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Numerik von Differentialgleichungen - Kreuzlübung 1

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Exercise 1:
For the given initial value problem $y'(t) = ty(t)$, $t \in [0, T]$, with $y(0) = 1$,
a) find a fixed-point formulation $y = \Phi(y)$ and use a fixed-point iteration of the form $y_{k+1} = \Phi(y_k)$ to find the solution.
b) approximate the inital value problem by an implementation of the explicit Euler method in a programming language of your choice. Use an equidistant partition of the interval $[0, 1]$. Analyze the error at the time $t = 1$ depending on the partition.

Exercise 2:
Let $A, M \in \mathbb{R}^{n\times n}$ be symmetric and positiv definite, and $f \in C([0, T], \mathbb{R}^n)$. Moreover, let $y_{y_0} \in C^1([0, T], \mathbb{R}^n)$ be a solution of the initial value problem

$$My'(t) = -Ay(t) + f(t), \qquad t \in [0, T], \qquad y(0) = y_0$$

for an arbitrary $y_0 \in \mathbb{R}^n$. Show that for any $t \in [0, T]$ the mapping $y_0 \mapsto y_{y_0}(t)$ is Lipschitz with constant 1 with respect to the norm induced by M, given by $\|\cdot\|_M : x \mapsto \sqrt{x^\top Mx}$. Is this problem well-conditioned in this sense?

Exercise 3:
Let $y \in C^1(\mathbb{R}_{\geq 0}, \mathbb{R})$ solve the initial value problem

$$y'(t) = \lambda y(t), \quad t > 0, \qquad y(0) = y_0$$

for some $\lambda < 0$. Let $h > 0$ be a constant step size, $t_j := jh$, $j \in \mathbb{N}_0$, and y^{e}_j, y^{i}_j the approximation to $y(t_j)$ by the explicit and implicit Euler method, respectively. Analyze the behavior of y^{e}_j and y^{i}_j for $j \to \infty$ depending on λ and h and compare the behavior to the one of the exact solution $y(t_j)$.

Exercise 4:
Prove the following version of Theorem 1.3: Let f be one-sided Lipschitz continuous with respect to the second argument, i.e. there exists $L_+ \in \mathbb{R}$ such that

$$\langle f(t, y) - f(t, z), y - z\rangle_2 \leq L_+\|y - z\|_2^2, \qquad (t, y),(t, z) \in J \times \Omega.$$

Moreover, let z be another solution of $z' = f(t, z)$ (i.e. $\delta = 0$ in Theorem 1.3). Then

$$\|y(t) - z(t)\|_2 \leq \|y(t_0) - z(t_0)\|_2 e^{L_+(t-t_0)}, \qquad t \geq t_0.$$

Exercise 5:
Let $A \in \mathbb{R}^{n \times n}$ be symmetric and $y \in C^1([0,T], \mathbb{R}^n)$ solves the initial value problem

$$y'(t) = Ay(t), \qquad t \in [0,T], \qquad y(0) = y_0.$$

Find the Lipschitz constant as well as the one-sided Lipschitz constant of the according function f and compare the statement of Theorem 1.3 to the one from exercise 4.

Hint: Symmetric matrices are diagonalizable.