HW9

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1. The GLRT for the normal variance - simple hypotheses

Derive the generalized likelihood ratio test (GLRT) for the normal variance: Assume $X_1, \ldots, X_n$ are i.i.d. $\mathcal{N}(\mu, \sigma^2)$, where both μ and σ are unknown. We want to test

$$H_0 : \sigma^2 = \sigma_0^2 \quad vs \quad H_1 : \sigma^2 \neq \sigma_0^2$$

Solution:

2. Most powerful test 1

Let $X_1, \ldots, X_n$ be i.i.d. Uniform$(0, \theta)$.

(a) Derive the most powerful (MP) test at level α for testing

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta = \theta_1, \ \theta_1 > 0.$$

(b) Calculate the power of the MP test.

Solution:

3. Most powerful test 2

Let $X_1, \ldots, X_n$ be i.i.d. from a distribution with density

$$f_\theta(x) = \frac{x}{\theta} e^{-\frac{x^2}{2\theta}}, \ x \geq 0, \ \theta > 0.$$

(a) Derive the MP test at level α for testing two simple hypotheses

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta = \theta_1, \ \theta_1 > \theta_0.$$

(b) Is there a uniformly most powerful (UMP) test at level α for testing the one-sided composite hypothesis

$$H_0 : \theta \leq \theta_0 \quad vs \quad H_1 : \theta > \theta_0$$

What is its power function?

Hint: Show $X_i^2 \sim \exp(1/2\theta)$, so that $\sum_i X_i^2 \sim \theta\chi^2(2n)$.

Solution:

4. Most powerful test for the normal variance - μ is known

Let $X_1, \dots, X_n$ be i.i.d. $\mathcal{N}(\mu, \sigma^2)$, where μ is known.

(a) Find an MP test at level α for testing two simple hypotheses

$$H_0 : \sigma^2 = \sigma_0^2 \quad vs \quad H_1 : \sigma^2 = \sigma_1^2,\ \sigma_1 > \sigma_0.$$

(b) Show that the MP test is a UMP test for testing

$$H_0 : \sigma^2 \leq \sigma_0^2 \quad vs \quad H_1 : \sigma^2 > \sigma_0^2.$$

Hint: $\sum_i (X_i - \mu)^2 \sim \sigma^2 \chi^2(n)$.

Solution:

5. Most powerful test for the normal variance - μ is unknown

Let $X_1, \dots, X_n$ be i.i.d. $\mathcal{N}(\mu, \sigma^2)$, where μ is unknown.

(a) Is there an MP test at level α for testing?

$$H_0 : \sigma^2 = \sigma_0^2 \quad vs \quad H_1 : \sigma^2 = \sigma_1^2,\ \sigma_1 > \sigma_0.$$

If not, find the corresponding GLRT.

(b) Is the above generalized likelihood ratio (GLR) test also a GLRT for testing the one-sided hypothesis?

$$H_0 : \sigma^2 \leq \sigma_0^2 \quad vs \quad H_1 : \sigma^2 > \sigma_0^2.$$

(c) Find the GLRT at level α for testing

$$H_0 : \sigma^2 \geq \sigma_0^2 \quad vs \quad H_1 : \sigma^2 < \sigma_0^2.$$

Solution: