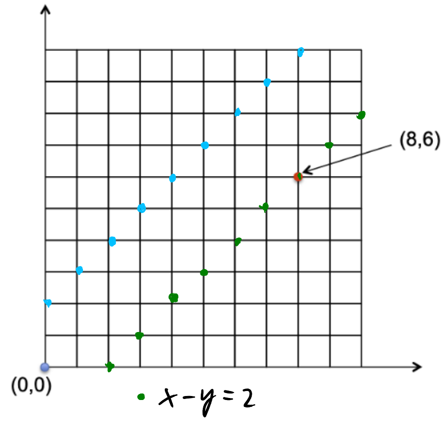
(1) **Random walk of a robot**

A robot is placed at the origin (the point $(0,0)$) on a two-dimension integer grid (see the figure below). Denote the position of the robot by (x, y). The robot can either move right to $(x+1, y)$ or move up to $(x, y+1)$.

(a) Suppose each time the robot randomly moves right or up with equal chance. What is the probability that the robot will ever reach the point $(8, 6)$?

(b) Suppose another robot has a $\frac{2}{3}$ chance to move right and a $\frac{1}{3}$ chance to move up when $x+y$ is even, otherwise it has a $\frac{1}{4}$ chance to move right and a $\frac{3}{4}$ chance to move up. It stops whenever $|x-y| \geq 2$. Find the probability that $x - y = 2$ when it stops.



a) The sum $x+y$ is exactly the number of total moves made so far. In order to reach the point $(8,6)$, it requires 14 moves in total, of which exactly 8 are to the right and 6 are up. The order in which these moves are made does not matter.

For all $i \in \mathbb{N}_0$ let X_i be the x-coordinate after i moves and Y_i the y-coordinate after i moves. Clearly, $X_i + Y_i = i$ for all $i \in \mathbb{N}_0$.

$$\mathbb{P}\left(\bigcup_{i=0}^{\infty}[(X_i,Y_i)=(8,6)]\right) = \mathbb{P}\left([(X_{14},Y_{14})=(8,6)]\right) = \mathbb{P}(X_{14}=8) = \binom{8}{14}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^{14-8} = \binom{8}{14}\left(\frac{1}{2}\right)^{14} \approx 0{,}183$$

b)
$$\mathbb{P}\left(\bigcup_{i=1}^{\infty}[X_i - Y_i = 2]\right) = (pq) + \left(p(1-q)pq + (1-p)qpq\right) +$$
$$\left(p(1-q)p(1-q)pq + p(1-q)(1-p)q\,pq + (1-p)q\,p(1-q)pq + (1-p)q(1-p)q\,pq\right) + \dots$$

$p = \frac{2}{3}$

$q = \frac{1}{4}$

$$= pq\left(1 + \left(p(1-q)+(1-p)q\right) + \left(p^2(1-q)^2 + 2pq(1-p)(1-q) + (1-p)^2q^2\right) + \dots\right)$$
$$= pq\sum_{i=0}^{\infty}\left(p(1-q)+(1-p)q\right)^i = pq\,\frac{1}{1-p(1-q)-(1-p)q}$$
$$= \frac{2}{3}\cdot\frac{1}{4}\cdot\frac{1}{1-\frac{2}{3}\cdot\frac{3}{4}-\frac{1}{3}\cdot\frac{1}{4}} = \frac{1}{6}\cdot\frac{1}{1-\frac{1}{2}-\frac{1}{12}} = \frac{1}{6}\cdot\frac{12}{5} = \frac{2}{5}$$

<u>A bit more formal:</u> R ... roboter gets to position ; $x \in \mathbb{N}_0$

$$\mathbb{P}(R=(x,x+2)) = q\,\mathbb{P}(R=(x,x+1)) = q\,p\,\mathbb{P}(R=(x,x)) = q\,p\,\mathbb{P}(R=(x-1,x-1))\left(p(1-q)+(1-p)q\right)$$

$$\mathbb{P}(R=(0,0)) = 1$$

Hence, we have a recursive formula that can easily be made explicit:

$\mathbb{P}(R=(x,x)) = (p(1-q)+(1-p)q)^x$, hence, just as above, we obtain the sum

$$\sum_{i=0}^{\infty}\mathbb{P}(R=(i,i+2)) = pq\sum_{i=0}^{\infty}\left(p(1-p)+(1-p)q\right)^i = \frac{2}{5}$$

(2) **Continuous two-dimensional random variable**

The joint pdf of two random variables X and Y is defined by

$$f(x, y) = \begin{cases} c(x + 2y), & 0 < y < 1 \text{ and } 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find the value of c and the marginal distribution of Y.

(b) Find the joint cdf of X and Y.

(c) Find the marginal distribution of X and the pdf of $Z = \frac{9}{(X+1)^2}$.

a) $1 \stackrel{!}{=} \int_{\mathbb{R}^2} f \, d\lambda^2 = \int_0^1 \int_0^2 c(x+2y)\, dx\, dy = c \int_0^1 (2+4y)\, dy = c\left(2+2\right) = 4c \Leftrightarrow c = \frac{1}{4}$

$\forall y =]0,1[: \int_0^2 f(x,y)\, dx = \frac{1}{2} + y = \frac{1+2y}{2}$

$$f_Y(y) = \begin{cases} y + \frac{1}{2} & , \text{if } 0 < y < 1 \\ 0 & , \text{otherwise} \end{cases}$$

b) $\int_0^y \int_0^x f(\xi, \eta)\, d\xi\, d\eta = \int_0^y \int_0^x \left(\frac{\xi}{4} + \frac{\eta}{2}\right) d\xi\, d\eta = \int_0^y \left(\frac{x^2}{8} + \frac{\eta x}{2}\right) d\eta = \frac{x^2 y}{8} + \frac{y^2 x}{4}$

$$F(x,y) = \begin{cases} 0 & , \text{if } x < 0 \text{ or } y < 0 \\ \frac{x^2 y}{8} + \frac{y^2 x}{4} & , \text{if } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 1 \\ \frac{x^2}{8} + \frac{x}{4} & , \text{if } 0 \leq x \leq 2 \text{ and } y > 1 \\ \frac{y}{2} + \frac{y^2}{2} & , \text{if } x > 2 \text{ and } 0 \leq y \leq 1 \\ 1 & , \text{if } x > 2 \text{ and } y > 2 \end{cases}$$

c) $\forall x \in]0,2[: \int_0^1 f(x,y)\, dy = \int_0^1 \left(\frac{x}{4} + \frac{y}{2}\right) dy = \frac{x}{4} + \frac{1}{4}$

$$f_X(x) = \begin{cases} \frac{1}{4}(x+1) & , \text{if } 0 < x < 2 \\ 0 & , \text{otherwise} \end{cases}$$

$g :]0,2[\to]1,9[: x \mapsto \frac{9}{(1+x)^2}$

$\forall x \in]0,2[\; \forall z \in]1,9[: \; z = \frac{9}{(1+x)^2} \Leftrightarrow z(1+x)^2 = 9 \Leftrightarrow 1+x = \sqrt{\frac{9}{z}} \Leftrightarrow x = \sqrt{\frac{9}{z}} - 1$

$h :]1,9[\to]0,2[: z \mapsto \sqrt{\frac{9}{z}} - 1 = 3z^{-\frac{1}{2}} - 1 \Rightarrow h'(z) = -\frac{3}{2} z^{-\frac{3}{2}}$

$f_X(h(z))\,|h'(z)| = \frac{1}{4}\left(-\frac{3}{2} z^{-\frac{3}{2}} + 1\right) \frac{3}{2} z^{-\frac{3}{2}} = \frac{3}{8} z^{-\frac{3}{2}} - \frac{9}{16} z^{-3} = \frac{3}{8}\left(z^{-\frac{3}{2}} - \frac{3}{2} z^{-3}\right)$

$$f_Z(z) = \begin{cases} \frac{3}{8}\left(z^{-3/2} - \frac{3}{2} z^{-3}\right) & , \text{if } 1 < z < 9 \\ 0 & , \text{otherwise} \end{cases}$$

(3) **Chi squared distribution**

Let X and Y be independent and identically distributed (i.i.d.) $\mathcal{N}(0,1)$ random variables. Define $Z = \min\{X, Y\}$. Show that $Z^2 \sim \chi_1^2$, i.e. show that the pdf of Z^2 is given by

$$f_{Z^2}(z) = \frac{1}{\sqrt{2\pi}} \cdot z^{-\frac{1}{2}} \cdot e^{-\frac{z}{2}} \cdot \mathbf{1}_{\{z>0\}},$$

$$\mathbb{P}(Z<z) = 1-\mathbb{P}(Z\geq z) = 1-\mathbb{P}(\min\{X,Y\}\geq z) = 1-\mathbb{P}(X\geq z \wedge Y\geq z)$$
$$= 1-\mathbb{P}(X\geq z)\,\mathbb{P}(Y\geq z) = 1-\frac{1}{2\pi}\left(\int_z^\infty e^{-s^2/2}\,ds\right)^2$$

$$f_Z(z) = -\frac{1}{\pi}\int_z^\infty e^{-\frac{s^2}{2}}\,ds\left(-e^{-\frac{z^2}{2}}\right) = \frac{1}{\pi}e^{-\frac{z^2}{2}}\int_z^\infty e^{-\frac{s^2}{2}}\,ds$$

$g:\mathbb{R}\setminus\{0\}\to\mathbb{R}^+ : z\mapsto z^2$ has got two right inverses, namely

$h_-:(0,\infty)\to(-\infty,0) : u\mapsto -\sqrt{u}$ and $h_+:(0,\infty)\to(0,\infty): u\mapsto\sqrt{u}$

$h_-'(u) = -\frac{1}{2}u^{-\frac{1}{2}}$ and $h_+(u) = \frac{1}{2}u^{-\frac{1}{2}}$, hence for all $u\in\mathbb{R}^+$, we obtain

$$f_{Z^2}(u) = f_Z(\sqrt{u})\frac{1}{2\sqrt{u}} + f_Z(-\sqrt{u})\frac{1}{2\sqrt{u}}$$
$$= \frac{1}{\pi}e^{-\frac{u}{2}}\int_{\sqrt{u}}^\infty e^{-\frac{s^2}{2}}\,ds\,\frac{1}{2\sqrt{u}} + \frac{1}{\pi}e^{-\frac{u}{2}}\int_{-\sqrt{u}}^\infty e^{-\frac{s^2}{2}}\,ds$$
$$= \frac{1}{\sqrt{2\pi}}u^{\frac{1}{2}}e^{-\frac{u}{2}}\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}\,ds = \frac{1}{\sqrt{2\pi}}u^{\frac{1}{2}}e^{-\frac{u}{2}}$$

Thus, $f_{Z^2}(u) = \frac{1}{\sqrt{2\pi}}u^{\frac{1}{2}}e^{-\frac{u}{2}}\,\mathbb{1}_{\mathbb{R}^+}(u)$

(4) **Random variables on the unit disk**
Let (X, Y) be uniformly distributed on the unit disk $\{f(x; y) : x^2 + y^2 \leq 1\}$. Let

$$R = \sqrt{X^2 + Y^2}.$$

Find the cdf, pdf, and the expectation the random variable R.

$$K := \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$$

$$1 \stackrel{!}{=} \int_K c \, d\lambda^2 = c \int_0^1 \int_0^{2\pi} r \, d\varphi \, dr = c\pi \Leftrightarrow c = \frac{1}{\pi}$$

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi}, & \text{if } (x,y) \in K \\ 0, & \text{otherwise} \end{cases}$$

$$h : \mathbb{R}^+ \times [0, 2\pi[\to \mathbb{R}^2 : (r, \varphi) \mapsto (r\cos(\varphi), r\sin(\varphi))$$

$$f_{R,\Phi}(r,\varphi) = f_{X,Y}(h(r,\varphi))\, r = \begin{cases} \frac{r}{\pi}, & \text{if } 0 \leq r \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_R(r) = \int_0^{2\pi} f_{R,\Phi}(r,\varphi) \, d\varphi \, \mathbb{1}_{[0,1]}(r) = 2r \, \mathbb{1}_{[0,1]}(r)$$

$$F_R(r) = \int_0^r f_R(\rho) \, d\rho = \begin{cases} 0, & \text{if } r \leq 0 \\ r^2, & \text{if } 0 < r < 1 \\ 1, & \text{otherwise} \end{cases}$$

$$\mathbb{E}(R) = \int_{-\infty}^{\infty} f_R(r)\, r \, dr = \int_0^1 2r^2 \, dr = \frac{2}{3}$$

(5) **Transformations**
Suppose X and Y are independent gamma distributed random variables with $X \sim Gamma(\alpha_1, \beta)$ and $Y \sim Gamma(\alpha_2, \beta)$. Consider the following two random variables

$$U = X + Y \qquad \text{and} \qquad V = \frac{X}{X+Y}.$$

(a) Show that $U \sim Gamma(\alpha_1 + \alpha_2, \beta)$.
(b) Show that U and V are also independent random variables.

$f_X(x) = \frac{1}{\Gamma(\alpha_1)\beta^{\alpha_1}} x^{\alpha_1 - 1} e^{-\frac{x}{\beta}} \mathbb{1}_{\mathbb{R}^+}(x)$ $\qquad \alpha_1, \alpha_2, \beta > 0$

$f_Y(y) = \frac{1}{\Gamma(\alpha_2)\beta^{\alpha_2}} y^{\alpha_2 - 1} e^{-\frac{y}{\beta}} \mathbb{1}_{\mathbb{R}^+}(y)$

a) As X and are independent, we have $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$g: \mathbb{R}^{+2} \to \mathbb{R}^+ \times (0,1) : (x,y) \mapsto (x+y, \frac{x}{x+y})$

$\forall x,y,u,v \in \mathbb{R}^+: \ (u,v) = (x+y, \frac{x}{x+y}) \Leftrightarrow u = x+y \wedge v = \frac{x}{x+y} \Leftrightarrow u = x+y \wedge v = \frac{x}{u} \Leftrightarrow y = u - x \wedge x = vu$
$\Leftrightarrow x = vu \wedge y = u(1-v)$

Hence, $h: \mathbb{R}^+ \times (0,1) \to \mathbb{R}^{+2}: (u,v) \mapsto (uv, u(1-v))$ is the inverse of g

$dh(u,v) = \begin{pmatrix} v & u \\ 1-v & -u \end{pmatrix}$, and $\det(dh(u,v)) = -uv - u(1-v) = -u$

$f_{U,V}(u,v) = f_{X,Y}(h(u,v)) \, |\det(dh(u,v))| = f_X(uv) f_Y(u(1-v)) u$

$C(\alpha) := \frac{1}{\Gamma(\alpha)\beta^\alpha}$

$f_U(u) = \int_{\mathbb{R}} f(u,v) dv = C(\alpha_1) C(\alpha_2) u^{(\alpha_1+\alpha_2)-1} \int_{\mathbb{R}} \mathbb{1}_{\mathbb{R}^+}(uv) \mathbb{1}_{\mathbb{R}^+}(u(1-v)) v^{\alpha_1 - 1} (1-v)^{\alpha_2 - 1} e^{-\frac{uv}{\beta}} e^{-\frac{u(1-v)}{\beta}} dv$

$e^{-\frac{uv}{\beta}} e^{-\frac{u(1-v)}{\beta}} = e^{-\frac{uv}{\beta}} e^{(\frac{uv}{\beta} - \frac{u}{\beta})} = e^{-\frac{u}{\beta}}$, and $uv > 0 \wedge u(1-v) > 0 \Leftrightarrow u > 0 \wedge 0 < v < 1$

$\int_0^1 v^{\alpha_1 - 1}(1-v)^{\alpha_2 - 1} dv = B(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}$, hence $\qquad \alpha := \alpha_1 + \alpha_2$

Finally, we obtain $f_U(u) = \frac{1}{\Gamma(\alpha_1+\alpha_2)\beta^{(\alpha_1+\alpha_2)}} u^{(\alpha_1+\alpha_2)-1} e^{-\frac{u}{\beta}} \mathbb{1}_{\mathbb{R}^+}(u)$

b) $f_V(v) = \int_{\mathbb{R}} f(u,v) du = C(\alpha_1) C(\alpha_2) v^{\alpha_1 - 1}(1-v)^{\alpha_2 - 1} \mathbb{1}_{(0,1)}(v) \int_{\mathbb{R}^+} u^{\alpha - 1} e^{-\frac{u}{\beta}} du$

$\frac{u}{\beta} = w \Leftrightarrow \frac{dw}{du} = \frac{1}{\beta} \Leftrightarrow du = \beta dw$

$\int_{\mathbb{R}^+} u^{\alpha-1} e^{-\frac{u}{\beta}} du = \int_{\mathbb{R}^+} (\beta w)^{\alpha - 1} e^{-w} \beta dw = \beta^\alpha \Gamma(\alpha)$

Thus, $f_V(v) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} v^{\alpha_1 - 1}(1-v)^{\alpha_2 - 1} \Gamma(\alpha) \mathbb{1}_{(0,1)}(v)$

$f_U(u) f_V(v) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} v^{\alpha_1 - 1}(1-v)^{\alpha_2 - 1} u^{\alpha_2 - 1} u^{\alpha_1 - 1} u \, \mathbb{1}_{(0,1)}(v) \mathbb{1}_{\mathbb{R}^+}(u) = f_{U,V}(u,v)$

Hence, U and V are independent!