CompMath: LaTeX-Übung 1

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Theorem (Brezzi 1974)**.** *Let X and Y be Hilbert spaces. Further, let $a : X \times X \to \mathbb{R}$ and $b : X \times Y \to \mathbb{R}$ be continuous bilinear forms and $X_0 \coloneqq \{x \in X : b(x, \cdot) = 0 \in Y^*\}$. Under the assumptions*

- $\alpha \coloneqq \inf\limits_{v \in X_0 \setminus \{0\}} \frac{a(v,v)}{\|v\|_X^2} > 0$, *i.e., $a(\cdot, \cdot)$ is coercive in X_0,*
- $\beta \coloneqq \inf\limits_{\substack{y \in Y \\ y \neq 0}} \sup\limits_{\substack{x \in X \\ x \neq 0}} \frac{b(x,y)}{\|x\|_X \|y\|_Y} > 0$

there holds the assertion: For each $(x^*, y^*) \in X^* \times Y^*$ there is a unique solution $(x, y) \in X \times Y$ of the so-called saddle point problem

$$\begin{aligned} a(x, \tilde{x}) + b(\tilde{x}, y) &= x^*(\tilde{x}) \textit{ for all } \tilde{x} \in X, \\ b(\tilde{x}, y) &= y^*(\tilde{y}) \textit{ for all } \tilde{y} \in Y. \end{aligned} \tag{SP}$$

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The Gamma function is defined as

$$\Gamma(x) \coloneqq \lim_{n \to \infty} \frac{n!\, n^x}{x(x+1) \cdots (x+n)}.$$

There holds the Weierstraß product representation

$$\frac{1}{\Gamma(x)} = x \cdot e^{Cx} \cdot \prod_{k=1}^{\infty} \left(1 + \frac{x}{k}\right) e^{-x/k} \text{ where } C \coloneqq \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right).$$

3

For given ***basis*** $b \in \mathbb{N}_{\geq 2}$, ***mantissa length*** $t \in \mathbb{N}$ and ***exponential bounds*** $e_{\min} < 0 < e_{\max}$ we define the set of ***normalized floating point numbers*** $F \coloneqq F(b, t, e_{\min}, e_{\max}) \subset \mathbb{R}$ by

$$F = \{0\} \cup \left\{ \left(\sigma \sum_{k=1}^{t} a_k b^{-k} \right) b^e \,\middle|\, \sigma \in \{\pm 1\}, a_j \in \{0, \ldots, b-1\}, a_1 \neq 0, e \in \mathbb{Z}, e_{\min} \leq e \leq e_{\max} \right\}.$$

The finite sum $a = \sum_{k=1}^{t} a_k b^{-k}$ is called ***normalized mantissa*** of a floating point number.

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For $q \in \mathbb{R}$, it holds that

$$\lim_{n\to\infty} q^n = \begin{cases} +\infty & \text{iff } q > 1, \\ 1 & \text{iff } q = 1, \\ 0 & \text{iff } -1 < q < 1, \\ \nexists & \text{iff } q \leq -1. \end{cases}$$

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$$V(x_1, \ldots, x_n) \coloneqq \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix} \in \mathbb{R}^{n\times n}$$

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In general,

$$\det V(x_1, \ldots, x_n) = \prod_{1\leq i<j\leq n} (x_j - x_i).$$

For $n = 6$, Maple gives us

$$\begin{aligned} \det V(x_1, \ldots, x_6) = \; & (x_5 - x_6) \\ & (x_4 - x_6)\,(x_4 - x_5) \\ & (x_3 - x_6)\,(x_3 - x_5)\,(x_3 - x_4) \\ & (x_2 - x_6)\,(x_2 - x_5)\,(x_2 - x_4)\,(x_2 - x_3) \\ & (x_1 - x_6)\,(x_1 - x_5)\,(x_1 - x_4)\,(x_1 - x_3)\,(x_1 - x_2)\,. \end{aligned}$$

7

Theorem. *The matrix $L \in \mathbb{R}^{n\times n}$ has the following form*

$$L = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix}$$

with $L_{11} \in \mathbb{R}^{k\times k}$ and $0 < k < n$. If L_{11} and L_{22} are regular, then L is regular as well, and the inverse is given by

$$L^{-1} = \begin{pmatrix} L_{11}^{-1} & 0 \\ -L_{22}^{-1}L_{21}L_{11}^{-1} & L_{22}^{-1} \end{pmatrix}.$$

Proof. Because L_{11}, L_{22} are regular, L_{11}^{-1}, L_{22}^{-1} and thus, L^{-1} are well-defined.
To verify that L^{-1} is indeed the correct inverse of L, we multiply block wise

$$L^{-1} \cdot L = \begin{pmatrix} L_{11}^{-1} & 0 \\ -L_{22}^{-1} L_{21} L_{11}^{-1} & L_{22}^{-1} \end{pmatrix} \cdot \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} E_k & 0 \\ 0 & E_{n-k} \end{pmatrix} = E_n.$$

This calculation shows that $L \in GL_n(\mathbb{R})$.

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Theorem. *Let $n \in \mathbb{N}, A, B \subset \mathbb{R}^n$ be open intervals with compact closure $\overline{A}, \overline{B}$ and $A \cap B = \emptyset$. We define the boundary of the sets as $\partial A := \overline{A} \backslash A$ and $\partial B := \overline{B} \backslash B$.*
Then, there holds for the distances of the two sets that dist$(A, B) =$ dist$(\partial A, \partial B)$, where we define for arbitrary sets $C, D \subset \mathbb{R}^n$

$$\textit{dist}(C, D) := \inf \left\{ \|x - y\|_2 : x \in C, y \in D \right\}. \tag{1}$$

Proof. We can write

$$A = \prod_{k=1}^{n} (a_k^-, a_k^+), B = \prod_{k=1}^{n} (b_k^-, b_k^+), \overline{A} = \prod_{k=1}^{n} [a_k^-, a_k^+] \text{ and } \overline{B} = \prod_{k=1}^{n} [b_k^-, b_k^+],$$

for bounded intervals $A, B \in \mathbb{R}^n$.
We notice, that $\text{dist}(A, B) = \text{dist}(\overline{A}, \overline{B})$:

"$\leq$" Because $\overline{A}, \overline{B}$ are compact and $\|x - y\|_2$ continuous, the set in (1) is compact and $\text{dist}(\overline{A}, \overline{B}) = \min \left\{ \|x - y\|_2 : x \in \overline{A}, y \in \overline{B} \right\}$. Let $x \in \overline{A}, y \in \overline{B}$, be such that $\text{dist}(\overline{A}, \overline{B}) = \|x - y\|_2$.
However, one can find $\tilde{x} \in A, \tilde{y} \in B$, sufficiently close to x, y. That is, for arbitrary $\epsilon > 0$, let $\|x - \tilde{x}\|_2, \|y - \tilde{y}\|_2 \leq \frac{\epsilon}{2}$. Then

$$\begin{aligned} &\Rightarrow \text{dist}(A, B) \leq \|\tilde{x} - \tilde{y}\|_2 \leq \|\tilde{x} - x\|_2 + \|x - y\|_2 + \|y - \tilde{y}\|_2 \leq \frac{\epsilon}{2} + \text{dist}(\overline{A}, \overline{B}) + \frac{\epsilon}{2} \\ &\Rightarrow \forall \epsilon > 0 : \text{dist}(A, B) \leq \text{dist}(\overline{A}, \overline{B}) + \epsilon. \end{aligned}$$

"$\geq$" This is trivial, because $A \subseteq \overline{A}, B \subseteq \overline{B}$.

In order to see, that $\text{dist}(A, B) = \text{dist}(\overline{A}, \overline{B}) = \text{dist}(\partial A, \partial B)$, we notice, that $x \in \partial A, y \in \partial B$.
If this were not the case, then w.l.o.g.

$$\exists \ell \in \{1, \ldots, n\} : x_\ell \in (a_\ell^-, a_\ell^+) \Rightarrow \exists \epsilon > 0 : K_\epsilon(x_\ell) \subseteq (a_\ell^-, a_\ell^+).$$

But, considering $\epsilon^2 < 2|x_\ell - y_\ell|\epsilon$, we would get the contradiction

$$\begin{aligned} \|x - y\|_2^2 - \sum_{\ell \neq k=1}^{n} |x_k - y_k|^2 &= |x_\ell - y_\ell|^2 \\ &> |(x_\ell - y_\ell) - \text{sgn}(x_\ell - y_\ell)\epsilon|^2 \\ &= |x_\ell - y_\ell|^2 - 2|x_\ell - y_\ell|\epsilon + \epsilon^2. \end{aligned}$$

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