(3) **Confidence interval 2**

Suppose $X_1, \dots, X_n$ are i.i.d. with pdf

$$f(x|\lambda,\eta) = \begin{cases} \lambda e^{-\lambda(x-\eta)} & x > \eta \\ 0, & \text{otherwise} \end{cases}$$

where λ and η positive parameters with η known but λ unknown. Find the MLE of λ and construct a $(1-\alpha)100\%$ confidence interval for λ when n is assumed to be large.

The likelihood function is

$$L(\lambda,\eta;x) = \begin{cases} \lambda^n \exp\left(-\lambda \sum_{i=1}^n (x_i-\eta)\right), & \text{if } \min\{x_i \mid 1\le i\le n\} > \eta \\ 0 & , \text{otherwise} \end{cases}$$

Thus, the log-likelihood for $x \in (\eta,\infty)^n$ is $\ell(\lambda,\eta;x) = n\log(\lambda) - \lambda \sum_{i=1}^n (x_i-\eta)$ and

$\frac{\partial \ell}{\partial \lambda}(\hat\lambda,\eta;x) = \frac{n}{\hat\lambda} - \sum_{i=1}^n (x_i-\eta) \overset{!}{=} 0 \Leftrightarrow \hat\lambda = n\left(\sum_{i=1}^n (x_i-\eta)\right)^{-1}$ is the MLE of λ.

$\frac{\partial^2 \ell}{\partial \lambda}(\lambda,\eta;x) = -\frac{n}{\lambda^2} < 0$ Hence, $I_n(\lambda) = -\mathbb{E}\left(\frac{\partial^2 \ell}{\partial \lambda}(\lambda,\eta;x)\right) = \frac{n}{\lambda^2}$

By slide 12 from lecture 7 we have $\sqrt{n}\left(\hat\lambda(X)-\lambda\right) \xrightarrow{d} \mathcal{N}\left(0,\frac{\lambda^2}{n}\right)$, hence

the MLE is approximately $\mathcal{N}\left(\hat\lambda, \frac{\hat\lambda^2}{n}\right)$ distributed.

Let $Z \sim \mathcal{N}\left(\hat\lambda, \frac{\hat\lambda^2}{n}\right)$. The confidence interval $[\hat\lambda-\delta, \hat\lambda+\delta)$ is such that

$$1-\alpha = \mathbb{P}\left(\hat\lambda-\delta \le Z < \hat\lambda+\delta\right) = 1-2\mathbb{P}\left(Z<\hat\lambda-\delta\right) = 1-2\mathbb{P}\left(\frac{(Z-\hat\lambda)\sqrt{n}}{\hat\lambda} < -\frac{\delta\sqrt{n}}{\hat\lambda}\right)$$

$$\Leftrightarrow \phi\left(-\frac{\delta\sqrt{n}}{\hat\lambda}\right) = \frac{\alpha}{2} \Leftrightarrow -\frac{\delta\sqrt{n}}{\hat\lambda} = \phi^{-1}\left(\frac{\alpha}{2}\right) \Leftrightarrow \delta = -\frac{\hat\lambda}{\sqrt{n}}\phi^{-1}\left(\frac{\alpha}{2}\right)$$