EXERCISES FOR "PARTIAL DIFFERENTIAL EQUATIONS" WS 2020
EXERCISE SHEET 10 (03. 12. 2020)

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1. *(Inhomogeneous boundary data)* Consider the initial value problem (IVP)

$$\begin{cases} u_t = u_{xx} - u + \sin(3\pi x) + x & \text{for } x \in (0,1)\,, \quad t > 0\,, \\ u(0,t) = 0 & \text{for } t > 0\,, \\ u(1,t) = 1 & \text{for } t > 0\,, \\ u(x,0) = \sin(\pi x) + x & \text{for } x \in (0,1). \end{cases}$$

i) Transform the IVP into a problem with homogeneous boundary data.
ii) Discuss the existence of a solution for the transformed problem.
iii) Find a complete orthonormal system consisting of eigenfunctions of the operator $Lu = -u_{xx} + u$ with suitable boundary conditions.
iv) Find a solution of the IVP using the orthonormal system from *iii)*.

2. *(Periodic Sobolev Spaces)* Let $\Omega = (0, 2\pi)$ and consider the complete orthonormal system of $L^2(\Omega)$ given by

$$\left\{ C_0 = \frac{1}{\sqrt{2\pi}},\, C_n(x) = \frac{1}{\sqrt{\pi}}\cos(nx),\, S_n(x) = \frac{1}{\sqrt{\pi}}\sin(nx) \,\middle|\, n \in \mathbb{N} \right\}.$$

i) Show that for $k \in \mathbb{N}$ the space

$$H^k_{per}(\Omega) := \left\{ f \in H^k(\Omega) \mid f^{(j)}(0) = f^{(j)}(2\pi) \text{ for } j = 0, \dots, k-1 \right\}$$

is a well-defined Hilbert space.

ii) Show that $f \in H^1_{per}(\Omega)$ if and only if

$$f = \sum_{m=1}^{\infty} a_m S_m + \sum_{m=0}^{\infty} b_m C_m \quad \text{with} \quad \sum_{m=1}^{\infty} m^2 \left(|a_m|^2 + |b_m|^2 \right) < \infty.$$

In this case, f can be differentiated "term-wise".

iii) For $n \in \mathbb{N}$ consider the projection

$$P_n \colon H^k_{per}(\Omega) \to H^k_{per}(\Omega),$$
$$f = \sum_{m=1}^{\infty} a_m S_m + \sum_{m=0}^{\infty} b_m C_m \mapsto P_n f = \sum_{m=1}^{n} a_m S_m + \sum_{m=0}^{n} b_m C_m.$$

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Show that for $f \in H^k_{per}(\Omega)$ it holds that

$$\|f - P_n f\|_{L^2(\Omega)} \leq \frac{1}{(n+1)^k} \|f^{(k)}\|_{L^2(\Omega)}.$$

3. *(Smoothing properties of the heat operator)* Let $\Omega = (0, 2\pi)$ and consider the initial value problem (IVP)

$$\begin{cases} u_t = u_{xx} & \text{for} (x,t) \in \Omega \times (0, \infty)\,, \\ u(x, t=0) = u_0 & \text{for } x \in \Omega\,, \\ u(x=0,t) = u(x=2\pi, t) & \text{for } t > 0\,, \end{cases}$$

with $u_0 \in L^2(\Omega)$. The set $\{\phi_n(x) = e^{\mathrm{i}nx} : n \in \mathbb{Z}\}$ is a complete orthogonal system in the complex Hilbert space $L^2(\Omega)$, corresponding to the eigenfunctions of the operator $Lu = -u_{xx}$ with periodic boundary conditions. We now consider the operator

$$e^{-Lt} : L^2(\Omega) \mapsto L^2(\Omega), \quad v \mapsto \sum_{n\in\mathbb{Z}} e^{-n^2 t} \left\langle v, \frac{\phi_n}{\|\phi_n\|} \right\rangle_{L^2} \frac{\phi_n}{\|\phi_n\|}.$$

In what sense is $u(\cdot, t) = e^{-Lt} u_0$ a solution of IVP? Furthermore, show that:

i) for $t \geq 0$ it holds that $\|e^{-Lt}\|_{L^2 \to L^2} = 1$;

ii) for $t > 0$ it holds that $\|e^{-Lt}\|_{L^2 \to H^1_{per}} \leq C(1 + t^{-1/2})$ for some constant $C > 0$;

iii) for $t > 0$ it holds that $\|e^{-Lt}\|_{L^2 \to H^k_{per}} \leq C_k(1 + t^{-a_k})$ for all $k \in \mathbb{N}$ and $a_k, C_k > 0$.

Notice that here $\|e^{-Lt}\|_{X \to Y}$ denotes the operator-norm of e^{-Lt} as an operator from X to Y.

4. *(Galerkin method for the Poisson equation)* Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with smooth boundary. For $f \in L^2(\Omega)$ construct a solution of the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1}$$

using a Galerkin method. To do this, let $\{\phi_k\}$ for $k \in \mathbb{N}$ denote the eigenfunctions of the Laplacian with homogeneous Dirichlet boundary data on Ω. Then prove that for any $m \in \mathbb{N}$ there exists

$$u_m = \sum_{k=1}^{m} \mathbf{d}^k_m \phi_k, \quad \text{for } \mathbf{d}^k_m \in \mathbb{R}$$

that satisfies

$$\int_\Omega \nabla u_m \cdot \nabla \phi_k \,\mathrm{d}x = \int_\Omega f \phi_k, \quad \text{for } k = 1, \dots, m.$$

To finish, show that the sequence $\{u_m\}_{m\in\mathbb{N}}$ converges weakly in $H^1_0(\Omega)$ to a weak solution of (1).

5. *(Exponential decay to equilibrium for the Fokker-Planck equation)* We consider smooth solutions of the Fokker-Planck equation

$$\partial_t u = \nabla \cdot (\nabla u + u \nabla V), \quad t > 0, \quad u(0) = u_0 \quad \text{in } \mathbb{R}^d, \tag{2}$$

where we assume that V satisfies the Bakry-Emery condition $\nabla^2 V \geq \lambda \mathrm{Id}$ for some $\lambda > 0$. Furthermore, assume that $\phi \in C^4((0,\infty))$ is convex with $\phi(1) = 0$, and $1/\phi''$ is well-defined and concave. Examples of admissible functions ϕ are $\phi(s) = s(\log(s) - 1) + 1$ and $\phi(s) = s^\alpha - 1$ for $1 < \alpha \leq 2$.

i) Compute u_∞, a stationary solution of (2) that is strictly positive ($u_\infty > 0$) and has unit mass ($\int_{\mathbb{R}^d} u_\infty = 1$).

ii) Define the relative entropy with respect to ϕ as

$$H_\phi[u] = \int_{\mathbb{R}^d} \phi\Big(\frac{u}{u_\infty}\Big) u_\infty \,\mathrm{d}x.$$

Notice that, setting $\rho = \frac{u}{u_\infty}$, we have that $\partial_t u = \nabla \cdot (u_\infty \nabla \rho)$. Using this, show that the *entropy production*, $-\frac{d}{dt} H_\phi[u]$, is nonnegative.

iii) Show that

$$\begin{aligned}\nabla \partial_t \rho &= \nabla \Delta \rho - \nabla^2 \rho \nabla V - \nabla^2 V \nabla \rho,\\ \nabla \rho \cdot \nabla \Delta \rho &= \nabla \cdot (\nabla^2 \rho \nabla \rho) - |\nabla^2 \rho|^2,\end{aligned}$$

where $|\nabla^2 \rho|^2 = \sum_{i,j=1,\dots,n} |\partial_i \partial_j \rho|^2$. Using these identities and the expression for $\frac{d}{dt} H_\phi[u]$ that you have obtained in *i*), show that

$$\begin{aligned}\frac{d^2}{dt^2} H_\phi[u] &\geq \int_{\mathbb{R}^d} \big(\phi''''(\rho)|\nabla \rho|^4 + 4\phi'''(\rho) \nabla \rho^T \nabla^2 \rho \nabla \rho + 2\phi''(\rho)|\nabla^2 \rho|^2\big) u_\infty \,\mathrm{d}x\\ &\quad - 2\lambda \frac{d}{dt} H_\phi[u]\\ &= 2\int_{\mathbb{R}^d} \phi''(\rho) \Big|\nabla^2 \rho + \frac{\phi'''(\rho)}{\phi''(\rho)} \nabla \rho \otimes \nabla \rho\Big|^2 u_\infty \,\mathrm{d}x\\ &\quad + \int_{\mathbb{R}^d} \Big(\phi''''(\rho) - 2\frac{\phi'''(\rho)^2}{\phi''(\rho)}\Big) |\nabla \rho|^4 u_\infty \,\mathrm{d}x\\ &\quad - 2\lambda \frac{d}{dt} H_\phi[u].\end{aligned}$$

iv) Using the concavity of $1/\phi''$ and convexity of ϕ, argue that the result of *iii*) yields that

$$\frac{d^2}{dt^2} H_\phi[u] \geq -2\lambda \frac{d}{dt} H_\phi[u], \qquad t > 0. \tag{3}$$

v) Argue that integrating (3) on the interval (s,∞) yields that

$$\frac{d}{dt} H_\phi[u(s)] \leq -2\lambda H_\phi[u(s)], \qquad s \geq 0.$$

Hint: For this use Gronwall's lemma applied to (3) and you may use (without proof) that $\lim_{t\to\infty} H_\phi[u(t)] = 0$.

vi) Using (without proof) that

$$\|u(t) - u_\infty\|_{L^1(\mathbb{R}^d)} \leq \frac{2}{\phi''(1)} H_\phi[u(t)],$$

which follows from the Csiszár-Kullback-Pinsker inequality, show that

$$\|u(t) - u_\infty\|^2_{L^1(\mathbb{R}^d)} \leq \frac{1}{\phi''(1)} H_\phi[u_0] e^{-2\lambda t}.$$