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Numerik von Differentialgleichungen - Kreuzlübung 5

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Exercise 21:
The following initial value problem models the evolution of three species x, y, z:

$$\begin{aligned} x'(t) &= -0.04x(t) + 10^4 y(t)z(t), \\ y'(t) &= 0.04x(t) - 10^4 y(t)z(t) - 3 \cdot 10^7 y(t)^2, \\ z'(t) &= 3 \cdot 10^7 y(t)^2, \end{aligned}$$

with initial values $x(0) = 1, y(0) = 0, z(0) = 0$ and $t \in [0, 1]$. Approximate the solution of above equation with the embedded RK5(4) method with different error tolerances and look at the resulting step sizes. Furthermore, approximate the solution with the explicit RK4 method and the implicit Euler method. Try to optimize the step sizes. Discribe your observations. Which of the used methods do you think is the most efficient for this example?

Hint: You can use the programs provided in TUWEL.

Exercise 22:
Let $x_0, \ldots, x_r \in [0, h]$ with $x_i \neq x_j$ für $1 \leq i < j \leq r$. For arbitrary $g \in C^r([a, b])$, let $p \in \Pi_r$ be the solution of

$$p(x_0) = g(x_0), \qquad p'(x_j) = g'(x_j), \quad j = 1, \ldots, r. \tag{1}$$

a) Show that this problem has is uniquely solveable.
b) Prove the following upper bound for the error:

$$\sup_{x \in [0,h]} |p(x) - g(x)| \leq \frac{h^{r+1}}{r!} \sup_{x \in [0,h]} \left| g^{(r)}(x) \right|. \tag{2}$$

Exercise 23:
Prove which of the following mappings define norms on the space Π_r of polynomials of maximum degree r.
a) $\|p\|_k := \sup_{x \in [a,b]} |p(x)| + \sup_{x \in [a,b]} |p^{(k)}(x)|$ for $k \in \mathbb{N}$.
b) $\|p\| := \sup_{j=0,\ldots,r} |p(x_j)|$ with pairwise different $x_j \in [a, b]$ for $j = 0, \ldots, r$.
c) $\|p\|_k := |p(x_0)| + \sup_{j=1,\ldots,r} |p^{(k)}(x_j)|$ for $k \in \mathbb{N}$, $x_0 \in [a, b]$ and pairwise different $x_j \in [a, b]$ for $j = 1, \ldots, r$.
d) $\|p\|_k := |p(x_0)| + |p(x_1)| + \sup_{j=2,\ldots,r} |p^{(k)}(x_j)|$ for $k \in \mathbb{N}$, $x_0 \neq x_1 \in [a, b]$ and pairwise different $x_j \in [a, b]$ for $j = 2, \ldots, r$.

Hint to the case $k = 1$ for d): Show that the mapping is a norm if and only if

$$\int_{x_0}^{x_1} \prod_{j=2}^{r} (\xi - x_j)\, d\xi \neq 0.$$

Exercise 24:
Compute the Butcher-tableaux of all 3-step, implicit Runge-Kutta methods which use the nodes of an open or closed Newton-Cotes Formula as collocation points.

Exercise 25:
Let $M \in \mathbb{R}^{n\times n}$ with $n \in \mathbb{N}$ be a diagonizable matrix and $y : [0,T] \to \mathbb{R}^n$ be the solution of the initial value problem

$$y' = My, \qquad y(0) = y_0 \in \mathbb{R}^n.$$

Show that a Runge-Kutta method applied to this initial value problem is the same (in which sense?) as the Runge-Kutta method applied to the separated initial value problems

$$\tilde{y}_j : [0,T] \to \mathbb{C} : \qquad \tilde{y}_j' = \lambda_j \tilde{y}_j, \qquad \tilde{y}_j(0) = \tilde{y}_{j,0} \in \mathbb{C}, \qquad j = 1, \ldots, n,$$

where λ_j for $j = 1, \ldots, n$ are the eigenvalues of M according to their multiplicity. How should the initial values $\tilde{y}_{j,0}$ be chosen?