(3) **Minimum variance estimator**

Let $W_1, \dots, W_k$ be unbiased estimators of a parameter θ with $\mathbb{V}ar = \sigma_i^2$ and $\mathbb{C}ov(W_i, W_j) = 0$ if $i \neq j$. Show that, of all estimators of the form $\sum a_i W_i$ where a_is are constant and $\mathbb{E}_\theta\Big(\sum a_i W_i\Big) = \theta$, the estimator

$$W^* = \frac{\sum W_i/\sigma_i^2}{\sum(1/\sigma_i^2)}$$

has minimum variance. Show that

$$\mathbb{V}ar\, W^* = \frac{1}{\sum(1/\sigma_i^2)}.$$

For $\theta \neq 0$, we have $\mathbb{E}\left(\sum_{i=1}^{k} a_i W_i\right) = \theta = \sum_{i=1}^{k} a_i \mathbb{E}(W_i) = \theta \sum_{i=1}^{k} a_i \Rightarrow \sum_{i=1}^{k} a_i = 1$,

hence we obtain for all $\theta \in \mathbb{R}$

$$\mathbb{E}\left(\left(\sum_{i=1}^{k} a_i W_i\right)^2\right) = \sum_{i=1}^{k}\sum_{j=1}^{k} a_i a_j \mathbb{E}(W_i W_j)$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{k} a_i a_j \Big(\underbrace{\mathbb{E}\big((W_i-\theta)(W_j-\theta)\big)}_{= Cov(W_i, W_j)} + \theta \underbrace{\mathbb{E}(W_i)}_{=\theta} + \theta \underbrace{\mathbb{E}(W_j)}_{=\theta} - \theta^2\Big)$$

$$= \theta^2 \sum_{i=1}^{k}\sum_{j=1}^{k} a_i a_j + \sum_{i=1}^{k} a_i^2 \sigma_i^2 = \theta^2 + \sum_{i=1}^{k} a_i^2 \sigma_i^2$$

For $\theta \neq 0$, we define $b_i := a_i - \left(\sigma_i^2 \sum_{j=1}^{k} \frac{1}{\sigma_j^2}\right)^{-1}$, which fulfill $\sum_{i=1}^{k} b_i = 0$ and obtain

$$\mathbb{V}ar\left(\sum_{i=1}^{k} a_i W_i\right) = \mathbb{E}\left(\left(\sum_{i=1}^{k} a_i W_i\right)^2\right) - \left(\mathbb{E}\left(\sum_{i=1}^{k} a_i W_i\right)\right)^2 = \sum_{i=1}^{k} a_i^2 \sigma_i^2$$

$$= \sum_{i=1}^{k}\left(b_i + \left(\sigma_i^2 \sum_{j=1}^{k}\frac{1}{\sigma_j^2}\right)^{-1}\right)^2 \sigma_i^2$$

$$= \sum_{i=1}^{k} b_i^2\sigma_i^2 + 2\sum_{i=1}^{k} b_i \sigma_i^2 \left(\sigma_i^2 \sum_{j=1}^{k}\frac{1}{\sigma_j^2}\right)^{-1} + \sum_{i=1}^{k}\sigma_i^2\left(\sigma_i^2\sum_{j=1}^{k}\frac{1}{\sigma_j^2}\right)^{-2}$$

$$= \sum_{i=1}^{k} b_i^2\sigma_i^2 + 2\left(\sum_{j=1}^{k}\frac{1}{\sigma_j^2}\right)^{-1}\underbrace{\sum_{i=1}^{k} b_i}_{=0} + \left(\sum_{i=1}^{k}\frac{1}{\sigma_i^2}\right)^{-1}$$

$$= \sum_{i=1}^{k} b_i^2\sigma_i^2 + \left(\sum_{i=1}^{k}\frac{1}{\sigma_i^2}\right)^{-1} \geq \left(\sum_{i=1}^{k}\frac{1}{\sigma_i^2}\right)^{-1} = \sum_{i=1}^{k}\left(\sigma_i^2\sum_{j=1}^{k}\frac{1}{\sigma_j^2}\right)^{-2}\sigma_i^2$$

$$= \mathbb{V}ar(W^*)$$

For $\theta = 0$, the values $a_i = 0$ are valid and W^* does not have minimum Variance