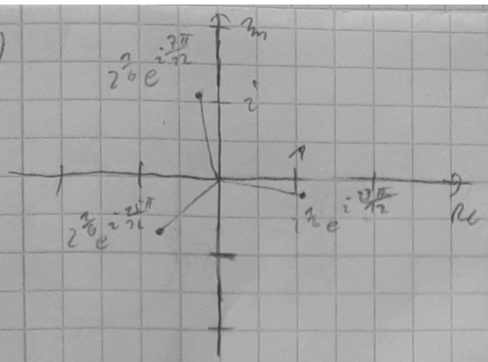
1.1

a)

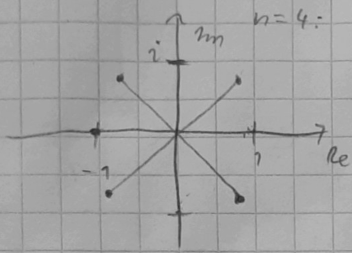


$$\sqrt[3]{1-i} = \sqrt[3]{\sqrt{2}\, e^{i\frac{7\pi}{4}}} = 2^{\frac{1}{6}} e^{i\frac{7\pi}{12}}$$

$$\sqrt[3]{1-i} = 2^{\frac{1}{6}} e^{i\frac{15\pi}{12}}$$

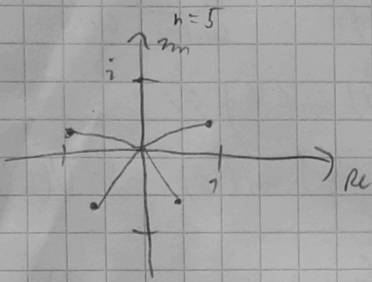
$$\sqrt[3]{1-i} = 2^{\frac{1}{6}} e^{i\frac{23\pi}{12}}$$

b)



$$\sqrt[n]{-1} = e^{i\frac{\pi}{n}} \cdot e^{i\frac{2\pi}{n}k} \text{ mit } k \in \{0, \dots, n-1\}$$

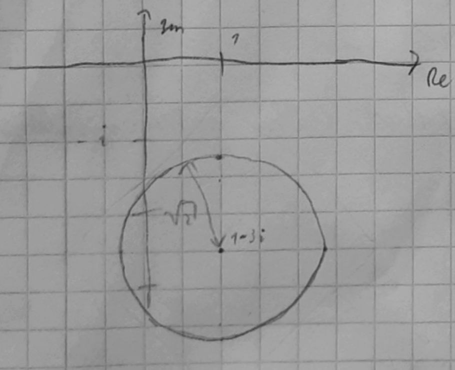
c)



$$\sqrt[n]{i} = \left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{n}} e^{i\frac{2\pi}{n}k} = e^{i\frac{\pi}{n}\left(\frac{1}{2}+2k\right)}$$

1.2

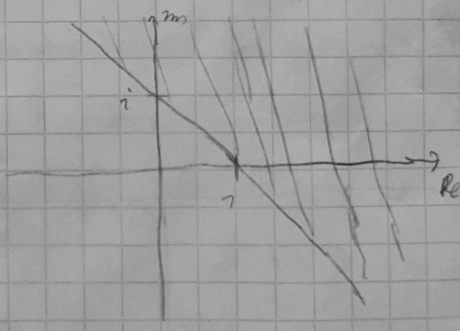
$\{z \in \mathbb{C} \mid |z-(1-3i)| \leq 2\}$



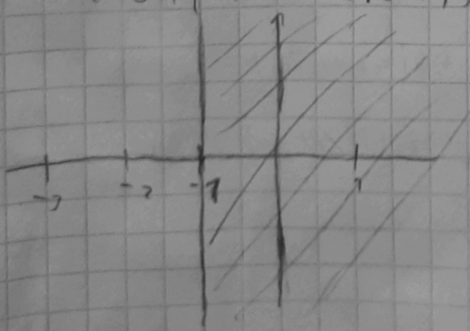
$\{z \in \mathbb{C} \mid \operatorname{Im}((z-i)(1+i)) > 0\}$

$(z-i)(1+i) = (x+i(y-1))(1+i) =$

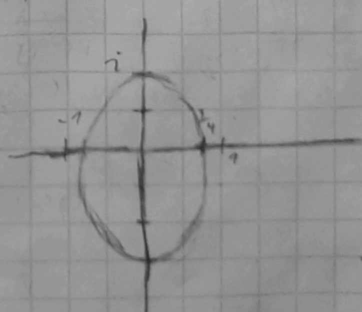
$= x-(y-1)+i(y-1+x)$



$\{z \in \mathbb{C} \mid |z-1| < |z+3|\}$



$\{z \in \mathbb{C} \mid |z| + |z+i| = 2\}$

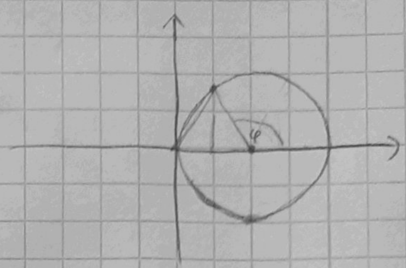


$$2\sqrt{x^2+\frac{1}{4}} = 2 \Leftrightarrow \sqrt{x^2+\frac{1}{4}} = 1 \Leftrightarrow x^2+\frac{1}{4} = 1 \Leftrightarrow$$

$$\Leftrightarrow x^2 = \frac{3}{4} \Leftrightarrow x = \sqrt{\frac{3}{4}}$$

1.3) • Sei $|z-1| = 1$

$$\Rightarrow \arg(z-1) \overset{(a)}{=} 2\arg(z) \overset{(b)}{=} \frac{2}{3}\arg(z^2-z)$$



Bew.:

a) $\exists \varphi \in [0, 2\pi[\qquad z-1 = e^{i\varphi}$

$$\Rightarrow z = 1+e^{i\varphi} = 1+\cos(\varphi)+i\sin(\varphi) =$$

$$= \sqrt{(1+\cos(\varphi))^2+\sin^2(\varphi)}\; e^{i\arg(1+\cos(\varphi)+i\sin(\varphi))}$$

Fall 1: $\varphi = \pi \Rightarrow \arg(z)$ undef. $\overset{?}{0}$

Fall 2: $\varphi \neq \pi \Rightarrow \arg(z) = \arg(1+\cos(\varphi)+i\sin(\varphi)) =$

$$= \arctan\left(\frac{\sin(\varphi)}{1+\cos(\varphi)}\right)$$

Wunsch: $\arctan\left(\frac{\sin(\varphi)}{1+\cos(\varphi)}\right) = \frac{\varphi}{2}$, also $\tan\left(\frac{\varphi}{2}\right) = \frac{\sin(\varphi)}{1+\cos(\varphi)} \Leftrightarrow$

$$\Leftrightarrow \sin\left(\frac{\varphi}{2}\right) + \sin\left(\frac{\varphi}{2}\right)\cos(\varphi) = \sin(\varphi)\cos\left(\frac{\varphi}{2}\right) \Leftrightarrow$$

$$\Leftrightarrow \sin\left(\frac{\varphi}{2}\right) = \sin(\varphi)\cos\left(\frac{\varphi}{2}\right) - \sin\left(\frac{\varphi}{2}\right)\cos(\varphi) = \sin\left(\varphi - \frac{\varphi}{2}\right)$$

b) $\frac{2}{3}\arg(z^2-z) = \frac{2}{3}\arg(z(z-1)) = \frac{2}{3}\left(\arg(z) + \arg(z-1)\right) =$

$$= \frac{2}{3}\left(\frac{\arg(z-1)}{2} + \arg(z-1)\right) = \arg(z-1)$$

1.4) $\mathbb{R}[x]$ Polynome mit reellen Koeffizienten, Ring

$P_1(x) \sim P_2(x) \Leftrightarrow P_1(x) - P_2(x)$ durch x^2+1 ohne Rest teilbar ist Äquivalenzrel.

$\mathbb{R}[x]/(x^2+1)$ ist Quotientenring mit $+, \cdot$

ges. Isomorphismus zwischen $\mathbb{R}[x]/(x^2+1)$ und $\mathbb{C}$

$\varphi: \mathbb{C} \longrightarrow \mathbb{R}[x]/(x^2+1) : a+ib \mapsto [a+bx]_{(x^2+1)}$

$w, z \in \mathbb{C}$ bel.

- $\varphi(w+z) = \varphi(w_1+z_1+i(w_2+z_2)) = w_1+z_1+(w_2+z_2)x = w_1+w_2x+z_1+z_2x = \varphi(w)+\varphi(z)$
- $\varphi(w\cdot z) = \varphi(w_1z_1 - w_2z_2 + i(w_1z_2+w_2z_1)) = w_1z_1 - w_2z_2 + x(w_1z_2+w_2z_1) =$

$$\overset{(1)}{=} w_1z_1 + x(w_1z_2+w_2z_1) + x^2 w_2z_2 = (w_1+xw_2)(z_1+xz_2) = \varphi(w)\varphi(z)$$

wobei (1) wegen $x^2 \sim -1$ gilt.

1.4) $p(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{R}[x]$ bel.

Forts.

Beh. $\exists\, b_0, \dots, b_n \in \mathbb{R}$: $p(x) = b_0 + b_1 x + (x^2+1) \sum_{j=0}^{n-2} b_{j+2} x^j = \sum_{j=0}^{n} b_j x^j + \sum_{j=0}^{n-2} b_{j+2} x^j$

also

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}}_{\in \mathbb{R}^{(n+1)\times(n+1)}} \underbrace{\begin{pmatrix} b_0 \\ \vdots \\ \vdots \\ b_n \end{pmatrix}}_{\in \mathbb{R}^{n+1}} = \underbrace{\begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}}_{\in \mathbb{R}^{n+1}}$$

Ist lösbar, also gibt es in jeder Äquivalenzklasse von $\mathbb{R}[x]/(x^2+1)$ einen Repräsentanten der Form $a+bx$

$\varphi^{-1}: \mathbb{R}[x]/(x^2+1) \to \mathbb{C}: [a+bx]_{\mathbb{R}[x]} \mapsto a+ib$

$\boxed{1.5}$ a) Wo ist $f(z) = (\bar{z})^2$ holomorph?

$$\frac{(\overline{z+h})^2 - (\bar{z})^2}{h} = \frac{2\bar{z}\bar{h} + (\bar{h})^2}{h}$$

Für $z \neq 0 \Rightarrow \bar{z} \neq 0$ und damit nicht komplex diffbar

weil für $x \in \mathbb{R}$ $\lim\limits_{x\to 0} \frac{2\bar{z}x + x^2}{x} = \lim\limits_{x\to 0} (2\bar{z} + x) = 2\bar{z}$ $\not\!\!\times$

und $\lim\limits_{x\to 0} \frac{2\bar{z}(-ix) + (-ix)^2}{ix} = \lim\limits_{x\to 0} (-2\bar{z} + ix) = -2\bar{z}$

Für $z = 0$ gilt $\lim\limits_{h\to 0} \left|\frac{(\bar{h})^2}{h}\right| = \lim\limits_{h\to 0} \frac{|h|^2}{|h|} = \lim\limits_{h\to 0} |h| = 0 \Rightarrow \lim\limits_{h\to 0} \frac{\bar{h}^2}{h} = 0$

b) $f(x+iy) = \underbrace{\log\sqrt{x^2+y^2}}_{u(z):=} + i\,\underbrace{\arctan\left(\frac{y}{x}\right)}_{v(z):=}$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2+y^2}}\,\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\,2x = \frac{x}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\,\frac{1}{x} = \frac{1}{x+\frac{y^2}{x}} = \frac{1}{\frac{x^2+y^2}{x}} = \frac{x}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} \qquad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$\Rightarrow f: \mathbb{C}\setminus\{x+iy \in \mathbb{C} \mid x = 0\} \to \mathbb{C}: x+iy \mapsto \log(\sqrt{x^2+y^2}) + i\arctan\left(\frac{y}{x}\right)$ ist holomorph,

weil sie die Cauchy-Riemannschen Diffglg. erfüllt sind

c) $f(z) = \underbrace{\sin^2(x+y)}_{u(z):=} + i\,\underbrace{\cos^2(x+y)}_{v(z):=}$

$$\frac{\partial u}{\partial x} = 2\sin(x+y)\cos(x+y) \qquad \frac{\partial v}{\partial y} = -2\cos(x+y)\sin(x+y)$$

$$\frac{\partial u}{\partial y} = 2\sin(x+y)\cos(x+y) \qquad \frac{\partial v}{\partial x} = -2\cos(x+y)\sin(x+y)$$

C-R Diffglg. erfüllt auf $\{x+iy \in \mathbb{C} \mid x+y \in \frac{k\pi}{2}, k \in \mathbb{Z}\}$, also

dort komplex diffbar.

1.6) Sei $U \subseteq \mathbb{C}$ offen und zusammenhängend

$u_g, u_f, v : U \to \mathbb{R}$, wobei $f, g : U \to \mathbb{C}$ holomorph

mit $f : z \mapsto u_f(z) + i\,v(z)$, $g : z \mapsto u_g(z) + i\,v(z)$

betrachte $h : U \to \mathbb{R} : z \mapsto f(z) - g(z)$

$u := Re(h)$, $v := Re(h) = 0$

$\Rightarrow \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$

h ist als Differenz zweier holomorpher Fkt. holomorph

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$ und $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0$

Da U ein Gebiet ist lassen sich zwei Punkte $a, b \in U$ durch

einen achsenparallelen Polygonzug $\overline{c_j c_{j+1}}$, $j \in \{0, \dots, n-1\}$ verbinden

, wobei $c_0 = a$ und $c_n = b$

$R := \{ j \in \{0, \dots, n-1\} \mid Im(c_{j+1}) = Im(c_j) \}$

$I := \{ j \in \{0, \dots, n-1\} \mid Re(c_{j+1}) = Re(c_j) \}$

Nach dem Hauptsatz der Integral- und Differentialrechnung gilt

$$u(b) - u(a) = \sum_{j \in R} \int_{Re(c_j)}^{Re(c_{j+1})} \underbrace{\frac{\partial u}{\partial x}(t + i\,Im(c_j))}_{=0} dt + i \sum_{k \in I} \int_{Im(c_k)}^{Im(c_{k+1})} \underbrace{\frac{\partial u}{\partial y}(Re(c_k) + is)}_{=0} ds =$$

$$= c \in \mathbb{R}$$

1.7) $u(x,y) := x^2 - y^2 + e^{-y}\sin(x) - e^{y}\cos(x)$

Suche v, so, dass die Cauchy-Riemannschen Diffglg. erfüllt sind

$$\frac{\partial u(x,y)}{\partial x} = 2x + e^{-y}\cos(x) + e^{y}\sin(x) \overset{!}{=} \frac{\partial v}{\partial y}(x,y)$$

$$-\frac{\partial u}{\partial y}(x,y) = +2y + e^{-y}\sin(x) + e^{y}\cos(x) \overset{!}{=} \frac{\partial v}{\partial x}(x,y)$$

$$v(x,y) := 2xy + e^{y}\sin(x) - e^{-y}\cos(x) + C \quad ; \quad C \in \mathbb{R}$$

Aus Aufgabe 1.6 wissen wir schon, dass das alle Lsg. sind.

1.8) $G \subseteq \mathbb{C}$ ein Gebiet, $f: G \to \mathbb{C}$ holomorph, $|f|^2 = c \in \mathbb{R}$

z.z.: $\exists\, a \in \mathbb{R}: f = a$

Fall 1: „$c = 0$"

$\Rightarrow |f|^2 = 0 \Rightarrow f = 0$

Fall 2: „$c \neq 0$"

$u := \mathrm{Re}(f) \quad v := \mathrm{Im}(f)$

$z \in G$ bel. $\quad (u(z))^2 + (v(z))^2 = c \neq 0 \Rightarrow u(z) \neq 0 \lor v(z) \neq 0$

o.B.d.A.: $u(z) \neq 0$

Es gilt: $u(z) \frac{\partial u}{\partial x}(z) = -v(z) \frac{\partial v}{\partial x}(z)$ und $u(z) \frac{\partial u}{\partial y}(z) = -v(z) \frac{\partial v}{\partial y}(z)$

Fall 1: „$v(z) = 0$" $\Rightarrow u(z) \frac{\partial u}{\partial x}(z) = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$, ebenso $\frac{\partial u}{\partial y} = 0$ und mit C-R Diffgl.

$\frac{\partial v}{\partial x}(z) = \frac{\partial v}{\partial y}(z) = 0$

Fall 2: „$v(z) \neq 0$" $\Rightarrow \frac{\partial u}{\partial x}(z) = -\frac{v(z)}{u(z)} \frac{\partial v}{\partial x}(z) = \frac{v(z)}{u(z)} \frac{\partial u}{\partial y}(z) = -\left(\frac{v(z)}{u(z)}\right)^2 \frac{\partial v}{\partial y}(z) = -\left(\frac{v(z)}{u(z)}\right)^2 \frac{\partial u}{\partial x}(z)$

$\Rightarrow \frac{\partial u}{\partial x}(z) = 0 \Rightarrow \frac{\partial v}{\partial x}(z) = \frac{\partial u}{\partial y}(z) = \frac{\partial v}{\partial y}(z) = 0$, da G ein Gebiet ~~ist~~ gilt $f = $ konst.