HW5

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1. Distribution of the maximum

Let $X_1, X_2, \dots$ be a sequence of i.i.d. with uniform $(0,1)$ distribution and let $X_{(n)} = \max_{1\le i\le n} X_i$. Show that the sequence

$$Y_n = n(1 - X_{(n)}), \quad n \in \mathbb{N}$$

converges to an exponential exp(1) random variable as $n \to \infty$.

Solution: We will show convergence in distribution:

$$\begin{aligned} F_n(y) &= \mathbb{P}(Y_n \le y) = \mathbb{P}\big(n(1 - X_{(n)}) \le y\big) \\ &= \mathbb{P}\big(X_{(n)} \ge 1 - \frac{y}{n}\big) \\ &= 1 - \mathbb{P}\big(\max_{1\le i\le n} X_i \le 1 - \frac{y}{n}\big) \\ &= 1 - \mathbb{P}\big(X_1 \le 1 - \frac{y}{n}\big)^n \\ &= \begin{cases} 1, & 1 - \frac{y}{n} \le 0 \\ 1 - (1 - \frac{y}{n})^n, & 0 < 1 - \frac{y}{n} < 1 \\ 0, & 1 \le 1 - \frac{y}{n} \end{cases} \end{aligned}$$

Here we used the fact that $\mathbb{P}(\max_{1\le i\le n} X_i \le x) = \mathbb{P}(X_1 \wedge X_2 \wedge \dots \wedge X_n \le x)$ as well as the independence of the X_i. We can now reformulate the boundary:

$$1 - \frac{y}{n} \le 0 \iff n \le y$$

So for $n \to \infty$ this case will never happen. The second one is

$$\begin{aligned} 0 < 1 - \frac{y}{n} < 1 &\iff 1 > \frac{y}{n} > 0 \iff n > y > 0 \\ 1 \le 1 - \frac{y}{n} &\iff y \le 0 \end{aligned}$$

If we know use our knowledge from calulus, namely that $\lim_{n\to\infty}(1 + \frac{y}{n})^n = e^y$, we now get

$$\lim_{n\to\infty} F_n(y) = \begin{cases} 1 - e^{-y}, & y \ge 0 \\ 0, & y < 0 \end{cases}$$

which is exactly the cdf of exp(1).

2. Coin throws

An unfair coin is thrown 600 times. The probability of getting a tail in each throw is $\frac{1}{4}$.

(a) Use a Binomial distribution to compute the probability that the number of heads obtained does not differ more than 10 from 450.

(b) Use a Normal approximation without a continuity correction to calculate the probability in (a). How does the result change if the approximation is provided with a continuity correction?

Solution:

(a) Suppose $X \sim bin(600, \frac{3}{4})$, what we want to calculate is

$$\mathbb{P}(440 \leq X \leq 460) = \mathbb{P}(X \leq 460) - \mathbb{P}(X \leq 439)$$

We can do this very easily in R. We note that this is not the same as

$$\mathbb{P}(X \leq 460) - \mathbb{P}(X \leq 440)$$

which we would get if we were working with continuous distributions. Nevertheless we will calculate this value, to compare in (b).

```
pbinom(460, 600,3/4) - pbinom(439, 600, 3/4)
```

```
## [1] 0.6778428
```

```
pbinom(460, 600,3/4) - pbinom(440, 600, 3/4)
```

```
## [1] 0.6540917
```

(b) We want to use the CLT for $X_i \sim bernoulli(\frac{3}{4})$, with $\mathbb{E}(X_i) = \frac{3}{4}$ and $\mathbb{V}ar(X_i) = \frac{3}{16}$. We however do not want to approximate a sample mean, but just the sum $S_n = \overline{X} \cdot n$. If we plug this into the CLT we see that

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \approx \mathcal{N}(0, 1)$$

So all in all we get (without continuity correction)

$$\mathbb{P}(440 \leq S_n \leq 460) = \mathbb{P}(\frac{440 - n\mu}{\sqrt{n\sigma^2}} \leq \frac{S_n - n\mu}{\sqrt{n\sigma^2}} \leq \frac{460 - n\mu}{\sqrt{n\sigma^2}}) \approx \Phi(\frac{460 - n\mu}{\sqrt{n\sigma^2}}) - \Phi(\frac{440 - n\mu}{\sqrt{n\sigma^2}})$$

In our case $n = 600$ and the expectation and variance we have already stated above. With continuity correction we get almost the same, just $460 + 0.5$ on the upper bound and $440 - 0.5$ on the lower one. We use R to calculate the values.

```
pnorm((460-600*0.75)/sqrt(600*(3/16))) - pnorm((440-600*0.75)/sqrt(600*(3/16)))
```

```
## [1] 0.6542214
```

```
pnorm((460+ 0.5-600*0.75)/sqrt(600*(3/16))) - pnorm((440-0.5-600*0.75)/sqrt(600*(3/16)))
```

```
## [1] 0.6778012
```

We see that the value with continuity correction is a better approximation. What we can also observe that the value without continuity correction is a good approximation to the second value in (a), which is not surprising since there we assumed that we deal with a continuous distribution there.

3. Simulations

(a) By applying the R- function `replicate()` generate a sample $X_1, \ldots, X_{10}$ of size 10 from an exponential distribution with a a rate parameter 0.2 and sum up its elements. Do this sum 10000 times and make a histogram of the simulation. Can you say something about the shape of the distribution?

(b) Use R to simulate 50 tosses of a fair coin (0 and 1). We call a *run* a sequence of all 1's or all 0's. Estimate the average length of the longest run in 10000 trials and report the result.

Hint: Use the commands `rbinom` and `rle`. The command `rle()` stands for run lenght endcoding. For example,

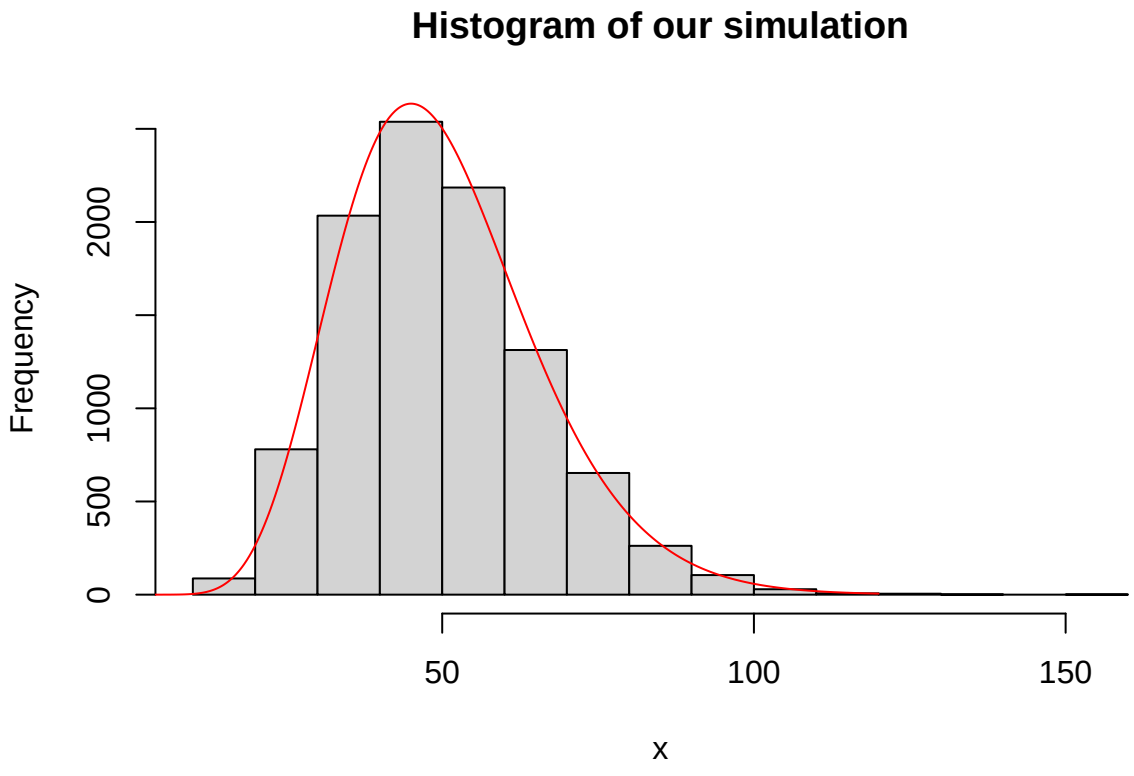
```
rle(rbinom(5, 1, 0.5))$lenghts
```

is a vector of the lengths of all the different runs in a trial of 5 flips of a fair coin.

Solution:

(a)

```
x = replicate(10000, sum(rexp(10,0.2)))
z =  seq(0,120,0.1)
y = 100000*dgamma(z,10,0.2)
hist(x, main = "Histogram of our simulation")
lines(z,y, col ="red")
```



As by 2(b) of HW4 we get a $Gamma(10, 0.2)$ distribution.

(b)

```
mean(replicate(10000,max(rle(rbinom(50, 1, 0.5))$lengths)))
```

```
## [1] 5.9641
```

4. Contitional variance

(a) Show that for any two random variables X and Y the conditional variance identity holds

$$\mathbb{V}arY = \mathbb{E}\big(\mathbb{V}ar(Y|X)\big) + \mathbb{V}ar\big(\mathbb{E}(Y|X)\big),$$

provided that the expectation exists. The law of total expectation (the tower property) $\mathbb{E}X = \mathbb{E}(\mathbb{E}(X|Y))$ should be applied.

(b) Suppose that the distribution of Y conditional on $X = x$ is $\mathcal{N}(x, x^2)$ and that the marginal distribution of X is uniform on $(0, 1)$. Compute $\mathbb{E}Y, \mathbb{V}arY$ and $\mathbb{C}ov(X, Y)$.

Solution: (a) We first remind of the definition

$$\mathbb{V}ar(Y|X) = \mathbb{E}\Big((Y - \mathbb{E}(Y|X))^2|X\Big) = \mathbb{E}(Y^2|X) + \mathbb{E}(Y|X)^2.$$

We will use the conditional parallel axis theorem (taking $a(X) = \mathbb{E}(Y)$)

$$\mathbb{E}\big((Y - \mathbb{E}(Y))^2|X\big) = \mathbb{V}ar(Y|X) + \big(\mathbb{E}(Y) - \mathbb{E}(Y|X)\big)^2.$$

With the law of total expectation we now get

$$\begin{aligned}
\mathbb{V}ar(Y) &= \mathbb{E}\big((Y - \mathbb{E}(Y))^2\big) = \mathbb{E}\Big(\mathbb{E}\big((Y - \mathbb{E}(Y))^2|X\big)\Big) \\
&= \mathbb{E}\Big(\mathbb{V}ar(Y|X) + \big(\mathbb{E}(Y) - \mathbb{E}(Y|X)\big)^2\Big) \\
&= \mathbb{E}\big(\mathbb{V}ar(Y|X)\big) + \mathbb{E}\Big(\big(\mathbb{E}(Y) - \mathbb{E}(Y|X)\big)^2\Big) \\
&= \mathbb{E}\big(\mathbb{V}ar(Y|X)\big) + \mathbb{E}\Big(\big(\mathbb{E}(\mathbb{E}(Y|X)) - \mathbb{E}(Y|X)\big)^2\Big) \\
&= \mathbb{E}\big(\mathbb{V}ar(Y|X)\big) + \mathbb{V}ar\big(\mathbb{E}(Y|X)\big)
\end{aligned}$$

(b) We use the law of total expectation and get

$$\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X)) = \mathbb{E}(X) = \frac{1}{2}$$

To calculate the variance we use the formula we derived in (a) as well as $\mathbb{E}(X^2) = \int_0^1 x^2 dx = 1/3$ to get

$$\mathbb{V}ar(Y) = \mathbb{E}\big(\mathbb{V}ar(Y|X)\big) + \mathbb{V}ar\big(\mathbb{E}(Y|X)\big) = \mathbb{E}(X^2) + \mathbb{V}ar(X) = \frac{1}{3} + \frac{1}{12} = \frac{5}{12}$$

To calculate the covariance we again use the law of total expectation as well as the fact that $\mathbb{E}(XY) = \mathbb{E}\big(X\mathbb{E}(Y|X)\big)$ and finally get

$$\mathbb{C}ov(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \mathbb{E}\big(X\mathbb{E}(Y|X)\big) - \mathbb{E}(X)^2 = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \mathbb{V}ar(X) = \frac{1}{12}$$

5. (a) Delta method

Let $X_1, \ldots, X_n$ be i.i.d. from normal distribution with unknown mean μ and known variance σ^2. Let $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$. Find the limiting distribution of $\sqrt{n}\Big(\overline{X}^3 - c\Big)$ for an appropriate constant c.

(b) Logit transformation

Let $X_n \sim bin(n, p)$. Consider the logit transformation, defined by

$$logit(y) = \ln \frac{y}{1-y}, \quad 0 < y < 1.$$

Determine the approximate distribution of $logit\Big(\frac{X_n}{n}\Big)$.

Solution: (a) We aim to use the lemma on page 10 of the slides to lecture five. We know from the CLT that

$$\sqrt{n}(\overline{X} - \mu) \to \mathcal{N}(0, \sigma^2)$$

We use the lemma with $g(x) = x^3$, so we get

$$\sqrt{n}\big(g(\overline{X}) - g(\mu)\big) \to \mathcal{N}\big(0, \sigma^2\big(g'(\mu)\big)^2\big)$$

So the limiting distribution is

$$\mathcal{N}\big(0, 9\sigma^2\mu^4\big)$$

with constant $c := \mu^3$.

(b) We define the new i.i.d. random variables $Y_1, \ldots, Y_n \sim bernoulli(p)$, then it holds that $\frac{X_n}{n} = \overline{Y_n}$. We again aim to use the same lemma as in (a) with $g(x) := logit(x)$ and derivative

$$g'(x) = \frac{1}{x(1-x)}$$

We know from the CLT that

$$\sqrt{n}(\overline{Y_n} - p) \to \mathcal{N}(0, p(1-p))$$

With our lemma we get

$$\sqrt{n}\big(logit(\overline{Y_n}) - logit(p)\big) \to \mathcal{N}\Big(0, \frac{1}{p(1-p)}\Big)$$

With the results from last week we get, for large n, that

$$logit(\overline{Y_n}) \approx \mathcal{N}\Big(logit(p), \frac{1}{np(1-p)}\Big).$$