A 9.3.A Gg.: $\zeta \in \mathrm{Aut}\, \mathbb{R}$

(a) Zz.: $\zeta|_{\mathbb{Q}} = \mathrm{id}_{\mathbb{Q}}$

Sei $\xi := \zeta|_{\mathbb{Q}}$, dann $\xi(0) = 0$, $\xi(\pm 1) = \pm 1$.

Sei $q \in \mathbb{Q}$, dann $\exists m \in \mathbb{Z}\ \exists n \in \mathbb{N}^{\times} : q = mn^{-1}$

$\xi(q) = \xi(m)\xi(n)^{-1} = mn^{-1} = q$.

(b) Zz.: $\forall x, y \in \mathbb{R} : x < y \Rightarrow \zeta(x) < \zeta(y)$

Ww.: $\exists z \in \mathbb{R}^{\times} : x + z^2 = y$

$\Rightarrow \zeta(y) = \zeta(x) + \zeta(z)^2 > \zeta(x)$

Ges.: Alle Elemente von Aut $\mathbb{R}$

Beh.: $\mathrm{Aut}\, \mathbb{R} = \{\mathrm{id}_{\mathbb{R}}\}$

Bew.: oBdA. sei $x \in \mathbb{R} \setminus \mathbb{Q}$, dann gilt

$\forall t \in \mathbb{Q}^{+}\ \exists y_t \in \mathbb{Q} : y_t < x < y_t + t$

$\overset{(b)}{\Rightarrow} y_t < \zeta(x) < y_t + t$

$\Rightarrow \zeta(x) = x$,

für $\zeta \in \mathrm{Aut}\, \mathbb{R}$.

A 9.3.1 Gg.: K Körper, $\forall t \in K$:

$M(t) := \begin{pmatrix} 0 & -1 \\ t & 0 \end{pmatrix} \in K^{2\times 2} \Rightarrow \square := M(1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$

(a) Zz.: $\varphi : \begin{cases} K^{2\times 2} \to K^{2\times 2} \\ X \mapsto \square \cdot X^T \end{cases}$ bij.

$$\square \cdot X^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{pmatrix} = \begin{pmatrix} -x_{12} & -x_{22} \\ x_{11} & x_{21} \end{pmatrix} = \mathcal{O}_{K^{2\times 2}}$$

$\Rightarrow X = \mathcal{O}_{K^{2\times 2}}$

$\Rightarrow$ ker φ trivial

$\Rightarrow$ φ inj. $\Rightarrow$ φ surj. $\Rightarrow$ φ bij. weil $\dim K^{2\times 2} = 4 < \infty$.

(b) Gg.: $A, B \in K^{2\times 2}$

Zz.: A, B Kongruent $\Leftrightarrow$
$\exists P \in GL_2(K)$: $\square \cdot B^T = P^{\#} \cdot (\square \cdot A^T) \cdot P$

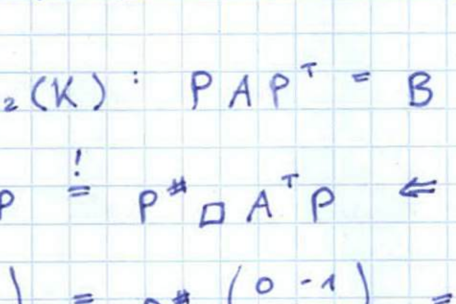
„$\Rightarrow$": Sei $P \in GL_2(K)$: $PAP^T = B \Leftrightarrow P^T A^T P = B^T$.

$$\square B^T = \square P^T A^T P \overset{!}{=} P^{\#} \square A^T P \Leftarrow$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{pmatrix}}_{P^T} = \underbrace{P^{\#}}_{} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -p_{12} & -p_{22} \\ p_{11} & p_{21} \end{pmatrix}$$

$$\begin{pmatrix} p_{11}^{\#} & p_{12}^{\#} \\ p_{21}^{\#} & p_{22}^{\#} \end{pmatrix} = \begin{pmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{pmatrix},$$

weil $p_{ek}^{\#} = (-1)^{k+e} \underbrace{\det P_{ke}^{-}}_{}$.



„⇔": $\square B^T = P^{\#} \square A^T P$

$$\Leftrightarrow \quad B^T = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{P^T} A^T P$$

$$\Leftrightarrow \quad B^T = PAP^T$$

(c) Ges.: $t_1, t_2 \in K$: $M(t_1), M(t_2)$ Kongruent

$$\Leftrightarrow \quad \square M(t_1)^T = P^{\#} \square M(t_2)^T P$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & t_1 \end{pmatrix} = P^{-1} \begin{pmatrix} \det P & 0 \\ 0 & \det P \, t_2 \end{pmatrix} P$$

$$\Leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & t_1 \end{pmatrix} \overset{P}{\approx} \begin{pmatrix} \det P & 0 \\ 0 & \det P \, t_2 \end{pmatrix}$$

Beide Matrizen sind in JNF. Damit sie ähnlich sind, müssen sie (bis auf Reihenfolge) gleich sein.

$\Rightarrow$ $\det P = 1$, $t_1 = t_2$ oder $\det P = t_1$, $t_2 = \frac{1}{\det P}$

$\Rightarrow$ $M(t_1)$, $M(t_2)$ müssen gleich oder invers zu einander sein.

A 9.3.2 Gg.: K Körper, $x, y \in K$ quadratisch äquivalent $:\Leftrightarrow$ $\exists c \in K^{\times}: y = c^2 x$, Äquivalenzklassen „Quadratklassen“ von K;

(a) Zz.: $\forall A, B \in K^{n \times n}$ kongruent: $\det A \sim \det B$

Ww.: $\exists P \in GL_n(K)$:

$$\det A = \det(PBP^T)$$
$$= \underbrace{\det P}_{\neq 0} \cdot \det B \cdot \underbrace{\det P^T}_{\det P} = (\det P)^2 \cdot \det B.$$

(b) Geg.: $A := \operatorname{diag}(\overline{1}, \overline{1}) \in \mathbb{Z}_3^{2\times 2}$

Ges.: $B, C := \operatorname{diag}(\overline{1}, \overline{2}), \operatorname{diag}(\overline{2}, \overline{2}) \in \mathbb{Z}_3^{2\times 2}$ kongruent zu A

$\det A = 1$

$\det B = 2 \nsim 1$

$\det C = 2 \cdot 2 = 4 = 1$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \overset{s}{\mapsto} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \overset{z}{\mapsto} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$
$$\overset{s}{\mapsto} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \overset{z}{\mapsto} \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = C.$$

A 9.4.2 Gg.: $\mathrm{Bil}(V) := L(V \times V, K)$,

$\mathrm{Sym}(V) := \{\sigma \in \mathrm{Bil}(V) : \forall x,y \in V : \sigma(x,y) = \sigma(y,x)\}$

$\mathrm{Alt}(V) := \{\sigma \in \mathrm{Bil}(V) : \forall x \in V : \sigma(x,x) = 0_K\}$

(a) Zz.: $\mathrm{Sym}(V), \mathrm{Alt}(V) \leq \mathrm{Bil}(V)$

(i) $\mathcal{O}_{\mathrm{Bil}(V)} \in \mathrm{Sym}(V), \mathrm{Alt}(V)$

(ii) Seien $\sigma_1, \sigma_2 \in \mathrm{Sym}(V)$, $c \in K$, dann gilt

$$\forall x,y \in V : (\sigma_1 + c\sigma_2)(x,y) = \\ \sigma_1(x,y) + c\sigma_2(x,y) = \\ \sigma_1(y,x) + c\sigma_2(y,x) = \\ (\sigma_1 + c\sigma_2)(y,x).$$

Seien nun $\sigma_1, \sigma_2 \in \mathrm{Alt}(V)$, dann muss

$$\forall x \in V : (\sigma_1 + c\sigma_2)(x,x) = \\ \sigma_1(x,x) + c\sigma_2(x,x) = 0.$$

(6) Gg.: $\mathrm{Char}\, K \neq 2$

Zz.: $\mathrm{Bil}(V) = \mathrm{Sym}(V) \oplus \mathrm{Alt}(V)$

„$\cap$“ $\forall x,y \in K : \underbrace{\sigma(x+y, x+y)}_{0_K} = \sigma(\cancel{x,x}) + \sigma(x,y) + \sigma(y,x) + \sigma(\cancel{y,y})$

$$\overset{\mathrm{Char}\,K \neq 2}{\Rightarrow} \quad \sigma(x,y) = 0_K \qquad = 2\sigma(x,y)$$

„$+$“ Weil $\mathrm{Char}\, K \neq 2$, definieren wir $\forall x,y \in V$:

$\sigma_1(x,y) := \sigma(y,x) + [\sigma(x,y) - \sigma(y,x)]/2$,

$\sigma_2(x,y) := [(\sigma(x,y) - \sigma(y,x)]/2$.

- $\Rightarrow \quad d = d_1 + d_2$

- $$\begin{aligned}\Rightarrow \quad d_1(x,y) - d_1(y,x) &= d(y,x) + [d(x,y) - d(y,x)]/2 \\ &\quad - d(x,y) - [d(y,x) - d(x,y)]/2 \\ &= 0\end{aligned}$$

  $\Rightarrow \quad d_1(x,y) = d_1(y,x)$

- $$\begin{aligned}\Rightarrow \quad d_2(x,y) &= [d(x,y) - d(y,x)]/2 \\ &= -[d(y,x) - d(x,y)]/2 \\ &= -d(y,x)\end{aligned}$$

(c) Gg.: $\mathrm{Char}\, K \neq 2$

Zz.: $\forall n \in \mathbb{N} \ \forall A \in K^{n \times n}: \ \exists!\, B \in \mathrm{Sym}_n(K) \ \exists!\, C \in \mathrm{Asym}(K):$
$A = B + C$

trivial

A 9.4.5 Gg.: $\sigma \in L_\omega(V \times V, K)$, $(g_{ij})_{i,j\in I} \in K^{I\times I}$ legt σ fest;

(a) Zz.: σ ω-symm. $\Leftrightarrow \forall i,j \in I: g_{ij} = \omega(g_{ji})$

Sei $B = (b_i)_{i\in I}$, sodass

$\forall i,j \in I: \sigma(b_i, b_j) = g_{ij}$

„$\Rightarrow$": $g_{ij} = \sigma(b_i, b_j) = \omega(\sigma(b_j, b_i)) = \omega(g_{ji})$

„$\Leftarrow$": Seien $x, y \in V$, dann $\exists! (x_i)_{i\in I}, (y_i)_{i\in I} \in K^I$:

$$\sigma(x,y) = \sigma\Big(\sum_{i\in I} x_i b_i, \sum_{j\in I} y_j b_j\Big) = \sum_{i,j\in I} x_i y_j \,\sigma(b_i, b_j) =$$

$$\sum_{i,j\in I} y_j x_i \,\omega(\sigma(b_j, b_i)) = \omega\Big(\sum_{i,j\in I} y_j x_i \,\sigma(b_j, b_i)\Big) =$$

$$\omega\Big(\sigma\Big(\sum_{j\in I} y_j b_j, \sum_{i\in I} x_i b_i\Big)\Big) = \omega(\sigma(y,x)).$$

(b) Zz.: σ ω-schiefsymm. $\Leftrightarrow \forall i,j \in I: g_{ij} = -\omega(g_{ji})$

Analog zu (a)

(c) Gg.: $\omega = \mathrm{id}_K$ d.h. σ bilinear

Zz.: σ alternierend $\Leftrightarrow \forall i,j \in I: g_{ij} = -g_{ji}, g_{ii} = 0_K$

„$\Rightarrow$": $0_K = \sigma(b_i + b_j, b_i + b_j)$

$$= \cancel{\sigma(b_i, b_i)} + \sigma(b_i, b_j) + \sigma(b_j, b_i) + \cancel{\sigma(b_j, b_j)}$$

„$\Leftarrow$": $\sigma(x,x) = \dots = \sum_{i,j\in I} x_i x_j \,\sigma(b_i, b_j) = 0_K$

A 9.8.2 Ges.: Kongruente Matrix in Normalform, Transformationsmatrix;

(α) Geg.: $\begin{bmatrix}0&1&2&1\\2&0&1&1\\1&2&0&1\\2&2&2&0\end{bmatrix} =: A \in \mathbb{Z}_3^{4\times 4}$

$$\underset{+2}{\left[\begin{array}{cccc}1&&&\\&1&&\\&&1&\\&&&1\\\hline 0&1&2&1\\2&0&1&1\\1&2&0&1\\2&2&2&0\end{array}\right]} \xmapsto{+2} \left[\begin{array}{cccc}1&2&&\\&1&&\\&&1&\\&&&1\\\hline 0&1&2&1\\2&1&1&1\\1&1&0&1\\2&0&2&0\end{array}\right] \mapsto \underset{+2}{\begin{bmatrix}0&1&2&1\\2&0&2&0\\1&1&0&1\\2&0&2&0\end{bmatrix}} \xmapsto{+2} \left[\begin{array}{cccc}1&2&2&\\&1&&\\&&1&\\&&&1\\\hline 0&1&2&1\\2&0&0&0\\1&1&2&1\\2&0&0&0\end{array}\right] \mapsto \begin{bmatrix}0&1&2&1\\2&0&0&0\\1&0&0&0\\2&0&0&0\end{bmatrix}$$

$$\mapsto \left[\begin{array}{cccc}2&1&2&\\1&&&\\&&1&\\&&&1\\\hline 1&0&2&1\\0&2&0&0\\0&1&0&0\\0&2&0&0\end{array}\right] \mapsto \underset{+1}{\begin{bmatrix}0&2&0&0\\1&0&2&1\\0&1&0&0\\0&2&0&0\end{bmatrix}} \xmapsto{+1} \left[\begin{array}{cccc}2&1&1&\\1&&1&\\&&1&\\&&&1\\\hline 0&2&0&0\\1&0&0&1\\0&1&0&0\\0&2&0&0\end{array}\right] \mapsto \underset{+2}{\begin{bmatrix}0&2&0&0\\1&0&0&1\\0&0&0&0\\0&2&0&0\end{bmatrix}}$$

$$\xmapsto{+2} \left[\begin{array}{cccc}2&1&1&1\\1&&1&2\\&&1&\\&&&1\\\hline 0&2&0&0\\1&0&0&0\\0&0&0&0\\0&2&0&0\end{array}\right] \mapsto \begin{bmatrix}0&2&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \mapsto \left[\begin{array}{cccc}1&2&1&1\\&1&1&2\\&&1&\\&&&1\\\hline 2&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{array}\right] \mapsto \underbrace{\begin{bmatrix}0&1&0&0\\2&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}}_{B \text{ in Normalform}}$$

P Transformationsmatrix

$$\Rightarrow P^T A P = B$$

A 9.10.3

Geg.: $A = \begin{pmatrix} 0 & i & 2i \\ -i & 0 & -i \\ -2i & i & 0 \end{pmatrix} \in \mathbb{C}^{3\times 3}$ alternierend, hermetisch

Ges.: $P, Q \in GL_3(\mathbb{C})$: $B := P^T A P$ zu A kongr.,

$C := \overline{Q}^T A Q$ zu A herm. Kongr.,

in Normalform

$$\underset{\circlearrowright\ +1/2}{\begin{array}{ccc} 1 & & \\ & 1 & \\ & & 1 \\ \hline 0 & i & 2i \\ -i & 0 & -i \\ -2i & i & 0 \end{array}} \mapsto {+1/2}\circlearrowleft \begin{array}{ccc} 1 & 1/2 & \\ & 1 & \\ & & 1 \\ \hline 0 & i & 2i \\ -i & -i/2 & -i \\ -2i & 0 & 0 \end{array} \mapsto \underset{\circlearrowright}{\begin{array}{ccc} & & \\ & & \\ & & \\ \hline 0 & i & 2i \\ -i & 0 & 0 \\ -2i & 0 & 0 \end{array}} \mapsto \begin{array}{ccc} 1/2 & 1 & \\ 1 & & \\ & & 1 \\ \hline i & 0 & 2i \\ 0 & -i & 0 \\ 0 & -2i & 0 \end{array} \mapsto \underset{\circlearrowright\ -2}{\begin{array}{ccc} & & \\ & & \\ & & \\ \hline 0 & -i & 0 \\ i & 0 & 2i \\ 0 & 2i & 0 \end{array}} \mapsto -2\circlearrowleft \begin{array}{ccc} 1/2 & 1 & -1 \\ 1 & & -2 \\ & & 1 \\ \hline 0 & -i & 0 \\ i & 0 & 0 \\ 0 & -2i & 0 \end{array}$$

(1)

$$\mapsto \underset{\substack{\uparrow \\ \cdot i}}{\begin{array}{ccc} & & \\ & & \\ & & \\ \hline 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{array}} \mapsto \cdot i \rightarrow \begin{array}{ccc} 1/2 & i & -1 \\ 1 & & -2 \\ & & 1 \\ \hline 0 & 1 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{array} \Big\} P \text{ Transformationsmatrix} \mapsto \underbrace{\begin{array}{ccc} & & \\ & & \\ & & \\ \hline 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{array}}_{B \text{ in Normalform}}$$

(2)

$$\mapsto \begin{array}{ccc} & & \\ & & \\ & & \\ \hline 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{array} \mapsto \cdot \overline{i} \rightarrow \begin{array}{ccc} 1/2 & i & -1 \\ 1 & & -2 \\ & & 1 \\ \hline 0 & 1 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{array} \mapsto \underset{\circlearrowright\ +1}{\begin{array}{ccc} & & \\ & & \\ & & \\ \hline 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}} \mapsto +1\circlearrowleft \begin{array}{ccc} 1/2+i & i & -1 \\ 1 & & -2 \\ & & 1 \\ \hline 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}$$

$$\mapsto \underset{\circlearrowright\ -1/2}{\begin{array}{ccc} & & \\ & & \\ & & \\ \hline 2 & 1 & 0 \\ 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}} \mapsto -1/2\circlearrowleft \begin{array}{ccc} 1/2+i & -1/4+i/2 & -1 \\ 1 & -1/2 & -2 \\ 0 & 0 & 1 \\ \hline 2 & 0 & 0 \\ 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array} \mapsto \cdot 1/\sqrt{2} \rightarrow \underset{\substack{\uparrow \\ \cdot 1/\sqrt{2}}}{\begin{array}{ccc} & & \\ & & \\ & & \\ \hline 2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}} \mapsto \begin{array}{ccc} 1/\sqrt{2}^3+i/\sqrt{2} & -1/4+i/2 & -1 \\ 1/\sqrt{2} & -1/2 & -2 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & \underset{\substack{\uparrow \\ \cdot i\sqrt{2}}}{0} & 0 \end{array}$$

$$\begin{array}{rccc} \mapsto & & & \\ \hline & 1 & 0 & 0 \\ \cdot i\sqrt{2} \rightarrow & 0 & -i/\sqrt{2} & 0 \\ & 0 & 0 & 0 \end{array} \quad \mapsto \quad \begin{array}{ccc} 1/\sqrt{2}^3 + i/\sqrt{2} & -1/\sqrt{2} - i/\sqrt{2}^3 & -1 \\ 1/\sqrt{2} & -i/\sqrt{2} & -2 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}$$

} Q Transformationsmatrix

A 9.10.5 (α)

Gg.: $G = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & 0 \\ i & 0 & 2 \end{pmatrix} \in \mathbb{C}^{3\times 3}$

(a) Zz.: G positiv definit (mittels Hauptminorenkriterium)

$$\left.\begin{array}{l} |1| = 1 \\[1em] \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \\[1em] \begin{vmatrix} 1 & 0 & -i \\ 0 & 1 & 0 \\ i & 0 & 2 \end{vmatrix} = 1 \end{array}\right\} \in \mathbb{R}^+$$

(b) Ges.: $P \in GL_3(\mathbb{C}) : \overline{P}^T P = G$

$$\underset{+i}{\begin{array}{ccc} 1 & & \\ & 1 & \\ & & 1 \\ \hline 1 & & -i \\ & 1 & \\ i & & 2 \end{array}} \xrightarrow{} \overset{}{\underset{+\bar{i}}{}}\begin{array}{ccc} 1 & & i \\ & 1 & \\ & & 1 \\ \hline 1 & & \\ & 1 & \\ i & & 1 \end{array} \mapsto \begin{array}{c} \\ \\ \\ \hline E_3 \\ \\ \end{array}$$

$$\Rightarrow \overline{\begin{pmatrix} 1 & & i \\ & 1 & \\ & & 1 \end{pmatrix}}^T \cdot G \cdot \begin{pmatrix} 1 & & i \\ & 1 & \\ & & 1 \end{pmatrix} = E_3$$

$$\Rightarrow G = \overline{\begin{pmatrix} 1 & & -i \\ & 1 & \\ & & 1 \end{pmatrix}}^T \cancel{E_3} \underbrace{\begin{pmatrix} 1 & & -i \\ & 1 & \\ & & 1 \end{pmatrix}}_{\begin{pmatrix} 1 & & i \\ & 1 & \\ & & 1 \end{pmatrix}^{-1} =: P}$$