

- ① Show that the one-parameter exponential family has a monotone likelihood ratio in a sufficient statistic $T(X)$ if $w(\eta)$ is a non-decreasing function in η .

One-parameter exp. family: $f(x|\eta) = \frac{1}{h(x)} \underbrace{c(\eta)}_{\geq 0} \underbrace{e^{w(\eta)t(x)}}_{\geq 0}$. Let $\eta_1 < \eta_2$.

$$\Rightarrow L(\eta|x) = \left(\prod_{i=1}^n h(x_i) \right) (c(\eta))^n e^{w(\eta) \sum_{i=1}^n t(x_i)} \Rightarrow \frac{L(\eta_2|x)}{L(\eta_1|x)} = \underbrace{\left(\frac{c(\eta_2)}{c(\eta_1)} \right)^n}_{>0} e^{\underbrace{(w(\eta_2) - w(\eta_1)) \sum_{i=1}^n t(x_i)}_{>0}}$$

We choose the statistic $T(X) := \sum_{i=1}^n t(x_i)$; obviously the LR is a non-dec. function of $T(x)$ if $\eta_1 < \eta_2$. It is also sufficient by the Fisher-Neyman factorization theorem.

② $\alpha = 0,1$; $\left[\bar{X} - t_{\alpha/2}(n-1) \cdot \frac{s}{\sqrt{n}}, \bar{X} + t_{\alpha/2}(n-1) \cdot \frac{s}{\sqrt{n}} \right] \stackrel{(\text{see R})}{=} [148, 166]$.

- ③ X_1, \dots, X_n iid, $f(x|\lambda, \eta) = \begin{cases} \lambda e^{-\lambda(x-\eta)}, & x > \eta \\ 0, & \text{else} \end{cases}$, $\eta, \lambda > 0$, η known, λ unknown.

• MLE of λ : $l(\lambda|x) = \sum_{i=1}^n \log(f(x_i|\lambda)) = \sum_{i=1}^n [\log(\lambda) - \lambda(x_i - \eta)]$

$$= n \log \lambda + \lambda n \eta - \lambda \sum_{i=1}^n x_i \Rightarrow \frac{\partial}{\partial \lambda} l(\lambda|x) \Big|_{\hat{\lambda}} = \frac{n}{\hat{\lambda}} + n \eta - \sum_{i=1}^n x_i \stackrel{!}{=} 0$$

$$\Rightarrow n + \hat{\lambda}(n \eta - \sum_{i=1}^n x_i) \stackrel{!}{=} 0 \Rightarrow \hat{\lambda} = \frac{n}{\sum x_i - n \eta} = \frac{1}{\bar{x} - \eta}.$$

$$\frac{\partial^2}{\partial \lambda^2} l(\lambda|x) = -\frac{n}{\lambda^2} < 0 \Rightarrow \hat{\lambda} \text{ is MLE.}$$

- Construct a $(1-\alpha)100\%$ confidence interval for λ when n is large.

For large n , we know that $\sqrt{n}(\hat{\lambda} - \lambda) \approx \mathcal{N}(0, \frac{1}{I(\lambda)})$.

By the calculation above, $I(\lambda) = -\mathbb{E}_\lambda(l''(\lambda|x)) = \frac{n}{\lambda^2}$. → continuous in λ

As λ is unknown, we can estimate $I(\lambda)$ by $I(\hat{\lambda})$.

As $I(\lambda)$ is continuous in λ ($\lambda > 0!$), we have $I(\hat{\lambda}) \xrightarrow{P} I(\lambda)$, therefore

$$\sqrt{n I(\hat{\lambda})}(\hat{\lambda} - \lambda) = \frac{n}{\hat{\lambda}}(\hat{\lambda} - \lambda) = \sqrt{\frac{I(\hat{\lambda})}{I(\lambda)}} \sqrt{n I(\lambda)}(\hat{\lambda} - \lambda) \xrightarrow{d} \mathcal{N}(0, 1)$$

$$\Rightarrow P_\lambda(-z_{\alpha/2} \leq \frac{n}{\hat{\lambda}}(\hat{\lambda} - \lambda) \leq z_{\alpha/2}) = P_\lambda(\hat{\lambda} - z_{\alpha/2} \frac{\hat{\lambda}}{n} \leq \lambda \leq \hat{\lambda} + z_{\alpha/2} \frac{\hat{\lambda}}{n}) \rightarrow 1 - \alpha$$

\Rightarrow the asymptotic confidence interval is given by $\hat{\lambda} \pm z_{\alpha/2} \frac{\hat{\lambda}}{n}$.

5. (a), (b) \rightarrow see R file

(c) Data: 2,2 3,5 7,6 9,0 13,7 23,7 34,2 44,3

~~Def~~ $g_p \in \mathbb{R}$ empirical p -quantile $\Leftrightarrow |(x_i \leq g_p)| \geq pm \wedge |(x_i \geq g_p)| \geq (1-p)m$.

\rightarrow 0,5-quantile: [9, 13,7]

\rightarrow 0,25-quantile = 1st quartile: $8 \cdot 0,25 = 2 \Rightarrow [3,5, 7,6]$

\rightarrow $\frac{2}{3}$ -quantile: $8 \cdot \frac{2}{3} = 5,3 \Rightarrow \{23,7\}$
 $8 - 5,3 = 2,7$

\rightarrow See R file.

4. We know that the $(1-\alpha)$ -CI is given by
 $\hat{\lambda} \pm z_{\alpha/2} \cdot \sqrt{\frac{\hat{\lambda}}{n}}$, where $\hat{\lambda} = \bar{X}$.