(3) **Real roots**

Let A, B and C be independent random variables, uniformly distributed on $(0, 1)$.

(a) What is the probability that the qudratic equation $Ax^2 + Bx + C = 0$ has real roots?

(b) Consider the following code in R.
What does it do and how is it related to your solution in part (a)?

```
n=10000
a=runif(n)
b=runif(n)
c=runif(n)
sum(b^2>4*a*c)/n
```

Hint: In HW2/ex. 3(b) we showed that if X has uniform $(0, 1)$ distribution then $-\log X$ has exponential distribution $\exp(1)$. In an analogue way, one can prove that $-s \log X \sim \exp(\frac{1}{s})$ for any $s > 0$. Also, in HW4/ex. 2(b) we proved that the sum of two independent exponential distributions is a gamma distribution. Namely, if $X \sim \exp(1)$ and $Y \sim \exp(1)$ are independent then $X + Y \sim Gamma(2, 1)$.

a) $Ax^2 + Bx + C = 0 \Leftrightarrow x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$, hence the quadratic equation has real roots if and only if $B^2 - 4AC \geq 0$.

$$\mathbb{P}(B^2 - 4AC \geq 0) = \mathbb{P}(B^2 \geq 4AC) = \mathbb{P}(\log(B^2) \geq \log(4AC)) = \mathbb{P}(\log(B^2) \geq \log(4) + \log(AC))$$
$$= \mathbb{P}(\log(B^2) - \log(AC) \geq \log(4)) = \mathbb{P}(-\log(AC) \geq -\log(B^2) + \log(4))$$

$-\log(A) \sim \exp(1)$, $-\log(C) \sim \exp(1)$, $-\log(AC) = (-\log(A)) + (-\log(C)) \sim Gamma(2,1)$

$g: (0,1) \to (0, \infty): x \mapsto -\log(x^2)$ has got the inverse $h: (0, \infty) \to (0,1): y \mapsto e^{-\frac{y}{2}}$

$f_B(x) = \mathbb{1}_{(0,1)}(x) \Rightarrow f_{-\log(B^2)}(x) = \mathbb{1}_{(0,1)}(e^{-\frac{x}{2}})\, e^{-\frac{x}{2}} = \mathbb{1}_{(0,\infty)}(x)\, \frac{1}{2} e^{-\frac{x}{2}}$, hence $-\log(B^2) \sim \exp(\frac{1}{2})$

Clearly, $-\log(B^2)$ and $-\log(AC)$ are independent and we conclude

$$\mathbb{P}(-\log(AC) \geq -\log(B^2) + \log(4)) = \int_{\log(4)}^{\infty} \int_0^{y-\log(4)} f_{-\log(AC)}(y)\, f_{-\log(B^2)}(x)\, dx\, dy$$
$$= \int_{\log(4)}^{\infty} y e^{-y} \int_0^{y-\log(4)} \frac{1}{2} e^{-\frac{x}{2}}\, dx\, dy$$
$$= \int_{\log(4)}^{\infty} y e^{-y} \left(1 - \exp\left(\frac{\log(4) - y}{2}\right)\right) dy$$
$$= \int_{\log(4)}^{\infty} y e^{-y}\, dy - \int_{\log(4)}^{\infty} 2y\, e^{-\frac{3y}{2}}\, dy$$
$$= \frac{1}{4}(1 + \log(4)) - \frac{1}{9}(1 + \log(8)) \approx 0{,}2544$$

b) The code gives an approximation of the Probability that was calculated in (a).