Stat 6. UE

① Let $X_1, \dots, X_n$ random sample with pdf $f_\vartheta(x) = \begin{cases} \frac{\vartheta x^{\vartheta-1}}{3^\vartheta}, & 0<x<3 \\ 0, & \text{else} \end{cases}$ where $\vartheta \in \mathbb{R}^+$ unknown.

(a) Show that the MM estimator for ϑ is $T_n = \frac{\overline{X}}{3-\overline{X}}$.

(b) Find the limiting distribution of $\frac{T_n - \vartheta}{\frac{1}{\sqrt{n}}}$ as $n \to \infty$.

(a) $\mu(\vartheta) = \int_{\mathbb{R}} x f_\vartheta(x)\, dx = \frac{1}{3^\vartheta} \int_0^3 \vartheta x^{\vartheta-1} x = \frac{\vartheta}{3^\vartheta} \int_0^3 x^\vartheta dx = \frac{\vartheta}{3^\vartheta}\left(\frac{3^{\vartheta+1}}{\vartheta+1}\right) = \frac{3\vartheta}{\vartheta+1}$

Method of moments: $\overline{X} = \mu(\hat{\vartheta}) = \frac{3\hat{\vartheta}}{\hat{\vartheta}+1} \Rightarrow \hat{\vartheta} = \frac{\overline{X}}{3-\overline{X}}$.

(b) By the CLT, $\sqrt{n}\left(\overline{X} - \frac{3\vartheta}{\vartheta+1}\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{9\vartheta^2}{(\vartheta+1)^2}\right)$.

Define $g(x) := \frac{x}{3-x}$, then we have $g'(x) = \frac{3}{(x-3)^2}$ and therefore, by the Delta method

$\sqrt{n}\left(g(\overline{X}) - g\left(\frac{3\vartheta}{\vartheta+1}\right)\right) = \sqrt{n}\left(\frac{\overline{X}}{3-\overline{X}} - \vartheta\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{9\vartheta^2}{(\vartheta+1)^2} \cdot \left(\frac{3}{(\vartheta-3)^2}\right)^2\right)$.

2. Box: red, blue candles; $\mathbb{P}(\text{blue}) = \frac{1}{1+2a}$, $a > 0$. Find MLE $\hat{a}$ of a.

Let $X \sim Bin\left(1, \frac{1}{1+2a}\right)$, then $L(a|x) = \prod_{i=1}^{n} f(x_i|\vartheta) = \left(\frac{1}{1+2a}\right)^{\#blue} \left(1 - \frac{1}{1+2a}\right)^{\#red}$.

We consider the log likelihood $\ell(a|x) = (\#blue) \cdot \log\left(\frac{1}{1+2a}\right) + (\#red) \cdot \log\left(1 - \frac{1}{1+2a}\right)$.

Taking the derivative, we obtain $\ell'(a|x) = -\frac{2\#blue}{1+2a} + \frac{\#red}{2a^2+a} = \frac{-2\#blue\, a + \#red}{2a^2+a}$

so $-2(\#blue)\hat{a} + \#red = 0 \Rightarrow \hat{a} = \frac{\#red}{2\#blue}$.

Second derivative: $\ell''(\hat{a}|x) = \frac{4\hat{a}^2\#blue - 4\hat{a}\#red - \#red}{\hat{a}^2(1+2\hat{a})^2} = -\frac{4(\#blue)^3}{\#red\,(\#blue + \#red)} < 0 \Rightarrow \hat{a}$ ... MLE.

③ Let $\vartheta > 0$ unknown, $X_1, \dots, X_n \sim \mathcal{U}(0, \vartheta)$ i.i.d. $\quad \left[f_\vartheta(x) = \mathbb{1}_{[0 \leq x \leq \vartheta]} \frac{1}{\vartheta} \right]$

(a) Find MME and MLE of ϑ.

<u>MM:</u> $\mu(\vartheta) = \int_0^\vartheta x \frac{1}{\vartheta} \, dx = \frac{\vartheta}{2}$, i.e. $\overline{X} = \frac{\hat{\vartheta}}{2} \Rightarrow \hat{\vartheta} = 2\overline{X}$.

<u>MLE:</u> $\underset{(x_1, \dots, x_n)}{L(\vartheta | \overset{\shortparallel}{x})} = \prod_{i=1}^n f(x_i | \vartheta) = \prod_{i=1}^n \frac{1}{\vartheta} \mathbb{1}_{[0 \leq x_i \leq \vartheta]} = \left(\frac{1}{\vartheta}\right)^n \mathbb{1}_{[\max_{i \in [n]} x_i \leq \vartheta]} = \begin{cases} \left(\frac{1}{\vartheta}\right)^n, & \vartheta \geq \max x_i \\ 0, & \text{else.} \end{cases}$

Since $\left(\frac{1}{\vartheta}\right)^n$ is decreasing for $\vartheta > 0$, $\max_{\vartheta > 0} L(\vartheta | x) = \max_{i \in [n]} x_i = X_{(n)} = \hat{\vartheta}$.

(b) Compare the mean square errors of the two estimators: $\quad \left[MSE(\hat{\vartheta}) = \mathbb{E}_\vartheta \left((\hat{\vartheta} - \vartheta)^2 \right) \right]$

- $\mathbb{E}_\vartheta \left((2\overline{X} - \vartheta)^2 \right) = \mathbb{E}_\vartheta \left(4 \left(\overline{X} - \frac{\vartheta}{2} \right)^2 \right) = 4 \, \mathbb{E}_\vartheta \left((\overline{X} - \mathbb{E}_\vartheta(\overline{X}))^2 \right) = 4 \, \mathbb{V}_\vartheta(\overline{X}) = \frac{4}{n} \mathbb{V}(X_1) = \frac{4}{n} \frac{1}{12} \vartheta^2$

- $\mathrm{MSE}(T) = \mathbb{V}(T) - (\mathbb{E}(T) - \vartheta)^2$.

$\mathbb{P}(\max x_i \le x) = \prod \mathbb{P}(x_i \le x) = \left(\frac{x}{\vartheta}\right)^n \mathbb{1}_{[0 \le x \le \vartheta]}$

$\mathbb{E}(\max x_i) = \int_0^\vartheta x n \left(\frac{x}{\vartheta}\right)^{n-1} \frac{1}{\vartheta}\, dx = \frac{\vartheta n}{n+1}$

$\mathbb{P}((\max x_i)^2 \le x) = \mathbb{P}(-\sqrt{x} \le \max x_i \le \sqrt{x}) = \left(\frac{\sqrt{x}}{\vartheta}\right)^n$ for $x > 0$ and $\sqrt{x} < \vartheta \Leftrightarrow x < \vartheta^2$.

$f(x) := \frac{d}{dx}\left(\frac{\sqrt{x}}{\vartheta}\right)^n = n\left(\frac{\sqrt{x}}{\vartheta}\right)^{n-1} \frac{1}{2\vartheta\sqrt{x}}$

$\Rightarrow \mathbb{E}((\max x_i)^2) = \int_0^{\vartheta^2} x f(x)\, dx \overset{\text{MAPLE}}{=} \vartheta^2 \frac{n}{n+2}$

$\Rightarrow \mathbb{V}(\max x_i) = \vartheta^2 \frac{n}{n+2} - \left(\frac{\vartheta n}{n+1}\right)^2 = \frac{\vartheta^2 n}{(n+2)(n+1)^2} = ⊛$

$\Rightarrow \mathrm{MSE} = ⊛ - \left(\frac{\vartheta n}{n+1} - \vartheta\right)^2 = -2\vartheta^2 \left(\frac{1}{(n+1)^2 (n+2)}\right) = O(n^{-3})$, so the second one is preferable.

(4.) Let a, b unknown and $\hat{a}, \hat{b}$ unbiased estimators. $[T \text{ unbiased} :\Leftrightarrow \mathbb{E}_\vartheta(T) = \vartheta]$

(a) $\alpha, \beta \in \mathbb{R} \overset{?}{\Rightarrow} \alpha\hat{a} + \beta\hat{b}$ unbiased estimator of $\alpha a + \beta b$. $\mathbb{E}_a(\hat{a}) = a$, $\mathbb{E}_b(\hat{b}) = b$.

Yes: $\mathbb{E}(\alpha\hat{a} + \beta\hat{b}) = \alpha\mathbb{E}(\hat{a}) + \beta\mathbb{E}(\hat{b}) = \alpha\hat{a} + \beta\hat{b}$ by assumption and linearity.

(b) $\hat{a}^2$ unbiased estimator of a^2?

No: $\mathbb{E}(\hat{a}^2) = \mathbb{V}(\hat{a}) + \mathbb{E}(\hat{a})^2 = \mathbb{V}(\hat{a}) + a^2$, so we would need $\mathbb{V}(\hat{a}^2) = 0$, which will not be the case in general.

(c) Measurements of side of square: 15, 17, 16, 16, 17, 14. Find unb. est. of the area.

We define $\hat{A} := \frac{1}{\binom{2}{n} - n} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} X_i X_j$ and get $\mathbb{E}(\hat{A}) = \frac{1}{\binom{2}{n} - n}\left(\binom{2}{n} - n\right) \mathbb{E}(X_1)\mathbb{E}(X_2) = \mu^2$.

With the concrete data, we estimate A to be $\frac{7514}{30} = 250,4\dot{7}$.

(5.) $f(x|\vartheta) = \frac{x}{\vartheta^2} e^{-\frac{x^2}{2\vartheta^2}} \mathbb{1}_{[x \geq 0]}$. ϑ unknown, $X_1, \dots, X_n$ random sample.

(a) • $\mathbb{E}(X) = \int_0^\infty \frac{x^2}{\vartheta^2} \exp(-x^2/2\vartheta^2)\, dx = \left[u = \frac{x}{\vartheta}\right] \vartheta \int_0^\infty u^2 e^{-\frac{u^2}{2}} du$

$= 2\vartheta \int_{\infty}^{\infty} u^2 e^{-\frac{u^2}{2}} du = 2\vartheta\sqrt{2\pi} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} u^2 e^{-\frac{u^2}{2}} du}_{= \mathbb{E}(X^2) \text{ for } X \text{ standard normal}} = \underline{\vartheta\sqrt{\pi/2}}.$

• $\mathbb{E}(X^2) = \int_0^\infty \frac{x^3}{\vartheta^2} \exp(-x^2/2\vartheta^2)\, dx = \left[u = \frac{x}{\vartheta^2}\right] \vartheta^2 \int_0^\infty u^3 e^{-\frac{u^2}{2}} du$

$= \vartheta^2 \left[-(u^2+2) e^{-\frac{u^2}{2}}\right]\Big|_0^\infty = \underline{2\vartheta^2}.$

• MME: $\overline{X} = \mu(\hat{\vartheta}) = \hat{\vartheta}\sqrt{\frac{\pi}{2}} \Rightarrow \hat{\vartheta} = \overline{X} \cdot \sqrt{\frac{2}{\pi}}.$

- MLE: We work with the log-likelihood function:

$$\ell(\vartheta \mid x) = \log L(\vartheta \mid x) = \sum_{i=1}^{n} \log f(x_i \mid \vartheta) = \sum_{i=1}^{n} \left(\log x_i - \log \vartheta^2 - \frac{x_i^2}{2\vartheta^2} \right)$$

$$= \sum_{i=1}^{n} \log(x_i) - (2 \log \vartheta) n - \frac{1}{2\vartheta^2} \sum_{i=1}^{n} x_i^2$$

$$\rightsquigarrow -2n \log \vartheta - \frac{1}{2\vartheta^2} \sum_{i=1}^{n} x_i^2 =: \tilde{\ell}(\vartheta \mid x)$$

$$\tilde{\ell}(\vartheta \mid x)' = -2n \frac{1}{\vartheta} + \frac{1}{\vartheta^3} \sum_{i=1}^{n} x_i^2 \overset{!}{=} 0 \Leftrightarrow 2n \hat{\vartheta}^2 = \sum_{i=1}^{n} x_i^2 \Leftrightarrow \hat{\vartheta} = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} x_i^2}.$$

$$\tilde{\ell}(\vartheta \mid x)'' = 2n \frac{1}{\vartheta^2} - \frac{3}{\vartheta^4} |x|^2 \Rightarrow \tilde{\ell}(\hat{\vartheta} \mid x)'' = -\frac{8n^2}{|x|^2} < 0$$

Def. γ^2 is called „as. variance" of $\hat{\vartheta}$ $:\Leftrightarrow \sqrt{n}(\hat{\vartheta} - \vartheta) \xrightarrow{d} \mathcal{N}(0, \gamma^2)$.

Define $g(x) := \sqrt{\frac{1}{2} \cdot x}$, then $\hat{\vartheta} = g\Big(\underbrace{\frac{1}{n} \sum_{i=1}^{n} x_i^2}_{=: \overline{X^2}}\Big)$, then $g'(2\vartheta^2) = \frac{1}{4\vartheta}$.

By the CLT, $\sqrt{n}\left(\overline{X^2} - \mathbb{E}(X^2)\right) = \sqrt{n}\left(\overline{X^2} - 2\vartheta^2\right) \xrightarrow{d} \mathcal{N}(0, \mathbb{V}(X^2))$

and therefore, by the Delta method,

$$\sqrt{n}\left(g(\overline{X^2}) - g(2\vartheta^2)\right) = \sqrt{n}(\hat{\vartheta} - \vartheta) \xrightarrow{d} \mathcal{N}\Big(0, \underbrace{\mathbb{V}(X^2) \frac{1}{16\vartheta^2}}_{=: \gamma^2}\Big).$$