1. (a) (i) $\forall n \in \mathbb{N}: \mu_n \geq 0$

(ii) $\forall n \in \mathbb{N}\ \forall (A_k)_{k\in\mathbb{N}} \in \mathfrak{A}^{\mathbb{N}}$ disjunkt:

$$\mu_n\left(\sum_{k\in\mathbb{N}} A_k\right) = \frac{1}{n}\left|\sum_{k\in\mathbb{N}} A_k \cap \{1,\dots,n\}\right| =$$

$$\sum_{k\in\mathbb{N}} \frac{1}{n}\left|A_k \cap \{1,\dots,n\}\right| = \sum_{k\in\mathbb{N}} \mu_n(A_k).$$

(iii) $\forall A \in \mathfrak{A}: \mu(A) = \lim\limits_{n\to\infty} \mu_n(A) = \lim\limits_{n\to\infty} \frac{1}{n}\left|A \cap \{1,\dots,n\}\right|$

$$\leq \lim_{n\to\infty} \frac{1}{n}\left|\{1,\dots,n\}\right| = 1.$$

(iv) Betrachte $(\{k\})_{k\in\mathbb{N}}$:

$$\lim_{n\to\infty} \frac{1}{n}\left|\sum_{k\in\mathbb{N}} \{k\} \cap \{1,\dots,n\}\right| = 1 \neq$$

$$\neq 0 = \sum_{k\in\mathbb{N}} \lim_{n\to\infty} \frac{1}{n}\left|\{k\} \cap \{1,\dots,n\}\right|.$$

(b) (i) ✓

(ii) … :

$$\mu_n\left(\sum_{k\in\mathbb{N}} A_k\right) = \frac{1}{n}\left|\sum_{k\in\mathbb{N}} A_k\right| = \sum_{k\in\mathbb{N}} \frac{1}{n}\left|A_k\right| = \sum_{k\in\mathbb{N}} \mu_n(A_k).$$

(iii) … : … $= \lim\limits_{n\to\infty} \frac{1}{n}|A| \in \{0, \infty\}$

(iv) … :

$$\lim_{n\to\infty} \frac{1}{n}\left|\sum_{k\in\mathbb{N}} \{k\}\right| = \infty \neq 0 = \sum_{k\in\mathbb{N}} \lim_{n\to\infty} \frac{1}{n}\left|\{k\}\right|.$$

2. Gg: $\mathfrak{A}$ σ-Algebra, μ endl. Inhalt auf $\mathfrak{A}$,
$\mathfrak{A} \ni C \subseteq \Omega$.

Zz: μ kann zu Inhalt auf $\mathfrak{A}_\sigma(\mathfrak{A} \cup \{C\}) =: \mathfrak{A}_\sigma$ fortgesetzt werden.

Ww: $\mathfrak{A}_\sigma = \{(A \cap C) \cup (B \setminus C) : A, B \in \mathfrak{A}\}$, also
$\forall M \in \mathfrak{A}_\sigma \, \exists A, B \in \mathfrak{A} : M = (A \cap C) \dot\cup (B \setminus C)$.

Wähle $\tilde\mu(M) := \tilde\mu((A \cap C) \dot\cup (B \setminus C)) :=$
$\bar\mu(A \cap C) + \bar\mu(B) - \bar\mu(B \cap C)$, $\overbrace{\hphantom{xx}}^{B \cap C^c}$ wobei
$\bar\mu(A \cap C) := \sup\{\mu(E) : \mathfrak{A} \ni E \subseteq A \cap C\}$, aber
$\bar\mu(B) = \bar\mu(B \cap B) = \mu(B)$.

Wohldefiniert: Seien $A, A', B, B' \in \mathfrak{A}$, sodass
$(A \cap C) \cup (B \setminus C) = M = (A' \cap C) \cup (B' \setminus C)$.

Zz: $\bar\mu(A \cap C) + \bar\mu(B \setminus C) = \bar\mu(A' \cap C) + \bar\mu(B' \setminus C)$.

Ww: $\Omega = C \dot\cup C^c$, also „$\dot\cup$" $\Rightarrow$
- $A \cap C = A' \cap C$
- $B \cap C^c = B' \cap C^c \Rightarrow B \setminus (B \cap C) = B' \setminus (B' \cap C)$.

Fortsetzung: Zz: $\forall N \in \mathfrak{A} : \tilde\mu(N) = \mu(N)$.

Ww: $\exists \tilde A, \tilde B := N : (\tilde A \cap C) \dot\cup (\tilde B \setminus C) = N$
$\Rightarrow \tilde\mu(N) = \cancel{\bar\mu(\tilde A \cap C)} + \bar\mu(N) - \cancel{\bar\mu(\tilde B \cap C)} = \mu(N)$.

Additivität: oBdA. Betrachte bloß den Induktionsanfang.
Seien $A, B \in \mathfrak{A}_\sigma$ disjunkt.

$\bar{\mu}(A) + \bar{\mu}(B) =$

$\sup\{\mu(E) : \mathcal{U} \ni E \subseteq A\} + \sup\{\mu(E) : \mathcal{U} \ni E \subseteq B\}$

$= \cdots$

Ww: $A \cap B = \emptyset$

$\Rightarrow \forall E \subseteq A \cup B \ \exists E_1 \subseteq A, E_2 \subseteq B : E = E_1 \dot{\cup} E_2.$

$\cdots = \sup\{\mu(E_1) + \cancel{\mu(E_2)} : E \subseteq A \subseteq A \dot{\cup} B\} +$

$\sup\{\cancel{\mu(E_1)} + \mu(E_2) : E \subseteq B \subseteq B \dot{\cup} A\} =$

$\sup\{\mu(E_1) + \mu(E_2) : E \subseteq A \dot{\cup} B\} = \bar{\mu}(A \cup B).$

3. Gg: μ endl. Maß auf $\mathcal{R}$ σ-Ring.

(a) Zz: μ beschränkt.

Angenommen, μ ist unbeschränkt, d.h.
$\forall C \in \mathbb{R}\ \exists A \in \mathcal{R} : \mu(A) \geq C$.

Sei $A_0 \in \mathcal{R}$ beliebig und $\forall i \in \mathbb{N} : A_i := B_i \setminus \sum_{k=0}^{i-1} A_k$
wobei B_i so sei, dass $\mu(B_i) \geq \sum_{k=0}^{i-1} \mu(A_k) + 1$.

Also muss wegen der Subadditivität $\forall i \in \mathbb{N}$

$$A_i = B_i \setminus \sum_{k=0}^{i-1} A_k \Rightarrow \sum_{k=0}^{i} A_k = B_i \cup \sum_{k=0}^{i-1} A_k \supseteq B_i ,$$

$$\Rightarrow \mu(B_i) \leq \mu\left(\sum_{k=0}^{i} A_k\right)$$

$$\Rightarrow \mu(A_i) \geq \mu(B_i) - \mu\left(\sum_{k=0}^{i-1} A_k\right) \geq 1.$$

Also müsste

$$\mu\Big(\underbrace{\sum_{i \in \mathbb{N}} A_i}_{\mathcal{R}}\Big) = \sum_{i \in \mathbb{N}} \mu(A_i) \geq \sum_{i \in \mathbb{N}} 1 = \infty .$$

(3) Zz: μ kann zum endl. Maß $\tilde{\mu}$ auf der erzeugten σ-Algebra $\mathfrak{A}_\sigma(\mathcal{R}) =: \mathfrak{A}_\sigma$ fortgesetzt werden.

Wegen Aufgabe 6. und (a), folgt
$\tilde{\mu}(\Omega) := \sup\{\mu(A) : A \in \mathcal{R}\} < \infty$.

Ww: $\mathfrak{A}_\sigma = \{A \subseteq \Omega : A \in \mathcal{R} \vee A^c \in \mathcal{R}\}$.

Wähle also
$$\tilde{\mu}(A) := \begin{cases} \mu(A), & \text{wenn } A \in \mathcal{R} \\ \tilde{\mu}(\Omega) - \mu(A^c), & \text{sonst} \end{cases}$$

Wohldefiniert: ✓

Maß: Zz: $\tilde{\mu} \geq 0$ ✓

Zz: $\tilde{\mu}$ σ-additiv.

Zz: $A \in \mathcal{R} \subseteq \mathfrak{A}_\sigma$, $B \in \mathfrak{A}_\sigma \Rightarrow A \cap B \in \mathcal{R}$.

oBdA. $B \notin \mathcal{R} \Rightarrow B^c \in \mathcal{R}$.

$A \cap B = A \setminus B^c \in \mathcal{R}$.

Sei $(A_n)_{n\in\mathbb{N}} \in \mathfrak{A}_\sigma^{\mathbb{N}}$ disjunkt.

oBdA. $\exists i \in \mathbb{N}: A_i \notin \mathcal{R} \Rightarrow A_i^c \in \mathcal{R}$.

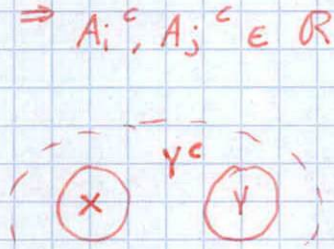
Fall I: „$\exists!$"

$$\tilde{\mu}\Big(\sum_{n\in\mathbb{N}} A_n\Big) = \tilde{\mu}\Big(\sum_{n\in\mathbb{N}\setminus\{i\}} A_n \,\dot\cup\, A_i\Big) \overset{!}{=} \sum_{n\in\mathbb{N}\setminus\{i\}} \tilde{\mu}(A_n) + \tilde{\mu}(A_i)$$
$$= \sum_{n\in\mathbb{N}} \tilde{\mu}(A_n).$$

Fall II: $\exists i,j \in \mathbb{N}, i \neq j: A_i, A_j \notin \mathcal{R} \Rightarrow A_i^c, A_j^c \in \mathcal{R}$.

Ww. $A_i \cap A_j = \emptyset \Rightarrow A_i \subseteq A_j^c$

$$\Rightarrow \underbrace{A_i}_{\in \mathfrak{A}_\sigma} \cap \underbrace{A_j^c}_{\in \mathcal{R}} = A_i \left.\begin{matrix} \\ \Rightarrow \end{matrix}\right\} \in \mathcal{R} \;\lightning$$



Additivität:

Ww: $A \subseteq B \Rightarrow \tilde{\mu}(B) = \tilde{\mu}(A \,\dot\cup\, B\setminus A) = \tilde{\mu}(A) + \tilde{\mu}(B\setminus A)$

$\Rightarrow \tilde{\mu}(B\setminus A) = \tilde{\mu}(B) - \tilde{\mu}(A)$.

Seien $A, B \in \mathfrak{A}_\delta$ disjunkt und oBdA. $B \notin \mathcal{R}$.

$$\tilde{\mu}(A) + \tilde{\mu}(B) = \tilde{\mu}(A) + \tilde{\mu}(\Omega) - \tilde{\mu}(B^c) =$$
$$\tilde{\mu}(\Omega) - \underbrace{(\tilde{\mu}(B^c) - \tilde{\mu}(A))}_{\tilde{\mu}(B^c \setminus A) \,=\, \tilde{\mu}((\underbrace{B \dot\cup A}_{\in \mathfrak{A}_\delta \setminus \mathcal{R}})^c)} = \tilde{\mu}(B \dot\cup A).$$

$\in \mathfrak{A}_\delta \setminus \mathcal{R}$, weil:

sonst $B = (B \dot\cup A) \setminus A \in \mathcal{R}$ ↯

4. Seien $A_1, \dots, A_n \in \mathcal{R}$ Ring mit μ Inhalt, und $\forall i = 1, \dots, n : \mu(A_i) < \infty$, dann

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{I \subseteq \{1, \dots, n\}} (-1)^{|I|-1} \mu\left(\bigcap_{i \in I} A_i\right).$$

$$\mu(A \cup B) = \mu(A) + \mu(B) + \mu(A \cap B) = 0.9.$$

$$\begin{aligned}\mu(A \cup B \cup C) &= \mu(A) + \mu(B) + \mu(C) \\ &\quad - (\mu(A \cap B) + \mu(B \cap C) + \mu(C \cap A)) \\ &\quad + \mu(A \cap B \cap C) \\ &= 1.\end{aligned}$$

5. $\mu(A \triangle B) = \mu(\underbrace{A \setminus B}_{A \setminus (A \cap B)}) + \mu(\underbrace{B \setminus A}_{B \setminus (B \cap A)}) =$

$\mu(A) + \mu(B) - 2\mu(A \cap B)$.

$\mu(A \triangle B \triangle C) = \mu((A \triangle B) \setminus C) + \mu(C \setminus (A \triangle B)) =$

$\mu(A \triangle B) - 2\underbrace{\mu(C \cap (A \triangle B))} + \mu(C) = \cdots$

$\mu(C \cap (A \setminus B)) + \mu(C \cap (B \setminus A)) =$

$\mu((C \cap A) \setminus B) + \mu((C \cap B) \setminus A) =$

$\mu(C \cap A) + \mu(C \cap B) - 2\mu(A \cap B \cap C$

$\cdots = \mu(A) + \mu(B) + \mu(C) -$

$2(\mu(A \cap B) + \mu(B \cap C) + \mu(C \cap A)) +$

$4\mu(A \cap B \cap C)$.

$$\mu\left(\bigtriangleup_{i=1} A_i\right) = \sum_{\substack{I \subseteq \{1, \ldots, n\} \\ I \neq \emptyset}} (-2)^{|I|-1} \mu\left(\bigcap_{i \in I} A_i\right).$$

6. Gg: $\mathcal{R}$ σ-Ring über Ω, μ Maß auf $\mathcal{R}$.

Zz: $\tilde{\mu}(A) = \sup\{\mu(B) : B \in \mathcal{R}, B \subseteq A\}$ Maß auf $\mathfrak{A}(\mathcal{R})$.

Ww: $\mathfrak{A}(\mathcal{R}) = \{A \subseteq \Omega : A \in \mathcal{R} \vee A^c \in \mathcal{R}\}$

Sei $(A_n)_{n\in\mathbb{N}} \in \mathfrak{A}^{\mathbb{N}}$ eine disjunkte Folge.
oBdA. $\exists i \in \mathbb{N} : A_i \notin \mathcal{R}_\sigma \Rightarrow A_i^c \in \mathcal{R}_\sigma$

Fall I: „$\exists!$".

$\tilde{\mu}(\sum_{k\in\mathbb{N}} A_k) = \sup\{\mu(B) : \mathcal{R} \ni B \subseteq \sum_{k\in\mathbb{N}\setminus\{i\}} A_k \,\dot{\cup}\, A_i\} =$

$\sup\{\mu(B_1) + \mu(B_2) : \mathcal{R} \ni B = B_1 \,\dot{\cup}\, B_2,$
$B_1 \subseteq \sum_{k\in\mathbb{N}\setminus\{i\}} A_k, B_2 \subseteq A_i\} =$

$\sup\{\mu(B_1) + \mu(B_2) : \mathcal{R} \ni B \subseteq \sum_{k\in\mathbb{N}\setminus\{i\}} A_i\} +$
$\sup\{\mu(B_1) + \mu(B_2) : \mathcal{R} \ni B \subseteq A_i\}$

$= \tilde{\mu}(\sum_{k\in\mathbb{N}\setminus\{i\}} A_k) + \tilde{\mu}(A_i) = \sum_{k\in\mathbb{N}} \tilde{\mu}(A_k).$

Fall II: Analog zu Fall II aus 3.