(2) **Box of candles**
There are blue and red candles in a box. Probability that a randomly chosen candle is blue is $\frac{1}{1+2a}$, for $a > 0$. Based on a sample of sample size n, find the maximum likelihood estimator (MLE) $\hat{a}$ of the parameter a.

$$p(a) = (1+2a)^{-1} \Rightarrow p'(a) = -2(1+2a)^{-2}$$

For $x \in \{0,1\}^n$:

$$L(a|x) = \prod_{i=1}^{n} (p(a))^{x_i} (1-p(a))^{1-x_i} = (p(a))^{\sum_{i=1}^n x_i} (1-p(a))^{\left(n - \sum_{i=1}^n x_i\right)}, \text{ hence}$$

$$\ell(a|x) = \log(p(a)) \sum_{i=1}^{n} x_i + \left(n - \sum_{i=1}^{n} x_i\right) \log(1-p(a))$$

$$\ell'(\hat{a}|x) = \frac{1}{p(\hat{a})} p'(\hat{a}) \sum_{i=1}^{n} x_i + \left(n - \sum_{i=1}^{n} x_i\right) \frac{1}{1-p(\hat{a})} \left(-p'(\hat{a})\right)$$

$$= -2(1+2\hat{a})^{-1} \sum_{i=1}^{n} x_i + \left(n - \sum_{i=1}^{n} x_i\right) \left(1 - (1+2\hat{a})^{-1}\right)^{-1} 2(1+2\hat{a})^{-2}$$

$$= -2(1+2\hat{a})^{-1} \sum_{i=1}^{n} x_i + \left(n - \sum_{i=1}^{n} x_i\right)(2\hat{a})^{-1}(1+2\hat{a})\, 2(1+2\hat{a})^{-2}$$

$$= \left((1+2\hat{a})\hat{a}\right)^{-1} \left(-2\hat{a} \sum_{i=1}^{n} x_i + \left(n - \sum_{i=1}^{n} x_i\right)\right) \overset{!}{=} 0$$

$$\Leftrightarrow -2\hat{a} \sum_{i=1}^{n} x_i + n - \sum_{i=1}^{n} x_i = 0 \Leftrightarrow \hat{a} = \frac{1}{2}\left(n\left(\sum_{i=1}^{n} x_i\right) - 1\right)$$

$$\ell''(\hat{a}|x) = 4(1+2\hat{a})^{-2} \sum_{i=1}^{n} x_i + \left(n - \sum_{i=1}^{n} x_i\right)\left((-1)(\hat{a})^{-2}(1+2\hat{a})^{-1} - \hat{a}^{-1}(1+2\hat{a})^{-2}\right)$$

$$= \left((1+2\hat{a})\hat{a}\right)^{-2} \left(4\hat{a}^2 \sum_{i=1}^{n} x_i - \left(n - \sum_{i=1}^{n} x_i\right)\left((1+2\hat{a}) + \hat{a}\right)\right.$$