The problems of this homework are to be presented on **April 20, 2021**. Students should tick in TUWEL problems they have solved and are prepared to present their detailed solutions. The problems should be ticked and solution paths uploaded by **23:59 on April 19, 2021**.

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(1) **The mean of independent normal distributions**

(a) Show that the moment generating function (mgf) of $X \sim \mathcal{N}(\mu, \sigma^2)$ is of the form

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

(b) Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and let $Y = aX + b$ with fixed real constants a and b. Show that $Y \sim \mathcal{N}(a\mu + b,\, a^2\sigma^2)$.

(c) Let $X_1, \dots X_n$ be independent identically distributed random variables with $X_1 \sim \mathcal{N}(\mu, \sigma^2)$. Show that the mean $\bar{X} = \frac{1}{n}(X_1 + \cdots + X_n)$ is also normally distributed and $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$.

(2) **Sum of two independent distributions**

(a) Let $X \sim \mathcal{P}(\lambda_1)$ and $Y \sim \mathcal{P}(\lambda_2)$ be two independent Poisson random variables. Show that

$$X + Y \sim \mathcal{P}(\lambda_1 + \lambda_2).$$

(b) Let U and V be two independent random variables with exponential distribution $\exp(\lambda)$. Show that

$$U + V \sim Gamma(2, \lambda) \qquad \text{and}$$
$$\min\{U, V\} \sim \exp(2\lambda).$$

Hint: It is useful to use moment generating functions. Recall, the pdf of a random variable $X \sim Gamma(\alpha, \beta)$ is

$$f(x) = \begin{cases} \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\, \beta^\alpha} & x > 0 \\ 0, & x \leq 0 \end{cases},$$

and its mgf is of the form $\left(\frac{1}{1-\beta t}\right)^\alpha$ for $t < \frac{1}{\beta}$. Particularly, the pdf of a random variable $X \sim \exp(\lambda) = Gamma(1, \frac{1}{\lambda})$ is of the form

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x \leq 0 \end{cases}.$$

(3) **Real roots**
Let A, B and C be independent random variables, uniformly distributed on $(0,1)$.

(a) What is the probability that the qudratic equation $Ax^2 + Bx + C = 0$ has real roots?

(b) Consider the following code in R.
What does it do and how is it related to your solution in part (a)?

```
n=10000
a=runif(n)
b=runif(n)
c=runif(n)
sum(b^2>4*a*c)/n
```

Hint: In HW2/ex. 3(b) we showed that if X has uniform $(0,1)$ distribution then $-\log X$ has exponential distribution $\exp(1)$. In an analogue way, one can prove that $-s \log X \sim \exp(\frac{1}{s})$ for any $s > 0$. Also, in HW4/ex. 2(b) we proved that the sum of two independent exponential distributions is a gamma distribution. Namely, if $X \sim \exp(1)$ and $Y \sim \exp(1)$ are independent then $X + Y \sim Gamma(2,1)$.

(4) **Sum and average**
Let X be a random variable with $\mathcal{N}(5, 2^2)$. Let $X_1, X_2, \ldots, X_{50}$ be independent identically distributed copies of X. Let S be their sum and $\bar{X}$ their average, i.e.

$$S = X_1 + \cdots + X_{50} \qquad \text{and} \qquad \bar{X} = \frac{1}{50}(X_1 + \cdots + X_{50}).$$

(a) Plot the density and the distribution function for X using R.

(b) What are the expectation and the standard deviation of S and of $\bar{X}$?

(c) Generate a sample of 50 numbers from $\mathcal{N}(5, 2^2)$. Plot the histogram for this sample.
Do the same for a sample of 500 numbers from $\mathcal{N}(5, 2^2)$.

(5) **Central Limit Theorem**
Let $\bar{X}_1$ and $\bar{X}_2$ be the means of two independent samples of size n from the same population with variance σ^2. Use the Central limit theorem to find a value for n so that

$$P(|\bar{X}_1 - \bar{X}_2| < \frac{\sigma}{50}) \approx 0.99.$$

Justify your calculations.