Stat. 3. UE

① $(x,y) \xrightarrow{p_1} (x+1,y)$, $(x,y) \xrightarrow{p_2} (x,y+1)$ Start at $(0,0)$

a) $p_1 = \frac{1}{2} = p_2$. $\mathbb{P}(\text{will ever reach } (8,6)) = ?$

Because x and y are increased by 1 in every step, $(8,6)$ must be reached in the 14^{th} step.

Possible ways to reach $(8,6)$: $\binom{14}{8} = 3003$

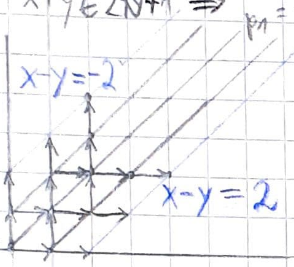
Possible ways with 14 steps: 2^{14}

$\Rightarrow \mathbb{P} = \frac{3003}{2^{14}} \approx \underline{18\,\%}$

b) $x+y \in 2\mathbb{N} \Rightarrow p_1 = \frac{2}{3},\ p_2 = \frac{1}{3}$

$x+y \in 2\mathbb{N}+1 \Rightarrow p_1 = \frac{1}{4},\ p_2 = \frac{3}{4}$

Stops whenever $|x-y| \geq 2$ $\qquad \mathbb{P}(x-y=2 \mid \text{it stopped}) = ?$



$$\mathbb{P}(\text{reaches } (x,x-2)) = \mathbb{P}(\text{reaches } (x,x-1)) \cdot \frac{1}{4} = \mathbb{P}(\text{reaches } (x,x)) \cdot \frac{2}{3} \cdot \frac{1}{4}$$

$$\mathbb{P}(\text{reaches } (x,x+2)) = \mathbb{P}(\text{reaches } (x,x+1)) \cdot \frac{3}{4} = \mathbb{P}(\text{reaches } (x,x)) \cdot \frac{1}{3} \cdot \frac{3}{4}$$

$$\mathbb{P} = \frac{\sum_{x=0}^{\infty} \mathbb{P}(\text{reaches } (x,x)) \cdot \frac{2}{3} \cdot \frac{1}{4}}{\sum_{x=0}^{\infty} \mathbb{P}(\text{reaches } (x,x)) \cdot \frac{1}{3} \frac{3}{4} + \sum_{x=0}^{\infty} \mathbb{P}(\text{reaches } (x,x)) \cdot \frac{2}{3} \cdot \frac{1}{4}} = \frac{\frac{2}{3} \cdot \frac{1}{4}}{\frac{1}{3} \cdot \frac{3}{4} + \frac{2}{3} \cdot \frac{1}{4}} = \underline{\underline{\frac{2}{5}}}$$

② X, Y random variables. $f(x,y) = \begin{cases} c(x+2y), & 0<y<1,\ 0<x<2 \\ 0, & \text{else.} \end{cases}$

a) $\rightsquigarrow \int_0^1 \int_0^2 x+2y \,dx\,dy = \int_0^1 4y+2 \,dy = 4 \Rightarrow c = \frac{1}{4}$

$\rightsquigarrow \int_0^2 x+2y \,dx = 2+4y \Rightarrow f(y) = \frac{1+2y}{2}$

b) $F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y) = \frac{1}{4} \int_0^y \int_0^x \xi + 2\eta \,d\xi\,d\eta = \frac{1}{4} \int_0^y \frac{x^2}{2} + 2\eta x \,d\eta = \left(\frac{x^2 y}{2} + y^2 x\right) \cdot \frac{1}{4}$

for $0<x<2,\ 0<y<1$.

c) $\rightsquigarrow \int_0^1 x+2y \,dy = x+1 \Rightarrow f(x) = \frac{x+1}{4}$

$\rightsquigarrow Z = \frac{9}{(x+1)^2}$: $\quad g: \begin{cases} [0,2] \to [1,9] \\ x \mapsto 9/(x+1)^2 \end{cases}$ has the inverse $h: \begin{cases} [1,9] \to [0,2] \\ y \mapsto \frac{3}{\sqrt{y}} - 1. \end{cases}$

h is continuously diff'able and so, by the theorem of the lecture, we have

$$f_Z(y) = f_X(h(y)) \cdot |h'(y)| = \frac{3}{4} y^{-\frac{1}{2}} \cdot \frac{3}{2} y^{-\frac{3}{2}} = \frac{9}{8} y^{-2} \quad \text{for } y \in [0,9].$$

3. $X, Y \sim \mathcal{N}(0,1)$ i.i.d., $Z = \min\{X, Y\}$. Show $f_{Z^2}(z) = \frac{1}{\sqrt{2\pi}} z^{-\frac{1}{2}} e^{-\frac{z}{2}} \mathbb{1}_{[z>0]}$.

- $$\begin{aligned}
\mathbb{P}(Z^2 \leq z) &= \mathbb{P}(Z \leq \sqrt{z}) - \mathbb{P}(Z < -\sqrt{z}) \\
&= \mathbb{P}(X \leq \sqrt{z})\,\mathbb{P}(Y \leq \sqrt{z}) - \mathbb{P}(X < -\sqrt{z})\,\mathbb{P}(Y < -\sqrt{z}) \\
&= \Phi(\sqrt{z})^2 - \Phi(-\sqrt{z})^2 \\
&= \underbrace{\left(\Phi(\sqrt{z}) + \Phi(-\sqrt{z})\right)}_{=1} \left(\Phi(\sqrt{z}) - \Phi(-\sqrt{z})\right) \\
&= \Phi(\sqrt{z}) - \Phi(-\sqrt{z})
\end{aligned}$$

- $\mathbb{P}(X^2 \leq x) = \Phi(\sqrt{x}) - \Phi(-\sqrt{x})$, so $Z^2 \sim X^2$, and therefore the result follows from what we already know from the lecture.

4\. Let (X,Y) uniformly distributed on $B_1(0)$ and $R=\sqrt{X^2+Y^2}$.

$f_{X,Y}(x,y)=\frac{1}{\pi}$. $\quad \mathbb{P}(R(x,y)\leq r)=\mathbb{P}(R^{-1}([0,r]))=\frac{r^2\pi}{\pi}=r^2,\ r\in[0,1] \Rightarrow F_R(r)=\begin{cases}0, & r<0\\ r^2, & r\in[0,1]\\ 1, & r>1.\end{cases}$

Because F_R is absolutely continuous, we get $f_R(r)=F_R'(r)=\begin{cases}0, & r<0\\ 2r, & r\in[0,1]\\ 0, & r>1\end{cases}$.

$$\mathbb{E}(R)=\int_{-\infty}^{\infty} r f_R(r)\,dr=\int_0^1 2r^2\,dr=\underline{\underline{\frac{2}{3}}}.$$

5. $X \sim \Gamma(\alpha_1, \beta)$, $Y \sim \Gamma(\alpha_2, \beta)$ independent, $U = X+Y$, $V = \frac{X}{X+Y}$.

(a) We have $f_X(x) = \frac{\beta^{\alpha_1} x^{\alpha_1 - 1} e^{-\beta x}}{\Gamma(\alpha_1)} \mathbb{1}_{x>0}$, $f_Y(y) = \frac{\beta^{\alpha_2} y^{\alpha_2-1} e^{-\beta y}}{\Gamma(\alpha_2)} \mathbb{1}_{y>0}$. $\left[\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx\right]$

Since X and Y are indep., we have $f_{X+Y}(z) = \int_0^\infty f_X(z-y) f_Y(y)\, dy$

$$= \mathbb{1}_{z>0} \int_0^z \frac{\beta^{\alpha_1}(z-y)^{\alpha_1-1} e^{-\beta(z-y)}}{\Gamma(\alpha_1)} \frac{\beta^{\alpha_2} y^{\alpha_2-1} e^{-\beta y}}{\Gamma(\alpha_2)}\, dy = \beta^{\alpha_1+\alpha_2} \frac{e^{-\beta z}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^z (z-y)^{\alpha_1-1} y^{\alpha_2-1}\, dy \quad \left|\begin{array}{l} t = y/z \\ dt = dy/z \\ dy = dt \cdot z \end{array}\right.$$

$$= \mathbb{1}_{z>0} \beta^{\alpha_1+\alpha_2} \frac{e^{-\beta z}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 z^{\alpha_1-1} (1-t)^{\alpha_1-1} z^{\alpha_2-1} t^{\alpha_2-1} z\, dt =$$

$$= \mathbb{1}_{z>0} \beta^{\alpha_1+\alpha_2} e^{-\beta z} z^{\alpha_1+\alpha_2-1} \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \underbrace{\int_0^1 (1-t)^{\alpha_1-1} t^{\alpha_2-1}\, dt}_{= B(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}}$$

$$= \mathbb{1}_{z>0} \beta^{\alpha_1+\alpha_2} e^{-\beta z} z^{\alpha_1+\alpha_2-1} \frac{1}{\Gamma(\alpha_1+\alpha_2)}$$

$$\Longrightarrow U \sim \Gamma(\alpha_1+\alpha_2, \beta).$$

(b) We know that $f_U \cdot f_V = f_{U,V} \rightarrow U, V$ independent.

Let $g: \binom{X}{Y} \mapsto \binom{X+Y}{X/(X+Y)}$, then $g^{-1}\binom{u}{v} = \binom{uv}{u-uv}$ and $J(u,v) = \begin{pmatrix} v & u \\ 1-v & -u \end{pmatrix}$,

By the change-of-variable theorem, we have $|J(u,v)| = u$.

$$f_{U,V}(u,v) = f_{X,Y}\binom{uv}{u-uv} u = f_X(uv) \cdot f_Y(u-uv) \cdot u$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (uv)^{\alpha_1-1} (u(1-v))^{\alpha_2-1} e^{-\beta(uv+u-uv)} \mathbb{1}_{(0,\infty)}(uv)\, \mathbb{1}_{(0,\infty)}(u(1-v)) \cdot u$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{\alpha_1+\alpha_2-1} v^{\alpha_1-1} (1-v)^{\alpha_2-1} e^{-\beta u}\, \mathbb{1}_{(0,\infty)}(u)\, \mathbb{1}_{(0,1)}(v)$$

$$= \underbrace{\left[\mathbb{1}_{(0,\infty)}(u) \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} e^{-\beta u} u^{\alpha_1+\alpha_2-1}\right]}_{\overset{(a)}{=} f_U(u)} \cdot \underbrace{\left[\mathbb{1}_{(0,1)}(v) v^{\alpha_1-1} (1-v)^{\alpha_2-1} \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\right]}_{\overset{*}{=} f_V(v)}. \quad \square$$

$$* \quad f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, du = \mathbb{1}_{(0,1)}(v)\, v^{\alpha_1-1} (1-v)^{\alpha_2-1} \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \underbrace{\int_0^\infty f_U(u)\, du}_{=1}$$