HW9

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1. The GLRT for the normal variance - simple hypotheses

Derive the generalized likelihood ratio test (GLRT) for the normal variance: Assume $X_1, \ldots, X_n$ are i.i.d. $\mathcal{N}(\mu, \sigma^2)$, where both μ and σ are unknown. We want to test

$$H_0 : \sigma^2 = \sigma_0^2 \quad vs \quad H_1 : \sigma^2 \neq \sigma_0^2$$

Solution:

Our test-regions are $\Theta_0 = \mathbb{R} \times \{\sigma_0^2\}$, $\Theta_1 = \mathbb{R} \times \mathbb{R}^+ \setminus \{\sigma_0^2\}$ and the GLR is given by

$$\lambda(\mathbf{x}) = \frac{\sup_{(\mu,\sigma^2)\in\Theta} L(\mu, \sigma^2; \mathbf{x})}{\sup_{(\mu,\sigma^2)\in\Theta_0} L(\mu, \sigma^2; \mathbf{x})}$$

We already know the MLEs of the normal

$$\begin{aligned}
\hat{\mu} &= \bar{X} \\
\hat{\sigma}^2 &= \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2
\end{aligned}$$

from the lecture. The MLE for Θ_0 is given by $\hat{\mu} = \bar{X}$ and $\hat{\sigma}^2 = \sigma_0^2$. This means the GLR reads as follows:

$$\begin{aligned}
\lambda(\mathbf{x}) &= \left(\frac{\sigma_0^2}{\hat{\sigma}^2}\right)^{n/2} \exp\left((\frac{1}{2\sigma_0^2} - \frac{1}{2\hat{\sigma}^2}) \sum_i (x_i - \bar{\mathbf{x}})^2 \right) \\
&= \left(\frac{\sigma_0^2}{\hat{\sigma}^2}\right)^{n/2} \exp\left((\frac{1}{2\sigma_0^2} - \frac{1}{2\hat{\sigma}^2}) n\hat{\sigma}^2 \right) \\
&= \left(\frac{\sigma_0^2}{\hat{\sigma}^2}\right)^{n/2} \exp\left((\frac{\hat{\sigma}^2}{\sigma_0^2} - 1)\frac{n}{2} \right)
\end{aligned}$$

We note the statistic $T(\mathbf{x}) = \frac{\hat{\sigma}^2}{\sigma_0^2}$ and the function

$$f(x) = \left(\frac{1}{x}\right)^n \exp\left((x-1)\frac{n}{2}\right)$$

fulfill $\lambda(\mathbf{x}) = f(T(\mathbf{x}))$. We now reject H_0 if

$$\lambda(\mathbf{x}) = f(T(\mathbf{x})) \geq C$$

where the critical value C at level α is given by

$$\alpha = \sup_{(\mu,\sigma^2)\in\Theta_0} \mathbb{P}(\lambda \geq C)$$

2. Most powerful test 1

Let $X_1, \ldots, X_n$ be i.i.d. Uniform$(0, \theta)$.

(a) Derive the most powerful (MP) test at level α for testing

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta = \theta_1,\ \theta_1 > 0.$$

(b) Calculate the power of the MP test.

Solution:

We use the test statistic $T(\mathbf{X}) = \max_{i=1}^n X_i$ with rejection region $\Omega_1 = \{\mathbf{x} : T(\mathbf{x}) \geq C\}$. We reject H_0 at level α when

$$\alpha = \mathbb{P}_{\theta_0}(\mathbf{X} \in \Omega_1) = \mathbb{P}_{\theta_0}(T(\mathbf{X}) \geq C) = 1 - \mathbb{P}_{\theta_0}(\mathbf{X}_{(n)} \leq C) = 1 - \mathbb{P}_{\theta_0}(X_1 \leq C)^n = 1 - \left(\frac{C}{\theta_0}\right)^n$$

so our critical value is

$$C = \theta_0 \sqrt[n]{1-\alpha}$$

The power of this test for fixed α is given by

$$\pi = \mathbb{P}_{\theta_1}(T(\mathbf{X}) \geq C) = 1 - \mathbb{P}_{\theta_1}(\mathbf{X}_{(n)} \leq C) = 1 - \mathbb{P}_{\theta_1}(X_1 \leq C)^n = 1 - (1-\alpha)\left(\frac{\theta_0}{\theta_1}\right)^n$$

If we now consider any other test at level $\alpha' \leq \alpha$ and let Ω_1' and π' be its rejection region and power, respectively

$$\alpha' = \int_{\Omega_1'} f_{\theta_0}(\mathbf{x}) d\mathbf{x}, \quad \pi' = \int_{\Omega_1'} f_{\theta_1}(\mathbf{x}) d\mathbf{x}$$

Then it holds

$$
\begin{aligned}
\pi' &= \int_{\Omega_1'} \left(\frac{1}{\theta_1}\right)^n \mathbf{1}_{[0,\theta_1]^n}(\mathbf{x}) d\mathbf{x} \\
&= \int_{\Omega_1' \cap [0,\theta_0]^n} \left(\frac{1}{\theta_1}\right)^n d\mathbf{x} + \int_{\Omega_1' \cap ([0,\theta_0]^n)^c} \left(\frac{1}{\theta_1}\right)^n \mathbf{1}_{[0,\theta_1]^n}(\mathbf{x}) d\mathbf{x} \\
&= \left(\frac{\theta_0}{\theta_0}\right)^n \int_{\Omega_1' \cap [0,\theta_0]^n} \left(\frac{1}{\theta_1}\right)^n d\mathbf{x} + \int_{\Omega_1' \cap ([0,\theta_0]^n)^c} \left(\frac{1}{\theta_1}\right)^n \mathbf{1}_{[0,\theta_1]^n}(\mathbf{x}) d\mathbf{x} \\
&\le \left(\frac{\theta_0}{\theta_1}\right)^n \alpha' + \int_{([0,\theta_0]^n)^c} \left(\frac{1}{\theta_1}\right)^n \mathbf{1}_{[0,\theta_1]^n}(\mathbf{x}) d\mathbf{x} \\
&= \left(\frac{\theta_0}{\theta_1}\right)^n \alpha' + \int_{[\theta_0,\theta_1]^n} \left(\frac{1}{\theta_1}\right)^n d\mathbf{x} \\
&= \left(\frac{\theta_0}{\theta_1}\right)^n \alpha' + 1 - \left(\frac{\theta_0}{\theta_1}\right)^n \\
&\le 1 - (1-\alpha)\left(\frac{\theta_0}{\theta_1}\right)^n = \pi
\end{aligned}
$$

3. Most powerful test 2

Let $X_1, \ldots, X_n$ be i.i.d. from a distribution with density

$$f_\theta(x) = \frac{x}{\theta} e^{-\frac{x^2}{2\theta}}, \; x \ge 0, \; \theta > 0.$$

(a) Derive the MP test at level α for testing two simple hypotheses

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta = \theta_1, \; \theta_1 > \theta_0.$$

(b) Is there a uniformly most powerful (UMP) test at level α for testing the one-sided composite hypothesis

$$H_0 : \theta \le \theta_0 \quad vs \quad H_1 : \theta > \theta_0$$

What is its power function?

Hint: Show $X_i^2 \sim \exp(1/2\theta)$, so that $\sum_i X_i^2 \sim \theta\chi^2(2n)$.

Solution:

We first show the hint, to do that we use the transformation theorem with function $g : \mathbb{R}^+ \to \mathbb{R}^+ : x \mapsto x^2$ with differentiable inverse $h(x) = \sqrt{x}$. The theorem now states that

$$f_{X_i^2}(x) = f_{X_i}(h(x))h'(x) = \frac{\sqrt{x}}{\theta} e^{-\frac{x}{2\theta}} \frac{1}{2\sqrt{x}} = \frac{1}{2\theta} e^{-\frac{x}{2\theta}}, \quad x \ge 0$$

which is exactly the PDF of the $\exp(\frac{1}{2\theta})$. With the same transformation theorem we can easily show that

$$\frac{X_i^2}{2\theta} \sim \exp(1) = \chi^2(2)/2,$$

therefore $\frac{1}{2\theta}\sum_i X_i^2 \sim \chi^2(2n)/2$ or

$$\sum_i X_i^2 \sim \theta\chi^2(2n).$$

To derive the MP test at level α we look at the likelihood ratio

$$\lambda(x) = \frac{L(\theta_1, \mathbf{x})}{L(\theta_0, \mathbf{x})} = \prod_{i=1}^{n} \frac{x_i}{\theta_1} e^{-\frac{x_i^2}{2\theta_1}} \frac{\theta_0}{x_i} e^{\frac{x_i^2}{2\theta_0}} = \Big(\frac{\theta_0}{\theta_1}\Big)^n \exp\big(\sum_{i=1}^{n} \frac{x_i^2}{2\theta_0} - \frac{x_i^2}{2\theta_1}\big) = \Big(\frac{\theta_0}{\theta_1}\Big)^n \exp\big((\frac{1}{2\theta_0} - \frac{1}{2\theta_1})\sum_{i=1}^{n} x_i^2\big)$$

we see that it is an increasing function in the statistic $T(\mathbf{x}) = \sum_{i=1}^{n} x_i^2$. We can now do the Test: reject H_0 if

$$\lambda(\mathbf{x}) \geq C^*$$

or equivalently

$$T(\mathbf{x}) \geq C$$

where C fulfills

$$\mathbb{P}_{\theta_0}(T(\mathbf{X}) \geq C) = \alpha$$

so

$$1 - \mathbb{P}_{\theta_0}(T(\mathbf{X}) \leq C) = \alpha \iff \mathbb{P}_{\theta_0}(T(\mathbf{X}) \leq C) = 1 - \alpha$$

we know the distribution of $T(\mathbf{X})$ so we can use the quantile-function

$$C = \theta_0 \chi^2_\alpha(2n)$$

(b) According to theorem on site 33, Lecture 10, the UMP at level α for testing this hypothesis is to reject H_0 if

$$T(\mathbf{X}) \geq C, \quad \text{and} \quad P_{\theta_0}(T(\mathbf{X}) \geq C) = \alpha$$

where $X \sim f_\theta(\mathbf{x})$ belongs to a family of distributions with monotone likelihood ratio in statistic $T(\mathbf{X})$. We have already seen in (a) that the likelihood ratio is monotone in statistic $T(\mathbf{x}) = \sum_{i=1}^{n} x_i^2$ for any θ that fulfills $\theta_0 < \theta$. So the UMP is the test we derived in (a). The power function is defined by

$$\pi(\theta) = \mathbb{P}_\theta(\text{reject } H_0) = \mathbb{P}_\theta(\mathbf{X} \in \Omega_1)$$

where $\Omega_1 = \{\mathbf{x} : T(\mathbf{x}) \geq C\}$ and C is given like above. With $Y \sim \chi^2(2n)$ the power function for fixed α is

$$\pi(\theta) = \mathbb{P}_\theta(T(\mathbf{X}) \geq C) = 1 - \mathbb{P}_\theta(T(\mathbf{X}) \leq C) = 1 - \mathbb{P}_\theta(Y \leq \frac{C}{\theta}) = 1 - F_Y\Big(\frac{\theta_0 \chi^2_\alpha(2n)}{\theta}\Big)$$

4. Most powerful test for the normal variance - μ is known

Let $X_1, \ldots, X_n$ be i.i.d. $\mathcal{N}(\mu, \sigma^2)$, where μ is known.

(a) Find an MP test at level α for testing two simple hypotheses

$$H_0 : \sigma^2 = \sigma_0^2 \quad vs \quad H_1 : \sigma^2 = \sigma_1^2,\ \sigma_1 > \sigma_0.$$

(b) Show that the MP test is a UMP test for testing

$$H_0 : \sigma^2 \leq \sigma_0^2 \quad vs \quad H_1 : \sigma^2 > \sigma_0^2.$$

Hint: $\sum_i (X_i - \mu)^2 \sim \sigma^2 \chi^2(n)$.

Solution:

We first show the hint again: We know that $\frac{(X_i - \mu)}{\sigma^2} \sim \mathcal{N}(0, 1)$ so $\frac{1}{\sigma^2} \sum_i (X_i - \mu)^2 \sim \chi^2(n)$. To find the MP test we look at the likelihood function

$$\lambda(x) = \frac{L(\sigma_1^2, \mathbf{x})}{L(\sigma_0^2, \mathbf{x})} = \Big(\frac{\sigma_0^2}{\sigma_1^2}\Big)^{n/2} \exp\Big(\big(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}\big) \sum_i (x_i - \mu)^2\Big)$$

Because $\sigma_1 > \sigma_0$ this is monotone in the statistic $T(\mathbf{X}) = \sum_i (X_i - \mu)^2$. We reject H_0 if

$$T(\mathbf{x}) \geq C$$

where C fulfills

$$\mathbb{P}_{\sigma_0^2}(T(\mathbf{x}) \geq C) = \alpha$$

so just as in the last exercise

$$C = \sigma_0^2 \chi_\alpha^2(n).$$

(b) As we have seen in (a), the likelihood is monotone in $T(\mathbf{X}) = \sum_i (X_i - \mu)^2$ so the fact that the MP test is a UMP test for testing this hypothesis follows from the theorem on site 33, Lecture 10.

5. Most powerful test for the normal variance - μ is unknown

Let $X_1, \dots, X_n$ be i.i.d. $\mathcal{N}(\mu, \sigma^2)$, where μ is unknown.

(a) Is there an MP test at level α for testing?

$$H_0 : \sigma^2 = \sigma_0^2 \quad vs \quad H_1 : \sigma^2 = \sigma_1^2,\ \sigma_1 > \sigma_0.$$

If not, find the corresponding GLRT.

(b) Is the above generalized likelihood ratio (GLR) test also a GLRT for testing the one-sided hypothesis?

$$H_0 : \sigma^2 \leq \sigma_0^2 \quad vs \quad H_1 : \sigma^2 > \sigma_0^2.$$

(c) Find the GLRT at level α for testing

$$H_0 : \sigma^2 \geq \sigma_0^2 \quad vs \quad H_1 : \sigma^2 < \sigma_0^2.$$

Solution: