(4) **Normal unbiased esimator of μ^2**
Let $X_1 \dots X_n$ be i.i.d. $\mathcal{N}(\mu, 1)$.

(a) Show that $\bar{X}^2 - \frac{1}{n}$ is unbiased esimator of μ^2.
(b) By using Stein's Lemma, calculate its variance and show that it is greater than the Cramér-Rao lower bound.

Hint: Recall, Stein's Lemma states that for $X \sim \mathcal{N}(\mu, \sigma^2)$ and a differentiable function g satisfying $E|g'(X)| < \infty$ it holds $\mathbb{E}\left(g(X)(X-\mu)\right) = \sigma^2\,\mathbb{E}g'(X)$.

a) For $i \neq j$ we have

$$\mathrm{Cov}(X_i, X_j) = \int_{\mathbb{R}}\int_{\mathbb{R}} xy \frac{1}{2\pi} e^{\frac{-x^2}{2}} e^{\frac{-y^2}{2}} dx\, dy = 0$$

Thus,

$$\mathbb{E}\left(\bar{X}^2 - \frac{1}{n}\right) = \frac{1}{n^2}\sum_{i=1}^n \sum_{j=1}^n \mathbb{E}(X_i X_j) - \frac{1}{n}$$

$$= \frac{1}{n^2}\sum_{i=1}^n \sum_{j=1}^n \left(\mathbb{E}((X_i-\mu)(X_j-\mu)) + \mu^2\right) - \frac{1}{n}$$

$$= \mu^2 + \frac{1}{n^2}\sum_{i=1}^n \sum_{j=1}^n \mathrm{Cov}(X_i, X_j) - \frac{1}{n} = \mu^2 + \frac{1}{n^2}\sum_{i=1}^n 1 - \frac{1}{n} = \mu^2$$

b) $$\mathbb{E}\left(\left(\bar{X}^2 - \frac{1}{n} - \mu^2\right)^2\right) = \mathbb{E}\left(\left((\bar{X}+\mu)(\bar{X}-\mu) - \frac{1}{n}\right)^2\right)$$

$$= \mathbb{E}\left((\bar{X}+\mu)^2(\bar{X}-\mu)^2\right) - \frac{2}{n}\mathbb{E}\left((\bar{X}+\mu)(\bar{X}-\mu)\right) + \frac{1}{n^2}$$

we define $g_1(x) := x + \mu$ and $g_2(x) := (x+\mu)^2(x-\mu)$ and have

$g_1'(x) = 1$ and $g_2'(x) = 2(x+\mu)(x-\mu) + (x+\mu)^2 = 2x^2 - 2\mu^2 + (x+\mu)^2$, hence

$\mathbb{E}(|g_1'(\bar{X})|) = \mathbb{E}(1) = 1$, and

$$\mathbb{E}(|g_2'(\bar{X})|) \leq 2\mathbb{E}(\bar{X}^2) + 2\,\mathbb{E}(\mu^2) + \mathbb{E}\left((\bar{X}+\mu)^2\right)$$

$$= 2\mathbb{E}(\bar{X}^2) + 2\mu^2 + \mathbb{E}(\bar{X}^2) + 2\mu\mathbb{E}(\bar{X}) + \mu^2 < \infty$$

We already know that $\bar{X} \sim \mathcal{N}\left(\frac{n\mu}{n}, \frac{n}{n^2}\right) = \mathcal{N}\left(\mu, \frac{1}{n}\right)$

By applying Stein's Lemma, we obtain

$$\mathrm{Var}\left(\bar{X}^2 - \frac{1}{n}\right) = \mathbb{E}\left(\left(\bar{X}^2 - \frac{1}{n} - \mu^2\right)^2\right) = \mathbb{E}\left(g_2(\bar{X})(\bar{X}-\mu)\right) - \frac{2}{n}\mathbb{E}\left(g_1(\bar{X})(\bar{X}-\mu)\right) + \frac{1}{n^2}$$

$$= \frac{1}{n}\mathbb{E}\left(g_2'(\bar{X})\right) - \frac{2}{n^2}\mathbb{E}\left(g_1'(\bar{X})\right) + \frac{1}{n^2}$$

$$= \frac{1}{n}\left(2\mathbb{E}(\bar{X}^2) - 2\mu^2 + \mathbb{E}(\bar{X}^2) + 2\mu\,\mathbb{E}(\bar{X}) + \mu^2\right) - \frac{1}{n^2}$$

$$= \frac{1}{n}\left(3\left(\frac{1}{n} + \mu^2\right) + \mu^2\right) - \frac{1}{n^2} = \frac{2}{n^2} + \frac{4\mu^2}{n}$$

We define $\theta := \mu^2$ and obtain the log-likelihood function

$$\ell_n(\theta) = \sum_{i=1}^{n} \log\left(f(x_i)\right) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(x_i - \sqrt{\theta})^2 ,$$ hence

$$\ell_n'(\theta) = -\frac{1}{2}\, 2 \sum_{i=1}^{n}(x_i - \sqrt{\theta})\left(-\frac{1}{2}\theta^{-\frac{1}{2}}\right) = \frac{1}{2}\sum_{i=1}^{n}\frac{x_i - \sqrt{\theta}}{\sqrt{\theta}} = \frac{1}{2}\sum_{i=1}^{n}\frac{x_i}{\sqrt{\theta}} - \frac{n}{2}$$

$$\ell_n''(\theta) = \frac{1}{2}\sum_{i=1}^{n} x_i \left(-\frac{1}{2}\right)\theta^{-\frac{3}{2}} = -\frac{1}{4}\theta^{-\frac{3}{2}}\sum_{i=1}^{n} x_i$$

We obtain

$$I_n(\theta) = \mathbb{E}\left(\ell_n''(\theta)\right) = -\frac{1}{4}\theta^{-\frac{3}{2}}\sum_{i=1}^{n}\mathbb{E}(X_i) = -\frac{1}{4}\mu^{-3} n\mu = -\frac{n}{4\mu^2} ,$$

and we conclude that

$$-\frac{1}{I_n(\theta)} = \frac{4\mu^2}{n} < \frac{2}{n^2} + \frac{4\mu^2}{n} = \mathbb{V}\mathrm{ar}\left(\bar{X}^2 - \frac{1}{n}\right)$$