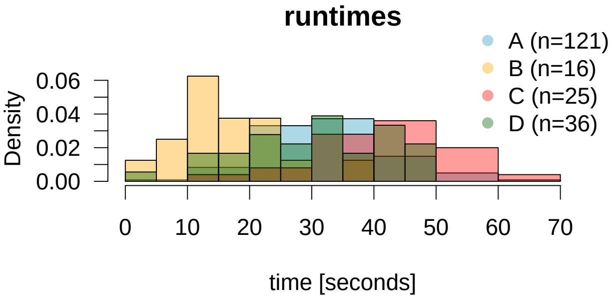
Boxplot

Comparison of four groups A, B, C and D



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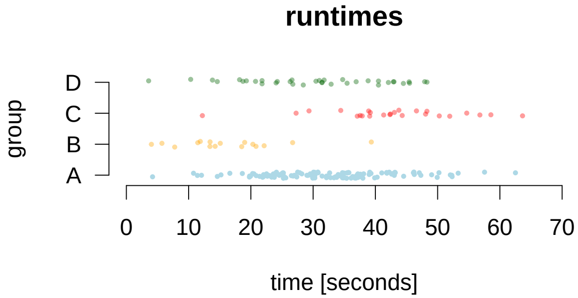

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  Could represent the data in a stripchart

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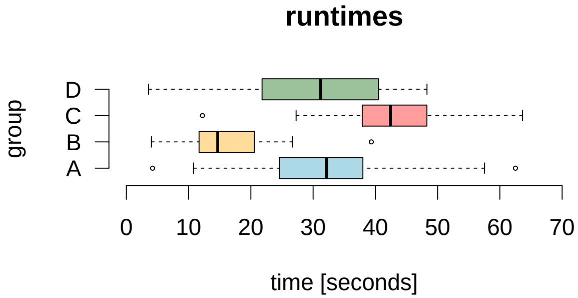
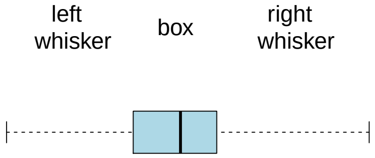


- Histograms overplotted
  Could represent the data in a stripchart
  Other possibility: the *box and whisker plot*, short boxplot

Boxplot

Boxplot



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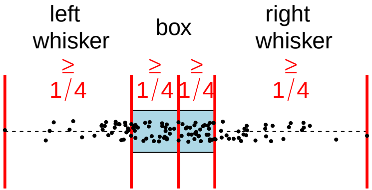
Boxplot



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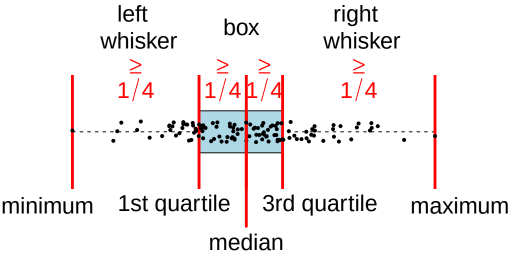
- Four sections, contain at least 1/4 of the data

Boxplot



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  - *Minimum*, smallest observation
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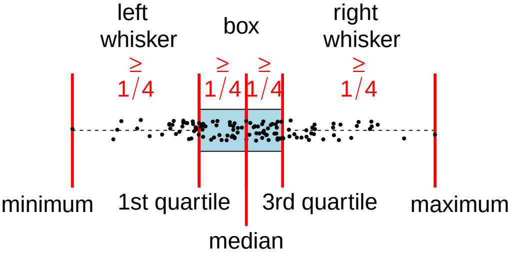
Boxplot



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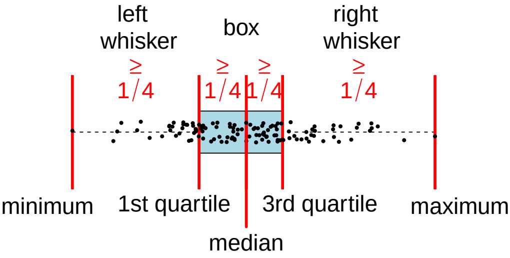

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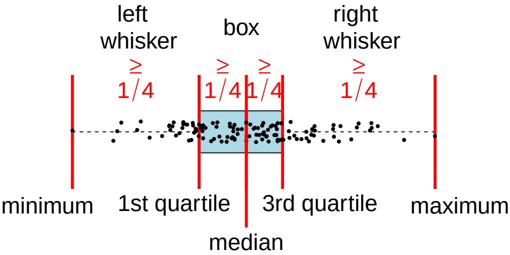
Boxplot



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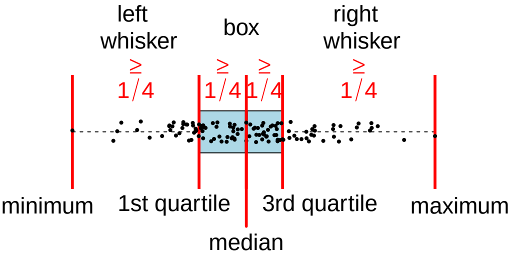

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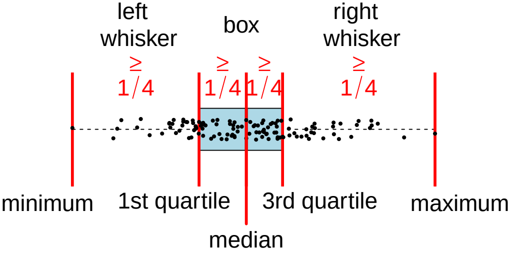

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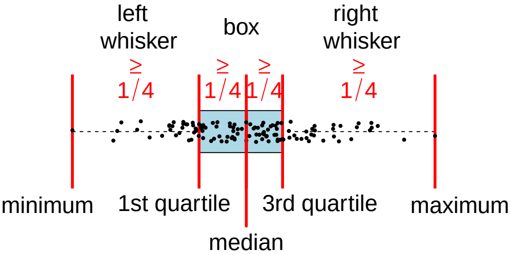
  - Interquartile range $q_{3/4} - q_{1/4}$ (width of the box) is a measure for the dispersion of the data ($\rightarrow$ how variable?)

Boxplot



left whisker
box
right whisker
≥ 1/4
≥ 1/4
≥ 1/4
≥ 1/4
minimum
1st quartile
3rd quartile
maximum
median

Empirical quantile (general)

- Definition: Given n data $x_1, \ldots, x_n$. Let $p \in (0, 1)$. A number $q_p \in \mathbb{R}$ is called an (empirical) $p-quan\underline{n}tile$, if
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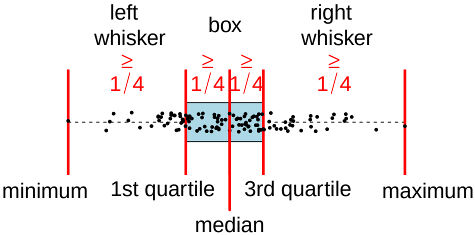

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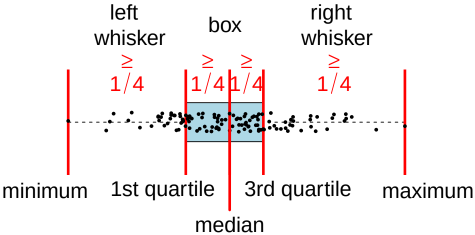

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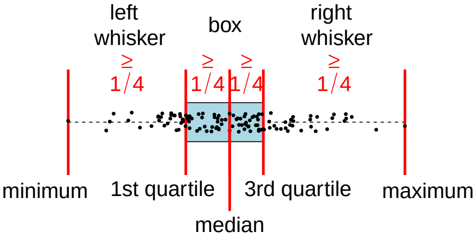

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superscript t denotes the transpose

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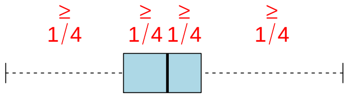
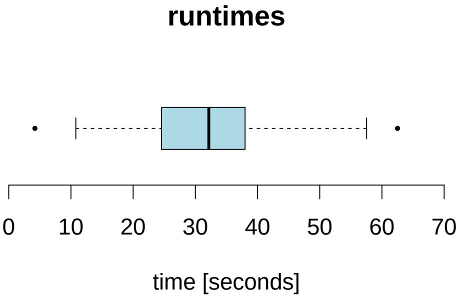
  - Many quantiles equal: The number 2 is a p-quantile for every p of $[0.25, 0.5]$
- Remark.: These kind of 'exotic' messages may support the understanding of the definition of a quantile. The main message however is, that the boxplot appropriately summarizes many data using only five simple statistics

  Take home: Many data $\rightarrow$ at first sight: "1/4, 1/4, 1/4, 1/4"



Boxplot in R

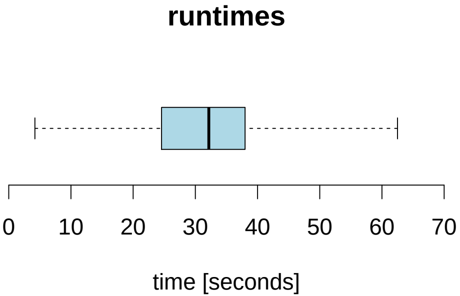
```
#Boxplot, horizontal representation
boxplot(x,horizontal=TRUE,...)
```



Attention: per default a whisker ranges to the observartion which is most far away from the box, but does not exceed 1.5 times the interquartile range. Extreme values ('outliers') are plotted seperately.

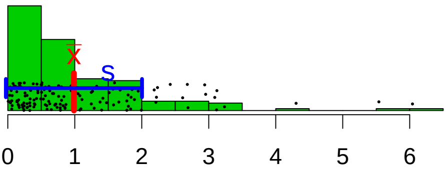
Boxplot in R

```
#Boxplot, Whisker range to extreme values
boxplot(x,horizontal=TRUE,range=0,...)
```



Through the argument `range=0` the whiskers are extended to the extreme values

Boxplot



Reminder

- due to the asymmetric distribution of the data, $\bar{x}$ and s should not be used for the description of the location and the dispersion

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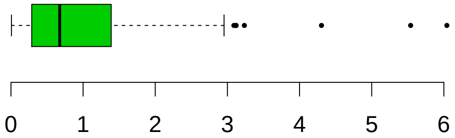


Reminder

- due to the asymmetric distribution of the data, $\bar{x}$ and s should not be used for the description of the location and the dispersion
- The five statistics of the boxplot are more appropriate for the description of the data

Most important message today

3

Most important message today

2

Most important message today

1

Most important message today

Always graphically visualize your data first

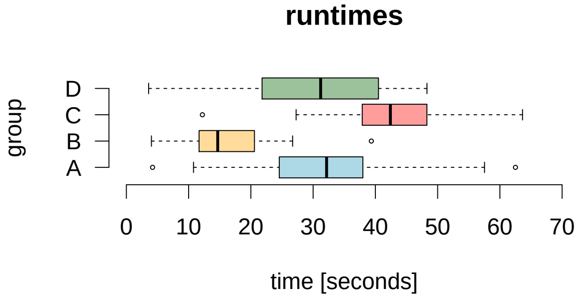
Most important message today

Always graphically visualize your data first

(and start computing afterwards)

Questions

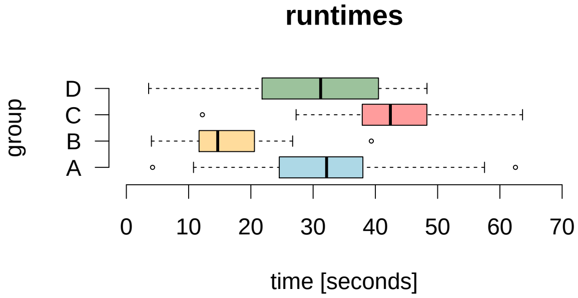
Comparison of four groups A, B, C und D



- The slowest runtime in C was about?

Questions

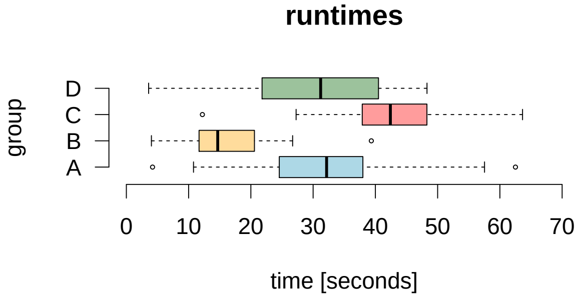
Comparison of four groups A, B, C und D



- The slowest runtime in C was about? 65

Questions

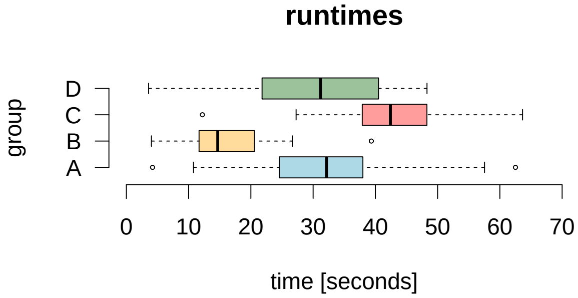
Comparison of four groups A, B, C und D



- The slowest runtime in C was about? 65
- The fastest runtime in A is about?

Questions

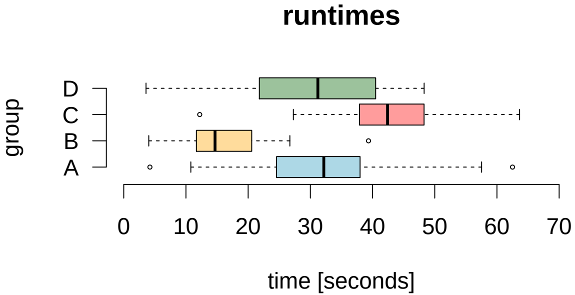
Comparison of four groups A, B, C und D



- The slowest runtime in C was about? 65
- The fastest runtime in A is about? 5

Questions

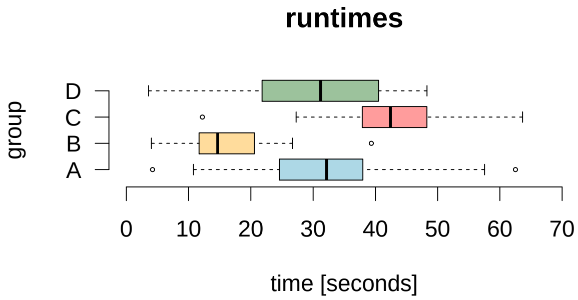
Comparison of four groups A, B, C und D



- The slowest runtime in C was about? 65
- The fastest runtime in A is about? 5
- The median runtime in D is about?

Questions

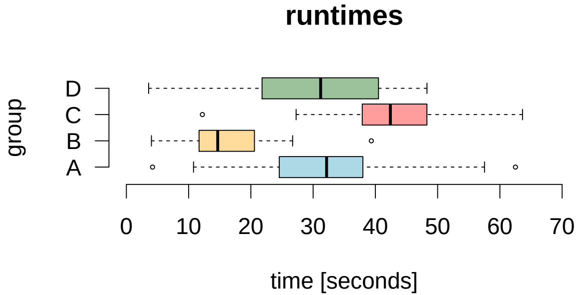
Comparison of four groups A, B, C und D



- The slowest runtime in C was about? 65
- The fastest runtime in A is about? 5
- The median runtime in D is about? 30

Questions

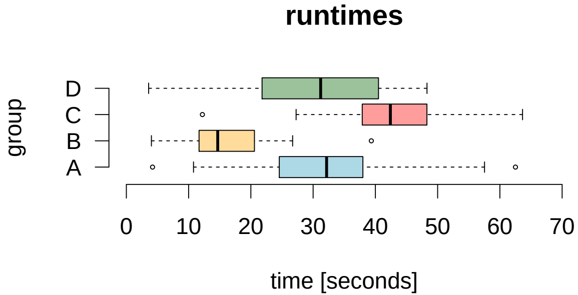
Comparison of four groups A, B, C und D



- The slowest runtime in C was about? 65
- The fastest runtime in A is about? 5
- The median runtime in D is about? 30
- What is the percentage of runtimes in group B that are smaller than 20?

Questions

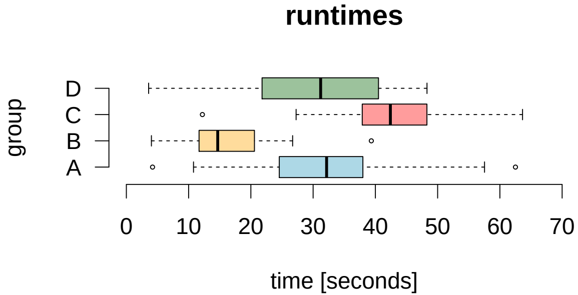
Comparison of four groups A, B, C und D



- The slowest runtime in C was about? 65
- The fastest runtime in A is about? 5
- The median runtime in D is about? 30
- What is the percentage of runtimes in group B that are smaller than 20? about 75%

Questions

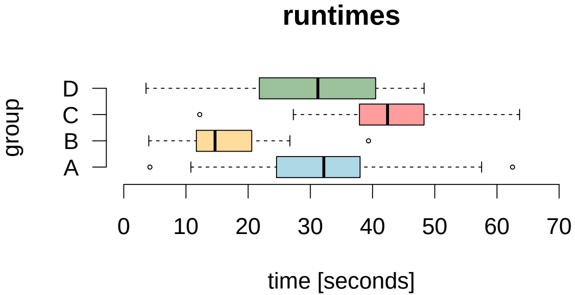
Comparison of four groups A, B, C und D



- The slowest runtime in C was about? 65
- The fastest runtime in A is about? 5
- The median runtime in D is about? 30
- What is the percentage of runtimes in group B that are smaller than 20? about 75%
- Were 50% of the runtimes in A faster than 75% of the times in C?

Questions

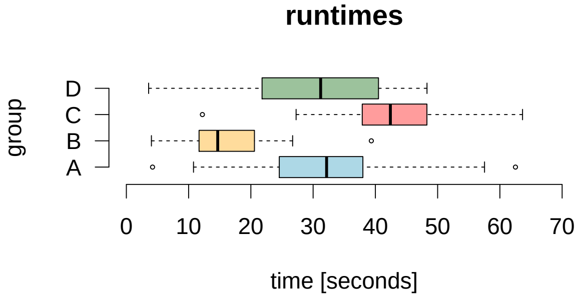
Comparison of four groups A, B, C und D



- The slowest runtime in C was about? 65
- The fastest runtime in A is about? 5
- The median runtime in D is about? 30
- What is the percentage of runtimes in group B that are smaller than 20? about 75%
- Were 50% of the runtimes in A faster than 75% of the times in C? yes

Questions

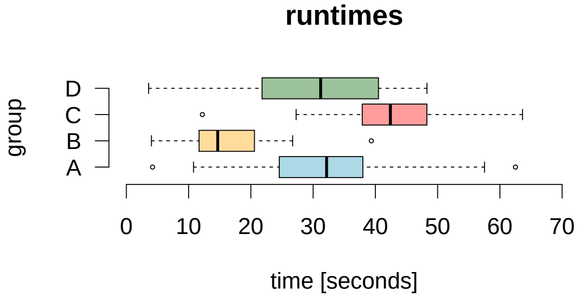
Comparison of four groups A, B, C und D



- The slowest runtime in C was about? 65
- The fastest runtime in A is about? 5
- The median runtime in D is about? 30
- What is the percentage of runtimes in group B that are smaller than 20? about 75%
- Were 50% of the runtimes in A faster than 75% of the times in C? yes
- In group B, apart from a single runtime all others were faster than half of those of group A, half of those of C and half of those of D.

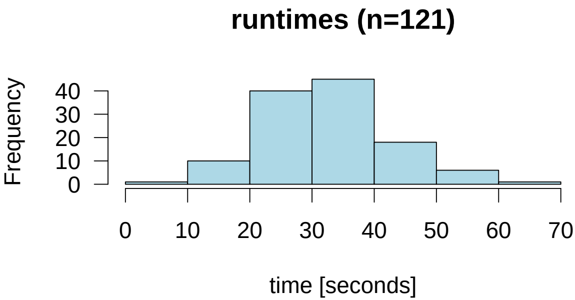
Questions

Comparison of four groups A, B, C und D



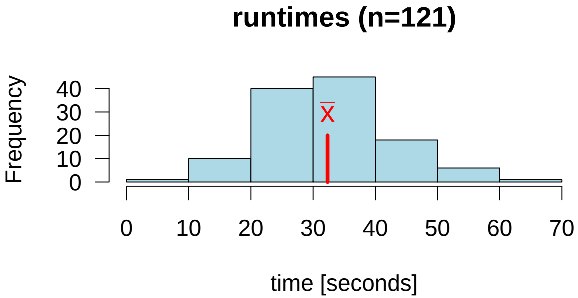
- The slowest runtime in C was about? 65
- The fastest runtime in A is about? 5
- The median runtime in D is about? 30
- What is the percentage of runtimes in group B that are smaller than 20? about 75%
- Were 50% of the runtimes in A faster than 75% of the times in C? yes
- In group B, apart from a single runtime all others were faster than half of those of group A, half of those of C and half of those of D. Correct

Questions



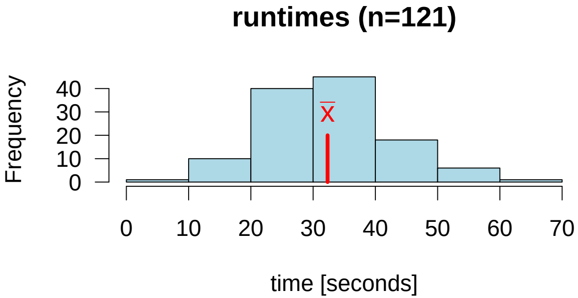
- What is the mean runtime?

Questions



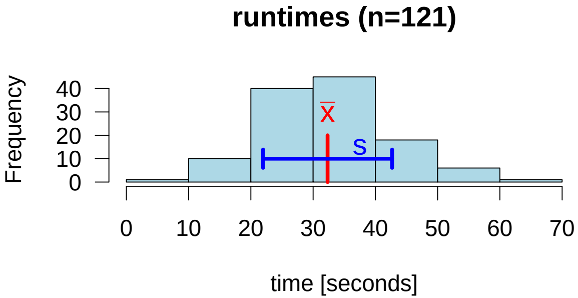
- What is the mean runtime? about 32

Questions



- What is the mean runtime? about 32
- The standard deviation of the runtimes is about?

Questions



- What is the mean runtime? about 32
- The standard deviation of the runtimes is about? 10

Thank you!