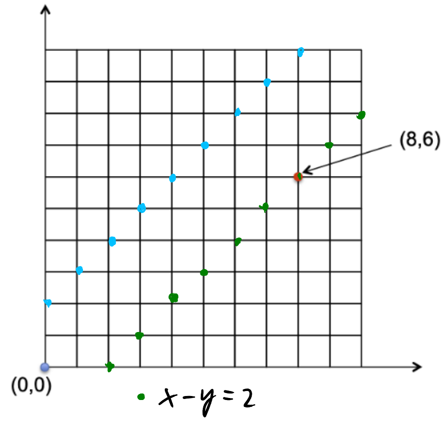
(1) **Random walk of a robot**
A robot is placed at the origin (the point $(0,0)$) on a two-dimension integer grid (see the figure below). Denote the position of the robot by $(x,y)$. The robot can either move right to $(x+1,y)$ or move up to $(x,y+1)$.




(a) Suppose each time the robot randomly moves right or up with equal chance. What is the probability that the robot will ever reach the point $(8,6)$?

(b) Suppose another robot has a $\frac{2}{3}$ chance to move right and a $\frac{1}{3}$ chance to move up when $x+y$ is even, otherwise it has a $\frac{1}{4}$ chance to move right and a $\frac{3}{4}$ chance to move up. It stops whenever $|x-y| \geq 2$. Find the probability that $x-y=2$ when it stops.

a) The sum $x+y$ is exactly the number of total moves made so far.
In order to reach the point $(8,6)$, it requires 14 moves in total, of which exactly 8 are to the right and 6 are up. The order in which these moves are made does not matter.
For all $i \in \mathbb{N}_0$ let $X_i$ be the x-coordinate after $i$ moves and $Y_i$ the y-coordinate after $i$ moves. Clearly, $X_i + Y_i = i$ for all $i \in \mathbb{N}_0$.

$$\mathbb{P}\left(\bigcup_{i=0}^{\infty}[(X_i,Y_i)=(8,6)]\right) = \mathbb{P}\left([(X_{14},Y_{14})=(8,6)]\right) = \mathbb{P}(X_{14}=8) = \binom{8}{14}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^{14-8} = \binom{8}{14}\left(\frac{1}{2}\right)^{14} \approx 0{,}183$$

b) $$\mathbb{P}\left(\bigcup_{i=1}^{\infty}[X_i - Y_i = 2]\right) = (pq) + \left(p(1-q)pq + (1-p)qpq\right) +$$
$$\left(p(1-q)p(1-q)pq + p(1-q)(1-p)q\,pq + (1-p)q\,p(1-q)pq + (1-p)q(1-p)q\,pq\right) + \dots$$

$p = \frac{2}{3}$

$q = \frac{1}{4}$

$$= pq\left(1 + \left(p(1-q)+(1-p)q\right) + \left(p^2(1-q)^2 + 2pq(1-p)(1-q) + (1-p)^2q^2\right) + \dots\right)$$
$$= pq\sum_{i=0}^{\infty}\left(p(1-q)+(1-p)q\right)^i = pq\,\frac{1}{1-p(1-q)-(1-p)q}$$
$$= \frac{2}{3}\cdot\frac{1}{4}\cdot\frac{1}{1-\frac{2}{3}\cdot\frac{3}{4}-\frac{1}{3}\cdot\frac{1}{4}} = \frac{1}{6}\cdot\frac{1}{1-\frac{1}{2}-\frac{1}{12}} = \frac{1}{6}\cdot\frac{12}{5} = \frac{2}{5}$$

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A bit more formal: $R$ ... roboter gets to position ; $x \in \mathbb{N}_0$

$$\mathbb{P}(R=(x,x+2)) = q\,\mathbb{P}(R=(x,x+1)) = q\,p\,\mathbb{P}(R=(x,x)) = q\,p\,\mathbb{P}(R=(x-1,x-1))\left(p(1-q)+(1-p)q\right)$$

$$\mathbb{P}(R=(0,0)) = 1$$

Hence, we have a recursive formula that can easily be made explicit:
$\mathbb{P}(R=(x,x)) = (p(1-q)+(1-p)q)^x$, hence, just as above, we obtain the sum

$$\sum_{i=0}^{\infty}\mathbb{P}(R=(i,i+2)) = pq\sum_{i=0}^{\infty}\left(p(1-p)+(1-p)q\right)^i = \frac{2}{5}$$