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Numerik von Differentialgleichungen - Kreuzlübung 11

Date: 17.6.2020 June 10, 2020

Exercise 51:
Show that the symplectic Euler methods have convergence order 1. Moreover, construct an example which shows that they do not have higher convergence order.

Exercise 52:
Consider the Hamiltonian $H(p,q) = \frac{1}{2}p^2 + \frac{1}{2}q^2$. Show that the symplectic Euler

$$\begin{pmatrix} p_{\ell+1} \\ q_{\ell+1} \end{pmatrix} = \begin{pmatrix} p_\ell \\ q_\ell \end{pmatrix} + h \begin{pmatrix} -\nabla_q H(p_{\ell+1}, q_\ell) \\ \nabla_p H(p_{\ell+1}, q_\ell) \end{pmatrix} \tag{1}$$

does *not*, in general, preserve the energy $H(p,q)$, i.e., show that there exist p_0, q_0, such that $H(p_\ell, q_\ell) \neq H(p_0, q_0)$ for the iterates p_ℓ, q_ℓ of the symplectic Euler method.

Furthermore, consider the perturbed Hamiltonian

$$H_h(p,q) := \frac{1}{2} \begin{pmatrix} p \\ q \end{pmatrix}^\top \begin{pmatrix} 1 & -h/2 \\ -h/2 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}. \tag{2}$$

Show that for all p, q with $|p|, |q| \leq R \in \mathbb{R}$ there holds $\|H(p,q) - H_h(p,q)\| = \mathcal{O}(h)$ and that the symplectic Euler method preserves H_h, i.e., show that $H_h(p_\ell, q_\ell) = H_h(p_0, q_0)$.

Exercise 53:
Let Ψ^h be the discrete flow of the explicit Euler method. Which well-known methods are obtained by $\left(\Psi^{h/2}\right)^* \circ \Psi^{h/2}$ and $\Psi^{h/2} \circ \left(\Psi^{h/2}\right)^*$? What convergence order do these methods have?

Exercise 54:
Consider an arbitrary m-stage Runge-Kutta method with Butcher tableau $\begin{array}{c|c} c & A \\ \hline & b^\top \end{array}$ and discrete flow Ψ^h. Provide the Butcher tableaux of

a) the corresponding adjoint method with adjoint flow $\left(\Psi^h\right)^*$ and

b) the reversible method $\left(\Psi^{h/2}\right)^* \circ \Psi^{h/2}$.

Exercise 55:
Let $f \in C^1(\mathbb{R}^{2d}, \mathbb{R}^{2d})$ be the right-hand side of the autonomous system

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = f\big(\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right)\big). \tag{3}$$

Furthermore, let the mapping R be defined by

$$R\big(\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right)\big) = \left(\begin{smallmatrix} -p \\ q \end{smallmatrix}\right). \tag{4}$$

Moreover, suppose that there holds that

$$R \circ f = -f \circ R. \tag{5}$$

a) Show that, for the continuous flow Φ^t of (3), there holds that

$$R \circ \Phi^t = \Phi^{-t} \circ R.$$

b) Show that, for the discrete flow Ψ^h of a Runge-Kutta method applied to (3), there holds that

$$R \circ \Psi^h = \Psi^{-h} \circ R.$$

c) Let M be a symmetric, positive definite matrix, U a two times continuously differentiable function and H a Hamiltonian with

$$H(p,q) := \frac{1}{2} p^\top M^{-1} p + U(q). \tag{6}$$

Show that the function f of the corresponding Hamiltonian system satisfies (5).