(1) **Cramér-Rao lower bound - Simulation**
In Homework 7 Exercise 2 a density $f(x|\theta) = \theta x^{\theta-1}$ for $0 < x < 1$ and $\theta > 0$ was given. The goal was to find a sutable function $g$ of the parameter $\theta$ such that there exists an unbiased estimator of $g(\theta)$ which attains the Cramér-Rao lower bound.

A unbiased statistic which attains the Cramér-Rao lower bound is for $g(\theta) = \frac{1}{\theta}$ given by

$$S_n(X_1, ..., X_n) = -\frac{1}{n}\sum_{i=1}^{n} \ln(X_i).$$

Implement the following steps in `R`:

(a) Write pdf `dhw`, cdf `phw`, quantile `qhw` and random sampling function `rhw` for the above distribution parameterized by $\theta$ (see for example `?runif, ?rnorm`).
*Hint: Given an strict monotone continuous cdf $F$, then $F^{-1}(U)$ is distributed with cdf $F$ for $U \sim U(0,1)$.*

(b) Fix an arbitrary $\theta$ and perform a simulation with growing sample size $n$ = 500, 1000, 1500, ..., 10000 each with 100 replications for the estimation of $g(\theta)$ with the statistic $S_n$.

(c) Create a scatter plot of all the estimates over the sample size, add the sample mean and standard deviation aggregated over the sample size to the plot. Finally, add the theoretical mean and standard deviation of the statistic $S_n$.

a) $\theta > 0$; $f_\theta(x) = \theta x^{\theta-1} \mathbb{1}_{(0,1)}(x)$, hence $F_\theta(x) = \begin{cases} 0, & \text{if } x \le 0 \\ x^\theta, & \text{if } 0 < x < 1 \\ 1, & \text{if } 1 \le x \end{cases}$

Consider any two numbers $x, p \in (0,1)$. We have $F_\theta(x) = p \Leftrightarrow x^\theta = p \Leftrightarrow x = p^{\frac{1}{\theta}}$.
Hence, the Quantile function $Q_\theta : [0,1] \to [0,1]$ is given by $Q_\theta(p) := p^{\frac{1}{\theta}}$

c) We know from last weeks Homework, that $\mathbb{E}(S_n) = \frac{1}{\theta}$ and $\mathbb{V}ar(S_n) = \frac{1}{n\theta^2}$.