(4) **Conditional variance**

(a) Show that for any two random variables X and Y the conditional variance identity holds

$$\mathbb{V}ar\, Y = \mathbb{E}\left(Var\,(Y|X)\right) + \mathbb{V}ar\,\left(\mathbb{E}\,(Y|X)\right),$$

provided that the expectations exist. The law of total expectation (the tower property) $\mathbb{E}X = \mathbb{E}(\mathbb{E}(X|Y))$ should be applied.

(b) Suppose that the distribution of Y conditional on $X = x$ is $\mathcal{N}(x, x^2)$ and that the marginal distribution of X is uniform on $(0,1)$. Compute $\mathbb{E}Y$, $\mathbb{V}ar\,Y$ and $\mathbb{C}ov(X,Y)$.

a) $\mathbb{E}(\mathbb{V}ar(Y|X)) + \mathbb{V}ar(\mathbb{E}(Y|X)) = \mathbb{E}\left(\mathbb{E}(Y^2|X) - (\mathbb{E}(Y|X))^2\right) + \mathbb{E}\left((\mathbb{E}(Y|X))^2\right) - \left(\mathbb{E}(\mathbb{E}(Y|X))\right)^2$

$= \mathbb{E}(\mathbb{E}(Y^2|X)) \underbrace{- \mathbb{E}\left((\mathbb{E}(Y|X))^2\right) + \mathbb{E}\left((\mathbb{E}(Y|X))^2\right)}_{=0} - \left(\mathbb{E}(\mathbb{E}(Y|X))\right)^2$

$= \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = \mathbb{V}ar(Y)$

b) $f(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} \Leftrightarrow f_{X,Y}(x,y) = f_X(x)\, f(y|x) = \begin{cases} 0 & , \text{ if } x < 0 \\ \frac{1}{\sqrt{2\pi}\, x} \exp\left(\frac{-(y-x)^2}{2x^2}\right) & , \text{ if } 0 \le x < 1 \\ 0 & , \text{ if } 1 \le x \end{cases}$

$f_Y(y) = \int_0^1 f_{X,Y}(x,y)\, dx$

$Z_x \sim \mathcal{N}(x, x^2)$

$\mathbb{E}(Y) = \int_{\mathbb{R}} y\, f_Y(y)\, dy \overset{\text{Fubini}}{=} \int_0^1 \int_{\mathbb{R}} y\, f_{X,Y}(x,y)\, dy\, dx = \int_0^1 \mathbb{E}(Z_x)\, dx \overset{f_{X,Y}(x,y) \text{ is normal distribution with Expectation } x}{=} \int_0^1 x\, dx = \boxed{\frac{1}{2}}$

$\mathbb{E}(Z^2) = \mathbb{V}ar(Z) + (\mathbb{E}(Z))^2 = 2x^2$

$\mathbb{E}(Y^2) = \int_{\mathbb{R}} y^2 f_Y(y)\, dy\, dx \overset{\text{Fubini}}{=} \int_0^1 \int_{\mathbb{R}} y^2 f(x,y)\, dy\, dx = \int_0^1 \mathbb{E}(Z_x^2)\, dx = \int_0^1 2x^2\, dx = \frac{2}{3}$

$\Rightarrow \mathbb{V}ar(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \boxed{\frac{5}{12}}$

$\mathbb{E}(XY) = \int_{\mathbb{R}^2} xy\, f_{X,Y}(x,y)\, dx\, dy = \int_0^1 x \int_{\mathbb{R}} y\, f_{X,Y}(x,y)\, dy\, dx = \int_0^1 x\, \mathbb{E}(Z_x)\, dx = \int_0^1 x^2\, dx = \frac{1}{3}$

$\Rightarrow \mathbb{C}ov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y) = \frac{1}{3} - \frac{1}{2}\frac{1}{2} = \frac{4-3}{12} = \boxed{\frac{1}{12}}$