(1) **The GLRT for the normal variance - simple hypotheses**
Derive the generalized likelihood ratio test (GLRT) for the normal variance: Assume $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma^2)$, where both μ and σ are unknown. We want to test

$$H_0 : \sigma^2 = \sigma_0^2 \quad vs \quad H_1 : \sigma^2 \neq \sigma_0^2.$$

The likelihood function is given by $L(\mu, \sigma^2; x) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right)$

We have $\Theta_0 := \mathbb{R} \times \{\sigma_0^2\}$, $\Theta := \mathbb{R} \times \mathbb{R}^+$ and $\Theta_1 := \Theta \setminus \Theta_0$

We can use the MLEs $\hat{\mu} = \bar{x}$ and $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$ to obtain the GLR

$$\lambda(x) = \frac{\sup\{L(\mu, \sigma^2; x) \mid (\mu, \sigma^2) \in \Theta\}}{\sup\{L(\mu, \sigma^2; x \mid (\mu, \sigma^2) \in \Theta_0\}} = \frac{L(\hat{\mu}, \hat{\sigma}^2; x)}{L(\hat{\mu}, \sigma_0^2; x)} = \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{-\frac{n}{2}} \exp\left(\left(\frac{1}{\sigma_0^2} - \frac{1}{\hat{\sigma}^2}\right)\frac{n}{2}\hat{\sigma}^2\right)$$

$$= \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{-\frac{n}{2}} \exp\left(\frac{n}{2}\left(\frac{\hat{\sigma}^2}{\sigma_0^2} - 1\right)\right)$$

We take $T(x) := \lambda(x)$ as our test statistic.

We reject H_0, if $\lambda(x) \geq c$, where $\alpha = \sup\{\mathbb{P}(\lambda(X) \geq c) \mid (\mu, \sigma^2) \in \Theta_0\}$

Since $T(X)$ does not depend on μ, we have to solve $\alpha = \mathbb{P}(\lambda(X) \geq c)$

We have
$$\lambda(X) \geq c \iff \left(\left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{-1} \exp\left(\frac{\hat{\sigma}^2}{\sigma_0^2} - 1\right)\right)^{\frac{n}{2}} \geq c$$

$$\iff \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{-1} \exp\left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right) \exp(-1) \geq c^{-n/2}$$

$$\iff c^{n/2} \exp\left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right) \geq \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right) \exp(1)$$

$$\iff c^{n/2} \exp\left(\frac{1}{n\sigma_0^2} \sum_{i=1}^{n} (X_i - \bar{X})^2\right) \geq \exp(1) \frac{1}{n\sigma_0^2} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

Since $X_i \sim N(\mu, \sigma^2)$ we have $\frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \sim \chi^2(n-1)$