(4) **Confidence inerval 3**

Use R to generate a random sample $X_1, \ldots, X_n$ from $Pois(1)$ distribution (for $n = 30$ and $n = 100$). Compute the 90% condidence interval for λ, check if it contains the true value of $\lambda = 1$, and repeat this 10000 times. What is the fraction of simulations for which the confidence interval covers λ?

We saw in lecture 7 on slide 16 that the MLE is $\hat{\lambda} = \overline{X}$ with variance $\frac{\hat{\lambda}}{n}$. So we consider

$$1-\alpha = \mathbb{P}\left(\hat{\lambda}-\sigma \leq Z < \hat{\lambda}+\sigma\right) = \mathbb{P}\left(-\sigma \leq Z-\hat{\lambda} < \sigma\right) = \mathbb{P}\left(-\frac{\sigma\sqrt{n}}{\sqrt{\hat{\lambda}}} \leq \frac{\sqrt{n}(Z-\hat{\lambda})}{\sqrt{\hat{\lambda}}} < \frac{\sigma\sqrt{n}}{\sqrt{\hat{\lambda}}}\right)$$

$$= 1 - 2\Phi\left(-\frac{\sigma\sqrt{n}}{\sqrt{\hat{\lambda}}}\right) \Leftrightarrow \Phi\left(-\frac{\sigma\sqrt{n}}{\sqrt{\hat{\lambda}}}\right) = \frac{\alpha}{2} \Leftrightarrow \sigma = -\frac{\sqrt{\hat{\lambda}}}{\sqrt{n}}\,\Phi^{-1}\left(\frac{\alpha}{2}\right).$$

$\alpha = \frac{1}{10}$