(5) **Central Limit Theorem**
Let $\bar{X}_1$ and $\bar{X}_2$ be the means of two independent samples of size n from the same population with variance σ^2. Use the Central limit theorem to find a value for n so that

$$P(|\bar{X}_1 - \bar{X}_2| < \frac{\sigma}{50}) \approx 0.99.$$

Justify your calculations.

We rename $\overline{Y_n} := \overline{X_1}$ and $\overline{Z_n} := \overline{X_2}$

$$\mathbb{E}(\overline{Y_n} - \overline{Z_n}) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i - \frac{1}{n}\sum_{i=1}^{n} Z_i\right) = \frac{1}{n}\sum_{i=1}^{n}\left(\mathbb{E}(Y_i) - \mathbb{E}(Z_i)\right) = 0$$

$$\mathbb{V}(\overline{Y_n} - \overline{Z_n}) = \mathbb{V}(\overline{Y_n}) + \mathbb{V}(-\overline{Z_n}) = \frac{1}{n^2}\sum_{i=1}^{n}\left(\mathbb{V}(Y_i) + \mathbb{V}(Z_i)\right) = \frac{2\sigma^2}{n}$$

Chebyshev's inequality (13.14 in Kusolitsch) says that

$$\mathbb{P}(|(\overline{Y_n} - \overline{Z_n}) - \underbrace{\mu}_{=0}| \geq \frac{\sigma}{50}) \leq \frac{\mathbb{V}(\overline{Y_n} - \overline{Z_n})}{\left(\frac{\sigma}{50}\right)^2} = \frac{\left(\frac{2\sigma}{n}\right)^2}{\left(\frac{\sigma}{50}\right)^2} = \frac{100^2}{n^2}, \text{ hence}$$

$$\mathbb{P}(|\overline{Y_n} - \overline{Z_n}| < \frac{\sigma}{50}) \geq 1 - \frac{100^2}{n^2} \stackrel{!}{=} \frac{99}{100} \Leftrightarrow \frac{1}{100} = \frac{100^2}{n^2} \Leftrightarrow n^2 = 100^3 \Leftrightarrow n = 100^{\frac{3}{2}} = \underline{\underline{1000}}$$