(1) **Uniform distribution**
Let $X_1, \dots, X_n$ be a random sample from uniform $(\theta, 1)$ distribution, where $\theta < 1$ is an unknown parameter.

(a) Find the MLE $\hat{\theta}$ of θ.

(b) Is $\hat{\theta}$ asymptotically normal? If yes, find the asymptotic mean and variance. Otherwise, find a sequence r_n and a_n such that $r_n(\hat{\theta} - a_n)$ converges in distribution to a non-degenerate (not pointmass) distribution.

a)

$$L(\theta|x) = \prod_{i=1}^{n} f_\theta(x_i) = \begin{cases} 0 & , \text{if } \exists i \in \{1,\dots,n\} : (x_i \leq \theta \vee x_i \geq 1) \Leftrightarrow (\min\{x_i | 1 \leq i \leq n\} \geq \theta) \wedge (\max\{x_i | 1 \leq i \leq n\} \leq 1) \\ (1-\theta)^{-n} & , \text{else} \end{cases}$$

Let $\theta_1 < \theta_2 < 1$, then $(1-\theta_2)^{-n} > (1-\theta_1)^{-n} \Leftrightarrow \left(\frac{1-\theta_1}{1-\theta_2}\right)^n > 1 \Leftrightarrow \frac{1-\theta_1}{1-\theta_2} > 1 \Leftrightarrow 1-\theta_1 > 1-\theta_2 \Leftrightarrow \theta_1 < \theta_2$,

hence $L(\theta, x)$ is an increasing function for $\theta \in (-\infty, \min\{x_i | 1 \leq i \leq n\})$ that clearly has it's maximum at $\hat{\theta}(x) := \min\{x_i | 1 \leq i \leq n\}$

b)

$$\mathbb{P}(\min\{X_i | 1 \leq i \leq n\} \leq x) = \prod_{i=1}^{n} \mathbb{P}(X_i \leq x) = \begin{cases} 0 & , \text{if } x < \theta \\ \left(\frac{x-\theta}{1-\theta}\right)^n & , \text{if } \theta \leq x < 1 \\ 1 & , \text{if } 1 \leq x \end{cases} \xrightarrow{n \to \infty} \begin{cases} 0 & , \text{if } x < 1 \\ 1 & , \text{if } x \geq 1 \end{cases}$$

hence, $\hat{\theta}(X)$ is not asymptotically normal distributed.

We choose $r_n := n$, $a_n := 1$ and obtain for all $x \in \mathbb{R}$

$r_n(\theta - a_n) = n(\theta - 1) \xrightarrow{n \to \infty} -\infty$, $\quad r_n(1 - a_n) = n(1-1) = 0$, and

$$\left(\frac{x}{r_n(1-\theta)} + \frac{a_n - \theta}{1-\theta}\right)^n = \left(1 + \frac{x/(1-\theta)}{n}\right)^n \xrightarrow{n \to \infty} \exp\left(\frac{x}{1-\theta}\right).$$

Since $\frac{x}{r_n} + a_n < \theta \Leftrightarrow x < r_n(\theta - a_n)$ and $\frac{x}{r_n} + a_n < 1 \Leftrightarrow x < r_n(1 - a_n)$, we have

$$\mathbb{P}(r_n(\min\{X_i | 1 \leq i \leq n\} - a_n) \leq x) = \begin{cases} 0 & , \text{if } \frac{x}{r_n} + a_n < \theta \\ \left(\frac{x}{r_n(1-\theta)} - \frac{a_n - \theta}{1-\theta}\right)^n & , \text{if } \theta \leq \frac{x}{r_n} + a_n < 1 \\ 1 & , \text{if } 1 \leq \frac{x}{r_n} + a_n \end{cases} \xrightarrow{n \to \infty} \begin{cases} \exp\left(\frac{x}{1-\theta}\right) & , \text{if } x < 0 \\ 1 & , \text{if } x \geq 0 \end{cases}$$

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