(1) **The GLRT for the normal variance - simple hypotheses**
Derive the generalized likelihood ratio test (GLRT) for the normal variance: Assume $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, \sigma^2)$, where both μ and σ are unknown. We want to test

$$H_0 : \sigma^2 = \sigma_0^2 \quad vs \quad H_1 : \sigma^2 \neq \sigma_0^2.$$

The likelihood function is given by $L(\mu, \sigma^2; x) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2\right)$

We have $\Theta_0 := \mathbb{R} \times \{\sigma_0^2\}$, $\Theta := \mathbb{R} \times \mathbb{R}^+$ and $\Theta_1 := \Theta \setminus \Theta_0$

We can use the MLEs $\hat{\mu} = \bar{x}$ and $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$ to obtain the GLR

$$\lambda(x) = \frac{\sup\{L(\mu, \sigma^2; x) \mid (\mu, \sigma^2) \in \Theta\}}{\sup\{L(\mu, \sigma^2; x \mid (\mu, \sigma^2) \in \Theta_0\}} = \frac{L(\hat{\mu}, \hat{\sigma}^2; x)}{L(\hat{\mu}, \sigma_0^2; x)} = \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{-\frac{n}{2}} \exp\left(\left(\frac{1}{\sigma_0^2} - \frac{1}{\hat{\sigma}^2}\right)\frac{n}{2}\hat{\sigma}^2\right)$$

$$= \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{-\frac{n}{2}} \exp\left(\frac{n}{2}\left(\frac{\hat{\sigma}^2}{\sigma_0^2} - 1\right)\right)$$

We take $T(x) := \lambda(x)$ as our test statistic.

We reject H_0, if $\lambda(x) \geq C$, where $\alpha = \sup\{\mathbb{P}(\lambda(X) \geq C) \mid (\mu, \sigma^2) \in \Theta_0\}$

Since $T(X)$ does not depend on μ, we have to solve $\alpha = \mathbb{P}(\lambda(X) \geq C)$

We have $\lambda(X) \geq C \Leftrightarrow \left(\left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{-1} \exp\left(\frac{\hat{\sigma}^2}{\sigma_0^2} - 1\right)\right)^{\frac{n}{2}} \geq C$

$$\Leftrightarrow \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{-1} \exp\left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right) \exp(-1) \geq C^{-n/2}$$

$$\Leftrightarrow C^{n/2} \exp\left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right) \geq \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right) \exp(1)$$

$$\Leftrightarrow C^{n/2} \exp\left(\frac{1}{n\sigma_0^2}\sum_{i=1}^n (x_i - \bar{x})^2\right) \geq \exp(1)\frac{1}{n\sigma_0^2}\sum_{i=1}^n (X_i - \bar{X})^2$$

Since $X_i \sim N(\mu, \sigma^2)$ we have $\frac{1}{\sigma^2}\sum_{i=1}^n (X_i - \bar{X})^2 \sim \chi^2(n-1)$

(2) **Most powerful test 1**
Let $X_1, \ldots, X_n$ be iid Uniform$(0, \theta)$.

(a) Derive the most powerful (MP) test at level α for testing

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta = \theta_1, \theta_1 > \theta_0.$$

(b) Calculate the power of the MP test.

a) $L(\theta; x) = \begin{cases} \frac{n}{\theta}, & \text{if } \forall i \in \{1, \ldots, n\}: x_i \in (0, \theta) \\ 0, & \text{else} \end{cases}$, Therefore we obtain for $x \in (0, \theta_1)$

$$\lambda(X) = \frac{L(\theta_1, x)}{L(\theta_0, x)} = \begin{cases} \frac{\theta_0}{\theta_1}, & \text{if } \max\{x_i \mid 1 \le i \le n\} < \theta_0 \\ \text{„}\infty\text{"}, & \text{else} \end{cases}$$

We presume ⊛ that $T(x) = \max\{x_i \mid 1 \le i \le n\}$ is an appropriate Test-statistic for an MP

Assuming that $X_i \sim U(0, \theta_0)$, we have

$$\mathbb{P}(T(x) \ge C) = 1 - \mathbb{P}(T(x) < C) = 1 - \prod_{i=1}^{n} \mathbb{P}(X_i < C) = \begin{cases} 1 & , \text{if } C \le 0 \\ 0 & , \text{if } C \ge \theta_0 \\ 1 - \left(\frac{C}{\theta_0}\right)^n & , \text{if } 0 < C < \theta_0 \end{cases}$$

If $0 < \alpha < 1$, then $\mathbb{P}(T(X) \ge C) = \alpha \Leftrightarrow \alpha = 1 - \left(\frac{C}{\theta_0}\right)^n \Leftrightarrow \left(\frac{C}{\theta_0}\right)^n = 1 - \alpha \Leftrightarrow C = \theta_0 (1-\alpha)^{\frac{1}{n}}$

Hence, our test rejects H_0, if $T(x) \ge \theta_0 (1-\alpha)^{\frac{1}{n}}$

b) The power q of the test is

$$q = \mathbb{P}(T(X) \ge \theta_0 (1-\alpha)^{\frac{1}{n}} \mid X_i \sim U(0, \theta_1)) = 1 - \prod_{i=1}^{n} \mathbb{P}(X_i < \theta_0 (1-\alpha)^{\frac{1}{n}}) = 1 - \left(\frac{\theta_0 (1-\alpha)^{\frac{1}{n}}}{\theta_1}\right)^n$$

$$= 1 - (1-\alpha)\left(\frac{\theta_0}{\theta_1}\right)^n$$

⊛ This presumption turns out to be true, since for any other test of level α' with rejection region R' we have

$$\mathbb{P}(X \in R' \mid X \sim U(0, \theta_1)) = \int_{R'} \frac{1}{\theta_1^n} \mathbb{1}_{[0,\theta_1]^n}(x)\, dx = \frac{\theta_0^n}{\theta_1^n} \int_{R'} \frac{1}{\theta_0^n} \mathbb{1}_{[0,\theta_0]^n}(x)\, dx + \int_{R'} \frac{1}{\theta_1^n} \mathbb{1}_{([0,\theta_0]^n)^c}(x)\, dx$$

$$\le \frac{\theta_0^n}{\theta_1^n} \alpha + \int_{([0,\theta_0]^n)^c} \frac{1}{\theta_1^n}\, dx = \frac{\theta_0^n}{\theta_1^n} \alpha + 1 - \frac{\theta_0^n}{\theta_1^n} = 1 - (1-\alpha)\left(\frac{\theta_0}{\theta_1}\right)^n = q$$

(3) **Most powerful test 2**
Let $X_1, \dots, X_n$ be iid from a distribution with density

$$f_\theta(x) = \frac{x}{\theta} e^{-\frac{x^2}{2\theta}}, \; x \geq 0, \, \theta > 0.$$

(a) Derive the MP test at level α for testing two simple hypoheses

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta = \theta_1, \, \theta_1 > \theta_0.$$

(b) Is there a uniformly most powerful (UMP) test at level α for testing the one-sided composite hypothesis

$$H_0 : \theta \leq \theta_0 \quad vs \quad H_1 : \theta > \theta_0$$

What is its power function?
Hint: Show $X_i^2 \sim \exp(1/2\theta)$, so that $\sum_i X_i^2 \sim \theta\chi^2(2n)$.

a)
$$L(\theta; x) = \begin{cases} \frac{1}{\theta^n} \prod_{i=1}^n x_i \exp\left(-\frac{1}{2\theta} \sum_{i=1}^n x_i^2\right), & \text{if } \min\{x_i \mid 1 \leq i \leq n\} \geq 0 \\ 0 & \text{, else} \end{cases}$$

For $x \in (\mathbb{R}^+)^n$ we have

$$\lambda(x) = \frac{L(\theta_1; x)}{L(\theta_0; x)} = \left(\frac{\theta_0}{\theta_1}\right)^n \exp\left(\left(\frac{1}{2\theta_0} - \frac{1}{2\theta_1}\right) \sum_{i=1}^n x_i^2\right)$$

Since $\theta_1 > \theta_0$, we obtain that the function $\lambda(x)$ is a monotone increasing function of $T(x) = \sum_{i=1}^n x_i^2$, which we choose as our test-statistic

We define $Y_i := X_i^2$, and have

$$f_{Y_i}(y) = f_{X_i}(\sqrt{y}) \frac{1}{2\sqrt{y}} + \overbrace{f_{X_i}(-\sqrt{y})}^{=0} \frac{1}{2\sqrt{y}} = \frac{\sqrt{y}}{\theta} e^{-\frac{y}{2\theta}} \frac{1}{2\sqrt{y}} = \frac{1}{2\theta} e^{-\frac{y}{2\theta}}$$

Thus $Y_i \sim \exp\left(\frac{1}{2\theta}\right)$, or equivalently $Y_i \sim \text{Gamma}(1, \frac{1}{2\theta})$, hence $T(X) \sim \text{Gamma}(n, \frac{1}{2\theta})$

Hence, $\sum_{i=1}^n Y_i \sim \text{Erlang}(n, \frac{1}{2\theta})$ and $\frac{1}{\theta} T(X) \sim \chi^2(2n)$, we write symbolically

$T(X) \sim \theta \chi^2(2n)$

We have $\alpha = \mathbb{P}(T(X) \geq c) \Leftrightarrow \alpha = 1 - \mathbb{P}(\frac{1}{\theta_0} T(X) < \frac{c}{\theta_0}) \Leftrightarrow \mathbb{P}(\frac{1}{\theta_0} T(X) < \frac{c}{\theta_0}) = 1 - \alpha$

$\Leftrightarrow F_{\chi^2(2n)}\left(\frac{c}{\theta_0}\right) = 1 - \alpha \Leftrightarrow \frac{c}{\theta_0} = F^{-1}_{\chi^2(2n)}(1-\alpha) \Leftrightarrow c = \theta_0 F^{-1}_{\chi^2(2n)}(1-\alpha)$

Our test rejects H_0, if $T(x) \geq c$.

b) The test from (a) is by the theorem at p. 33 from Lecture 10 an UMP

the power q is given by

$$q = \mathbb{P}\left(T(X) \geq c \mid \tfrac{1}{\theta} T(X) \sim \chi^2(2n)\right) = \mathbb{P}\left(\tfrac{1}{\theta} T(X) \geq \tfrac{c}{\theta}\right) = 1 - F_{\chi^2(2n)}\left(\tfrac{c}{\theta}\right)$$

$$= 1 - F_{\chi^2(2n)}\left(\frac{\theta_0}{\theta} F^{-1}_{\chi^2(2n)}(1-\alpha)\right)$$

(4) **Most powerful test for the normal variance - μ is known**
Let $X_1, \dots, X_n$ be iid $\mathcal{N}(\mu, \sigma^2)$, where μ is known.

(a) Find an MP test at level α for testing two simple hypoheses
$$H_0 : \sigma^2 = \sigma_0^2 \quad vs \quad H_1 : \sigma^2 = \sigma_1^2,\ \sigma_1 > \sigma_0 .$$

(b) Show that the MP test is a UMP test for testing
$$H_0 : \sigma^2 \leq \sigma_0^2 \quad vs \quad H_1 : \sigma^2 > \sigma_0^2.$$

Hint: $\sum_i (X_i - \mu)^2 \sim \sigma^2 \chi^2(n)$.

a) $L(\mu, \sigma; x) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right)$

$\lambda(x) = \frac{L(\mu, \sigma_1; x)}{L(\mu, \sigma_0; x)} = \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{n/2} \exp\left(\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) \frac{1}{2} \sum_{i=1}^{n} (x_i - \mu)^2\right)$ is an MP-test

$T(x) := \sum_{i=1}^{n} (x_i - \mu)^2$, we reject H_0 if $T(x) \geq C$, where $\mathbb{P}(T(X) \geq C) = \alpha$.

$\frac{X_i - \mu}{\sigma} \sim \mathcal{N}(0, 1)$, hence $\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 = \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2 \sim \chi^2(n)$

We write symbolically $T(X) \sim \sigma^2 \chi^2(n)$

Therefore, $\alpha = \mathbb{P}(T(X) \geq C) = 1 - \mathbb{P}\left(\frac{1}{\sigma_0^2} T(X) < \frac{C}{\sigma_0^2}\right) = 1 - F_{\chi^2(n)}\left(\frac{C}{\sigma_0^2}\right)$

$\Leftrightarrow F_{\chi^2(n)}\left(\frac{C}{\sigma_0^2}\right) = 1 - \alpha \Leftrightarrow C = \sigma_0^2 F^{-1}_{\chi^2(n)}(1 - \alpha)$

Our test rejects H_0, if $T(x) \geq C$.

b) The test from (a) is by the theorem at p. 33 from Lecture 10 also an UMP

For all $\sigma \in (0, \sigma_0]$ we have

$\mathbb{P}_\sigma(T(X) \geq C) = \mathbb{P}_\sigma\left(\frac{1}{\sigma^2} T(X) \geq \frac{C}{\sigma^2}\right) = 1 - F_{\chi^2(n)}\left(\frac{C}{\sigma^2}\right) = 1 - F_{\chi^2(n)}\left(\frac{\sigma_0^2}{\sigma^2} F^{-1}_{\chi^2(n)}\left(\frac{C}{\sigma^2}\right)\right)$

$\leq 1 - \frac{C}{\sigma_0^2} \Leftrightarrow \frac{C}{\sigma_0^2} \leq F_{\chi^2(n)}\left(\frac{\sigma_0^2}{\sigma^2} F^{-1}_{\chi^2(n)}\left(\frac{C}{\sigma^2}\right)\right)$

and since $\sigma_0^2 > \sigma^2$ we have $F_{\chi^2(n)}\left(\frac{\sigma_0^2}{\sigma^2} F^{-1}_{\chi^2(n)}\left(\frac{C}{\sigma^2}\right)\right) \geq F_{\chi^2(n)}\left(F^{-1}_{\chi^2(n)}\left(\frac{C}{\sigma^2}\right)\right) = \frac{C}{\sigma^2}$

Hence $\sup\{\mathbb{P}_\sigma(T(X) \geq C) \mid \sigma \in (0, \sigma_0]\} = \mathbb{P}_{\sigma_0}(T(X) \geq C) = \alpha$

To show the remaining property we consider $\sigma_1 > \sigma_0$ and any test at level $\alpha' \leq \alpha$ with rejection region R'. Since our test is an MP for (a), we have

$\mathbb{P}_{\sigma_1}(T(X) \geq C) \geq \mathbb{P}(X \in R)$

(5) **Most powerful test for the normal variance - μ is unknown**
Let $X_1, \dots, X_n$ be iid $\mathcal{N}(\mu, \sigma^2)$, where μ is unknown.

(a) Is there an MP test at level α for testing?

$$H_0 : \sigma^2 = \sigma_0^2 \quad vs \quad H_1 : \sigma^2 = \sigma_1^2,\ \sigma_1 > \sigma_0 .$$

If not, find the corresponding GLRT.

(b) Is the above generalized likelihood ratio (GLR) test also a GLRT for testing the one-sided hypothesis?

$$H_0 : \sigma^2 \leq \sigma_0^2 \quad vs \quad H_1 : \sigma^2 > \sigma_0^2 .$$

(c) Find the GLRT at level α for testing

$$H_0 : \sigma^2 \geq \sigma_0^2 \quad vs \quad H_1 : \sigma^2 < \sigma_0^2 .$$

a) Assume there is an MP test at level α with the rejection region R, then

$$\sum_{i=1}^n (x_i-\mu_1)^2 = \sum_{i=1}^n (x_i^2 - 2x_i\mu_1 + \mu_1^2) = \sum_{i=1}^n \left((x_i-\mu)^2 + 2x_i\mu - \mu^2 - 2x_i\mu_1 + \mu_1^2\right)$$

$$\mathbb{P}(X \in R) = \int_R (2\pi\sigma_1^2)^{-n/2} \exp\left(-\frac{1}{2\sigma_1^2}\sum_{i=1}^n (x_i-\mu)^2\right) dx$$

We have MPs for the simple problems (μ_1, σ_1) and (μ_1', σ_1)

$$= \exp\left(-\frac{1}{2\sigma_1^2}\sum_{i=1}^n \left(2x_i(\mu-\mu_1) + \mu_1^2 - \mu^2\right)\right) \int_R L(\mu_1, \sigma_1^2; x)\, dx$$

$$\leq \exp\left(-\frac{1}{2\sigma_1^2}\sum_{i=1}^n \left(2x_i(\mu-\mu_1) + \mu_1^2 - \mu^2\right)\right) \alpha$$

$$< \exp\left(-\frac{1}{2\sigma_1^2}\sum_{i=1}^n \left(2x_i(\mu-\mu_1') + \mu_1'^2 - \mu^2\right)\right) \alpha$$

$$= \mathbb{P}(X \in R')$$

$\Theta_0 = \mathbb{R} \times \{\sigma_0\}$, $\quad \Theta_1 = \mathbb{R} \times \{\sigma_1\}$, $\quad \Theta := \Theta_0 \cup \Theta_1$

The likelihood function is given by $L(\mu, \sigma; x) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2\right)$

The MLEs are $\hat{\mu} = \bar{x}$ and $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$, hence the GLRT reads

$$\lambda(x) = \frac{L(\hat{\mu}, \sigma_1^2, x)}{L(\hat{\mu}, \sigma_0^2, x)} = \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{n/2} \exp\left(\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\frac{1}{2}\sum_{i=1}^n (x_i - \bar{x})^2\right)$$

$$\frac{1}{\sigma_0^2} \geq \frac{1}{\sigma_1} \Leftrightarrow \sigma_1^2 \geq \sigma_0^2 \Leftrightarrow \sigma_1^2 \geq \sigma_0^2$$

We choose $T(x) := \sum_{i=1}^n (x_i - \bar{x})^2$ as a simpler test statistic

Since $X_i \sim N(\mu, \sigma^2)$ we have $\frac{1}{\sigma^2} T(X) \sim \chi^2(n-1)$

b) $$\lambda(x) = \frac{\sup\{L(\mu,\sigma^2;x) \mid (\mu,\sigma^2) \in \mathbb{R}\times\mathbb{R}^+\}}{\sup\{L(\mu,\sigma^2;x \mid (\mu,\sigma^2) \in \mathbb{R}\times(0,\sigma_0^2]\}} = \begin{cases} 1 & , \text{if } \hat{\sigma}^2 \leq \sigma_0^2 \\ \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{-\frac{n}{2}} \exp\left(\left(\frac{1}{\sigma_0^2} - \frac{1}{\hat{\sigma}^2}\right)\frac{n}{2}\hat{\sigma}^2\right) & , \text{if } \hat{\sigma}^2 > \sigma_0^2 \end{cases}$$

c) $$\lambda(x) = \frac{\sup\{L(\mu,\sigma^2;x) \mid (\mu,\sigma^2) \in \mathbb{R}\times\mathbb{R}^+\}}{\sup\{L(\mu,\sigma^2;x \mid (\mu,\sigma^2) \in \mathbb{R}\times[\sigma_0,\infty)\}} = \begin{cases} 1 & , \text{if } \hat{\sigma}^2 \geq \sigma_0^2 \\ \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{-\frac{n}{2}} \exp\left(\underbrace{\left(\frac{1}{\sigma_0^2} - \frac{1}{\hat{\sigma}^2}\right)\frac{n}{2}\hat{\sigma}^2}_{=\left(\frac{\hat{\sigma}^2}{\sigma_0^2}-1\right)\frac{n}{2}}\right) & , \text{if } \hat{\sigma}^2 < \sigma_0^2 \end{cases}$$