(3) **Minimal sufficient statistic 1**
Let $X_1, \ldots, X_n$ be a random sample from a population with $\mathcal{N}(\mu, \mu)$ distribution, where $\mu > 0$ is unknown.

(a) Show that the statistic $\sum X_i^2$ is minimal sufficient in the $\mathcal{N}(\mu, \mu)$ family.

(b) Show that the statistic $(\sum X_i, \sum X_i^2)$ is sufficient but not minimal sufficient in the $\mathcal{N}(\mu, \mu)$ family.

a) $L(x|\mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\mu}} \exp\left(\frac{-(x_i-\mu)^2}{2\mu}\right) = (2\pi\mu)^{-\frac{n}{2}} \exp\left(\sum_{i=1}^{n} \frac{1}{2\mu}\left(2x_i\mu - x_i^2 - \mu^2\right)\right)$

$= (2\pi\mu)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\mu}\sum_{i=1}^{n} x_i^2 - \frac{n\mu}{2}\right) \exp\left(\sum_{i=1}^{n} x_i\right) = g(T(x)|\mu)\, h(x)$, where

$g(z|\mu) = (2\pi\mu)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\mu} z - \frac{n\mu}{2}\right)$ and $h(x) = \exp\left(\sum_{i=1}^{n} x_i\right)$

Hence, $T(x) = \sum_{i=1}^{n} x_i^2$ is sufficient.

$\frac{L(x|\mu)}{L(y|\mu)} = \exp\left(\frac{1}{2\mu}\left(\sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{n} x_i^2\right)\right) \frac{h(x)}{h(y)}$ is constant as a function of μ, if and only if $T(y) = \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} x_i^2 = T(x)$, hence $T(X)$ is minimal sufficient.

b) $S(x) = \left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i^2\right)$ is clearly sufficient, since one component is $T(X)$, simply take $\tilde{g}(z|\mu) = (2\pi\mu)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\mu} z_2 - \frac{n\mu}{2}\right)$

Assume there was a function $r: \mathbb{R} \to \mathbb{R}^2$ such that $r(T(X)) = S(X)$.

We have $1 = 1 + \sum_{i=2}^{n} 0 = \pi_1\left(r\left(1^2 + \sum_{i=2}^{n} 0^2\right)\right) = \pi_2\left(r\left((-1)^2 + \sum_{i=2}^{n} 0^2\right)\right) = -1$ ↯

Hence, such a function r can not exist and $S(X)$ is bigger than $T(X)$.