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Numerik von Differentialgleichungen - Kreuzlübung 7

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Exercise 31:
Solve the initial value problem

$$\begin{aligned} u' &= u + v \\ \epsilon v' &= 2u - v \end{aligned} \tag{1}$$

with initial values $(u(0), v(0)) = (1, 4)$ using the Radau-IIA method, a Gauss-method and the RK4-method numerically for different $\epsilon > 0$. Use methods with comparable orders of convergence. Analyze the dependence of the component-wise error at $t = 0.1$ on the parameter ϵ.

Exercise 32:
We use the implicit trapezoidal rule from Exa. 3.6.
a) Show that this method has the same stability function as the implicit midpoint rule from Exa. 3.5.
b) Show that the implicit trapezoidal rule is A-stable.
c) Show that the implicit trapezoidal rule is not B-stable.

Hint for c: Show first that the function

$$f(y) := \begin{cases} -y^3, & y \leq 0 \\ -y^2, & y > 0 \end{cases} \tag{2}$$

is dissipative. Then, apply the implicit trapezoidal rule to an initial value problem with this right-hand side and the initial values $y_0 = 0$ and $\tilde{y}_0$. Use, e.g., the step size $h = 1$ and find $\tilde{y}_0 < 0$ such that $\tilde{y}_1 > -\tilde{y}_0$.

Exercise 33:
Show, by using the initial value problem

$$y' = f_\epsilon(y), \qquad y(0) = 1 \tag{3}$$

with the smooth non-increasing function

$$f_\epsilon(y) = \begin{cases} -1, & |y-1| \leq \epsilon \\ -y, & |y-1| \geq 2\epsilon \end{cases}, \tag{4}$$

that the linear-implicit RK-Methods from Exercise 20 cannot be B-stable. Note the updated exercise sheet in TUWEL to Exercise 20.

Exercise 34:
Consider an implicit m-stage RK-method with Butcher tableau $\begin{array}{c|c} c & A \\ \hline & b^\top \end{array}$ and a problem with dimension $n = 1$. Instead of solving the (implicit) equation for the vector of stages $k \in \mathbb{R}^m$ exactly, we employ m steps of the Banach fixpoint iteration to obtain approximate stages. We set $k^{(0)} := f(t_\ell, y_\ell)(1, \dots, 1)^\top \in \mathbb{R}^m$, define $k^{(s)}$ for $s = 0, \dots, m$ as the s-th fixpoint iterate and set $\widetilde{k} := k^{(m)}$. This gives rise to a one-step method

$$y_{\ell+1} = y_\ell + h \sum_{j=1}^{m} b_j \widetilde{k}_j. \tag{5}$$

Compute the stability function of this method. Is this method A-stable?

Exercise 35:
Consider an m-stage collocation method with collocation nodes $c_1, \dots, c_m$. Define the polynomial

$$M(x) := \frac{1}{m!} \prod_{i=1}^{m} (x - c_i). \tag{6}$$

a) Show that the stability function $R(z)$ with $z = \lambda h$ for the collocation method is the rational polynomial $R(z) = P(z)/Q(z)$, where, $P, Q \in \mathbb{P}_m$ are given by

$$P(z) = M^{(m)}(1) + M^{(m-1)}(1)z + \ldots + M(1)z^m, \tag{7}$$
$$Q(z) = M^{(m)}(0) + M^{(m-1)}(0)z + \ldots + M(0)z^m. \tag{8}$$

b) Use this explicit representation of $R(z)$ to show that Gauss-methods are not L-stable.

Hint for a: In order to obtain the representation for $R(z)$, consider the usual model problem and $h = 1$ (which implies $z = \lambda$). From the definition of the collocation polynomial $q \in \mathbb{P}_m$ infer that

$$q'(x) - zq(x) = KM(x) \tag{9}$$

for a constant $K \neq 0$. Differentiate equation (9) $s = 0, \dots, m$ times to obtain an expression for $q(x)$. Finally, there holds $R(z) = q(1)/q(0)$ (why?).