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Numerik von Differentialgleichungen - Kreuzlübung 4

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Exercise 16:
Apply the method of (two step) Richardson extrapolation from Section 2.7 to the explicit Euler method.

a) Which method do you get?

b) Independently of Section 2.7, prove that this method has convergence order 2.

Exercise 17:
Implement the implicit Euler method by using the Newton-method to solve the arising nonlinear system of equations. As input parameters, the algorithm should get a vector of nodes t, an initial value y_0, the right-hand side f and its derivative $\frac{\partial}{\partial y} f$, and an appropriate stopping criterion for the Newton-method (tolerance and/or maximal number of iterations).

Test the method with the following initial value problems: Let $Y = (y_1, y_2)^\top$ be the solution of

$$Y'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} Y(t) + \begin{pmatrix} 2\sin t \\ 2(\cos t - \sin t) \end{pmatrix}, \quad t \geq 0, \qquad Y(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \tag{1}$$

Let $Z = (z_1, z_2)^\top$ be the solution of

$$Z'(t) = \begin{pmatrix} -2 & 1 \\ 998 & -999 \end{pmatrix} Z(t) + \begin{pmatrix} 2\sin t \\ 999(\cos t - \sin t) \end{pmatrix}, \quad t \geq 0, \qquad Z(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \tag{2}$$

Compare your results and stepsizes with the embedded Runge-Kutta method RK5(4) from Exercise 15. To this end, use the parameters $t \in [0, 10], \rho = 0.7, \eta = 1.5, \mathrm{tol} = 10^{-6}, h_{min} = 10^{-10}$.

Exercise 18:
Consider an implicit s-stage Runge-Kutta method of the form

$$\begin{array}{c|c} c & A \\ \hline 0 & b^\top \end{array}. \tag{3}$$

Show that: If the method is applied to the initial value problem $y' = \lambda y$ with $y(0) = y_0$ and $\lambda \in \mathbb{C}$ there holds for sufficiently small h that

$$y_{i+1} = R(\lambda h) y_i \tag{4}$$

with a rational function $R = P/Q$ and polynomials $P, Q \in \Pi_s$ of maximal degree s.

Exercise 19:
Consider the implicit Runge-Kutta methods from Example 3.11 (implicit Euler), Example 3.12 (implicit trapezoidal rule), and Example 3.13 (implicit midpoint rule). Prove their respective convergence orders.

Exercise 20:
Implicit Runge-Kutta methods lead to a nonlinear system of equations, which can be very costly to solve. As a simplification, one can use the following method to solve autonomous differential equations $y'(t) = f(y(t))$.

Consider $b \in \mathbb{R}^m$, $A = (A_{ij}) \in \mathbb{R}^{m\times m}$ with $A_{ij} = 0$ for $i \leq j$ and $B = (B_{ij}) \in \mathbb{R}^{m\times m}$ with $B_{ij} = 0$ for $i < j$. Let further J be the Jacobi-matrix of f, i.e., $J := \partial_y f$. Then, the following equations define an implicit one-step method:

$$k_i = J\left\{y_\ell + h\sum_{j=1}^{i}(B_{ij} - A_{ij})\,k_j\right\} + f\left(y_\ell + h\sum_{j=1}^{i-1}A_{ij}k_j\right), \qquad i = 1,\dots,m \tag{5a}$$

$$y_{\ell+1} := y_\ell + h\sum_{j=1}^{m} b_j k_j. \tag{5b}$$

a) Show that, for this method, only m linear systems of equations have to be solved (in particular, no nonlinear system of equations has to be solved).
b) What is the overall cost for solving these linear systems of equations if $B_{ii} = \beta$ for all $i = 1,\dots,m$?
c) Show that the linear systems of equations are uniquely solveable for all $h > 0$ if $B_{ii} = \beta > 0$ for all $i = 1,\dots,m$ and J only has negative eigenvalues.
d) Show that (5) defines an implicit Runge-Kutta method for linear functions f.