(1) **Card game**
A deck of 52 cards has 13 ranks $(2, 3, 4, 5, 6, 7, 8, 9, 10, \text{J}, \text{Q}, \text{K}, \text{A})$ and 4 suits $(\heartsuit, \spadesuit, \diamondsuit, \clubsuit)$.
Three cards are drawn randomly without replacement from a deck of 52 cards.

(a) What is the probability that all three cards are in the same suit?

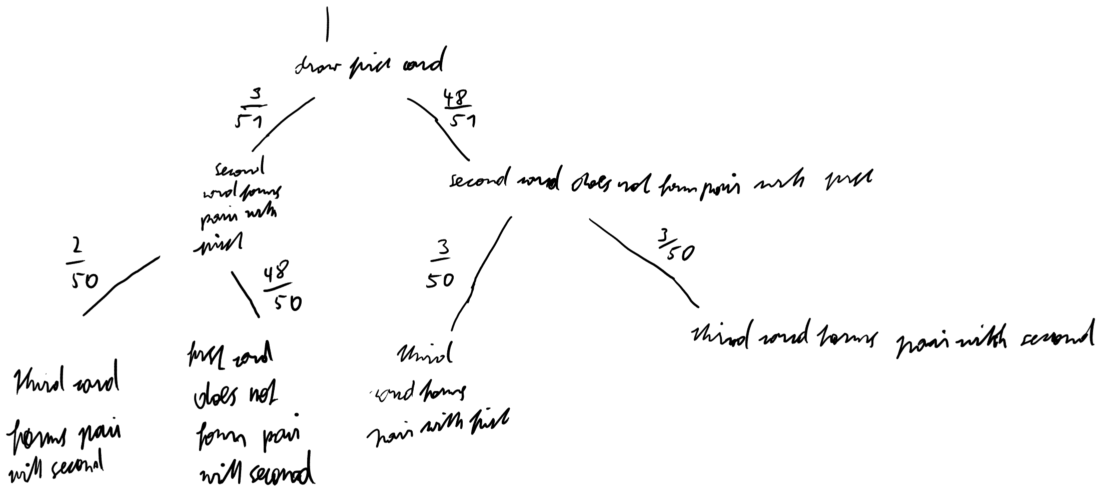
(b) What is the probability that all three cards have the same rank?

(c) What is the probability that the three cards contain exactly one pair (a pair means two cards with the same rank from two different suits)?

a) $\mathbb{P}(\text{all three cards are in the same suit}) = 4 \frac{13}{52} \frac{12}{51} \frac{11}{50} = 4 \frac{1}{4} \frac{4}{17} \frac{11}{50} = \frac{2}{17} \frac{11}{25}$

b) $\mathbb{P}(\text{all three cards have the same rank}) = 13 \frac{4}{52} \frac{3}{51} \frac{2}{50} = 13 \frac{1}{13} \frac{1}{17} \frac{1}{25} = \frac{1}{17} \frac{1}{25}$

c) $\mathbb{P}(\text{the three cards contain exactly one pair}) = \frac{48}{50} \frac{3}{51} + 2 \frac{48}{51} \frac{3}{50} = 3 \frac{1}{17} \frac{24}{25} = \frac{72}{425}$



(2) **Coin game**

Two players, A and B, alternately and independently flip a coin and the first player to obtain a head wins. Assume player A flips first. Suppose that $P(\text{head}) = p$, not necessarily $\frac{1}{2}$. What is the probability that the player B wins?

$$\mathbb{P}(B \text{ wins}) = \sum_{n=0}^{\infty} (1-p)^{2n+1} p = p(1-p) \sum_{n=0}^{\infty} \left((1-p)^2\right)^n = p(1-p) \frac{1}{1-(1-p)^2} = \frac{p(1-p)}{2p-p^2} = \frac{1-p}{2-p}$$

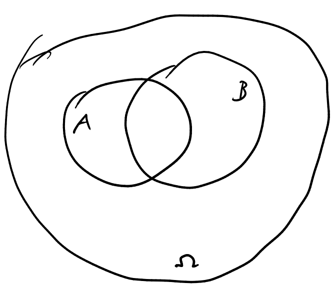
(3) **Independence**
Let A and B be two independent events.

(a) Prove that A^c and B^c are also independent.

(b) If we additionally know that $P(A|B) = 0.6$ and $P(B|A) = 0.3$, compute the probabilities of the following two events

  (i) at most one of A or B

  (ii) either A or B but not both.



a) $\mathbb{P}(A^c \cap B^c) = \mathbb{P}((A \cup B)^c) = \mathbb{P}(\Omega \setminus (A \cup B)) = \mathbb{P}(\Omega) - \mathbb{P}(A \cup B)$

$= 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A \cap B) = 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(B)$

$= 1 - \mathbb{P}(B) + \mathbb{P}(A)(\mathbb{P}(B) - 1) = (1 - \mathbb{P}(B))(1 - \mathbb{P}(A)) = \mathbb{P}(A^c)\,\mathbb{P}(B^c)$

b) (i) $\mathbb{P}(\text{at most one of } A \text{ or } B) = 1 - \mathbb{P}(A \cap B) = 1 - \mathbb{P}(A)\mathbb{P}(B) = 1 - \mathbb{P}(A|B)\,\mathbb{P}(B|A) = 1 - \frac{6}{10}\frac{3}{10} = 1 - \frac{18}{100}$

$= 1 - \frac{9}{50} = \frac{41}{50}$

(ii) $\mathbb{P}(\text{either } A \text{ or } B \text{ but not both}) = \mathbb{P}(A) + \mathbb{P}(B) - 2\,\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - 2\,\mathbb{P}(A)\,\mathbb{P}(B)$

$= \frac{3}{10} + \frac{6}{10} - 2\,\frac{3}{10}\,\frac{6}{10} = \frac{9}{10} - \frac{36}{100}$

$= \frac{9}{10} - \frac{18}{50} = \frac{45-18}{50} = \frac{27}{50}$

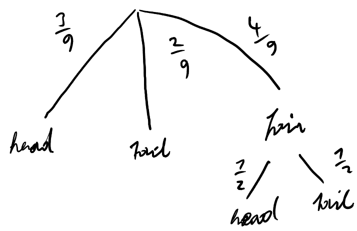
(4) **Box with coins**

A box contains three coins with a head on each side, two coins with a tail on each side, and four fair coins.

(a) One of these nine coins is selected at random and tossed once. What is the probability of getting a tail?

(b) If we get a tail, what is the probability that the selected coin has a tail on both side? If we get a tail, what is the probability that it is a fair coin?

(c) If the first toss is tail, and another coin is selected at random from the remaining eight coins and tossed once, what is the probability of getting a tail again?



a) $\mathbb{P}(\text{tail}) = \frac{2}{9} + \frac{4}{9}\frac{1}{2} = \frac{4}{9}$

b) $$\mathbb{P}(\text{tail on both sides} \mid \text{tail}) = \frac{\mathbb{P}(\text{tail on both sides} \cap \text{tail})}{\mathbb{P}(\text{tail})} = \frac{2/9}{4/9} = \frac{1}{2}$$

c) $$\mathbb{P}(\text{second is tail} \mid \text{first was tail}) = \frac{\mathbb{P}(\text{first \& second tail})}{\mathbb{P}(\text{first tail})}$$

$$= \frac{\mathbb{P}(\text{first only tail})\,\mathbb{P}(\text{second tail} \mid \text{first only tail}) + \mathbb{P}(\text{first fair \& tail})\,\mathbb{P}(\text{second tail} \mid \text{first fair \& tail})}{\mathbb{P}(\text{first tail})}$$

$$= \frac{\frac{2}{9}\left(\frac{1}{8} + \frac{4}{8}\frac{1}{2}\right) + \frac{4}{9}\frac{1}{2}\left(\frac{2}{8} + \frac{3}{8}\frac{1}{2}\right)}{\frac{2}{9} + \frac{4}{9}\frac{1}{2}}$$

$$= \frac{\frac{2}{9}\frac{3}{8} + \frac{2}{9}\frac{7}{16}}{\frac{4}{9}} = \frac{\frac{2}{9}}{\frac{4}{9}}\left(\frac{6}{16} + \frac{7}{16}\right) = \frac{1}{2}\frac{13}{16} = \frac{13}{32}$$

(5) **Cumulative distribution function**

Let a cumulative distribution function (cdf) F of a continuous random variable Y be given by

$$F(y) = \begin{cases} 0, & y \leq 0 \\ \frac{2}{5}y, & 0 < y \leq 1 \\ ay - b, & 1 < y \leq 2 \\ 1, & y > 2 \end{cases},$$

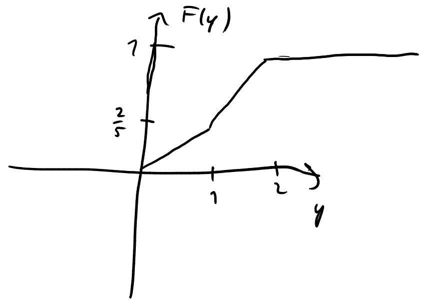
where a and b are real constants.

(a) Find out the values of a and b.

(b) Write down the probability density function (pdf) of Y.

(c) What is the probability that an observed random variable Y is greater than 1.8, given that it is greater than 1?

a) $\frac{2}{5} = F(1) \overset{!}{=} \lim_{y \to 1+} F(y) = \lim_{y \to 1+} ay - b = a - b \Rightarrow a = b + \frac{2}{5}$



$2a - b = F(2) \overset{!}{=} \lim_{y \to 2+} F(y) = \lim_{y \to 2+} 1 = 1$

$\Rightarrow 2\left(b + \frac{2}{5}\right) - b = 1 \Leftrightarrow b + \frac{4}{5} = 1 \Leftrightarrow b = \frac{1}{5} \Rightarrow a = \frac{3}{5}$

b)

$$f(y) = \begin{cases} 0, & \text{if } y \leq 0 \vee y > 2 \\ \frac{2}{5}, & \text{if } 0 < y \leq 1 \\ \frac{3}{5}, & \text{if } 1 < y \leq 2 \end{cases}$$

c) $\mathbb{P}(Y > \frac{18}{10} \mid Y > 1) = \frac{\mathbb{P}(Y > \frac{18}{10} \wedge Y > 1)}{\mathbb{P}(Y > 1)} = \frac{1 - F(\frac{18}{10})}{1 - F(1)} = \frac{1 - \frac{3}{5}\frac{18}{10} + \frac{1}{5}}{1 - \frac{2}{5}}$

$= \frac{1 - \frac{27}{25} + \frac{1}{5}}{\frac{3}{5}} = \frac{5 - \frac{27}{5} + 1}{3} = \frac{\frac{30-27}{5}}{3}$

$= \frac{\frac{3}{5}}{3} = \frac{1}{5}$