5.8. ADDITIONAL TOPICS

5.8.1. Poincaré's inequalities.

We now illustrate how the compactness assertion in §5.7 can be used to generate new inequalities.

Notation. $(u)_U = \fint_U u\,dy$ = average of u over U. □

THEOREM 1 (Poincaré's inequality). *Let U be a bounded, connected, open subset of $\mathbb{R}^n$, with a C^1 boundary ∂U. Assume $1 \le p \le \infty$. Then there exists a constant C, depending only on n, p and U, such that*

$$\|u - (u)_U\|_{L^p(U)} \le C\|Du\|_{L^p(U)} \tag{1}$$

for each function $u \in W^{1,p}(U)$.

The significance of (1) is that only the gradient of u appears on the right hand side.

Proof. We argue by contradiction. Were the stated estimate false, there would exist for each integer $k = 1, \ldots$ a function $u_k \in W^{1,p}(U)$ satisfying

$$\|u_k - (u_k)_U\|_{L^p(U)} > k\|Du_k\|_{L^p(U)}. \tag{2}$$

We renormalize by defining

$$v_k := \frac{u_k - (u_k)_U}{\|u_k - (u_k)_U\|_{L^p(U)}} \quad (k = 1, \ldots). \tag{3}$$

Then

$$(v_k)_U = 0, \ \|v_k\|_{L^p(U)} = 1;$$

and (2) implies

$$\|Dv_k\|_{L^p(U)} < \frac{1}{k} \quad (k = 1, 2, \ldots). \tag{4}$$

In particular the functions $\{v_k\}_{k=1}^\infty$ are bounded in $W^{1,p}(U)$.

In view of the Remark after the Rellich–Kondrachov Theorem in §5.7, there exists a subsequence $\{v_{k_j}\}_{j=1}^\infty \subset \{v_k\}_{k=1}^\infty$ and a function $v \in L^p(U)$ such that

$$v_{k_j} \to v \quad \text{in } L^p(U). \tag{5}$$

From (3) it follows that

$$(v)_U = 0, \ \|v\|_{L^p(U)} = 1. \tag{6}$$

On the other hand, (4) implies for each $i = 1, \dots, n$ and $\phi \in C_c^\infty(U)$ that

$$\int_U v\phi_{x_i}\,dx = \lim_{k_j\to\infty}\int_U v_{k_j}\phi_{x_i}\,dx = -\lim_{k_j\to\infty}\int_U v_{k_j,x_i}\phi\,dx = 0.$$

Consequently $v \in W^{1,p}(U)$, with $Dv = 0$ a.e. Thus v is constant, since U is connected (see Problem 10). However this conclusion is at variance with (6): since v is constant and $(v)_U = 0$, we must have $v \equiv 0$; in which case $\|v\|_{L^p(U)} = 0$. This contradiction establishes estimate (1). □

A particularly important special case follows.

Notation. $(u)_{x,r} = ⨍_{B(x,r)} u\,dy$ = average of u over the ball $B(x,r)$. □

THEOREM 2 (Poincaré's inequality for a ball). *Assume* $1 \le p \le \infty$. *Then there exists a constant* C, *depending only on* n *and* p, *such that*

$$\|u - (u)_{x,r}\|_{L^p(B(x,r))} \le Cr\|Du\|_{L^p(B(x,r))} \tag{7}$$

for each ball $B(x,r) \subset \mathbb{R}^n$ *and each function* $u \in W^{1,p}(B^0(x,r))$.

Proof. 1. The case $U = B^0(0,1)$ follows from Theorem 1. In general, if $u \in W^{1,p}(B^0(x,r))$ write

$$v(y) := u(x+ry) \quad (y \in B(0,1)).$$

Then $v \in W^{1,p}(B^0(0,1))$, and we have

$$\|v - (v)_{0,1}\|_{L^p(B(0,1))} \le C\|Dv\|_{L^p(B(0,1))}.$$

Changing variables, we recover estimate (7). □

Remark. Assume $u \in W^{1,n}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, and let $B(x,r)$ be any ball. Then Theorem 2 with $p = 1$ implies

$$\begin{aligned} ⨍_{B(x,r)} |u - (u)_{x,r}|\,dy &\le Cr ⨍_{B(x,r)} |Du|\,dy \\ &\le Cr\left(⨍_{B(x,r)} |Du|^n\,dy\right)^{1/n} \le C\left(\int_{\mathbb{R}^n} |Du|^n\,dy\right)^{1/n}. \end{aligned}$$

Thus $u \in BMO(\mathbb{R}^n)$, the space of functions of *bounded mean oscillation* in $\mathbb{R}^n$, with the seminorm

$$[u]_{BMO(\mathbb{R}^n)} := \sup_{B(x,r)\subset\mathbb{R}^n}\left\{⨍_{B(x,r)} |u - (u)_{x,r}|\,dy\right\}.$$

See Stein [**SE,** Chapter IV] for the theory of BMO. □