

unknown

(1) X_1, \dots, X_n i.i.d. $\mathcal{N}(\mu, \sigma^2)$. Find GLRT for $H_0: \sigma^2 = \sigma_0^2$, $H_1: \sigma^2 \neq \sigma_0^2$.

Define $\Theta = \{(\mu, \sigma^2) \mid \mu \in \mathbb{R}, \sigma^2 > 0\}$, $\Theta_0 = \{(\mu, \sigma_0^2) \mid \mu \in \mathbb{R}\}$.

We have $L(\mu, \sigma^2 \mid x) = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2}$.

From lecture 6, we know that $\sup_{(\mu, \sigma^2) \in \Theta} L(\mu, \sigma^2 \mid x) = L(\hat{\mu}, \hat{\sigma}^2 \mid x)$ with $\hat{\mu} = \bar{x}$ and $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$.

Obviously, $\sup_{(\mu, \sigma^2) \in \Theta_0} L(\mu, \sigma^2 \mid x) = L(\bar{x}, \sigma_0^2 \mid x)$. Therefore,

$$\lambda(x) = \frac{\sup_{\Theta} L(\mu, \sigma^2 \mid x)}{\sup_{\Theta_0} L(\mu, \sigma^2 \mid x)} = \frac{(2\pi\hat{\sigma}^2)^{-\frac{n}{2}} e^{-\frac{1}{2\hat{\sigma}^2} n\hat{\sigma}^2}}{(2\pi\sigma_0^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma_0^2} n\hat{\sigma}^2}} = \left(\frac{\sigma_0^2}{\hat{\sigma}^2}\right)^{\frac{n}{2}} e^{\frac{n}{2} \left(\frac{\hat{\sigma}^2}{\sigma_0^2} - 1\right)}.$$

We define $f(x) = \left(\frac{1}{x}\right)^{\frac{n}{2}} e^{\frac{n}{2}(x-1)}$ and calculate with Maple: $f'(x) = 0 \Leftrightarrow x = 1$,

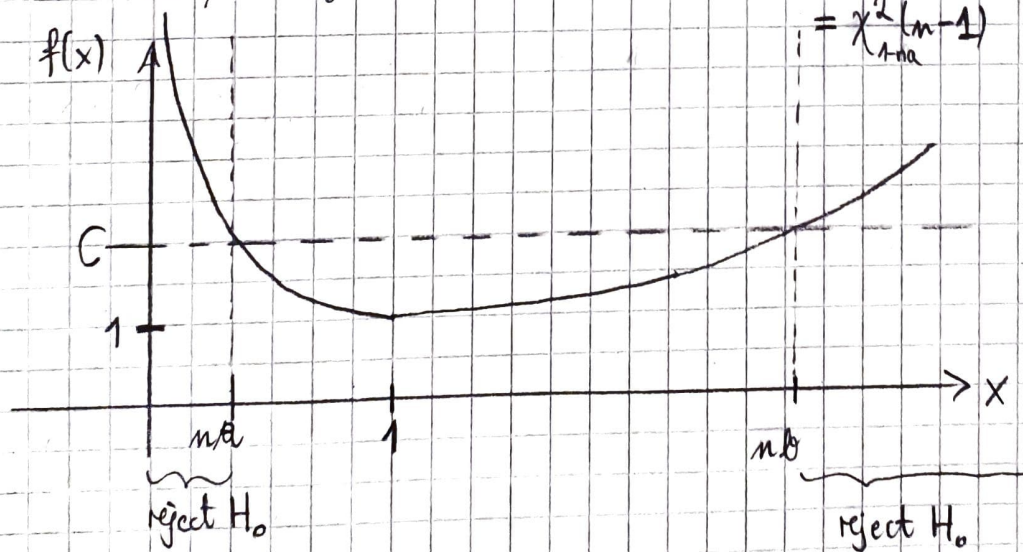
$f'(x) > 0 \Leftrightarrow x > 1$ and $f'(x) < 0 \Leftrightarrow x < 1$. Therefore, we will reject H_0

if $f\left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right) \geq c \Leftrightarrow \frac{\hat{\sigma}^2}{\sigma_0^2} \leq a$ or $\frac{\hat{\sigma}^2}{\sigma_0^2} \geq b \Leftrightarrow n\frac{\hat{\sigma}^2}{\sigma_0^2} \leq na$ or $n\frac{\hat{\sigma}^2}{\sigma_0^2} \geq nb$

for some constants a, b .

From lecture 5, we know that $n\frac{\hat{\sigma}^2}{\sigma_0^2} \sim \chi^2(n-1)$ if H_0 holds.

$$\text{We have } \alpha = \sup_{(\mu, \sigma^2) \in \Theta_0} P_{(\mu, \sigma^2)}(\lambda(x) \geq c) = \underbrace{P_{(\mu, \sigma_0^2)}\left(n\frac{\hat{\sigma}^2}{\sigma_0^2} \leq na\right)}_{= \chi^2_{na}(n-1)} + \underbrace{P_{(\mu, \sigma_0^2)}\left(n\frac{\hat{\sigma}^2}{\sigma_0^2} \geq nb\right)}_{= \chi^2_{nb}(n-1)}.$$



(2) Let X_1, \dots, X_n i.i.d. $\mathcal{N}(0, \vartheta)$. Derive MP test at level α for $H_0: \vartheta = \vartheta_0$ vs. $H_1: \vartheta = \vartheta_1 > \vartheta_0$.



We have $\mathbb{P}_{\vartheta_0}(\max X_i \geq z) = 1 - \mathbb{P}_{\vartheta_0}(\max X_i \leq z) \stackrel{\text{i.i.d.}}{=} 1 - \left(\frac{z}{\vartheta_0}\right)^n = \alpha \Leftrightarrow z = \vartheta_0 \sqrt[n]{1-\alpha}$

and claim that a MP test is given by $T(X) = \max x_i$ with rejection region $\tilde{\Omega}_\alpha = [z; \infty)$.

The power of this test is $\mathbb{P}_{\vartheta_1}(T(X) \geq z) = 1 - \mathbb{P}_{\vartheta_1}(T(X) \leq z) = 1 - \left(\frac{z}{\vartheta_1}\right)^n =$
 $= 1 - \left(\frac{\vartheta_0 \sqrt[n]{1-\alpha}}{\vartheta_1}\right)^n = 1 - \left(\frac{\vartheta_0}{\vartheta_1}\right)^n (1-\alpha).$

We consider another α -test $T_1(X)$ with rejection region $\tilde{\Omega}_\alpha$, then we have

$$1-\beta = \mathbb{P}_{\vartheta_1}(X \in \tilde{\Omega}_\alpha) = \int_{\tilde{\Omega}_\alpha} \left(\frac{1}{\vartheta_1}\right)^n \mathbb{1}_{[0, \vartheta_1]}(\max x_i) d\lambda^n(x)$$

$$= \left(\frac{\vartheta_0}{\vartheta_1}\right)^n \underbrace{\int_{\tilde{\Omega}_\alpha \cap [0, \vartheta_0]} \left(\frac{1}{\vartheta_0}\right)^n d\lambda^n(x)}_{\leq \mathbb{P}_{\vartheta_0}(X \in \tilde{\Omega}_\alpha) = \alpha} + \int_{\tilde{\Omega}_\alpha \setminus [0, \vartheta_0]} \left(\frac{1}{\vartheta_1}\right)^n \mathbb{1}_{[0, \vartheta_1]}(\max x_i) d\lambda^n(x)$$

$$\leq \left(\frac{\vartheta_0}{\vartheta_1}\right)^n \alpha + \int_{[0, \vartheta_1]} \left(\frac{1}{\vartheta_1}\right)^n \underbrace{\mathbb{1}_{[0, \vartheta_1]}(\max x_i)}_{=1} d\lambda^n(x) - \int_{[0, \vartheta_0]} \left(\frac{1}{\vartheta_1}\right)^n \underbrace{\mathbb{1}_{[0, \vartheta_1]}(\max x_i)}_{=1} d\lambda^n(x)$$

$$\leq \left(\frac{\vartheta_0}{\vartheta_1}\right)^n \alpha + 1 - \left(\frac{\vartheta_0}{\vartheta_1}\right)^n.$$

(4) X_1, \dots, X_n i.i.d. $\mathcal{N}(\mu, \sigma^2)$, μ known. $f_{\sigma^2}(x) = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2}$

(a) Find MP test at level α for testing $H_0: \sigma^2 = \sigma_0^2$, $H_1: \sigma^2 = \sigma_1^2$, $\sigma_1 > \sigma_0$.

The likelihood ratio is $\lambda(x) = \frac{f_{\sigma_1^2}(x)}{f_{\sigma_0^2}(x)} = \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{\frac{n}{2}} e^{\underbrace{\left(-\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_0^2}\right) \sum_{i=1}^n (x_i - \mu)^2}_{>0}}$,

which is a non-decreasing function in $T(X) := \sum_{i=1}^n (x_i - \mu)^2$.

We have $X_i \sim \mathcal{N}(\mu, \sigma^2) \Rightarrow X_i - \mu \sim \mathcal{N}(0, \sigma^2) \Rightarrow (X_i - \mu)^2 \sim \sigma^2 \chi^2(1) \Rightarrow T(X) \sim \sigma^2 \chi^2(n)$

and therefore $\alpha \stackrel{!}{=} \mathbb{P}_{\sigma_0^2}(T(X) \geq C) \Rightarrow C = \frac{1}{\sigma_0^2} \chi_\alpha^2(n).$

(b) Show that the MP test is a UMP test for $H_0: \sigma^2 \leq \sigma_0^2$, $H_1: \sigma^2 > \sigma_0^2$.

We have already observed that $\lambda(x)$ is non-decreasing in $T(X)$ if $\sigma_1^2 > \sigma_0^2$.

By the theorem from the lecture, our MP test is a UMP test for $\sigma^2 > \sigma_0^2$ vs. $\sigma^2 \leq \sigma_0^2$.

③ X_1, \dots, X_n i.i.d. with $f_{\nu}(x) = \frac{x}{\nu} e^{-\frac{x^2}{2\nu}}$, $x \geq 0$, $\nu > 0$.

(a) Derive MP test at level α for $H_0: \nu = \nu_0$, $H_1: \nu = \nu_1$, $\nu_1 > \nu_0$.

$$f_{\nu}(\vec{x}) = \prod_{i=1}^n \frac{x_i}{\nu} e^{-\frac{x_i^2}{2\nu}}$$

$$\lambda(x) = \frac{f_{\nu_1}(\vec{x})}{f_{\nu_0}(\vec{x})} = \frac{\prod_{i=1}^n \frac{x_i}{\nu_1} e^{-\frac{x_i^2}{2\nu_1}}}{\prod_{i=1}^n \frac{x_i}{\nu_0} e^{-\frac{x_i^2}{2\nu_0}}} = \left(\frac{\nu_0}{\nu_1}\right)^n e^{-\sum_{i=1}^n \frac{x_i^2}{2\nu_1} + \sum_{i=1}^n \frac{x_i^2}{2\nu_0}} = \left(\frac{\nu_0}{\nu_1}\right)^n e^{\overbrace{\left(\sum_{i=1}^n x_i^2\right) \cdot \left(\frac{1}{2\nu_0} - \frac{1}{2\nu_1}\right)}^{= T(X)}}$$

$$f_{X_i^2}(x) = f_{X_i}(\sqrt{x}) \frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{\nu} e^{-\frac{x}{2\nu}} \frac{1}{2\sqrt{x}} = \frac{1}{2\nu} e^{-\frac{x}{2\nu}} \Rightarrow X_i^2 \sim \exp\left(\frac{1}{2\nu}\right).$$

$$g(X_i) = X_i^2 \Rightarrow g^{-1}(y) = \sqrt{y} \Rightarrow \frac{d}{dy} g^{-1}(y) = \frac{1}{2\sqrt{y}}$$

$$\text{We know that } \sum_{i=1}^n X_i^2 \sim \Gamma\left(n, \frac{1}{2\nu}\right) = \Gamma\left(\frac{2n}{2}, \frac{1}{2\nu}\right) = \nu \Gamma\left(\frac{2n}{2}, \frac{1}{2}\right) = \nu \chi^2(2n).$$

$$\Rightarrow \frac{1}{\nu_0} \sum_{i=1}^n X_i^2 \sim \chi^2(2n). \quad \chi^2(\nu) = \Gamma\left(\frac{\nu}{2}, \frac{1}{2}\right)$$

$$\alpha = \mathbb{P}_{\nu_0} \left(\sum_{i=1}^n X_i^2 \geq C \right) = \mathbb{P}_{\nu_0} \left(\frac{1}{\nu_0} \sum_{i=1}^n X_i^2 \geq \frac{1}{\nu_0} C \right) \Rightarrow \frac{1}{\nu_0} C = \chi_{\alpha}^2(2n) \Rightarrow \underline{\underline{C = \nu_0 \chi_{\alpha}^2(2n)}}.$$

(b) Is there a UMP test at level α for testing $H_0: \nu \leq \nu_0$ vs. $H_1: \nu > \nu_0$?

We observe that the likelihood ratio $\lambda(x)$ is monotone in $T(X) = \sum_{i=1}^n X_i^2$, therefore

we have an UMP test by the theorem from the lecture.

$$\text{Power: } \pi(\nu) = \mathbb{P}_{\nu}(\text{reject } H_0) = \mathbb{P}_{\nu} \left(\sum_{i=1}^n X_i^2 \geq C \right) = \mathbb{P}_{\nu} \left(\frac{1}{\nu} \sum_{i=1}^n X_i^2 \geq \frac{C}{\nu} \right) = \chi_{\frac{C}{\nu}}^2(2n).$$