(1) **Method of moment estimator**
Let $X_1, \dots, X_n$ be a random sample from a population with pdf

$$f(x) = \begin{cases} \frac{\theta x^{\theta-1}}{3^\theta}, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

where $\theta \in \mathbb{R}^+$ is unknown parameter.

(a) Show that the method of moments estimator for θ is $T_n = \frac{\bar{X}}{3-\bar{X}}$.

(b) Find the limiting distribution of $\frac{T_n - \theta}{\frac{1}{\sqrt{n}}}$ as $n \to \infty$.

a) $\mu(\theta) = \int_0^3 \frac{\theta x^{\theta-1}}{3^\theta} x \, dx = \theta 3^{-\theta} \left[\frac{x^{\theta+1}}{\theta+1} \right]_{x=0}^{3} = \frac{\theta}{\theta+1} 3 \overset{!}{=} \bar{X} \Leftrightarrow 3\theta = \bar{X}\theta + \bar{X} \Leftrightarrow \theta(3-\bar{X}) = \bar{X}$

Thus, $T_n = \frac{\bar{X}}{3-\bar{X}}$.

b) $g:]0,3[\longrightarrow \mathbb{R}: y \mapsto \frac{y}{3-y}$; $\quad g'(y) = (3-y)^{-1} + y(3-y)^{-2} = \frac{3-y+y}{(3-y)^2} = \frac{3}{(3-y)^2}$

$(\sigma(\theta))^2 - (\mu(\theta))^2 = \int_0^3 \frac{\theta x^{\theta-1}}{3^\theta} x^2 \, dx = \theta 3^{-\theta} \left[\frac{x^{\theta+2}}{\theta+2} \right]_0^3 = \frac{9\theta}{\theta+2} \Rightarrow (\sigma(\theta))^2 = \frac{9\theta}{\theta+2} - \frac{9\theta^2}{(\theta+1)^2}$

$\Rightarrow (\sigma(\theta))^2 = \left((\theta+2)(\theta+1)^2\right)^{-1} \left(9\theta(\theta+1)^2 - 9\theta^2(\theta+2) \right)$

$= \left((\theta+2)(\theta+1)^2\right)^{-1} \left(9\theta^3 + 18\theta^2 + 9\theta - 9\theta^3 - 18\theta^2 \right) = 9\theta \left((\theta+2)(\theta+1)^2 \right)$

By CLT, we have $\sqrt{n}\left(\bar{X} - \frac{3\theta}{\theta+1}\right) \xrightarrow{d} Y \sim \mathcal{N}(0, (\sigma(\theta))^2) = \mathcal{N}\left(0, \frac{9\theta}{(\theta+2)(\theta+1)^2}\right)$.

$g\left(\frac{3\theta}{\theta+1}\right) = \frac{3\theta / \cancel{\theta+1}}{3 - \frac{3\theta}{\cancel{\theta+1}}} = \frac{3\theta}{3(\theta+1) - 3\theta} = \frac{3\theta}{3\theta + 3 - 3\theta} = \theta$

We apply the delta method and obtain

$\sqrt{n}(T_n - \theta) = \sqrt{n}\left(g(\bar{X}) - g\left(\frac{3\theta}{\theta+1}\right)\right) \longrightarrow \mathcal{N}\left(0, (\sigma(\theta))^2 \left(g'\left(\frac{3\theta}{\theta+1}\right)\right)^2\right)$

$g'\left(\frac{3\theta}{\theta+1}\right) = \frac{3}{\left(3 - \frac{3\theta}{\theta+1}\right)^2} = \frac{3(\theta+1)^2}{(3(\theta+1) - 3\theta)^2} = \frac{(\theta+1)^2}{3}$

$\Rightarrow (\sigma(\theta))^2 \left(g'\left(\frac{3\theta}{\theta+1}\right)\right)^2 = \frac{9\theta}{(\theta+2)(\theta+1)^2} \frac{(\theta+1)^4}{9} = \frac{\theta(\theta+1)^2}{\theta+2}$

Thus, $\sqrt{n}(T_n - \theta) \longrightarrow \mathcal{N}\left(0, \frac{\theta(\theta+1)}{\theta+2}\right)$

(2) **Box of candles**

There are blue and red candles in a box. Probability that a randomly chosen candle is blue is $\frac{1}{1+2a}$, for $a > 0$. Based on a sample of sample size n, find the maximum likelihood estimator (MLE) $\hat{a}$ of the parameter a.

$$p(a) = (1+2a)^{-1} \Rightarrow p'(a) = -2(1+2a)^{-2}$$

For $x \in \{0,1\}^n$:

$$L(a|x) = \prod_{i=1}^{n} (p(a))^{x_i} (1-p(a))^{1-x_i} = (p(a))^{\sum_{i=1}^{n} x_i} (1-p(a))^{\left(n-\sum_{i=1}^{n} x_i\right)}, \text{ hence}$$

$$\ell(a|x) = \log(p(a)) \sum_{i=1}^{n} x_i + \left(n - \sum_{i=1}^{n} x_i\right) \log(1-p(a))$$

$$\ell'(\hat{a}|x) = \frac{1}{p(\hat{a})} p'(\hat{a}) \sum_{i=1}^{n} x_i + \left(n - \sum_{i=1}^{n} x_i\right) \frac{1}{1-p(\hat{a})} \left(-p'(\hat{a})\right)$$

$$= -2(1+2\hat{a})^{-1} \sum_{i=1}^{n} x_i + \left(n - \sum_{i=1}^{n} x_i\right) \left(1-(1+2\hat{a})^{-1}\right)^{-1} 2(1+2\hat{a})^{-2}$$

$$= -2(1+2\hat{a})^{-1} \sum_{i=1}^{n} x_i + \left(n - \sum_{i=1}^{n} x_i\right)(2\hat{a})^{-1}(1+2\hat{a})\, 2(1+2\hat{a})^{-2}$$

$$= \left((1+2\hat{a})\hat{a}\right)^{-1} \left(-2\hat{a} \sum_{i=1}^{n} x_i + \left(n - \sum_{i=1}^{n} x_i\right)\right) \overset{!}{=} 0$$

$$\Leftrightarrow -2\hat{a} \sum_{i=1}^{n} x_i + n - \sum_{i=1}^{n} x_i = 0 \Leftrightarrow \hat{a} = \frac{1}{2}\left(n\left(\sum_{i=1}^{n} x_i\right) - 1\right)$$

$$\ell''(\hat{a}|x) = 4(1+2\hat{a})^{-2} \sum_{i=1}^{n} x_i + \left(n - \sum_{i=1}^{n} x_i\right)\left((-1)(\hat{a})^{-2}(1+2\hat{a})^{-1} - \hat{a}^{-1}(1+2\hat{a})^{-2}\right)$$

$$= \left((1+2\hat{a})\hat{a}\right)^{-2} \left(4\hat{a}^2 \sum_{i=1}^{n} x_i - \left(n - \sum_{i=1}^{n} x_i\right)\left((1+2\hat{a}) + \hat{a}\right)\right.$$

(2) **Box of candles**
There are blue and red candles in a box. Probability that a randomly chosen candle is blue is $\frac{1}{1+2a}$, for $a > 0$. Based on a sample of sample size n, find the maximum likelihood estimator (MLE) $\hat{a}$ of the parameter a.

Für $x \in \{0,1\}^n$: $\quad S_n := \sum_{i=1}^{n} x_i$

$$L_n(a) = \prod_{i=1}^{n} \left(\frac{1}{1+2a}\right)^{x_i} \left(1 - \frac{1}{1+2a}\right)^{1-x_i} = \left(\frac{1}{1+2a}\right)^{\sum_{i=1}^n x_i} \left(\frac{2a}{1+2a}\right)^{n - \sum_{i=1}^n x_i}$$

$$\ell_n(a) = -S_n \log(1+2a) + (n-S_n)\left(\log(2a) - \log(1+2a)\right) = \log(2a)\,(n-S_n) - n\log(1+2a)$$

$$\ell_n'(a) = \frac{n-S_n}{a} - \frac{2n}{1+2a} \overset{!}{=} 0 \Leftrightarrow (n-S_n)(1+2a) = 2na \Leftrightarrow n + 2na - S_n - 2aS_n = 2na$$
$$\Leftrightarrow n - S_n = 2aS_n \Leftrightarrow a = \frac{n-S_n}{2S_n}$$

$0 < S_n < n$

$$\ell_n''(a) = -\frac{n-S_n}{a^2} + \frac{4n}{(1+2a)^2}$$

$$\ell_n''\left(\frac{n-S_n}{2S_n}\right) = -\frac{4S_n^2}{n-S_n} + 4n\left(\frac{S_n + n - S_n}{S_n}\right)^{-2} = \frac{4S_n^2}{n} - \frac{4S_n^2}{n-S_n} = 4S_n^2\left(\frac{1}{n} - \frac{1}{n-S_n}\right) < 0$$

If $S_n = 0$, then $L_n(a) = \left(\frac{2a}{1+2a}\right)^n$ is infinitely increasing, hence „$\hat{a} = \infty$"

If $S_n = n$, then $L_n(a) = \left(\frac{1}{1+2a}\right)^n$ is decreasing, hence $\hat{a} = 0$

In general: $\hat{a} = \frac{n-S_n}{2S_n}$

(3) **Point estimator statistics: Comparison**
Let $X_1 \dots X_n$ be i.i.d. uniform $(0, \theta)$, with unknown parameter $\theta > 0$.

(a) Show that the method of moments estimator of θ is $2\bar{X}$ and the MLE of θ is $X_{(n)} = \max_{1\le i\le n} X_i$.

(b) Compare the mean square errors of the two estimators. Which of the estimators should be preferred if any? Explain your reasoning.

a) $\mu(\theta) = \int_0^\theta \frac{1}{\theta} x \, dx = \frac{1}{\theta} \frac{\theta^2}{2} = \frac{\theta}{2} \overset{!}{=} \bar{X} \Leftrightarrow \theta = 2\bar{X}$ ... method of moments estimator

For $x \in [0, \theta]^n$:

$$L_n(\theta) = \prod_{i=1}^n \frac{1}{\theta} = \frac{1}{\theta^n},$$

For $\theta_1, \theta_2 \in [\max\{X_i \mid 1 \le i \le n\}, \infty)$: $L_n(\theta_1) > L_n(\theta_2) \Leftrightarrow \frac{1}{\theta_1^n} > \frac{1}{\theta_2^n} \Leftrightarrow \theta_2 > \theta_1$, hence $L_n(\theta)$ is decreasing. Therefore, it has it's maximum at $\theta = \max\{X_i \mid 1 \le i \le n\}$, which is, consequently, the MLE.

b) $$2\bar{X} - \theta = \frac{2}{n}\sum_{i=1}^n X_i - \frac{2}{n}\sum_{i=1}^n \frac{\theta}{2} = \frac{2}{n}\sum_{i=1}^n \left(X_i - \frac{\theta}{2}\right)$$

independence

$$MSE_\theta(2\bar{X}) = \mathbb{E}\left((2\bar{X} - \theta)^2\right) = \frac{4}{n^2}\mathbb{E}\left(\left(\sum_{i=1}^n \left(X_i - \frac{\theta}{2}\right)\right)^2\right) = \frac{4}{n^2}\sum_{i=1}^n \mathbb{V}\mathrm{ar}(X_i) = \frac{4}{n^2}\frac{n\theta^2}{12} = \frac{\theta^2}{3n}$$

$$MSE_\theta(X_{(n)}) = \mathbb{E}\left((X_{(n)} - \theta)^2\right) = \int_{[0,\theta]^n} (\max(x) - \theta)^2 \frac{1}{\theta^n} \, dx = \frac{n}{\theta^n} \int_0^\theta \underbrace{\int_0^{x_n} \cdots \int_0^{x_n}}_{n-1 \text{ times}} (x_n - \theta)^2 \, dx_1 \cdots dx_n$$

$$= \frac{n}{\theta^n} \int_0^\theta (x_n - \theta)^2 x_n^{n-1} \, dx_n = \frac{n}{\theta^n} \int_0^\theta \left(x_n^{n+1} - 2x_n^n\theta + \theta^2 x_n^{n-1}\right) dx_n$$

$$= \frac{n}{\theta^n}\left(\frac{\theta^{n+2}}{n+2} - 2\frac{\theta^{n+2}}{n+1} + \frac{\theta^{n+2}}{n}\right)$$

$$= \theta^2 n \left(\frac{n(n+1) - 2n(n+2) + (n+1)(n+2)}{n(n+1)(n+2)}\right)$$

$$= \theta^2\left(\frac{n^2 + n - 2n^2 - 4n + n^2 + 3n + 2}{(n+1)(n+2)}\right) = \frac{2\theta^2}{(n+1)(n+2)}$$

The MLE converges faster.

(4) **Unbiased estimators**

Let $\hat{a}$ and $\hat{b}$ be unbiased estimators of unknown parameters a and b respectively.

(a) Check if $\alpha\hat{a} + \beta\hat{b}$ is an unbiased estimator of the parameter $\alpha\, a + \beta\, b$, where $\alpha, \beta \in \mathbb{R}$.

(b) Is $\hat{a}^2$ an unbiased estimator of a^2?

(c) Based on the following measurements of a side of a square (in milimeters)

$$15, 17, 16, 16, 17, 14$$

find an unbiased estimator of the area.

a) Let X be a random variable corresponding to the data, such that

$\mathbb{E}(\hat{a}(X)) = a$ and $\mathbb{E}(\hat{b}(X)) = b$.

Since $\mathbb{E}(\alpha\hat{a}(X) + \beta\hat{b}(X)) = \alpha\,\mathbb{E}(\hat{a}(X)) + \beta\,\mathbb{E}(\hat{b}(X)) = \alpha a + \beta b$, $\alpha\hat{a} + \beta\hat{b}$ is an unbiased estimator of $\alpha a + \beta b$.

b) $\mathbb{E}((\hat{a}(X))^2) = \mathbb{V}\mathrm{ar}(\hat{a}(X)) + (\mathbb{E}(\hat{a}(X)))^2 = \mathbb{V}\mathrm{ar}(\hat{a}(X)) + a^2$

$\Leftrightarrow \mathbb{E}((\hat{a}(X))^2) - a^2 = \mathbb{V}\mathrm{ar}(\hat{a}(X)) \geqslant 0$, equality holds only if $\hat{a}(X)$ is constant almost everywhere.

c) $\hat{a} : \mathbb{R}^n \to \mathbb{R} : X \longmapsto \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{\substack{j \in \{1,\dots,n\} \\ j \neq i}} x_i x_j$

$$\mathbb{E}(\hat{a}(X)) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{\substack{j \in \{1,\dots,n\} \\ j \neq i}} \mathbb{E}(X_i X_j) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{\substack{j \in \{1,\dots,n\} \\ j \neq i}} \mathbb{E}(X_i)\mathbb{E}(X_j) = \ell^2 =: a$$

where $\ell = \mathbb{E}(X_i)$ for all $i \in \{1,\dots,n\}$, hence ℓ is the true length of a side of the square.

In our case, $n = 6$, and we obtain $\hat{a}(x) = \frac{7514}{30} = \frac{3757}{15} \approx 250{,}5$ (mm^2).

(5) **Rayleigh distribution**

Let $X_1, \ldots, X_n$ be a random sample with Rayleigh distribution

$$f(x|\theta) = \begin{cases} \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}}, & x \geq 0 \\ 0, & x < 0 \end{cases},$$

where $\theta > 0$ is unknown.

(a) Find the method of moments estimator of θ.

(b) Find the MLE of θ and its asymptotic variance.

Hint: Show that the first two moments are $\mathbb{E}X = \theta\sqrt{\frac{\pi}{2}}$ and $\mathbb{E}X^2 = 2\theta^2$.

$$\mathbb{E}(X_i) = \int_0^\infty \frac{x^2}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) dx = \theta \int_0^\infty u^2 \exp\left(-\frac{u^2}{2}\right) du = \frac{\theta}{2}\sqrt{2\pi} \int_{\mathbb{R}} u^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du$$

$$= \theta \sqrt{\frac{\pi}{2}}\, \mathbb{E}(Y^2) = \theta\sqrt{\frac{\pi}{2}}\left(\mathrm{Var}(Y) - (\mathbb{E}(Y))^2\right) = \theta\sqrt{\frac{\pi}{2}}, \text{ where } Y \sim \mathcal{N}(0,1)$$

$$\mathbb{E}(X_i^2) = \int_0^\infty \frac{x^3}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) dx = \int_0^\infty 2\theta^2 u \exp(-u)\, du = \theta^2 \int_0^\infty u e^{-u} du$$

$$\boxed{\begin{aligned} u &= \frac{x^2}{2\theta^2} \\ \frac{du}{dx} &= \frac{x}{\theta^2} \\ dx &= du \frac{\theta^2}{x} \end{aligned}}$$

$$= 2\theta^2\left(\left[-ue^{-u}\right]_0^\infty + \int_0^\infty e^{-u} du\right) = 2\theta^2\left[-e^{-u}\right]_0^\infty = 2\theta^2$$

a) $\overline{X} = \hat{\theta}\sqrt{\frac{\pi}{2}} \Leftrightarrow \hat{\theta} = \sqrt{\frac{2}{\pi}}\,\overline{X}$

b) $L(\theta|x) = \prod_{i=1}^n \frac{1}{\theta^2} \exp\left(-\frac{x_i^2}{2\theta^2}\right) \longrightarrow \ell(\theta|x) = -2n\log(\theta) + \sum_{i=1}^n \left(-\frac{x_i^2}{2\theta^2}\right)$

$$= -2n\log(\theta) - \frac{1}{2}\theta^{-2}\sum_{i=1}^n x_i^2$$

$$\ell'(\theta, x) = \frac{-2n}{\theta} + \theta^{-3}\sum_{i=1}^n x_i^2 \overset{!}{=} 0 \Leftrightarrow \theta^2 = \frac{1}{2n}\sum_{i=1}^n x_i^2 \Leftrightarrow \theta = \sqrt{\frac{1}{2n}\sum_{i=1}^n x_i^2}$$

$$\ell''(\theta, x) = 2n\theta^{-2} - 3\theta^{-4}\sum_{i=1}^n x_i^2 \quad \text{and}$$

$$\boxed{\begin{aligned} &\mathbb{E}(X_i^4) \overset{\text{computer}}{=} 8\theta^4 \\ &\Rightarrow \mathrm{Var}(X_i^2) = 8\theta^4 - 4\theta^4 = 4\theta^4 \end{aligned}}$$

$$\ell''\left((2n)^{-\frac{1}{2}}|x|\right) = \frac{4n^2}{|x|^2} - \frac{12n^2}{|x|^2} = \frac{-8n^2}{|x|^2} < 0$$

By the CLT, $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n x_i^2 - 2\theta^2\right) \xrightarrow{d} \mathcal{N}(0, \mathrm{Var}(X_i^2))$

$g: \mathbb{R}_0^+ \to \mathbb{R}: x \mapsto \sqrt{\frac{x}{2}}$, $\quad g'(x) = \frac{1}{\sqrt{2}}\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{4}\sqrt{\frac{2}{x}}$

Applying the delta method, we obtain

$$\sqrt{n}\left(\sqrt{\frac{1}{2n}\sum_{i=1}^n x_i^2} - \theta\right) = \sqrt{n}\left(g\left(\frac{1}{n}\sum_{i=1}^n x_i^2\right) - g(2\theta^2)\right) \longrightarrow \frac{1}{4}\frac{1}{\theta}\mathcal{N}(0, \mathrm{Var}(x_i^2))$$

Thus, the asymptotic variance is $\frac{\mathrm{Var}(X_i^2)}{16\theta^2} = \frac{\theta^2}{4}$