(4) **Unbiased estimators**

Let $\hat{a}$ and $\hat{b}$ be unbiased estimators of unknown parameters a and b respectively.

(a) Check if $\alpha\hat{a} + \beta\hat{b}$ is an unbiased estimator of the parameter $\alpha\, a + \beta\, b$, where $\alpha, \beta \in \mathbb{R}$.

(b) Is $\hat{a}^2$ an unbiased estimator of a^2?

(c) Based on the following measurements of a side of a square (in milimeters)

$$15, 17, 16, 16, 17, 14$$

find an unbiased estimator of the area.

a) Let X be a random variable corresponding to the data, such that

$\mathbb{E}(\hat{a}(X)) = a$ and $\mathbb{E}(\hat{b}(X)) = b$.

Since $\mathbb{E}(\alpha\hat{a}(X) + \beta\hat{b}(X)) = \alpha\,\mathbb{E}(\hat{a}(X)) + \beta\,\mathbb{E}(\hat{b}(X)) = \alpha a + \beta b$, $\alpha\hat{a} + \beta\hat{b}$ is an unbiased estimator of $\alpha a + \beta b$.

b) $\mathbb{E}((\hat{a}(X))^2) = \mathbb{V}\mathrm{ar}(\hat{a}(X)) + (\mathbb{E}(\hat{a}(X)))^2 = \mathbb{V}\mathrm{ar}(\hat{a}(X)) + a^2$

$\Leftrightarrow \mathbb{E}((\hat{a}(X))^2) - a^2 = \mathbb{V}\mathrm{ar}(\hat{a}(X)) \geqslant 0$, equality holds only if $\hat{a}(X)$ is constant almost everywhere.

c) $\hat{a} : \mathbb{R}^n \to \mathbb{R} : x \longmapsto \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{\substack{j \in \{1,\dots,n\} \\ j \neq i}} x_i x_j$

$$\mathbb{E}(\hat{a}(X)) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{\substack{j \in \{1,\dots,n\} \\ j \neq i}} \mathbb{E}(X_i X_j) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{\substack{j \in \{1,\dots,n\} \\ j \neq i}} \mathbb{E}(X_i)\mathbb{E}(X_j) = \ell^2 =: a$$

where $\ell = \mathbb{E}(X_i)$ for all $i \in \{1, \dots, n\}$, hence ℓ is the true length of a side of the square.

In our case, $n = 6$, and we obtain $\hat{a}(x) = \frac{7514}{30} = \frac{3757}{15} \approx 250{,}5$ (mm^2).