CompMath: LaTeX-Übung 2

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Theorem (Lemma of Lax-Milgram). *Let X be a finite dimensional vector space over $\mathbb{R}$ with the basis $\{v_1, \ldots, v_n\}$, $F : X \to \mathbb{R}$ linear and $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ a bilinear form on X, i.e. $a(\cdot,\cdot)$ is linear in both components. Further, we assume $a(v,v) > 0$ for all $v \in X$.*
Then there exists a unique $u \in X$ with $a(u,v) = F(v)$ for all $v \in X$.

Proof. $a(v,v) > 0$ for all $v \in X$, implies that a is not degenerate, i.e.

1. $\forall u \in X \backslash \{\vec{0}\}\, \exists\, v \in X : a(u,v) \neq 0$,
2. $\forall v \in X \backslash \{\vec{0}\}\, \exists\, u \in X : a(u,v) \neq 0$.

So, now we know, that the linear mapping

$$d_a : \begin{cases} X \to X^* \\ u \mapsto a(u,\cdot) \end{cases}$$

is bijective, which lets us find a unique u for any given $F \in X^*$.

$\blacksquare$

2

$$p(t) = \det(A - t \cdot \mathrm{Id}) = \begin{vmatrix} a_{11} - t & a_{12} & \cdots & a_{1n} \\ a21 & a_{22} - t & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - t \end{vmatrix}$$

3

Theorem. *If $A \in \mathbb{R}^{n \times n}$ is a matrix with $\sum_{j,k=1}^{n} x_j A_{jk} x_k > 0$ for all $x \in \mathbb{R}^n$, then A is regular.*

Proof. Note, that $\sum_{j,k=1}^{n} x_j A_{jk} x_k = x^T A x$, lets A induce a positive definite bilinear form. Due to the principal minor criterium, $\det A \neq 0$, which implies that A is indeed regular.

$\blacksquare$

4

Theorem. *Let I be a nonempty open interval. Then it holds for $f, g \in C^\infty(I)$ and $n \in \mathbb{N}$ that*

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}. \tag{1}$$

Proof. (1) is trivial for $n = 0$. Let (1) be true for n, then one calculates

$$\begin{aligned}
(fg)^{(n+1)} = \left((fg)^{(n)}\right)' &\overset{(1)}{=} \\
&\left(\sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}\right)' = \\
&\sum_{k=0}^{n} \binom{n}{k} \left(f^{(k)} g^{(n-k)}\right)' = \\
&\sum_{k=0}^{n} \binom{n}{k} \left(f^{(k+1)} g^{(n-k)} + f^{(k)} g^{(n-k+1)}\right) = \\
&\sum_{k=0}^{n} \binom{n}{k} f^{(k+1)} g^{(n-k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k+1)} = \\
&\sum_{k=0}^{n-1} \binom{n}{k} f^{(k+1)} g^{(n-k)} + f^{(n+1)} g + f g^{(n+1)} + \sum_{k=1}^{n} \binom{n}{k} f^{(k)} g^{(n-k+1)} = \\
&f g^{(n+1)} + \sum_{k=1}^{n} \binom{n}{k-1} f^{(k)} g^{(n-k+1)} + \sum_{k=1}^{n} \binom{n}{k} f^{(k)} g^{(n-k+1)} + f^{(n+1)} g = \\
&f g^{(n+1)} + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k}\right) f^{(k)} g^{(n-k+1)} + f^{(n+1)} g = \\
&f g^{(n+1)} + \sum_{k=1}^{n} \binom{n+1}{k} f^{(k)} g^{(n+1-k)} + f^{(n+1)} g = \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)} g^{(n+1-k)}
\end{aligned}$$

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5

Theorem. *Let $n \in \mathbb{N}$ It holds:*

$$\sqrt{n} \in \begin{cases} \mathbb{N}, & \text{if } n \text{ is a square number,} \\ \mathbb{R}\backslash\mathbb{Q}, & \text{otherwise.} \end{cases}$$

Proof. Let $m, n \in \mathbb{N}$ be arbitrary and $\sqrt[m]{n} \in \mathbb{Q}$. Then $\exists\, a, b \in \mathbb{N} : \gcd(a, b) = 1,\ \sqrt[m]{n} = \frac{a}{b}$. This gives us $a^m = b\left(nb^{m-1}\right)$ and thus, $\gcd(a^m, b) = b$. However, because of $\gcd(a, b) = 1$, $\frac{a}{b}$ is irreducible, and so is $\frac{a^m}{b}$. Therefore, $\gcd(a^m, b) = 1 = b$.
We conclude, that all natural roots of an integer are either natural or irrational.

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6

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \Big[x\sqrt{1-x^2}\Big]_{x=-1}^{1} - \int_{-1}^{1} \frac{x(-2x)}{2\sqrt{1-x^2}}\,dx$$
$$= \Big[x\sqrt{1-x^2}\Big]_{x=-1}^{1} + \int_{-1}^{1} \frac{dx}{2\sqrt{1-x^2}}\,dx - \int_{-1}^{1} \frac{1-x^2}{2\sqrt{1-x^2}}\,dx$$
$$= \Big[x\sqrt{1-x^2} + \arcsin x\Big]_{x=-1}^{1} - \int_{-1}^{1} \sqrt{1-x^2}\,dx.$$

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Formally, a triangle T with vertices $x, y, z \in \mathbb{R}^2$ is defined as convex hull of these points

$$\operatorname{conv}(x,y,z) := \{ax + by + cz : a, b, c \geq 0 \text{ with } a + b + c = 1\}.$$

The triangle T is called non-degenerated if the vectors $y - x$ and $z - x$ are linearly independent.

Theorem. *Let $T = conv(x, y, z)$ and $\widetilde{T} = conv(\widetilde{x}, \widetilde{y}, \widetilde{z})$ be two non-degenerated triangles. Then, there exists an affine bijection $\Phi : T \to \widetilde{T}$, i.e., a bijective mapping of the form $\Phi(v) = Av + b$ with a matrix $A \in \mathbb{R}^{2\times 2}$ and a vector $b \in \mathbb{R}^2$.*

Proof. Because $T, \widetilde{T}$ are Triangles, their vertices $(x, y, z), (\widetilde{x}, \widetilde{y}, \widetilde{z})$ are affine independent respectively. Therefore, each tuple forms an affine basis and spans out an affine sub room.
Now, the continuation theorem for affine mappings gives us a unique affine bijection, if

$$\Phi : \begin{cases} x \mapsto \widetilde{x} \\ y \mapsto \widetilde{y} \\ z \mapsto \widetilde{z}. \end{cases}$$

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Theorem. *Let $\Omega \subseteq \mathbb{R}^d$ (with $d \geq 3$) be a bounded domain with Lipschitz-boundary and $u \in C^2(\Omega)$ a solution of the Laplace equation $\Delta u = 0$ in Ω.[1] Then there holds the representation formula*

$$\forall x \in \Omega : \quad u(x) = \frac{1}{4\pi} \int_{\partial\Omega} \frac{1}{|x-y|} \frac{\partial}{\partial \nu(y)} u(y)\,dy - \frac{1}{4\pi} \int_{\partial\Omega} \left(\frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} \right) u(y)\,dy.$$

[1] Recall that the Laplace operator Δ is defined for all $x \in \Omega$ by $(\Delta u)(x) := \sum_{i=1}^{d} \frac{\partial}{\partial x_i} u(x) = 0$.