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Numerik von Differentialgleichungen - Kreuzlübung 2

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Exercise 6:
Let $y : [0, T] \to \mathbb{R}^n$ be a solution to the initial value problem $y' = f(y)$ with $y(0) = y_0$. Implement a general solver for such kind of problems. The input data should be the function f, a partition of the interval $[0, T]$, the initial value y_0, and the Butcher-tableau of an arbitrary explicit Runge-Kutta-method. The output should be the approximations y_i to $y(t_i)$ of the corresponding Runge-Kutta-method. Use your code to numerically solve the following predator-prey-model.

Model: For each time t, let $y_\mathrm{J}(t)$ and $y_\mathrm{B}(t)$ be the size of a predator and prey population, respectively. The growth rate of the populations is the difference of birth rate and death rate. Here, we assume that there is enough food for the prey population such that their birth rate $\alpha > 0$ is constant. On every encounter of predator and prey, the prey gets eaten with constant rate $\beta > 0$. In comparison, the natural death rate of the prey is negligible. These assumptions yield the differential equation

$$y_\mathrm{B}'(t) = \alpha y_\mathrm{B}(t) - \beta y_\mathrm{J}(t) y_\mathrm{B}(t) \tag{1a}$$

for the prey population. For the predator population, we assume that it grows proportionally to the number of encounters with rate $\gamma > 0$. The natural death rate $\delta > 0$ of the predators is not negligible in this model. Overall, this yields the differential equation

$$y_\mathrm{J}'(t) = \gamma y_\mathrm{J}(t) y_\mathrm{B}(t) - \delta y_\mathrm{J}(t) \tag{1b}$$

for the predator population.

For testing, you can e.g. use the model parameters $\alpha = 2$, $\beta = \gamma = 0.01$, $\delta = 1$, $y_\mathrm{J}(0) = 150$, and $y_\mathrm{B}(0) = 300$. For the discretization you can start with an equidistant partition of $t \in [0, 100]$ with $h = 0.01$ and the explicit Runge-Kutta-method from the lecture (Example 2.23 and Example 2.25). However, you should vary the model parameters and the discretization to study their effects on the numerical results.

Exercise 7:
An initial value problem is called autonomous if the right hand side of the differential equation does not explicitly depend on time, i.e, if Y is the solution to the initial value problem

$$Y'(t) = F(Y(t)), \qquad t \in J, \qquad Y(t_0) = Y_0. \tag{2}$$

Every "normal" initial value problem

$$y = f(t, y(t)), \qquad t \in J, \qquad y(t_0) = y_0 \tag{3}$$

can be equivalently reformulated as an autonomous initial value problem with $Y(t) := (t, y(t))^\top$, $Y_0 := (t_0, y_0)^\top$ and $F(x) := (1, f(x))^\top$. A one-step method is called invariant under autonomization, if it gives exactly the same approximations when applied to (2) and (3) for arbitrary f.

Show that an explicit s-stage Runge-Kutta-method is invariant under autonomization if and only if there holds

$$c_j = \sum_{i=1}^{j-1} a_{ji}, \qquad j = 1, \dots, s. \tag{4}$$

Exercise 8:
Show that an explicit s-stage Runge-Kutta-method, which is invariant under autonomization, has order 3 if there holds

$$\sum_{j=1}^{s} b_j = 1, \tag{5a}$$

$$\sum_{j=1}^{s} b_j c_j = \frac{1}{2}, \tag{5b}$$

$$\sum_{j=1}^{s} b_j c_j^2 = \frac{1}{3}, \tag{5c}$$

$$\sum_{j=1}^{s} \sum_{i=1}^{j-1} b_j a_{ji} c_i = \frac{1}{6}. \tag{5d}$$

Compare this to Proposition 2.17 from the lecture notes.

Hint: Use Taylor's Theorem on k_j, $j = 1, \dots, s$. With the notation from Proposition 2.17 you should get

$$k_j = f + h\left(c_j f_t + f_y \sum_{i=1}^{j-1} a_{ji} k_i\right) + h^2 \left(\frac{c_j^2}{2} f_{tt} + c_j f_{yt} \sum_{i=1}^{j-1} a_{ji} k_i + \frac{1}{2} f_{yy} \left(\sum_{i=1}^{j-1} a_{ji} k_i\right)^2\right) + \mathcal{O}(h^3). \tag{6}$$

Subsitute the k_i in this formula by their corresponding Taylor approximation of sufficient order. Now you can proceed as in Proposition 2.17 to obtain the order conditions.

Exercise 9:
Construct all explicit 3-stage Runge-Kutta-methods which are invariant under autonomization and which use the quadrature weights (not the quadrature points) of the Simpson-rule.

Exercise 10:
Compute an upper bound for the stability constant C_{stab} for the classcial Runge-Kutta-method from Example 2.25. To this end, compute the coefficients μ_j from (2.38) explicitly.