Wigner's Theorem

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Abstract

This paper presents a detailed proof of Wigner's Theorem. The proof here was given by Daniel D. Spiegel in 2018. As this is a seminar-paper it comprises a lot of detailed calculations that are needed for the proof.

1 Introduction

Wigner's theorem has its motivation in physics. It plays a role in the mathematical formulation of quantum mechanics. In this paper we proof a rather general form of Wigner's theorem. As already mentioned in the abstract most of the ideas in this paper were taken from [4]. The present paper is a seminar paper. Therefore, it was written with the intention of practicing the writing process and not with the intention to present new results. Nevertheless the paper might be interesting, especially for less experienced mathematicians, because everything is presented in great detail. Furthermore the paper comprises some additional ideas from [2] or [1].

2 Complex numbers $\mathbb{C}$

As we will have to work a lot with the complex numbers $\mathbb{C}$ we want to start with some of their properties.

Definition 2.1. Let K be a field and $\zeta : K \to K$ a bijective function. We call ζ an *automorphism* on K if for all $\lambda, \mu \in K$ the equalities

$$\zeta(\lambda + \mu) = \zeta(\lambda) + \zeta(\mu) \quad \text{and} \quad \zeta(\lambda\mu) = \zeta(\lambda)\zeta(\mu)$$

hold true.

Definition 2.2. Throughout this paper $\bar{\cdot} : \mathbb{C} \to \mathbb{C} : \lambda_1 + i\lambda_2 \mapsto \lambda_1 - i\lambda_2$ will be the *complex conjugation.*

Lemma 2.3. There exist only two continuous automorphisms on $\mathbb{C}$, namely the identity function and the complex conjugation. These two functions both conincide with their own inverse and act as the identity function on the real line.

Proof. The identity function and the complex conjugation are both isometries. Thus, they are continuous and they are clearly automorphisms on $\mathbb{C}$. They both coincide with their own inverse and act as the identity function on the real line.
For any continuous automorphism ζ on $\mathbb{C}$ we have $\zeta(0) = 0$ and $\zeta(1) = 1$. Assume $\zeta(\alpha) = \alpha$ for some $\alpha \in \mathbb{N}$. We conclude $\zeta(\alpha + 1) = \zeta(\alpha) + \zeta(1) = \alpha + 1$. Hence, we showed by induction that for all $\alpha \in \mathbb{N}$ the equality

$\zeta(\alpha) = \alpha$ holds true. For an arbitrary $\lambda \in \mathbb{C} \setminus \{0\}$ we have

$$0 = \zeta(0) = \zeta(\lambda - \lambda) = \zeta(\lambda) + \zeta(-\lambda) \quad \text{and} \quad 1 = \zeta(1) = \zeta\left(\frac{\lambda}{\lambda}\right) = \zeta(\lambda)\zeta(\lambda^{-1}).$$

Thus, $\zeta(-\lambda) = -\zeta(\lambda)$ and $\zeta\left(\lambda^{-1}\right) = \zeta(\lambda)^{-1}$. We conclude $\zeta(\beta) = \beta$ for all $\beta \in \mathbb{Z}$ and, in turn, $\zeta(\gamma) = \gamma$ for all $\gamma \in \mathbb{Q}$. For any $\delta \in \mathbb{R}$ there exists a sequence of rational numbers $(\gamma_n)_{n\in\mathbb{N}}$ that converges to δ. Due to continuity of ζ we have

$$\zeta(\delta) = \zeta\left(\lim_{n\to\infty} \gamma_n\right) = \lim_{n\to\infty} \zeta(\gamma_n) = \lim_{n\to\infty} \gamma_n = \delta.$$

From

$$-1 = \zeta(-1) = \zeta\left(i^2\right) = \zeta(i)^2$$

we conclude that $\zeta(i) \in \{i, -i\}$. For a complex number $\mu = \mu_1 + i\mu_2$, where $\mu_1, \mu_2 \in \mathbb{R}$, we obtain

$$\zeta(\mu) = \zeta(\mu_1 + i\mu_2) = \zeta(\mu_1) + \zeta(i)\zeta(\mu_2) = \mu_1 + \zeta(i)\mu_2.$$

Thus, ζ is either the identity function or the complex conjugation. □

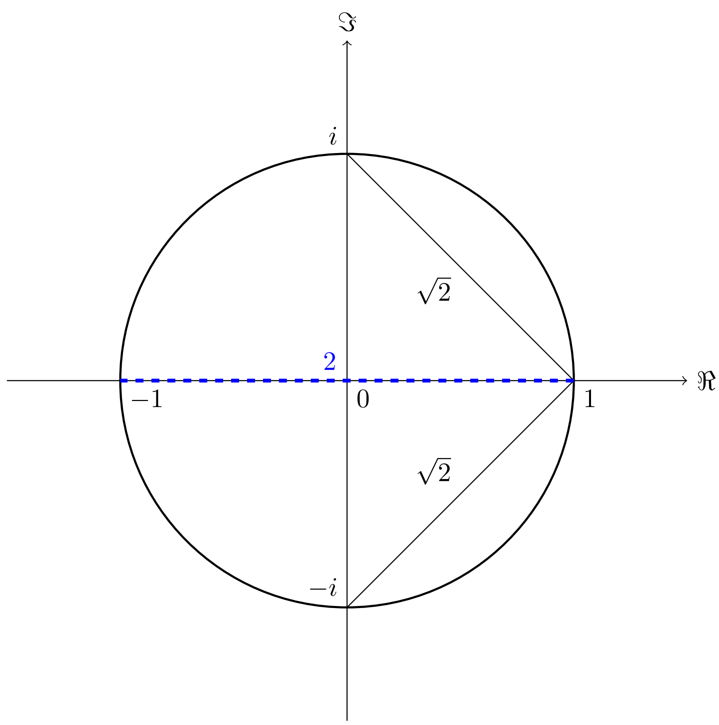

Figure 1: Geometric interpretaion of Lemma 2.4

Lemma 2.4. For $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ the following statements are true.

1. If $-\lambda$ has euclidean distance 2 from the complex number 1 then $\lambda = 1$. Thus,

$$|1 + \lambda| = 2 \Rightarrow \lambda = 1.$$

2. If $-\lambda$ has euclidean distance $\sqrt{2}$ from the complex number 1 then either $\lambda = i$ or $\lambda = -i$ holds true, i.e.

$$|1+\lambda| = \sqrt{2} \Rightarrow \lambda = i \vee \lambda = -i.$$

Proof. For any $\lambda \in \mathbb{C}$, $|\lambda| = 1$, we have

$$|1+\lambda|^2\lambda = (1+\lambda)\big(1+\overline{\lambda}\big)\lambda = (1+\lambda)^2 = 1 + 2\lambda + \lambda^2.$$

1. If $|1+\lambda|^2 = 4$, then we obtain

$$0 = 1 - 2\lambda + \lambda^2 = (1-\lambda)^2,$$

   and in turn $\lambda = 1$.

2. If $|1+\lambda|^2 = 2$, then we obtain

$$0 = 1 + \lambda^2.$$

   Hence, $\lambda \in \{i, -i\}$.

□

Lemma 2.5. Let $\mu \in \mathbb{C} \setminus \{0\}$. If a function $\zeta : \mathbb{C} \to \mathbb{C}$ satisfies $|\zeta(\mu)| = |\mu|$, then there exists a unique $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $|\mu| = \lambda\zeta(\mu)$.

Proof. Let $\mu \in \mathbb{C} \setminus \{0\}$ be an arbitrary number. Defining

$$\lambda := \frac{|\mu|}{\zeta(\mu)} \quad \text{we have} \quad |\lambda| = \frac{|\mu|}{|\zeta(\mu)|} = \frac{|\mu|}{|\mu|} = 1$$

and

$$|\mu| = \frac{|\mu|}{\zeta(\mu)}\zeta(\mu) = \lambda\zeta(\mu).$$

For another $\nu \in \mathbb{C}$ with $|\nu| = 1$ and $|\mu| = \nu\zeta(\mu)$ we obtain

$$\nu = \frac{|\mu|}{\zeta(\mu)} = \lambda.$$

□

Lemma 2.6. Let $\lambda, \mu, \nu \in \mathbb{C}$ where $\lambda \neq 0$ and $|\mu| = |\nu|$. Then the following implication holds true.

$$|\lambda + \nu| = |\lambda + \mu| \wedge |\lambda - i\nu| = |\lambda - i\mu| \Rightarrow \nu = \mu.$$

Proof. We have

$$|\lambda|^2 + \lambda\overline{\mu} + \overline{\lambda}\mu + |\mu|^2 = (\lambda+\mu)\big(\overline{\lambda} + \overline{\mu}\big) = |\lambda+\mu|^2 = |\lambda+\nu|^2 = (\lambda+\nu)\big(\overline{\lambda}+\overline{\nu}\big) = |\lambda|^2 + \lambda\overline{\nu} + \overline{\lambda}\nu + |\nu|^2$$

and because of $|\mu| = |\nu|$ we conclude $\lambda(\overline{\nu} - \overline{\mu}) = -\overline{\lambda}(\nu - \mu)$. Furthermore,

$$|\lambda|^2 + i\lambda\overline{\mu} - i\overline{\lambda}\mu + |\mu|^2 = (\lambda - i\mu)\big(\overline{\lambda} + i\overline{\mu}\big) = |\lambda - i\mu|^2 = |\lambda - i\nu|^2 = (\lambda - i\nu)\big(\overline{\lambda} + i\overline{\nu}\big) = |\lambda|^2 + i\lambda\overline{\nu} - i\overline{\lambda}\nu + |\nu|^2.$$

Again because of the assumption $|\mu| = |\nu|$ we obtain

$$\overline{\lambda}(\nu - \mu) = \lambda(\overline{\nu} - \overline{\mu}) = -\overline{\lambda}(\nu - \mu).$$

$\overline{\lambda} \neq 0$ finally implies $\nu = \mu$. □

3 Hilbert spaces

Definition 3.1. Let H be a vector space over $\mathbb{C}$. A function $(\cdot, \cdot) : H \times H \to \mathbb{C}$ is called *inner product* if

1. $(x, x) > 0$ for all $x \in H \setminus \{0\}$.
2. $(x, y) = \overline{(y, x)}$ for all $x, y \in H$.
3. $(x + y, z) = (x, z) + (y, z)$ for all $x, y, z \in H$, and $(\lambda x, y) = \lambda(x, y)$ for all $\lambda \in \mathbb{C}$, $x, y \in H$.

Remark 3.2. We know from [5, p.41] that an inner product induces a norm $\|x\| = \sqrt{(x, x)}$. Throughout this paper a vector space H provided with an inner product will always be normed with this norm.

Remark 3.3. Let V be a vector space and $(\cdot, \cdot)$ an inner product on V. Then for all $x, y \in V$ the inequality $|(x, y)| \leq \|x\| \|y\|$ holds true. Equality holds if and only if x and y are linearly dependent. This inequality is called *Cauchy-Schwarz inequality*. The proof can be found in [5, p. 41].

Remark 3.4. For a vector space with inner product $(\cdot, \cdot) : V \times V \to \mathbb{C}$ the inner product is continuous when V is endowed with the topology induced by the norm and $V \times V$ is endowed with the product topology. Furthermore for every $y \in V$ the linear functional $f_y : V \to \mathbb{C}$ defined by $f_y(x) = (x, y)$, is continuous. The proof of these facts can be found in [5, p.43]

Definition 3.5. A vector space H over $\mathbb{C}$ with a scalar product that is complete as a normed space endowed with the norm induced by the scalar product is called *Hilbert space*.

In this paper a Hilbert space shall always be a vector space over the field $\mathbb{C}$ and not over $\mathbb{R}$.

Definition 3.6. Let V be a vector space with an inner product $(\cdot, \cdot)$. We call two subsets $M, N \subseteq V$ *orthogonal*, denoted by $M \perp N$, if for all $x \in M$ and all $y \in N$ we have $(x, y) = 0$. Two vectors $v, w \in V$ are called *orthogonal* if $(v, w) = 0$.

Definition 3.7. Let H be a Hilbert space. A subset $M \subseteq H$ is called an *orthonormal system* if for all $u, v \in M$

$$(u, v) = \begin{cases} 1 & , \text{if } u = v, \\ 0 & , \text{if } u \neq v. \end{cases}$$

If M is an orthonormal system and every orthonormal system $\tilde{M}$ with $\tilde{M} \supseteq M$ satisfies $\tilde{M} = M$, then M is called an *orthonormal basis* of H.

Remark 3.8. Whenever we write an orthonormal system M as an indexed set $M = \{e_j \mid j \in J\}$ in this paper, we require that $e_j \neq e_k$ for $j, k \in J$ with $j \neq k$.

Lemma 3.9. Let H be a Hilbert space and M an orthonormal system. Then there exists an orthonormal basis $\tilde{M} \supseteq M$. In particular, there exists an orthonormal basis of H.

The proof can be found in [5, p.52].

Theorem 3.10. Let H be a Hilbert space and $M = \{e_j \mid j \in J\}$ an orthonromal system. Then the following statements are equivalent.

1. M is an orthonormal basis.
2. For every $x \in H$

$$\sum_{j \in J} |(x, e_j)|^2 = \|x\|^2. \tag{1}$$

3. For all $x, y \in H$ the equality

$$\sum_{j\in J}(x, e_j)\overline{(y, e_j)} = (x, y)$$

holds true.

4. For every $x \in H$

$$x = \sum_{j\in J}(x, e_j)e_j. \tag{2}$$

The proof can be found in [5, p. 54].

Definition 3.11. For a Hilbert space H, an orthonormal basis $M = \{e_j \mid j \in J\}$ of H and $x \in H$, equality (1) is called *Parseval's equality.* The series in (2) is called *Fourier series* of x with respect to the orthonormal basis M.

Lemma 3.12. Let H be a Hilbert space and $M := \{e_j \mid j \in J\}$ be a non-empty orthonormal system. Then for every $x \in H$ we have

$$\|x\|^2 = \sum_{j\in J}|(x, e_j)|^2 \Leftrightarrow x = \sum_{j\in J}(x, e_j)e_j. \tag{3}$$

Proof. We consider an orthonormal basis $\{f_k \mid k \in K\} \supseteq M$; see Lemma 3.9.

„$\Rightarrow$“ Using Parzeval's equality (1) we obtain

$$\sum_{j\in J}|(x, e_j)|^2 = \|x\|^2 = \sum_{k\in K}|(x, f_k)|^2.$$

Hence, for all $k \in K$ with $f_k \notin M$ the equality $(x, f_k) = 0$ must hold true. Finally, using the representation as a Fourier series (2) we obtain

$$x = \sum_{k\in K}(x, f_k)f_k = \sum_{j\in J}(x, e_j)e_j.$$

„$\Leftarrow$“ We observe that for all $k \in K$ with $f_k \notin M$ we have

$$(x, f_k) = \left(\sum_{j\in J}(x, e_j)e_j, f_k\right) = \sum_{j\in J}(x, e_j)(e_j, f_k) = 0.$$

Hence, Parseval's equality yields

$$\sum_{j\in J}|(x, e_j)|^2 = \sum_{k\in K}|(x, f_k)|^2 = \|x\|^2.$$

$\square$

Definition 3.13. Let V and W be two vector spaces over the same field K and ζ be an automorphism on K. A function $f : V \to W$ is called *semilinear* with respect to ζ or *ζ-linear*, if for all $x, y \in V$ and all $\lambda \in K$

$$f(x + y) = f(x) + f(y) \quad \text{and} \quad f(\lambda x) = \zeta(\lambda)f(x).$$

If $K = \mathbb{C}$ and ζ is the complex conjugation, then f is called an *antilinear function.*

Remark 3.14. If f is a ζ-linear function and $\zeta = id_K$, then f is simply a *linear function.* The properties of ζ-linear functions are very similar to the ones we know from linear function. See [3, p. 138] for these results. We will use the property that a ζ-linear function f is injective if $\ker f = \{0\}$. Furthermore, a scalar product in this paper is linear in the first and antilinear in the second argument, as can be found in [5, p. 41].

It is not necessary to precisely define a topological vector space here. We only need to know that every normed space is a topological vector space. This result can be found in [5, p. 18]

Definition 3.15. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological vector spaces. We denote the set of all ζ-linear and continuous functions from X to Y with ζ-$L_b(X,Y)$. In the case $(X, \mathcal{T}_X) = (Y, \mathcal{T}_Y)$ we write ζ-$L_b(X) = \zeta$-$L_b(X,Y)$. If ζ is the identity function, then we write $L_b(X,Y)$ and $L_b(X)$.

Definition 3.16. If $(X, \mathcal{T})$ is a topological vector space over $\mathbb{C}$, then we denote by $(X, \mathcal{T})'$ the set of all linear and continuous functions from X into the field $\mathbb{C}$. We call this set the *continuous dual space* of $(X, \mathcal{T})$.

Remark 3.17. Let X be a normed space. Then X' provided with the operator norm

$$\|f\| = \sup\{|f(x)| : x \in X \wedge \|x\|_X \leq 1\}, \quad f \in X',$$

is a Banach space. See [5, p. 25] for this result.

Proposition 3.18. Let H be a Hilbert space. Then the function

$$\Phi : \begin{cases} H \to H' \\ y \mapsto f_y \end{cases}$$

where $f_y : H \to \mathbb{C}$ defined by $f_y(x) = (x, y)$ is an isometric and antilinear bijection from H onto H' endowed with the norm introduced in Remark 3.17.

The proof can be found in [5, p. 50]

Definition 3.19. Let A be an algebra with an identity element e. This is a vector space additionally provided with a bilinear and associative multiplication $\cdot : A \times A \to A$, where $e \in A$ satisfies $ea = ae = a$ for all $a \in A$. See [5, p.121-122] for this definition and some properties of an algebra. An element $a \in A$ is called *inveritble*, if there exists $b \in A$ with $ab = ba = e$. We define

$$\mathrm{Inv}(A) := \{a \in A \mid a \text{ is invertible}\}$$

and based on this the *spectrum* of an element $a \in A$ as

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid (a - \lambda e) \notin \mathrm{Inv}(A)\}.$$

Furthermore, we define the *spectral radius* of an element $a \in A$ by

$$r(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\},$$

where $\sup \emptyset := 0$. If A is a Banach space with a norm $\|\cdot\|$ that satisfies $\|ab\| \leq \|a\| \|b\|$ for all $a, b \in A$, then we call it a Banach algebra with identity element.

Remark 3.20. For a Banach space X the space $L_b(X)$ is a Banach algebra with the identity mapping as the identity element, see [5, p.121-122] for this result.

Definition 3.21. Let X be a Banach space and $T \in L_b(X)$. Then $\lambda \in \mathbb{C}$ is called *eigenvalue* of T if $\ker(T - \lambda I) \neq \{0\}$.

Definition 3.22. Let X, Y be Banach spaces. A linear function $T : X \to Y$ is called compact, if $T(\{x \in X : |x| \leq 1\})$ is relatively compact in Y.

Remark 3.23. If X, Y are Banach spaces and $T \in L_b(X, Y)$ with $\dim \operatorname{ran} T < \infty$, then T is compact. This result can be found in [5, p. 133].

Remark 3.24. Let X be a Banach space and $T : X \to X$ compact. Then every $\lambda \in \sigma(T) \setminus \{0\}$ is an eigenvalue of T. This result can be found in [5, p.138].

Lemma 3.25. Let H_1 and H_2 be Hilbert spaces and $\zeta : \mathbb{C} \to \mathbb{C}$ either the identity mapping or the complex conjugation. If $T \in \zeta\text{-}L_b(H_1, H_2)$, then there exists a unique function $T^*_\zeta : H_2 \to H_1$ such that for all $x \in H_1$ and $y \in H_2$ the equation

$$(Tx, y)_{H_2} = \zeta\big((x, T^*_\zeta y)_{H_1}\big)$$

holds true. The function T^*_ζ is ζ-linear.

Proof. For an arbitrary $y \in H_2$ we define $f_y : H_1 \to \mathbb{C}$ by $f_y(x) := \zeta^{-1}((Tx, y)_{H_2})$. For $u, v \in H_1$ and $\lambda, \mu \in \mathbb{C}$ we obtain

$$\begin{aligned} f_y(\mu u + \lambda v) &= \zeta^{-1}((T(\mu u + \lambda v), y)_{H_2}) = \zeta^{-1}(\zeta(\mu)(Tu, y)_{H_2} + \zeta(\lambda)(Tv, y)_{H_2}) \\ &= \mu\zeta^{-1}((Tu, y)_{H_2}) + \lambda\zeta^{-1}((Tv, y)_{H_2}) = \mu f_y(u) + \lambda f_y(v). \end{aligned}$$

Hence, f_y is a linear function. Furthermore, by Remark 3.4 the function $(\cdot,\cdot)_{H_2} : H_2 \times H_2 \to \mathbb{C}$ is continuous and by Lemma 2.3 we have continuity of ζ^{-1}. As f_y is a composition function of these two functions it is continuous as well. Using Proposition 3.18 there exists a unique $z_y \in H_1$ which fulfills $f_y(x) = (x, z_y)_{H_1}$ for all $x \in H_1$. This allows us to uniquely define a function

$$T^*_\zeta : H_2 \to H_1, \quad y \mapsto z_y$$

that satisfies

$$(Tx, y)_{H_2} = \zeta\big(\zeta^{-1}((Tx, y)_{H_2})\big) = \zeta(f_y(x)) = \zeta\big((x, T^*_\zeta y)_{H_1}\big).$$

for all $x \in H_1$ and all $y \in H_2$.
Consider arbitrary $y, z \in H_2$ and $\lambda, \mu \in \mathbb{C}$. For every $x \in H_1$ we have

$$\begin{aligned} \big(x, T^*_\zeta(\mu y + \lambda z)\big)_{H_1} &= \zeta^{-1}\big((Tx, \mu y + \lambda z)_{H_2}\big) = \zeta^{-1}(\overline{\mu})\zeta^{-1}\big((Tx, y)_{H_2}\big) + \zeta^{-1}\big(\overline{\lambda}\big)\zeta^{-1}\big((Tx, z)_{H_2}\big) \\ &= \zeta^{-1}(\overline{\mu})\big(x, T^*_\zeta y\big)_{H_1} + \zeta^{-1}\big(\overline{\lambda}\big)\big(x, T^*_\zeta z\big)_{H_1} = \Big(x, \overline{\zeta^{-1}(\overline{\mu})}T^*_\zeta y + \overline{\zeta^{-1}\big(\overline{\lambda}\big)}T^*_\zeta z\Big)_{H_1}. \end{aligned}$$

We conclude $T^*_\zeta(\mu y + \lambda z) = \overline{\zeta^{-1}(\overline{\mu})}T^*_\zeta y + \overline{\zeta^{-1}\big(\overline{\lambda}\big)}T^*_\zeta z$. By assumption $\zeta = id_{\mathbb{C}}$ or $\zeta = \overline{\cdot}$. In both cases we see that T^*_ζ is a ζ-linear function. □

Definition 3.26. Let H be a Hilbert space and $T \in L_b(H)$. Then T is called *normal* if $TT^* = T^*T$.

Remark 3.27. If H is a Hilbert space and $N : H \to H$ is normal, then $r(N) = \|N\|$. The proof of this statement can be found in [5, p.142].

Definition 3.28. Let H_1 and H_2 be Hilbert spaces, $\zeta : \mathbb{C} \to \mathbb{C}$ either the identity mapping or the complex conjugation and $U \in \zeta\text{-}L_b(H_1, H_2)$ satisfying $U^*_\zeta U = I_{H_1}$ and $UU^*_\zeta = I_{H_2}$. If ζ is the identity mapping, then U is called *unitary*. If ζ is the complex conjugation, then U is called *antiunitary*.

Remark 3.29. If H is a Hilbert space and if $T \in L_b(H)$ satisfies $(Tx, x)_H = 0$ for all $x \in H$, then $T = 0$. The proof of this can be found in [5, p.142].

Proposition 3.30. Let H_1 and H_2 be Hilbert spaces $U \in \zeta\text{-}L_b(H_1, H_2)$, where $\zeta : \mathbb{C} \to \mathbb{C}$ is either the identity mapping or the complex conjugation. Then the following statements are equivalent.

(i) U is ζ-unitary.

(ii) $\operatorname{ran} U = H_2$ and $(Ux, Uy)_{H_2} = \zeta((x,y)_{H_1})$ for all $x, y \in H_1$.

(iii) $\operatorname{ran} U = H_2$ and $\|Ux\|_{H_2} = \|x\|_{H_1}$ for all $x \in H_1$.

Proof.
„(i) $\Rightarrow$ (ii)“. Due to the fact that $UU_\zeta^* = I_{H_2}$ we have $\operatorname{ran} U = H_2$. Because of the assumption $U_\zeta^* U = I_{H_1}$ we obtain for $x, y \in H_1$

$$(Ux, Uy)_{H_2} = \zeta\big((x, U_\zeta^* Uy)_{H_1}\big) = \zeta((x,y)_{H_1}).$$

„(ii) $\Rightarrow$ (iii)“. By assumption the function ζ is either the identity function or the complex conjugation. Hence, $\zeta(\alpha) = \alpha$ for all $\alpha \in \mathbb{R}$. Given $x \in H_1$ we have

$$\|Ux\|_{H_2}^2 = (Ux, Ux)_{H_2} = \zeta\big((x,x)_{H_1}\big) = \zeta\left(\|x\|_{H_1}^2\right) = \|x\|_{H_1}^2.$$

„(iii) $\Rightarrow$ (i)“. For every $x \in H_1$ we have

$$(x, U_\zeta^* Ux)_{H_1} = \zeta^{-1}((Ux, Ux)_{H_2}) = \zeta^{-1}\left(\|Ux\|_{H_2}^2\right) = \|Ux\|_{H_2}^2 = \|x\|_{H_1}^2 = (x,x)_{H_1}.$$

Using Remark 3.29 we obtain $U_\zeta^* U = I_{H_1}$. The function U is surjective by assumption and injective as a consequence of $\|Ux\|_{H_2} = \|x\|_{H_1}$. Hence, U is bijective and $U_\zeta^* U = I_{H_1}$ implies

$$UU_\zeta^* = UU_\zeta^* UU^{-1} = UI_{H_1}U^{-1} = UU^{-1} = I_{H_2}.$$

□

Definition 3.31. Let V be a vector space with an inner product $(\cdot,\cdot)$. We call a linear function $P : V \to V$ an *orthogonal projection*, if $P = P^2$ and $\operatorname{ran} P \perp \ker P$.

Remark 3.32. In a vector space V with an inner product $(\cdot,\cdot)$ a linear function $P : V \to V$ with $P^2 = P$ is an orthogonal projection if and only if for all $x, y \in V$

$$(Px, y) = (x, Py).$$

Moreover, orthogonal projections are bounded with norm one in case $P \neq 0$. This result can be found in [5, p. 47].

Remark 3.33. Let H be a Hilbert space. If $M \subseteq H$ is a closed subspace, then there exists a unique orthogonal projection P with $\operatorname{ran} P = M$. The proof of this statement can be found in [5, p. 48].

4 Projective Hilbert spaces

Definition 4.1. Let V be a vector space over the field K. The set $\mathcal{P}(V) = \{Kx \mid x \in V \setminus \{0\}\}$ consisting of all onedimensional subspaces of V is called the *projective space* of V. If V is a Hilbert space then we call $\mathcal{P}(V)$ *projective Hilbert space* and its elements *rays*.

Lemma 4.2. If R_1 and R_2 are rays of a projective Hilbert space $\mathcal{P}(H)$, then there exists a unique $\rho \in [0,1]$ such that for all $x_1 \in R_1 \setminus \{0\}$ and $x_2 \in R_2 \setminus \{0\}$

$$\frac{|(x_1, x_2)|}{\|x_1\|\|x_2\|} = \rho.$$

Proof. Let $x_1, y_1 \in R_1 \setminus \{0\}$ and $x_2, y_2 \in R_2 \setminus \{0\}$. Then we can write $y_1 = \lambda_1 x_1$ and $y_2 = \lambda_2 x_2$ for some $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$. Hence,

$$\rho := \frac{|(y_1, y_2)|}{\|y_1\|\|y_2\|} = \frac{|(\lambda_1 x_1, \lambda_2 x_2)_H|}{\|\lambda_1 x_1\|\|\lambda_2 x_2\|} = \frac{|\lambda_1 \lambda_2||(x_1, x_2)|}{|\lambda_1 \lambda_2|\|x_1\|\|x_2\|} = \frac{|(x_1, x_2)|}{\|x_1\|\|x_2\|}.$$

Because of the Cauchy-Schwarz inequality we have $\rho \in [0, 1]$. □

Definition 4.3. The previous Lemma 4.2 allows us to define the *ray-product* $(\cdot, \cdot)_{\mathcal{P}(H)} : \mathcal{P}(H) \times \mathcal{P}(H) \to [0, 1]$ on a projective Hilbert space $\mathcal{P}(H)$ by

$$(\mathbb{C}x, \mathbb{C}y)_{\mathcal{P}(H)} := \frac{|(x, y)|}{\|x\|\|y\|}.$$

Lemma 4.4. Let $\mathcal{P}(H)$ be a projective Hilbert space and let $f : \mathcal{P}(H) \to L_b(H)$ be defined by $f(R)(x) := (x, v_R)v_R$ for $x \in H$ and $R \in \mathcal{P}(H)$, where $v_R \in R$ is a vector of norm one. Then for all $R \in \mathcal{P}(H)$ the operator $f(R)$ is the orthogonal projection with $\operatorname{ran} f(R) = R$. Moreover, $d : \mathcal{P}(H) \times \mathcal{P}(H) \to [0, \infty)$ defined by $(R, S) \mapsto \|f(R) - f(S)\|$ is a metric.

Proof. Given $u, v \in R$ with $\|u\| = \|v\|$ we have $v = \lambda u$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Hence,

$$(x, v)v = (x, \lambda u)\lambda u = \lambda\overline{\lambda}(x, u)u = (x, u)u$$

for $x \in H$. Thus, $f(R) : H \to H$ is an obviously linear operator which does not depend on the choice of v_R. Moreover,

$$P^2 x = P((x, v_R)v_R) = ((x, v_R)v_R, v_R)v_R = (x, v_R)v_R = Px.$$

Since also for $x, y \in H$ we have

$$(Px, y) = ((x, v_R)v_R, y) = (x, v_R)(v_R, y) = (x, (y, v_R)v_R) = (x, Py),$$

the operator $P = f(R)$ is an orthogonal projection. The map d is a metric, because the operator norm induces a metric in the well known way. □

Remark 4.5. Throughout this paper a projective Hilbert space will be endowed with the metric from Lemma 4.4.

Lemma 4.6. In a projective Hilbert space $\mathcal{P}(H)$ the equality

$$d(R, S) = \sqrt{1 - (R, S)^2_{\mathcal{P}(H)}}$$

holds true for all rays $R, S \in \mathcal{P}(H)$.

Proof. Let $R, S \in \mathcal{P}(H)$ be arbitrary rays and let $P := f(R)$ and $Q := f(S)$ be the orthogonal projections onto R and S respectively as in Lemma 4.4 and set $u := v_R$ and $v := v_S$. If $R = S$ then the equation holds true because of Remark 3.3. Thus, from now on we assume $R \neq S$. We are going to have a look at the spectrum of $T : H \to H$ defined by $Tx = Px - Qx = (x,u)u - (x,v)v$. From

$$\begin{aligned}
T\left(u - \frac{(u,v)}{1+\sqrt{1-|(u,v)|^2}}v\right) &= Tu - \frac{(u,v)}{1+\sqrt{1-|(u,v)|^2}}Tv = u - (u,v)v - \frac{(u,v)}{1+\sqrt{1-|(u,v)|^2}}((v,u)u - v)\\
&= \left(1 - \frac{|(u,v)|^2}{1+\sqrt{1-|(u,v)|^2}}\right)u - \left((u,v) - \frac{(u,v)}{1+\sqrt{1-|(u,v)|^2}}\right)v\\
&= \frac{1-|(u,v)|^2+\sqrt{1-|(u,v)|^2}}{1+\sqrt{1-|(u,v)|^2}}u - \frac{(u,v)\sqrt{1-|(u,v)|^2}}{1+\sqrt{1-|(u,v)|^2}}v\\
&= \sqrt{1-|(u,v)|^2}\left(u - \frac{(u,v)}{1+\sqrt{1-|(u,v)|^2}}v\right)
\end{aligned}$$

we conclude that $\sqrt{1-|(u,v)|^2} = \sqrt{1-(R,S)^2_{\mathcal{P}(H)}}$ is an eigenvalue of T. As

$$T^* = P^* - Q^* = P - Q = T$$

the operator T is selfadjoint and therefore normal. We also observe that $\operatorname{ran} T \subseteq \operatorname{span}\{u,v\}$. According to Remark 3.23 this implies that T is compact. Let us now assume that $\lambda \in \mathbb{C} \setminus \{0\}$ belongs to the spectrum of T. Due to the fact that T is compact and by Remark 3.24 the complex number λ is eigenvalue of T, which gives $Tx = \lambda x$ for some $x \in H \setminus \{0\}$. Hence, $x \in \operatorname{ran} T$ which implies the existence of $\mu, \nu \in \mathbb{C}$ with $x = \mu u + \nu v$. From $x \neq 0$ we conclude $\mu \neq 0$ or $\nu \neq 0$. With no loss of generality we assume $\mu \neq 0$.
As $R \neq S$, the vectors u and v are linearly independent. We conclude

$$\lambda\mu = \mu + \nu(v,u) \tag{4}$$
$$\lambda\nu = -\nu - \mu(u,v) \tag{5}$$

from

$$\lambda\mu u + \lambda\nu v = \lambda x = Tx = \mu Tu + \nu Tv = \mu(u-(u,v)v) + \nu((v,u)u - v) = (\mu + \nu(v,u))u - (\nu + \mu(u,v))v.$$

If $(v,u) = 0$, then (4) yields $\lambda\mu = \mu$ and hence $\lambda = 1$. In this case we have $(R,S)_{\mathcal{P}(H)} = |(u,v)_H| = 0$ which, according to Remark 3.27, yields

$$d(R,S) = \|P - Q\| = r(P-Q) = 1 = \sqrt{1-(R,S)_{\mathcal{P}(H)}}.$$

Assuming $(v,u) \neq 0$ we can do further calculations. From (4) we conclude that

$$\lambda = \frac{\mu + \nu(v,u)}{\mu} = 1 + \frac{\nu}{\mu}(v,u)$$

and hence

$$\frac{\nu}{\mu} = \frac{\lambda - 1}{(v,u)}. \tag{6}$$

Using (5) we obtain

$$(\lambda+1)\frac{\nu}{\mu} = -(u,v),$$

what together with (6) implies

$$\frac{\lambda^2-1}{(v,u)} = (\lambda+1)\frac{\lambda-1}{(v,u)} = -(u,v).$$

With a simple transformation we find

$$\lambda = \pm\sqrt{1-|(u,v)|^2} = \pm\sqrt{1-(R,S)^2_{\mathcal{P}(H)}}.$$

Although we do not know for sure, whether 0 belongs to the spectrum of T, we know its spectral radius $|\lambda|$. Finally, by Remark 3.27 we obtain

$$d(R,S) = \|P-Q\| = r(P-Q) = \sqrt{1-(R,S)^2_{\mathcal{P}(H)}}.$$

□

Lemma 4.7. Let $\mathcal{P}(H_1)$ and $\mathcal{P}(H_2)$ be two projective Hilbert spaces and $g : \mathcal{P}(H_1) \to \mathcal{P}(H_2)$ an isometry with respect to the metric from Lemma 4.4. Let $M := \{e_j \mid j \in J\}$ be an orthonormal basis of H_1 and $x, y \in H_1 \setminus \{0\}$. Let $\tilde{x} \in g(\mathbb{C}x)$ and $\tilde{y} \in g(\mathbb{C}y)$ be vectors satisfying $\|x\|_{H_1} = \|\tilde{x}\|_{H_2}$ and $\|y\|_{H_1} = \|\tilde{y}\|_{H_2}$. Lastly, for every $j \in J$ let $\tilde{e}_j \in g(\mathbb{C}e_j)$ be a normalized vector. Then

$$|(\tilde{x},\tilde{y})_{H_1}| = |(x,y)_{H_2}|, \tag{7}$$

the set $L := \{\tilde{e}_j \mid j \in J\}$ is an orthonormal system in H_2, and

$$\tilde{x} = \sum_{j\in J}(\tilde{x},\tilde{e}_j)_{H_2}\tilde{e}_j. \tag{8}$$

Proof. Employing Lemma 4.6 we obtain

$$\begin{aligned}\sqrt{1-\frac{|(x,y)_{H_1}|^2}{\|x\|^2_{H_1}\|y\|^2_{H_1}}} &= \sqrt{1-(\mathbb{C}x,\mathbb{C}y)^2_{\mathcal{P}(H_1)}} = d(\mathbb{C}x,\mathbb{C}y)\\ &= d(g(\mathbb{C}x),g(\mathbb{C}y)) = \sqrt{1-(g(\mathbb{C}x),g(\mathbb{C}y))^2_{\mathcal{P}(H_2)}} = \sqrt{1-\frac{|(\tilde{x},\tilde{y})_{H_2}|^2}{\|\tilde{x}\|^2_{H_2}\|\tilde{y}\|^2_{H_2}}}\end{aligned}$$

which immediately implies (7). Using this equation and the fact that M is an orthonormal basis of H_1 we obtain for every $i, j \in J$

$$|(\tilde{e}_i,\tilde{e}_j)_{H_2}| = |(e_i,e_j)_{H_1}| = \begin{cases} 0 & \text{, if } i \neq j,\\ 1 & \text{, if } i = j.\end{cases}$$

Hence, L is an orthonormal system. Using (7) and Parzeval's equality (1) we obtain

$$\|\tilde{x}\|^2_{H_2} = \|x\|^2_{H_1} = \sum_{j\in J}|(x,e_j)_{H_1}|^2 = \sum_{j\in J}|(\tilde{x},\tilde{e}_j)_{H_2}|^2.$$

Due to the fact that L is an orthonormal system by (3) we obtain (8). □

5 Statement and proof of Wigner's Theorem

If for Hilbert spaces H_1 and H_2 we consider an isometry $g : \mathcal{P}(H_1) \to \mathcal{P}(H_2)$, we mean in this section a function from $\mathcal{P}(H_1)$ to $\mathcal{P}(H_2)$ which is isometric with respect to the metric d from Lemma 4.4.

Example 5.1. Let H_1 and H_2 be Hilbert spaces where $H_1 = \{0\}$ and let $g : \mathcal{P}(H_1) \to \mathcal{P}(H_2)$ be an isometry. Obviously, we have $\mathcal{P}(H_1) = \emptyset$. By defining $U : H_1 \to H_2$ as the zero operator, we observe that U is linear as well as antilinear and both unitary and antiunitary. Furthermore, the statement $Ux \in g(\mathbb{C}x)$ is true for every $x \in H_1 \setminus \{0\}$, simply because no such x exists.

Example 5.2. Let H_1 and H_2 be Hilbert spaces where $\dim H_1 = 1$ and let $g : \mathcal{P}(H_1) \to \mathcal{P}(H_2)$ be an isometry. Due to the fact that $\mathcal{P}(H_1) = \{H_1\}$ there exists only one ray in $\mathcal{P}(H_1)$. Take a normalized $x \in H_1$ and a normalized $y \in g(H_1)$. We define $U : H_1 \to H_2$ by $\lambda x \mapsto \lambda y$ and $T : H_1 \to H_2$ by $\lambda x \mapsto \overline{\lambda} y$. For any $\lambda \in \mathbb{C}$ we have

$$\|U\lambda x\|_{H_2} = \|\lambda y\|_{H_2} = |\lambda|\|y\|_{H_2} = |\lambda| = |\lambda|\|x\|_{H_1} = \|\lambda x\|_{H_1}$$

and

$$\|T\lambda x\|_{H_2} = \left\|\overline{\lambda} y\right\|_{H_2} = |\lambda|\|y\|_{H_2} = |\lambda| = |\lambda|\|x\|_{H_1} = \|\lambda x\|_{H_1}.$$

Hence, U is unitary and T is antiunitary. Furthermore, we clearly have $Tu, Uu \in g(\mathbb{C}u)$ for every $u \in H_1 \setminus \{0\}$.

Lemma 5.3. Let H_1 and H_2 be Hilbert spaces and let $g : \mathcal{P}(H_1) \to \mathcal{P}(H_2)$ be an isometry. For two vectors $x, y \in H_1$ with $(y, x)_{H_1} \neq 0$ and a vector $\tilde{x} \in g(\mathbb{C}x)$ with $\|\tilde{x}\|_{H_2} = \|x\|_{H_1}$ there exists a unique $\tilde{y} \in g(\mathbb{C}y)$ that satisfies $\|\tilde{y}\|_{H_2} = \|y\|_{H_1}$ and $(\tilde{y}, \tilde{x})_{H_2} = |(\tilde{y}, \tilde{x})_{H_2}|$.

Proof. Take an arbitrary vector $\tilde{w} \in g(\mathbb{C}y)$ with $\|\tilde{w}\|_{H_2} = \|\tilde{x}\|_{H_1}$ and define $\mu := (\tilde{w}, \tilde{x})_{H_2}$. By (7) we have $|\mu| = |(\tilde{w}, \tilde{x})_{H_2}| = |(y, x)_{H_1}| \neq 0$. According to Lemma 2.5 there exists a unique $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $|\mu| = \lambda\mu$. For $\tilde{y} := \lambda\tilde{w}$ we obtain

$$(\tilde{y}, \tilde{x})_{H_2} = \lambda(\tilde{w}, \tilde{x})_{H_2} = \lambda\mu = |\mu| = |\lambda||(\tilde{w}, \tilde{x})_{H_2}| = |(\tilde{y}, \tilde{x})_{H_2}|$$

and $\|\tilde{y}\|_{H_2} = |\lambda|\|\tilde{w}\|_{H_2} = \|y\|_{H_1}$.
For another vector $\tilde{z} \in g(\mathbb{C}y)$ with $\|\tilde{z}\|_{H_2} = \|y\|_{H_1}$ and $(\tilde{z}, \tilde{x})_{H_2} = \left|(\tilde{z}, \tilde{x})_{H_2}\right|$ we have $\tilde{z} = \nu\tilde{y}$ for some $\nu \in \mathbb{C} \setminus \{0\}$. From

$$\|y\|_{H_1} = \|\tilde{z}\|_{H_2} = \|\nu\tilde{y}\|_{H_2} = |\nu|\|\tilde{y}\|_{H_2} = |\nu|\|y\|_{H_1}.$$

we conclude $|\nu| = 1$. Furthermore, from

$$|\nu|\left|(y, x)_{H_1}\right| = |\nu|\left|(\tilde{y}, \tilde{x})_{H_2}\right| = \left|(\tilde{z}, \tilde{x})_{H_2}\right| = (\tilde{z}, \tilde{x})_{H_2} = \nu(\tilde{y}, \tilde{x})_{H_2} = \nu\left|(\tilde{y}, \tilde{x})_{H_2}\right| = \nu\left|(y, x)_{H_1}\right|$$

and the assumption $(y, x) \neq 0$ we obtain $\nu = |\nu| = 1$. Thus, $\tilde{z} = \nu\tilde{y} = \tilde{y}$. ☐

Lemma 5.4. Let H_1 and H_2 be Hilbert spaces and $g : \mathcal{P}(H_1) \to \mathcal{P}(H_2)$ an isometry. Let $M = \{e_j \mid j \in J\}$ be an orthonormal basis of H_1 and for every $j \in J$ let $\tilde{e}_j \in g(\mathbb{C}e_j)$ be a normalized vector. Furthermore, let $x \in H_1$ with

$$x = \sum_{j \in J} \lambda_j e_j.$$

If $k \in J$ such that $\lambda_k \in \mathbb{C} \setminus \{0\}$ and $\zeta : \mathbb{C} \to \mathbb{C}$ with $|\lambda_k| = |\zeta(\lambda_k)|$, then there exists a unique $\tilde{x} \in g(\mathbb{C}x)$ such that

$$\tilde{x} = \zeta(\lambda_k)\tilde{e}_k + \sum_{j \in J \setminus \{k\}} (\tilde{x}, \tilde{e}_j)_{H_2} \tilde{e}_j.$$

This vector $\tilde{x} \in g(\mathbb{C}x)$ satisfies $\|\tilde{x}\|_{H_2} = \|x\|_{H_1}$. Moreover, $\left|(\tilde{x}, \tilde{e}_j)_{H_2}\right| = |\lambda_j|$ for $j \in J \setminus \{k\}$.

Proof. We first observe that for every $l \in J$

$$(x, e_l)_{H_1} = \left(\sum_{j \in J} \lambda_j e_j, e_l \right)_{H_1} = \sum_{j \in J} \lambda_j (e_j, e_l)_{H_1} = \lambda_l. \tag{9}$$

As $\lambda_k \neq 0$ we can employ Lemma 5.3 and (7) in order to obtain a unique $\tilde{w} \in g(\mathbb{C}x)$ with $\|\tilde{w}\|_{H_2} = \|x\|_{H_1}$ and

$$(\tilde{w}, \tilde{e}_k)_{H_2} = |(\tilde{w}, \tilde{e}_k)_{H_2}| = |(x, e_k)_{H_1}| = |\lambda_k|.$$

By Lemma 2.5 there exists a unique $\nu \in \mathbb{C}$ with $|\nu| = 1$ and $|\lambda_k| = \nu\zeta(\lambda_k)$. We define

$$\tilde{x} := \frac{1}{\nu}\tilde{w} \quad \text{and find} \quad (\tilde{x}, \tilde{e}_k)_{H_2} = \frac{1}{\nu}(\tilde{w}, \tilde{e}_k)_{H_2} = \frac{1}{\nu}|\lambda_k| = \frac{1}{\nu}\nu\zeta(\lambda_k) = \zeta(\lambda_k).$$

Finally, by (8) we obtain

$$\tilde{x} = \sum_{j \in J} (\tilde{x}, \tilde{e}_j)_{H_2} \tilde{e}_j = \zeta(\lambda_k)\tilde{e}_k + \sum_{j \in J \setminus \{k\}} (\tilde{x}, \tilde{e}_j)_{H_2} \tilde{e}_j \quad \text{and} \quad \|\tilde{x}\|_{H_2} = \left\| \frac{1}{\nu}\tilde{w} \right\|_{H_2} = \left| \frac{1}{\nu} \right| \|\tilde{w}\|_{H_2} = \|x\|_{H_1}.$$

Consider another $\tilde{y} \in g(\mathbb{C}x)$ with

$$\tilde{y} = \zeta(\lambda_k)\tilde{e}_k + \sum_{j \in J \setminus \{k\}} (\tilde{y}, \tilde{e}_j)_{H_2} \tilde{e}_j.$$

We have $\tilde{y} = \mu\tilde{x}$ for some $\mu \in \mathbb{C} \setminus \{0\}$. From

$$\mu\zeta(\lambda_k) = \mu(\tilde{x}, \tilde{e}_k)_{H_2} = (\tilde{y}, \tilde{e}_k)_{H_2} = \zeta(\lambda_k)$$

we conclude $\mu = 1$. Thus, $\tilde{x} = \tilde{y}$.
Finally, by (7) for all $j \in J \setminus \{k\}$ we have

$$|\mu_j| = \left| (\tilde{x}, \tilde{e}_j)_{H_2} \right| = \left| (x, e_j)_{H_1} \right| = |\lambda_j|.$$

□

Lemma 5.5. Let H_1 and H_2 be Hilbert spaces with $\dim H_1 > 1$ and $g : \mathcal{P}(H_1) \to \mathcal{P}(H_2)$ an isometry. Furthermore, let $M = \{e_j \mid j \in J\}$ be an orthonormal basis of H_1. We fix $q \in J$ and define for all $j \in J \setminus \{q\}$

$$v_{qj} := \frac{1}{\sqrt{2}}(e_q + e_j), \qquad w_{qj} := \frac{1}{\sqrt{2}}(e_q + ie_j), \qquad w_{jq} := \frac{1}{\sqrt{2}}(e_j + ie_q).$$

Assume that $\tilde{e}_q \in g(\mathbb{C}e_q)$ is a given normalized vector. Then for every $k \in J \setminus \{q\}$ there exists a normalized $\tilde{e}_k \in g(\mathbb{C}e_k)$, a normalized $\tilde{v}_{qk} \in g(\mathbb{C}v_{qk})$, a normalized $\tilde{w}_{qk} \in g(\mathbb{C}w_{qk})$, a normalized $\tilde{w}_{kq} \in g(\mathbb{C}w_{kq})$, and $\lambda_k \in \{i, -i\}$ such that

$$\tilde{v}_{qk} = \frac{1}{\sqrt{2}}(\tilde{e}_q + \tilde{e}_k), \qquad \tilde{w}_{qk} = \frac{1}{\sqrt{2}}(\tilde{e}_q + \lambda_k\tilde{e}_k), \qquad \tilde{w}_{kq} = \frac{1}{\sqrt{2}}(\tilde{e}_k + \lambda_k\tilde{e}_q).$$

Proof. First observe that for every $j \in J$ and every $k \in J \setminus \{q\}$ we have

$$(v_{qk}, e_j)_{H_1} = \left(\frac{1}{\sqrt{2}}(e_q + e_k), e_j \right)_{H_1} = \begin{cases} \frac{1}{\sqrt{2}} & \text{, if } j \in \{q, k\}, \\ 0 & \text{, else.} \end{cases}$$

Hence, we can employ Lemma 5.3 and (7) in order to obtain a unique, normalized $\tilde{v}_{qk} \in g(\mathbb{C}v_{qk})$ with

$$(\tilde{v}_{qk}, \tilde{e}_q)_{H_2} = |(\tilde{v}_{qk}, \tilde{e}_q)_{H_2}| = |(v_{qk}, e_q)_{H_1}| = \frac{1}{\sqrt{2}}.$$

Again by Lemma 5.3 together with (7) we find a unique, normalized $\tilde{e}_k \in g(\mathbb{C}e_k)$ such that

$$(\tilde{v}_{qk}, \tilde{e}_k)_{H_2} = \left|(\tilde{v}_{qk}, \tilde{e}_k)_{H_2}\right| = \left|(v_{qk}, e_k)_{H_1}\right| = \frac{1}{\sqrt{2}}.$$

Since for all $j \in J \setminus \{q, k\}$

$$\left|(\tilde{v}_{qk}, \tilde{e}_j)_{H_2}\right| = \left|(v_{qk}, e_j)_{H_1}\right| = 0,$$

we derive (8) from

$$\tilde{v}_{qk} = \sum_{j \in J} (\tilde{v}_{qk}, \tilde{e}_j)_{H_2} \tilde{e}_j = \frac{1}{\sqrt{2}}(\tilde{e}_q + \tilde{e}_k).$$

By Lemma 5.4 there exist $\tilde{w}_{qk} \in g(\mathbb{C}w_{qk})$ and $\tilde{w}_{kq} \in g(\mathbb{C}w_{kq})$ with

$$\tilde{w}_{qk} = \frac{1}{\sqrt{2}}(\tilde{e}_q + \lambda_k \tilde{e}_k), \quad \tilde{w}_{kq} = \frac{1}{\sqrt{2}}(\tilde{e}_k + \lambda_q \tilde{e}_q)$$

and $|\lambda_q| = |\lambda_k| = 1$. We have

$$\begin{aligned} |1 + \lambda_k| &= \left|(\tilde{e}_q + \lambda_k \tilde{e}_k, \tilde{e}_q + \tilde{e}_k)_{H_2}\right| = 2\left|(\tilde{w}_{qk}, \tilde{v}_{qk})_{H_2}\right| \\ &= 2\left|(w_{qk}, v_{qk})_{H_1}\right| = \left|(e_q + ie_k, e_q + e_k)_{H_1}\right| = |1 + i| = \sqrt{2} \end{aligned}$$

and similarly $|1 + \lambda_q| = \sqrt{2}$. By Lemma 2.4 we obtain $\lambda_k, \lambda_q \in \{i, -i\}$. From

$$\begin{aligned} \frac{1}{2}\left|\lambda_k + \overline{\lambda_q}\right| &= \left|\left(\frac{1}{\sqrt{2}}(\tilde{e}_q + \lambda_k \tilde{e}_k), \frac{1}{\sqrt{2}}(\tilde{e}_k + \lambda_q \tilde{e}_q)\right)_{H_2}\right| = \left|(\tilde{w}_{qk}, \tilde{w}_{kq})_{H_2}\right| \\ &= \left|(w_{qk}, w_{kq})_{H_1}\right| = \left|\left(\frac{1}{\sqrt{2}}(e_q + ie_k), \frac{1}{\sqrt{2}}(e_k + ie_q)\right)_{H_1}\right| = \frac{1}{2}|i - i| = 0 \end{aligned}$$

together with $\lambda_k, \lambda_q \in \{i, -i\}$ we conclude that $\lambda_k = \lambda_q$. $\square$

Example 5.6. Let H_1 and H_2 be Hilbert spaces with $\dim H_1 = 2$ and $g : \mathcal{P}(H_1) \to \mathcal{P}(H_2)$ an isometry. We consider an orthonormal basis $M = \{e_1, e_2\}$ of H_1 and define

$$v := \frac{1}{\sqrt{2}}(e_1 + e_2), \qquad w_{12} := \frac{1}{\sqrt{2}}(e_1 + ie_2), \qquad w_{21} := \frac{1}{\sqrt{2}}(e_2 + ie_1).$$

By Lemma 5.5 there exist $\tilde{e}_1 \in g(\mathbb{C}e_1)$, $\tilde{e}_2 \in g(\mathbb{C}e_2)$, $\tilde{v} \in g(\mathbb{C}v)$, $\tilde{w}_{12} \in g(\mathbb{C}w_{12})$, $\tilde{w}_{21} \in g(\mathbb{C}w_{21})$ and $\lambda \in \{i, -i\}$ with

$$\tilde{v} = \frac{1}{\sqrt{2}}(\tilde{e}_1 + \tilde{e}_2), \qquad \tilde{w}_{12} = \frac{1}{\sqrt{2}}(\tilde{e}_1 + \lambda \tilde{e}_2), \qquad \tilde{w}_{21} = \frac{1}{\sqrt{2}}(\tilde{e}_2 + \lambda \tilde{e}_1).$$

If $\lambda = i$, then we define $\zeta := \mathrm{id}_{\mathbb{C}}$. If $\lambda = -i$, then we define ζ as the complex conjugation. Either way we have $\lambda = \zeta(i)$.

We are going to define $U : H_1 \to H_2$. First of all set $U0 := 0$. For an arbitrary $z \in H_1 \setminus \{0\}$ there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ with $z = \lambda_1 e_1 + \lambda_2 e_2$ where $\lambda_j \neq 0$ for some $j \in \{1, 2\}$. We choose $r = 1$ if $\lambda_1 \neq 0$ and $r = 2$ otherwise. Let $s \in \{1, 2\} \setminus \{r\}$. By Lemma 5.4 there exists a unique $\tilde{z} \in g(\mathbb{C}z)$ with

$$\tilde{z} = \zeta(\lambda_r)\tilde{e}_r + (\tilde{z}, \tilde{e}_s)_{H_2}\tilde{e}_s$$

and this vector satisfies $\|\tilde{z}\|_{H_2} = \|z\|_{H_1}$. First we define $\nu_s := \zeta^{-1}\big((\tilde{z}, \tilde{e}_s)_{H_2}\big)$ and then

$$Uz := \tilde{z} = \zeta(\lambda_r)\tilde{e}_r + \zeta(\nu_s)\tilde{e}_s.$$

We find

$$\frac{1}{\sqrt{2}}|\lambda_r + \nu_s| = \frac{1}{\sqrt{2}}|\zeta(\lambda_r + \nu_s)| = \big|(\tilde{z}, \tilde{v})_{H_2}\big| = \big|(z, v)_{H_1}\big| = \frac{1}{\sqrt{2}}|\lambda_r + \lambda_s|$$

and for

$$y := \frac{1}{\sqrt{2}}(e_r + ie_s) \quad \text{and} \quad \tilde{y} := \frac{1}{\sqrt{2}}(\tilde{e}_r + \zeta(i)\tilde{e}_s)$$

we have

$$\begin{aligned}\frac{1}{\sqrt{2}}|\lambda_r - i\nu_s| &= \frac{1}{\sqrt{2}}\Big|\zeta(\lambda_r) + \overline{\zeta(i)}\zeta(\nu_s)\Big| = \big|(\tilde{z}, \tilde{y})_{H_2}\big| \\ &= \big|(z, y)_{H_1}\big| = \left|\left(\lambda_r e_r + \lambda_s e_s, \frac{1}{\sqrt{2}}(e_r + ie_s)\right)_{H_1}\right| = \frac{1}{\sqrt{2}}|\lambda_r - i\lambda_s|.\end{aligned}$$

As

$$|\nu_s| = \big|\zeta^{-1}\big((\tilde{z}, \tilde{e}_s)_{H_2}\big)\big| = \big|(\tilde{z}, \tilde{e}_s)_{H_2}\big| = \big|(z, e_s)_{H_1}\big| = |\lambda_s|$$

we can employ Lemma 2.6 in order to obtain $\lambda_s = \nu_s$. Hence, we have

$$Uz = \zeta(\lambda_1)\tilde{e}_1 + \zeta(\lambda_2)\tilde{e}_2.$$

and

$$\|Uz\|^2_{H_2} = \|\zeta(\lambda_1)\tilde{e}_1 + \zeta(\lambda_2)\tilde{e}_2\|^2_{H_2} = |\zeta(\lambda_1)|^2 + |\zeta(\lambda_2)|^2 = |\lambda_1|^2 + |\lambda_2|^2 = \|\lambda_1 e_1 + \lambda_2 e_2\|^2_{H_1} = \|z\|^2_{H_1}.$$

Lemma 5.7. Let H_1 and H_2 be Hilbert spaces with $\dim H_1 > 2$ and $g : \mathcal{P}(H_1) \to \mathcal{P}(H_2)$ an isometry. Let $\{e_j \mid j \in J\}$ be an orthonormal basis of H_1 and for all distinct $k, l \in J$ set

$$v_{kl} := \frac{1}{\sqrt{2}}(e_k + e_l) \quad \text{and} \quad w_{kl} := \frac{1}{\sqrt{2}}(e_k + ie_l).$$

Then for all $j \in J$ there exists $\tilde{e}_j \in g(\mathbb{C}e_j)$, for all distinct $k, l \in J$ there exists a normalized $\tilde{v}_{kl} \in g(\mathbb{C}v_{kl})$, a normalized $\tilde{w}_{kl} \in g(\mathbb{C}w_{kl})$ and $\lambda_{kl} \in \{-i, i\}$ such that

$$\tilde{v}_{kl} = \frac{1}{\sqrt{2}}(\tilde{e}_k + \tilde{e}_l) \quad \text{and} \quad \tilde{w}_{kl} = \frac{1}{\sqrt{2}}(\tilde{e}_k + \lambda_{kl}\tilde{e}_l),$$

where $\lambda_{kl} = \lambda_{lk}$.

Proof. Choose $q \in J$ and a normalized $\tilde{e}_q \in g(\mathbb{C}e_q)$. By Lemma 5.5 there exist $\tilde{e}_j \in g(\mathbb{C}e_j)$, $\tilde{v}_{qj} \in g(\mathbb{C}v_{qj})$, $\tilde{w}_{qj} \in g(\mathbb{C}w_{qj})$, $\tilde{w}_{jq} \in g(\mathbb{C}w_{jq})$ and $\lambda_{jq} = \lambda_{qj} \in \{i, -i\}$ with

$$\tilde{v}_{qj} = \frac{1}{\sqrt{2}}(\tilde{e}_q + \tilde{e}_j), \qquad \tilde{w}_{qj} = \frac{1}{\sqrt{2}}(\tilde{e}_q + \lambda_{qj}\tilde{e}_j), \qquad \tilde{w}_{jq} = \frac{1}{\sqrt{2}}(\tilde{e}_j + \lambda_{jq}\tilde{e}_q).$$

for every $j \in J \setminus \{q\}$. We set $\tilde{v}_{jq} := \tilde{v}_{qj}$. For distinct $k, l \in J \setminus \{q\}$ we define

$$x_{kl} := \frac{1}{\sqrt{3}}(e_q + e_k + e_l).$$

Note that because of $\dim H_1 > 2$ such a choice of k, l is possible. By Lemma 5.4 there exists $\tilde{x}_{kl} \in g(\mathbb{C}x_{kl})$ such that

$$\tilde{x}_{kl} = \frac{1}{\sqrt{3}}(\tilde{e}_q + \mu_k \tilde{e}_k + \mu_l \tilde{e}_l)$$

and $|\mu_k| = |\mu_l| = 1$. Next, observe that for $j \in \{k, l\}$

$$\begin{aligned}\frac{1}{\sqrt{6}}|1 + \mu_j| &= \left| \left(\frac{1}{\sqrt{3}}(\tilde{e}_q + \mu_k \tilde{e}_k + \mu_l \tilde{e}_l), \frac{1}{\sqrt{2}}(\tilde{e}_q + \tilde{e}_j) \right)_{H_2} \right| = |(\tilde{x}_{kl}, \tilde{v}_{qj})_{H_2}| \\ &= |(x_{kl}, v_{qj})_{H_1}| = \left| \left(\frac{1}{\sqrt{3}}(e_q + e_k + e_l), \frac{1}{\sqrt{2}}(e_q + e_j) \right)_{H_1} \right| = \frac{2}{\sqrt{6}}.\end{aligned}$$

Employing Lemma 2.4 we obtain $\mu_k = \mu_l = 1$ and therefore

$$\tilde{x}_{kl} = \frac{1}{\sqrt{3}}(\tilde{e}_q + \tilde{e}_k + \tilde{e}_l).$$

Making use of Lemma 5.4 again we obtain $\tilde{v}_{kl} \in g(\mathbb{C}v_{kl})$ and $\tilde{w}_{kl} \in g(\mathbb{C}w_{kl})$ satisfying

$$\tilde{v}_{kl} = \frac{1}{\sqrt{2}}(\tilde{e}_k + \nu_l \tilde{e}_l) \quad \text{and} \quad \tilde{w}_{kl} = \frac{1}{\sqrt{2}}(\tilde{e}_k + \lambda_{kl} \tilde{e}_l),$$

where $|\nu_l| = |\lambda_{kl}| = 1$. Due to

$$\begin{aligned}\frac{1}{\sqrt{6}}|1 + \nu_l| &= \left| \left(\frac{1}{\sqrt{2}}(\tilde{e}_k + \nu_l \tilde{e}_l), \frac{1}{\sqrt{3}}(\tilde{e}_q + \tilde{e}_k + \tilde{e}_l) \right)_{H_2} \right| = \left|(\tilde{v}_{kl}, \tilde{x}_{kl})_{H_2}\right| \\ &= \left|(v_{kl}, x_{kl})_{H_1}\right| = \left| \left(\frac{1}{\sqrt{2}}(e_k + e_l), \frac{1}{\sqrt{3}}(e_q + e_k + e_l) \right)_{H_1} \right| = \frac{2}{\sqrt{6}}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\sqrt{6}}|1 + \lambda_{kl}| &= \left| \left(\frac{1}{\sqrt{2}}(\tilde{e}_k + \lambda_{kl} \tilde{e}_l), \frac{1}{\sqrt{3}}(\tilde{e}_q + \tilde{e}_k + \tilde{e}_l) \right)_{H_2} \right| = \left|(\tilde{w}_{kl}, \tilde{x}_{kl})_{H_2}\right| \\ &= \left|(w_{kl}, x_{kl})_{H_1}\right| = \left| \left(\frac{1}{\sqrt{2}}(e_k + ie_l), \frac{1}{\sqrt{3}}(e_q + e_k + e_l) \right)_{H_1} \right| = \frac{\sqrt{2}}{\sqrt{6}}\end{aligned}$$

we conclude from Lemma 2.4 that $\nu_l = 1$ and $\lambda_{kl} \in \{i, -i\}$. Finally, for any $k, l \in J$

$$\begin{aligned}\frac{1}{2}|\lambda_{kl} - \lambda_{lk}| = \frac{1}{2}\left|\lambda_{kl} + \overline{\lambda_{lk}}\right| &= \left| \left(\frac{1}{\sqrt{2}}(\tilde{e}_k + \lambda_{kl} \tilde{e}_l), \frac{1}{\sqrt{2}}(\tilde{e}_l + \lambda_{lk} \tilde{e}_k) \right)_{H_2} \right| = \left|(\tilde{w}_{kl}, \tilde{w}_{lk})_{H_2}\right| \\ = \left|(w_{kl}, w_{lk})_{H_1}\right| &= \left| \left(\frac{1}{\sqrt{2}}(e_k + ie_l), \frac{1}{\sqrt{2}}(e_l + ie_k) \right)_{H_1} \right| = \frac{1}{2}\left|i + \bar{i}\right| = 0\end{aligned}$$

yields $\lambda_{kl} = \lambda_{lk}$. □

Theorem 5.8 (Wigner)**.** Let H_1 and H_2 be Hilbert spaces and $g : \mathcal{P}(H_1) \to \mathcal{P}(H_2)$ be an isometry. Then there exists an isometry $U : H_1 \to H_2$ that is either linear or antilinear and satisfies $Ux \in g(\mathbb{C}x)$ for every $x \in H_1$.

Proof. We already showed the theorem for $\dim H_1 = 0$ in Example 5.1, for $\dim H_1 = 1$ in Example 5.2 and for $\dim H_1 = 2$ in Example 5.6. Thus we assume $\dim H_1 > 2$. We know from Lemma 3.9 that there exists an orthonormal basis $M := \{e_j \mid j \in J\}$ of H_1. We assume that J is well-ordered and note that by the well-ordering theorem every set can be well-ordered. For all distinct $k, l \in J$ we define

$$v_{kl} := \frac{1}{\sqrt{2}}(e_k + e_l) \quad \text{and} \quad w_{kl} := \frac{1}{\sqrt{2}}(e_k + ie_l).$$

By Lemma 5.7 there exist normalized $\tilde{e}_j \in g(\mathbb{C}e_j)$, $j \in J$, normalized $\tilde{v}_{kl} \in g(\mathbb{C}v_{kl})$, normalized $\tilde{w}_{kl} \in g(\mathbb{C}w_{kl})$ and $\lambda_{kl} \in \{-i, i\}$, $k, l \in J$, $k \neq l$, such that

$$\tilde{v}_{kl} = \frac{1}{\sqrt{2}}(\tilde{e}_k + \tilde{e}_l) \quad \text{and} \quad \tilde{w}_{kl} = \frac{1}{\sqrt{2}}(\tilde{e}_k + \lambda_{kl}\tilde{e}_l)$$

and $\lambda_{kl} = \lambda_{lk}$.
For distinct $k, l, m \in J$ we define

$$y_{klm} := \frac{1}{\sqrt{3}}(e_k + e_l + ie_m).$$

By Lemma 5.4 there exists $\tilde{y}_{klm} \in g(\mathbb{C}y_{klm})$ with

$$\tilde{y}_{klm} = \frac{1}{\sqrt{3}}(\tilde{e}_k + \mu_l\tilde{e}_l + \mu_m\tilde{e}_m)$$

and $|\mu_l| = |\mu_m| = 1$. For $j \in \{l, m\}$ we find

$$\begin{aligned}\frac{1}{\sqrt{6}}|1 + \mu_j| &= \left|\left(\frac{1}{\sqrt{3}}(\tilde{e}_k + \mu_l\tilde{e}_l + \mu_m\tilde{e}_m), \frac{1}{\sqrt{2}}(\tilde{e}_k + \tilde{e}_j)\right)\right| = \left|(\tilde{y}_{klm}, \tilde{v}_{kj})_{H_2}\right| \\ &= \left|(y_{klm}, v_{kj})_{H_1}\right| = \left|\left(\frac{1}{\sqrt{3}}(e_k + e_l + ie_m), \frac{1}{\sqrt{2}}(e_k + e_j)\right)_{H_1}\right| = \begin{cases} \frac{2}{\sqrt{6}} & \text{, if } j = l, \\ \frac{\sqrt{2}}{\sqrt{6}} & \text{, if } j = m. \end{cases}\end{aligned}$$

Employing Lemma 2.4 we obtain $\mu_l = 1$ and $\mu_m \in \{-i, i\}$. Next, for $j \in \{k, l\}$ we have

$$\begin{aligned}\frac{1}{\sqrt{6}}|\mu_m - \lambda_{mj}| = \frac{1}{\sqrt{6}}|\mu_m + \overline{\lambda_{mj}}| &= \left|\left(\frac{1}{\sqrt{3}}(\tilde{e}_k + \tilde{e}_l + \mu_m\tilde{e}_m), \frac{1}{\sqrt{2}}(\tilde{e}_m + \lambda_{mj}\tilde{e}_j)\right)_{H_2}\right| = \left|(\tilde{y}_{klm}, \tilde{w}_{mj})_{H_2}\right| \\ &= \left|(y_{klm}, w_{mj})_{H_1}\right| = \left|\left(\frac{1}{\sqrt{3}}(e_k + e_l + ie_m), \frac{1}{\sqrt{2}}(e_m + ie_j)\right)_{H_1}\right| = \frac{1}{\sqrt{6}}|i + \bar{i}| = 0\end{aligned}$$

and conclude that $\lambda_{mk} = \mu_m = \lambda_{ml}$. For an additional $n \in J \setminus \{k, m\}$ we derive

$$\lambda_{kl} = \lambda_{kn} = \lambda_{nk} = \lambda_{nm} = \lambda_{mn}.$$

Thus, we have $\lambda_{kl} = \lambda_{mn}$ for all $k, l, m, n \in J$ with $k \neq l$ and $m \neq n$.
We define $\zeta : \mathbb{C} \to \mathbb{C}$ as the identitiy function if $\lambda_{kl} = i$ and as the complex conjugation if $\lambda_{kl} = -i$. For all distinct $k, l \in J$ we then have

$$\tilde{w}_{kl} = \frac{1}{\sqrt{2}}(e_k + \zeta(i)e_l).$$

We are going to define $U : H_1 \to H_2$. First we set $U0 := 0$. For $z \in H_1 \setminus \{0\}$ we have

$$z = \sum_{j \in J} \lambda_j e_j$$

where $\lambda_j := (z, e_j)_{H_1}$ for all $j \in J$. As $z \neq 0$ there exists a least element $k \in J$ with $\lambda_k \neq 0$. By Lemma 5.4 there exists a unique vector $\tilde{z} \in g(\mathbb{C}z)$ such that

$$\tilde{z} = \zeta(\lambda_k)\tilde{e}_k + \sum_{j \in J \setminus \{k\}} (\tilde{z}, \tilde{e}_j)_{H_2} \tilde{e}_j.$$

This vector satisfies $\|\tilde{z}\|_{H_2} = \|z\|_{H_1}$. First we define $\nu_j := \zeta^{-1}\Big((\tilde{z}, \tilde{e}_j)_{H_2}\Big)$ for all $j \in J \setminus \{k\}$ and then

$$Uz := \tilde{z} = \zeta(\lambda_k)\tilde{e}_k + \sum_{j \in J \setminus k} \zeta(\nu_j)\tilde{e}_j.$$

For $l \in J \setminus \{k\}$ we have

$$\begin{aligned} \frac{1}{\sqrt{2}}|\lambda_k + \nu_l| = \frac{1}{\sqrt{2}}|\zeta(\lambda_k) + \zeta(\nu_l)| &= \left| \left(\zeta(\lambda_k)\tilde{e}_k + \sum_{j \in J \setminus k} \zeta(\nu_j)\tilde{e}_j, \frac{1}{\sqrt{2}}(\tilde{e}_k + \tilde{e}_l) \right)_{H_2} \right| = |(Uz, \tilde{v}_{kl})_{H_2}| \\ &= \left|(z, v_{kl})_{H_1}\right| = \left| \left(\sum_{j \in J} \lambda_j e_j, \frac{1}{\sqrt{2}}(e_k + e_l) \right)_{H_1} \right| = \frac{1}{\sqrt{2}}|\lambda_k + \lambda_l| \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\sqrt{2}}|\lambda_k - i\nu_l| = \frac{1}{\sqrt{2}}\Big|\zeta(\lambda_k) + \overline{\zeta(i)}\zeta(\nu_l)\Big| &= \left| \left(\zeta(\lambda_k)\tilde{e}_k + \sum_{j \in J \setminus \{k\}} \zeta(\nu_j)\tilde{e}_j, \frac{1}{\sqrt{2}}(e_k + \zeta(i)e_l) \right)_{H_2} \right| = |(Uz, \tilde{w}_{kl})_{H_2}| \\ &= \left|(z, w_{kl})_{H_1}\right| = \left| \left(\sum_{j \in J} \lambda_j e_j, \frac{1}{\sqrt{2}}(e_k + ie_l) \right)_{H_1} \right| = \frac{1}{\sqrt{2}}|\lambda_k - i\lambda_l|. \end{aligned}$$

As

$$|\nu_l| = \left|\zeta^{-1}\big((\tilde{z}, \tilde{e}_l)_{H_2}\big)\right| = \left|(\tilde{z}, \tilde{e}_l)_{H_2}\right| = \left|(z, e_l)_{H_1}\right| = |\lambda_l|$$

we can employ Lemma 2.6 in order to obtain $\nu_l = \lambda_l$. Thus,

$$Uz = \sum_{j \in J} \zeta(\lambda_j)\tilde{e}_j = \sum_{j \in J} \zeta\Big((z, e_j)_{H_1}\Big)\tilde{e}_j.$$

By definition $Uz = \tilde{z} \in g(\mathbb{C}z)$. For arbitrary $x, y \in H_1$ and $\mu, \lambda \in \mathbb{C}$ we obtain

$$\begin{aligned} U(\mu x + \lambda y) &= U\left(\sum_{j \in J} \Big(\mu(x, e_j)_{H_1} + \lambda(y, e_j)_{H_1} \Big) e_j \right) \\ &= \zeta(\mu) \sum_{j \in J} \zeta\Big((x, e_j)_{H_1}\Big)\tilde{e}_j + \zeta(\lambda) \sum_{j \in J} \zeta\Big((y, e_j)_{H_1}\Big)\tilde{e}_j = \zeta(\mu)Ux + \zeta(\lambda)Uy. \end{aligned}$$

Thus, U is ζ-linear and

$$\begin{aligned}\|Uz\|_{H_2}^2 &= \left\|\sum_{j\in J}\zeta\Big((z,e_j)_{H_1}\Big)\tilde{e}_j\right\|_{H_2}^2 = \sum_{j\in J}\left|\zeta\Big((z,e_j)_{H_1}\Big)\right|^2 \\ &= \left\|\sum_{j\in J}\zeta\Big((z,e_j)_{H_1}\Big)e_j\right\|_{H_1}^2 = \|z\|_{H_1}^2.\end{aligned}$$

□

Corollary 5.9. Let H_1 and H_2 be Hilbert spaces and $g : \mathcal{P}(H_1) \to \mathcal{P}(H_2)$ be a surjective isometry. Then there exists a function $U : H_1 \to H_2$ that is either linear and unitary or antilinear and antiunitary and satisfies $Ux \in g(\mathbb{C}x)$ for every $x \in H_1$.

Proof. By Theorem 5.8 there exists a ζ-linear isometry $U : H_1 \to H_2$ that satisfies $Ux \in g(\mathbb{C}x)$ for all $x \in H_1 \setminus \{0\}$, where $\zeta : \mathbb{C} \to \mathbb{C}$ is either the identity mapping or the complex conjugation. Let us consider some arbitrary $y \in H_2$. If $y = 0$, then $U0 = 0 = y$. Thus, assume $y \neq 0$. Due to the fact that g is surjective we know there exists $R \in \mathcal{P}(H_1)$ with $g(R) = \mathbb{C}y$. From this we conclude that $U(R) = \mathbb{C}y$. Hence, there exists $x \in R$ with $Ux = y$ which implies $\operatorname{ran} U = H_2$. From Proposition 3.30 we obtain that U is ζ-unitary. □

6 Concluding remarks

The proof given here is not particularly short and it involves quite a few calculations. Despite these drawbacks it has the merit that it proves a very general form of Wigner's theorem where the two Hilbert spaces involved can be different ones and do not have to be separable. Furthermore, we constructed the desired function step by step in the proof which might be very insightful and we did not have to use very deep mathematical results. Finally, it is worth mentioning that the paper does give a lot of detailed calculations which should make it easy to read.

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