(1) **Distribution of the maximum**
Let $X_1, X_2, \ldots$ be a sequence of i.i.d. with uniform $(0,1)$ distribution and let $X_{(n)} = \max_{1\le i\le n} X_i$.
Show that the sequence

$$Y_n = n(1 - X_{(n)}), \quad n \in \mathbb{N}$$

converges to an exponential exp(1) random variable as $n \to \infty$.

$$\bullet)\ F_n(x) := \mathbb{P}(X_{(n)} \le x) = \mathbb{P}\left(\max_{1\le i\le n} X_i \le x\right) = \mathbb{P}\left(\bigcap_{i=1}^{n} [X_i \le x]\right) = \prod_{i=1}^{n} \mathbb{P}(X_i \le x) = \begin{cases} 0, & \text{if } x < 0 \\ x^n, & \text{if } 0 \le x < 1 \\ 1, & \text{if } 1 \le x \end{cases}$$

$$G_n(y) := \mathbb{P}(Y_n \le y) = \mathbb{P}(n(1 - X_{(n)}) \le y) = \mathbb{P}\left(1 - X_{(n)} \le \frac{y}{n}\right) = \mathbb{P}\left(1 - \frac{y}{n} \le X_{(n)}\right)$$

$$= 1 - \mathbb{P}\left(X_{(n)} < 1 - \frac{y}{n}\right) = \begin{cases} 0 & , \text{if } y \le 0 \\ 1 - \left(1 - \frac{y}{n}\right)^n, & \text{if } 0 < y \le n \\ 1 & , \text{if } n < y \end{cases}$$

We know from Analysis, that $\left(1 + \frac{(-y)}{n}\right)^n \xrightarrow{n \to \infty} e^{-y}$ pointwise in $\mathbb{R}$, hence

$$G_n(y) \xrightarrow{n \to \infty} \begin{cases} 0 & , \text{if } y < 0 \\ 1 - e^{-y}, & \text{if } y \ge 0 \end{cases},$$ which is the distribution function

of an exp(1) random Variable, hence $Y_n \xrightarrow{d} Z \sim \exp(1)$