(1) **Exponential family**

Show that the one-parameter exponential family has a monotone likelihood ratio in a sufficient statistic $T(\mathbf{X})$ if the natural parameter $w(\theta)$ is a non-decreasing function in θ.

The pdf is $f_\theta(x) = h(x)\, c(\theta)\, e^{w(\theta) t(x)}$, where $h, c \geq 0$, hence, the likelihood is

$L(\theta; x) = (c(\theta))^n \exp\left(w(\theta) \sum_{i=1}^n t(x_i) \right) \prod_{i=1}^n h(x_i)$. We obtain the likelihood ratio

$$\lambda(x) = \frac{L(\theta_1; x)}{L(\theta_0; x)} = \left(\frac{c(\theta_1)}{c(\theta_0)} \right)^n \exp\left((w(\theta_1) - w(\theta_0)) \sum_{i=1}^n t(x_i) \right).$$

The statistic $T(X) = \sum_{i=1}^n t(X_i)$ is sufficient by lecture 8 slide 48. Since w is non-decreasing, we have $w(\theta_1) - w(\theta_0) \geq 0$, because $\theta_1 > \theta_0$. Therefore, λ is a non-decreasing function of $T(x)$.