(1) **A left-turn lane problem**

A civil engineer is studying a left-turn lane that is long enough to hold six cars. Let X be the number of cars in the lane at the end of a randomly chosen red light. The engineer believes that the probability that $X = x$ is proportional to $(x+1)(7-x)$.

(a) Find the probability mass function (pmf) of X.

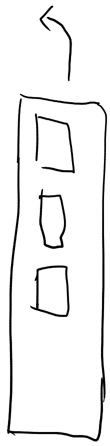
(b) Compute the probability that X will be at least 4.

(c) Calulate the expectation and standard deviation of X.
Note: `R` might be useful.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$$



a) $f(x) = \mathbb{P}(X=x) = a\,(x+1)(7-x)$

$$1 \overset{!}{=} \sum_{x=0}^{6} f(x) = a \sum_{x=0}^{6} (x+1)(7-x) = a\left(\sum_{x=0}^{6} 7 + \sum_{x=0}^{6} 6x - \sum_{x=0}^{6} x^2\right)$$

$$= a\left(49 + 6\,\frac{7\cdot 6}{2} - \frac{6\cdot 7\cdot 13}{6}\right)$$

$$= a\left(49 + 6\cdot 21 - 7\cdot 13\right) = a\left(49 + 126 - 91\right)$$

$$= a\,84 \Rightarrow a = \frac{1}{84}$$

b) $$\mathbb{P}(X \geq 4) = 1 - \mathbb{P}(X<4) = 1 - \sum_{k=0}^{3} \mathbb{P}(X=k) = 1 - \frac{1}{84}\left(\sum_{k=0}^{3} 7 + \sum_{k=0}^{3} 6k - \sum_{k=0}^{3} k^2\right)$$

$$= 1 - \frac{1}{84}\left(28 + 6\,\frac{3\cdot 4}{2} - \frac{3\cdot 4\cdot 7}{6}\right) = 1 - \frac{1}{84}\left(28 + 36 - 14\right) = 1 - \frac{1}{84}\,50 = 1 - \frac{25}{42} = \frac{17}{42}$$

c) $$\mathbb{E}(X) = \sum_{k=0}^{6} k\,\mathbb{P}(X=k) = \frac{1}{84}\sum_{k=0}^{6} k(k+1)(7-k) = \frac{1}{84}\left(7\sum_{k=0}^{6} k + 6\sum_{k=0}^{6} k^2 - \sum_{k=0}^{6} k^3\right)$$

$$= \frac{1}{84}\left(7\,\frac{6\cdot 7}{2} + 6\,\frac{6\cdot 7\cdot 13}{6} - 21^2\right) = \frac{1}{84}\left(7\cdot 21 + 2\cdot 13\cdot 21 - 21^2\right)$$

$$= \frac{21}{84}\left(7 + 26 - 21\right) = \frac{21}{84}\,12 = \frac{7}{28}\,12 = \frac{7}{14}\,6 = \frac{6}{2} = 3$$

$$\mathbb{V}(X) = \mathbb{E}\left(\left(X - \mathbb{E}(X)\right)^2\right) = \mathbb{E}\left((X-3)^2\right) = \sum_{k=0}^{6} (k-3)^2\,\mathbb{P}(X=k) = 3 \ldots$$ calculation in R

Thus, $s = \sqrt{\mathbb{V}(X)} = \sqrt{3}$

(2) **Basketball free throws**

Two professional basketball players, Tom and John, each throw ten free throws with a basketball. Tom makes 80% of the free throws he tries, while John makes 85% of the free throws he tries.

(a) What is the probability that the number of free throws that Tom will make is exactly 7?

(b) What is the probability that the number of free throws that John will make is at least 8?

(c) Player who achives the highest score wins the game. It is assumed that the two players do not influence each other when throwing. What is the probability that neither Tom or John will win the game?

Hint: Use R-function `dbinom()` to calculate the probability mass functions.

X ... number of throws Tom makes out of 10

Y ... number of throws John makes out of 10

a) $$\mathbb{P}(X=7) = \binom{10}{7}\left(\frac{8}{10}\right)^7\left(\frac{2}{10}\right)^3 = \frac{10!}{7!\cdot 3!}\left(\frac{8}{10}\right)^7\left(\frac{2}{10}\right)^3 = \frac{10\cdot 9\cdot 8}{3\cdot 2}\left(\frac{8}{10}\right)^7\left(\frac{2}{10}\right)^3 = 15\,\frac{8^8}{10^7}\,\frac{8}{10^3}$$

$$= \frac{15\cdot 8^9}{10^{10}} = \frac{15}{10}\left(\frac{8}{10}\right)^9 = \frac{3}{2}\left(\frac{4}{5}\right)^9 \approx 0{,}2$$

b) $$\mathbb{P}(X\geq 8) = \mathbb{P}(X=8)+\mathbb{P}(X=9)+\mathbb{P}(X=10) = \binom{10}{8}\left(\frac{85}{100}\right)^8\left(\frac{15}{100}\right)^2 + \binom{10}{9}\left(\frac{85}{100}\right)^9\left(\frac{15}{100}\right) + \binom{10}{10}\left(\frac{85}{100}\right)^{10}$$

$$= \frac{10\cdot 9}{2}\left(\frac{17}{20}\right)^8\left(\frac{3}{20}\right)^2 + 10\left(\frac{17}{20}\right)^9\frac{3}{20} + \left(\frac{17}{20}\right)^{10}$$

$$= \left(\frac{17}{20}\right)^8\left(5\cdot 9\cdot\frac{9}{20\cdot 20} + 10\cdot\frac{17}{20}\,\frac{3}{20} + \frac{17^2}{20^2}\right)$$

$$= \left(\frac{17}{20}\right)^8\left(\frac{81}{80} + \frac{51}{40} + \frac{17^2}{20^2}\right) \approx 0{,}82$$

c) $$\mathbb{P}(X=Y) = \sum_{k=0}^{10}\mathbb{P}(X=k)\,\mathbb{P}(Y=k) \approx 0{,}23$$

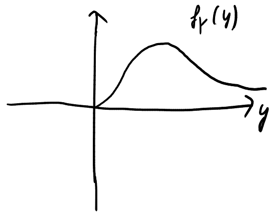
(3) **Uniform-exponential relationship**

(a) Let Y be an exponential random variable $Y \sim \exp(\lambda)$, i.e. its pdf is given by

$$f_Y(y) = \begin{cases} \lambda\, e^{-\lambda y}, & y \geq 0 \\ 0, & \text{else} \end{cases}$$

and its mean equals $\frac{1}{\lambda}$. Compute $P(Y > y)$.

(b) Let X be a random variable, uniformlly distributed on $(0, 1)$. Find the cumulative distribution function of X. What is the distribution of $Z = -\log X$?



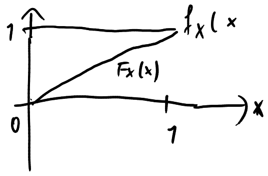
$z := \max\{y, 0\}$

a) $\mathbb{P}(Y > y) = \int_z^\infty \lambda e^{-\lambda \eta}\, d\eta = \lambda \int_z^\infty e^{-\lambda \eta}\, d\eta = \lambda \left[-\frac{1}{\lambda} e^{-\lambda \eta} \right]_{\eta = z}^{\infty}$

$\left(-\frac{1}{\lambda} e^{-\lambda \eta}\right)' = e^{-\lambda \eta}$

$= \lambda \frac{1}{\lambda} e^{-\lambda z} = \begin{cases} 1, & \text{if } y \leq 0 \\ e^{-\lambda y}, & \text{else} \end{cases}$

b)



$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x > 1 \\ x, & \text{else} \end{cases}$

$\mathbb{P}(Z \leq z) = \mathbb{P}(-\log(X) \leq z) = \mathbb{P}(X \geq e^{-z}) = 1 - \mathbb{P}(X < e^{-z}) = 1 - e^{-z} = F_Z(z)$

$f_Z(z) = F_Z'(z) = e^{-z}$ Hence, $Z \sim \exp(1)$

(4) **Hurricane insurance**

An insurance company needs to asses the risk associated with providing hurricane insurance. During 22 years from 1990 through 2011, Florida was hit by 27 major hurricanes (level 3 and above). The insurance company assumed Poisson distribution for modeling number of hurricanes.

(a) If hurricanes are independent and the mean has not changed, what is the probability of having a year in Florida with each of the following?

(1) No hits.

(2) Exactly one hit.

(3) More than two hits.

(b) Use `R` to estimate the number of hurricane hits that will occur with the probability 99.5%.

Hint: One of the following R-commands: `dpois()`, `ppois()`, `qpois()`, `rpois()` is applicable.

a) X ... number of Hurricanes hitting Florida in a year

$X \sim \mathcal{P}\left(\frac{27}{22}\right),\ \lambda := \frac{27}{22} \qquad \forall k \in \mathbb{N}_0 : p_X(k) = \mathbb{P}(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$

(1) $\mathbb{P}(X=0) = e^{-\lambda} \approx 0{,}29$

(2) $\mathbb{P}(X=1) = \lambda e^{-\lambda} \approx 0{,}36$

(3) $\mathbb{P}(X>2) = 1 - \sum_{k=0}^{2} \mathbb{P}(X=k) = 1 - \sum_{k=0}^{2} \frac{\lambda^k}{k!} e^{-\lambda} \approx 0{,}13$

b) $\mathbb{P}(X \le n) = \frac{995}{1000} = \frac{199}{200} \Leftarrow n \geq 5$, mit qpois$\left(\frac{199}{200}, \lambda\right)$

(5) **Drug company**

Manufacturing and selling drugs that claim to reduce an individual's cholesterol level is big business. A company would like to market their drug to women if their cholesterol is in the top 15%. Assume the cholesterol levels of adult American women can be desribed by a Normal model with a mean of 188 mg/dL and a standard deviation of 24 mg/dL.

(a) Use `R` to draw and label the Normal model.

(b) What percent of adult women do you expect to have cholesterol levels over 200 mg/dL?

(c) What percent of adult women do you expect to have cholesterol levels between 150 mg/dL and 170 mg/dL?

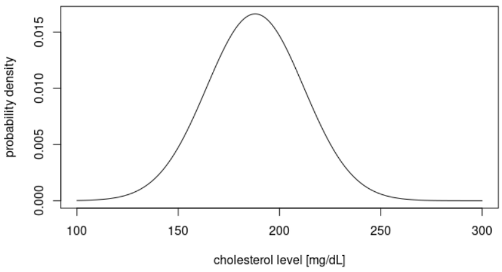
(d) Calculate the interquartile range of the cholesterol levels. Recall, the interquartile range is the diference between upper and lower quartile, i.e.

$$IQR = x_{0.75} - x_{0.25}.$$

(e) Above what value are the highest 15% of women's cholesterol levels?

Hint: If using `R` for all computations the following commands `pnorm()`, `qnorm()` and `dnorm()` are useful. Otherwise values from Table of standard Normal distribution should be used.

a)



b) X... Cholesterol level in mg/dL

$\mathbb{P}(X > 200) = 1 - \mathbb{P}(X \leq 200) \approx 0{,}31$ pnorm(200, μ, σ, lower.tail = FALSE)

c) $\mathbb{P}(150 < X < 170) = 1 - \mathbb{P}(X \leq 150 \vee 170 \leq X) = 1 - \mathbb{P}(X \leq 150) - \mathbb{P}(X \geq 170) \approx 0{,}17$

d) $IQR \approx 32{,}38 \ \frac{mg}{dL}$ qnorm($\frac{75}{100}$, μ, σ) - qnorm($\frac{25}{100}$, μ, σ)

e) above $212.87 \ \frac{mg}{dL}$ qnorm($\frac{15}{100}$, μ, σ, lower.tail = FALSE)