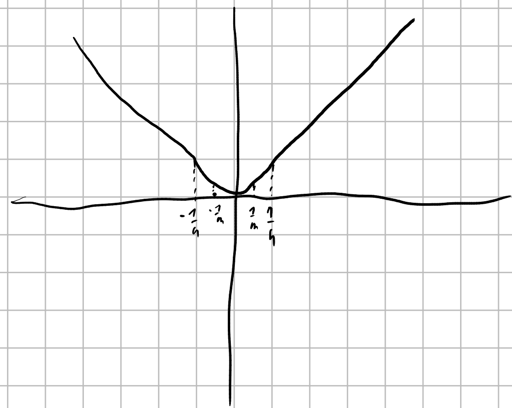
8) $p_n(x) = -\frac{n^3}{2}x^4 + \frac{3n}{2}x^2 = x^2\left(-\frac{n^3}{2}x^2 + \frac{3n}{2}\right)$



$$f_n(x) := \begin{cases} p_n(x) & , |x| < \frac{1}{n} \\ |x| & , |x| \geq \frac{1}{n} \end{cases}$$

$p_n(\frac{1}{n}) = -\frac{1}{2n} + \frac{3}{2n} = \frac{2}{2n} = \frac{1}{n}$

$p_n'(\frac{1}{n}) = -\frac{4}{2} + \frac{6}{2} = -2+3 = 1$

$f_n \in C^1(\mathbb{R})$, $f_n(0) = 0$, $\sup_{y \in \mathbb{R}} |f_n'(y)| < \infty$

Nach Aufgabe 7 ist $f_n \circ u \in H^1(\Omega)$ und $\partial_i(f_n \circ u) = f_n' \circ u \, \partial_i u$

$$\|f\circ u - f_n \circ u\|^2_{L^2(\Omega)} = \int_\Omega \left(f(u(x)) - f_n(u(x))\right)^2 d\lambda^n(x) = \int_\Omega \mathbb{1}_{[-\frac{1}{n},\frac{1}{n}]}(u(x)) \left(|u(x)| - p_n(u(x))\right)^2 d\lambda^n(x) = \ldots$$

$$g_n(x) := x - p_n(x) = x - x^2\left(-\frac{n^3}{2}x^2 + \frac{3n}{2}\right) = x\left(1 - x\left(\frac{3n}{2} - \frac{n^3}{2}x^2\right)\right) = x\left(1 + \frac{n^3}{2}x^3 - \frac{3n}{2}x\right)$$

$$= x\left(1 + \frac{n^3}{2}x^3 - \frac{3n}{2}x\right) \Leftrightarrow x = 0 \vee x = \frac{1}{n} \vee x = -\frac{2}{n}$$

$g_n(\frac{1}{2n}) = \frac{1}{2n}\left(1 + \frac{1}{16} - \frac{3}{4}\right) = \frac{1}{2n}\left(\frac{16+1-12}{16}\right) = \frac{1}{2n}\,\frac{5}{16} > 0$ also $\forall x \in [0, \frac{1}{n}]: g_n(x) \geq 0 \Leftrightarrow x \geq p_n(x)$

Wegen der Symmetrie von p_n gilt $\forall x \in [-\frac{1}{n}, \frac{1}{n}]: |x| \geq p_n(x)$

daher ist $\left|\mathbb{1}_{[-\frac{1}{n},\frac{1}{n}]}(u(x)) \cdot (|u(x)| - p_n(u(x)))^2\right| \leq |u(x)|^2$ integrierbare Majorante daher

$$\lim_{n\to\infty} \int_\Omega \mathbb{1}_{[-\frac{1}{n},\frac{1}{n}]}(u(x)) \left(|u(x)| - p_n(u(x))\right)^2 d\lambda^n(x) = \int_\Omega \underbrace{\lim_{n\to\infty} \mathbb{1}_{[-\frac{1}{n},\frac{1}{n}]}(u(x))}_{=0} \left(|u(x)| - p_n(u(x))\right)^2 d\lambda^n(x) = 0$$

$f_n \circ u$ Cauchy-Folge in H^1 und $f_n \circ u \to f\circ u$ in $L^2(\Omega)$ $\Rightarrow$ $f_n \circ u \to f \circ u$ in $H^1(\Omega)$

$n < m$

$$\|\partial_i(f_n \circ u) - \partial_i(f_m \circ u)\|^2_{L^2(\Omega)} = \|f_n' \circ u \, \partial_i u - f_m' \circ u \, \partial_i u\|^2_{L^2(\Omega)} = \int_\Omega \left(f_n'(u(x)) - f_m'(u(x))\right)^2 (\partial_i u(x))^2 d\lambda^d(x)$$

$$= \int_\Omega \left(f_n'(u(x)) - f_m'(u(x))\right)^2 (\partial_i u(x))^2 d\lambda^d(x) = \int_\Omega \mathbb{1}_{[-\frac{1}{n},\frac{1}{n}]}(u(x)) \left(p_n'(u(x)) - f_m'(u(x))\right)^2 (\partial_i u(x))^2 d\lambda^d(x)$$

$p_n(x) = x^2\left(-\frac{n^3}{2}x^2 + \frac{3n}{2}\right) \Rightarrow p_n'(x) = \frac{6n}{2}x - \frac{4n^3}{2}x^3 = 3nx - 2n^3x^3$

$\frac{1}{m} \leq x \leq \frac{1}{n}$

$h_n(x) := \underbrace{1}_{=f_m'(x)} - p_n'(x) = 1 - 3nx + 2n^3x^3 \leq 1 + 2n^3x^3 \leq 3$

$p_n'(x) - 1 = 3nx - 2n^3x^3 - 1 \leq 3nx \leq 3$

$0 \leq x \leq \frac{1}{m}$

$k_{n,m}(x) := p_m'(x) - p_n'(x) = 3mx - 2m^3x^3 - 3nx + 2n^3x^3 = 2x^3\underbrace{(n^3 - m^3)}_{\leq 0} + 3x\underbrace{(m-n)}_{\geq 0} \leq 3\frac{m-n}{m} = 3\left(1 - \frac{n}{m}\right) \leq 3$

$p_n'(x) - p_m'(x) = \ldots = 2x^3(m^3 - n^3) + 3x(n-m) \leq 2x^3(m^3 - n^3) \leq 2\frac{m^3 - n^3}{m^3} \leq 2 \leq 3$

$$\int_\Omega \mathbb{1}_{[-\frac{1}{n},\frac{1}{n}]}(u(x)) \left(p_n'(u(x)) - f_n'(u(x))\right)^2 (\partial_i u(x))^2 \, d\lambda^d(x) \leq 9 \int_\Omega \underbrace{\mathbb{1}_{[-\frac{1}{n},\frac{1}{n}]}(u(x)) \, (\partial_i u(x))^2}_{\leq (\partial_i u(x))^2} \, d\lambda^d(x) < \varepsilon \text{ für } n \text{ hinr. groß}$$

Majorante

$$\max\{u,0\} = \frac{u+0+|u-0|}{2} \in H^1(\Omega)$$

$$\min\{u,0\} = \frac{u+0-|u-0|}{2} \in H^1(\Omega)$$