Spectral Graph Theory

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There are many different ways to associate a matrix with a graph (an introduction of which can be found in Chapter 28 on Matrices and Graphs). The focus of spectral graph theory is to examine the eigenvalues (or spectrum) of such a matrix and use them to determine structural properties of the graph; or conversely, given a structural property of the graph determine the nature of the eigenvalues of the matrix.

Some of the different matrices we can use for a graph G include the adjacency matrix $\mathcal{A}_G$, the (combinatorial) Laplacian matrix L_G, the signless Laplacian matrix $|L_G|$, and the normalized Laplacian matrix $\mathcal{L}_G$. The eigenvalues of these different matrices can be used to find out different information about the graph. No one matrix is best because each matrix has its own limitations in that there is some property which the matrix cannot always determine (this shows the importance of understanding all the different matrices).

In the table below we summarize the four matrices and four different structural properties of a graph and indicate which properties can be determined by the eigenvalues. A "No" answer indicates the existence of two non-isomorphic graphs which have the same spectrum but differ in the indicated structure.

Matrix	bipartite	# components	# bipartite components	# edges
Adjacency — $\mathcal{A}_G$	Yes	No	No	Yes
(combinatorial) Laplacian — L_G	No	Yes	No	Yes
Signless Laplacian — $\lvert L_G \rvert$	No	No	Yes	Yes
Normalized Laplacian — $\mathcal{L}_G$	Yes	Yes	Yes	No

In this chapter we will introduce the normalized Laplacian and give some basic properties of this matrix and some additional applications of the eigenvalues of graphs. This chapter will focus more on analytic aspects, whereas Chapter 28 focused more on the algebraic aspects. Both sides are important to spectral graph theory and powerful when used correctly.

In addition to Chapter 28 and the references contained therein there are a number of excellent books [Chu97, Van11] and surveys [HLW06, KS06] that can be consulted.

1 The normalized Laplacian matrix

In this section we introduce the normalized Laplacian matrix alluded to in the introduction. This has intimate ties to random walks and so has many useful properties; unfortunately it also is poor at counting certain structures (i.e., edges).

Definitions:

A **weighted graph** G is a graph with vertices $v_1, v_2, \ldots, v_n$ and a **weight function**, w, satisfying $w(v_i, v_j) = w(v_j, v_i)$ (symmetric) and $w(v_i, v_j) \geq 0$ (nonnegative). In this setting edges correspond to when $w(v_i, v_j) > 0$ and non-edges when $w(v_i, v_j) = 0$; note that we allow loops but do not allow multiple edges (opting instead to modify the weight).

A **simple graph** is a weighted graph where $w(v_i, v_j) \in \{0, 1\}$ and $w(v_i, v_i) = 0$ for all v_i.

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The **degree** of a vertex v_i in a weighted graph is $d_i = \sum_{v_j} w(v_i, v_j)$. A vertex is **isolated** if $d_i = 0$.

Given a subset X of the vertices the **volume of** X is $\mathrm{vol}(X) = \sum_{v_i \in X} d_i$.

We let $\mathrm{vol}(G)$ denote the sum of all the degrees.

Given subsets X and Y of the vertices (not necessarily disjoint), $e(X, Y) = \sum_{x \in X, y \in Y} w(x, y)$.

The **adjacency matrix** of a weighted graph, $\mathcal{A}_G$, is the real symmetric $n \times n$ matrix with entries $(\mathcal{A}_G)_{i,j} = w(v_i, v_j)$.

The **diagonal degree matrix** of a weighted graph, D_G, is the diagonal matrix with $(D_G)_{i,i} = d_i$.

The **(combinatorial) Laplacian** of a weighted graph, L_G, is defined as $L_G = D_G - \mathcal{A}_G$.

The **signless Laplacian** of a weighted graph, $|L_G|$, is defined as $|L_G| = D_G + \mathcal{A}_G$.

The **normalized Laplacian matrix** of a weighted graph, $\mathcal{L}_G$, is the symmetric $n \times n$ matrix with entries defined as follows:

$$(\mathcal{L}_G)_{i,j} = \begin{cases} 1 - \frac{w(v_i, v_i)}{d_i} & \text{if } i = j \text{ and } d_i \neq 0 \\ -\frac{w(v_i, v_j)}{\sqrt{d_i d_j}} & \text{if } i \neq j \text{ and } w(v_i, v_j) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Facts:

If no reference is given, the results are elementary or can be found in [Chu97].

1. If the weighted graph G has no isolated vertices then $\mathcal{L}_G = D_G^{-1/2} L_G D_G^{-1/2}$.

2. If the weighted graph G has each vertex of degree $k > 0$, then $\mathcal{L}_G = I - \frac{1}{k}\mathcal{A}_G = \frac{1}{k}L_G$. In particular, if α is an eigenvalue of $\mathcal{A}_G$ then $1 - \frac{1}{k}\alpha$ is an eigenvalue of $\mathcal{L}_G$.

3. [But08, Section 1.5] Let δ and Δ denote the minimum and maximum degrees of a weighted graph G without isolated vertices respectively. Further, let $\phi_0 \leq \phi_1 \leq \cdots \leq \phi_{n-1}$, $q_0 \leq q_1 \leq \cdots \leq q_{n-1}$ and $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ denote the eigenvalues of L_G, $|L_G|$ and $\mathcal{L}_G$ respectively. Then the following holds for $0 \leq i \leq n-1$:[1]
   - $\dfrac{1}{\Delta}\phi_i \leq \lambda_i \leq \dfrac{1}{\delta}\phi_i$
   - $\dfrac{1}{\Delta}q_{n-1-i} \leq 2 - \lambda_i \leq \dfrac{1}{\delta}q_{n-1-i}$
   - $2\delta \leq \phi_i + q_{n-1-i} \leq 2\Delta$

4. The eigenvalues of $\mathcal{L}_G$ for a weighted graph G are real numbers satisfying $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1} \leq 2$. Further, the multiplicity of 0 as an eigenvalue is the number of connected components of G. In particular G is connected if and only if $\lambda_1 > 0$. The multiplicity of 2 as an eigenvalue is the number of bipartite connected components of G with at least two vertices.

5. A weighted graph is bipartite with no isolated vertices if and only if the spectrum of $\mathcal{L}_G$ is symmetric around 1.

6. For $n \geq 2$ the eigenvalues of the normalized Laplacian of the complete simple graph are 0 and $\frac{n}{n-1}$ (with multiplicities 1 and $n-1$ respectively); for $p + q = n$ the eigenvalues of the simple graph $K_{p,q}$ are 0, 1 and 2 (with multiplicities 1, $n-2$ and 1 respectively). In particular, all complete bipartite simple graphs on n vertices are cospectral.

7. For a simple graph without isolated vertices and $n \geq 2$ vertices then $\lambda_1 = \frac{n}{n-1}$ if and only if the graph is the complete graph on n vertices; if the graph is not the complete graph then $\lambda_1 \leq 1$. Further, if G is a simple graph with no isolated vertices then $\lambda_{n-1} \geq \frac{n}{n-1}$.

[1] Only the proof of the first item is found in [But08, Section 1.5], the second item has an identical proof and the third item is an easy combination of the first two.

8. If the weighted graph G has a set of two or more nonadjacent vertices with the same neighbors and corresponding edge weights then 1 is an eigenvalue of $\mathcal{L}_G$ and 0 is an eigenvalue of $\mathcal{A}_G$. If the weighted graph G has a set of two or more adjacent vertices with degree d and the same (closed) neighborhood and corresponding edge weights then $\frac{d+1}{d}$ is an eigenvalue of $\mathcal{L}_G$ and -1 is an eigenvalue of $\mathcal{A}_G$.

9. Let $\mathbf{1}$ denote the vector of all 1s. Then for a connected weighted graph G with eigenvalues $\lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$ for $\mathcal{L}_G$ the following hold:
   (a) [But08, Section 1.6] $\lambda_{n-1} - \lambda_1 \ge \dfrac{2}{n-1}\sqrt{(n-1)\mathbf{1}^T D_G^{-1}\mathcal{A}_G D_G^{-1}\mathbf{1} - n}$.
   (b) [But08, Section 1.6] $\sqrt{\dfrac{\lambda_{n-1}}{\lambda_1}} + \sqrt{\dfrac{\lambda_1}{\lambda_{n-1}}} \ge 2\sqrt{(1-\frac{1}{n})(1+\frac{1}{n}\mathbf{1}^T D_G^{-1}\mathcal{A}_G D_G^{-1}\mathbf{1})}$.
   (c) [But08, Section 5.6.2] If $X, Y \subseteq V$ and $e(X,Y) = 0$ (i.e., no edges joining X and Y), then
   $$\frac{\operatorname{vol}(X)\operatorname{vol}(Y)}{\operatorname{vol}(\overline{X})\operatorname{vol}(\overline{Y})} \le \left(\frac{\lambda_{n-1} - \lambda_1}{\lambda_{n-1} + \lambda_1}\right)^2.$$

10. Let D be the diameter of the weighted graph G.
    (a) The matrix $\mathcal{L}_G$ has at least $D+1$ distinct eigenvalues.
    (b) $\lambda_1 \ge \dfrac{1}{D\operatorname{vol}(G)}$.
    (c) [Gro05] If Δ is the maximum degree of the graph then $\lambda_1 \ge \dfrac{1}{(\Delta+1)\Delta^{\lceil D/2\rceil - 1}}$.
    (d) If Δ is the maximum degree of the graph then
    $$\lambda_1 \le 1 - 2\frac{\sqrt{\Delta - 1}}{\Delta}\left(1 - \frac{2}{D}\right) + \frac{2}{D}.$$

11. [Kir07] Suppose G a simple graph other than the complete graph. Let X and Y be subsets of the vertices of G with $X \ne Y$ and $X \ne \overline{Y}$, and let $d(X,Y)$ denote the minimum distance between some vertex in X and some vertex in Y. Then
$$d(X,Y) \le \max\left\{ \left\lceil \frac{\log\sqrt{\frac{\operatorname{vol}(\overline{X})\operatorname{vol}(\overline{Y})}{\operatorname{vol}(X)\operatorname{vol}(Y)}}}{\log\frac{\lambda_{n-1}+\lambda_1}{\lambda_{n-1}-\lambda_1}} \right\rceil, 2 \right\}.$$

12. [Chu97, Theorem 8.3] Recall $\mathcal{L}_G(k)$ denotes the sub-matrix of $\mathcal{L}_G$ with the kth row and column removed. For any $1 \le k \le n$, the number of spanning trees in a simple graph G is $\det\big(\mathcal{L}_G(k)\big)\prod_{i\ne k} d_i$.

Examples:

1. The eigenvalues of the normalized Laplacian of the path on n vertices are $1 - \cos\dfrac{\pi k}{n-1}$ for $k = 0, 1, \ldots, n-1$.

2. The eigenvalues of the normalized Laplacian of the n-cube on 2^n vertices are $\frac{2k}{n}$ (with multiplicity $\binom{n}{k}$) for $k = 0, 1, \ldots, n$.

3. [BG11] The two pairs of simple graphs shown in Figure 1 have the same spectra for the normalized Laplacian. These pairs shows that the spectrum of the normalized Laplacian cannot determine the number of edges, cannot determine if a graph is regular, and cannot always distinguish a graph from its subgraph.

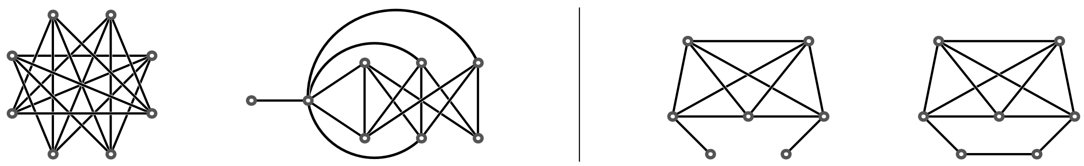

Figure 1: Two examples of pairs of graphs which are cospectral with respect to $\mathcal{L}_G$.

4. Consider the paw graph shown in Figure 2. The normalized Laplcian for this graph is

$$\mathcal{L}_G = \begin{bmatrix} 1 & -\frac{1}{\sqrt{3}} & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 1 & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{\sqrt{6}} & -\frac{1}{2} & 1 \end{bmatrix}$$

This graph has three spanning trees and

$$\det\big(\mathcal{L}_G(1)\big)\prod_{i\neq 1} d_i = \frac{1}{4}\cdot 12 = 3; \qquad \det\big(\mathcal{L}_G(2)\big)\prod_{i\neq 2} d_i = \frac{3}{4}\cdot 4 = 3.$$

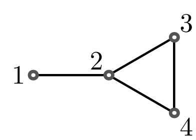

Figure 2: The (labelled) paw graph.

2 Random walks

Random walks on graphs have a variety of applications from mixing and searching to finding cuts and so on. One of the main tools to help understand random walks comes from spectral graph theory. This material is closely related to Markov Chains (see Chapter 54) and Stochastic Matrices (see Section 9.4).

Definitions:

A **probability distribution** on a weighted graph G is an assignment of nonnegative values to the vertices which sum to 1, i.e., a function f with $f(v) \geq 0$ and $\sum_{v\in V} f(v) = 1$. For convenience we will treat f as a row vector.

A **random walk** on a weighted graph consists of starting at a vertex (usually chosen from some initial probability distribution) and then taking a specified number of steps. Each step in the random walk consists of moving from a current vertex to an adjacent vertex where the probability of using an edge is proportional to the weight of that edge.

Given an initial probability distribution $f^{(0)}$, then a **probability distribution of a random walk at time** t, denoted $f^{(t)}$, is the probability distribution reflecting the likelihood of being at some given vertex given that a random walk started with the initial distribution $f^{(0)}$ and took t steps.

A **stationary distribution** of a random walk is a probability distribution π where the distribution is unchanged after one step of the random walk.

A random walk is **ergodic** if it converges to a unique stationary distribution.

A **lazy random walk** with parameter p is a random walk where at each time step we either stay at the current vertex (with probability p) or take one step of the random walk (with probability $1-p$).

If G has no isolated vertices, the **probability transition matrix** $\mathcal{P}_G$ of a random walk (also known as the **walk transition matrix**) is the matrix with entries defined as follows:

$$(\mathcal{P}_G)_{i,j} = \begin{cases} \frac{w(v_i,v_j)}{d_i} & \text{if } v_i \text{ is adjacent to } v_j; \\ 0 & \text{otherwise.} \end{cases}$$

Facts:

The graphs on which we will consider random walks will be weighted graphs.

1. Given a probability distribution $f^{(i)}$, then

$$f^{(i+1)}(v) = \sum_u f^{(i)}(u)\frac{w(u,v)}{d_u}.$$

2. The probability distribution of a random walk after taking t steps is given by $f^{(t)} = f^{(0)}\mathcal{P}_G^t$. A stationary distribution satisfies $\pi = \pi\mathcal{P}_G$ (i.e., corresponds to a left eigenvector of $\mathcal{P}_G$ for the eigenvalue 1).

3. If the graph G has no isolated vertices then $\mathcal{P}_G = D_G^{-1/2}(I-\mathcal{L}_G)D_G^{1/2}$, where D_G is the diagonal degree matrix. In particular, when G has no isolated vertices then $I - \mathcal{L}_G$ and $\mathcal{P}_G$ are cospectral.

4. A lazy random walk is equivalent to taking a random walk on the graph where we either add a loop of weight $pd_i/(1-p)$ or if a loop is already present by increasing the weight of the loop by $pd_i/(1-p)$.

5. [Chu97, Section 1.5] If G is connected and not bipartite then the random walk on the graph is ergodic. (For directed graphs the condition of being bipartite is replaced by being aperiodic.) In particular, for any initial probability distribution $f^{(0)}$ we have

$$\lim_{t\to\infty} f^{(0)}\mathcal{P}_G^t = \pi = \frac{\mathbf{1}^T D_G}{\mathrm{vol}(G)},$$

where $\mathbf{1}$ is the all 1s vector and D_G is the diagonal degree matrix. If G is connected then the lazy random walk on the graph is ergodic.

6. [Chu97, Section 1.5] If $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_{n-1} < 2$ are the eigenvalues of $\mathcal{L}_G$ and we let $\widehat{\lambda} = \max\{1-\lambda_1, \lambda_{n-1}-1\}$ then

$$\|f^{(0)}\mathcal{P}_G^t - \pi\| \leq \widehat{\lambda}^t \frac{\max_x \sqrt{d_x}}{\min_y \sqrt{d_y}}.$$

Examples:

1. [DGM90] Let G be the n-cube graph and let $N = \frac{1}{4}n\log n + cn$. Then for f being the probability distribution with initial weight all at some vertex

$$\left\|f\mathcal{P}_G^N - \frac{1}{2^n}\mathbf{1}^T\right\| \approx \frac{2}{\sqrt{\pi}}\int_0^{e^{-2c}/\sqrt{8}} e^{-t^2}\,dt.$$

This bound is much better than the spectral bound in Fact 6 applied to the cube graph, thus showing some limitations of spectral techniques.

3 Graphs with shared eigenvalues

Cospectral graphs have the same eigenvalues including multiplicities and help us understand weaknesses in identifying structures only using the spectrum. A graph can also share some of its eigenvalues with a smaller related graph.

Definitions:

An **equitable partition** of the vertices of a simple graph G is a partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$ such that for each $v \in V_i$ the number of adjacent vertices in V_j does not depend on v, this includes the situation with $i = j$. In a weighted graph, this is similarly defined but now we need the sum of the weights from $v \in V_i$ to vertices in V_j to be independent of v.

An **automorphism** of a weighted graph G is a mapping $\pi : V(G) \to V(G)$ (i.e., a relabeling of the vertices) such that for each $u, v \in V(G)$ we have $w(u, v) = w\big(\pi(u), \pi(v)\big)$.

A weighted graph G is a **cover** of a weighted graph H if there is a surjective map $\pi : V(G) \to V(H)$ which preserves edge weights so that for each vertex $v \in V(G)$ the restriction of π to the neighborhood of v is a bijection.

Let G be a simple graph with $V = C_1 \cup C_2 \cup \cdots \cup C_k \cup E$ where the graph restricted to $C_1 \cup \cdots \cup C_k$ forms an equitable partition and for each $v \in E$ the number of neighbors of v in C_i is either 0, $\frac{1}{2}|C_i|$ or $|C_i|$. **Godsil-McKay switching** on one of the C_i is the operation taking all of the vertices $v \in E$ which have $\frac{1}{2}|C_i|$ neighbors in C_i and replaces edges by non-edges and vice-versa between E and C_i.

A **distance transitive graph** is a graph where whenever the distance between x and y equals the distance between z and w then there is an automorphism that sends x to z and y to w.

The **join** of two simple graphs G_1 and G_2 is formed by taking disjoint copies of G_1 and G_2 and adding an edge between each vertex of G_1 and each vertex of G_2.

Facts:

1. An automorphism of a weighted graph produces an equitable partition by grouping vertices into orbits.
2. [BH12, Section 6.4] If the weighted graph G is a cover of the weighted graph H then the eigenvalues of $\mathcal{A}_H$ (L_H, $|L_H|$, and $\mathcal{L}_H$ respectively), counting multiplicity, are also eigenvalues of $\mathcal{A}_G$ (L_G, $|L_G|$, and $\mathcal{L}_G$ respectively).
3. [GR01, Section 9.3] Given an equitable partition $V = V_1 \cup \cdots \cup V_k$ of a weighted graph G we can form a matrix $\mathcal{B}$ by letting $\mathcal{B}_{ij} = e(u, V_j)$ where $u \in V_i$ (the matrix $\mathcal{B}$ need not be symmetric and so can be interpreted as the adjacency matrix of a weighted directed graph). Then the eigenvalues of $\mathcal{B}$ are also eigenvalues of $\mathcal{A}_G$ (including multiplicity).
4. Given an equitable partition $V = V_1 \cup \cdots \cup V_k$ of a weighted graph G we can form a new graph H with vertices $v_1, \ldots, v_k$ and edge weights $w(v_i, v_j) = e(V_i, V_j)$. Then the eigenvalues of $\mathcal{L}_H$ are also eigenvalues of $\mathcal{L}_G$ (including multiplicity).[2]
5. [Chu97, Theorem 7.11] The eigenvalues for the normalized Laplacian (not counting multiplicity) of a weighted distance transitive graph of diameter D are the eigenvalues of a normalized Laplacian for a weighted path on $D + 1$ vertices. In particular, such a graph has precisely $D + 1$ distinct eigenvalues.
6. [BH12, Section 1.8.3] Let G be a simple graph and let G' be a graph related to G under Godsil-McKay switching. Then G and G' are cospectral with respect to $\mathcal{A}_G$.
7. [Cav10, Section 2.5] Let G be a simple graph and let G' be a graph related to G under Godsil-McKay switching. If the degrees of G and G' agree, then G and G' are cospectral with respect to $\mathcal{L}_G$.
8. [But08, Section 2.3] Let G_1 be a weighted graph on n vertices with each vertex of degree r and G_2 be a weighted graph on m vertices with each vertex of degree s and let G be the join of G_1 and G_2.

[2] See Example 1 for a proof.

- If $0=\lambda_0^{(1)}\le\lambda_1^{(1)}\le\cdots\le\lambda_{n-1}^{(1)}$ and $0=\lambda_0^{(2)}\le\lambda_1^{(2)}\le\cdots\le\lambda_{m-1}^{(2)}$ are the eigenvalues of $\mathcal{L}_{G_1}$ and $\mathcal{L}_{G_2}$ respectively then the eigenvalues for $\mathcal{L}_G$ are

$$0,\frac{m+r\lambda_1^{(1)}}{m+r},\ldots,\frac{m+r\lambda_{n-1}^{(1)}}{m+r},\frac{n+s\lambda_1^{(2)}}{n+s},\ldots,\frac{n+s\lambda_{m-1}^{(2)}}{n+s},2-\frac{r}{m+r}-\frac{s}{n+s}.$$

- If $\alpha_0^{(1)}\le\alpha_1^{(1)}\le\cdots\le\alpha_{n-1}^{(1)}=r$ and $\alpha_0^{(2)}\le\alpha_1^{(2)}\le\cdots\le\alpha_{m-1}^{(2)}=s$ are the eigenvalues of $\mathcal{A}_{G_1}$ and $\mathcal{A}_{G_2}$ respectively then the eigenvalues for $\mathcal{A}_G$ are

$$\alpha_0^{(1)},\ldots,\alpha_{n-2}^{(1)},\alpha_0^{(2)},\ldots,\alpha_{m-2}^{(2)},\frac{r+s\pm\sqrt{(r-s)^2+4mn}}{2}.$$

9. [But08, Section 4.6.1] Let G be a bipartite simple graph with no isolated vertices where the vertices are partitioned into $V=V_1\cup V_2$ with $|V_1|=p\le|V_2|=q$. Let the graphs G_1 and G_2 be formed as follows (in the notation below we let V_i' denote a copy of V_i):

$$\begin{aligned}V(G_1)&=V_1\cup V_1'\cup V_2\\E(G_1)&=\big\{\{v_1,v_2\},\{v_1',v_2\}\mid\{v_1,v_2\}\in E(G)\big\}\\V(G_2)&=V_1\cup V_2\cup V_2'\\E(G_2)&=\big\{\{v_1,v_2\},\{v_1,v_2'\}\mid\{v_1,v_2\}\in E(G)\big\}\end{aligned}$$

Then the spectrum of $\mathcal{L}_{G_1}$ and $\mathcal{L}_{G_2}$ differ by $q-p$ eigenvalues of 1; the spectrum of $\mathcal{A}_{G_1}$ and $\mathcal{A}_{G_2}$ differ by $q-p$ eigenvalues of 0. In particular if $q=p$ the graphs G_1 and G_2 are cospectral both with respect to the adjacency matrix and the normalized Laplacian matrix.

Examples:

1. Fact 4 shares similarities with Fact 3. However where in Fact 3 the "weight" function from the equitable partition is generally not symmetric, in Fact 4 the resulting weight function will always be symmetric. The two results can be proved in a similar way. Since Fact 4 has not previously appeared we give a proof for completeness. Before we begin we note without loss of generality that we may assume there are no isolated vertices.

Let S be the $n\times k$ matrix defined by $S=D_G^{1/2}\widehat{S}D_H^{-1/2}$ where $\widehat{S}_{ij}=1$ if $v_i\in V_j$ and 0 otherwise. In particular, $S_{ij}=\sqrt{d_i/\operatorname{vol}(V_j)}=\sqrt{1/|V_j|}$ if $v_i\in V_j$; so the columns of S are orthonormal.

We have
$$\mathcal{L}_GS=D_G^{-1/2}(D_G-\mathcal{A}_G)D_G^{-1/2}D_G^{1/2}\widehat{S}D_H^{-1/2}=D_G^{-1/2}(D_G-\mathcal{A}_G)\widehat{S}D_H^{-1/2}.$$

This gives

$$\begin{aligned}\big(\mathcal{L}_GS\big)_{ij}&=\begin{cases}\frac{1}{\sqrt{d_i}}\big(d_i-\sum_{v_k\in V_j}w(v_i,v_k)\big)\frac{1}{\sqrt{\operatorname{vol}(V_j)}}&\text{if } v_i\in V_j\\-\frac{1}{\sqrt{d_i}}\sum_{v_k\in V_j}w(v_i,v_k)\frac{1}{\sqrt{\operatorname{vol}(V_j)}}&\text{if } v_i\in V_\ell\ne V_j\end{cases}\\&=\begin{cases}\frac{1}{\sqrt{|V_j|}}\big(1-\frac{e(V_j,V_j)}{\operatorname{vol}(V_j)}\big)&\text{if } v_i\in V_j\\\frac{1}{\sqrt{|V_\ell|}}\big(-\frac{e(V_\ell,V_j)}{\sqrt{\operatorname{vol}(V_\ell)\operatorname{vol}(V_j)}}\big)&\text{if } v_i\in V_\ell\ne V_j\end{cases}\\&=\frac{1}{\sqrt{|V_\ell|}}\big(\mathcal{L}_H\big)_{\ell j}\qquad(\text{where } v_i\in V_\ell)\\&=(S\mathcal{L}_H)_{ij}.\end{aligned}$$

In going from the first to the second line we used that we have an equitable partition and so for $v_i\in V_\ell$ we have $d_i=\operatorname{vol}(V_\ell)/|V_\ell|$ and also $\sum_{v_k\in V_j}w(v_i,v_k)=e(V_\ell,V_j)/|V_\ell|$, the rest is simplifying. We can conclude that $\mathcal{L}_GS=S\mathcal{L}_H$.

Let T be an $n \times (n-k)$ matrix so that the matrix $(S\ T)$ is invertible (i.e., the matrix formed by first putting in the columns of S followed by the columns of T). For some matrices B and C we have

$$\mathcal{L}_G(S\ T) = (S\ T)\begin{pmatrix} \mathcal{L}_H & B \\ O & C \end{pmatrix}.$$

Since $(S\ T)$ is invertible it follows that the eigenvalues of $\mathcal{L}_H$, including multiplicity, are eigenvalues of $\mathcal{L}_G$.

2. As an example of Fact 2, for $p \geq 3$ the cycle graph C_{pq} is a cover of C_p by the map $\pi(v_i) = v_{(i \mod p)}$. So the eigenvalues of the cycle graph C_p are also eigenvalues of C_{pq}. The collection of pre-images of vertices $u \in C_p$ under the map π form an equitable partition of C_{pq}.

3. As an example of Fact 6, two graph related under Godsil-McKay switching, is shown in Figure 3. (The central vertex corresponds to E and the outer cycle corresponds to a single set C.)

Figure 3: Two graphs related by Godsil-McKay switching.

4. [But08, Section 2.3.1] As an example of Fact 8, the wheel graph on $n+1$ vertices is formed by taking the join of an isolated vertex with a cycle on n vertices. So the eigenvalues of the wheel graph for the normalized Laplacian are:

$$\Big\{0, \underbrace{1 - \frac{2}{3}\cos\frac{2\pi k}{n}}_{k=1,\ldots,n-1}, \frac{4}{3}\Big\}.$$

5. [But08, Section 4.6.1] As an example of Fact 9, the graphs shown in Figure 4 are cospectral by duplicating different sides of a bipartite graphs.

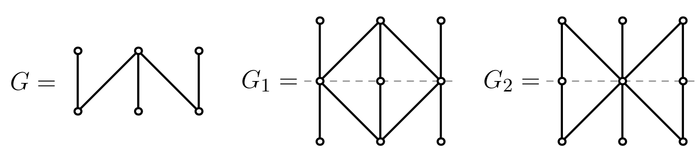

Figure 4: An example of constructing cospectral graphs by duplicating the two different sides of the bipartite graph.

4 Interlacing eigenvalues

Graphs can be related to one another via some small perturbation such as removal of edges or vertices. The result of such a small perturbation changes the eigenvalues; however the different spectra are still closely related and often interlace. This can allow us to make conclusions about the eigenvalues of a graph when some small structure is removed (or added). In particular if a graph has an eigenvalue with high multiplicity then removing a small enough substructure results in a graph with the same eigenvalue.

Definitions:

A **subgraph** of a weighted graph G is a weighted graph H on a subset of the vertices of G and satisfying $w_H(v_i, v_j) \leq w_G(v_i, v_j)$ for all vertices (with the convention that $w_H(v_i, v_j) = 0$ if v_i or v_j is not a vertex in H). If H is a subgraph of G then $G - H$ is the weighted graph with weight function $w_{G-H}(v_i, v_j) = w_G(v_i, v_j) - w_H(v_i, v_j)$.

A weighted graph G is a **weak cover** of the weighted graph H if there is some mapping of the vertices $\pi : V(G) \to V(H)$ so that for all $v_i, v_j \in V(H)$:

$$w_H(v_i, v_j) = \sum_{\substack{x \in \pi^{-1}(v_i) \\ y \in \pi^{-1}(v_j)}} w_G(x, y).$$

Facts:

1. [BH12, Section 3.2] Let G be a weighted graph with vertex v. If $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ are the eigenvalues of $\mathcal{A}_G$ and $\widehat{\alpha}_1 \leq \widehat{\alpha}_2 \leq \cdots \leq \widehat{\alpha}_{n-1}$ are the eigenvalues of A_{G-v} then
$$\alpha_i \leq \widehat{\alpha}_i \leq \alpha_{i+1}.$$

2. [BH12, Section 3.2] Let G be a weighted graph with edge e. If $0 = \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n$ are the eigenvalues of L_G and $0 = \widehat{\theta}_1 \leq \widehat{\theta}_2 \leq \cdots \leq \widehat{\theta}_n$ are the eigenvalues of L_{G-e} then
$$\theta_{i-1} \leq \widehat{\theta}_i \leq \theta_i.$$

3. [Lot07] Let G be a simple graph and with vertex v. If $0 = \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n$ are the eigenvalues of L_G and $0 = \widehat{\theta}_1 \leq \widehat{\theta}_2 \leq \cdots \leq \widehat{\theta}_{n-1}$ are the eigenvalues of L_{G-v} then
$$\theta_i - 1 \leq \widehat{\theta}_i \leq \theta_{i+1}.$$

4. [But07] Let G be a weighted graph and let H be a subgraph of G where H has t vertices none of which are isolated. If $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ are the eigenvalues of $\mathcal{L}_G$ and $\widehat{\lambda}_0 \leq \widehat{\lambda}_1 \leq \cdots \leq \widehat{\lambda}_{n-1}$ are the eigenvalues of $\mathcal{L}_{G-H}$ then
$$\lambda_{k-t+1} \leq \widehat{\lambda}_k \leq \begin{cases} \lambda_{k+t-1} & \text{if } H \text{ is bipartite} \\ \lambda_{k+t} & \text{otherwise} \end{cases}$$
where $\lambda_{-t+1} = \cdots = \lambda_{-1} = 0$ and $\lambda_n = \cdots = \lambda_{n+t-1} = 2$.

5. [But07] Suppose the weighted graph G is a weak cover of the weighted graph H with n and m vertices respectively, and let $\lambda_0 \leq \cdots \leq \lambda_{n-1}$ be the eigenvalues of $\mathcal{L}_G$ and $\widehat{\lambda}_0 \leq \cdots \leq \widehat{\lambda}_{m-1}$ be the eigenvalues of $\mathcal{L}_H$. Then for $k = 0, 1, \ldots, m-1$ we have
$$\lambda_k \leq \lambda_k \leq \lambda_{k+(n-m)}.$$

Examples:

1. As an illustration of Fact 4, if G is a simple graph on n vertices and more than $n/2$ of the vertices has degree $n-1$ then $\frac{n}{n-1}$ is an eigenvalue of the normalized Laplacian of G.

2. An example of a graph and a weak covering are shown in Figure 5.

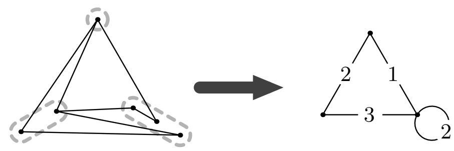

Figure 5: An example of a graph G (on left) which is a weak cover of a graph H (on right).

5 Isoperimetric problems

One of the most useful ways to understand a graph is to understand how to cut it into smaller graphs. In this section we will look at some of the connections between cuts and the spectrum of graphs. This section is closely related to Chapter 36 on Algebraic Connectivity.

Definitions:

Notation: G denotes a weighted graph.
Given subsets S and T of the vertices we let $E(S,T) = \{\text{edges with one end in } S \text{ and one end in } T\}$. By convention, if $S \cap T \neq \emptyset$, then an edge with both ends in $S \cap T$ will appear twice in $E(S,T)$.

The **edge boundary** of a subset of the vertices is $\partial S = E(S, \overline{S})$, i.e., the minimal set of edges needed to be removed which disconnects S from the rest of the graph.

For a graph that is not the complete graph, the **vertex boundary**, denoted δS, is the minimal set of vertices outside of S whose removal disconnects S from the rest of the graph.

The **Cheeger constant** of a set $S \subseteq V(G)$, denoted $h_G(S)$ is

$$h_G(S) = \frac{|\partial S|}{\mathrm{vol}(S)}.$$

The **Cheeger constant of the graph**, denoted h_G, is given by the following

$$h_G = \min_{S \subset V} \left(\frac{|\partial S|}{\min\{\mathrm{vol}(S), \mathrm{vol}(\overline{S})\}} \right) = \min_{\substack{S \subset V \\ \mathrm{vol}(S) \leq \frac{1}{2}\mathrm{vol}(G)}} \{h_G(S)\}.$$

(Roughly, this gives a measure of how well we can find a small edge-cut to break the graph into two large pieces.) Similarly,

$$g_G = \min_{S \subset V} \left(\frac{|\delta S|}{\min\{\mathrm{vol}(S), \mathrm{vol}(\overline{S})\}} \right).$$

The **isoperimetric number** of a simple graph is defined as

$$h'_G = \min_{S \subset V} \left(\frac{|\partial S|}{\min\{|S|, |\overline{S}|\}} \right).$$

Facts:

1. If the simple graph is k-regular then $h_G = \frac{1}{k} h'_G$.
2. [Chu97, Sections 2.3-2.4] Let the weighted graph G be connected but not the complete graph and let $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_{n-1}$ be the eigenvalues of $\mathcal{L}_G$, let $0 = \phi_0 < \phi_1 \leq \cdots \leq \phi_{n-1}$ be the eigenvalues of L_G, and let Δ denote the maximum degree of G. Then the following hold:
   (a) $\frac{1}{2} h_G^2 \leq 1 - \sqrt{1 - h_G^2} < \lambda_1 \leq 2h_G$.
   (b) $\frac{1}{2\Delta} (h'_G)^2 < \phi_1 \leq 2h'_G$.

(c) $\dfrac{g_G^2}{4\Delta + 2\Delta g_G^2} \le \lambda_1$.

3. [Chu97, Section 2.3] Let $\mathbf{x}$ be the eigenvector associated with eigenvalue λ_1 of $\mathcal{L}_G$ for the weighted graph G. Further suppose that the vertices are ordered such that $\mathbf{x}(v_1) \le \mathbf{x}(v_2) \le \cdots \le \mathbf{x}(v_n)$. Let

$$\alpha = \min_{1 \le i \le n-1} \left(\frac{e\big(\{v_1, \ldots, v_i\}, \{v_{i+1}, \ldots, v_n\}\big)}{\min\big(\sum_{j \le i} d_j, \sum_{j > i} d_j\big)} \right).$$

Then $1 - \sqrt{1 - h_G^2} \le \alpha \le \lambda_1$.

4. [But08, Section 5.6.1] Let G be a weighted graph with no isolated vertices and let $0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$ be the eigenvalues of $\mathcal{L}_G$. Then for every partitioning of the vertices into $N_1 \cup \cdots \cup N_m$

$$\sum_{i=0}^{m-1} \lambda_i \le \sum_{i=0}^{m-1} \frac{e(N_i, \overline{N_i})}{\operatorname{vol}(N_i)} = \sum_{i=0}^{m-1} h_G(N_i) \le \sum_{i=0}^{m-1} \lambda_{n-1-i}.$$

Examples:

1. The Cheeger constant of the (simple) path on n vertices is $1/(n-1)$ for n even and $1/(n-2)$ for n odd. The Cheeger constant for the n-cube is $2/n$. These show that the inequalities relating the Cheeger constants and λ_1 are best possible up to a constant.

2. The use of eigenvectors to find good cuts, as alluded to in Fact 3, is the basis of spectral clustering and spectral cuts.

6 Quasirandom graphs

Random graphs have been shown to have many useful and interesting properties. It is of interest to know if a particular graph has some of these properties; this has led to the theory of quasirandom graphs which give a way to certify that a graph has random-like properties through an examination of the eigenvalues or other properties. This is also connected to what are known as pseudorandom graphs.

Definitions:

A **graph property** $P = P(\varepsilon)$ is a property which a graph might satisfy involving some tolerance $\varepsilon > 0$. By $P \Rightarrow Q$ we mean that for each $\varepsilon > 0$ there is a $\delta > 0$ so that if G satisfies the property $P(\delta)$ then G must also satisfy the property $Q(\varepsilon)$. Two graph properties P and Q are equivalent if $P \Rightarrow Q$ and $Q \Rightarrow P$.

The **discrepancy of a graph**, denoted $\operatorname{disc}(G)$, is the minimal α so that for all $X, Y \subseteq V$ we have

$$\left| e(X,Y) - \frac{\operatorname{vol}(X)\operatorname{vol}(Y)}{\operatorname{vol}(G)} \right| \le \alpha\sqrt{\operatorname{vol}(X)\operatorname{vol}(Y)}.$$

(Roughly this is a measurement of how randomly edges are placed.)

The **square product** of two graphs $G = \big(V(G), E(G)\big)$ and $H = \big(V(H), E(H)\big)$, denoted $G \Box H$, is the graph with vertex set $V(G \Box H) = V(G) \times V(H) = \{(x,y) \mid x \in V(G), y \in V(H)\}$ and edge set

$$E(G \Box H) = \big\{\{(x_i, y_i), (x_j, y_j)\} \mid \text{if exactly one of } (x_i, x_j) \text{ and } (y_i, y_j) \text{ is an edge}\big\}.$$

(The square product of two graphs is not the same as the Cartesian product, which in some sources also uses the "$\Box$" notation.)

Facts:

1. [Chu97, CGW89] The following properties are all equivalent for simple graphs and are known as quasirandom graph properties.

   (a) P_1: Let $0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n-1}$ be the eigenvalues of $\mathcal{L}_G$. Then we have $\max\{|1-\lambda_1|, |\lambda_{n-1} - 1|\} \le \varepsilon$ and all but εn vertices have degree $(1/2 \pm \varepsilon)n$.

   (b) P_2: For each $S \subseteq V$, the number of edges with both ends in S is $\frac{1}{4}|S|^2 \pm \varepsilon n^2$.

   (c) P_3: For each $S \subseteq V$ with $|S| = \lfloor n/2 \rfloor$, the number of edges with both ends in S is $(\frac{1}{16} \pm \varepsilon)n^2$.

   (d) P_4: For $u \in V$ let N_u denote the neighbors of u. Then

$$\sum_{u,v} \left| N_u \cap N_v - \frac{n}{4} \right| \le \varepsilon n^2.$$

   (e) P_5: The number of edges in G is $\ge (1-\varepsilon)n^2/4$ and the number of occurrences of C_4 (the cycle on 4 vertices) in G as a labelled subgraph is $(1 \pm \varepsilon)n^4/16$.

   (f) $P_6(s)$ for $s \ge 4$: For *all* graphs H on s vertices, the number of labelled induced subgraphs of G isomorphic to H is $(1 \pm \varepsilon)n^s 2^{-\binom{s}{2}}$.

2. [CG91] If G and H are simple quasirandom graphs then $G \Box H$ is a simple quasirandom graph.

3. [Chu04] Let $0 \le \phi_1 \le \phi_2 \le \cdots \le \phi_{n-1}$ be the eigenvalues of L_G for a simple graph G. Further suppose that the graph has average degree d and that $|d - \phi_i| \le \theta$ for $i \ne 0$. Then for all $X, Y \subseteq V$

$$\left| e(X,Y) - \frac{d}{n}|X||Y| \right| \le \frac{\theta}{n}\sqrt{|X|(n-|X|)|Y|(n-|Y|)}.$$

4. [But06] Let $0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n-1}$ be the eigenvalues of $\mathcal{L}_G$ for a weighted graph G. Then

$$\operatorname{disc}(G) \le \max\{|1-\lambda_1|, |\lambda_{n-1}-1|\} \le 150\operatorname{disc}(G)\big(1 - 8\log\operatorname{disc}(G)\big).$$

   In particular, $\operatorname{disc}(G)$ is small if and only if $\max\{|1-\lambda_1|, |\lambda_{n-1}-1|\}$ is small.

Examples:

1. The graph $P_3 \Box P_3$ is shown in Figure 6.

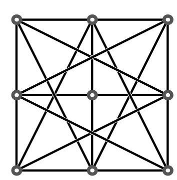

Figure 6: The graph $P_3 \Box P_3$.

2. [CGW89] For a prime $p \equiv 1 \pmod 4$ construct the graph G with vertices $0, 1, \ldots, p-1$ and an edge joins vertex i with vertex j if and only if $i - j$ is a quadratic residue modulo p. This is the Paley graph and is known to be an example of a quasirandom graph family which satisfies the above properties with $\varepsilon \to 0$ as $p \to \infty$.

3. [CG91] Let G be the graph with vertices indexed by the subsets of $\{1, 2, \ldots, n\}$ and two vertices X and Y are adjacent if and only if $\big|X \cap Y\big| \equiv 0 \pmod 2$. Then G is quasirandom and this family satisfies the above properties with $\varepsilon \to 0$ as $n \to \infty$.

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