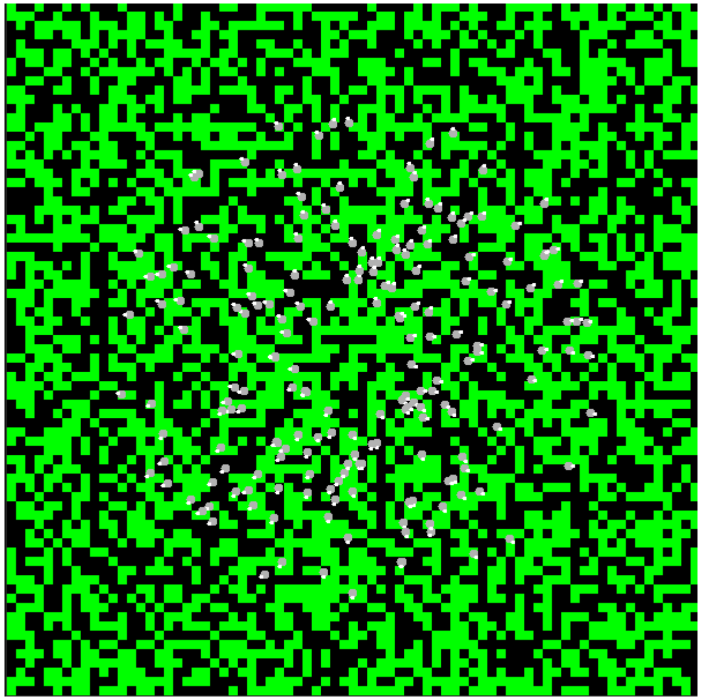
Mathematical Modelling and Simulation

Introduction to Cellular Automata

BASIC CONCEPTS

Cellular Automata

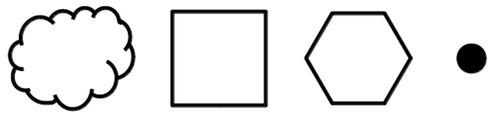
- Modelling using „cellular automata“, short CA, is a microscopic simulation method
- Cellular automata can be imagined as a coloured grid observed dynamically



Although this is a very simplified image of a CA, keep it in mind to understand the formal details of this concept

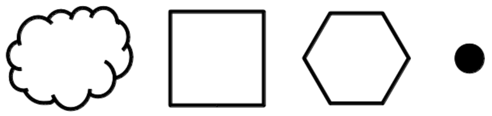
Components of a CA

- Cells



Components of a CA

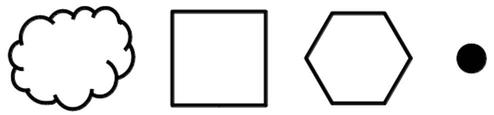
- Cells



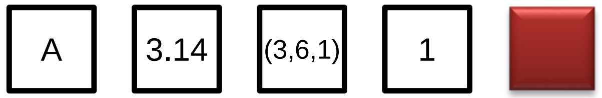
- Notations: cell, entity, node
- Cells are passive: no internal dynamic, only container for some information
- Each cell has some state.

Components of a CA

- Cells



- States



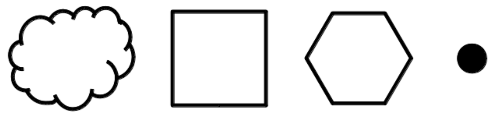
- State-space

$\mathbb{R}$ $\mathbb{Z}^4$ $\{1, 2, 3\}$

▪ Every Cell has a state
▪ There is always some space $\mathbb{S}$ that contains all possible states. It is usually called state-space.

Components of a CA

- Cells

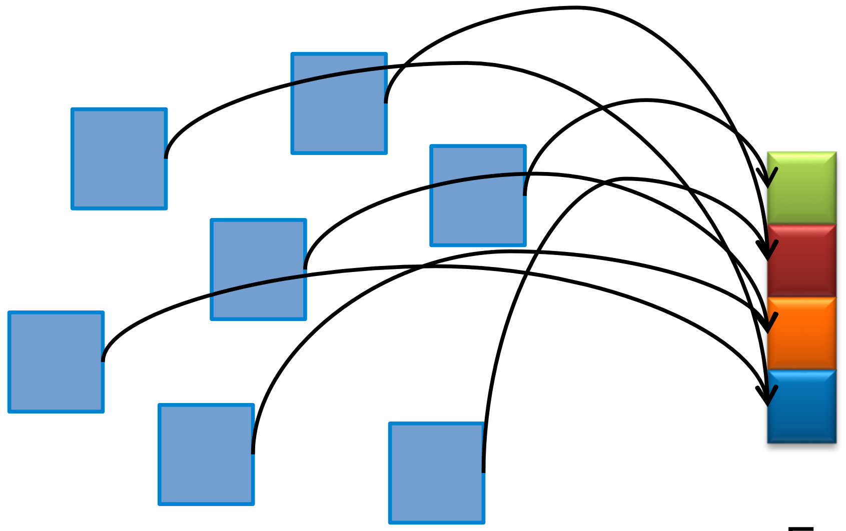


- States

$\boxed{\text{A}}$ $\boxed{3.14}$ $\boxed{(3,6,1)}$ $\boxed{1}$ 🟥

- State-space

$\mathbb{R}$ $\mathbb{Z}^4$ $\{1,2,3\}$ 🟩🟥🟧🟦

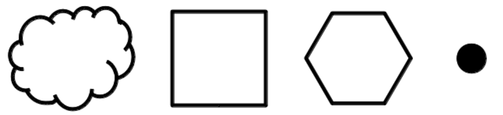


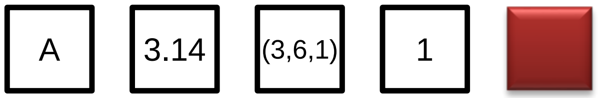
Every cell has a state from a common state-space

Components of a CA

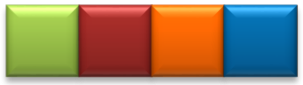
Mathematical Modelling and Simulation

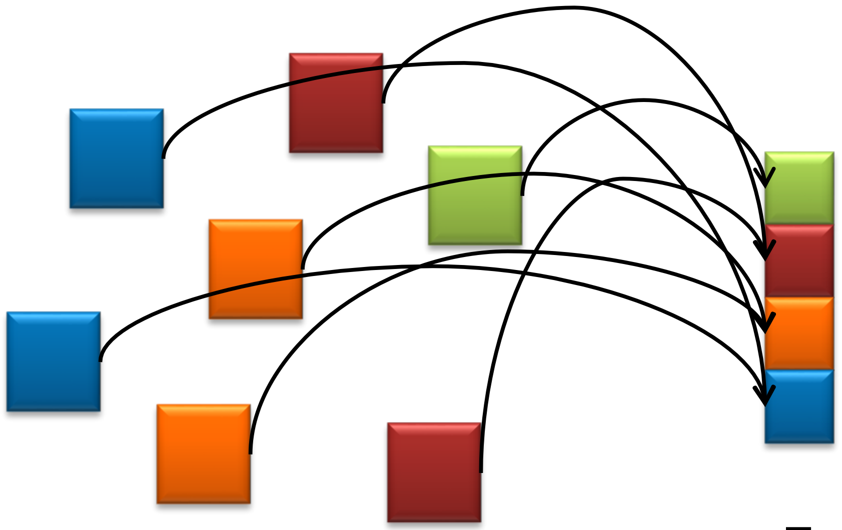
- Cells
- States
- State-space





$\mathbb{R}$ $\mathbb{Z}^4$ $\{1, 2, 3\}$

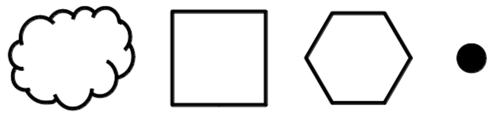




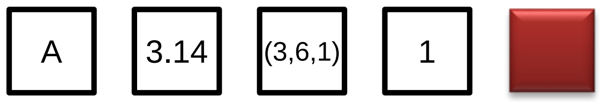
Every cell has a state from a common state-space

Components of a CA

- Cells



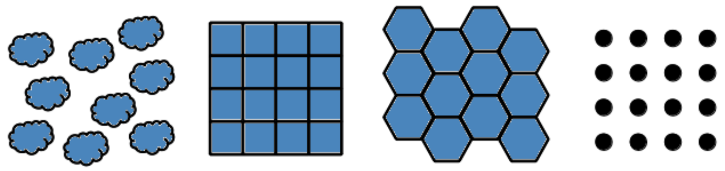
- States



- State-space

$\mathbb{R}$ $\mathbb{Z}^4$ $\{1, 2, 3\}$

- Arrangement (Cell-space)

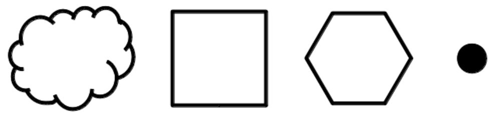


- All cells are arranged on some lattice structure: the „cell-space“ – in the simplest case, a rectangular grid.
- There is some index mapping that maps some subset of $\mathrm{I} \subset \mathbb{Z}^d$ onto each cell

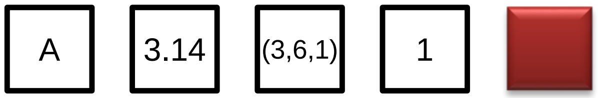
Components of a CA

Mathematical Modelling and Simulation

- Cells



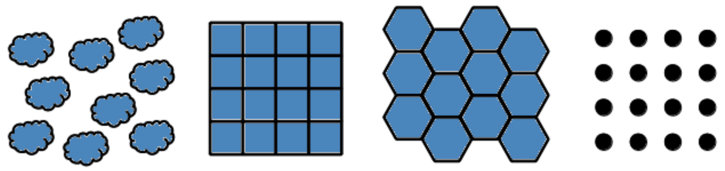
- States

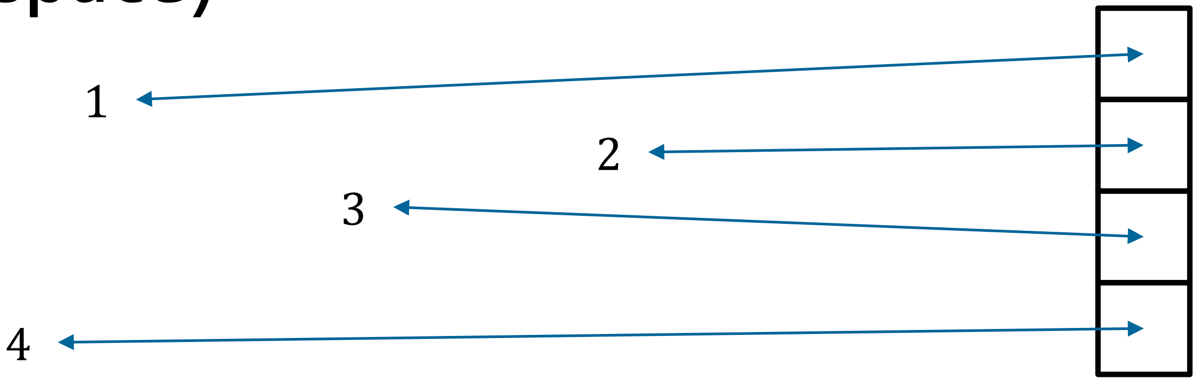


- State-space

$\mathbb{R}$ $\quad \mathbb{Z}^4$ $\quad \{1, 2, 3\}$

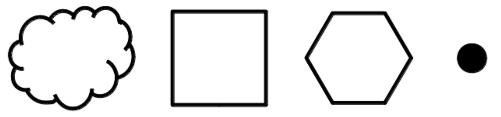
- Arrangement (Cell-space)



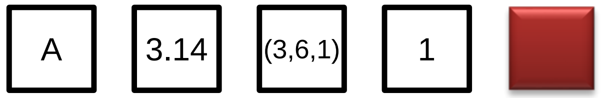


Components of a CA

- Cells



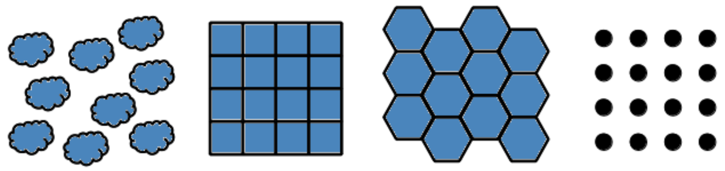
- States

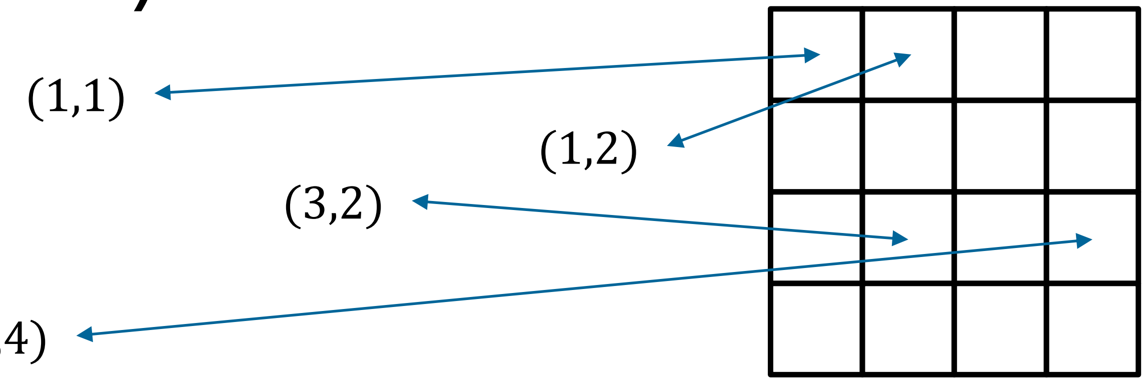


- State-space

$\mathbb{R}$ $\mathbb{Z}^4$ $\{1, 2, 3\}$ 🟩🟥🟧🟦

- Arrangement (Cell-space)

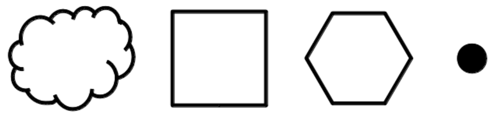




Components of a CA

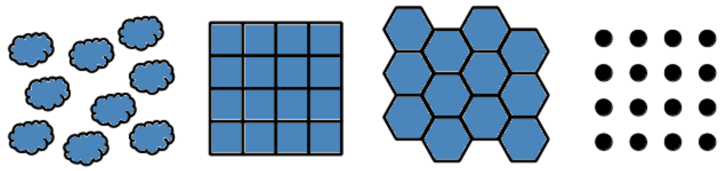
Mathematical Modelling and Simulation

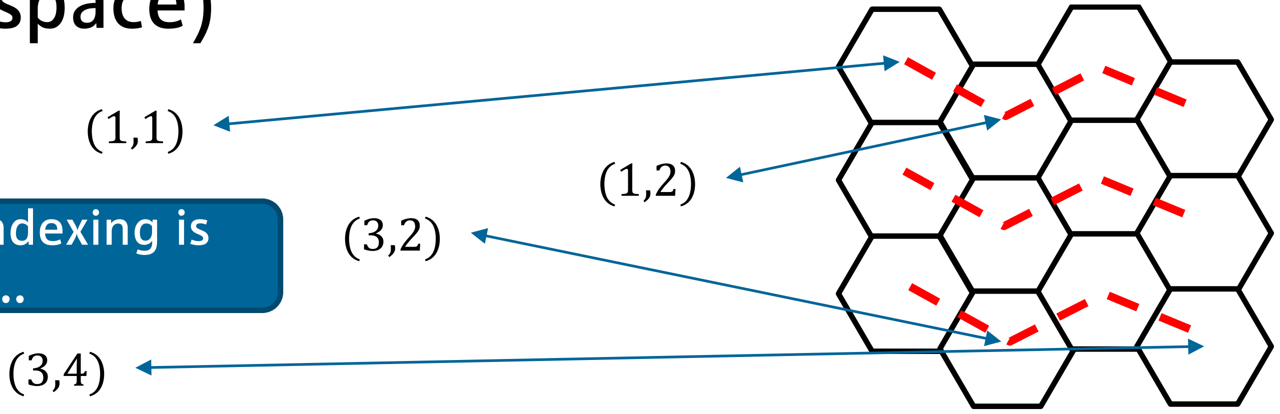
- Cells
- States
- State-space
- Arrangement (Cell-space)



A	3.14	(3,6,1)	1	🟥

$\mathbb{R}$ $\mathbb{Z}^4$ $\{1, 2, 3\}$ 🟩🟥🟧🟦



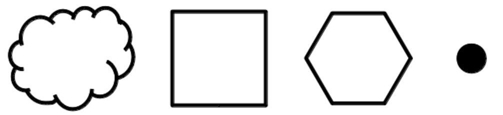


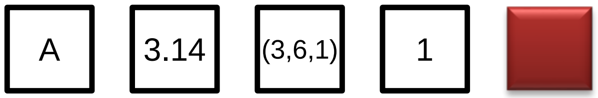
Sometimes indexing is not so trivial...

Components of a CA

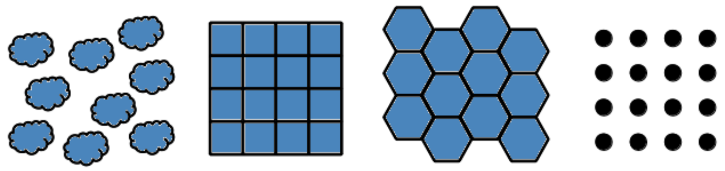
Mathematical Modelling and Simulation

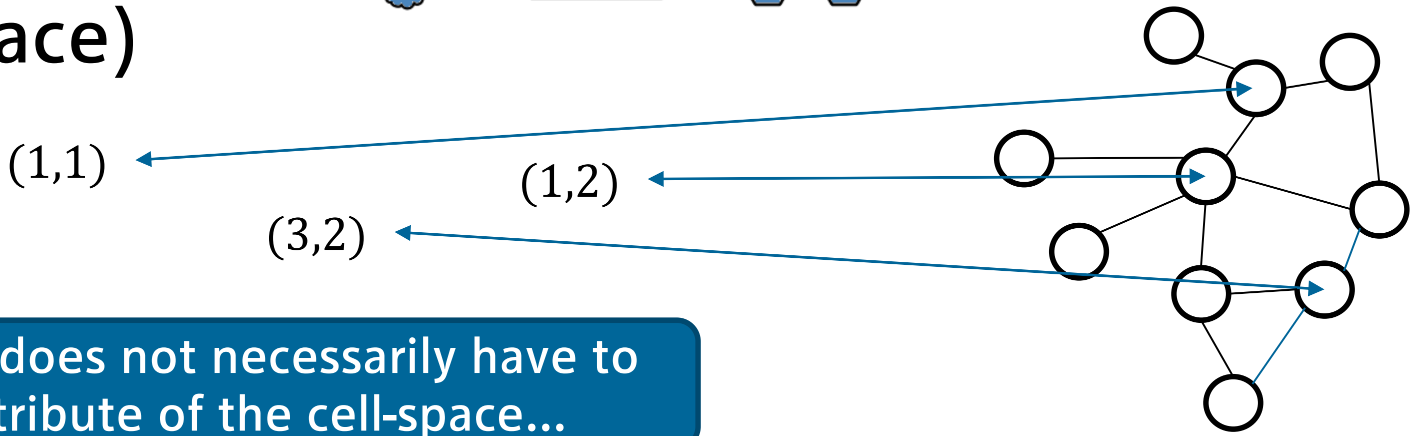
- Cells
- States
- State-space
- Arrangement (Cell-space)





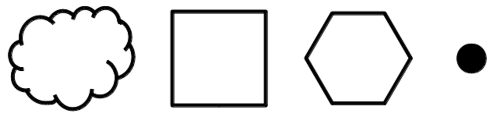
$\mathbb{R}$ $\mathbb{Z}^4$ $\{1,2,3\}$





It often is, but does not necessarily have to be a natural attribute of the cell-space...

Components of a CA

- Cells 
- States $\boxed{\text{A}}$ $\boxed{3.14}$ $\boxed{(3,6,1)}$ $\boxed{1}$ 🟥
- State-space $\mathbb{R}$ $\mathbb{Z}^4$ $\{1,2,3\}$ 🟩🟥🟧🟦

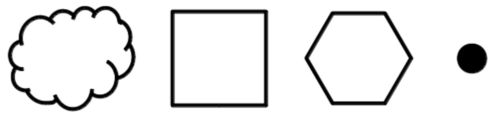
▪ Possible characteristics of the index set:
  - regular
  - finite or infinite
  - connected
  - multi-dimensional

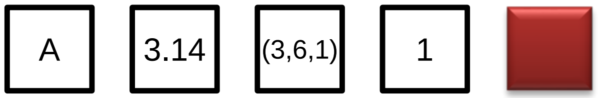
▪ Interpretation of the index set: discretisation of a space or spatial arrangement of entities

Components of a CA

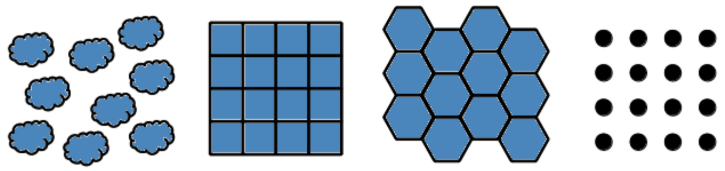
Mathematical Modelling and Simulation

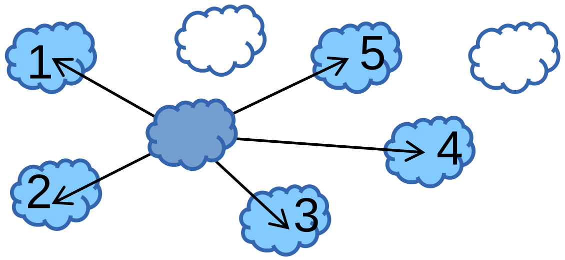
- Cells
- States
- State-space
- Arrangement (Cell-space)
- Neighbourhood





$\mathbb{R}$ $\quad \mathbb{Z}^4$ $\quad \{1, 2, 3\}$

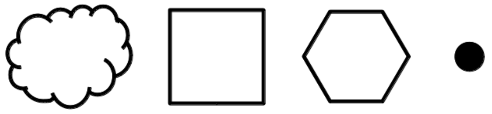




The neighborhood of a cell *z* is an ordered set of *n* other cells $(z_1, \dots, z_n)$.

Components of a CA

- Cells



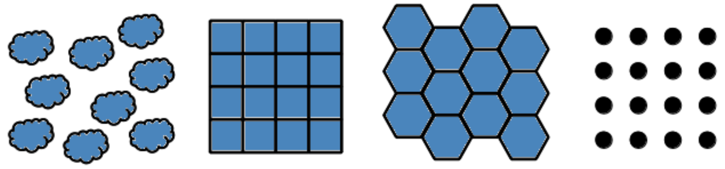
- States

A | 3.14 | (3,6,1) | 1 | 🟥

- State-space

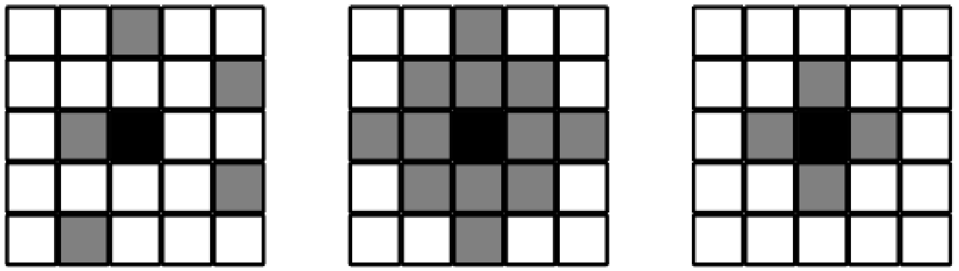
$\mathbb{R}$ $\mathbb{Z}^4$ $\{1,2,3\}$ 🟩🟥🟧🟦

- Arrangement (Cell-space)



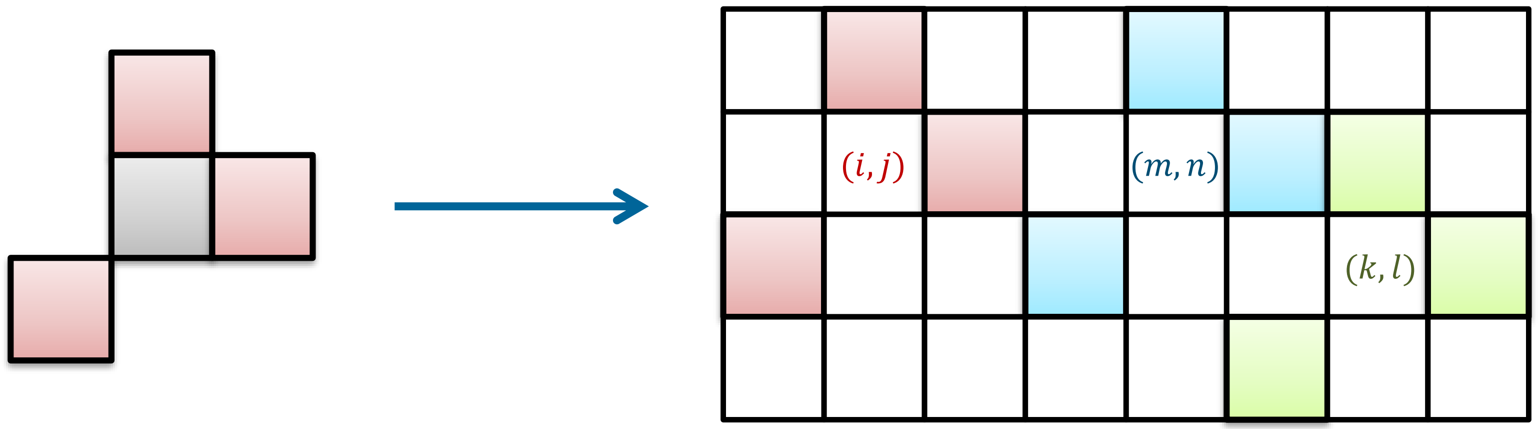
- Neighbourhood

Some examples:



Neighbourhood

- The neighbourhood mapping is relative to the cell's position (= index)



- Calculation of neighbouring cells by stencil: Index translations yield the positions (index) of *n* neighboring cells: $\vec{i} \mapsto (\vec{i} + \vec{t_1}, \ldots \vec{i} + \vec{t_n})$

Neighbourhood

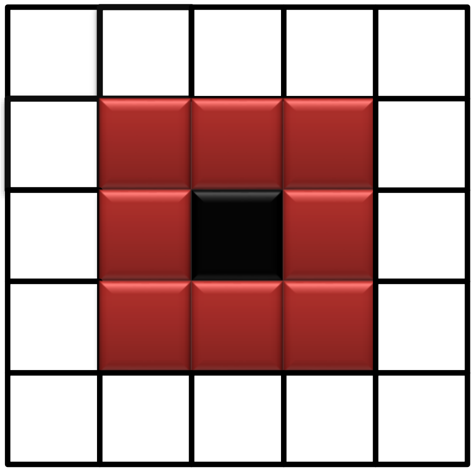
Mathematical Modelling and Simulation

- Possible characteristics of neighbourhoods:
  - **local:** the neighbourhood consists of cells of neighboring points on the grid
  - **symmetric:** the neighborhood of cell A contains cell B if and only if the neighborhood of cell B contains cell A

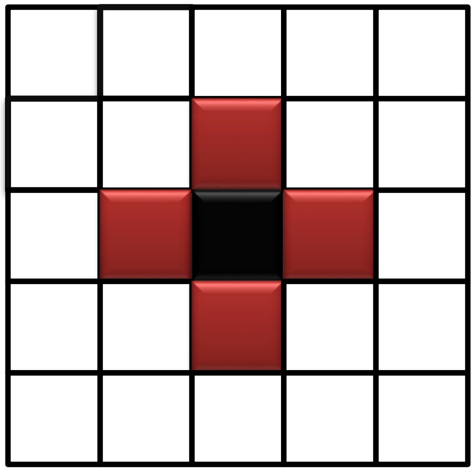
Neighbourhood

- Classic, popular neighborhoods

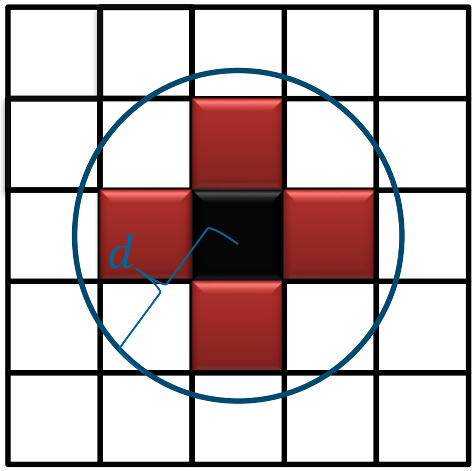
Moore neighborhood



Von-Neumann neighborhood



Neighbourhood by distance:
$\vec{i} \rightarrow \{\vec{j} : |\vec{i} - \vec{j}| < d\}$

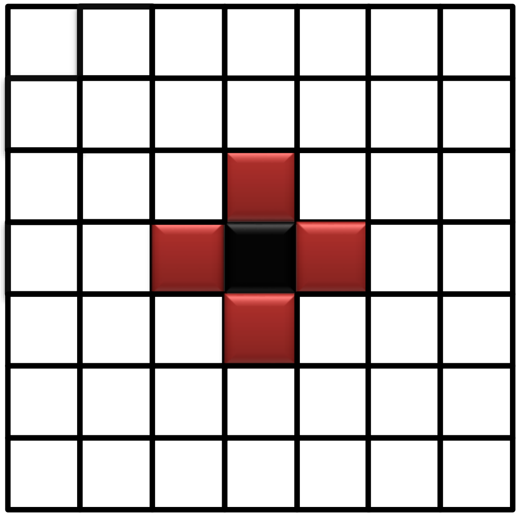


Neighbourhood

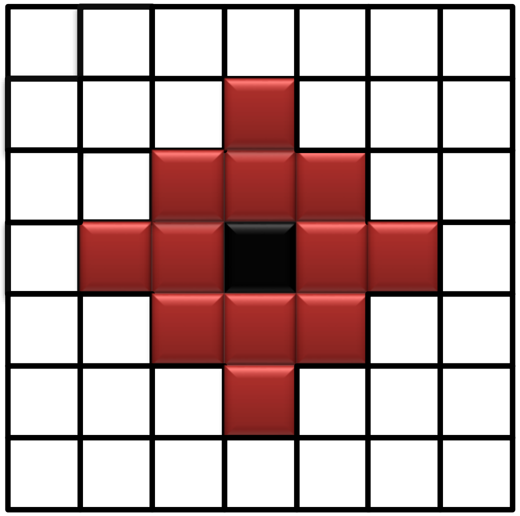
Mathematical Modelling and Simulation

- Von Neumann/Moore Neighbourhood of higher order

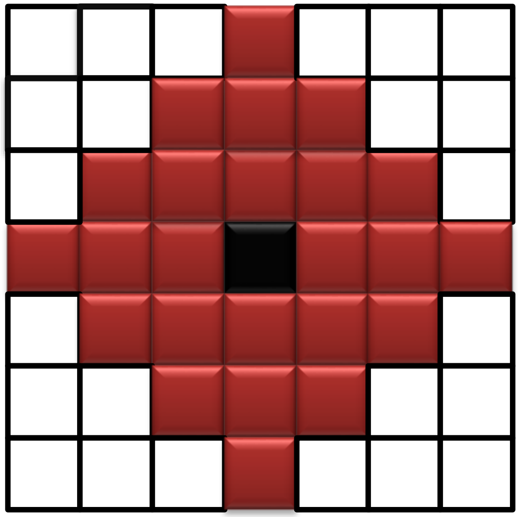
Von-Neumann neighborhood 1st order



Von-Neumann neighborhood 2nd order

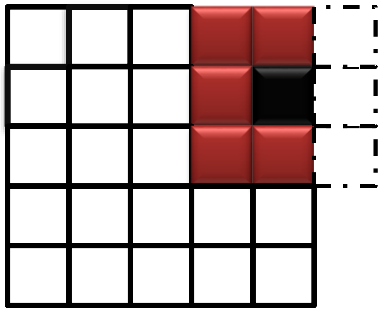


Von-Neumann neighborhood 3rd order

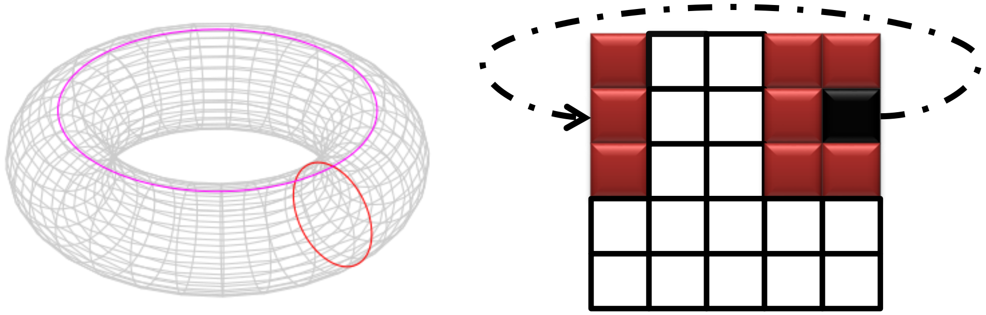


Neighbourhood

- The index set is limited → either incomplete neighborhoods for cells near the borders $(z_1, z_2, \emptyset, z_4 \ldots, z_n)$....



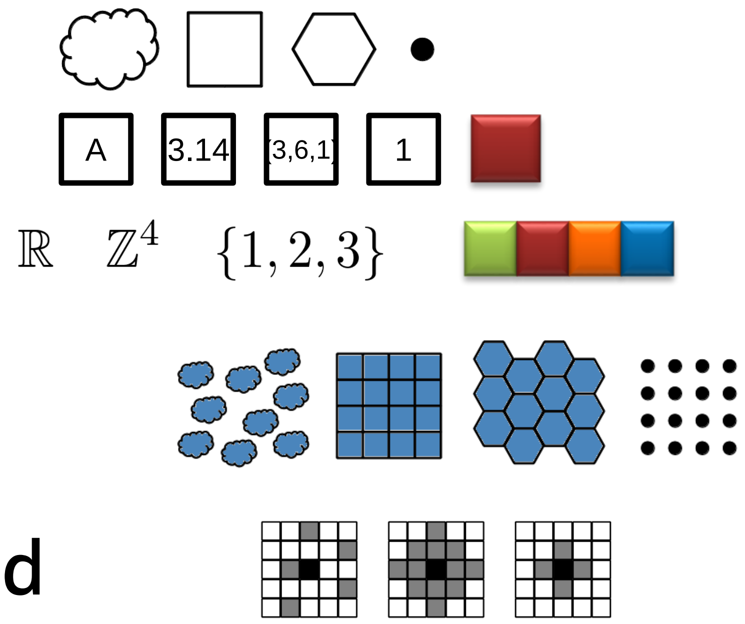
...or other compensation ideas.



Periodic Boundary Conditions (Torus)

Components of a CA

- Cells
- States
- State-space
- Arrangement (Cell-space)
- Neighbourhood
- Update Rule



Some rule, that simultaneously updates all states of all cells of the CA.
Maps all states of a cell's neighbourhood to a new state for the cell.

Components of a CA

Mathematical Modelling and Simulation

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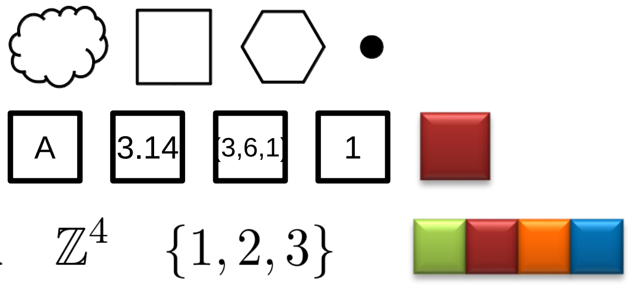
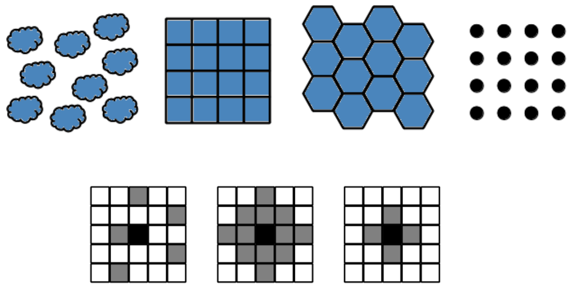




$$f(s, s_1, \ldots, s_n) = s_{new}$$

state of the cell

state of the (ordered) neighbors

new state of the cell

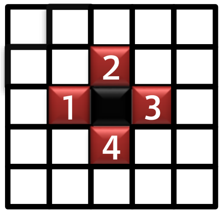
Stochastic CAs have stochastic updates!

Update Rule

- **Example:**

Neighbourhood = Von Neumann

$$f(s, s_1, s_2, s_3, s_4) = \sum s \ (mod\ 4)$$



Old state of the CA

1	2	1	0	2	1	1
2	3	1	2	3	1	1
0	2	1	2	0	2	1
3	2	1	1	1	5	1
0	1	2	0	2	2	0
2	0	1	1	1	3	1
2	2	0	2	3	1	2

New state of the CA

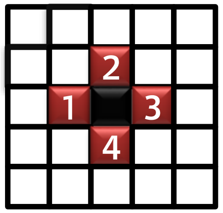
Update Rule

Mathematical Modelling and Simulation

- **Example:**

Neighbourhood = Von Neumann

$$f(s, s_1, s_2, s_3, s_4) = \sum s \ (mod\ 4)$$



Old state of the CA

1	2	1	0	2	1	1
2	3	1	2	3	1	1
0	2	1	2	0	2	1
3	2	1	1	1	5	1
0	1	2	0	2	2	0
2	0	1	1	1	3	1
2	2	0	2	3	1	2

$$f(s, s_1, s_2, s_3, s_4) =$$
$$= 1 + 1 + 2 + 1 + 0\ (mod 4) =$$
$$= 5\ (mod\ 4) = 1$$

New state of the CA

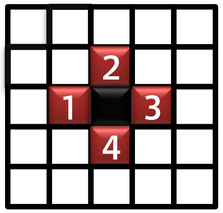
			1			

Update Rule

- **Example:**

Neighbourhood = Von Neumann

$$f(s, s_1, s_2, s_3, s_4) = \sum s \ (mod\ 4)$$



Old state of the CA

1	2	1	0	2	1	1
2	3	1	2	3	1	1
0	2	1	2	0	2	1
3	2	1	1	1	5	1
0	1	2	0	2	2	0
2	0	1	1	1	3	1
2	2	0	2	3	1	2

$$f(s, s_1, s_2, s_3, s_4) =$$
$$= 1 + 1 + 2 + 3 + 3\ (mod 4) =$$
$$= 10\ (mod\ 4) = 2$$

New state of the CA

			1			
					2	

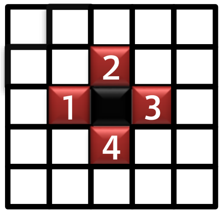
Update Rule

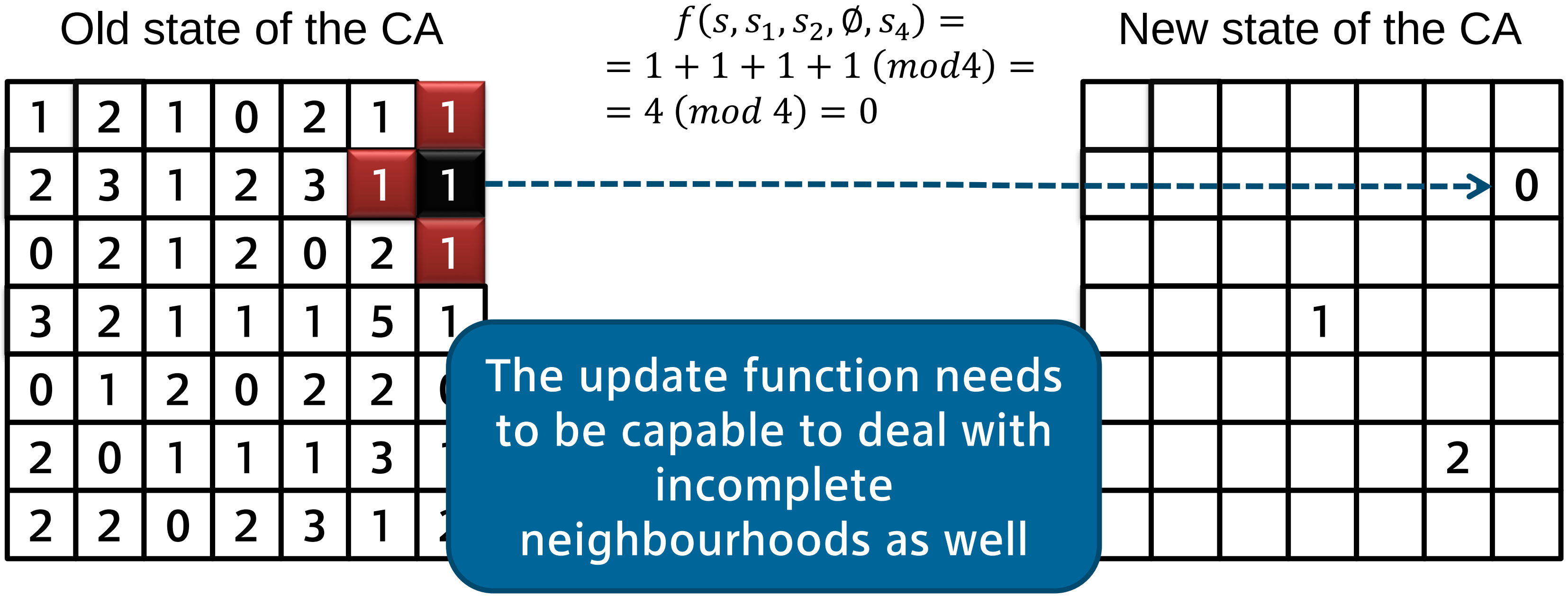
Mathematical Modelling and Simulation

- **Example:**

Neighbourhood = Von Neumann

$$f(s, s_1, s_2, s_3, s_4) = \sum s \pmod 4$$





Old state of the CA

1	2	1	0	2	1	🟥 1
2	3	1	2	3	🟥 1	■ 1
0	2	1	2	0	2	🟥 1
3	2	1	1	1	5	1
0	1	2	0	2	2	0
2	0	1	1	1	3	1
2	2	0	2	3	1	2

$$f(s, s_1, s_2, \emptyset, s_4) =$$
$$= 1 + 1 + 1 + 1 \ (mod 4) =$$
$$= 4 \pmod 4 = 0$$

New state of the CA

						0
			1			
					2	

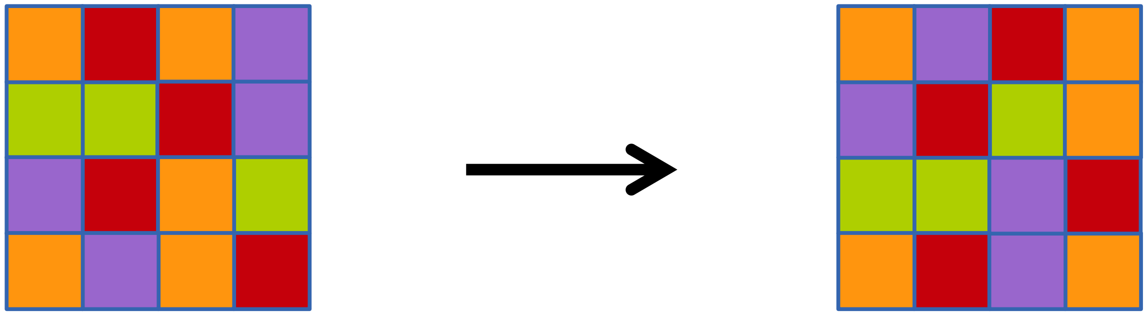
The update function needs to be capable to deal with incomplete neighbourhoods as well

Updates

- Updates happen for all cells simultaneously.

Why?

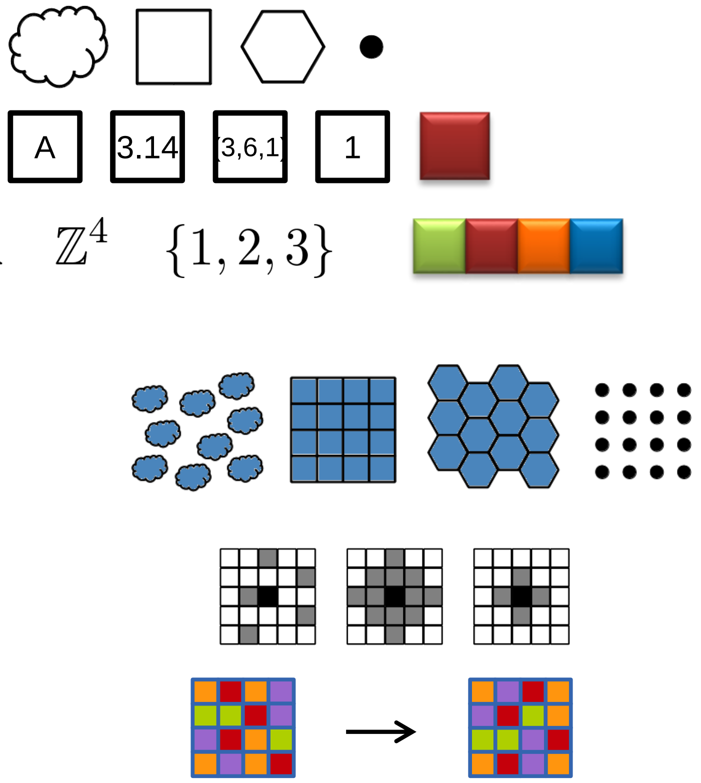
  – Neighborhoods are all computed from the same system state
  – Update order of cells is irrelevant



Components of a CA

Mathematical Modelling and Simulation

- Cells
- States
- State-space
- Arrangement (Cell-space)
- Neighbourhood
- Update Rule
- Iterations



Iteratively apply the update rule on the complete CA finally leads to a simulation model

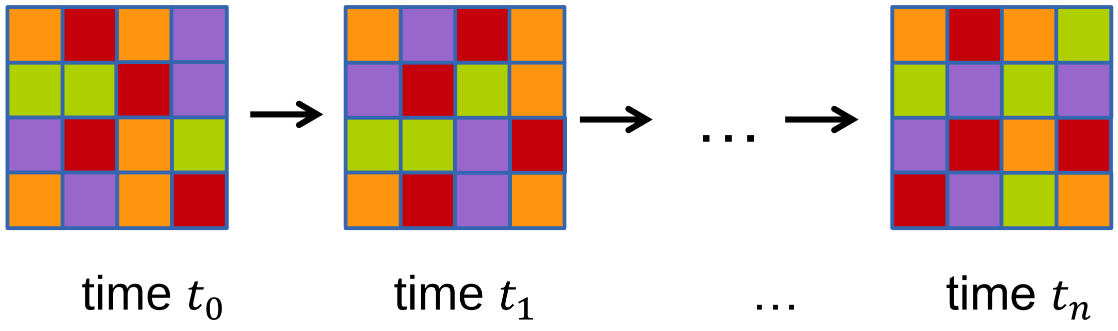
Iterations

- Define discrete, equidistant time points (all time steps between time points are of the same length): $t_0, t_1, \ldots, t_n$
- Every update of states brings the model to the next time point

➢ Cellular Automaton (CA)

Iterations

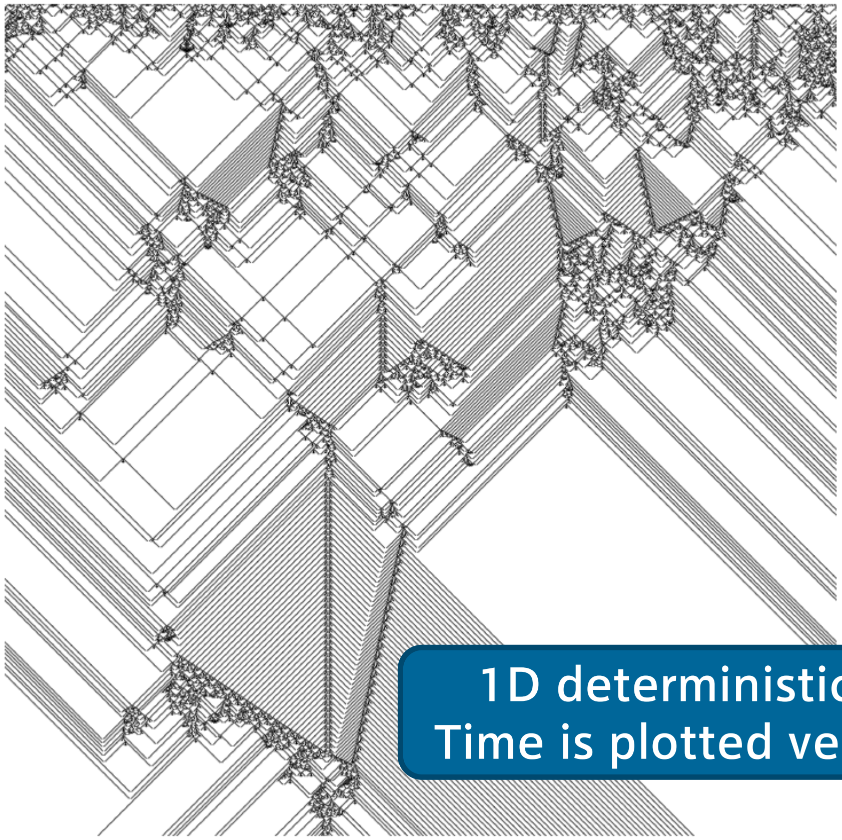
- Tasks for one iterations
  - Compute the neighbors of all cells
  - Determine states of all cells, and states of all neighbours of all cells
  - Compute state updates for all cells and store them
  - Apply the updated states for all cells



Properties of CA models

Cellular Automata are microscopic simulation models that are capable of producing almost arbitrarily complex, up to chaotic, behaviour.

They are, hence, not only a very powerful, but also a very dangerous modelling approach with respect to validity.



1D deterministic CA!
Time is plotted vertically

Properties of CA models

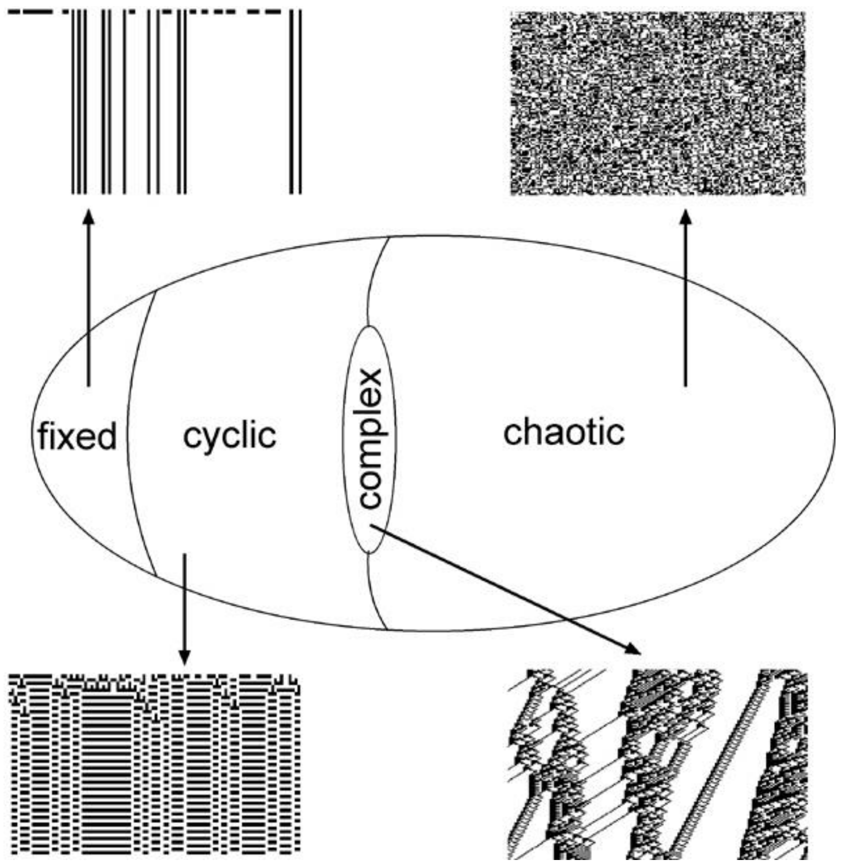
Mathematical Modelling and Simulation

Cellular Automata are microscopic simulation models that are capable of producing almost arbitrarily complex, up to chaotic, behaviour.

Stephen Wolfram
(A New Kind of Science, 2002)
stated that CAs may have one of the four types of behavour:

fixed, cyclic, complex, chaotic

Chris Langton developed the schematic to the right.



Mathematical Modelling and Simulation

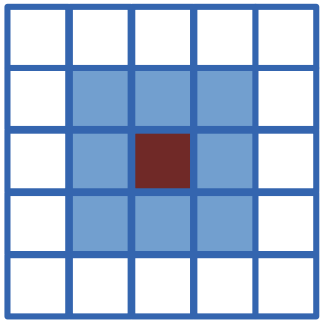
Example:

CONWAY'S GAME OF LIFE

Conway's Game of Life

Mathematical Modelling and Simulation

- Cells on a 2-dimensional, rectangular or infinite lattice: $I = (1,2,\dots a) \times (1,2,\dots,b)$ or on $I = \mathbb{Z}^2$.
- Set of states: $\mathbb{S} = (alive \quad , dead \quad)$
- Moore neighborhood

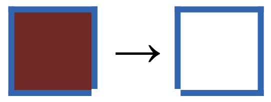
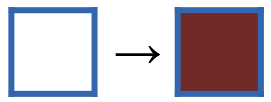
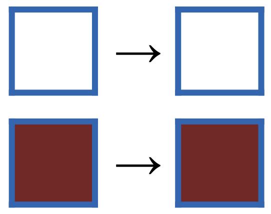


Index translations:

$$\left(\binom{1}{0},\binom{1}{1},\binom{0}{1},\binom{-1}{1},\binom{-1}{0},\binom{-1}{-1},\binom{0}{-1},\binom{1}{-1}\right)$$

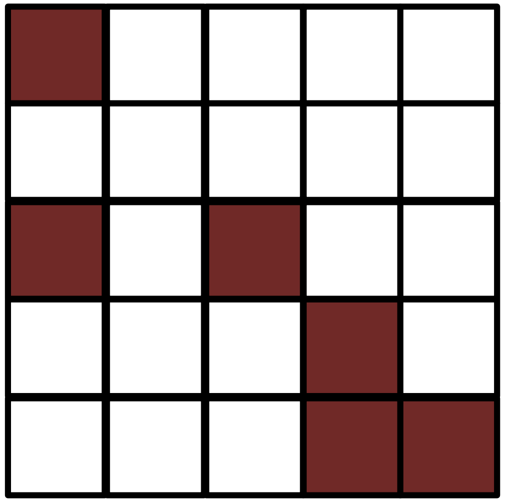
Conway's Game of Life

- Update rules:
  - An alive cell with fewer than two or more than three alive neighbors dies (“under-population” or “overcrowding”)
  - A dead cell with exactly three alive neighbors becomes alive (“reproduction”)
  - Cells keep their state in any other case

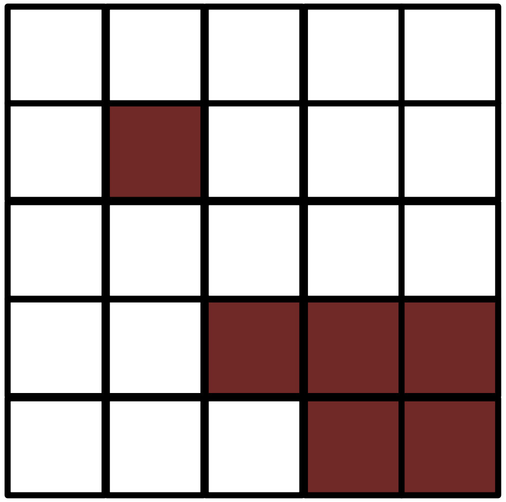




Conway's Game of Life

Mathematical Modelling and Simulation



time *t=0*



time *t=1*

Conway's Game of Life

- Designed by John Horton Conway, 1970
- Why "Game of Life"?
  - Teaching purposes
  - Academic competitions
  - Fundamental/methodological research
  - Game → figures

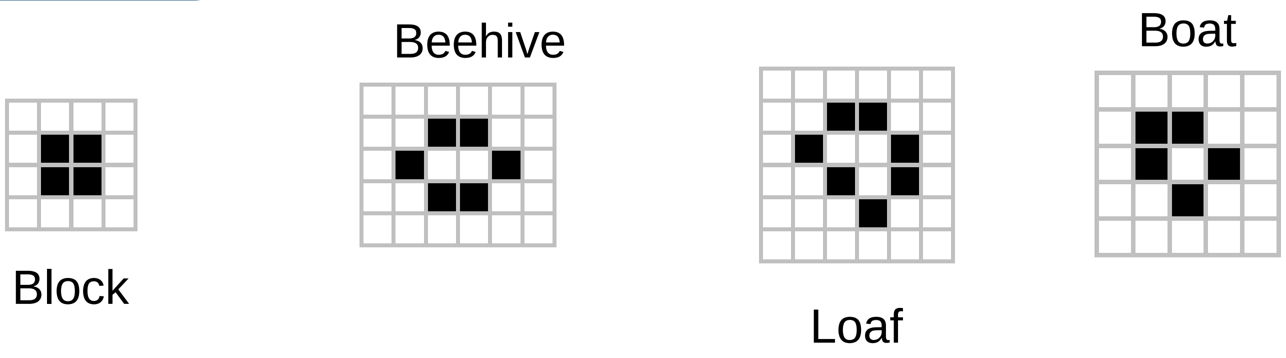
Probably ***worst*** example for a Cellular Automata simulation model,...

...but probably the ***best*** example to show the concepts of CAs.

Conway's Game of Life

Pattern analysis of the Game of Life became its own science (although its applicability can be doubted).

Static figures



Source: http://en.wikipedia.org/wiki/Conway%27s_Game_of_Life

Conway's Game of Life

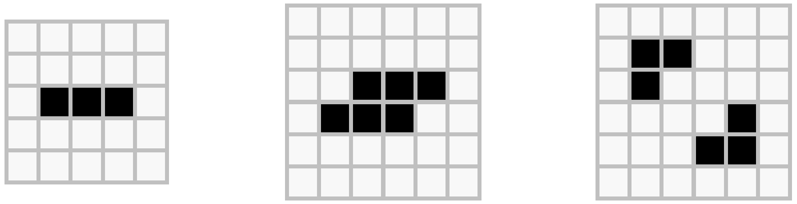
Mathematical Modelling and Simulation

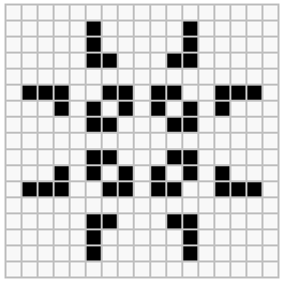
Pattern analysis of the Game of Life became its own science (although its applicability can be doubted).

Oscillators

Pulsar (period 3)

Toad (period 2)





Blinker (period 2)

Beacon (period 2)

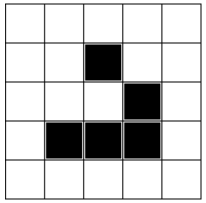
Source: http://en.wikipedia.org/wiki/Conway%27s_Game_of_Life

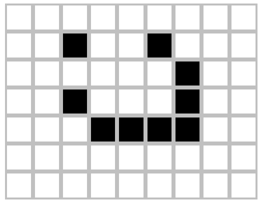
Conway's Game of Life

Pattern analysis of the Game of Life became its own science (although its applicability can be doubted).

Gliders (moving objects)

Lightweight spaceship (LWSS)


Glider



Source: http://en.wikipedia.org/wiki/Conway%27s_Game_of_Life

Conway's Game of Life

Pattern analysis of the Game of Life became its own science (although its applicability can be doubted).

As it seemed as if any starting configuration of the GoL resulted in a static or oscillating end-configuration, Conway offered a price of 50$ for a pattern that resulted in an infinitely growing population.

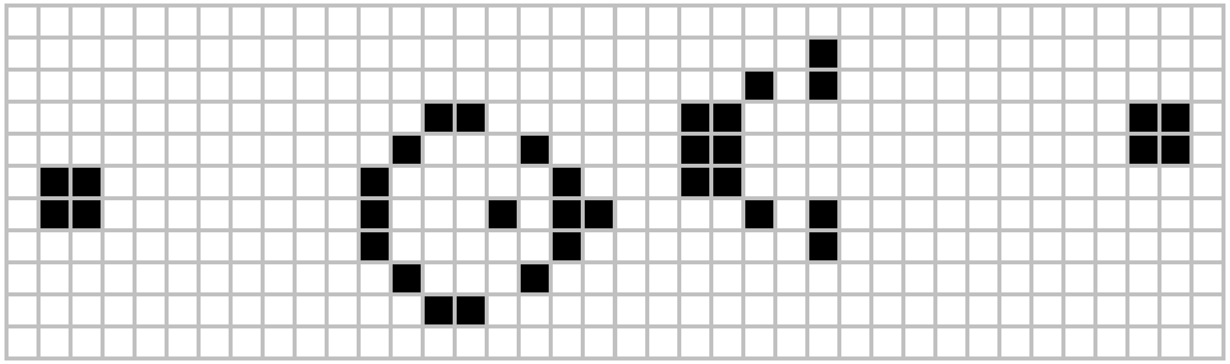
Source: http://en.wikipedia.org/wiki/Conway%27s_Game_of_Life

Conway's Game of Life

Pattern analysis of the Game of Life became its own science (although its applicability can be doubted).

Bill Gosper's answer:

Gosper Glider Gun



Source: http://en.wikipedia.org/wiki/Conway%27s_Game_of_Life

Example

LATTICE GAS CELLULAR AUTOMATA (LGCA)

Lattice Gas Cellular Automata

- Lattice Gas Cellular Automata (LGCA)
- Extension of the CA concept
- Intention: Simulate fluids and gases
- Invented by Hardy, Pomeau and de Pazzis (HPP automaton on square lattice), 1973
- Improved by Frisch, Hasslacher and Pomeau (FHP automaton on hexagonal grid), 1986

Lattice Gas Cellular Automata

- **Ideas**
  - Cells do not have states but instead can contain particles
  - A particle can only proceed to a cell in the neighborhood
  - Instead of state updates, particles move to other cells
  - Particles represent the fluid or the gas

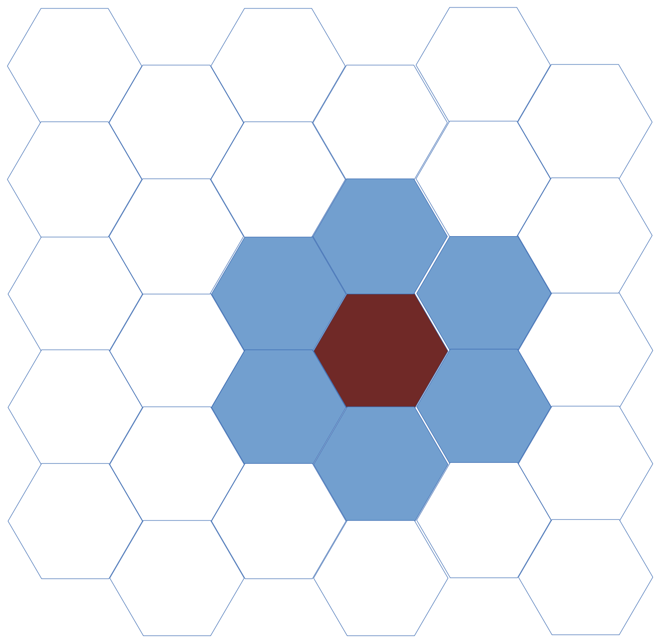
Lattice Gas Cellular Automata

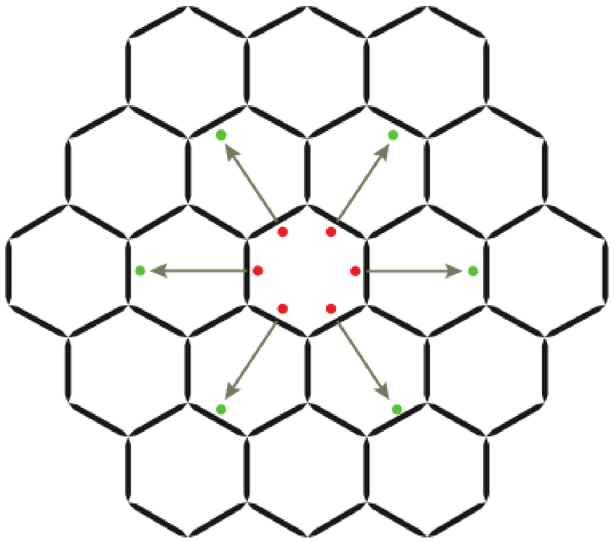
- **HPP**
  - square grid, Von-Neumann neighborhood, max. 4 particles per cell so that max. 1 particle goes to each neighbor
  - several issues when it comes to real interpretations (comparison with real fluids, validation)

Lattice Gas Cellular Automata

- **FHP**
  - hexagonal grid
  - neighborhood = surrounding cells
  - max. 6 particles per cell, each going into a different direction → consistent definition
  - Corresponds to the Navier-Stokes-Equations → valid representation of fluid dynamics

FHP Model


neighbourhood

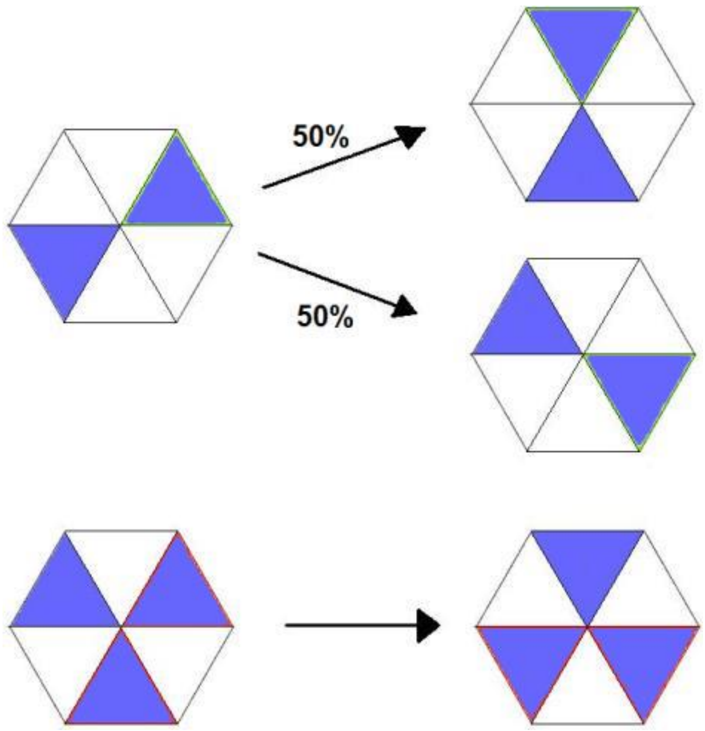

particles and directions

FHP Model

- Particle movements consist of two phases
  - Rotation of cells for special configurations
  - Movements of particles into their direction
- Developed by Wolf-Gladrow (2000)
- Different variations (FHP-I, FHP-II, FHP-III)

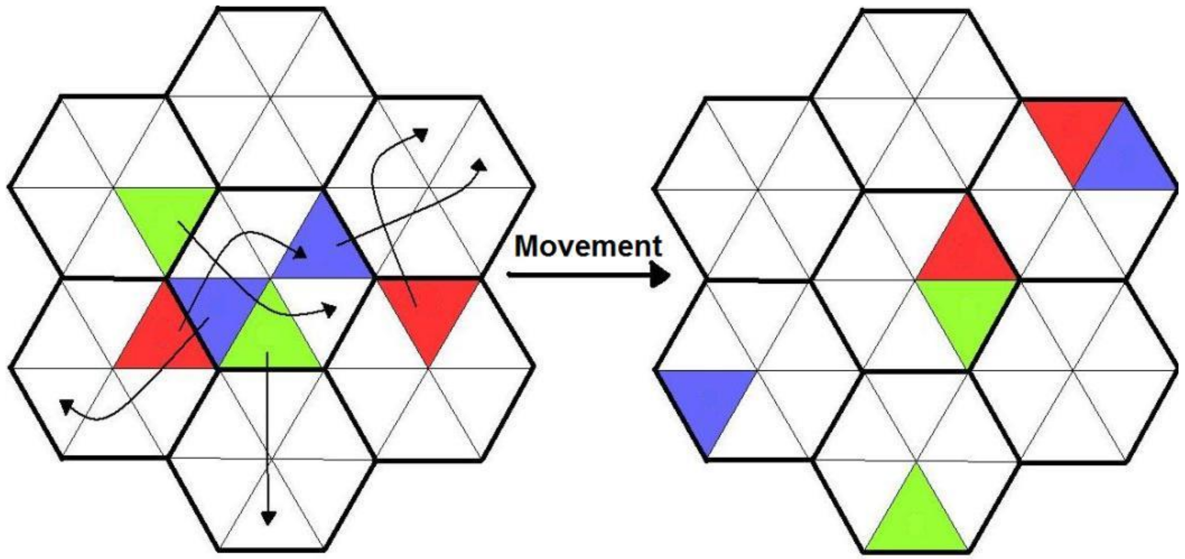
FHP Model

- Rotations
  - In the most simple case of FHP-I only for two situations
  - Provide a randomness



FHP Model

- Movements into designated directions

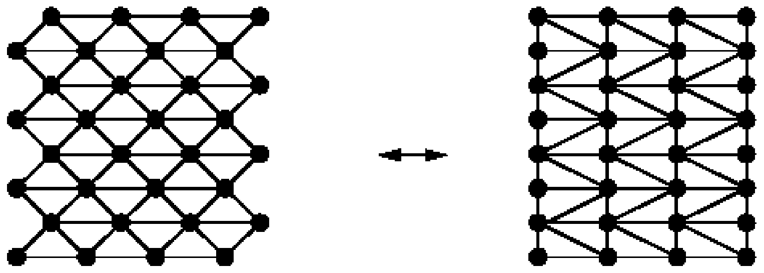


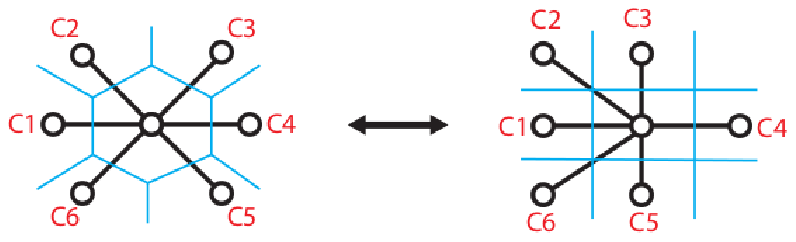
LGCAs

- **Simulations & Visualizations**
- **HPP**
- http://en.wikipedia.org/wiki/File:Gas_velocity.gif
- **FHP**
- http://www.youtube.com/watch?v=HluQpDFOceg
- http://www.youtube.com/watch?v=00W6H7BGZ94

Remark: Implementation of a hexagonal grid

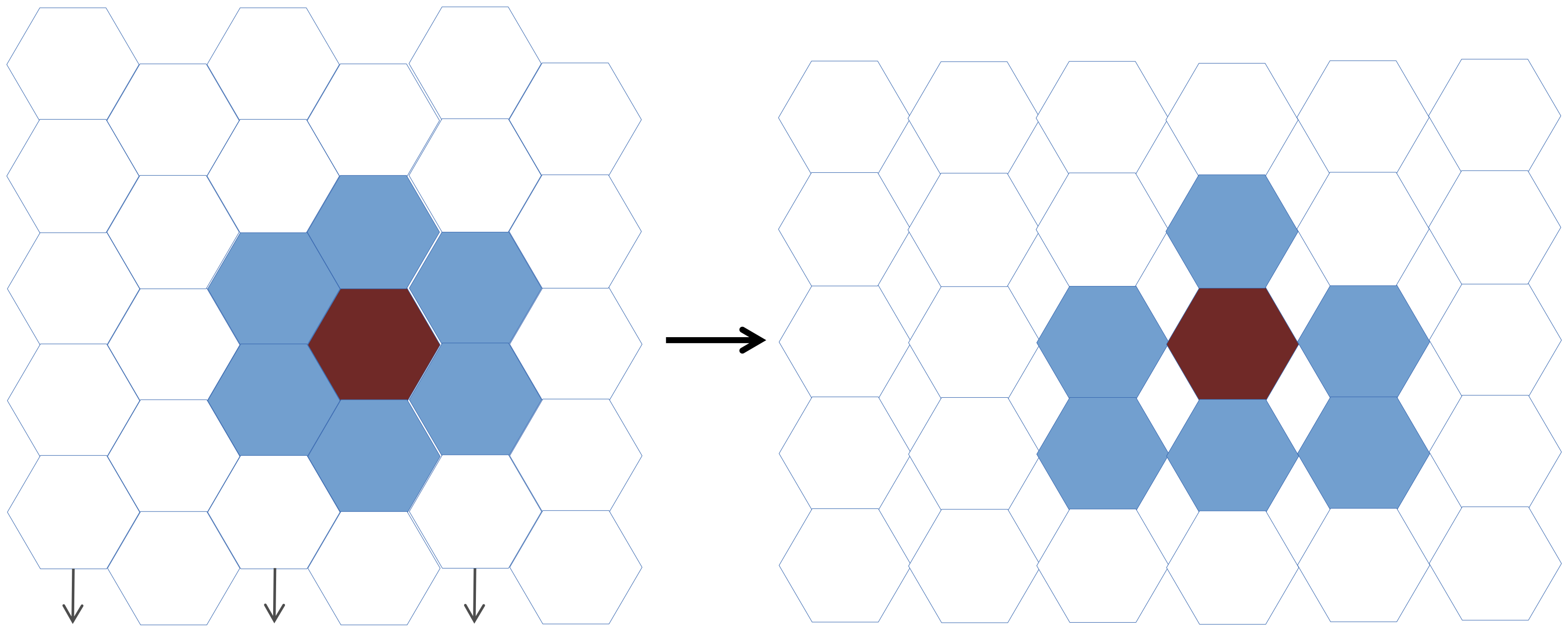
- **Implementation**
  - hexagonally arranged grid → assign to a square lattice
  - conditional neighborhoods





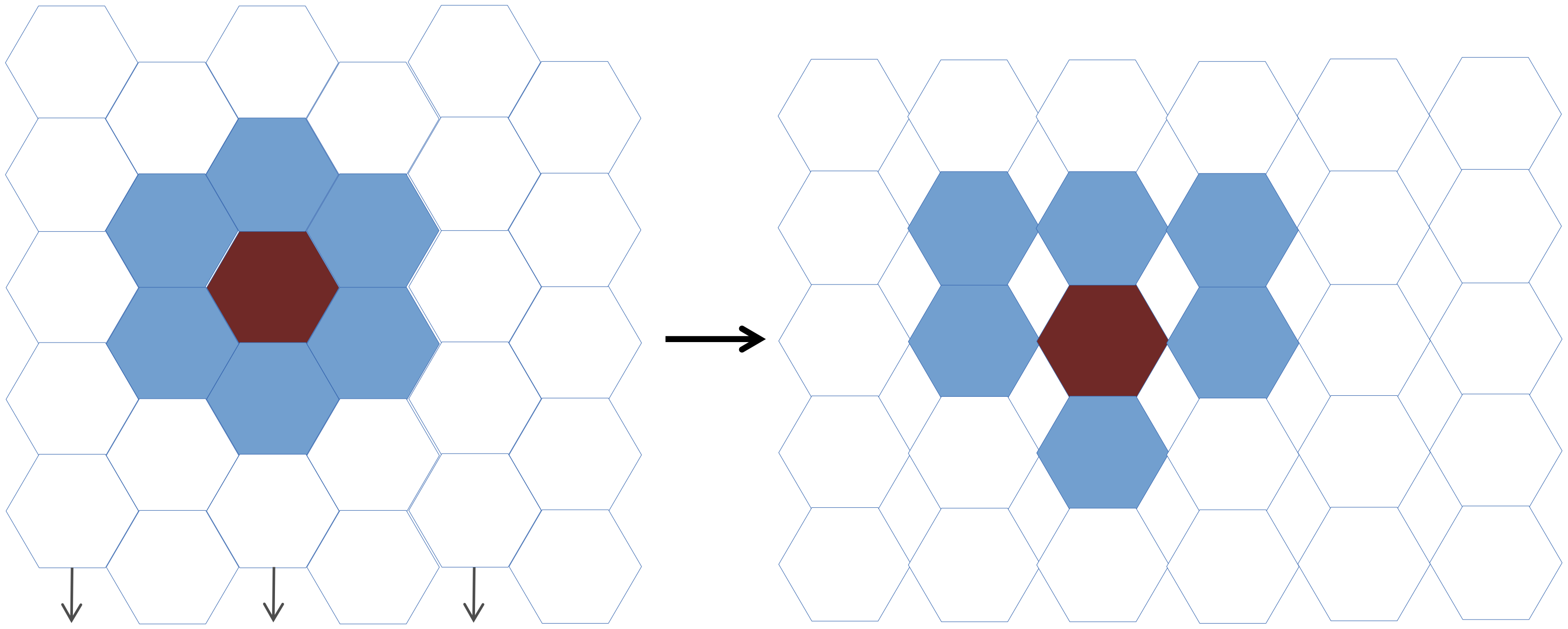
Remark: Implementation of a hexagonal grid

Mathematical Modelling and Simulation



Remark: Implementation of a hexagonal grid

Mathematical Modelling and Simulation



Example

EPIDEMIC SIMULATION WITH CA AND LGCA

SIR concept

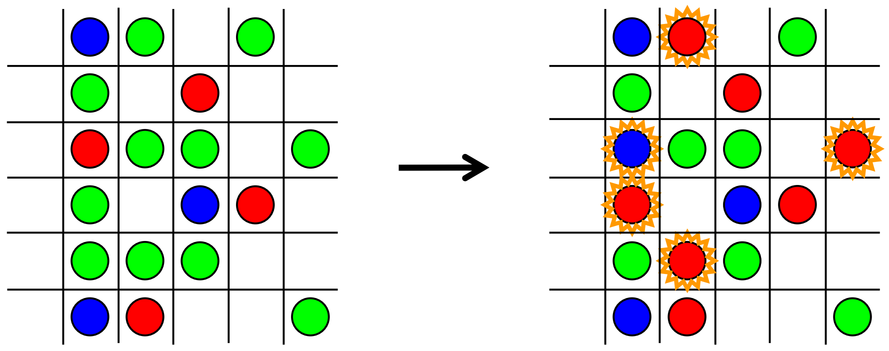
- Simulate the spread of an epidemics
- Susceptible (S) people become infected by infectous (I) and become resistant/recovered (R) after some time.
- Resistant persons cannot be infected again.

$$\begin{tikzcd} S \arrow[r] & I \arrow[r] & R \end{tikzcd}$$

CA Implementation

CA Implementation of SIR epidemics:

- Every cell in a rectangular (hexagonal..) lattice represents a person/group of persons/household/...
- Infecious cells recover after some time (with some probability).
- Infectious cells may spread the disease to their neighbours (e.g. Moore neighbourhood)

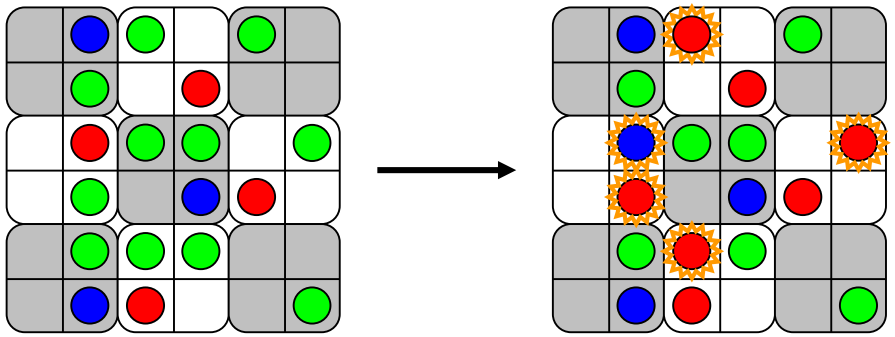


LGCA Implementation

Mathematical Modelling and Simulation

LGCA Implementation of SIR epidemics:

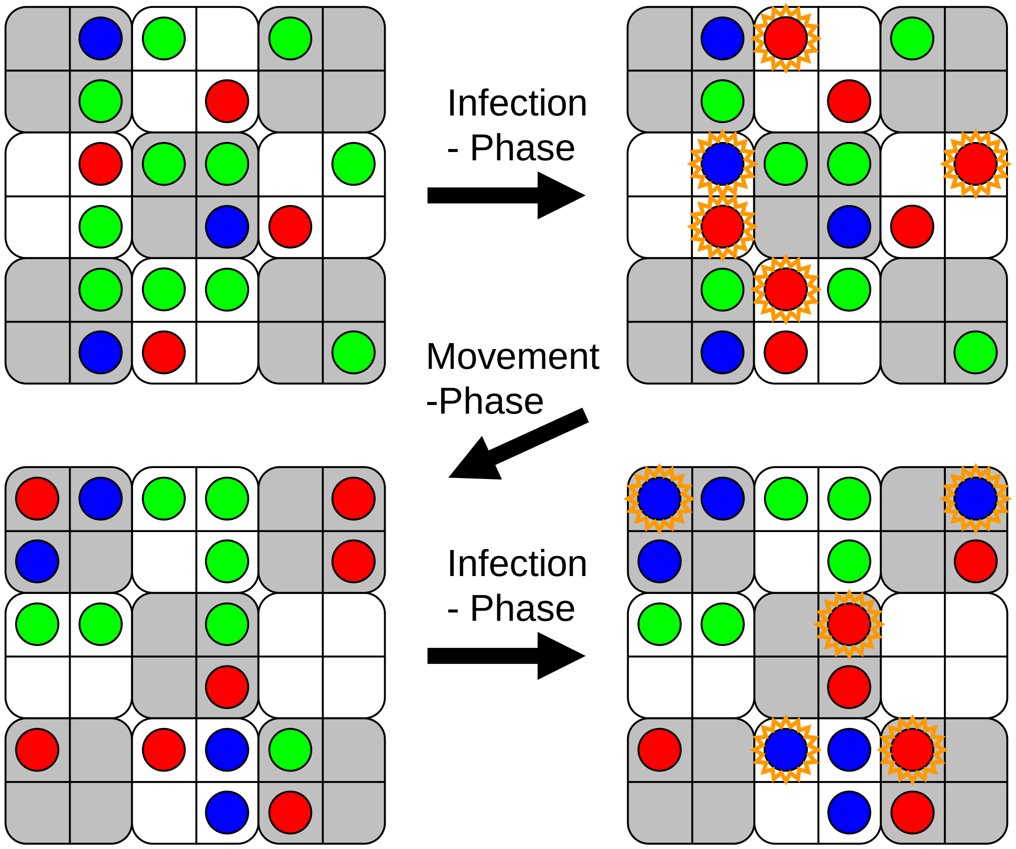
- Every cell in a rectangular (hexagonal..) lattice contains a number of persons (e.g. 4)
- Infecious persons recover after some time (with some probability).
- Infectious persons may spread the disease to all other persons in the cell



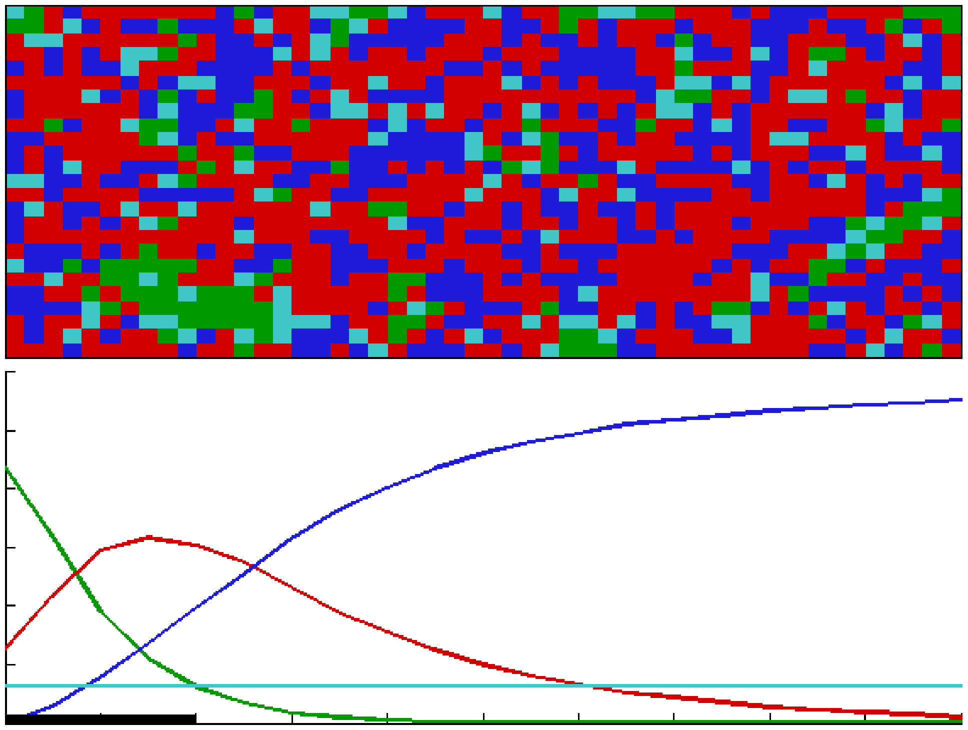
LGCA Implementation

Mathematical Modelling and Simulation

LGCA Implementation of SIR epidemics:

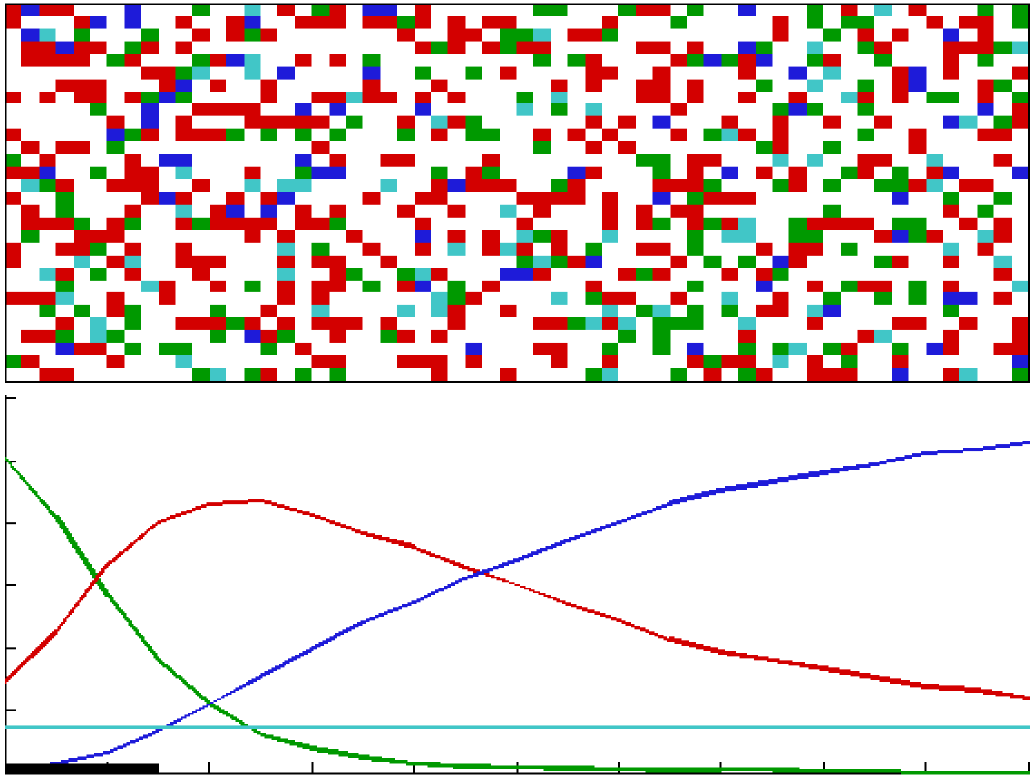


Epidemic simulation with CA



Epidemic simulation with HPP-LGCA

Mathematical Modelling and Simulation



Mathematical Modelling and Simulation

HISTORY OF CELLULAR AUTOMATA

History of Cellular Automata

- 1925: Ising Modell
  - ferromagnetism, discrete model
- 1950: Von Neumann, Ulam
  - term “cellular automaton”
  - self reproductive, Von-Neumanns theory on logic automata

- 1950-1970: Zuse, u.a.
  - parallel algorithms
  - discrete processes (e.g. PDEs)
- 1970s: Hardy, Pomeau, de Pazzis
  - Lattice Gas Cellular Automata
- 1979: Conway's Game of Life

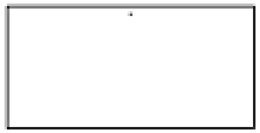
History of Cellular Automata

- 1980+: different applications
- 2002: Wolfram
  - complete classifications of 1-dimensional cellular automata

History of Cellular Automata

Mathematical Modelling and Simulation

Stephen Wolfram, „A new Kind of Science“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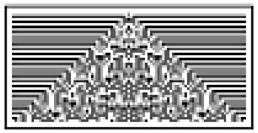
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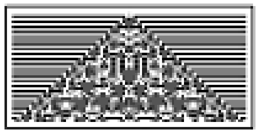
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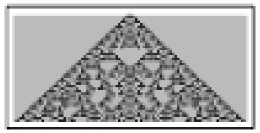
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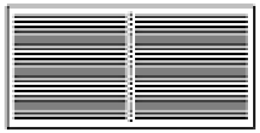
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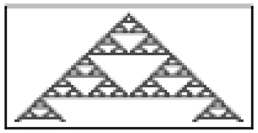
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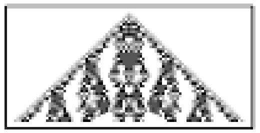
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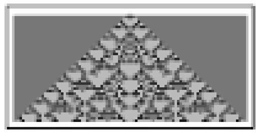
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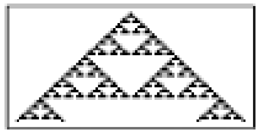
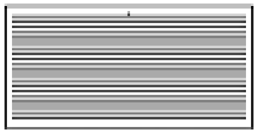

2-color code 2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4-color code 107397

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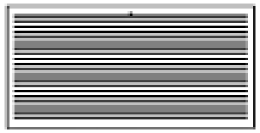
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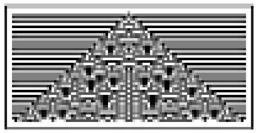
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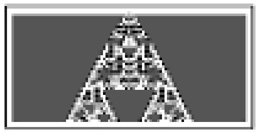
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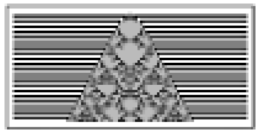
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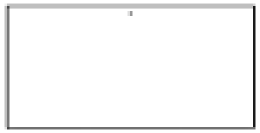
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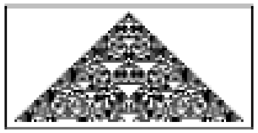
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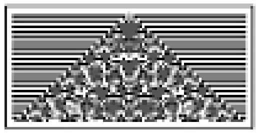
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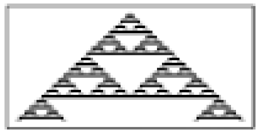

5-color code 160197745

History of Cellular Automata

- a spatially extended decentralized system made up of a number of individual components […] local interaction […] depending on the states of its local neighbors […] parallel processing [Ganguly]
- regular grid of cells, each in one of a finite number of states […] neighbourhood […] new generation is created according to some fixed rule [Wikipedia]

History of Cellular Automata

Mathematical Modelling and Simulation

- regular arrangements of single cells […] each cell holds a finite number of discrete states […] updated simultaneously […] the rules for the evolution of a cell depend only on a local neighborhood [Gladrow]

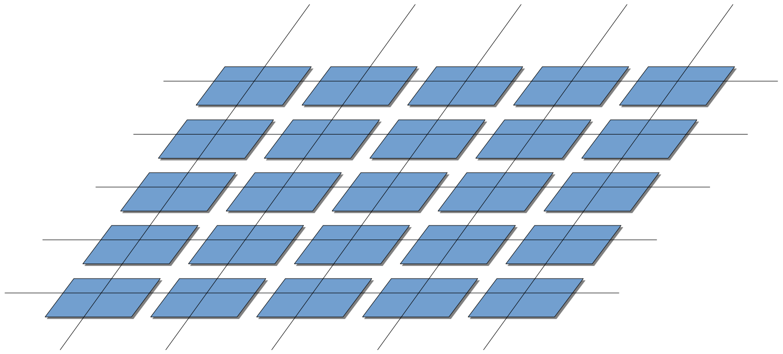
5. Conclusions

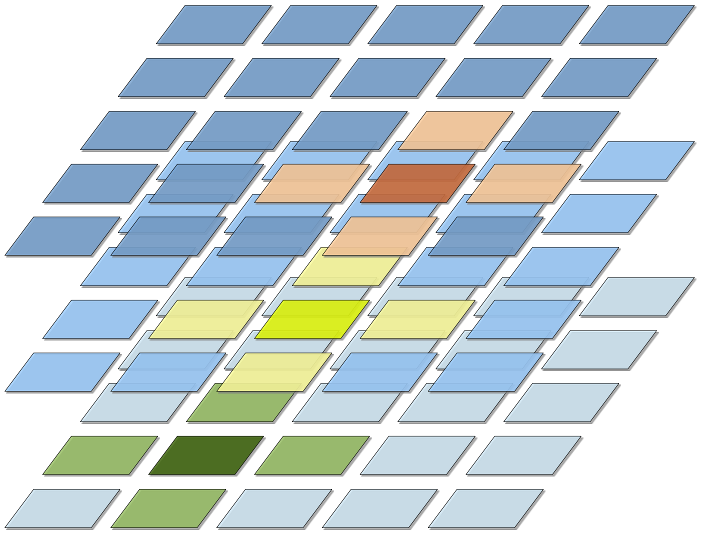
CONCLUSIONS

Characteristics of CA

Mathematical Modelling and Simulation

- Regular lattice, same kind of neighbourhoods

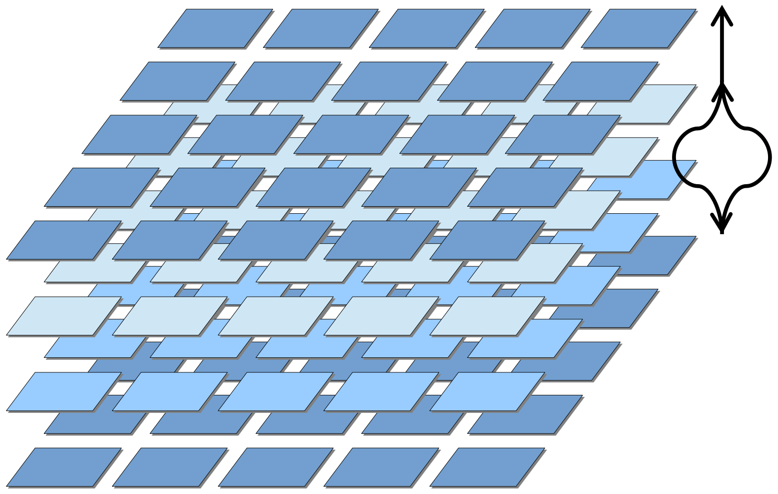




Characteristics of CA

Mathematical Modelling and Simulation

- Discrete time, equidistant time steps



Characteristics of CA

Mathematical Modelling and Simulation

- Spatial representation, locality