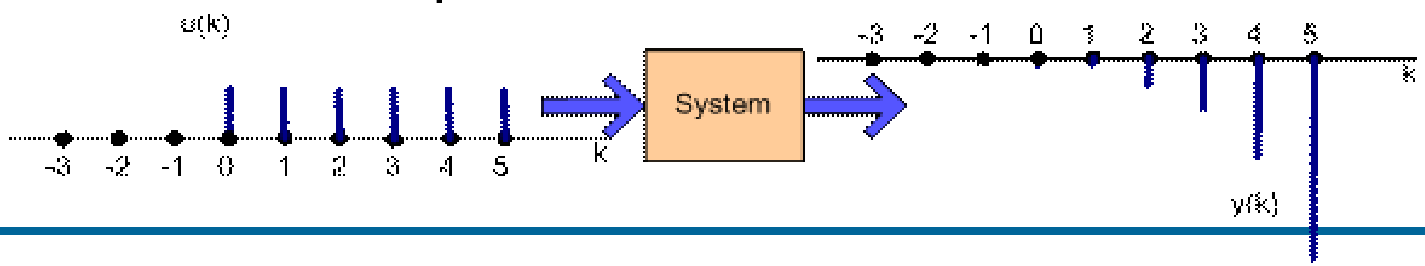
Discrete Modelling
Difference Equations

Part 1

Difference Equation

- equations involving differences of inputs and outputs
- three points of views
  - sequence of number
  - discrete dynamical system
  - iterated function

Difference equation - is a sequence of numbers that generated recursively using a rule to relate each number in the (output) sequence to previous (output) numbers and input numbers in the sequence.

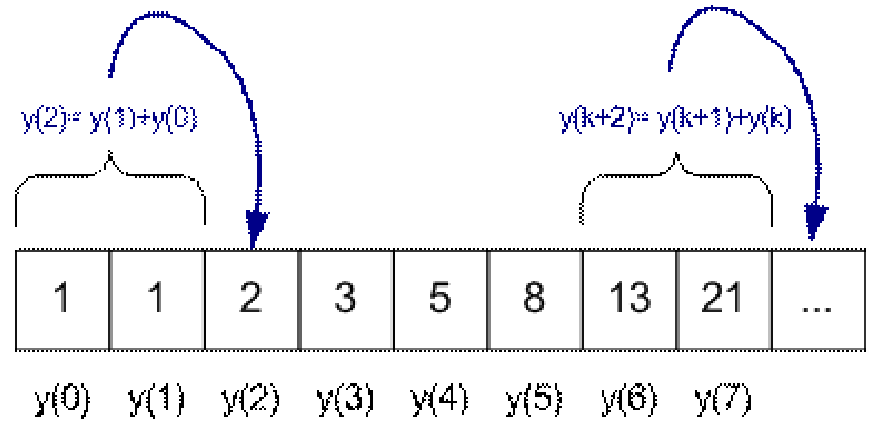


Difference Equations

Mathematical Modelling and Simulation

- **Fibonacci Sequence :**

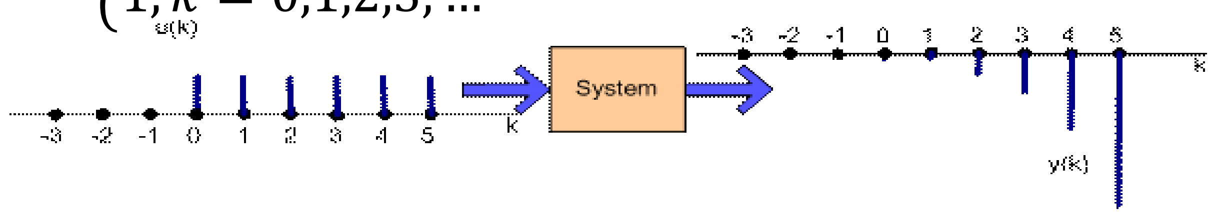
  $\{1,1,2,3,5,8,13,21,34\}$
  $y(k+2) = y(k+1) + y(k)$
  $y(0) = y(1) = 1, k = 0,1, \ldots$



- **Growth model**

- **Dynamical System with unit step input**

$$y(k) = 2y(k-1) + \frac{3}{2}u(k)$$
$$u(k) = \begin{cases} 0, k = -1,-2,-3,\ldots \\ 1, k = 0,1,2,3,\ldots \end{cases} \quad \Rightarrow \quad y(k) = \frac{3}{2}(1 - 2^{k+1})$$



Difference Equations

Mathematical Modelling and Simulation

- Iterated map $f(k)$
  $y(k+2) = f\big(y(k)\big), y(0) = y_0, k = 0,1,2,3, \dots$

  orbit $\{y_0, f(y_0), f\big(f(y_0)\big), f\Big(f\big(f(y_0)\big)\Big), \dots\}$
  dependent on y_0

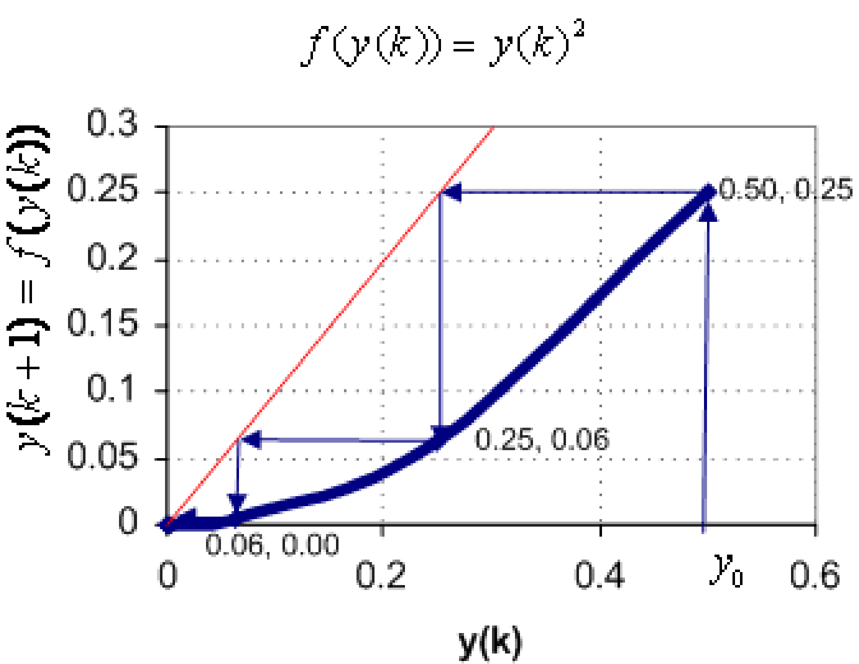
- Example: $y(k+1) = f\big(y(k)\big) \coloneqq y(k)^2, y(0) = y_0, k = 0,1,2,3 \dots$

$$y(0) = 1, \Rightarrow orbit\ \{1,1,1,1, \dots\}$$
$$y(0) = -1 \Rightarrow orbit\ \{-1,1,1,1, \dots\}$$
$$y(0) = 2 \Rightarrow \ \ orbit\ \ \{2,4,16,256,65536, \dots\}$$
$$y(0) = \frac{1}{2} \Rightarrow \ \ orbit\ \{0.5,0.25,0.0625, 0.00390625, \dots\}$$

Difference Equations

- Example $y(k+1) = f\big(y(k)\big) = y(k)^2, y(0) = y_0, k = 0,1,2,3$

$$y(0) = \frac{1}{2} \Rightarrow \quad orbit\ \{0.5, 0.25, 0.0625, 0.00390625, \dots\}$$



Cobweb Function:

$$\big(y(0), 0\big) \to \big(y(0), y(1)\big) \to$$
$$\to \big(y(1), y(1)\big) \to \big(y(1), y(2)\big) \to$$
$$\to \big(y(2), y(2)\big) \to \big(y(2), y(3)\big) \to$$
$$\to \big(y(3), y(3)\big) \to \big(y(3), y(4)\big) \to$$
$$\dots$$

„oscillates“ between $y = f(x)$ and $y = x$

Difference Equations

TU WIEN Mathematical Modelling and Simulation

- **Equlibria – Fixed Points**
  $y(k+2) = f\big(y(k)\big), y(0) = y_0, k = 0,1,2,3, \dots$
  $Equilibrium\ y^*: \quad y^* = f(y^*) \Leftrightarrow y(k+1) = f\big(y(k)\big) = y(k)$
- **Attractive/stable:** $y_0, y_1, y_2, y_3, \dots.\ converge\ to\ y^*$
- **Repelling/unstable:** $y_0, y_1, y_2, y_3, \dots.\ diverge\ from\ \ y^*$

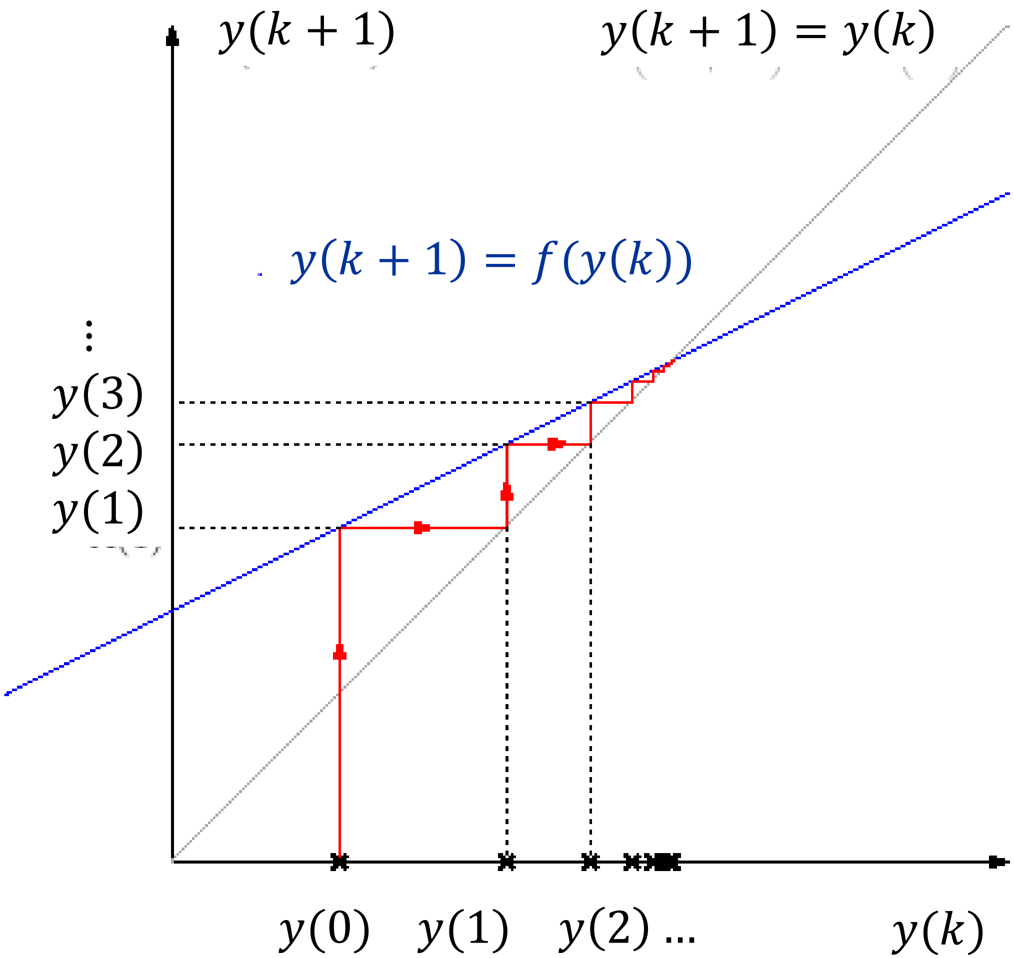
- **Graphic Test for stability / instability:**

  Cobweb-function stable/attractive:
  $(y(0), 0) \rightarrow \big(y(0), y(1)\big) \rightarrow (y(1), y(1)) \rightarrow \big(y(1), y(2)\big)$
  $\rightarrow (y(2), y(2)) \rightarrow \big(y(2), y(3)\big) \rightarrow \cdots \rightarrow (y*, y*)$

  Cobweb–function stable/attractive:
  $(y(0), 0) \rightarrow \big(y(0), y(1)\big) \rightarrow (y(1), y(1)) \rightarrow \big(y(1), y(2)\big)$
  $\rightarrow (y(2), y(2)) \rightarrow \big(y(2), y(3)\big) \rightarrow \cdots diverge$

Difference Equations

TU WIEN
Mathematical Modelling and Simulation



$$y(k+1) = \frac{4}{9}y(k) + \frac{7}{2}$$

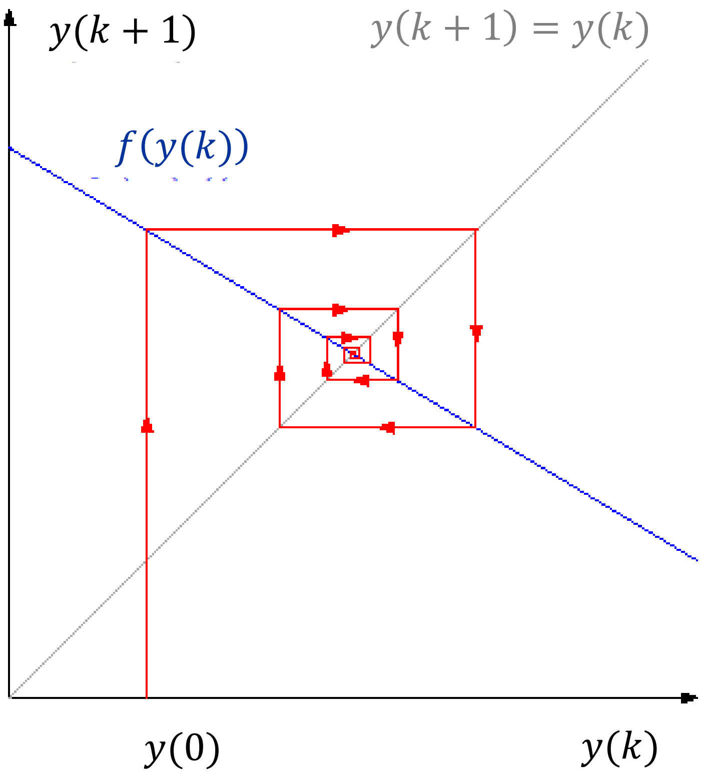
Cobweb Diagram

- Graphical technique to investigate iterated functions
- Iteration is performed graphically
- Consists of
  - Iterated Function $f(y)$
  - 1.Mediane $y(k+1) = y(k)$
  - Cobweb path

Cobweb Functions

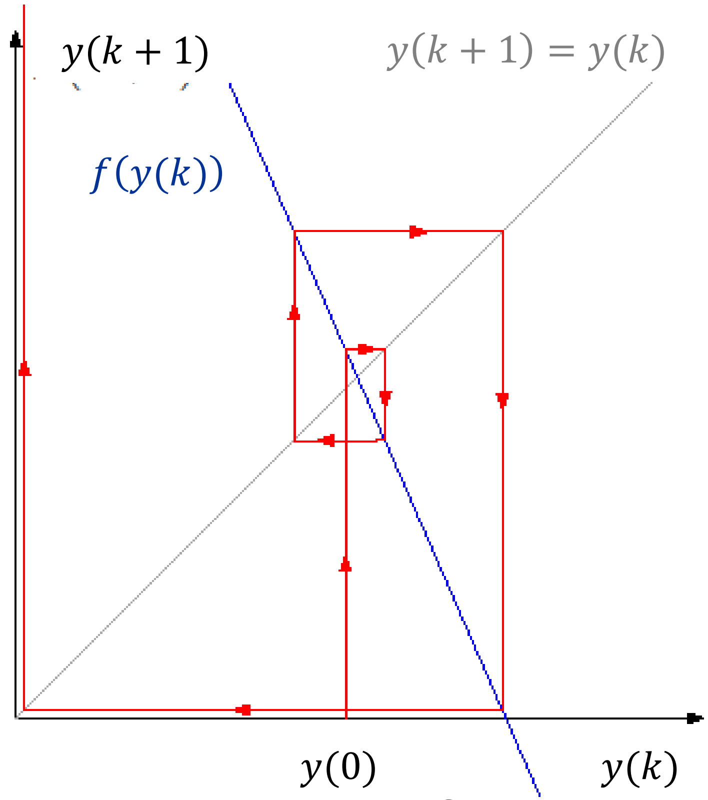
Mathematical Modelling and Simulation

$$y(k+1) = -0.6y(k) + 8$$



Inward spirals lead to attracting fixed points

$$y(k+1) = -3.5y(k) + 17.5$$



Outward spirals from repelling fixed points

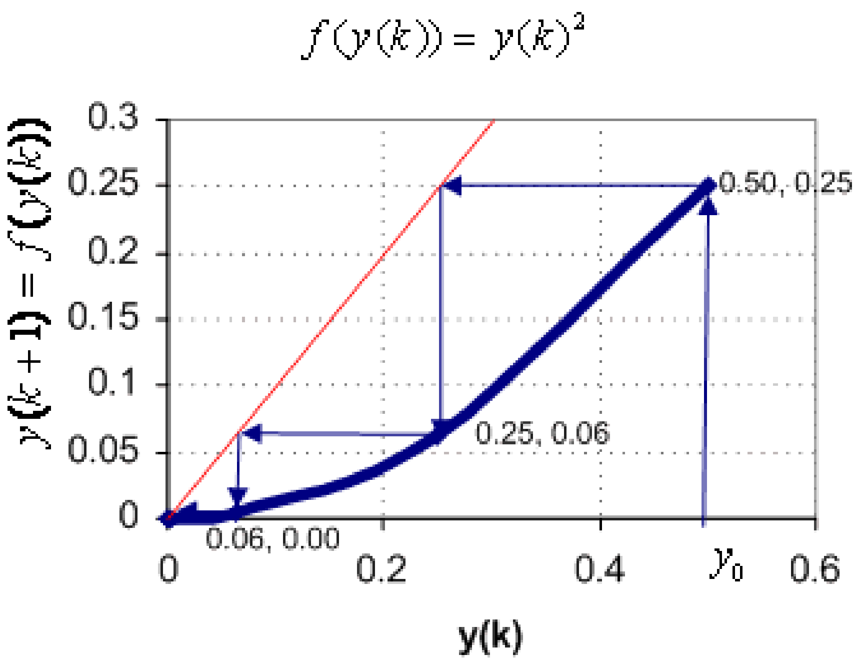
Cobweb Functions

Mathematical Modelling and Simulation

- **Example**

  $$y(k+1) = f\big(y(k)\big) := y(k)^2, y(0) = y_0, k = 0,1,2,3 \dots$$

  $$\Rightarrow Equilibria\ \ y^* = f(y^*) = {y^*}^2 \Rightarrow y^* \ \in \{0,1\}$$



Cobweb Function:

$$(y(0), 0) \to \big(y(0), y(1)\big) \to$$
$$\to (y(1), y(1)) \to \big(y(1), y(2)\big) \to$$
$$\to (y(2), y(2)) \to \big(y(2), y(3)\big) \to$$
$$\to (y(3), y(3)) \to \big(y(3), y(4)\big) \to$$
$$\dots \to (0,0)$$

attracts $y^* = 0$

Linear Affine Difference Equations

Mathematical Modelling and Simulation

$$y(k+1) = f\big(y(k)\big) = ay(k) + b, y(0) = y_0, k = 0,1,2, \dots$$

- Examples in Finance
  - Actual balance y(n)
    - after n compounding periods
    - with annual interest I
    - compounded m times a year
    - and constant amount b added at the end of every compounding period:

$$y(n+1) = \left(1 + \frac{I}{m}\right) y(n) + b$$

Linear Affine Difference Equations

$$y(k+1) = f\big(y(k)\big) = ay(k) + b, y(0) = y_0, k = 0,1,2, \dots$$

- **Solution**

  $y(1) = ay(0) + b = ay_0 + b$

  $y(2) = ay(1) + b = a(ay_0 + b) + b = a^2y_0 + ab + b$

  $y(3) = ay(2) + b = a(a^2y_0 + ab + b) + b$

  $= a^3y_0 + (a^2 + a + 1)b$

  …

  $$y(k) = a^ky_0 + \left(1 + a + a^2 + \cdots + a^{k-1}\right)b = a^ky_0 + b\sum_{i=0}^{k-1} a^i$$

Linear Affine Difference Equations

TU WIEN Mathematical Modelling and Simulation

$$y(k+1) = f\big(y(k)\big) = ay(k) + b, y(0) = y_0, k = 0,1,2,\dots$$

- **Solution**

$$y(k) = a^k y_0 + \left(1 + a + a^2 + \cdots + a^{k-1}\right)b = a^k y_0 + b\sum_{i=0}^{k-1} a^i$$

$\sum_{i=0}^{k-1} a^i$ geometric series for $a \neq 1$

$$\rightarrow \sum_{i=0}^{k-1} a^i = \frac{1-a^k}{1-a}$$

and for $a = 1 \rightarrow \sum_{i=0}^{k-1} a^i = \sum_{i=0}^{k-1} 1 = k$

- Hence

$$y(k) = \begin{cases} a^k y_0 + b\dfrac{1-a^k}{1-a}, & a \neq 1 \\ y_0 + kb, & a = 1 \end{cases}$$

Linear Affine Difference Equations

Mathematical Modelling and Simulation

$$y(k+1) = f\big(y(k)\big) = ay(k) + b, y(0) = y_0, k = 0,1,2, \ldots$$

- **Solution**

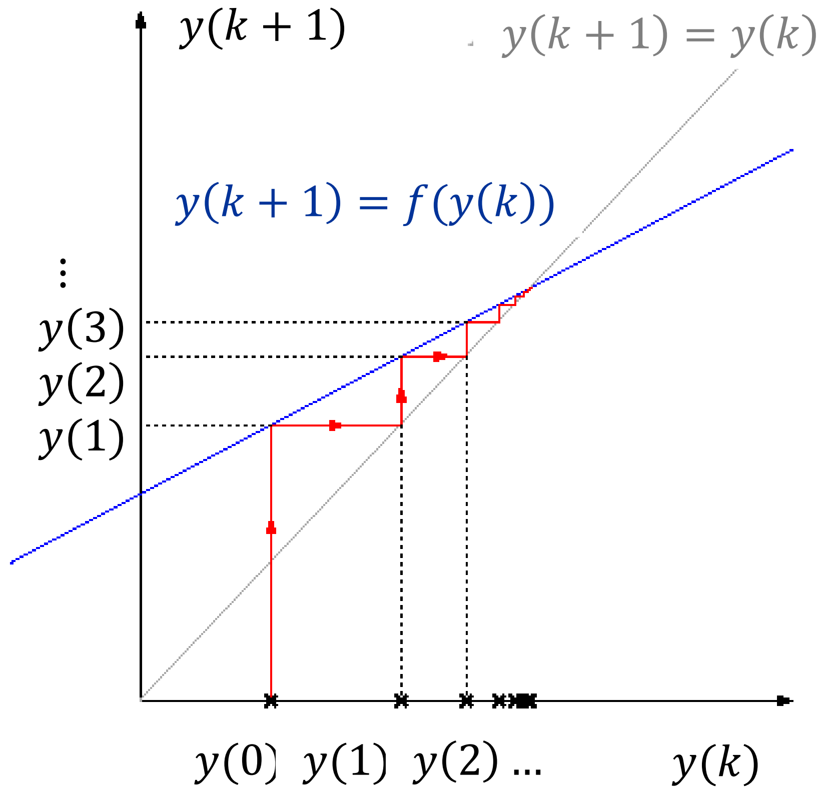
$$y(k) = \begin{cases} a^k y_0 + b\,\dfrac{1-a^k}{1-a}, a \neq 1 \\ y_0 + bk, \qquad a = 1 \end{cases}$$

Example:

$$y(k+1) = \frac{4}{9}y(k) + \frac{7}{2},$$

$$y(0) = 2.25 = \frac{9}{4}$$

$$y(k) = \frac{9}{4}\frac{4^k}{9^k} + \frac{7}{2}\frac{1-\frac{4^k}{9^k}}{1-\frac{4}{9}} = \frac{\left(7\cdot 3^{2k-2} - 2^{2n-1}\right)}{10\cdot 3^{2n-4}}$$



Linear Affine Difference Equations

$$y(k+1) = f\big(y(k)\big) = ay(k) + b, y(0) = y_0, k = 0,1,2, \ldots$$

- **Equilibrium / Fixed Point**
- $y^* = f(y^*) \leftrightarrow y^* = ay^* + b$

$$y^* = \frac{b}{1-a}, a \neq 1$$

  - Attractive/stable: $y_0, y_1, y_2, y_3, \ldots.\, converge\ to\ y^*$
  - Repelling/unstable: $y_0, y_1, y_2, y_3, \ldots.\, diverge\ from\ \ y^*$

- Solution with Equilibrium

$$y(k) = a^k y_0 + b\frac{1-a^k}{1-a} = a^k\left(y_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}$$

$$= a^k(y_0 - y^*) + y^*, a \neq 1$$

$$y(k) = y_0 + k, a = 1$$

Linear Affine Difference Equations

Mathematical Modelling and Simulation

$$y(k+1) = f\big(y(k)\big) = ay(k) + b, y(0) = y_0, k = 0,1,2, \dots$$

- **Solution**

$$y(k) = a^k y_0 + b\,\frac{1-a^k}{1-a} =$$

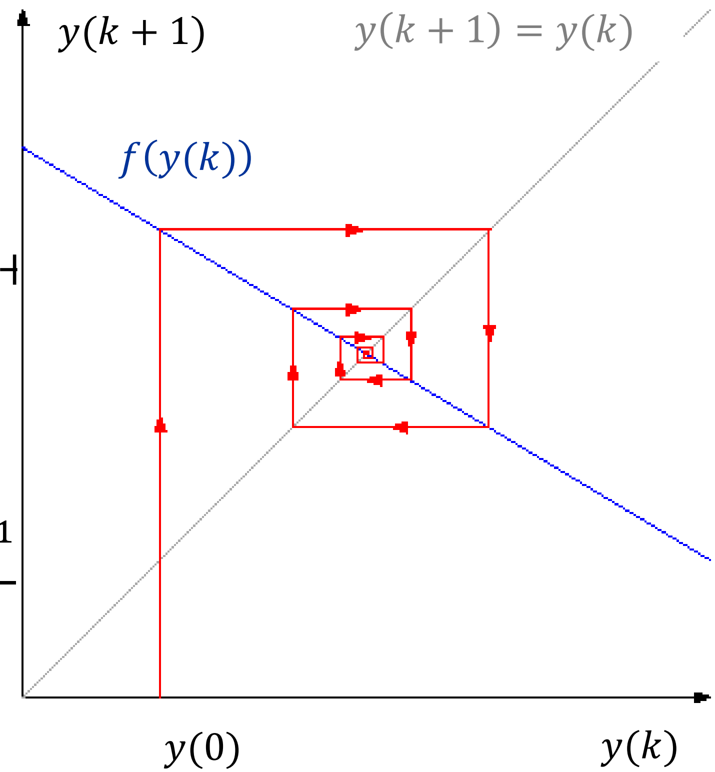
$$a^k(y_0 - y^*) + y^*, y^* = \frac{b}{1-a}, a \neq 1$$

- Example

$$y(k+1) = -0.6y(k) +$$

- $y^* = \frac{b}{1-a} = \frac{8}{1+0.6} = 5$
- $y(k) = \left(-\frac{3}{5}\right)^k (2-5) + 5 =$

$$\frac{(-1)^{k+1}3^{k+1}}{5^k}$$



Linear Affine Difference Equations

TU WIEN Mathematical Modelling and Simulation

$$y(k+1) = f\big(y(k)\big) = ay(k) + b, y(0) = y_0, k = 0,1,2, \dots$$

- **Solution**

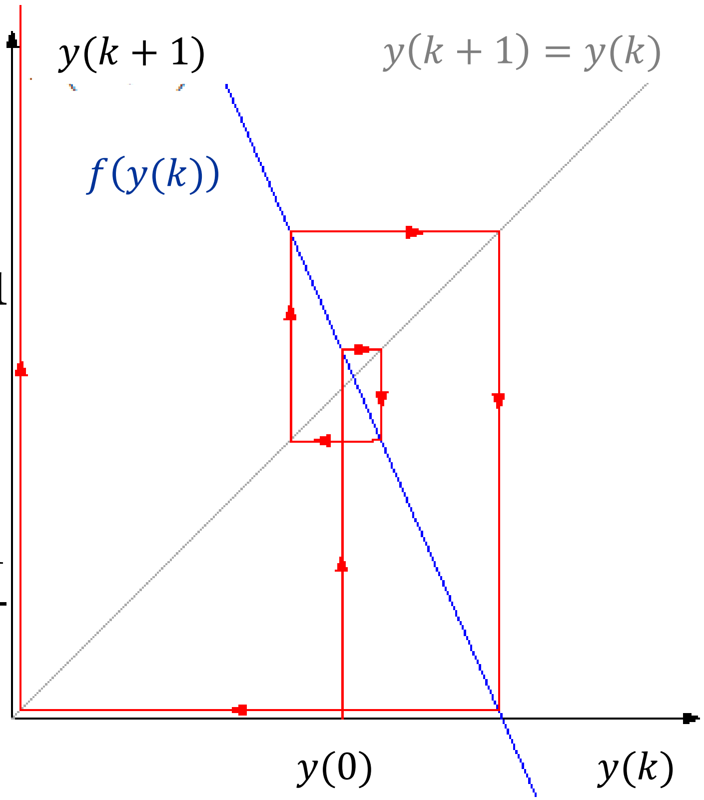
$$y(k) = a^k y_0 + b\frac{1-a^k}{1-a} =$$

$$a^k(y_0 - y^*) + y^*, y^* = \frac{b}{1-a}, a \neq 1$$

- **Example**

$$y(k+1) = -2.5y(k) + 1$$

- $y^* = \frac{b}{1-a} = \frac{17.5}{1+2.5} = 5$
- $y(k) = \left(-\frac{5}{2}\right)^k \left(\frac{24}{5} - 5\right) + 5 = \frac{(-1)^{k+1}5^{k-1}}{2^k}$



Linear Affine Difference Equations

$$y(k+1) = f\big(y(k)\big) = ay(k) + b, y(0) = y_0, k = 0,1,2,\ldots$$

- **Solution**

$$y(k) = a^k y_0 + b\,\frac{1-a^k}{1-a} =$$

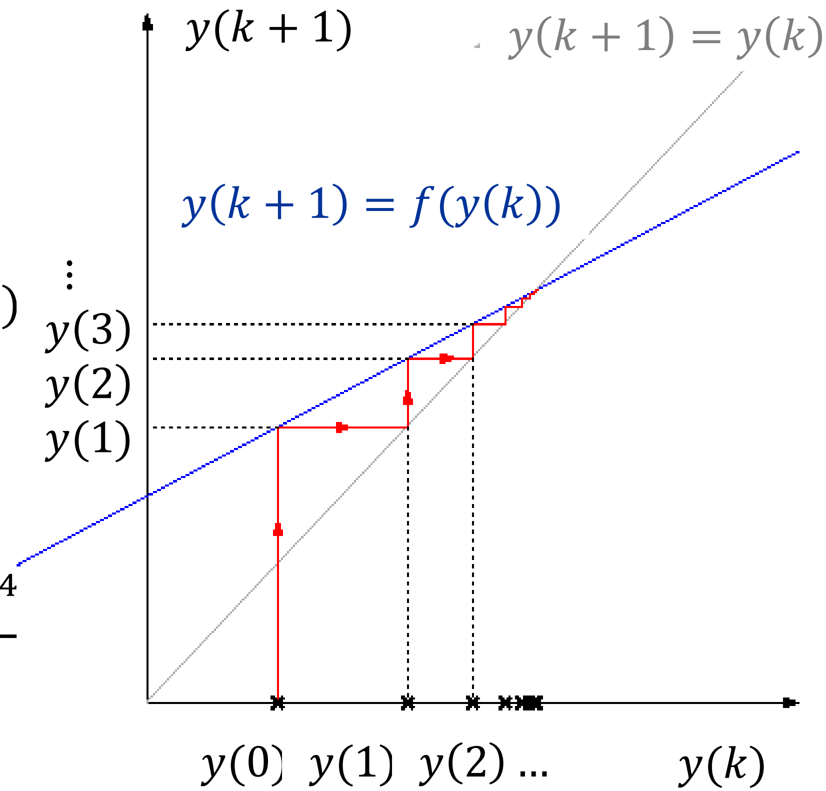
$$a^k(y_0 - y^*) + y^*, y^* = \frac{b}{1-a}, a \neq 1$$

- **Example**

$$y(k+1) = \frac{4}{9}y(k)$$

- $y^* = \frac{b}{1-a} = 6.3$
- $y(k) = \left(-\frac{4}{9}\right)^k \left(\frac{4}{9} - 6.3\right) + 6.3 =$

$$-\frac{2^{2k-2}3^4}{5}$$



Linear Affine Difference Equations

Mathematical Modelling and Simulation

$$y(k+1) = f\big(y(k)\big) = ay(k) + b, y(0) = y_0, k = 0,1,2, \ldots$$

- **Solution**

$$y(k) = a^k y_0 + b\frac{1-a^k}{1-a} =$$
$$a^k(y_0 - y^*) + y^*, y^* = \frac{b}{1-a}, a \neq 1$$

- **Equilibrium – Fixed Point**
  one (or no) fixed point

$$y^* = \frac{b}{1-a}, a \neq 1$$
$$y^* = y_0, a = 1, b = 0$$
$$no\ equilibrium\ for\ a = 1, b \neq 0$$

- **Stability:**

$$stable\ iff\ |a| < 1, \qquad y^*\ attracting$$
$$\text{unstable}\ iff\ |a| \geq 1, \qquad y^*\ repelling$$

Linear Affine Difference Equations

Mathematical Modelling and Simulation

$$y(k+1) = f\big(y(k)\big) = ay(k) + b, y(0) = y_0, k = 0,1,2, \ldots$$

- **Solution**

$$y(k) = a^k y_0 + b\frac{1-a^k}{1-a} =$$

$$a^k(y_0 - y^*) + y^*, y^* = \frac{b}{1-a}, a \neq 1$$

- Classification of Solutions
  Typ of solution depends on a, b and y_0

Main classification

1. $a > 1$
2. $a = 1$
3. $0 < a < 1$
4. $-1 < a < 0$
5. $a = -1$
6. $a < -1$

Sub-classification

1. $y_0 = \frac{b}{1-a}$
2. $y_0 > \frac{b}{1-a}$
3. $y_0 < \frac{b}{1-a}$

	Parameters	Solution Type
1	$a > 1, y_0 = y^*$	Constant
2	$a > 1, y_0 > y^*$	Exponentially increasing without bound
3	$a > 1, y_0 < y^*$	Exponentially decreasing without bound
4	$a = 1, \; b = 0$	Constant
5	$a = 1, \; b > 0$	Linearly increasing without bound
6	$a = 1, \; b < 0$	Linearly decreasing without bound
7	$0 < a < 1, y_0 = y^*$	Constant
8	$0 < a < 1, y_0 > y^*$	Exponentially decreasing to a bound
9	$0 < a < 1, y_0 < y^*$	Exponentially increasing to a bound
10	$-1 < a < 0, y_0 = y^*$	Constant
11	$-1 < a < 0, y_0 > y^*$	Oscillating with decreasing amplitude
12	$-1 < a < 0, y_0 < y^*$	Oscillating with decreasing amplitude
13	$a = -1, y_0 = b/2$	Constant
14	$a = -1, y_0 > b/2$	Oscillating with constant amplitude
15	$a = -1, y_0 < b/2$	Oscillating with constant amplitude
16	$a < -1, y_0 = y^*$	Constant
17	$a < -1, y_0 > y^*$	Oscillating with increasing amplitude
18	$a < -1, y_0 < y^*$	Oscillating with increasing amplitude

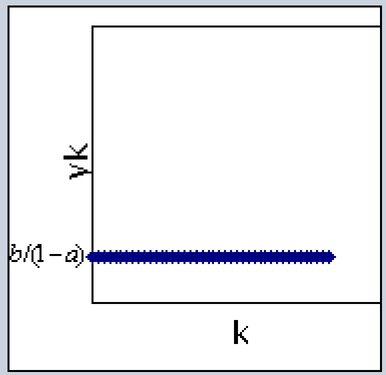
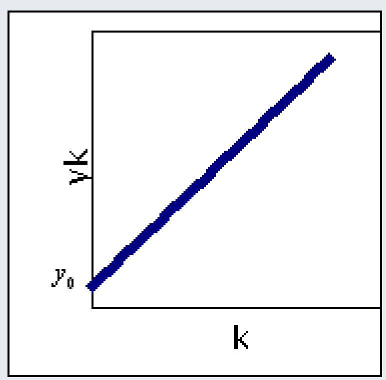
Linear Affine Difference Equations

$$y(k+1) = ay(k) + b$$
$$y(0) = y_0$$
$$y^* = \frac{b}{1-a}, a \neq 1$$

Linear Affine Difference Equations - Classification of Solutions

Mathematical Modelling and Simulation

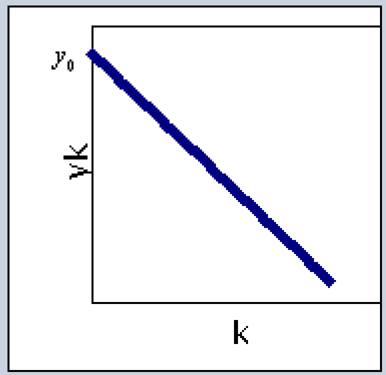
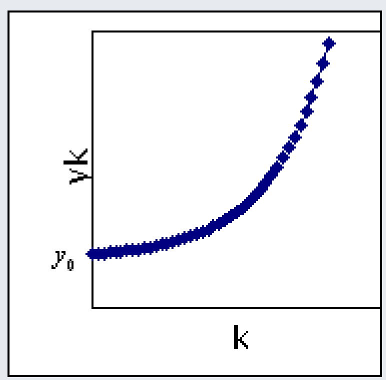
$$y(k+1) = ay(k) + b, y(0) = y_0, y^* = \frac{b}{1-a}, a \neq 1$$

No	Solution Type	Solution Sketch	Parameters
1	Constant		$a > 1, y_0 = y^*$ $a = 1,\ b = 0$ $0 < a < 1, y_0 = y^*$ $-1 < a < 0, y_0 = y^*$ $a = -1, y_0 = b/2$ $a < -1, y_0 = y^*$
2	Linearly increasing without bound		$a = 1,\ b > 0$

Linear Affine Difference Equations - Classification of Solutions

TU WIEN Mathematical Modelling and Simulation

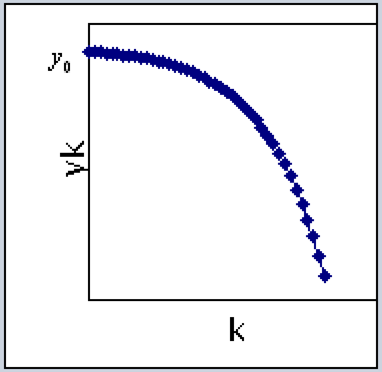
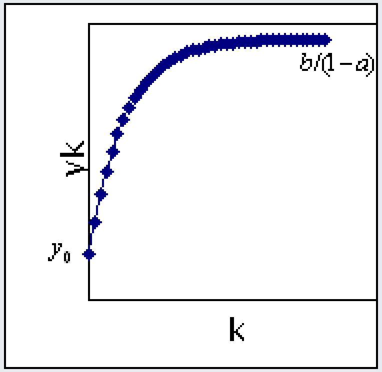
$$y(k+1) = ay(k) + b, y(0) = y_0, y^* = \frac{b}{1-a}, a \neq 1$$

No	Solution Type	Solution Sketch	Parameters
3	Linearly decreasing without bound		$a = 1, \; b < 0$
4	Exponentially increasing without bound		$a > 1, y_0 > y^*$

Linear Affine Difference Equations - Classification of Solutions

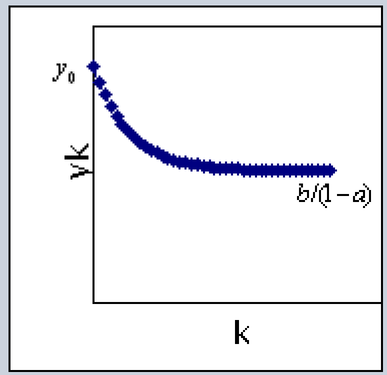
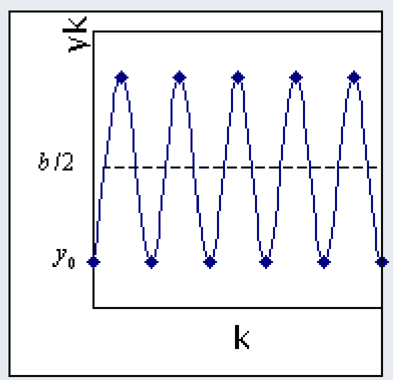
Mathematical Modelling and Simulation

$$y(k+1) = ay(k) + b, y(0) = y_0, y^* = \frac{b}{1-a}, a \neq 1$$

No	Solution Type	Solution Sketch	Parameters
5	Exponentially decreasing without bound		$a > 1, y_0 < y^*$
6	Exponentially increasing to a bound		$0 < a < 1, y_0 < y^*$

Linear Affine Difference Equations - Classification of Solutions

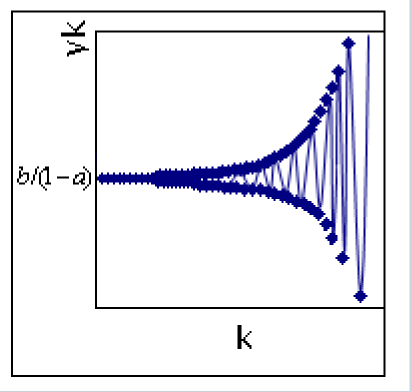
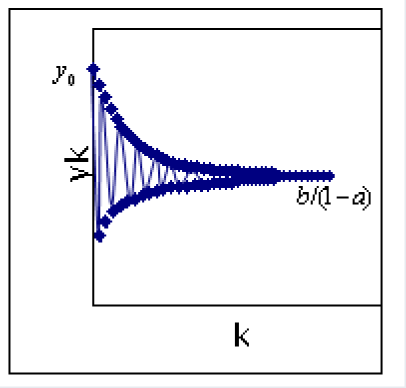
$$y(k+1) = ay(k) + b, y(0) = y_0, y^* = \frac{b}{1-a}, a \neq 1$$

No	Solution Type	Solution Sketch	Parameters
7	Exponentially decreasing to a bound		$0 < a < 1, y_0 > y^*$
8	Oscillating with constant amplitude		$a = -1, y_0 > b/2$ $a = -1, y_0 < b/2$

Linear Affine Difference Equations - Classification of Solutions

Mathematical Modelling and Simulation

$$y(k+1) = ay(k) + b, y(0) = y_0, y^* = \frac{b}{1-a}, a \neq 1$$

No	Solution Type	Solution Sketch	Parameters
9	Oscillating with increasing amplitude		$a < -1, y_0 > y^*$ $a < -1, y_0 < y^*$
10	Oscillating with decreasing amplitude		$-1 < a < 0, y_0 > y^*$ $-1 < a < 0, y_0 < y^*$

Applications to finance

- Actual balance $y(n)$ after n compounding periods with annual interest I, compounded m times a year and constant amount b added at the end of every compounding period:

$$y(k+1) = \left(1 + \frac{I}{m}\right) y(k) + b$$

Solution:

$$y^* = \frac{b}{1-a} = \frac{mb}{I}\ , y(k) = \left(1 + \frac{I}{m}\right)^k \left(y_0 - \frac{mb}{I}\right) + \frac{mb}{I}$$

Applications to economics

Mathematical Modelling and Simulation

- Supply and Demand
  - $S(n)$, $D(n)$, $P(n)$ ... supply, demand, price in the year n
  - Set of assumptions:

- $S(k+1) = sP(k) + a, a > 0$ s sensitivity of producers to price
- $D(k+1) = -dP(k+1) + b$ d sensitivity of consumers to price
- $S(k+1) = D(k+1)$ via adjustment of price/bargaining

first order affine dynamical system

$$\rightarrow -dP(k+1) + b = sP(k) + a$$

$$\rightarrow P(n+1) = -\frac{s}{d}P(n) + \frac{(b-a)}{d}, P^* = \frac{b-a}{d+s}$$

Applications to economics

- Supply and Demand
  - $S(n)$, $D(n)$, $P(n)$ ... supply, demand, price in the year n
  - Set of assumptions:

- $S(k+1) = sP(k) + a, a > 0$ s sensitivity of producers to price
- $D(k+1) = -dP(k+1) + b$ d sensitivity of consumers to price
- $S(k+1) = D(k+1)$ via adjustment of price/bargaining

first order affine dynamical system

$$\rightarrow -dP(k+1) + b = sP(k) + a$$

$$\rightarrow P(n+1) = -\frac{s}{d}P(n) + \frac{(b-a)}{d}, P^* = \frac{b-a}{d+s}$$

Applications to economics

- Supply and Demand
  - $S(n)$, $D(n)$, $P(n)$ ... supply, demand, price in the year n

- $P(k+1) = \frac{s}{d} P(k) + \frac{b-a}{d}$

first order affine dynamical system

- Fixed Point: $P^* = \frac{b-a}{s+d}$

- General Solution:

$$P(k) = c\left(-\frac{s}{d}\right)^k + p$$

stable for

$$-1 < -\frac{s}{d} < 1$$

Cobweb theorem of economics