1. $X_1, X_2, \dots$ i.i.d. $\mathcal{U}(0,1)$, $X_{(n)} = \max_{i=1}^{n} X_i$. $\left[f(x) = 1 - e^{-x} \mathbb{1}_{[x \geq 0]} \right]$

Show that $Y_{(n)} = n(1 - X_{(n)})$ converges to an $\exp(1)$ r.v.

$$\mathbb{P}(X_{(n)} \leq x) = \prod_{i=1}^{n} \mathbb{P}(X_i \leq x) = x^n \mathbb{1}_{[0 \leq x \leq 1]} + \mathbb{1}_{[x > 1]}$$

$$\mathbb{P}(Y_{(n)} \leq y) = \mathbb{P}(n - X_{(n)} \cdot n \leq y) = \mathbb{P}(X_{(n)} \geq 1 - \tfrac{y}{n}) = 1 - (1 - \tfrac{y}{n})^n \mathbb{1}_{[0 \leq 1 - \frac{y}{n} \leq 1]} - \mathbb{1}_{[1 - \frac{y}{n} > 1]}$$

$$= 1 - (1 - \tfrac{y}{n})^n \mathbb{1}_{[n \geq y]} \mathbb{1}_{[0 \leq y]} - \mathbb{1}_{[y \leq 0]}$$

$$\xrightarrow{n \to \infty} 1 - e^{-y} \mathbb{1}_{[0 \leq y]} - \mathbb{1}_{[y \leq 0]}, \text{ so } Y_{(n)} \xrightarrow{d} Z \sim \exp(1). \quad \square$$

2. Unfair coin, $\mathbb{P}(\text{tail}) = \frac{1}{4}$

(a) $\mathbb{P}(440 \leq \underbrace{\#\text{heads}}_{=: X} \leq 460) = \sum_{i=440}^{460} \binom{600}{i} \left(\frac{1}{4}\right)^{600-i} \left(\frac{3}{4}\right)^i \approx 67{,}8\,\%$

pbinom(460, 600, 0.75) − pbinom(439, 600, 0.75)

(b) • without continuity correction: $\mathbb{P}(440 \leq X \leq 460) = ⊛$

$$\mathbb{P}(X \leq 460) = \mathbb{P}(X_1 + \dots + X_{600} \leq 460) = \mathbb{P}\left(\frac{X_1 + \dots + X_{600} - 600 \cdot \frac{3}{4}}{\sqrt{600 \cdot \frac{3}{16}}} \leq \frac{460 - 600 \cdot \frac{3}{4}}{\sqrt{600 \cdot \frac{3}{16}}} \right) \approx \Phi(0{,}9428)$$

$$\mathbb{P}(X \leq 440) \approx \Phi(-0{,}9428)$$

$$⊛ \approx \Phi(0{,}9428) - \Phi(-0{,}9428) = 2\Phi(0{,}9428) - 1 \approx 65{,}4\,\%$$

• with continuity correction: We do the same calculation, but this time with 439,5 and 460,5 instead of 440 and 460.

We get $⊛ \approx 2\Phi(0{,}9899) - 1 \approx 67{,}8\,\%$, which is closer to the true value.

3. → see R file

4. • Tower property: $\mathbb{E}X = \mathbb{E}\,\mathbb{E}(X|Y)$. Show $\mathbb{V}(Y) = \mathbb{E}(\operatorname{Var}(Y|X)) + \mathbb{V}(\mathbb{E}(Y|X))$.

a) • By definition, $\mathbb{V}(X|Y) = \mathbb{E}((Y - \mathbb{E}(Y|X))^2 \mid Y) = \mathbb{E}(Y^2|X) - \mathbb{E}(Y|X)^2$.

• $\mathbb{E}(\operatorname{Var}(Y|X)) + \mathbb{V}(\mathbb{E}(Y|X)) = \mathbb{E}(\mathbb{E}(Y^2|X) - \mathbb{E}(Y|X)^2) + \mathbb{E}\left[\mathbb{E}(Y|X) - \underbrace{\mathbb{E}\,\mathbb{E}(Y|X)}_{=\mathbb{E}(Y)}\right]^2$

$$= \mathbb{E}(\mathbb{E}(Y^2|X)) - \cancel{\mathbb{E}(\mathbb{E}(Y|X)^2)} + \cancel{\mathbb{E}(\mathbb{E}(Y|X)^2)}$$

$$-2\mathbb{E}\left[\mathbb{E}(Y|X)\,\mathbb{E}(Y)\right] + \mathbb{E}\,\mathbb{E}(Y)^2$$

$$= \mathbb{E}(Y^2) - 2\mathbb{E}\,\mathbb{E}(Y|X)\,\mathbb{E}\mathbb{E}(Y) + \mathbb{E}(Y)^2 = \mathbb{E}(Y^2) - 2\mathbb{E}(Y)^2 + \mathbb{E}(Y)^2 =$$

$$= \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = \mathbb{V}(Y).$$

b) $f(y|x) \triangleq \mathcal{N}(x, x^2)$, $X \sim \mathcal{U}(0,1)$. Compute $\mathbb{E}(Y)$, $\mathbb{V}(Y)$, $\mathrm{Cov}(X,Y)$.

- $\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X)) = \mathbb{E}(X) = \underline{\frac{1}{2}}$
- $\mathbb{V}(Y) = \mathbb{E}(\mathbb{V}(Y|X)) + \mathbb{V}(\mathbb{E}(Y|X)) = \mathbb{E}(X^2) + \mathbb{V}(X) = \frac{1}{3} + \frac{1}{12} = \underline{\frac{5}{12}}$

  $\Big[\mathbb{E}((X-\mathbb{E}(X))^2) = \mathbb{E}((X-\frac{1}{2})^2) = \mathbb{E}(X^2 - X + \frac{1}{4}) = \mathbb{E}(X^2) - \frac{1}{2} + \frac{1}{4} = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$,

  $\mathbb{P}(X^2 \le x) = \mathbb{P}(X \le \sqrt{x}) = \sqrt{x} \implies f_{X^2}(x) = \frac{1}{2\sqrt{x}} \implies \mathbb{E}(X^2) = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{x}} x \, dx = \frac{1}{2}\int_0^1 \sqrt{x}\, dx = \frac{1}{3}\Big]$
- $\mathrm{Cov}(X,Y) = \mathbb{E}\big[(X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\big] = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

  $= \mathbb{E}(\mathbb{E}(XY|X)) - \frac{1}{4} = \mathbb{E}(X\,\mathbb{E}(Y|X)) - \frac{1}{4} = \mathbb{E}(X^2) - \frac{1}{4} = \frac{1}{3} - \frac{1}{4} = \underline{\frac{1}{12}}$

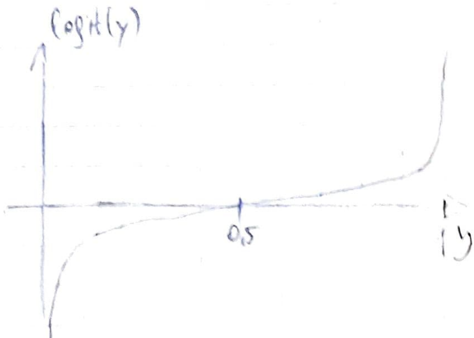
5. a) $X_1, \dots, X_n$ i.i.d. normal, μ unknown, σ^2 known. $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$.

Find the limiting distribution of $\sqrt{n}(\bar{X}^3 - c)$ for an appropriate constant c.

We know that $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$. We have $\sqrt{n}(\bar{X} - \mu) = \frac{\bar{X}-\mu}{\sqrt{n}} \longrightarrow \mathcal{N}(0, \sigma^2)$

by the central limit theorem and therefore, by the delta method (with $g(x) := x^3$),

$$\sqrt{n}(\bar{X}^3 - \mu^3) \xrightarrow{d} \mathcal{N}(0, \sigma^2 \cdot \underbrace{9\mu^4}_{= g'(\mu)^2}).$$



b) Let $X_n \sim \mathrm{Bin}(n, p)$, $\mathrm{logit}(y) = \log\frac{y}{1-y}$, $0 < y < 1$.

Determine the approximate distribution of $\mathrm{logit}(\frac{X_n}{n})$.

Let $Y_i \sim \mathrm{Bin}(1, p)$, $i \in [n]$, then $\frac{X_n}{n} = \bar{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_i$.

We have $\sqrt{n}(\bar{Y}_n - p) = \frac{\sum Y_i - p}{\sqrt{n}} \xrightarrow{d} Z \sim \mathcal{N}(0, \overbrace{\sigma^2}^{p(1-p)})$ by the CLT and therefore

by the delta method (with $g(\vartheta) = \mathrm{logit}(\vartheta)$) $[g'(\vartheta) = \frac{1}{\vartheta(1-\vartheta)} \neq 0$ for $\vartheta \notin \{0,1\}]$

$$\sqrt{n}(\mathrm{logit}(\bar{Y}_n) - \mathrm{logit}(p)) \xrightarrow{d} Z \sim \mathcal{N}(0, \frac{1}{(1-p)p}).$$

We apply the affine transformation $Z \mapsto \sqrt{n} Z + \mathrm{logit}(p)$ and get

$$\mathrm{logit}(\bar{Y}_n) \approx \sqrt{n} Z + \mathrm{logit}(p) \sim \mathcal{N}(\mathrm{logit}(p), \frac{1}{np(1-p)}).$$