(3) **Point estimator statistics: Comparison**
Let $X_1 \dots X_n$ be i.i.d. uniform $(0, \theta)$, with unknown parameter $\theta > 0$.

(a) Show that the method of moments estimator of θ is $2\bar{X}$ and the MLE of θ is $X_{(n)} = \max_{1 \le i \le n} X_i$.

(b) Compare the mean square errors of the two estimators. Which of the estimators should be preferred if any? Explain your reasoning.

a) $\mu(\theta) = \int_0^\theta \frac{1}{\theta} x \, dx = \frac{1}{\theta} \frac{\theta^2}{2} = \frac{\theta}{2} \overset{!}{=} \bar{X} \Leftrightarrow \theta = 2\bar{X}$ ... method of moments estimator

For $x \in [0, \theta]^n$:

$$L_n(\theta) = \prod_{i=1}^n \frac{1}{\theta} = \frac{1}{\theta^n},$$

For $\theta_1, \theta_2 \in [\max\{X_i \mid 1 \le i \le n\}, \infty)$: $L_n(\theta_1) > L_n(\theta_2) \Leftrightarrow \frac{1}{\theta_1^n} > \frac{1}{\theta_2^n} \Leftrightarrow \theta_2 > \theta_1$, hence $L_n(\theta)$ is decreasing. Therefore, it has it's maximum at $\theta = \max\{X_i \mid 1 \le i \le n\}$, which is, consequently, the MLE.

b) $$2\bar{X} - \theta = \frac{2}{n} \sum_{i=1}^n X_i - \frac{2}{n} \sum_{i=1}^n \frac{\theta}{2} = \frac{2}{n} \sum_{i=1}^n \left(X_i - \frac{\theta}{2}\right)$$

independence

$$MSE_\theta(2\bar{X}) = \mathbb{E}\left((2\bar{X} - \theta)^2\right) = \frac{4}{n^2} \mathbb{E}\left(\left(\sum_{i=1}^n \left(X_i - \frac{\theta}{2}\right)\right)^2\right) \overset{\text{independence}}{=} \frac{4}{n^2} \sum_{i=1}^n \mathbb{V}\mathrm{ar}(X_i) = \frac{4}{n^2} \frac{n\theta^2}{12} = \frac{\theta^2}{3n}$$

$$MSE_\theta(X_{(n)}) = \mathbb{E}\left((X_{(n)} - \theta)^2\right) = \int_{[0,\theta]^n} (\max(x) - \theta)^2 \frac{1}{\theta^n} \, dx = \frac{n}{\theta^n} \int_0^\theta \underbrace{\int_0^{x_n} \cdots \int_0^{x_n}}_{n-1 \text{ times}} (x_n - \theta)^2 \, dx_1 \cdots dx_n$$

$$= \frac{n}{\theta^n} \int_0^\theta (x_n - \theta)^2 x_n^{n-1} \, dx_n = \frac{n}{\theta^n} \int_0^\theta \left(x_n^{n+1} - 2x_n^n \theta + \theta^2 x_n^{n-1}\right) dx_n$$

$$= \frac{n}{\theta^n} \left(\frac{\theta^{n+2}}{n+2} - 2\frac{\theta^{n+2}}{n+1} + \frac{\theta^{n+2}}{n}\right)$$

$$= \theta^2 n \left(\frac{n(n+1) - 2n(n+2) + (n+1)(n+2)}{n(n+1)(n+2)}\right)$$

$$= \theta^2 \left(\frac{n^2 + n - 2n^2 - 4n + n^2 + 3n + 2}{(n+1)(n+2)}\right) = \frac{2\theta^2}{(n+1)(n+2)}$$

The MLE converges faster.