The problems are to be presented on **May 18, 2021**. They should be ticked and solution paths uploaded by **23:59 on May 17, 2021**.

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(1) **Cramér-Rao lower bound - Simulation**
In Homework 7 Exercise 2 a density $f(x|\theta) = \theta x^{\theta-1}$ for $0 < x < 1$ and $\theta > 0$ was given. The goal was to find a sutable function g of the parameter θ such that there exists an unbiased estimator of $g(\theta)$ which attains the Cramér-Rao lower bound.

A unbiased statistic which attains the Cramér-Rao lower bound is for $g(\theta) = \frac{1}{\theta}$ given by

$$S_n(X_1, ..., X_n) = -\frac{1}{n}\sum_{i=1}^{n} \ln(X_i).$$

Implement the following steps in `R`:

(a) Write pdf `dhw`, cdf `phw`, quantile `qhw` and random sampling function `rhw` for the above distribution parameterized by θ (see for example `?runif, ?rnorm`).
Hint: Given an strict monotone continuous cdf F, then $F^{-1}(U)$ is distributed with cdf F for $U \sim U(0,1)$.

(b) Fix an arbitrary θ and perform a simulation with growing sample size $n =$ 500, 1000, 1500, ..., 10000 each with 100 replications for the estimation of $g(\theta)$ with the statistic S_n.

(c) Create a scatter plot of all the estimates over the sample size, add the sample mean and standard deviation aggregated over the sample size to the plot. Finally, add the theoretical mean and standard deviation of the statistic S_n.

(2) **Sufficient statistic and point estimator statistics**
Let $X_1, \ldots X_n$ be a random sample from a population with pdf

$$f(x|\theta) = \begin{cases} \frac{\theta}{x^2}, & \theta \leq x \\ 0, & \text{otherwise} \end{cases}$$

with unknown $\theta > 0$. Use the Factorization theorem to obtain a sufficient statistic for θ.

(3) **Minimal sufficient statistic 1**
Let $X_1, \ldots, X_n$ be a random sample from a population with $\mathcal{N}(\mu, \mu)$ distribution, where $\mu > 0$ is unknown.

(a) Show that the statistic $\sum X_i^2$ is minimal sufficient in the $\mathcal{N}(\mu, \mu)$ family.

(b) Show that the statistic $(\sum X_i, \sum X_i^2)$ is sufficient but not minimal sufficient in the $\mathcal{N}(\mu, \mu)$ family.

(4) **Minimal sufficient statistic 2**

Let $X_1, \ldots, X_n$ be a random sample from a population with pdf

$$f(x|\theta) = \begin{cases} \frac{2x}{\theta}, & 0 < x < \theta \\ 0, & \text{otherwise} \end{cases},$$

with unknown parameter $\theta > 0$. Find a minimal sufficient statistic for θ.

(5) **Sufficiency, bias, Rao-Blackwell theorem**

Let $X_1, \ldots, X_n$ be i.i.d. $Poi(\lambda)$, with unknown $\lambda > 0$.

(a) Show that $Y = \sum_{n=1}^{n} X_i$ is a sufficient statistic for λ.

(b) Find an unbiased estimator of $p_r = P(X = r)$, which depends only on X_1.
Find $P(X_1 = r|Y = k)$ both for $k \geq r$ and $k < r$.
Hence use the Rao-Blackwell theorem to improve your estimator of p_r.