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Numerik von Differentialgleichungen - Kreuzlübung 6

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Exercise 26:
Let $c_1 = 0$, $c_3 = 1$ and $c_2 \in (0, 1)$ be arbitrary.
a) Which order of convergence is achievable by 3-stage Runge-Kutta methods which are created by collocation with these collocation nodes?
b) Write down the Butcher Tableaux of these methods.

Exercise 27:
Let $[a, b] = [-1, 1]$ and $\omega(x) \equiv 1$.
a) Show that the orthogonal polynomials $(q_s)_{s\in\mathbb{N}_0}$ from Remark B.13 of the lecture notes are even or odd polynomials, if s is even or odd, respectively. To this end, you can use the construction of these polynomials from the monomial basis by the Gram-Schmidt orthogonalization process.
b) Let $\begin{array}{c|c} c & A \\ \hline & b^\top \end{array}$ be the Butcher Tableau of an m-stage Runge-Kutta method which was created by collocation from Gauss-quadrature. Proof that the collocation nodes and the weights are symmetric in the following sense:

$$\left|c_j - \frac{1}{2}\right| = \left|c_{m+1-j} - \frac{1}{2}\right|, \qquad b_j = b_{m+1-j}, \qquad j = 1, \ldots, m. \tag{1}$$

Exercise 28:
Let

$$\mathbf{M}_h := \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)} \tag{2}$$

with $N \in \mathbb{N}$ and $h := 1/N$ the matrix from Example 4.2 of the lecture notes.
a) Show that the eigenvalues λ_j with their corresponding eigenvectors v_j of $\mathbf{M}_h$ are given by

$$\lambda_j = \frac{2}{h^2}\left(-1 + \cos\left(\frac{j\pi}{N}\right)\right), \qquad j = 1, \ldots, N-1 \tag{3a}$$

and

$$v^{(j)} := \left(\sin\left(\frac{j\pi}{N}\right), \sin\left(\frac{2j\pi}{N}\right), \ldots, \sin\left(\frac{(N-1)j\pi}{N}\right)\right)^\top. \tag{3b}$$

b) Justify why the initial value problem

$$U_h' = M_h U_h, \qquad U_h(0) = G \tag{4}$$

for $G \in \mathbb{R}^{N-1}$ is stiff. To this end, proof $\lim_{N\to\infty} \lambda_1 = -\pi^2$ and $\lim_{N\to\infty} \lambda_{N-1} = -\infty$.

Exercise 29:
Compute the solution of the initial value problem (4) on the interval $[0,1]$ numerically. The initial value G should approximate the function $g(x) := \exp(-30(x-1/2)^2)$ on the nodes, i.e., $G_i = g(i/N)$ for $i = 1, \dots, N-1$.

a) Let the spatial step size h be given. Use (3a) to compute how large the time step size τ can be depending on h, so that the explicit Euler method produces exponentially decreasing solutions.

b) Verify this numerically. To this end, use the explicit Euler method for different spatial and time step sizes (e.g., $h = 2^{-1}, \dots, 2^{-10}$, $\tau = 2^1, \dots, 2^{10}$). You can then, e.g., visualize $\|U_n(t)\|_\infty$ for times $t \in [0,T]$.

c) Test also with the implicit Euler method. For the solution of the arising linear systems, please use LU-factorization and forward / backward substitution (e.g., in Python with the functions `lu_factor` and `lu_solve` in the library `scipy.linalg`).

Exercise 30:
Write a program that, for a given Runge-Kutta method with stability function R and a rectangle $Q = \{z \in \mathbb{C} : (\Re(z), \Im(z)) \in [a,b] \times [c,d]\}$, visually highlights those $z \in Q$, für for which there holds $|R(z)| \leq 1$.

Test your program with explicit and implicit Runge-Kutta methods which you already know. Among others, use the methods from Remark 4.25 and Definition 4.31. Which rectangles Q are interesting to look at?