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Numerik von Differentialgleichungen - Kreuzlübung 3

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Exercise 11:
Let $a, b, c \in \mathbb{R}$ and y the solution of the initial value problem

$$y'(t) = |a - y(t)| + b, \quad t \geq 0, \qquad y(0) = c. \tag{1}$$

a) Solve the initial value problem analytically. Which behavior of the solution do you get for different parameters a, b, c? How smooth is the solution?
b) Solve the initial value problem numerically using explicit Runge-Kutta-methods of different order. What convergence rate do you get for different parameters a, b, c? Justify your results.

Hint: You may use the program from Exercise 6 which is available on TUWEL.

Exercise 12:
Let y be the solution of the initial value problem $y'(t) = f(t, y)$ with $t \in [0, T]$, $y(0) = y_0$ and arbitrarily smooth f. Let further y_i for $i = 0, \ldots, N$ be the approximations to $y(t_i)$, which are obtained by a one-step method of order p with the nodes $t_0 = 0, \ldots, t_N = T$. Moreover, let $\tilde{y}$ be the linear spline with $\tilde{y}(t_i) = y_i$ for $i = 0, \ldots, N$.

Show that there exists a constant $C > 0$ independent of h_i and N such that

$$\sup_{t \in [0,T]} |\tilde{y}(t) - y(t)| \leq C \max\{h_0, \ldots, h_{N-1}\}. \tag{2}$$

Exercise 13:
Consider an explicit one-step method with increment function $\Phi(t, y, h)$. Define the so-called discrete evolution by

$$\Psi^{t,t+h} y := y + h\Phi(t, y, h). \tag{3}$$

Then, the one-step method can be formulated by

$$y_{j+1} = \Psi^{t_j, t_j + h_j} y_j. \tag{4}$$

The method is called reversible if there holds $\Psi^{t+h,t}\Psi^{t,t+h} y = y$ for all admissible (t, y) and all sufficiently small h. Show that there is no consistent, explicit Runge-Kutta-method that is reversible for every initial value problem.

Hint: First, show that $\Psi^{0,h} y_0$ for an s-step, explicit Runge-Kutta-method is a polynomial of order s in h, if the method is applied to the differential equation

$$y'(t) = y(t), \qquad y(0) = y_0. \tag{5}$$

Additional information: For reversible one-step methods, one step with positive step size h followed by a step with negative step size $-h$ leads to the same value with which you started.

Exercise 14:
Explicit Runge-Kutta-methods were defined in (2.27) and (2.28) in the lecture notes. For an implicit Runge-Kutta-method, equation (2.28) is replaced by

$$k_i = f\left(t + c_i h, y + h \sum_{j=1}^{m} A_{ij} k_j\right), \qquad i = 1, \dots, m. \tag{6}$$

In this equation, the matrix A is not strictly lower triangular anymore, but every entry of A can be non-zero.

Generalize Theorem 2.27 to implicit Runge-Kutta-methods. To this end, prove that such methods are stable in the sense of Theorem 2.27.

Hint: You may assume that a step of an implicit Runge-Kutta-methods is well defined and unique. This question is non-trivial because (6) is a non-linear system of equations.

Exercise 15:
Expand the program from Exercise 6 by a step-size control with embedded Runge-Kutta-methods (RK5(4)) and apply it to the predator-prey model. Compare the step-size with the solution.

Hint: There is an error in the lecture notes in the RK5(4)-scheme. Please use the following scheme:

$$\begin{array}{c|ccccccc}
0 & & & & & & & \\
1/5 & 1/5 & & & & & & \\
3/10 & 3/40 & 9/40 & & & & & \\
4/5 & 44/45 & -56/15 & 32/9 & & & & \\
8/9 & 19372/6561 & -25360/2187 & 64448/6561 & -212/729 & & & \\
1 & 9017/3168 & -355/33 & 46732/5247 & 49/176 & -5103/18656 & & \\
1 & 35/384 & 0 & 500/1113 & 125/192 & -2187/6784 & 11/84 & \\
\hline
 & 35/384 & 0 & 500/1113 & 125/192 & -2187/6784 & 11/84 & 0 \\
 & 5179/57600 & 0 & 7571/16695 & 393/640 & -92097/339200 & 187/2100 & 1/40
\end{array}$$