The problems of this homework are to be presented on **May 4, 2021**. Students should tick in TUWEL problems they have solved and are prepared to present their detailed solutions. The problems should be ticked and solution paths uploaded by **23:59 on May 3, 2021**.

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(1) **Method of moment estimator**
Let $X_1, \dots, X_n$ be a random sample from a population with pdf

$$f(x) = \begin{cases} \frac{\theta\, x^{\theta-1}}{3^\theta}, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

where $\theta \in \mathbb{R}^+$ is unknown parameter.

(a) Show that the method of moments estimator for θ is $T_n = \frac{\bar{X}}{3-\bar{X}}$.

(b) Find the limiting distribution of $\frac{T_n - \theta}{\frac{1}{\sqrt{n}}}$ as $n \to \infty$.

(2) **Box of candles**
There are blue and red candles in a box. Probability that a randomly chosen candle is blue is $\frac{1}{1+2a}$, for $a > 0$. Based on a sample of sample size n, find the maximum likelihood estimator (MLE) $\hat{a}$ of the parameter a.

(3) **Point estimator statistics: Comparison**
Let $X_1 \dots X_n$ be i.i.d. uniform $(0, \theta)$, with unknown parameter $\theta > 0$.

(a) Show that the method of moments estimator of θ is $2\bar{X}$ and the MLE of θ is $X_{(n)} = \max_{1 \le i \le n} X_i$.

(b) Compare the mean square errors of the two estimators. Which of the estimators should be preferred if any? Explain your reasoning.

(4) **Unbiased estimators**
Let $\hat{a}$ and $\hat{b}$ be unbiased estimators of unknown parameters a and b respectively.

(a) Check if $\alpha\hat{a} + \beta\hat{b}$ is an unbiased estimator of the parameter $\alpha\, a + \beta\, b$, where $\alpha, \beta \in \mathbb{R}$.

(b) Is $\hat{a}^2$ an unbiased estimator of a^2?

(c) Based on the following measurements of a side of a square (in milimeters)

$$15, 17, 16, 16, 17, 14$$

find an unbiased estimator of the area.

(5) **Rayleigh distribution**

Let $X_1, \ldots, X_n$ be a random sample with Rayleigh distribution

$$f(x|\theta) = \begin{cases} \frac{x}{\theta^2}\, e^{-\frac{x^2}{2\theta^2}}, & x \geq 0 \\ 0, & x < 0 \end{cases},$$

where $\theta > 0$ is unknown.

(a) Find the method of moments estimator of θ.

(b) Find the MLE of θ and its asymptotic variance.

Hint: Show that the first two moments are $\mathbb{E}X = \theta\sqrt{\frac{\pi}{2}}$ and $\mathbb{E}\,X^2 = 2\theta^2$.