(1) **The GLRT for the normal variance - simple hypotheses**
Derive the generalized likelihood ratio test (GLRT) for the normal variance: Assume $X_1, \dots, X_n$ are iid $\mathcal{N}(\mu, \sigma^2)$, where both μ and σ are unknown. We want to test

$$H_0 : \sigma^2 = \sigma_0^2 \quad vs \quad H_1 : \sigma^2 \neq \sigma_0^2.$$

The likelihood function is given by $L(\mu, \sigma; x) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2\right)$

The MLEs are $\hat{\mu} = \bar{x}$ and $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$, hence

$$L(\hat{\mu}, \hat{\sigma}; x) = (2\pi\hat{\sigma}^2)^{-n/2} \exp\left(-\frac{n}{2}\right)$$

Under the null hypothesis we have $\hat{\sigma}_0^2 = \sigma_0^2$ and $\hat{\mu}_0 = \bar{x}$

The GLR is

$$\lambda(x) = \frac{L(\hat{\mu}, \hat{\sigma}, x)}{L(\hat{\mu}_0, \hat{\sigma}_0, x)} = (2\pi\hat{\sigma}^2)^{-n/2} \exp\left(\frac{-n}{2}\right) (2\pi\sigma_0^2)^{n/2} \exp\left(\frac{1}{2\sigma_0^2}\sum_{i=1}^n (x_i - \bar{x})^2\right)$$

$$= \left(\frac{\sigma_0^2}{\hat{\sigma}^2}\right)^{n/2} \exp\left(\frac{1}{2\sigma_0^2}\sum_{i=1}^n (x_i - \bar{x})^2 - \frac{n}{2}\right)$$

$$= \left(\frac{\sigma_0^2}{\sigma^2}\right)^{n/2} \exp\left(\frac{1}{2n}\sum_{i=1}^n \left(\frac{x_i - \bar{x}}{\sigma_0/\sqrt{n}}\right)^2 - \frac{n}{2}\right)$$

We define $T(x) := \sum_{i=1}^n \left(\frac{x_i - \bar{x}}{\sigma_0/\sqrt{n}}\right)$ and obtain, that λ is an increasing function of $T(x)$.

We have $\frac{X_i - \bar{X}}{\sigma_0/\sqrt{n}} \sim \mathcal{N}(0,1)$, hence $\sum_{i=1}^n \left(\frac{x_i - \bar{x}}{\sigma_0/\sqrt{n}}\right)^2 \sim \chi^2(n)$

(2) **Most powerful test 1**
Let $X_1, \ldots, X_n$ be iid Uniform$(0, \theta)$.

(a) Derive the most powerful (MP) test at level α for testing

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta = \theta_1, \theta_1 > \theta_0.$$

(b) Calculate the power of the MP test.

a) $L(\theta; x) = \begin{cases} \frac{n}{\theta}, & \text{if } \forall i \in \{1, \ldots, n\}: x_i \in (0, \theta) \\ 0, & \text{else} \end{cases}$, Therefore we obtain for $x \in (0, \theta_1)^n$

$$\lambda(X) = \frac{L(\theta_1, x)}{L(\theta_0, x)} = \begin{cases} \frac{\theta_0}{\theta_1}, & \text{if } \max\{x_i \mid 1 \leq i \leq n\} < \theta_0 \\ 0, & \text{else} \end{cases}$$

By NP Lemma, the MP test of level α rejects H_0, if $\lambda(x) \geq C$ with C satisfying $\alpha = \mathbb{P}(\lambda(X) \geq C)$

The rejection region Ω_1 is $\Omega_1 = \{x \in (0, \theta_1)^n \mid \lambda(x) \geq C\}$

$$= \{x \in (0, \theta_0)^n \mid \theta_0 \geq C\theta_1\} \cup \{x \in (0, \theta_1)^n \setminus (0, \theta_0)^n \mid 0 \geq C\}$$

b) The power q is defined as $q = \mathbb{P}(X \in \Omega_1 \mid H_0 \text{ is false}) =$

(3) **Most powerful test 2**
Let $X_1, \dots, X_n$ be iid from a distribution with density

$$f_\theta(x) = \frac{x}{\theta} e^{-\frac{x^2}{2\theta}},\ x \geq 0,\ \theta > 0.$$

(a) Derive the MP test at level α for testing two simple hypoheses

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta = \theta_1,\ \theta_1 > \theta_0.$$

(b) Is there a uniformly most powerful (UMP) test at level α for testing the one-sided composite hypothesis

$$H_0 : \theta \leq \theta_0 \quad vs \quad H_1 : \theta > \theta_0$$

What is its power function?
Hint: Show $X_i^2 \sim \exp(1/2\theta)$, so that $\sum_i X_i^2 \sim \theta\chi^2(2n)$.

a)
$$L(\theta; x) = \begin{cases} \frac{1}{\theta^n} \prod_{i=1}^n x_i \exp\left(-\frac{1}{2\theta} \sum_{i=1}^n x_i^2\right), & \text{if } \min\{x_i \mid 1 \leq i \leq n\} \geq 0 \\ 0, & \text{else} \end{cases}$$

For $x \in (\mathbb{R}^+)^n$ we have

$$\lambda(x) = \frac{L(\theta_1; x)}{L(\theta_0; x)} = \left(\frac{\theta_0}{\theta_1}\right)^n \exp\left(\left(\frac{1}{2\theta_0} - \frac{1}{2\theta_1}\right) \sum_{i=1}^n x_i^2\right)$$

Since $\theta_1 > \theta_0$, we obtain that the function $\lambda(x)$ is a monotone increasing function of $T(x) = \sum_{i=1}^n x_i^2$

H_0 is rejected if $T(x) > C$, where $\mathbb{P}(T(X) \geq C) = \alpha$

We have $T(X) \sim \text{Gamma}(n, \frac{1}{2\theta})$, hence for $C \geq 0$

$$\mathbb{P}(T(X) \geq C) = \int_C^\infty \left(\frac{1}{2\theta}\right)^n \frac{1}{\Gamma(n)} x^{n-1} e^{-\frac{x}{2\theta}} dx = \left(\frac{1}{2\theta}\right)^n \frac{1}{(n-1)!} \int_C^\infty x^{n-1} e^{-\frac{x}{2\theta}} dx$$

$C = \theta_0 \chi^2_\alpha(2n)$ $\quad$ def $\Omega_1 := \{x \in (\mathbb{R}^+)^n \mid T(x) \geq C\}$

b) the power q is given by

$$\mathbb{P}(X \in \Omega_1 \mid H_0 \text{ is false}) = \int_{\Omega_1} \frac{1}{\theta^n} \prod_{i=1}^n x_i \exp\left(-\frac{1}{2\theta} \sum_{i=1}^n x_i^2\right) dx$$

$$= \int_{\Omega_1} -\prod_{i=1}^n \frac{d}{dx_i} \left(\exp\left(-\frac{1}{2\theta} \sum_{i=1}^n x_i^2\right)\right) dx = 1 - F_{\chi^2}\left(\frac{C}{\theta}\right)$$

We define $Y_i := X_i^2$, and have

$$f_{Y_i}(y) = f_{X_i}(\sqrt{y}) \frac{1}{2\sqrt{y}} + \underbrace{f_{X_i}(-\sqrt{y})}_{=0} \frac{1}{2\sqrt{y}} = \frac{\sqrt{y}}{\theta} e^{-\frac{y}{2\theta}} \frac{1}{2\sqrt{y}} = \frac{1}{2\theta} e^{-\frac{y}{2\theta}}$$

Thus $Y_i \sim \exp\left(\frac{1}{2\theta}\right)$. We know that $\sum_{i=1}^n Y_i \sim \text{Gamma}\left(n, \frac{1}{2\theta}\right)$

$\chi^2(\nu) = \text{Gamma}\left(\frac{\nu}{2}, \frac{1}{2}\right)$

(4) **Most powerful test for the normal variance - μ is known**
Let $X_1, \ldots, X_n$ be iid $\mathcal{N}(\mu, \sigma^2)$, where μ is known.

(a) Find an MP test at level α for testing two simple hypoheses

$$H_0 : \sigma^2 = \sigma_0^2 \quad vs \quad H_1 : \sigma^2 = \sigma_1^2,\ \sigma_1 > \sigma_0\,.$$

(b) Show that the MP test is a UMP test for testing

$$H_0 : \sigma^2 \leq \sigma_0^2 \quad vs \quad H_1 : \sigma^2 > \sigma_0^2.$$

Hint: $\sum_i (X_i - \mu)^2 \sim \sigma^2 \chi^2(n)$.

a) $L(\mu, \sigma; x) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right)$

$\lambda(x) = \frac{L(\mu,\sigma_1;x)}{L(\mu,\sigma_0;x)} = \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{n/2} \exp\left(\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\right)$ is an MP-test

$T(x) := \sum_{i=1}^{n}(x_i-\mu)^2$, we reject H_0 if $T(x) \geq C$, where $\mathbb{P}(T(x) \geq C) = \alpha$.

$X_i - \mu \sim \mathcal{N}(0, \sigma^2) \Rightarrow (X_i-\mu)^2 \sim \sigma^2\chi^2(1) \Rightarrow \sum_{i=1}^{n}(X_i-\mu)^2 \sim \sigma^2\chi^2(n)$

Therefore, $C = \frac{1}{\sigma^2}\chi^2_\alpha(n)$.

b) By a theorem from the lecture

(5) **Most powerful test for the normal variance - μ is unknown**
Let $X_1, \dots, X_n$ be iid $\mathcal{N}(\mu, \sigma^2)$, where μ is unknown.

(a) Is there an MP test at level α for testing?

$$H_0 : \sigma^2 = \sigma_0^2 \quad vs \quad H_1 : \sigma^2 = \sigma_1^2,\ \sigma_1 > \sigma_0 .$$

If not, find the corresponding GLRT.

(b) Is the above generalized likelihood ratio (GLR) test also a GLRT for testing the one-sided hypothesis?

$$H_0 : \sigma^2 \leq \sigma_0^2 \quad vs \quad H_1 : \sigma^2 > \sigma_0^2 .$$

(c) Find the GLRT at level α for testing

$$H_0 : \sigma^2 \geq \sigma_0^2 \quad vs \quad H_1 : \sigma^2 < \sigma_0^2 .$$

a) For any given μ we know from (4) that $T(x) = \sum_{i=1}^n (X_i - \mu)^2$ is an MP

$\Theta_0 = \mathbb{R} \times \{\sigma_0\}$, $\quad \Theta_1 = \mathbb{R} \times \{\sigma_1\}$, $\quad \Theta := \Theta_0 \cup \Theta_1$

The likelihood function is given by $L(\mu, \sigma; x) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2\right)$

The MLEs are $\hat{\mu} = \bar{x}$ and $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$, hence

$$\frac{L(\hat{\mu}, \sigma_1^2, x)}{L(\hat{\mu}, \sigma_0^2, x)} = \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{n/2} \exp\left(\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) \frac{1}{2} \sum_{i=1}^n (x_i - \bar{x})^2\right)$$

$$\frac{1}{\sigma_0^2} \geq \frac{1}{\sigma_1} \Leftrightarrow \sigma_1^2 \geq \sigma_0^2 \Leftrightarrow \sigma_1^2 \geq \sigma_0^2$$

We choose $T(x) := \sum_{i=1}^n (x_i - \bar{x})^2$ as a simpler test statistic

$X_i \sim N(\mu, \sigma_0^2)$

$$\frac{\partial}{\partial \mu}\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2\right) \qquad \boxed{\hat{\mu} = \bar{x}}$$

$$= \frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu)$$

$$= \frac{1}{\sigma^2}\left(\sum_{i=1}^n x_i - n\mu\right) \stackrel{!}{=} 0 \Leftrightarrow \mu = \bar{x}$$

b) $$\frac{\partial}{\partial \sigma^2}\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 - \frac{n}{2} \log(\sigma^2)\right) = \frac{1}{2\sigma^4} \sum_{i=1}^n (x_i - \mu)^2 - \frac{n}{2\sigma^2} \stackrel{!}{=} 0$$

$$\hat{\sigma}^2 = \begin{cases} \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2, & \text{if } \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 > \sigma_0 \\ \sigma_0, & \text{else} \end{cases} \qquad \tilde{\sigma} := \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

$$\lambda(x) = \begin{cases} \left(\frac{\sigma_0^2}{\tilde{\sigma}^2}\right)^{n/2} \exp\left(\left(\frac{\tilde{\sigma}^2}{\sigma_0^2} - 1\right) \frac{n}{2}\right), & \text{if } \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 > \sigma_0 \\ 1, & \text{else} \end{cases}$$

$$\left(\frac{\sigma_0^2}{\tilde{\sigma}^2}\right)^{n/2} \exp\left(\left(\frac{\tilde{\sigma}^2}{\sigma_0^2} - 1\right) \frac{n}{2}\right) < 1 \Leftrightarrow (\sigma_0^2 n)^{n/2} \exp\left(\left(\frac{\tilde{\sigma}}{\sigma_0^2} - 1\right) \frac{n}{2}\right) < (T(x))^{n/2}$$

c)

$$\lambda(x) = \begin{cases} \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{\frac{n}{2}} \exp\left(\left(\frac{\sigma_0^2}{\hat{\sigma}^2} - 1\right)\frac{n}{2}\right) & , \text{if } \hat{\sigma} < \sigma_0 \\ 1 & , \text{if } \hat{\sigma} \geq \sigma_0 \end{cases}$$

$$\lambda(x) = \frac{\sup\{L(\mu, \sigma^2, x) \mid \mu \in \mathbb{R}, \sigma^2 \in \mathbb{R}^+\}}{\sup\{L(\mu, \sigma^2, x) \mid \mu \in \mathbb{R}, \sigma^2 \geq \sigma_0^2\}}$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$\lambda(x) \geq C \Leftrightarrow \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{\frac{n}{2}} \exp\left(\frac{\sigma_0^2}{\hat{\sigma}^2}\right) \geq C \exp\left(\frac{n}{2}\right) =: C' \Leftrightarrow \frac{\hat{\sigma}^2}{\sigma_0^2} > a \vee \frac{\hat{\sigma}^2}{\sigma_0^2} < b$$

$$\frac{n\hat{\sigma}^2}{\sigma_0^2} \overset{?}{\sim} \frac{1}{\sigma_0}\chi^2(n) \; ?$$

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