unknown

1. $X_1,\dots,X_n$ i.i.d. $\mathcal{N}(\mu,\sigma^2)$. Find GLRT for $H_0: \sigma^2=\sigma_0^2$, $H_1: \sigma^2\neq\sigma_0^2$.

Define $\Theta=\{(\mu,\sigma^2)\mid \mu\in\mathbb{R},\ \sigma^2>0\}$, $\Theta_0=\{(\mu,\sigma_0^2)\mid \mu\in\mathbb{R}\}$.

We have $L(\mu,\sigma^2|x)=(2\pi\sigma^2)^{-\frac{n}{2}}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$.

From lecture 6, we know that $\sup_{(\mu,\sigma^2)\in\Theta}L(\mu,\sigma^2|x)=L(\hat\mu,\hat\sigma^2|x)$

with $\hat\mu=\bar{x}$ and $\hat\sigma^2=\frac{1}{n}\sum(x_i-\bar{x})^2$.

Obviously, $\sup_{(\mu,\sigma^2)\in\Theta_0}L(\mu,\sigma^2|x)=L(\bar{x},\sigma_0^2|x)$. Therefore,

$$\lambda(x)=\frac{\sup_{\Theta}L(\mu,\sigma^2|x)}{\sup_{\Theta_0}L(\mu,\sigma^2|x)}=\frac{(2\pi\hat\sigma^2)^{-\frac{n}{2}}e^{-\frac{1}{2\hat\sigma^2}n\hat\sigma^2}}{(2\pi\sigma_0^2)^{-\frac{n}{2}}e^{-\frac{1}{2\sigma_0^2}n\hat\sigma^2}}=\left(\frac{\sigma_0^2}{\hat\sigma^2}\right)^{\frac{n}{2}}e^{\frac{n}{2}\left(\frac{\hat\sigma^2}{\sigma_0^2}-1\right)}.$$

We define $f(x)=\left(\frac{1}{x}\right)^{\frac{n}{2}}e^{\frac{n}{2}(x-1)}$ and calculate with Maple: $f'(x)=0 \Leftrightarrow x=1$,

$f'(x)>0$ for $x>1$ and $f(x)=1 \Leftrightarrow x=1$.

$\lambda(x)$ is always >1, hence $\frac{\hat\sigma^2}{\sigma_0^2}$ is >1 and therefore $\lambda(x)$

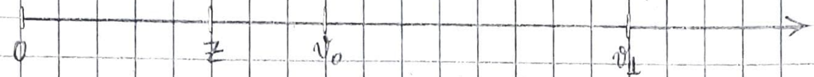
non-decreasing for $T(X):=\frac{\hat\sigma^2}{\sigma_0^2}$.

From lecture 5, we know that $n\frac{\hat\sigma^2}{\sigma_0^2}\sim\chi^2(n-1)$.

We have $\alpha=\sup_{(\mu,\sigma^2)\in\Theta_0}P_{(\mu,\sigma^2)}\left(\frac{\hat\sigma^2}{\sigma_0^2}\geq C\right)=\sup_{\mu\in\mathbb{R}}P_{(\mu,\sigma_0^2)}\left(n\frac{\hat\sigma^2}{\sigma_0^2}\geq nC\right)=$

$\Rightarrow nC=\chi^2_\alpha(n-1) \Rightarrow C=\frac{1}{n}\chi^2_\alpha(n-1)$.

2.) Let $X_1,\dots,X_n$ iid. $\mathcal{U}(0,\vartheta)$. Derive MP test at level α for $H_0: \vartheta=\vartheta_0$ vs. $H_1: \vartheta=\vartheta_1>\vartheta_0$.



We have $\mathbb{P}_{\vartheta_0}(\max x_i \geq z) = 1-\mathbb{P}_{\vartheta_0}(\max x_i \leq z) \overset{\text{i.i.d.}}{=} 1-\left(\frac{z}{\vartheta_0}\right)^n = \alpha \Leftrightarrow z=\vartheta_0\sqrt[n]{1-\alpha}$
and claim that a MP test is given by $T(X)=\max x_i$ with rejection region $\Omega_1=[z;\infty)$.
The power of this test is $\mathbb{P}_{\vartheta_1}(T(X)\geq z)=1-\mathbb{P}_{\vartheta_1}(T(X)\leq z)=1-\left(\frac{z}{\vartheta_1}\right)^n =$

$= 1-\left(\frac{\vartheta_0\sqrt[n]{1-\alpha}}{\vartheta_1}\right)^n = 1-\left(\frac{\vartheta_0}{\vartheta_1}\right)^n(1-\alpha).$

We consider another α-test $T_1(X)$ with rejection region $\tilde\Omega_1$, then we have

$$1-\beta = \mathbb{P}_{\vartheta_1}(X\in\tilde\Omega_1) = \int_{\tilde\Omega_1}\left(\frac{1}{\vartheta_1}\right)^n \mathbb{1}_{[0,\vartheta_1]}(\max x_i)\,d\lambda^n(x)$$

$$=\left(\frac{\vartheta_0}{\vartheta_1}\right)^n \underbrace{\int_{\tilde\Omega_1\cap[0,\vartheta_0]^n}\left(\frac{1}{\vartheta_0}\right)^n d\lambda^n(x)}_{\leq \mathbb{P}_{\vartheta_0}(X\in\tilde\Omega_1)=\alpha} + \int_{\tilde\Omega_1\setminus[0,\vartheta_0]^n}\left(\frac{1}{\vartheta_1}\right)^n \mathbb{1}_{[0,\vartheta_1]}(\max x_i)\,d\lambda^n(x)$$

$$\leq \left(\frac{\vartheta_0}{\vartheta_1}\right)^n\alpha + \underbrace{\int_{[0,\vartheta_1]^n}\left(\frac{1}{\vartheta_1}\right)^n\mathbb{1}_{[0,\vartheta_1]}(\max x_i)\,d\lambda^n(x)}_{=1} - \int_{[0,\vartheta_0]^n}\left(\frac{1}{\vartheta_1}\right)^n \underbrace{\mathbb{1}_{[0,\vartheta_1]}(\max x_i)}_{=1}\,d\lambda^n(x)$$

$$\leq\left(\frac{\vartheta_0}{\vartheta_1}\right)^n\alpha+1-\left(\frac{\vartheta_0}{\vartheta_1}\right)^n.$$

4.) $X_1,\dots,X_n$ i.i.d. $\mathcal{N}(\mu,\sigma^2)$, μ known. $\qquad f_{\sigma^2}(x)=(2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n(x_i-\mu)^2}$

(a) Find MP test at level α for testing $H_0:\sigma^2=\sigma_0^2$, $H_1:\sigma^2=\sigma_1^2$, $\sigma_1>\sigma_0$.

The likelihood ratio is $\lambda(x)=\frac{f_{\sigma_1^2}(x)}{f_{\sigma_0^2}(x)} = \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{\frac{n}{2}} e^{\overbrace{\left(-\frac{1}{2\sigma_1^2}+\frac{1}{2\sigma_0^2}\right)}^{}_{>0}\sum_{i=1}^n(x_i-\mu)^2}$,

which is a non-decreasing function in $T(X):=\sum_{i=1}^n(X_i-\mu)^2$.

We have $X_i\sim\mathcal{N}(\mu,\sigma^2)\Rightarrow X_i-\mu\sim\mathcal{N}(0,\sigma^2)\Rightarrow (X_i-\mu)^2\sim\sigma^2\chi^2(1)\Rightarrow T(X)\sim\sigma^2\chi^2(n)$

and therefore $\alpha\overset{!}{=}\mathbb{P}_{\sigma_0^2}(T(X)\geq C)\Rightarrow C=\frac{1}{\sigma_0^2}\chi^2_\alpha(n)$.

(b) Show that the MP test is a UMP test for $H_0:\sigma^2\leq\sigma_0^2$, $H_1:\sigma^2>\sigma_0^2$.

We have already observed that $\lambda(x)$ is non-decreasing in $T(X)$ if $\sigma_1^2>\sigma_0^2$.

By the theorem from the lecture, our MP test is a UMP test for $\sigma^2>\sigma_0^2$ vs. $\sigma^2\leq\sigma_0^2$, ③ $X_1, \dots, X_n$ i.i.d. with $f_\vartheta(x) = \frac{x}{\vartheta} e^{-\frac{x^2}{2\vartheta}}$, $x \geq 0$, $\vartheta > 0$.

(a) Derive MP test at level α for $H_0: \vartheta = \vartheta_0$, $H_1: \vartheta = \vartheta_1$, $\vartheta_1 > \vartheta_0$.

$$f_\vartheta(\vec{x}) = \prod_{i=1}^{n} \frac{x_i}{\vartheta} e^{-\frac{x_i^2}{2\vartheta}}$$

$$\lambda(x) = \frac{f_{\vartheta_1}(\vec{x})}{f_{\vartheta_0}(\vec{x})} = \frac{\prod_{i=1}^{n} \frac{x_i}{\vartheta_1} e^{-\frac{x_i^2}{2\vartheta_1}}}{\prod_{i=1}^{n} \frac{x_i}{\vartheta_0} e^{-\frac{x_i^2}{2\vartheta_0}}} = \left(\frac{\vartheta_0}{\vartheta_1}\right)^n e^{-\sum_{i=1}^{n} \frac{x_i^2}{2\vartheta_1} + \sum_{i=1}^{n} \frac{x_i^2}{2\vartheta_0}} = \left(\frac{\vartheta_0}{\vartheta_1}\right)^n e^{\overbrace{\left(\sum_{i=1}^{n} x_i^2\right)}^{=:T(X)} \cdot \left(\frac{1}{2\vartheta_0} - \frac{1}{2\vartheta_1}\right)}$$

$$f_{X_i^2}(x) = f_{X_i}(\sqrt{x}) \frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{\vartheta} e^{-\frac{x}{2\vartheta}} \frac{1}{2\sqrt{x}} = \frac{1}{2\vartheta} e^{-\frac{x}{2\vartheta}} \Rightarrow X_i^2 \sim \exp\left(\frac{1}{2\vartheta}\right).$$

$$g(X_i) = X_i^2 \Rightarrow g^{-1}(y) = \sqrt{y} \Rightarrow \frac{d}{dy} g^{-1}(y) = \frac{1}{2\sqrt{y}}$$

We know that $\sum_{i=1}^{n} X_i^2 \sim \Gamma\left(n, \frac{1}{2\vartheta}\right) = \Gamma\left(\frac{2n}{2}, \frac{1}{2\vartheta}\right) = \vartheta\, \Gamma\left(\frac{2n}{2}, \frac{1}{2}\right) = \vartheta \chi^2(2n)$.

$\chi^2(\vartheta) = \Gamma\left(\frac{\vartheta}{2}, \frac{1}{2}\right)$

$\Rightarrow \frac{1}{\vartheta_0} \sum X_i^2 \sim \chi^2(2n)$.

$$\alpha \overset{!}{=} \mathbb{P}_{\vartheta_0}\left(\sum_{i=1}^{n} X_i^2 \geq C\right) = \mathbb{P}_{\vartheta_0}\left(\frac{1}{\vartheta_0} \sum X_i^2 \geq \frac{1}{\vartheta_0} C\right) \Rightarrow \frac{1}{\vartheta_0} C = \chi^2_\alpha(2n) \Rightarrow \underline{\underline{C = \vartheta_0 \chi^2_\alpha(2n)}}.$$

(b) Is there a UMP test at level α for testing $H_0: \vartheta \leq \vartheta_0$ vs. $H_1: \vartheta > \vartheta_0$?

We observe that the likelihood ratio $\lambda(x)$ is monotone in $T(X) = \sum_{i=1}^{n} X_i^2$, therefore we have an UMP test by the theorem from the lecture.

Power: $\pi(\vartheta) = \mathbb{P}_\vartheta(\text{reject } H_0) = \mathbb{P}_\vartheta\left(\sum_{i=1}^{n} X_i^2 \geq C\right) = \mathbb{P}_\vartheta\left(\frac{1}{\vartheta} \sum_{i=1}^{n} X_i^2 \geq \frac{C}{\vartheta}\right) = \chi^2_{\frac{C}{\vartheta}}(2n)$.