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Numerik von Differentialgleichungen - Kreuzlübung 8

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Exercise 36:
Construct an Adams-method with $k = 2$, $s = 1$, and $r = 0$, as well as variable step-sizes $h_j := t_j - t_{j-1}$ of the form

$$y_{i+1} = y_i + \sum_{j=0}^{2} \beta_{i,j}(h_{i-1}, h_i, h_{i+1}) f_{i-j}, \qquad f_{i-j} := f(t_{i-j}, y_{i-j}). \tag{1}$$

To obtain the constants $\beta_{i,j}$ you can use a computer algebra system (e.g., Maple). Compare your method to the one for equidistant step-sizes from the lecture notes.

Exercise 37:
For the construction of linear k-step methods for the differential equation $y'(t) = f(t, y(t))$ one can use the following approach: Let $\{t_0, \ldots, t_N\}$ be a mesh with uniform step-size h. Let the values $y_\ell, \ldots, y_{\ell+1-k}$ be given. For the computation of $y_{\ell+1}$ let $p_\ell \in \Pi_k$ be the polynomial which satisfies the conditions

$$p_\ell(t_{\ell+1-j}) = y_{\ell+1-j}, \qquad j = 0, \ldots, k, \tag{2a}$$

and

$$p'_\ell(t_{\ell+1}) = f\left(t_{\ell+1}, y_{\ell+1}\right). \tag{2b}$$

Then, we define $y_{\ell+1} := p_\ell(t_{\ell+1})$.

a) Write down the constants in equation (5.2) of the lecture notes for this method and show that they are independent of h. Which assumptions are necessary for the method to be well-defined?

b) Write down the constants for this method explicitly for $k = 1, 2, 3$.

c) Show that, for this method,

$$\tilde{\eta}_\ell(p, h) := \sum_{j=0}^{k} \left\{ h\beta_j p'(t_{\ell+1-k} + jh) - \alpha_j p(t_{\ell+1-k} + jh) \right\} \tag{3}$$

vanishes for all $p \in \Pi_k$. What is the order of convergence for this method?

Exercise 38:
Write a program which, for $k \in \mathbb{N}$, computes the explicit and implicit k-step method with maximum consistency order. Compute the explicit and implicit 2- and 3-step method with maximum consistency order explicitly.

Exercise 39:
Let

$$\rho(\lambda) := (\lambda - \lambda_1)(\lambda - \lambda_2)^2(\lambda - \lambda_3)^3 \tag{4}$$

with pair-wise different $\lambda_1, \lambda_2, \lambda_3$.

a) Write down all solutions for the corresponding difference equation explicitly.

b) Show that the solutions found in a) in fact are all solutions to the difference equation. To this end, conduct the steps in the proof of Theorem 5.25 of the lecture notes explicitly for this problem.

Exercise 40:
Implement an ODE-solver based on an arbitrary, explicit linear k-step method. Input parameters should be the coefficients $\alpha_0, \dots, \alpha_{k-1}$ and $\beta_0, \dots, \beta_{k-1}$, the step-size h, and the necessary data for the definition of the initial value problem. Test your program on some of the initial value problems which occured in the exercise sheets so far and the Adams-methods from the lecture notes. For the computation of the necessary initial values for the multistep method, use an appropriate RK-method. Also test Example 5.17 from the lecture notes.