Goal driven error estimates

Classical error estimators estimate the error $u - u_h$ in the energy norm V for the problem: Find $u \in V$ such that

$$A(u, v) = f(v) \qquad \forall v \in V \tag{1}$$

and $u_h \in V_h \subset V$ is the corresponding Galerkin solution.

Some applications require to compute certain values (such as point values, average values, line integrals, fluxes through surfaces, ...). These values are described by linear functionals $b : V \to \mathbb{R}$. We want to design a method such that the error in this goal, i.e.,

$$b(u) - b(u_h) \tag{2}$$

is small. The technique is to solve additionally the dual problem, where the right hand side is the goal functional: Find $w \in V$ such that

$$A(v, w) = b(v) \qquad \forall v \in V. \tag{3}$$

Usually, one cannot solve the dual problem either, and one applies a Galerkin method also for the dual problem: Find $w_h \in V_h$ such that

$$A(v_h, w_h) = b(v_h) \qquad \forall v_h \in V_h. \tag{4}$$

In the case of point values, the solution of the dual problem is the Green function (which is not in H^1). The error in the goal is

$$b(u - u_h) = A(u - u_h, w) = A(u - u_h, w - w_h). \tag{5}$$

A rigorous upper bound for the error in the goal is obtained by using continuity of the bilinear-form, and energy error estimates η^1 and η^2 for the primal and dual problem, respectively:

$$|b(u - u_h)| \leq c\|u - u_h\|_V \|w - w_h\|_V \leq C\eta^1(u_h, f)\, \eta^2(w_h, b). \tag{6}$$

A good heuristic is the following (unfortunately, not correct) estimate

$$b(u - u_h) = A(u - u_h, w - w_h) \leq c \sum_{T \in \mathcal{T}} \|u - u_h\|_{H^1(T)} \|w - w_h\|_{H^1(T)} \leq C \sum_{T \in \mathcal{T}} \eta_T^1(u_h, f)\, \eta_T^2(w_h, b) \tag{7}$$

The last step would require a local reliability estimate. But, this is not true. We can interpret (7) that way: The local estimators $\eta_T^2(w_h)$ provide a way for weighting the primal local estimators according to the desired goal.