(5) **Transformations**
Suppose X and Y are independent gamma distributed random variables with $X \sim Gamma(\alpha_1, \beta)$ and $Y \sim Gamma(\alpha_2, \beta)$. Consider the following two random variables

$$U = X + Y \qquad \text{and} \qquad V = \frac{X}{X+Y}.$$

(a) Show that $U \sim Gamma(\alpha_1 + \alpha_2, \beta)$.
(b) Show that U and V are also independent random variables.

$$f_X(x) = \frac{1}{\Gamma(\alpha_1)\beta^{\alpha_1}} x^{\alpha_1 - 1} e^{-\frac{x}{\beta}} \mathbb{1}_{\mathbb{R}^+}(x) \qquad \alpha_1, \alpha_2, \beta > 0$$

$$f_Y(y) = \frac{1}{\Gamma(\alpha_2)\beta^{\alpha_2}} y^{\alpha_2 - 1} e^{-\frac{y}{\beta}} \mathbb{1}_{\mathbb{R}^+}(y)$$

a) As X and are independent, we have $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$g: \mathbb{R}^{+2} \to \mathbb{R}^+ \times (0,1) : (x,y) \mapsto (x+y, \frac{x}{x+y})$

$\forall x,y,u,v \in \mathbb{R}^+ : (u,v) = (x+y, \frac{x}{x+y}) \Leftrightarrow u = x+y \wedge v = \frac{x}{x+y} \Leftrightarrow u = x+y \wedge v = \frac{x}{u} \Leftrightarrow y = u - x \wedge x = vu$
$\Leftrightarrow x = vu \wedge y = u(1-v)$

Hence, $h: \mathbb{R}^+ \times (0,1) \to \mathbb{R}^{+2} : (u,v) \mapsto (uv, u(1-v))$ is the inverse of g

$dh(u,v) = \begin{pmatrix} v & u \\ 1-v & -u \end{pmatrix}$, and $\det(dh(u,v)) = -uv - u(1-v) = -u$

$f_{U,V}(u,v) = f_{X,Y}(h(u,v)) \, |\det(dh(u,v))| = f_X(uv) f_Y(u(1-v)) u$

$c(\alpha) := \frac{1}{\Gamma(\alpha)\beta^\alpha}$

$f_U(u) = \int_{\mathbb{R}} f(u,v) dv = c(\alpha_1) c(\alpha_2) u^{(\alpha_1+\alpha_2)-1} \int_{\mathbb{R}} \mathbb{1}_{\mathbb{R}^+}(uv) \mathbb{1}_{\mathbb{R}^+}(u(1-v)) v^{\alpha_1 - 1} (1-v)^{\alpha_2 - 1} e^{-\frac{uv}{\beta}} e^{-\frac{u(1-v)}{\beta}} dv$

$e^{-\frac{uv}{\beta}} e^{-\frac{u(1-v)}{\beta}} = e^{-\frac{uv}{\beta}} e^{(\frac{uv}{\beta} - \frac{u}{\beta})} = e^{-\frac{u}{\beta}}$, and $uv > 0 \wedge u(1-v) > 0 \Leftrightarrow u > 0 \wedge 0 < v < 1$

$\int_0^1 v^{\alpha_1 - 1}(1-v)^{\alpha_2 - 1} dv = B(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}$, hence $\qquad \alpha := \alpha_1 + \alpha_2$

Finally, we obtain $f_U(u) = \frac{1}{\Gamma(\alpha_1+\alpha_2)\beta^{(\alpha_1+\alpha_2)}} u^{(\alpha_1+\alpha_2)-1} e^{-\frac{u}{\beta}} \mathbb{1}_{\mathbb{R}^+}(u)$

b) $f_V(v) = \int_{\mathbb{R}} f(u,v) du = c(\alpha_1) c(\alpha_2) v^{\alpha_1 - 1} (1-v)^{\alpha_2 - 1} \mathbb{1}_{(0,1)}(v) \int_{\mathbb{R}^+} u^{\alpha - 1} e^{-\frac{u}{\beta}} du$

$\frac{u}{\beta} = w \Leftrightarrow \frac{dw}{du} = \frac{1}{\beta} \Leftrightarrow du = \beta dw$

$\int_{\mathbb{R}^+} u^{\alpha-1} e^{-\frac{u}{\beta}} du = \int_{\mathbb{R}^+} (\beta w)^{\alpha - 1} e^{-w} \beta dw = \beta^\alpha \Gamma(\alpha)$

Thus, $f_V(v) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} v^{\alpha_1 - 1}(1-v)^{\alpha_2 - 1} \Gamma(\alpha) \mathbb{1}_{(0,1)}(v)$

$f_U(u) f_V(v) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} v^{\alpha_1 - 1} (1-v)^{\alpha_2 - 1} u^{\alpha_2 - 1} u^{\alpha_1 - 1} u \, \mathbb{1}_{(0,1)}(v) \mathbb{1}_{\mathbb{R}^+}(u) = f_{U,V}(u,v)$

Hence, U and V are independent!