(5) **Most powerful test for the normal variance - μ is unknown**
Let $X_1, \dots, X_n$ be iid $\mathcal{N}(\mu, \sigma^2)$, where μ is unknown.

(a) Is there an MP test at level α for testing?

$$H_0 : \sigma^2 = \sigma_0^2 \quad vs \quad H_1 : \sigma^2 = \sigma_1^2,\ \sigma_1 > \sigma_0 .$$

If not, find the corresponding GLRT.

(b) Is the above generalized likelihood ratio (GLR) test also a GLRT for testing the one-sided hypothesis?

$$H_0 : \sigma^2 \le \sigma_0^2 \quad vs \quad H_1 : \sigma^2 > \sigma_0^2 .$$

(c) Find the GLRT at level α for testing

$$H_0 : \sigma^2 \ge \sigma_0^2 \quad vs \quad H_1 : \sigma^2 < \sigma_0^2 .$$

a) Assume there is an MP test at level α with the rejection region R, then

$$\sum_{i=1}^n (x_i - \mu_1)^2 = \sum_{i=1}^n (x_i^2 - 2x_i\mu_1 + \mu_1^2) = \sum_{i=1}^n \left((x_i-\mu)^2 + 2x_i\mu - \mu^2 - 2x_i\mu_1 + \mu_1^2\right)$$

$$\mathbb{P}(X \in R) = \int_R (2\pi\sigma_1^2)^{-n/2} \exp\left(-\frac{1}{2\sigma_1^2}\sum_{i=1}^n (x_i-\mu)^2\right) dx$$

We have MPs for the simple problems (μ_1, σ_1) and (μ_1', σ_1)

$$= \exp\left(-\frac{1}{2\sigma_1^2}\sum_{i=1}^n \left(2x_i(\mu-\mu_1) + \mu_1^2 - \mu^2\right)\right) \int_R L(\mu_1, \sigma_1^2; x)\, dx$$

$$\le \exp\left(-\frac{1}{2\sigma_1^2}\sum_{i=1}^n \left(2x_i(\mu-\mu_1) + \mu_1^2 - \mu^2\right)\right) \alpha$$

$$< \exp\left(-\frac{1}{2\sigma_1^2}\sum_{i=1}^n \left(2x_i(\mu-\mu_1') + \mu_1'^2 - \mu^2\right)\right) \alpha$$

$$= \mathbb{P}(X \in R')$$

$\Theta_0 = \mathbb{R} \times \{\sigma_0\}$, $\quad \Theta_1 = \mathbb{R} \times \{\sigma_1\}$, $\quad \Theta := \Theta_0 \cup \Theta_1$

The likelihood function is given by $L(\mu, \sigma; x) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2\right)$

The MLEs are $\hat{\mu} = \bar{x}$ and $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$, hence the GLRT reads

$$\lambda(x) = \frac{L(\hat{\mu}, \sigma_1^2, x)}{L(\hat{\mu}, \sigma_0^2, x)} = \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{n/2} \exp\left(\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\frac{1}{2}\sum_{i=1}^n (x_i - \bar{x})^2\right)$$

$$\frac{1}{\sigma_0^2} \ge \frac{1}{\sigma_1} \Leftrightarrow \sigma_1^2 \ge \sigma_0^2 \Leftrightarrow \sigma_1^2 \ge \sigma_0^2$$

We choose $T(x) := \sum_{i=1}^n (x_i - \bar{x})^2$ as a simpler test statistic

Since $X_i \sim N(\mu, \sigma^2)$ we have $\frac{1}{\sigma^2} T(X) \sim \chi^2(n-1)$