4. *(Galerkin method for the Poisson equation)* Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with smooth boundary. For $f \in L^2(\Omega)$ construct a solution of the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1}$$

using a Galerkin method. To do this, let $\{\phi_k\}$ for $k \in \mathbb{N}$ denote the eigenfunctions of the Laplacian with homogeneous Dirichlet boundary data on Ω. Then prove that for any $m \in \mathbb{N}$ there exists

$$u_m = \sum_{k=1}^{m} \mathbf{d}_m^k \phi_k, \quad \text{for } \mathbf{d}_m^k \in \mathbb{R}$$

that satisfies

$$\int_\Omega \nabla u_m \cdot \nabla \phi_k \,\mathrm{d}x = \int_\Omega f\phi_k, \quad \text{for } k = 1, \dots, m.$$

To finish, show that the sequence $\{u_m\}_{m\in\mathbb{N}}$ converges weakly in $H_0^1(\Omega)$ to a weak solution of (1).

Wir betrachten den Operator $K : L^2(\Omega) \to L^2(\Omega) : f \mapsto u$, wobei u die eindeutig bestimmte schwache Lsg. von $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{auf } \partial\Omega \end{cases}$, ist.

Folgen wir dem Skriptum, so sehen wir: K ist injektiv (Kern ist 0) also definieren wir $L : \overbrace{K(L^2(\Omega))}^{D(L):=} \to L^2(\Omega) : u \mapsto K^{-1}(u)$

Aus Lemma 6.14 wissen wir $H^2(\Omega) \cap H_0^1(\Omega) \subseteq D(L) \subseteq H_0^1(\Omega)$

wenn $\partial\Omega \in C^2$, dann gilt sogar $D(L) = H^2(\Omega) \cap H_0^1(\Omega)$ und vermutlich $Lu = -\Delta u$

$0 \notin \sigma_p(L)$, da L injektiv, also $Lu = 0u = 0 \Rightarrow u = 0$

Sei $\lambda \in \sigma_p(L)$, also $\phi \in D(L)\setminus\{0\}: L\phi = \lambda\phi$, es gilt $\psi \in L^2(\Omega): \phi = K\psi \Rightarrow \lambda K\psi = LK\psi = \psi \Rightarrow K\psi = \frac{1}{\lambda}\psi$

also ist $\frac{1}{\lambda} \in \sigma_p(K) = \{\mu_k \mid k \in \mathbb{N}\}$, umgekehrt $\mu \in \sigma_p(K) \Rightarrow \frac{1}{\mu} \in \sigma_p(L)$

Wir können o.B.d.A sagen, dass $\{\phi_k \mid k \in \mathbb{N}\}$ ein ONS

$Kf = \sum_{k\in\mathbb{N}} \mu_k \langle f, \phi_k\rangle_{L^2} \phi_k$; $d_m^k := \mu_k = \frac{1}{\lambda_k}$, wobei $L\phi_k = \lambda_k \phi_k$

Sei $w \in H_0^1$ bel.

$$|\langle Kf - u_m, w\rangle_{L^2(\Omega)}| = \Big|\sum_{k=m+1}^{\infty} \mu_k \langle f, \phi_k\rangle_{L^2} \langle \phi_k, w\rangle_{L^2}\Big| \leq \sum_{k=m+1}^{\infty} |\sqrt{\mu_k}\langle f, \phi_k\rangle \sqrt{\mu_k}\langle \phi_k, w\rangle|$$
$$\leq \sqrt{\sum_{k=m+1}^{\infty} \mu_k |\langle f, \phi_k\rangle|^2}\sqrt{\sum_{k=m+1}^{\infty} \mu_k |\langle \phi_k, w\rangle|^2} \xrightarrow{m\to\infty} 0$$

$$|\langle \nabla(Kf - u_m), \nabla w\rangle_{L^2}| = |\langle \nabla Kf, \nabla w\rangle_{L^2} - \langle \nabla u_m, \sum_{k=1}^{\infty} \langle w, \phi_k\rangle \nabla\phi_k\rangle|$$
$$= |\langle \nabla Kf, \nabla w\rangle_{L^2} - \sum_{k\in\mathbb{N}} \langle w, \phi_k\rangle \langle \nabla u_m, \nabla\phi_k\rangle|$$
$$= \Big|\sum_{k\in\mathbb{N}} \langle w, \phi_k\rangle \langle \nabla Kf, \nabla\phi_k\rangle - \sum_{k=m+1}^{\infty} \langle w, \phi_k\rangle \langle \nabla u_m, \nabla\phi_k\rangle - \sum_{k=1}^{m} \langle w, \phi_k\rangle\langle f, \phi_k\rangle\Big|$$
$$= \Big|\sum_{k\in\mathbb{N}} \langle w, \phi_k\rangle \sum_{j\in\mathbb{N}} \mu_j \langle f, \phi_j\rangle \langle \nabla\phi_j, \nabla\phi_k\rangle - \sum_{k=m+1}^{\infty} \langle w, \phi_k\rangle \langle \nabla u_m, \nabla\phi_k\rangle - \sum_{k=1}^{m} \langle w, \phi_k\rangle\langle f, \phi_k\rangle\Big|$$
$$= \Big|\sum_{k\in\mathbb{N}} \langle w, \phi_k\rangle \langle f, \phi_k\rangle - \sum_{k=m+1}^{\infty} \langle w, \phi_k\rangle \langle \nabla u_m, \nabla\phi_k\rangle - \sum_{k=1}^{m} \langle w, \phi_k\rangle\langle f, \phi_k\rangle\Big|$$
$$= \Big|\sum_{k=m+1}^{\infty} \langle w, \phi_k\rangle \big(\langle f, \phi_k\rangle - \langle \nabla u_m, \nabla\phi_k\rangle\big)\Big|$$

$$\int_\Omega \nabla\phi_j \nabla\phi_k \,dx = -\int_\Omega \phi_j \Delta\phi_k \,dx + \overbrace{\int_{\partial\Omega} \phi_j (\nabla\phi_k \cdot \nu)\,dx}^{=0} = -\lambda_k \int_\Omega \phi_j \phi_k \,dx$$

$$\langle \nabla u_m, \nabla \phi_k \rangle = \sum_{j=1}^{m} d_m^j \langle \nabla \phi_j, \nabla \phi_k \rangle = \begin{cases} 0, & \text{falls } k > m \\ \dots, & \text{sonst} \end{cases}$$

$$= \left| \sum_{k=m+1}^{\infty} \langle w, \phi_k \rangle \langle f, \phi_k \rangle \right| \leq \sqrt{\sum_{k=m+1}^{\infty} |\langle w, \phi_k \rangle|^2} \sqrt{\sum_{k=m+1}^{\infty} |\langle f, \phi_k \rangle|^2} \xrightarrow{m \to \infty} 0$$

$\dots$

z.z.: $\nabla \underbrace{\sum_{k \in \mathbb{N}} \mu_k \langle f, \phi_k \rangle \phi_k}_{= Kf} = \sum_{k \in \mathbb{N}} \mu_k \langle f, \phi_k \rangle \nabla \phi_k$

$$\sum_{k \in \mathbb{N}} \| \mu_k \langle f, \phi_k \rangle \phi_k \|_{H_0^1}$$

$$\left\| \nabla \left(\sum_{k \in \mathbb{N}} \mu_k \langle f, \phi_k \rangle \phi_k \right) - \sum_{k \in \mathbb{N}} \mu_k \langle f, \phi_k \rangle \nabla \phi_k \right\|_{L^2}^2$$

$$= \| \nabla Kf \|_{L^2}^2 - 2 \left\langle \nabla Kf, \sum_{k \in \mathbb{N}} \mu_k \langle f, \phi_k \rangle \nabla \phi_k \right\rangle_{L^2} + \left\| \sum_{k \in \mathbb{N}} \mu_k \langle f, \phi_k \rangle \nabla \phi_k \right\|_{L^2}^2$$

$$\langle \nabla Kf, \nabla \phi_k \rangle \overset{\text{Gauß}}{=} \int_\Omega \nabla Kf \, \nabla \phi_k \, dx = \int_\Omega \phi_k (-\Delta Kf) \, dx + \int_{\partial\Omega} \underbrace{\phi_k}_{=0 \text{ am Rand}} (\nabla Kf \cdot \nu) \, ds = \langle (-\Delta) Kf, \phi_k \rangle$$

$$\sum_{k \in \mathbb{N}} \mu_k \langle f, \phi_k \rangle \langle \nabla Kf, \nabla \phi_k \rangle = \sum_{k \in \mathbb{N}} \mu_k \langle f, \phi_k \rangle \langle \underbrace{(-\Delta) Kf}_{= f}, \phi_k \rangle = \sum_{k \in \mathbb{N}} |\mu_k \langle f, \phi_k \rangle|^2 = \| Kf \|_{L^2}^2$$

z.z.: $\exists v \in H_0^1 : \forall w \in H_0^1 : \lim_{m\to\infty} \langle u_m, w\rangle = \langle v, w\rangle$

$$|\langle u_m, w\rangle_{H_0^1} - \langle u_n, w\rangle_{H_0^1}| = |\langle u_m - u_n, w\rangle_{H_0^1}| = |\langle u_m - u_n, w\rangle_{L^2} + \int_{\Omega} \nabla(u_m - u_n)\cdot\nabla w \, dx| =$$

Sei v die schwache Lsg.

$$|\langle v - u_m, w\rangle_{H_0^1}| = |\langle v - \sum_{k=1}^{m} d_m^k \phi_k, w\rangle_{H_0^1}| = |\langle v - \sum_{k=1}^{m} \frac{1}{\lambda_k}\langle f, \phi_k\rangle \phi_k, w\rangle_{H_0^1}| =$$

$$= |\langle v, w\rangle - \langle u_m, w\rangle|$$

$$\langle v, w\rangle_{H_0^1} = \langle v, w\rangle_{L^2(\Omega)} + \langle \nabla v, \nabla w\rangle_{L^2(\Omega)} = \langle v, w\rangle_{L^2(\Omega)} + \langle f, w\rangle_{L^2(\Omega)} = \langle v, w\rangle + \sum_{j\in\mathbb{N}} \langle w, \phi_j\rangle\langle f, \phi_j\rangle$$

$$\langle u_m, w\rangle_{H_0^1} = \langle u_m, w\rangle_{L^2(\Omega)} + \langle \nabla u_m, \nabla w\rangle_{L^2(\Omega)}$$

$$= \langle u_m, w\rangle_{L^2(\Omega)} + \langle \nabla u_m, \nabla \sum_{j\in\mathbb{N}} \langle w, \phi_j\rangle \phi_j\rangle_{L^2(\Omega)}$$

$$\overset{?}{=} \langle u_m, w\rangle_{L^2(\Omega)} + \sum_{j\in\mathbb{N}} \langle w, \phi_j\rangle \langle \nabla u_m, \nabla\phi_j\rangle_{L^2(\Omega)}$$

$$= \langle u_m, w\rangle_{L^2(\Omega)} + \sum_{j=m+1}^{\infty} \langle w, \phi_j\rangle \langle \nabla u_m, \nabla\phi_j\rangle_{L^2(\Omega)} + \sum_{j=1}^{m} \langle w, \phi_j\rangle \langle f, \phi_j\rangle \cdot$$

$$= \langle u_m, w\rangle_{L^2(\Omega)} + \sum_{j=m+1}^{\infty} \langle w, \phi_j\rangle \langle \nabla u_m, \nabla\phi_j\rangle_{L^2(\Omega)} + \langle f, \sum_{j=1}^{m} \langle w, \phi_j\rangle, \phi_j\rangle$$

?

also $\lim_{m\to\infty} \langle \nabla u_m, \nabla w\rangle_{L^2} \to \langle \nabla v, \nabla w\rangle$

$$|\langle v - u_m, w\rangle_{H_0^1}| = |\langle v - u_m, w\rangle_{L^2(\Omega)} + \sum_{j=m+1}^{\infty} \langle w, \phi_j\rangle_{L^2} (\langle f, \phi_j\rangle_{L^2} - \langle \nabla u_m, \nabla\phi_j\rangle_{L^2})| \leq$$

$$\leq |\langle v - u_m, w\rangle_{L^2(\Omega)}| + |\sum_{j=m+1}^{\infty} \langle w, \phi_j\rangle_{L^2} (\langle f, \phi_j\rangle_{L^2} - \langle \nabla u_m, \nabla\phi_j\rangle_{L^2})|$$