Introduction to Statistics

Common Distributions

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Outline

1. Density and Mass Functions
   - Probability Mass Functions
   - Probability Density Functions
2. Common Familes of Distributions
   - Discrete distributions

Probability Mass Function

- Suppose X is a random variable with cdf $F_X(x)$.
- An associated function is the probability mass function (pmf) for a discrete X and
- the probability density function (pdf) for a continuous X.

Definition

The probability mass function (pmf) of a discrete random variable X is given by

$$f_X(x) = \mathbb{P}(X = x), \quad \forall x$$

Probability Mass Function

Example: Geometric Probabilities

- Let X = number of coin tosses required to get a head.
- Let p = the probability of getting a head in any toss.
- We have seen that the pmf is

$$f_X(x) = \mathbb{P}(X = x) = (1-p)^{x-1}p, \quad x = 1, 2, \ldots$$

- The cdf is

$$P(X \leq x) = \sum_{j=1}^{x} \mathbb{P}(X = j) = \sum_{j=1}^{x} (1-p)^{j-1}p$$

and since the partial sum of the geometric series is

$$\sum_{k=1}^{n} t^{k-1} = \frac{1-t^n}{1-t}, \quad t \neq 1$$

(by induction), we have

$$F_X(x) = 1 - (1-p)^x, \quad x = 1, 2, \ldots$$

Probability Density Function

The definition of the corresponding function for a continuous r.v. X is trickier: Since

$$\{X = x\} \subset \{x - \epsilon < X \leq x\} \text{ for any } \epsilon > 0$$

we have that

$$\mathbb{P}(X = x) \leq \mathbb{P}(x - \epsilon < X \leq x\} = F_X(x) - F_X(x - \epsilon) \text{ for any } \epsilon > 0$$

Therefore,

$$0 \leq \mathbb{P}(X = x) \leq \lim_{\epsilon \to 0} (F_X(x) - F_X(x - \epsilon)) = 0$$

by the right continuity of F.

Probability Density Function

To avoid this, we substitute integrals for sums:

$$\mathbb{P}(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

Using the *Fundamental Theorem of Calculus*, if $f_X(x)$ is continuous,

$$\frac{d}{dx}F_X(x) = f_X(x)$$

Direct analogy with the discrete case: we "add up" the "point probabilities" $f_X(x)$ to obtain interval probabilities

Probability Density Function

Definition

The probability density function (pdf) of a continuous random variable X is the function that satisfies

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt \quad \forall x \tag{1}$$

- (1) does not always hold because F_X may be continuous but not differentiable: these cases are rare and we ignore them; i.e., we consider only *absolutely continuous* random variables for which (1) holds
- Notation: "X has distribution given by $F_X(x)$" is abbreviated by $X \sim F_X(x)$, or $X \sim f_X(x)$
- Since $\mathbb{P}(X = x) = 0$ when X is continuous,

$$\mathbb{P}(a < X < b) = \mathbb{P}(a < X \leq b) = \mathbb{P}(a \leq X < b) = \mathbb{P}(a \leq X \leq b)$$

Example: Logistic Probabilities

The logistic distribution has cdf

$$F_X(x) = \frac{1}{1+e^{-x}},\ x \in \mathbb{R}$$

Hence,

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{e^{-x}}{(1+e^{-x})^2}$$

Interval probabilities:

$$\begin{aligned} \mathbb{P}(a < X < b) &= F_X(b) - F_X(a) \\ &= \int_{-\infty}^{b} f_X(t)dt - \int_{-\infty}^{a} f_X(t)dt \\ &= \int_{a}^{b} f_X(t)dt \end{aligned}$$

pmf and pdf

Theorem

A function $f_X(x)$ is a pdf (pmf) of a random variable X if and only if

1. $f_X(x) \geq 0$ *for all* x
2. $\sum_x f_X(x) = 1$ *(pmf), or* $\int_{-\infty}^{\infty} f_X(x)dx = 1$ *(pdf)*

Proof.

If $f_X(x)$ is a pdf (pmf), then the two properties are immediate from the definition. In particular, for a pdf,

$$1 = \lim_{x\to\infty} F_X(x) = \int_{-\infty}^{\infty} f_X(x)dx$$

The converse is also easy, since once we have $f_X(x)$, we can define $F_X(x)$. □

- Any nonnegative function with a finite positive integral or sum can be turned into a pdf or pmf.
- For example, if $h(x)$ is any nonnegative function that is posititve on a set A and 0 elsewhere with

$$\int_{\{x \in A\}} h(x)dx = k < \infty$$

for some $k > 0$, then

$$f_X(x) = \frac{h(x)}{k}$$

is a pdf of a random variable X taking values in A.

Common distributions

1. Discrete distributions
    1. Bernoulli
    2. Binomial
    3. Geometric
    4. Poisson

2. Continuous distributions
    - Uniform
    - Exponential
    - Normal (Gaussian)

    - χ^2-distribution
    - t-distribution ...will be introduced later in the course ..

Bernoulli distribution

$X \sim \text{Bernoulli}(p)$

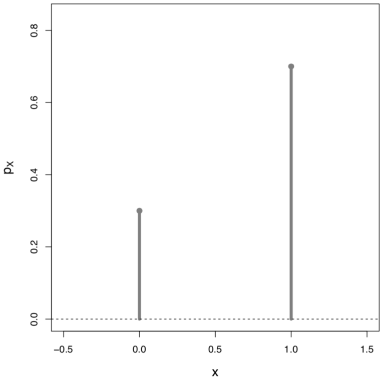
1. Models one trial in an experiment that can result in either success or failure.
   1. .. The associated X is only a indicator for the experiment
   2. Example: tossing a fair coin.
2. A random variable X has a Bernoulli distribution with parameter p if:
   - X takes the values 1 (for success) and 0 (for failure).
   - $P(X = 1) = p$ and $\mathbb{P}(X = 0) = 1 - p = q$

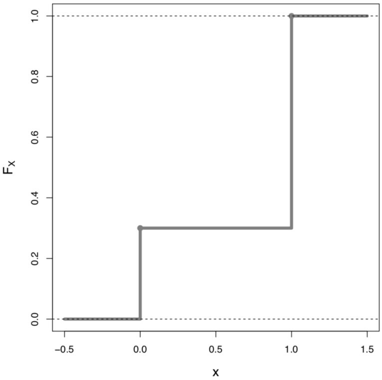
	x	0	1
pmf	$p(x)$	$1 - p$	p
cdf	$F(x)$	$1 - p$	1

3. Expectation/Variance $\mathbb{E}(X) = p, \quad \mathbb{V}ar(X) = p(1 - p)$

Example

- $X \sim Bernoulli(0.7)$





Probability mass function Cumulative distribution function

Binomial distribution

$X \sim \mathrm{Bin}(n,p)$ or $X \sim b(n,p)$

1. The binomial distribution models the number of successes in n independent $Bernoulli(p)$ trials.
    1. ... $Bin(1,p)$ is the same as Bernoulli(p)
    2. Example: The number of heads in n flips of a coin with probability p of heads follows a $Bin(n,p)$ distribution

Binomial distribution

1. X follows a binomial distribution if
   - X takes the values $0, 1, 2, 3, \ldots, n$
   - its probability mass function is given by

     $$p(x) = \mathbb{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

     for $x \in \{0, 1, \ldots, n\}$.
   - ... by the Binomial formula we get

     $$\sum_{x=0}^{n} p(x) = \sum_{x=0}^{n} \binom{n}{x} p^x \cdot (1-p)^{n-x} = (p + (1-p))^n = 1$$
2. Expectation/Variance: $\mathbb{E}(X) = np, \quad \mathbb{V}ar(X) = np(1-p)$

Example

- $X \sim Bin(10, 0.3)$

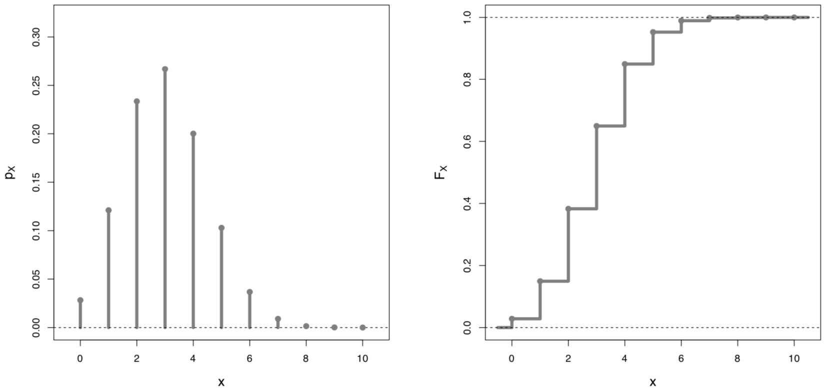


Figure: Probability mass function Cumulative distribution function

Example

Dice probabilities: Find the probability of obtaining at least one 6 in four rolls of a die.

- This experiment can be modeled as a sequence of four Bernoulli trials with success probability $p = \frac{1}{6} = P(\text{die shows } 6)$.
- Define the random variable

$$X = \text{Total number of sixes in four rolls}$$

- Then, $X \sim B(4, \frac{1}{6})$ and

$$\begin{aligned} P(\text{at least one } 6) &= P(X \geq 1) = 1 - P(X = 0) \\ &= 1 - \binom{4}{0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^4 = 0.518 \end{aligned}$$

1. In R: use `dbinom()`

```
1-dbinom(0,4,1/6)
[1] 0.5177469
```

Example

Consider another game; throw a pair of dice 24 times and compute the probability of at least two double 6.

- This experiment can be modeled by the binomial distribution with success probability p, with $p = P(\text{roll two sixes}) = \frac{1}{36}$.
- If Y = number of double 6s in 24 rolls, then $Y \sim B(24, \frac{1}{36})$ and

$$P(\text{at least two double sixes}) = P(Y \geq 2) = 1 - P(Y = 0) - P(Y = 1)$$
$$= 1 - \binom{24}{0}\left(\frac{1}{36}\right)^0\left(\frac{35}{36}\right)^{24} - \binom{24}{1}\left(\frac{1}{36}\right)^1\left(\frac{35}{36}\right)^{23} \approx 0.1427$$

1. In R: use `dbinom()`

```
1-sum(dbinom(0:1,24,1/36))
[1] 0.1426522
```

oder `pbinom()`

```
pbinom(1,24,1/36,lower.tail = FALSE)
[1] 0.1426522
```

More examples

- HW Standardized tests provide an interesting application of probability theory. Suppose that a test consists of 20 multiple-choice questions, each with 4 possible answers, of which exactly one is correct. If 17 questions are answered correctly, the exam is passed. A student comes unprepared for this test and randomly crosses one of the four possible answers. (If the student guesses on each question, then taking the exam can be modeled as a sequence of 20 independent events.) Use R to compute the probability that the student will pass the test.
- HW From a list of 15 households, 9 are homeowners and 6 households live in rental housing. Four households are randomly selected from these 15. Find the probability that the number of households that do not own a home is at least three.

Geometric distribution with parameter p

1. A geometric distribution models the total number of attempts before a success.
   1. Example: The number of tails before the first head in a sequence of coin flips (Bernoulli trials).
2. The random variable X follows a geometric distribution with parameter p if
   - X takes the values $0, 1, 2, 3, \dots$
   - its probability mass function is given by

$$p(x) = P(X = x) = (1-p)^x \cdot p \quad \text{for} \quad x \in \{0, 1, 2, \dots\}$$

   - ... through the geometric series $\sum_{x=0}^{\infty} k^x = \frac{1}{1-k}$, for $|k| < 1$, we get $\sum_{x=0}^{\infty} p(x) = p \cdot \sum_{x=0}^{\infty} (1-p)^x = \frac{p}{1-(1-p)} = 1$
3. Expectation/Variance $\mathbb{E}(X) = \frac{1-p}{p}, \quad \mathbb{V}ar(X) = \frac{1-p}{p^2}$

Example

- $X \sim$ Geometric distribution with $p = 0.1$

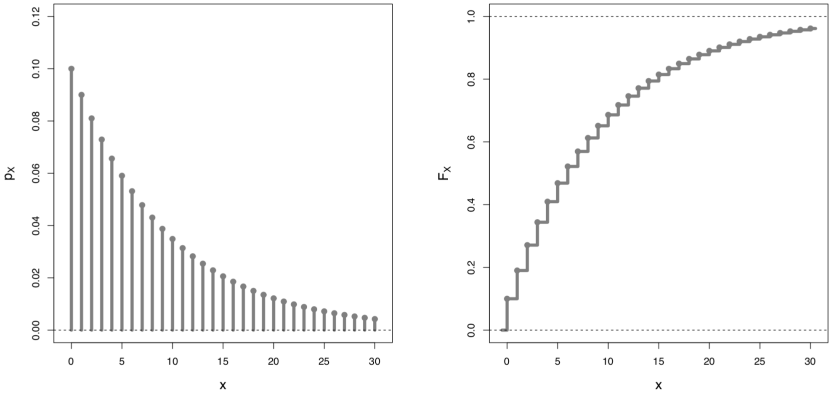


Figure: Probability mass function Cumulative distribution function

Example: Failure times

- Geometric distribution is the simplest of the waiting time distributions (used to model lifetimes of devices or components): Waiting for a success.
- For example, if the probability is 0.001 that a light bulb will fail on any given day, then the probability that it will last more than 30 days is

$$P(X > 30) = \sum_{x=31}^{\infty} 0.001(1 - 0.001)^{x-1} = 0.970$$

  - In R: use `dgeom()`

```
1-sum(dgeom(1:29, 0.001))
[1] 0.9704605
```

Example

HW Suppose that the inhabitants of an island plan their families by having babies until the first girl is born. Assume the probability of having a girl with each pregnancy is 0.4 independent of other pregnancies, that all babies survive and there are no multiple births. What is the probability that a family has 6 boys?

Poisson distribution

$X \sim \mathcal{P}(\lambda)$ or $X \sim Poi(\lambda)$

1. X has a Poisson distribution with parameter $\lambda > 0$, if
   - X takes the values $0, 1, 2, 3, \ldots$
   - its probability mass function is given by

$$p(k) = \mathbb{P}(X = k) = \frac{\lambda^k}{k!} \cdot e^{-\lambda} \quad \text{for} \quad k \in \{0, 1, 2, \ldots\}$$

   - ... λ is the intensity parameter
   - ... using the exponential series $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$, for all $\lambda \in \mathbb{R}$, we obtain $\sum_{x=0}^{\infty} p(x) = e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = 1$
   - Usually, λ is interpreted as the number of times an event occurs in an interval
2. Expectation/Variance: $\mathbb{E}(X) = \lambda, \quad \mathbb{V}ar(X) = \lambda$

Example

- $X \sim \mathcal{P}(3)$

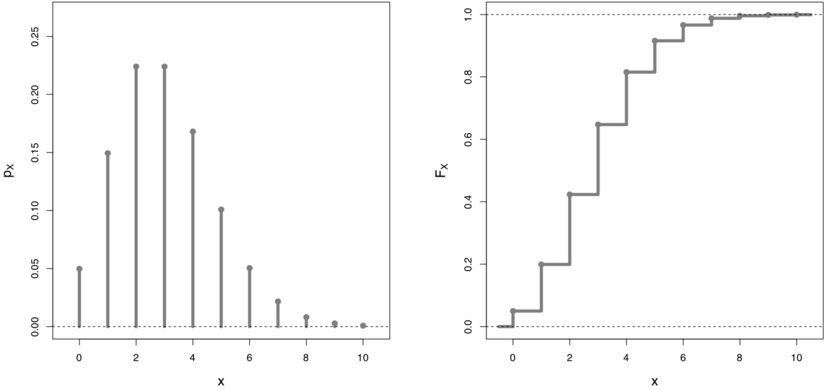


Figure: Probability mass function   Cumulative distribution function

Example: Waiting times

Consider a telephone operator who, on the average, handles five calls every 3 minutes. What is the probability that there will be no calls in the next minute? At least three calls?

- Let X = number of calls in a minute. Then $X \sim \mathcal{P}(\lambda)$, where $EX = \lambda = \frac{5}{3}$. Then,

$$P(\text{no calls in the next minute}) = P(X = 0) = \frac{e^{-\frac{5}{3}}\left(\frac{5}{3}\right)^0}{0!} = 0.189$$

$$P(\text{at least three calls in the next minute}) = P(X \geq 3)$$
$$= 1 - \sum_{x=0}^{2} P(X = k) = 1 - \sum_{x=0}^{2} \frac{e^{-\frac{5}{3}}\left(\frac{5}{3}\right)^x}{x!} = 0.234.$$

- In R: we use `dpois()`

```
1-sum(dpois(0:2, 5/3))
[1] 0.2340045
```

Example

HW Web visitors

A website manager has noticed that during the evening hours, about three people per minute check out from their shopping cart and make an online purchase. She believes that each purchase is independent of the others and wants to model the number of purchases per minute.

(1) What model might you suggest to model the number of purchases per minute?
(2) What is the probability that in any one minute at least one purchase is made?
(3) What is the probability that no one makes a purchase in the next two minutes?

Binomial-Poisson relationship

$\mathcal{P}(\lambda)$ distribution as a limit of the $Bin(n,p)$ distribution

1. If $X_n \sim Bin(n, \frac{\lambda}{n})$,

$$\lim_{n\to\infty} P(X_n = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

2. Interpretation: If there are many independent and identical Bernoulli experiments with a low probability of success, the number of successes can be approximated with a Poisson distribution

3. Rule of thumb
    1. For $n \geq 50$, $p \leq \frac{1}{10}$ and $np \leq 10$ a random variable $X \sim Bin(n,p)$ can be approximated by

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \approx \frac{(np)^x e^{-np}}{x!}$$

... we will return to this later in the course...

Recursion Relations

1. If $X \sim Poi(\lambda)$,

$$\mathbb{P}(X = x) = \frac{\lambda}{x}\mathbb{P}(X = x - 1), \quad x = 1, 2, \ldots$$

2. If $Y \sim Bin(n, p)$,

$$\mathbb{P}(Y = y) = \frac{n - y + 1}{y}\frac{p}{1 - p}\mathbb{P}(Y = y - 1)$$

These can be used to establish the relationship between the Binomial and Poisson:

1. Set $\lambda = np$, and if p is small, we can write

$$\frac{n - y + 1}{y}\frac{p}{1 - p} = \frac{np - p(y - 1)}{y - py} \approx \frac{\lambda}{y}$$

   since the terms $p(y - 1)$ and py can be ignored.

Recursion Relations

1. Therefore,

$$\mathbb{P}(Y = y) \approx \frac{\lambda}{y}\mathbb{P}(Y = y - 1) \tag{2}$$

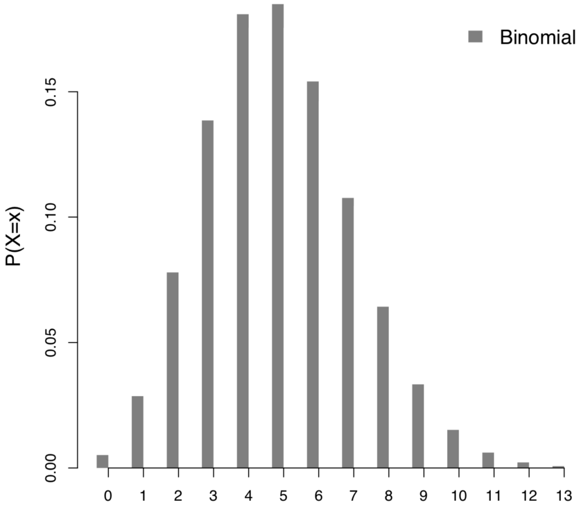
   which is the Poisson recursion.
2. It now suffices to show that $\mathbb{P}(X = 0) \approx \mathbb{P}(Y = 0)$, since all other probabilities will follow from (2).
3. Now,

$$\mathbb{P}(Y = 0) = (1 - p)^n = \left(1 - \frac{np}{n}\right)^n = \left(1 - \frac{\lambda}{n}\right)^n \approx e^{-\lambda}$$

   since $\lim_{n\to\infty}(1 - (\lambda/n))^n = e^{-\lambda}$.

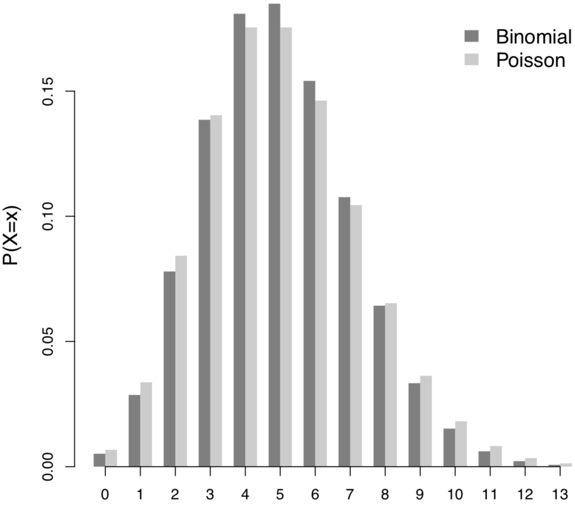
Example

- $X \sim Bin(50, \frac{1}{10})$



Example

- $X \sim Bin(50, \frac{1}{10}) \approx \mathcal{P}(5)$



Uniform distribution

$X \sim \mathcal{U}(a, b)$

1. X is a random variable with uniform distribution over the interval (a, b) $(a < b,\ a, b \in \mathbb{R})$ if its pdf is of the form

$$f(x) = \begin{cases} \frac{1}{b-a}, & x \in (a, b) \\ 0, & \text{otherwise} \end{cases}$$

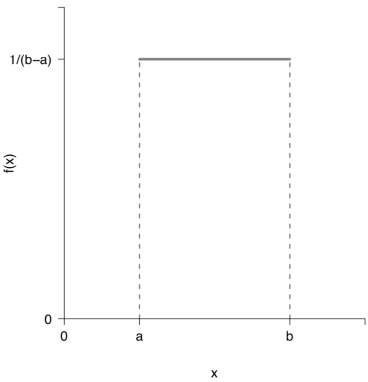
2. The cdf is given by

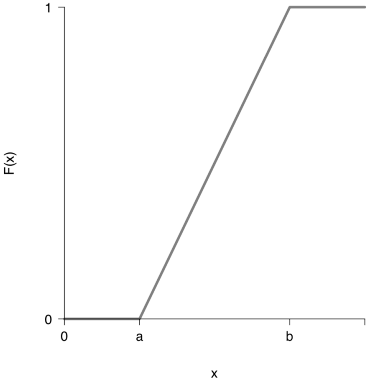
$$F(x) = \begin{cases} 0 & x \leq a \\ \dfrac{x-a}{b-a}, & a < x < b \\ 1, & x \geq b \end{cases}$$

3. Expectation/Variance: $\mathbb{E}(X) = \frac{a+b}{2}$, $\mathbb{V}ar(X) = \frac{(b-a)^2}{12}$

Example

- $X \sim \mathcal{U}(a, b)$





Probability density function   Cumulative distribution function

Exponential distribution

$X \sim Exp(\lambda)$ or $X \sim Exp(\tau)$, $\quad \tau = \frac{1}{\lambda}$

- X has a exponential distribution with the parameter $\lambda > 0$ if its pdf is of the form

$$f(x) = \begin{cases} \lambda\, e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

1. or when $\lambda = \frac{1}{\tau}$, the pdf is of the form

$$f(x) = \begin{cases} \frac{1}{\tau}\, e^{-\frac{1}{\tau}\, x} & x \geq 0 \\ 0 & x < 0. \end{cases}.$$

- ... models waiting time (continuous analogue of geometric distribution)

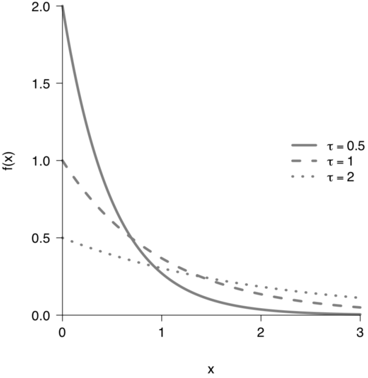
- The cdf is of the form

$$F(x) = \begin{cases} 1 - e^{-\lambda\, x} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

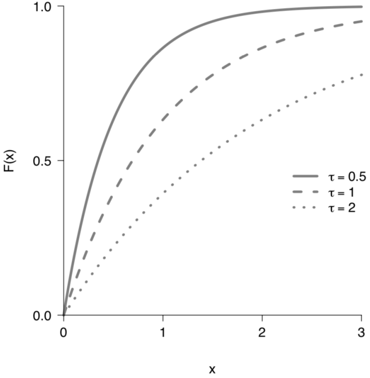
- Expectation/Variance: $\quad \mathbb{E}(X) = \tau = \frac{1}{\lambda}, \quad \mathbb{V}ar(X) = \tau^2 = \frac{1}{\lambda^2}$

Example

- $X \sim \exp(\tau)$, for $\tau = 0.5$, $\tau = 1$ and $\tau = 2$



Probability density function



Cumulative distribution function

Example: Waiting time

- Anna noticed that taxies drive past her street once every 10 minutes on average. Suppose time spent waiting for a taxi is modeled by an exponential random variable

$$X \sim \exp(\frac{1}{10}); \qquad f(x) = \frac{1}{10}e^{-\frac{x}{10}}, \quad \text{for } x \geq 0$$

- Calculate the probability of waiting for a taxi between 3 and 7 minutes.
  1. $P(3 < X < 7) = \int_3^7 f(x)\,dx = F(7) - F(3) \approx 0.244$
  2. In R: we use `pexp()`

```
pexp(7,0.1)-pexp(3,0.1)
[1] 0.2442329
```

The Memoryless Property of the Exponential Distribution

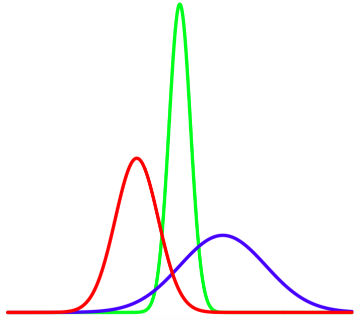
Suppose $X \sim Exp(\lambda)$. Then, for $s > t \geq 0$,

$$
\begin{aligned}
\mathbb{P}(X > s \mid X > t) &= \frac{\mathbb{P}(X > s, X > t)}{\mathbb{P}(X > t)} \\
&= \frac{\mathbb{P}(X > s)}{\mathbb{P}(X > t)} \\
&= \frac{\int_s^\infty \lambda e^{-\lambda x} dx}{\int_t^\infty \lambda e^{-\lambda x} dx} \\
&= \frac{e^{-\lambda s}}{e^{-\lambda t}} \\
&= e^{-(s-t)\lambda} \\
&= \mathbb{P}(X > s - t)
\end{aligned}
$$

- The probability of surviving an additional time $s - t$, having already survived up to time t, is the same as surviving past $s - t$ time in the beginning.
- dependence only on length of interval but not its position!

Normal (Gaussian) distribution

- $X \sim \mathcal{N}(\mu, \sigma^2)$
  - One of the most important distributions in statistics and probability theory.



1. Normal distributions are symmetric with relatively more values at the center of the distribution and relatively few in the tails.

Normal distribution

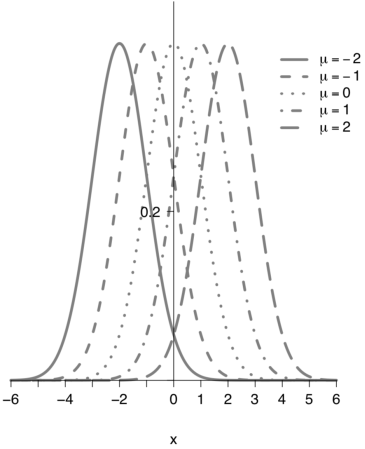
- $X \sim \mathcal{N}(\mu, \sigma^2)$
  - X follows a Normal distribution (or Gaussian distribution) with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$, if its pdf is of the form

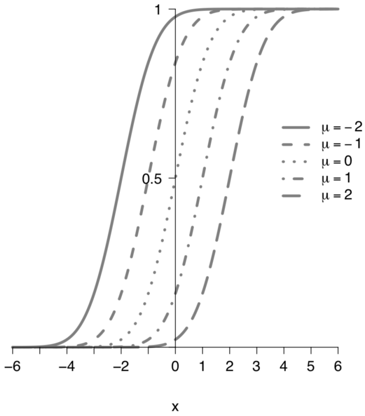
$$f(x) = \frac{1}{\sqrt{2\pi\,\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for} \quad x \in \mathbb{R}$$

  - Expectation/Variance $\qquad \mathbb{E}(X) = \mu, \quad \mathbb{V}ar(X) = \sigma^2$

Examples

$X \sim \mathcal{N}(\mu, 1)$, for $\mu = -2$, $\mu = -1$, $\mu = 0$, $\mu = 1$ and $\mu = 2$

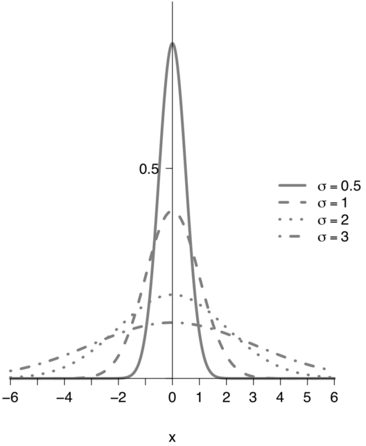


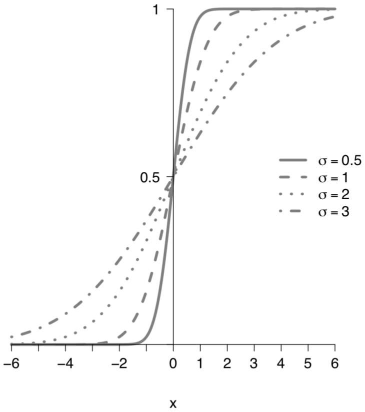


Probability density function Cumulative distribution function

Examples

$X \sim \mathcal{N}(0, \sigma^2)$, for $\sigma = 0.5$, $\sigma = 1$, $\sigma = 2$ and $\sigma = 3$

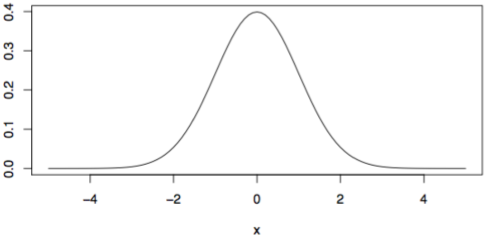




Probability density function Cumulative distribution function

Standard Normal distribution

$Z \sim \mathcal{N}(0,1)$ is the Standard Normal distribution



1. the cdf is

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} f(t)\, dt = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{t^2}{2}}\, dt$$

$$\Phi(-z) = 1 - \Phi(z), \quad \text{for } z \in \mathbb{R}$$

2. ... the values of Φ can be read from the `normal probability table`
3. or obtained in R by applying `pnorm()`

$\Phi(z) = P(Z \leq z)$

- The Table of $\mathcal{N}(0,1)$-distribution

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9958	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

Table of $\Phi(z)$: Examples

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.
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0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.

1. $\Phi(0.43) = 0.6664$ $\qquad\qquad$ $\Phi(0.76) = 0.7764$
2. $\Phi(-0.43) = 1 - 0.6664 = 0.3336$
   $\Phi(-0.76) = 1 - 0.7764 = 0.2236$
3. $P(-0.76 \leq Z \leq 0.43) = \Phi(0.43) - \Phi(-0.76) = 0.4428$

Table of $\Phi(z)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
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1. Computation of $P(|Z| \leq \frac{1}{2})$:

$$\begin{aligned} P(|Z| \leq \frac{1}{2}) &= P(-0.5 \leq Z \leq 0.5) \\ &= \Phi(0.5) - \Phi(-0.5) \\ &= \Phi(0.5) - (1 - \Phi(0.5)) \\ &= 2 \cdot \Phi(0.5) - 1 \\ &= 2 \cdot 0.6915 - 1 = 0.3830 \end{aligned}$$

$\Phi(z)$ in R

1. pnorm(x, mean, sd)
   1. ... pnorm (x, μ, σ) uses σ while in the notation of the normal distribution $\mathcal{N}(\mu, \sigma^2)$ is used σ^2...

```
#Values of the cdf of  N(0,1) distribution
pnorm(0.5)
[1] 0.6914625
# it is the same as
pnorm(0.5, mean=0, sd=1)
[1] 0.6914625
pnorm(1.3)
[1] 0.9031995
# Values of the cdf of  N(3,4) distribution
pnorm(2.5, mean=3, sd=2)
[1] 0.4012937
# it is the same as pnorm((2.5-3)/2)
pnorm(-0.25)
[1] 0.4012937
```

- We compute $P(|Z| \leq \frac{1}{2}) \approx 0.3830$ using R:

```
diff(pnorm(c(-0.5,0.5)))
[1] 0.3829249
```

Questions

(1) What is the probability that $Z \sim \mathcal{N}(0, 1)$ is at most one /two/three standard deviations away from the expectation $\mu = 0$?

(2) What is the probability that $X \sim \mathcal{N}(0, 3^2)$ is at most one /two/three standard deviations away from the expectation $\mu = 0$?

(3) How do the answers to question (2) change if we change μ?

... The following probabilities should be calculated

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(3) How do the answers to question (2) change if we change μ?

... The following probabilities should be calculated

$$P(X \in [\mu - \sigma, \mu + \sigma])$$
$$P(X \in [\mu - 2\sigma, \mu + 2\sigma])$$
$$P(X \in [\mu - 3\sigma, \mu + 3\sigma])$$

Probabilities of the normal distribution

(1) Let $Z \sim \mathcal{N}(0,1)$. Then,

$$P(-1 \leq Z \leq 1) \approx 0.68$$
$$P(-2 \leq Z \leq 2) \approx 0.95$$
$$P(-3 \leq Z \leq 3) \approx 0.997$$

```
# one sigma-away from mu=0
diff(pnorm(c(-1,1)))
[1] 0.6826895
# two sigmas-away from mu=0
diff(pnorm(c(-2,2)))
[1] 0.9544997
# three sigmas-away from mu=0
diff(pnorm(c(-3,3)))
[1] 0.9973002
```

Features of the normal distribution

(2) Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

- Standardization

$$Z = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$$

... Standardizing any normal random variable X produces the standard normal Z

... We use Z for a $\mathcal{N}(0, 1)$

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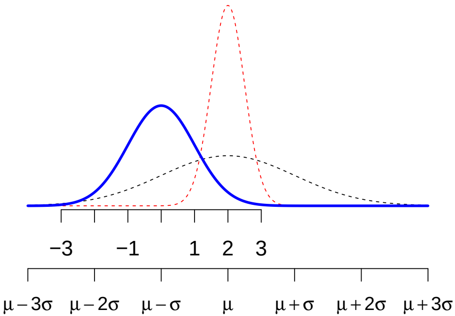

The standard normal distribution

Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$. Then, $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$

Easy to see:

$$\begin{aligned}
\mathbb{P}(Z \leq z) &= \mathbb{P}\left(\frac{X-\mu}{\sigma} \leq z\right) \\
&= \mathbb{P}(X \leq z\sigma + \mu) \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{z\sigma+\mu} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z\sigma+\mu} e^{-\frac{t^2}{2}} dt, \quad t = \frac{x-\mu}{\sigma}
\end{aligned}$$

so that $\mathbb{P}(Z \leq z)$ is the standard normal cdf.

Probabilities of the normal distribution

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- In general, let Y be a random variable with $E(Y) = \mu$ and standard deviation $\sigma = \sqrt{\mathrm{Var}\,(Y)}$. Then, the standardized random variable

$$U = \frac{Y - \mu}{\sigma}$$

has $\mathbb{E}(U) = 0$ and $\sqrt{\mathbb{V}ar}(U) = 1$. ... HW

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- Then,

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The 68-95-99.7-Rule applies to probabilities of the normal distribution

- 68% of the area of a normal distribution is within one standard
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- Approximately 95% of the area of a normal distribution is within two standard deviations of the mean.
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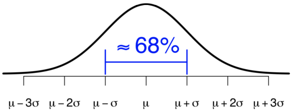

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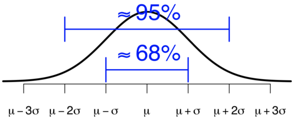

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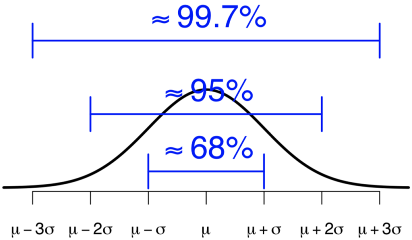

Properties

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and let F_X be its cdf.

- Affine transformation

$$Y = a + bX \sim \mathcal{N}(a + b\mu,\, b^2\sigma^2)$$

really important!

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where $\quad z_p = \Phi^{-1}(p) \quad$ denotes the p-quantile of $\mathcal{N}(0,1)$

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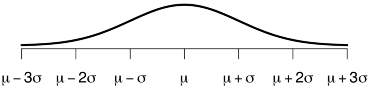
- in R: `qnorm()`

Properties

- Let $X_1, X_2, \ldots, X_n$ be independent identically distributed (iid) random variables with $X_i \sim \mathcal{N}(\mu, \sigma^2)$



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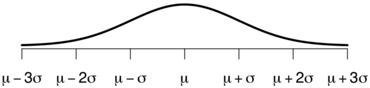

- The mean of these random variables satisfies

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

  - $\bar{X}$ is also normally distributed
  - the expectation of $\bar{X}$ is $\mu_{\bar{X}} = \mu$ (the same as the expectation of X_i)
  - the standard deviation of $\bar{X}$ equals $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$ (reduction by factor $\frac{1}{\sqrt{n}}$)

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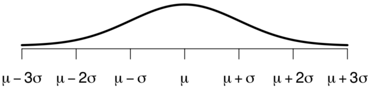

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... we are going to prove this!

The mean of iid normally distributed rvs

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$$

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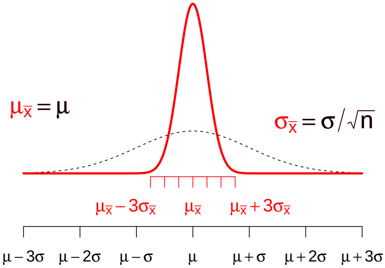

Examples

(1) As a group, the Dutch are among the tallest people in the world. The average Dutsch man is 184 cm tall. If a normal model is appropriate and the standard deviation for men is about 8 cm, what percentage of all Dutsch men will be over two meters?

Examples

(1) As a group, the Dutch are among the tallest people in the world. The average Dutsch man is 184 cm tall. If a normal model is appropriate and the standard deviation for men is about 8 cm, what percentage of all Dutsch men will be over two meters?

Answer:

- Let X denotes the height of a Dutch man. Then, $X \sim \mathcal{N}(184, 8^2)$.

Examples

(1) As a group, the Dutch are among the tallest people in the world. The average Dutsch man is 184 cm tall. If a normal model is appropriate and the standard deviation for men is about 8 cm, what percentage of all Dutsch men will be over two meters?

Answer:

- Let X denotes the height of a Dutch man. Then, $X \sim \mathcal{N}(184, 8^2)$.
- We want to calculate $P(X > 200)$.

  We also know

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 184}{8} \sim \mathcal{N}(0, 1)$$

Examples

(1) As a group, the Dutch are among the tallest people in the world. The average Dutsch man is 184 cm tall. If a normal model is appropriate and the standard deviation for men is about 8 cm, what percentage of all Dutsch men will be over two meters?

Answer:

- Let X denotes the height of a Dutch man. Then, $X \sim \mathcal{N}(184, 8^2)$.
- We want to calculate $P(X > 200)$.

I way: We use the table and calculate

$$
\begin{aligned}
P(X > 200) &= P\left(\frac{X - 184}{8} > \frac{200 - 184}{8}\right) \\
&= P\left(Z > 2\right) \\
&= 1 - P\left(Z \leq 2\right) \\
&= 1 - \Phi(2) \\
&= 1 - 0.9772 = 0.0228 = 2.28\%
\end{aligned}
$$

Examples

(1) As a group, the Dutch are among the tallest people in the world. The average Dutsch man is 184 cm tall. If Normal model is appropriate and the standard deviation for men is about 8 cm, what percentage of all Dutsch men will be over two meters?

Answer:

- Let X denotes the height of a Dutch man. Than $X \sim \mathcal{N}(184, 8^2)$.
- We want to calculate $P(X > 200)$.

II way: Use R

```
pnorm(184,200,8)
[1] 0.02275013
# or
1-pnorm(184,200,8, lower.tail=FALSE)
[1] 0.02275013
 # or
pnorm(184,200,8, lower.tail=TRUE)
[1] 0.02275013
```

Examples

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$$P(X < 22) = P\left(\frac{X - 20}{2} < \frac{22 - 20}{2}\right) = P\left(Z < 1\right) = \Phi(1) = 0.8413$$

$$P(X > 24) = P\left(Z > 2\right) = 1 - \Phi(2) = 1 - 0.9772 = 0.0228$$

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and

$$\frac{y_{0.7} - 10}{3} = \Phi^{-1}(0.7) \quad \Longrightarrow \quad y_{0.7} = 3 \cdot \Phi^{-1}(0.7) + 10$$

Examples

- We find the value $\Phi^{-1}(0.7)$ in Table of standard normal distribution.

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772
0.5	0.6915	0.6950	0.6985	$\boxed{0.7019}$	0.7054	0.7088	0.7123
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764
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Then, the 0.7-quantile of Y is

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- in R we use `qnorm()`
  $\Phi^{-1}(0.7) =$ `qnorm`$(0.7$, mean$=0$, sd$=1) = 0.5244$

Examples

(b) We look for y_0 such that it satisfies $P(Y \geq y_0) = 0.1$, i.e.

$$\begin{aligned} 0.1 &= P(Y \geq y_0) = 1 - P(Y < y_0) = 1 - P\left(Z < \frac{y_0 - 10}{3}\right) \\ &= 1 - \Phi\left(\frac{y_0 - 10}{3}\right) \end{aligned}$$

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- We use R:

$$y_0 = 3 \cdot \texttt{qnorm (0.9, 0, 1)} + 10 = 13.8447$$

or

$$y_0 = \texttt{qnorm(0.1, 10,3, lower.tail = FALSE)} = 13.84465$$

Examples

HW Miraculin is a protein naturally produced in a rare tropical fruit. It can convert a sour taste into a sweet taste. Consequently, miraculin has the potential to be an alternative low-calorie sweetener.
A group of Japanese environmental scientists investigated the ability of a hybrid tomato plant to produce miraculin. For a particular generation of the potato plant, the amount X of miraculin produced (measured in micrograms per gram of fresh weight) had a mean 105.3 and a standard deviation of 8.0. Assume that X is normally distributed.

(a) Find $P(X > 120)$.
(b) Find $P(100 < X < 110)$.
(c) Find the values a for which $P(X < a) = 0.25$.

R-functions

- The following functions in R may be used:

`pbinom()`	`dbinom()`	`qbinom()`	`rbinom()`
`pexp()`	`dexp()`	`qexp()`	`rexp()`
`pnorm()`	`dnorm()`	`qnorm()`	`rnorm()`
`punif()`	`dunif()`	`qunif()`	`runif()`
`ppois()`	`dpois()`	`qpois()`	`rpois()`

- The prefix `d` means the probability density function or the probability mass function, `p` means the cumulative distribution function, `q` stays for the quantile value and `r` returns a random simulation.
- The root `binom` stays for binomial distribution, while `exp` stays for exponential distribution, `norm` for normal distribution, `unif` for uniform distribution and `pois` for Poisson distribution.
  - For example, `dnorm` is the height of the density of a normal curve while `dbinom` gives the probability of an outcome of a binomial distribution.

Various examples

HW High temperatures in Vienna for the month of August follow a uniform distribution over the interval 22 to 27 degrees Celsius. Find the temperature which 90% of the August days exceed.

HW A construction zone on a highway has a posted speed limit of 40 miles per hour. The speeds of vehicles passing through this construction zone are normally distributed with a mean of 46 miles per hour and a standard deviation of 4 miles per hour. Find the percentage of vehicles passing through this construction zone that is exceeding the posted speed limit.

HW The distribution of cholesterol levels for patients in a cardiology practice follows a normal distribution with a mean of 210 and a standard deviation of 40. In this practice, the probability that a patient has a cholesterol level reading more than 290 is the same as the probability that a patient has a cholesterol level reading less than ______________ .

Various examples

HW It is known from a certain mailbox advertisement that in two out of 1000 people a purchase contract is concluded based on this advertisement. What is the probability that out of 800 people who find the advertisement in your mailbox,

(1) no one
(2) at most three
(3) at least four

will conclude a purchase contract?

HW The length of a workpiece is normally distributed with $\mu = 3$ and $\sigma = 0.2$. All workpieces that are shorter than 2.8 or longer than 3.2 are considered to be rejects. What is the probability that a workpiece is rejected?

HW Let Z a standard normal random variable and let $X = 5Z + 1$.

(1) Calculate $P(|X| \leq 1)$.
(2) Recall that the probability that Z is within one standard deviation of its mean is approximately 68%. What is the probability that X is within one standard deviation of its mean?

Various examples

HW It is known that the time X (in hours), that a technician needs to repair a machine, follows an exponential distribution with parameter $\lambda = 2$.

1. Calculate the associated distribution function F and sketch it as well as its density.
2. What is the probability that the technician needs
    1. at most half an hour
    2. between 0.2 and 0.4 hours
    3. more than 12 minutes

    for the repair?
3. How many hours are required on average for the repair of a machine? Also determine the variance of the repair time.

HW Let $X_1, X_2, \ldots, X_{25}$ be independent and identically distributed random variables and $X_1 \sim \mathcal{N}(1, 4)$. Find the probability $P(X_1 + X_2 + \ldots \ldots X_{25} \geq 26)$.