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Übungsaufgaben zur VU Computermathematik

Serie 2

Exercise 2.1: ***Oops.***

Compute the derivative w.r.t. x of

$$\frac{5\sin(3x+b\sqrt{x^2+e^{2x}})\,\tan\big(\frac{k^2x^2}{1+u^2x^2}\big)+\sqrt[3]{\frac{ax-\ln x}{a^2+x^2}}}{\arccos\big(\frac{x}{\sqrt{3+x}}\big)+\frac{3a^2x^3}{\arctan(1/x)}+e^{-\frac{x^2-b^2}{2}}\arcsin\sqrt{\frac{3x}{1-x^2}}}$$

Exercise 2.2: ***Two computer-assisted proofs.***

a) *Provide a computer-assisted proof of the identity*

$$\arctan x+\arctan y \;=\; \arctan\Big(\frac{x+y}{1-x\,y}\Big).$$

Hint: Differentiate the left- and right-hand sides with respect to x and simplify.[1] Argue why the outcome of this computation provides a proof.

(For $x=0$ the identity is trivially valid. What about the special case $x\,y=1$?)

b) *Provide a computer-assisted proof of the fact that for all $n\in\mathbb{N}$ there exists $x>0$ such that*

$$(1+x)^n < \mathrm{e}\,n x\,.$$

Find such an x (depending on n).

Remark: It may be necessary to manipulate the result by hand to finish the proof.

Exercise 2.3: ***A formula due to Gauß.***

The following formula required 20 pages of factorization tables[2] in the edition of Gauß' works (cf. Werke, ed. Königl. Ges. d. Wiss., Göttingen, vol. 2, pp. 477–502):

$$\frac{\pi}{4} \;=\; 12\arctan\frac{1}{38}+20\arctan\frac{1}{57}+7\arctan\frac{1}{239}+24\arctan c$$

with $c=\ldots$ (?).

a) *Determine the number c.*

b) *Use* `?taylor` *(about $x=0$) to derive a series representation for $\frac{\pi}{4}$ based on* **a)**.

Exercise 2.4: ***Another computer-assisted proof.***

a) *Provide a computer-assisted proof of* Young's inequality

$$x\,y \;\le\; \frac{x^p}{p}+\frac{y^q}{q}$$

for all $x,y\ge 0$, where $p>1$ and q such that $\frac{1}{p}+\frac{1}{q}=1$.

Hint: Consider $f(x)=\frac{x^p}{p}+\frac{y^q}{q}-x\,y$ (for fixed y) and investigate the zeros of f'.

b) In **a)**, $p=q=\frac{1}{2}$ is a special case. *Design a plot displaying the left- and right-hand sides in Youngs's inequality for this special case.*

Hint: Two variables x,y: use `?plot3d`.

[1] Sometimes simplification does not work (in this example it should work); a pragmatic approach in such cases is to test numerical values or produce a plot.

[2] Gauß was using Exercise **2.2 a)**.

Exercise 2.5: ***Analyzing a real function.***

Use Maple as a computational tool for analyzing the real function (Kurvendiskussion)

$$f(x) = x^2 \ln(x^2),$$

including nice plots of the function and its first and second derivatives.

Hint: Pay special attention to $x = 0$.

Exercise 2.6:

The Bernstein polynomials of degree n are defined as

$$B_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k}, \quad k = 0 \ldots n.$$

a) *Design a function* `B(k,n,t)` *implementing the Bernstein polynomials of degree* `n`.

b) *Use* `plot` *and* `plots[display]` *to plot the $B_{k,n}(t)$ together for some n on the interval $[0,1]$.*

c) *Design a function* `Bapprox(f,n,t)` *which, for a given function* `f` *to be approximated, returns the Bernstein-type approximation*[3] *of* `f` *of degree* `n`,

$$B_n(f)(t) = \sum_{k=0}^{n} B_{k,n}(t)\, f(\tfrac{k}{n})$$

in form of a function, using the functions `B(k,n,t)` *from* **c)**.

d) *Use* `plot` *and* `plots[display]` *to plot $f(t)$ and $B_n(f)(t)$ for some n together on the interval $[0,1]$. Use two different colors for these curves.*

Exercise 2.7: ***A parametric integral.***

Consider the integral (as a function of x)

$$I(x) = \int_0^x (x-t)\sin(t^2)\, dt.$$

a) *Check what answer is delivered by* Maple, *and plot the function $I(x)$.*

b) From **a)** you see that, apparently, $I(x)$ cannot be represented via standard functions like $\sin, \cos, \ldots$.
But: *What about $I'(x)$?*

c) *What about $I''(x)$?* [4]

d) Now we ask ourselves whether the result delivered by Maple under **c)** is indeed correct.
Assume you are doubting the result, but you do not know the correct answer. Suggest and realize a strategy to 'verify' the result delivered by Maple.

Exercise 2.8: ***Newton, schau owa.***

Newton iteration is a standard approach for determining numerically (in floating point arithmetic) a zero (Nullstelle) of a differentiable real function $f(x)$, starting from an initial guess x_0,

$$x_i := x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})}, \quad i = 1, 2, 3, \ldots$$

a) *Design a procedure* `Isaac(...)` *which expects a function f, an initial guess*[5] *$x_0 \in \mathbb{R}$ and a tolerance parameter tol as its arguments and which iterates until $|f(x_i)| \le tol$. Your procedure returns the list* `[`i, x_i`]`. *If the criterion $|f(x_i)|$ is not satisfied after* 100 *steps, we consider the iteration having failed. In this case, return* `[NULL]`. (`NULL` is the Maple symbol for 'nothing'.)

Example: Computation of $1/x$ for $x > 0$. This sounds awkward, but the point is that Newton iteration requires no division (/) if f is chosen in an appropriate way. *Explain how to do this and realize a numerical example.*

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[3] It can be shown that for a function f continuous on $[0,1]$, the sequence $\{B_n(f)\}$ converges uniformly to f for $n \to \infty$, i.e., $\lim_{n\to\infty} \max_{t\in[0,1]} |B_n(f)(t) - f(t)| = 0$.

[4] This can also be easily be computed by hand (assuming you know how to differentiate a parametric integral).

[5] Convergence is only guaranteed if x_0 is sufficiently close to the exact solution. Do not worry about this here – just choose 'harmless' data, where no convergence problems are to be expected.

b) An improved iteration involving the second derivative f'' reads

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})}\left(1 + \frac{1}{2}\,\frac{f(x_{i-1})\,f''(x_{i-1})}{f'(x_{i-1})^2}\right).$$

Proceed as in **a)** *and compare both versions: Which one converges faster?*

Hint: For testing, choose e.g. `Digits` $= 50$ and $tol = 1\text{E-}40$.