(4) **Random variables on the unit disk**

Let (X, Y) be uniformly distributed on the unit disk $\{f(x; y) : x^2 + y^2 \le 1\}$. Let

$$R = \sqrt{X^2 + Y^2}.$$

Find the cdf, pdf, and the expectation the random variable R.

$$K := \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\}$$

$$1 \overset{!}{=} \int_K c \, d\lambda^2 = c \int_0^1 \int_0^{2\pi} r \, d\varphi \, dr = c\pi \Leftrightarrow c = \frac{1}{\pi}$$

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi}, & \text{if } (x,y) \in K \\ 0, & \text{otherwise} \end{cases}$$

$$h: \mathbb{R}^+ \times [0, 2\pi[\to \mathbb{R}^2 : (r, \varphi) \mapsto (r\cos(\varphi), r\sin(\varphi))$$

$$f_{R,\Phi}(r,\varphi) = f_{X,Y}(h(r,\varphi))\, r = \begin{cases} \frac{r}{\pi}, & \text{if } 0 \le r \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_R(r) = \int_0^{2\pi} f_{R,\Phi}(r,\varphi) \, d\varphi \, \mathbb{1}_{[0,1]}(r) = 2r \, \mathbb{1}_{[0,1]}(r)$$

$$F_R(r) = \int_0^r f_R(\rho) \, d\rho = \begin{cases} 0, & \text{if } r \le 0 \\ r^2, & \text{if } 0 < r < 1 \\ 1, & \text{otherwise} \end{cases}$$

$$\mathbb{E}(R) = \int_{-\infty}^{\infty} f_R(r) \, r \, dr = \int_0^1 2r^2 \, dr = \frac{2}{3}$$