Einführung in die Statistik, 1. UE

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1. Kartendeck, 13 Werte, 4 Farben; 3x ziehen ohne Zurücklegen.

(a) $\mathbb{P}(\text{selbe Farbe}) = \sum_{i=1}^{4} \mathbb{P}(\text{alle Farbe } i) = \sum_{i=1}^{4} \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} = \frac{22}{425} \approx 5{,}2\,\%$

(b) $\mathbb{P}(\text{selber Rang}) = 13 \cdot \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \approx 0{,}24\,\%$

(c) $\mathbb{P}(\text{enthält genau ein Paar}) = 13 \cdot \left(\frac{4}{52} \cdot \frac{3}{51} \cdot \frac{48}{50} + \frac{4}{52} \cdot \frac{48}{51} \cdot \frac{3}{50} + \frac{48}{52} \cdot \frac{4}{51} \cdot \frac{3}{50} \right) \approx 17\,\%$

2. First with head wins, A starts, players flip alternately, $\mathbb{P}(\text{head}) = p$.

$$\mathbb{P}(B \text{ wins}) = \sum_{j=1}^{\infty} \mathbb{P}(B \text{ wins in } j\text{th round}) = \sum_{j=1}^{\infty} \left((1-p)^2\right)^{i-1} (1-p)\, p$$

$$= p \frac{1}{1-p} \sum_{j=1}^{\infty} \big(\underbrace{(1-p)^2}_{<1}\big)^{i} = p \frac{1}{1-p} \left(\frac{1}{1-(1-p)^2} - 1 \right) = \frac{\cancel{p}}{(1-p)\,\cancel{p}\,(2-p)} - \frac{p}{1-p}$$

$$= \frac{1-(2-p)p}{(1-p)(2-p)} = \frac{(1-p)^{\cancel{2}}}{\cancel{(1-p)}(2-p)} = \frac{1-p}{2-p}.$$

③ A, B independent.

(a) A^c, B^c indep.: $\mathbb{P}(A^c \cap B^c) = \mathbb{P}((A \cup B)^c) = 1 - \mathbb{P}(A \cup B) =$

$= 1 - \mathbb{P}(A) - \mathbb{P}(B) + \underbrace{\mathbb{P}(A \cap B)}_{\mathbb{P}(A)\,\mathbb{P}(B)} = 1 - \mathbb{P}(A) - \mathbb{P}(B)\,(1 - \mathbb{P}(A)) = \mathbb{P}(A^c) - \mathbb{P}(B)\,\mathbb{P}(A^c)$

$= \mathbb{P}(A^c)(1 - \mathbb{P}(B)) = \mathbb{P}(A^c)\,\mathbb{P}(B^c).$

(b) $\mathbb{P}(A|B) = 0{,}6$, $\mathbb{P}(B|A) = 0{,}3$. $\leadsto$ $\dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = 0{,}6$, $\dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = 0{,}3 \Rightarrow \dfrac{\mathbb{P}(A \cap B)^{\cancel{2}}}{\cancel{\mathbb{P}(A \cap B)}} = 0{,}18$

(i) $\mathbb{P}(\text{at most one of } A \text{ or } B) = 1 - \mathbb{P}(A \cap B) = 1 - 0{,}18 = 0{,}82.$

(ii) $\mathbb{P}(\text{either } A \text{ or } B, \text{ not both}) = \mathbb{P}(A)\,\mathbb{P}(B^c) + \mathbb{P}(A^c)\,\mathbb{P}(B) =$

$\overset{*}{=} \mathbb{P}(A)\,(1 - \mathbb{P}(B)) + (1 - \mathbb{P}(A))\,\mathbb{P}(B) = \mathbb{P}(A) - 2\mathbb{P}(A)\,\mathbb{P}(B) + \mathbb{P}(B) =$

$= 0{,}6 - 2 \cdot 0{,}18 + 0{,}3 = 0{,}54.$

$*$ because A and B are independent, $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(A)\,\cancel{\mathbb{P}(B)}}{\cancel{\mathbb{P}(B)}} = \mathbb{P}(A).$

(4.) 2 coins with 2 tails $\overset{\frac{2}{9}}{/\!/}$, 3 coins with 2 heads $\overset{\frac{3}{9}=\frac{1}{3}}{/\!/}$, 4 fair coins $\overset{\frac{4}{9}}{/\!/}$

(a) one selected randomly and tossed.

$$\mathbb{P}(\text{tail}) = \frac{3}{9}\cdot 0 + \frac{2}{9}\cdot 1 + \frac{4}{9}\cdot\frac{1}{2} = \underline{\underline{\frac{4}{9}}}$$

[Law of total probability: $\mathbb{P}(A) = \sum_{i\in I} \mathbb{P}(H_i)\,\mathbb{P}(A\mid H_i)$]

(b) • $\mathbb{P}(\text{selected coin has 2 tails} \mid \text{we got a tail}) \overset{\text{(Bayes)}}{=}$

$= \mathbb{P}(\text{we get tail} \mid \text{coin has two tails})\,\mathbb{P}(\text{coin has two tails}) / \mathbb{P}(\text{we get a tail})$

$\overset{(a)}{=} 1\cdot\frac{2}{9} / \frac{4}{9} = \underline{\underline{\frac{1}{2}}}$

• $\mathbb{P}(\text{selected coin is fair} \mid \text{we got a tail}) \overset{\text{(Bayes)}}{=} \mathbb{P}(\text{we get tail} \mid \text{coin fair})\,\mathbb{P}(\text{coin fair}) / \mathbb{P}(\text{tail})$

$\overset{(a)}{=} \frac{1}{2}\cdot\frac{4}{9} / \frac{4}{9} = \underline{\underline{\frac{1}{2}}}$

(c) If the first toss is tail, and another coin is selected, what is $\mathbb{P}(\text{getting tail again})$?

By (b), $\begin{cases} \mathbb{P}(\text{first coin had 2 tails}) = \frac{1}{2} \\ \mathbb{P}(\text{first coin was fair}) = \frac{1}{2} \end{cases}$, and $\mathbb{P}(\text{first coin had 2 heads}) = 0$.

It holds that $\mathbb{P}(\text{tail again} \mid \text{first coin had 2 tails}) = \underbrace{\frac{1}{8}}_{\text{remaining coins with 2 tails}}\cdot 1 + \frac{3}{8}\cdot 0 + \frac{4}{8}\cdot\frac{1}{2} = \frac{3}{8}$

and $\mathbb{P}(\text{tail again} \mid \text{first coin fair}) = \frac{2}{8}\cdot 1 + \frac{3}{8}\cdot 0 + \frac{3}{8}\cdot\frac{1}{2} = \frac{7}{16}$,

so all in all $\mathbb{P}(\text{tail again}) = \frac{1}{2}\cdot\frac{3}{8} + \frac{1}{2}\cdot\frac{7}{16} = \underline{\underline{\frac{13}{32}}}$.

5. Y continuous random variable, cdf $F(y) = \mathbb{P}(Y \leq x) = \begin{cases} 0, & y \leq 0 \\ 2/5 \cdot y, & y \in [0,1] \\ ay-b, & y \in (1,2] \\ 1, & y > 2 \end{cases}$, $a, b \in \mathbb{R}$.

(a) Find a, b: By definition, a cdf has to be right-continuous, i.e. $\forall x: \lim_{z \searrow x} F(z) = F(x)$.

Since $F(1) = \frac{2}{5}$, it must apply that $\lim_{y \searrow 1} ay - b = a - b \overset{!}{=} \frac{2}{5} \Rightarrow a = b + \frac{2}{5}$

as well as $\lim_{y \searrow 2} 1 = 1 \overset{!}{=} 2a - b \Rightarrow 2(b + \frac{2}{5}) - b = 1$,

so $\underline{\underline{b = 0{,}2}}$ and $\underline{\underline{a = 0{,}6}}$.

(b) Find the pdf of Y, i.e. a function f s.t. $F(y) = \int_{-\infty}^{y} f(t)\, dt$.

We define $f(t) = \begin{cases} 0, & y \leq 0 \\ 2/5, & y \in [0,1] \\ 0{,}6, & y \in (1,2] \\ 0, & y > 2. \end{cases}$ Since $f = \frac{d}{dx} F$ almost everywhere,

f has to be the pdf of the absolutely continuous function F.

(c) $\mathbb{P}(Y > 1{,}8 \mid Y > 1) = \underbrace{\mathbb{P}(Y > 1 \mid Y > 1{,}8)}_{=1} \mathbb{P}(Y > 1{,}8) / \mathbb{P}(Y > 1) =$

$$= \frac{\int_{1{,}8}^{\infty} f(t)\, dt}{\int_{1}^{\infty} f(t)\, dt} = \frac{1 - F(1{,}8)}{1 - F(1)} = \frac{0{,}12}{\frac{3}{5}} = \underline{\underline{0{,}2}}.$$