(1) **A left-turn lane problem**

A civil engineer is studying a left-turn lane that is long enough to hold six cars. Let X be the number of cars in the lane at the end of a randomly chosen red light. The engineer believes that the probability that $X = x$ is proportional to $(x+1)(7-x)$.

(a) Find the probability mass function (pmf) of X.

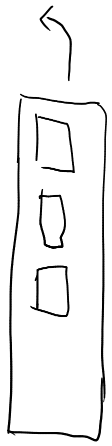
(b) Compute the probability that X will be at least 4.

(c) Calulate the expectation and standard deviation of X.
Note: `R` might be useful.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$$



a) $f(x) = \mathbb{P}(X=x) = a\,(x+1)(7-x)$

$$1 \overset{!}{=} \sum_{x=0}^{6} f(x) = a \sum_{x=0}^{6} (x+1)(7-x) = a\left(\sum_{x=0}^{6} 7 + \sum_{x=0}^{6} 6x - \sum_{x=0}^{6} x^2\right)$$

$$= a\left(49 + 6\,\frac{7\cdot 6}{2} - \frac{6\cdot 7\cdot 13}{6}\right)$$

$$= a\left(49 + 6\cdot 21 - 7\cdot 13\right) = a\left(49 + 126 - 91\right)$$

$$= a\,84 \Rightarrow a = \frac{1}{84}$$

b) $$\mathbb{P}(X \geq 4) = 1 - \mathbb{P}(X < 4) = 1 - \sum_{k=0}^{3} \mathbb{P}(X=k) = 1 - \frac{1}{84}\left(\sum_{k=0}^{3} 7 + \sum_{k=0}^{3} 6k - \sum_{k=0}^{3} k^2\right)$$

$$= 1 - \frac{1}{84}\left(28 + 6\,\frac{3\cdot 4}{2} - \frac{3\cdot 4\cdot 7}{6}\right) = 1 - \frac{1}{84}\left(28 + 36 - 14\right) = 1 - \frac{1}{84}\,50 = 1 - \frac{25}{42} = \frac{17}{42}$$

c) $$\mathbb{E}(X) = \sum_{k=0}^{6} k\,\mathbb{P}(X=k) = \frac{1}{84}\sum_{k=0}^{6} k(k+1)(7-k) = \frac{1}{84}\left(7\sum_{k=0}^{6} k + 6\sum_{k=0}^{6} k^2 - \sum_{k=0}^{6} k^3\right)$$

$$= \frac{1}{84}\left(7\,\frac{6\cdot 7}{2} + 6\,\frac{6\cdot 7\cdot 13}{6} - 21^2\right) = \frac{1}{84}\left(7\cdot 21 + 2\cdot 13\cdot 21 - 21^2\right)$$

$$= \frac{21}{84}\left(7 + 26 - 21\right) = \frac{21}{84}\,12 = \frac{7}{28}\,12 = \frac{7}{14}\,6 = \frac{6}{2} = 3$$

$$\mathbb{V}(X) = \mathbb{E}\left((X - \mathbb{E}(X))^2\right) = \mathbb{E}\left((X-3)^2\right) = \sum_{k=0}^{6} (k-3)^2\,\mathbb{P}(X=k) = 3 \ldots$$ calculation in R

Thus, $s = \sqrt{\mathbb{V}(X)} = \sqrt{3}$