Difference Equations with MATLAB

Case Study: Logistic Equation

Repetition: Difference Equation

- Problems defined by
$$x_{n+1} = f(n, x_n, x_{n-1}, \dots, x_{n-d})$$
$$x_0 = k$$
are called difference-equations.
- Solution of these equations is given by a sequence of, probably vector-valued, numbers x_n with a certain initial value k.

Repetition: Connection between Difference E. and Differential E.

Mathematical Modelling and Simulation

- $x_{n+1} = f(n, x_n, x_{n-1}, \dots, x_{n-d}) \Rightarrow$
$$x_{n+1} - x_n = g(n, x_n, x_{n-1}, \dots, x_{n-d})$$

Difference!

Solutions of difference equations are gained by the <u>sum of all differences</u> starting at a specific value!

Solutions of differential equations are gained by the <u>sum of all infinitesmial differencials</u> starting at a specific value! In this case, the sum is called <u>integral</u>!

Repetition: Connection between Difference E. and Differential E.

Mathematical Modelling and Simulation

A solution of a difference equation is a sequence. We receive a value for each interation step!
$\{0,1,2,\ldots,n\}$
This is usually called explicit representation of the sequence in contrast to a recursive one.

A solution of a differential equation is a „very infinite“ sequence“. We receive a value for **each** timepoint
$[0, t_{end}]$
Those kind of „sequences“ are usually called **functions**!

Repetition: Connection between Difference E. and Differential E.

Mathematical Modelling and Simulation

We differ between linear and nonlinear difference equations. E.g.:
Linear: $x_{n+1} = 4x_n + 2$
Nonlinear: $x_{n+1} = x_n^2 + x_n$

We differ between linear and nonlinear differentiale equations. E.g.:
Linear: $x' = 3x + 2$
Nonlinear: $x' = x^2$

Repetition: Connection between Difference E. and Differential E.

Mathematical Modelling and Simulation

We can perform a z-Transformation

$$x_{n+1} - x_n = 3x_n + 2$$
$$a(z) = \frac{2}{\frac{1}{z} - 3}$$

We can perform a Laplace-Transformation

$$x' = 3x + 2$$
$$\mathrm{t}(s) = \frac{2}{\frac{1}{s} - 3}$$

Repetition: Connection between Difference E. and Differential E.

Finding a explicit solution is usually very tricky! Sometimes comparisons with geometric sequences can lead to sucess.

Anyway values can be calculated directly through the recursive formula.

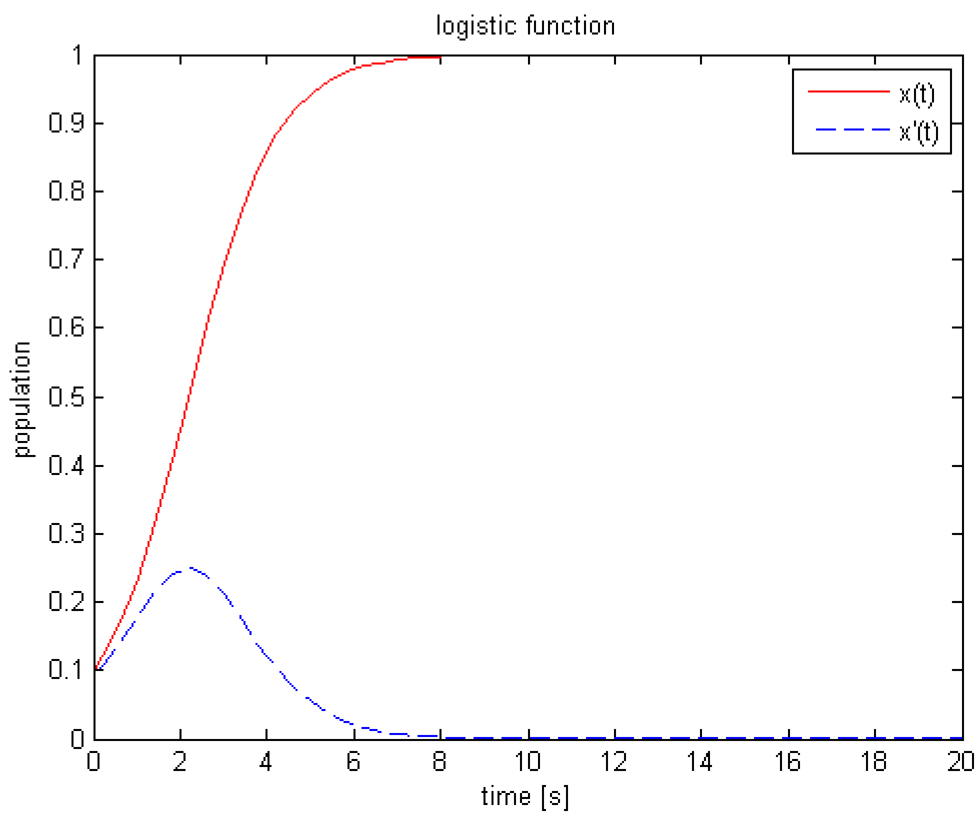
Finding an analytic solution can be performed with analytical methods. If no solutions can be found this way a numerical approximation method needs to be used usually leading to difference equations.

Repetition: Comparison Logistic Difference Equation and Logistic Differential Equation

- Logistic differential equation is given by
$$x' = ax(b - x)$$
- The corresponding logistic-difference equation is given by
$$x_{n+1} = x_n + ax_n(b - x_n)$$

Repetition: Comparison Logistic Difference Equation and Logistic Differential Equation

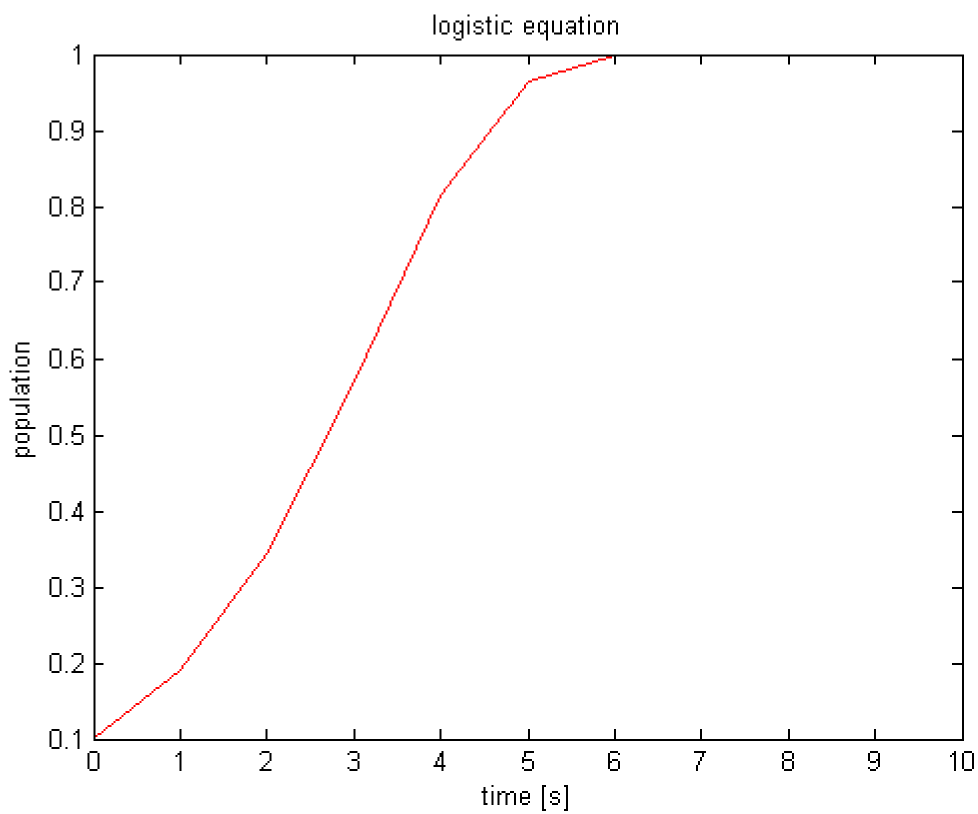
Solutions of the logistic differential equation are steady, and behave similar for all parameters.



Repetition: Comparison Logistic Difference Equation and Logistic Differential Equation

Mathematical Modelling and Simulation

Solutions of the logistic difference equation are unsteady and seem to differ extremely for different parameters



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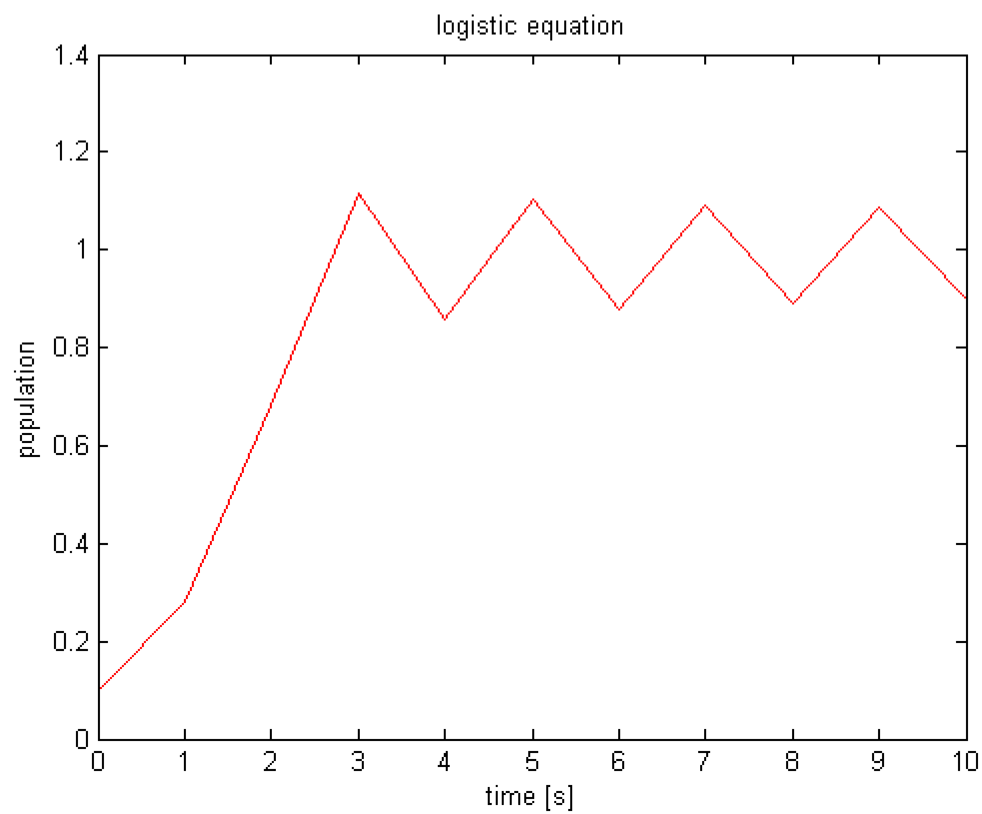


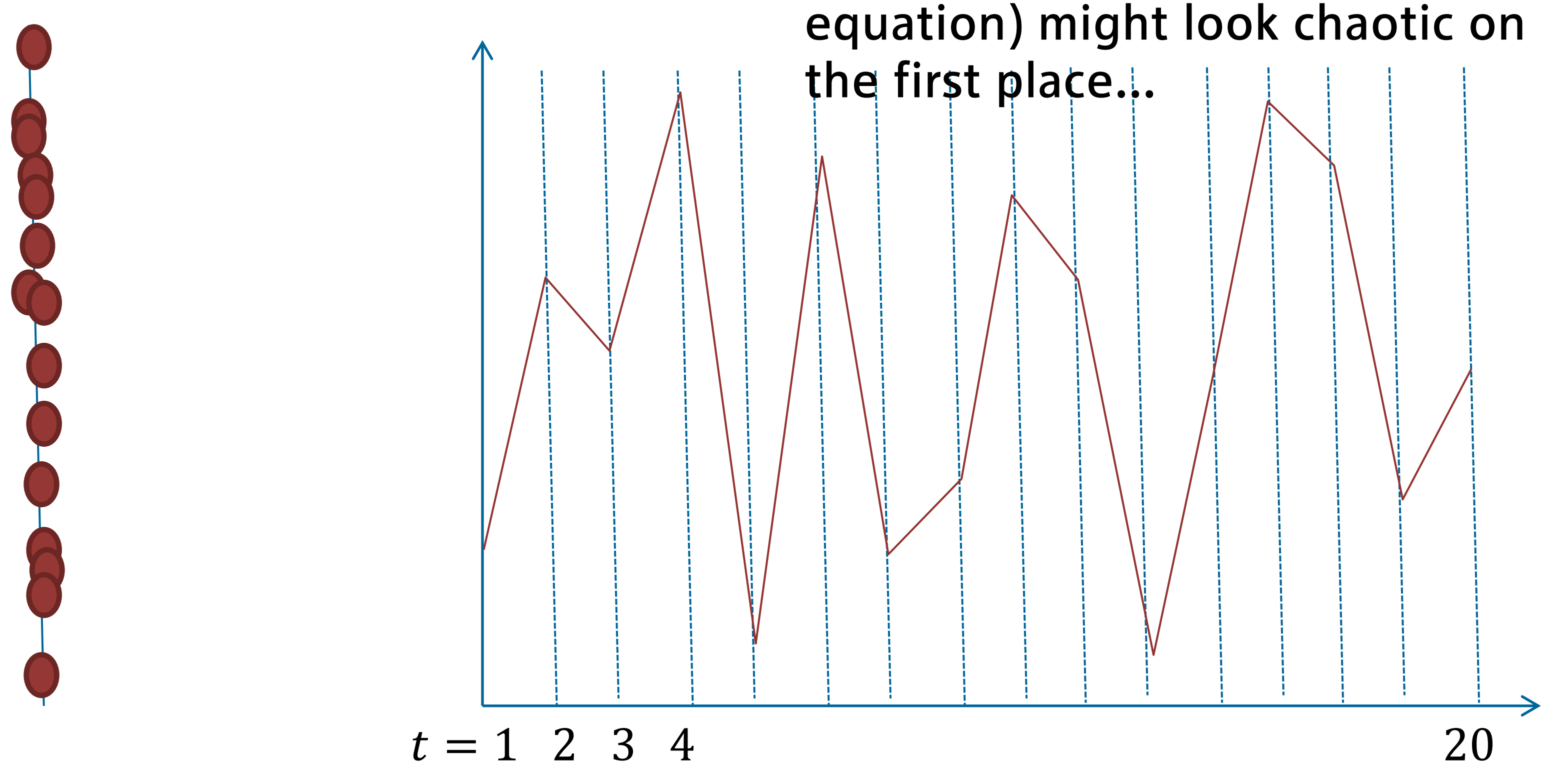
Conclusion

Mathematical Modelling and Simulation

difference equations are a lot more than just discrete versions of differential equations!

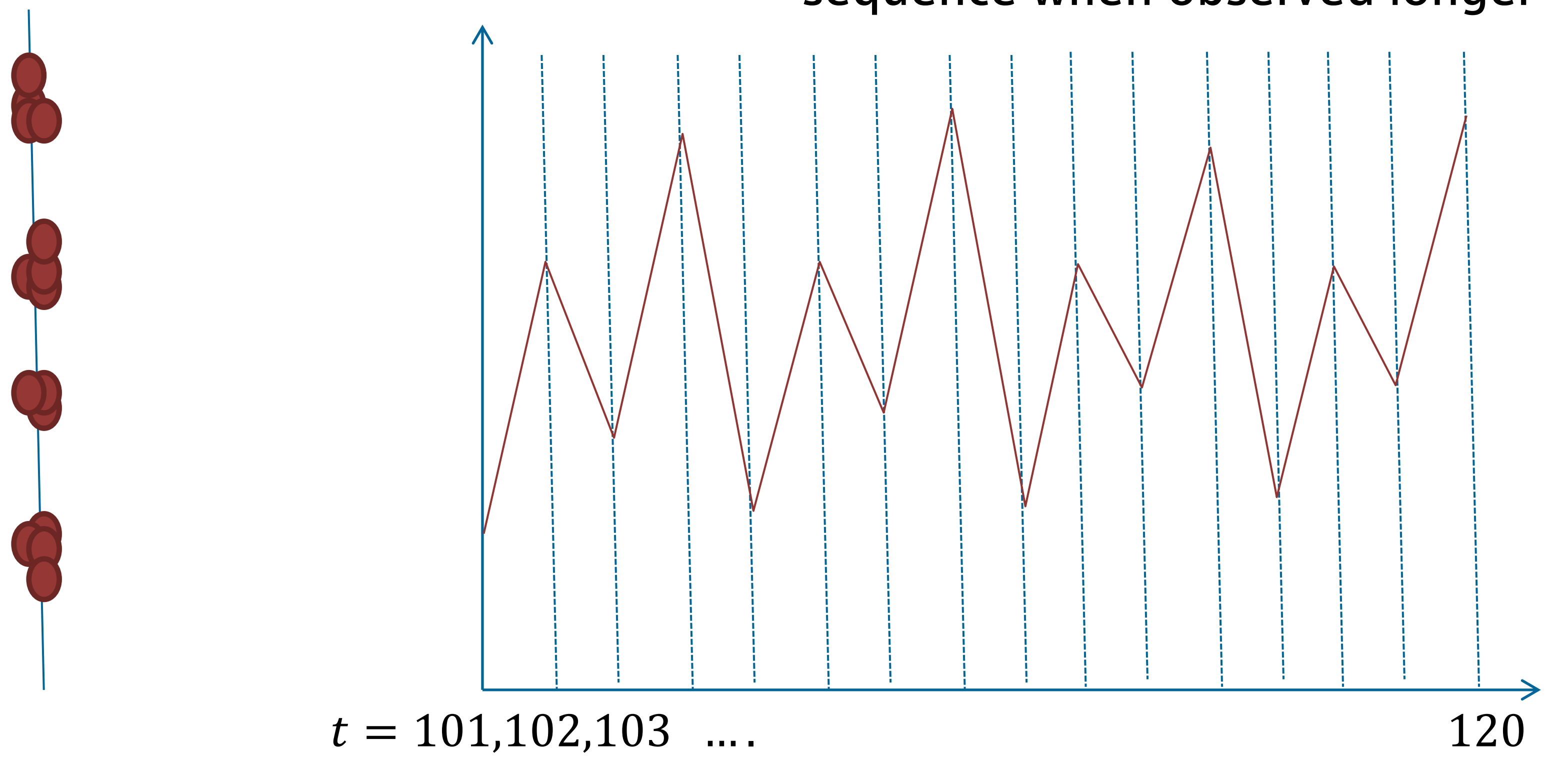
- What is an accumulation point?

Although a sequence (i.e. a solution of a difference equation) might look chaotic on the first place…



- **What is an accumulation point?**

... one might observe a „convergence“ to a periodic sub-sequence when observed longer



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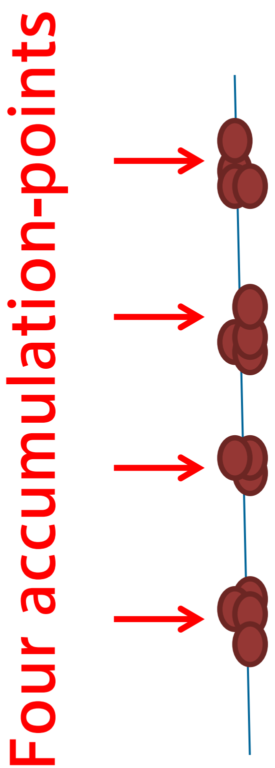
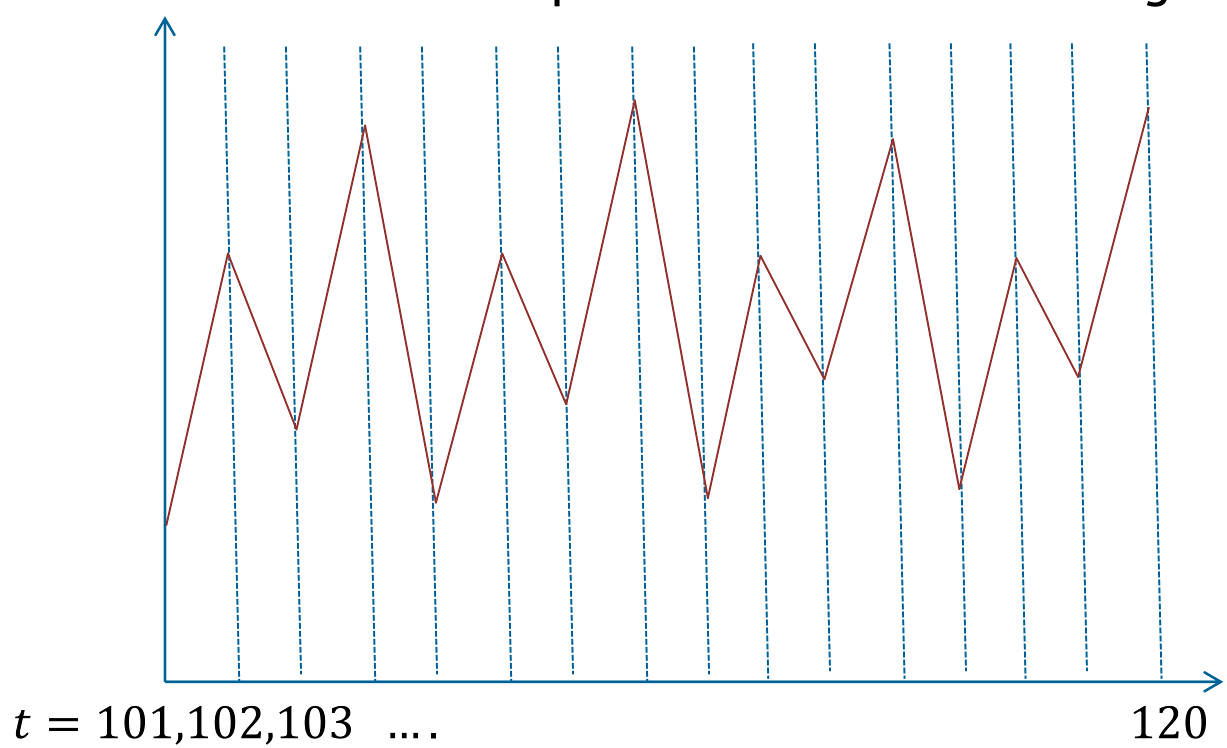


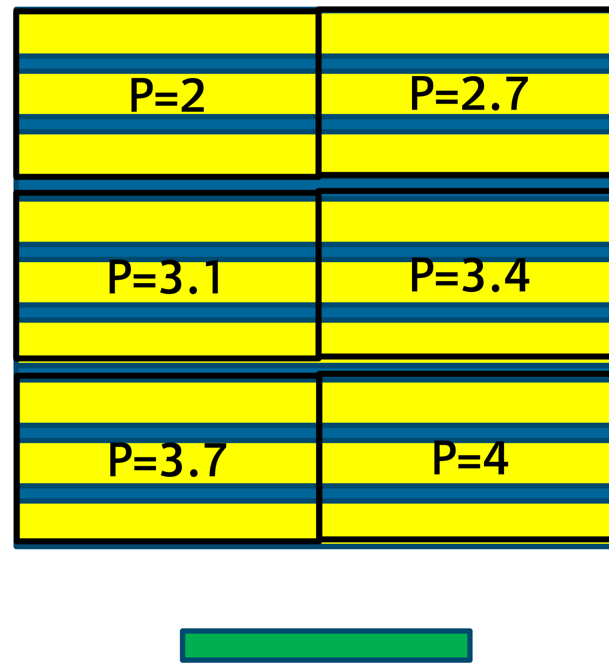


$$x_{n+1} = p x_n (1 - x_n)$$

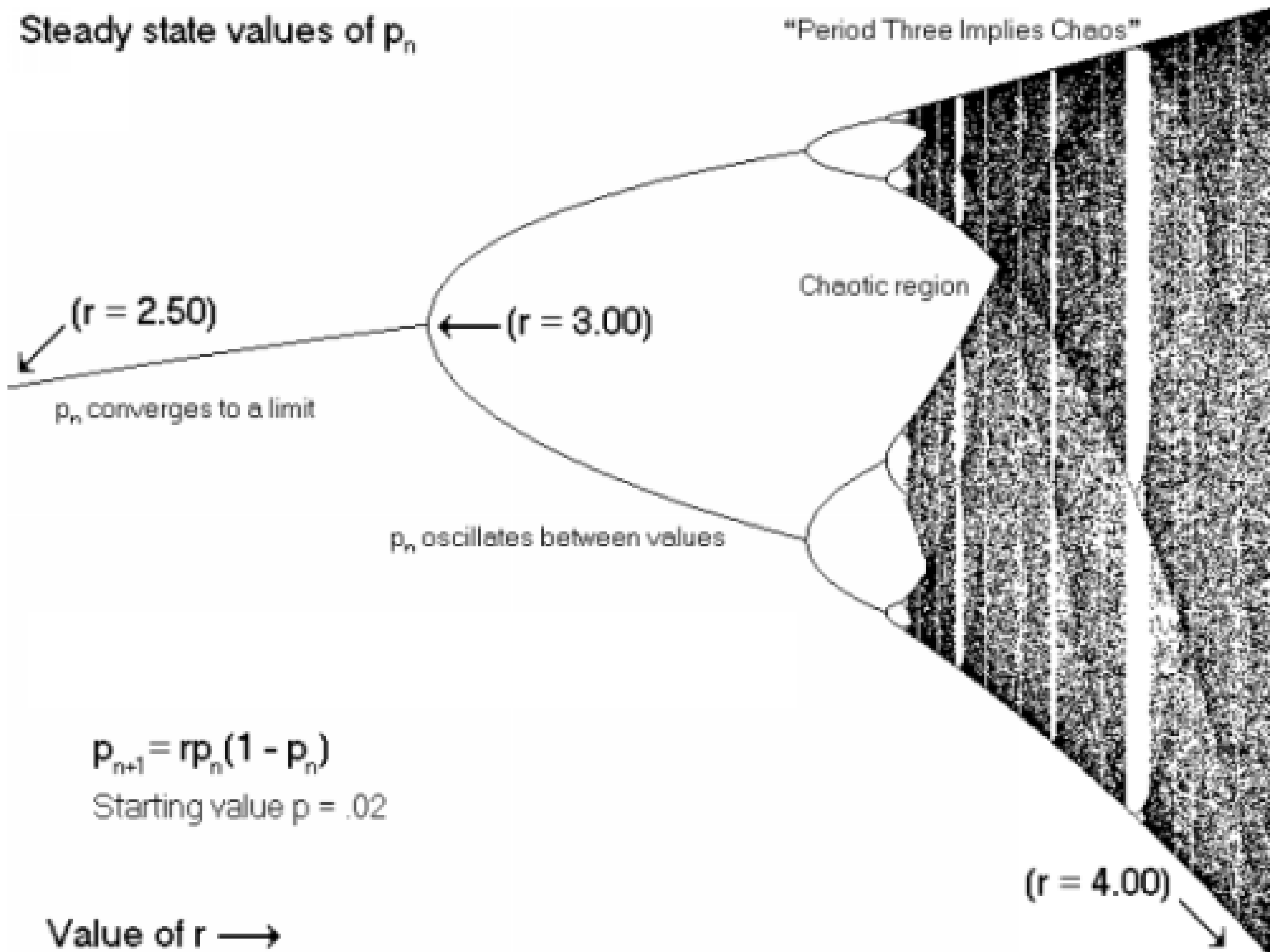
Experiments with MATLAB/Simulink

Mathematical Modelling and Simulation

How many accumulation points??



Bifurcation



Case Study: „Baby Planner"
Problem Definition

Mathematical Modelling and Simulation

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- Costs per week after birth are approximated with 150€.
- Income of the couple is saved with interest rate of 0.1%/month.

Research Question:

Mathematical Modelling and Simulation

Does the money last for 18 years?

Difference Equation Model

Mathematical Modelling and Simulation

- We observe that the type of the recursion depends on the division of the index by 4
- $x_{n+1} = x_n - 150$, if $n \equiv 1(4)$ or $n \equiv 3(4)$
- $x_{n+1} = x_n - 150 - 1150$, if $n \equiv 2(4)$
- $x_{n+1} = (x_n - 150) \cdot (1 + \frac{0.1}{100}) + 1700$, if $n \equiv 0(4)$

Implementation in Simulink

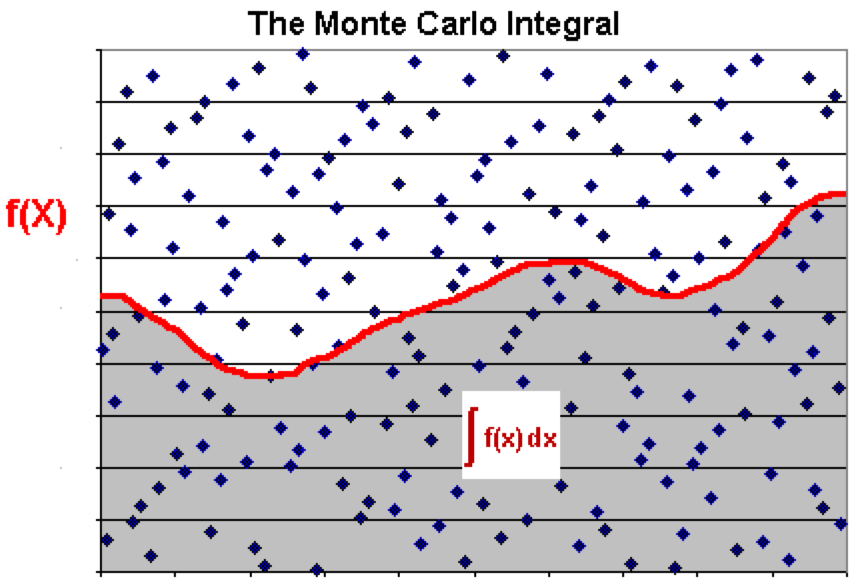
Adaption of the Model

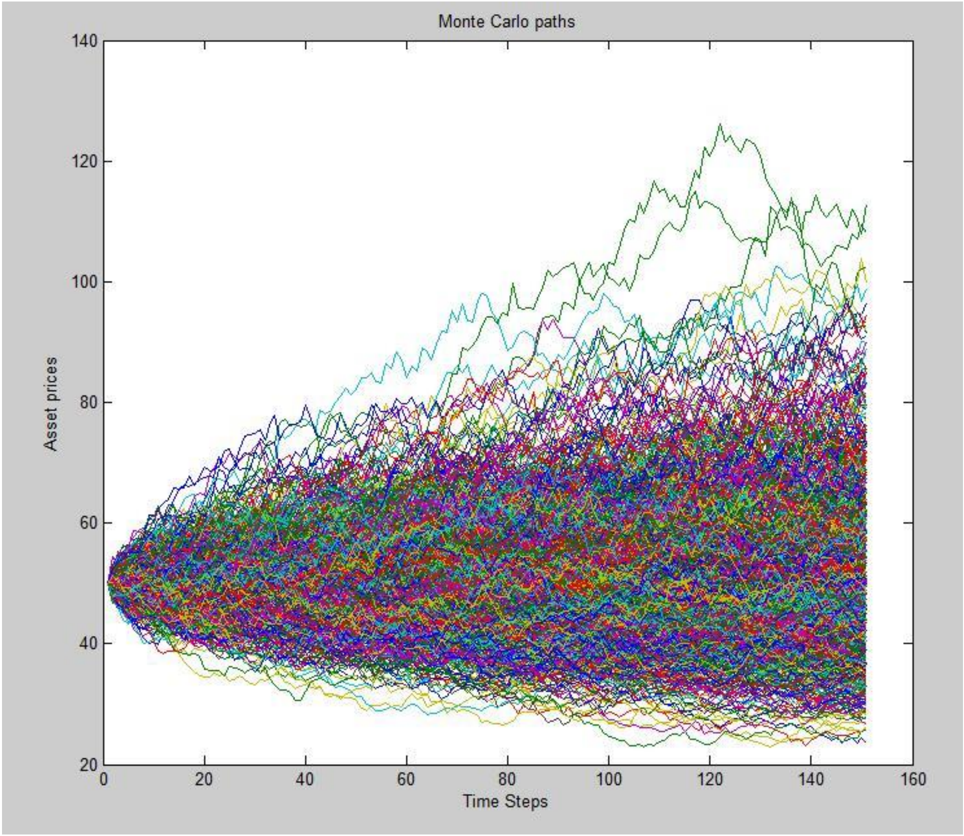
Mathematical Modelling and Simulation

- Unfortunately the anount of money spent each week is not known perfectly.
- We introduce a random variable making the simulation stochastic. This raises new questions:

Can I expect that the money will last for 18 years?
How confident is this assumption?
Variance? Mean? Quantiles?

Monte Carlo Simulation





Buffon's Needle Problem