(1) **The mean of independent normal distributions**

(a) Show that the moment generating function (mgf) of $X \sim \mathcal{N}(\mu, \sigma^2)$ is of the form

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

(b) Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and let $Y = aX + b$ with fixed real constants a and b. Show that $Y \sim \mathcal{N}(a\mu + b,\, a^2\sigma^2)$.

(c) Let $X_1, \dots X_n$ be independent identically distributed random variables with $X_1 \sim \mathcal{N}(\mu, \sigma^2)$. Show that the mean $\bar{X} = \frac{1}{n}(X_1 + \cdots + X_n)$ is also normally distributed and $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$.

a) $M_X(t) = \mathbb{E}(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left(\frac{2\sigma^2 tx - x^2 + 2x\mu - \mu^2}{2\sigma^2}\right) dx$

$\left| \begin{array}{l} u = \frac{x}{\sqrt{2}\sigma} - \left(\frac{\sigma t}{\sqrt{2}} + \frac{\mu}{\sqrt{2}\sigma}\right) \\ \frac{du}{dx} = \frac{1}{\sqrt{2}\sigma} \\ dx = \sqrt{2}\sigma\, du \end{array} \right| = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-u^2 + \frac{\sigma^2 t^2}{2} + t\mu\right) du = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$\left(\frac{x}{\sqrt{2}\sigma} - \left(\frac{\sigma t}{\sqrt{2}} + \frac{\mu}{\sqrt{2}\sigma}\right)\right)^2 = \frac{x^2}{2\sigma^2} - 2\frac{x}{\sqrt{2}\sigma}\left(\frac{\sigma t}{\sqrt{2}} + \frac{\mu}{\sqrt{2}\sigma}\right) + \frac{\sigma^2 t^2}{2} + t\mu + \frac{\mu^2}{2\sigma^2}$

b) Case 1: $a = 0$, then $Y = b$, hence $P(Y \leq c) = \begin{cases} 0, & \text{if } c < b \\ 1, & \text{else} \end{cases}$, could maybe be viewed as the limit „$\mathcal{N}(b, 0)$"

Case 2: $y = ax + b \Leftrightarrow x = \frac{y-b}{a}$ and $\left(\frac{y-b}{a} - \mu\right)^2 = \frac{(y - (\mu a + b))^2}{a^2}$

$f_Y(y) = f_X\left(\frac{y-b}{a}\right)\frac{1}{a} = \frac{1}{\sqrt{2\pi}\sigma a} \exp\left(-\frac{(y-(\mu a+b))^2}{2\sigma^2 a^2}\right)$, hence $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

c) $Y_n := \sum_{i=1}^{n} X_i$

Claim: $\forall n \in \mathbb{N}: Y_n \sim \mathcal{N}(n\mu, n\sigma^2)$

$n = 1$: $Y_1 = X_1 \sim \mathcal{N}(\mu, \sigma^2)$

$n \rightsquigarrow n+1$: Assume that $Y_n \sim \mathcal{N}(n\mu, n\sigma^2)$ for $n \in \mathbb{N}$, then

$Y_{n+1} = Y_n + X_{n+1} \sim \mathcal{N}((n+1)\mu, (n+1)\sigma^2)$, because

$f_{Y_n + X_{n+1}}(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(z-t-\mu)^2}{2\sigma^2}\right) \frac{1}{\sqrt{2\pi n}\sigma} \exp\left(\frac{(t-n\mu)^2}{2n\sigma^2}\right) dt$

$= \frac{1}{2\pi\sigma^2\sqrt{n}} \int_{-\infty}^{\infty} \exp\left(-\left(\frac{z^2}{2\sigma^2} - \frac{zt}{\sigma^2} + \frac{t^2}{2\sigma^2} - \frac{z\mu}{\sigma^2} + \frac{t\mu}{\sigma^2} + \frac{\mu^2}{2\sigma^2} + \frac{t^2}{2n\sigma^2} - \frac{\mu t}{\sigma^2} - \frac{n\mu^2}{2\sigma^2}\right)\right) dt$

$= \frac{\sqrt{2}\sigma(n+1)}{2\pi\sigma^2\sqrt{n}} \int_{-\infty}^{\infty} \exp\left(-u^2 + \frac{z^2(n+1)^2 - 1}{2\sigma^2} + \frac{z\mu}{\sigma^2} + \frac{\mu^2}{2\sigma^2} + \frac{n\mu^2}{2\sigma^2}\right) du$

$= \frac{n+1}{\sqrt{2}\pi\sigma} \exp\left(-\frac{(z-(n+1)\mu)^2}{2n^2\sigma^2}\right) \int_{-\infty}^{\infty} e^{-u^2} du = \frac{n+1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(z-(n+1)\mu)^2}{2n^2\sigma^2}\right)$

$\left(\frac{t}{\sqrt{2}\sigma(n+1)} - \frac{z(n+1)}{\sqrt{2}\sigma}\right)^2 = \frac{t^2}{2\sigma^2(n+1)^2} - \frac{zt}{\sigma^2} + \frac{z^2(n+1)^2}{2\sigma^2}$ $\Rightarrow \frac{1}{n} Y_n \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$