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Numerik von Differentialgleichungen - Kreuzlübung 9

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Exercise 41:
Prove that the k-step BDF-formulas from Exercise 37 satisfy the root condition $k = 1, \dots 6$. Furthermore, prove that they do not satisfy the root condition for $k = 7, \dots, 10$.

Hint: The coefficients of the k-step BDF-formulas can be computed with a computer algebra system. The methods take the form

$$\sum_{j=0}^{k} \alpha_{k-j} y_{\ell+1-j} = h f_{\ell+1}, \quad \alpha_{k-j} = h L_j'(t_{\ell+1}), \quad L_j(t) := \prod_{\substack{m=0 \\ m \neq j}}^{k} \frac{t - t_{\ell+1-m}}{t_{\ell+1-j} - t_{\ell+1-m}}. \tag{1}$$

Additional information: One can show that all BDF-formulas with $k \geq 7$ do not satisfy the root condition.

Exercise 42:
Generalize the Proof of Theorem 5.35 of the lecture notes for the general case $n \in \mathbb{N}$. To this end, you must essentially show that:

a) With the adapted definitions from the lecture notes there holds

$$E_{\ell+1} = \left(A_\rho^\top \otimes I\right) E_\ell + F_\ell, \tag{2}$$

where $I \in \mathbb{R}^{n \times n}$ denotes the identity matrix and $A \otimes B$ denotes the Kronecker-product of two matrices $A \in \mathbb{R}^{k \times k}$ and $B \in \mathbb{R}^{n \times n}$, i.e.,

$$A \otimes B := \begin{pmatrix} A_{11}B & \dots & A_{1k}B \\ \vdots & & \vdots \\ A_{k1}B & \dots & A_{kk}B \end{pmatrix} \in \mathbb{R}^{kn \times kn}. \tag{3}$$

b) From the root condition there follows

$$\sup_{k \in \mathbb{N}_0} \left\| \left(A_\rho^\top \otimes I\right)^k \right\|_\infty < \infty. \tag{4}$$

You should be able to explain why these are the essential changes compared to the scalar case!

Exercise 43:
Consider two Adams-Bashforth methods with k and $k+1$ steps, respectively, and the same step-size h. According to Definition 5.9. of the lecture notes, let $y_{\ell+1}$ be the solution of the k-step method

and $\tilde{y}_{\ell+1}$ the solution of the $(k+1)$-step method with exact initial values in both cases. Derive a computable error estimator μ for the consistency error $\tau_\ell(h)$ of the first method. There should hold

$$\tau_\ell(h) = \mu + \mathcal{O}(h^{k+3}).$$

To this end, use the expansion of the truncation error from the proof of Theorem 5.15 and retrace the construction of the error estimator in Section 2.6 and 2.7.

Exercise 44:
Use different linear multi-step methods to solve the initial value problem for the heat equation from Exercise 29. To this end use:
a) The Adams-Bashforth method with $k = 3$.
b) The Adams-Moulton method with $k = 2, 3$.
c) The BDF-method

$$y_{\ell+1} - \frac{48}{25} y_\ell + \frac{36}{25} y_{\ell-1} - \frac{16}{25} y_{\ell-2} + \frac{3}{25} y_{\ell-3} = h \frac{12}{25} f\left(t_{\ell+1}, y_{\ell+1}\right).$$

For each method, optimize the step-size h, i.e., choose h as large as possible, but $\|U(t)\|_\infty$ should be monotonically decreasing in time. What do you observe?

Hint: For the implicit methods you don't need the Newton iteration, but you can solve the arising equations explicitely. Pay attention to Exercise 29c.

Exercise 45:
Show that the stability region of an explicit, linear multi-step method is bounded and therefore the underlying method is not A-stable.

Hint: Assume that the moduli of the roots of $\xi \mapsto \rho(\xi) - z\sigma(\xi)$ are smaller than one and find an estimate which is uniform in z. Use this to construct a contradiction.