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Numerik von Differentialgleichungen - Kreuzlübung 12

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This is the last exercise. As a reminder: In total, you need 24 correctly worked out exercises to pass the exercise class. This corresponds to 40% of all exercises. The grade of the exercise class is a result of the number of correctly worked out exercises and the quality of your uploads and presentations.

Exercise 56:
Based on Newton's method, formulate an algorithm for the iterated shooting method (end of Section 7.2 of the lecture notes). Proceed in the same was as for Alg. 7.4 for the simple shooting method. Write down the Jacobian explicitly.

Exercise 57:
Consider the solution u of the Poisson problem

$$-\partial_x \left(k(x)\partial_x u(x)\right) = \cos(x), \qquad x \in (0, 2\pi) \tag{1}$$

with $u(0) = u(2\pi) = 1$ and given thermal conductivity k. This equation models the temperature distribution in a thin rod.
a) Solve the problem analytically for $k(x) = k_0$ for all $x \in [0, 2\pi]$.
b) Solve the problem from **a)** wit the finite difference method and compare the error for different values of h.
c) How can (1) be interpreted if the termal conductivity is only piecewise constant, i.e., if, for instance, $k(x) = k_1$ for $x \in [0, \pi)$ and $k(x) = k_2 \neq k_1$ for $x \in (\pi, 2\pi]$? Solve this problem analytically and suggest a suitable discretization method.

Exercise 58:
Consider the two-dimensional Poisson problem with homogeneous Dirichlet boundary conditions (cf. (7.19) in the lecture notes). Explicitly write down the system matrix A_h and the right-hand side g_h of the resulting linear system, if this problem is discretized by finite differences with step-size $h_1 = h_2 = h > 0$. For simplicity, start with the mesh from Fig. 1. To this end, arrange the unknowns $y_{j,k} \approx y(x_{j,k})$ in the following order:

$$y_{1,1}, \dots, y_{1,N_2-1}, \quad y_{2,1}, \dots, y_{2,N_2-1}, \quad y_{3,1}, \dots. \tag{2}$$

Exercise 59:
In the lecture, we have proved that, for sufficiently smooth functions, there holds that

$$u''(x) = \frac{1}{h^2}\left[1 \quad -2 \quad 1\right] u(x) + \mathcal{O}(h^2)$$

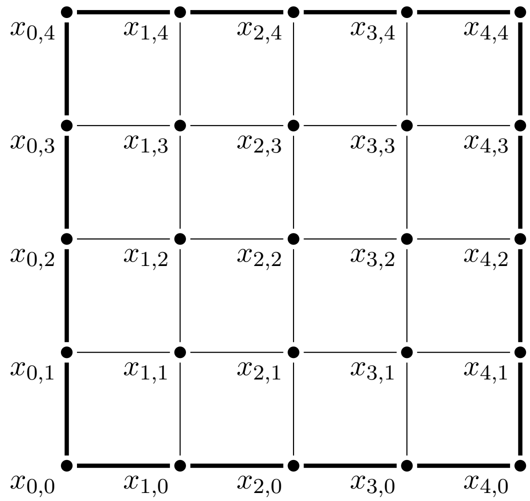

Figure 1: Finite difference mesh in 2D

with the 3-point stencil $[1 \quad -2 \quad 1]\, u(x) := 1u(x-h) - 2u(x) + 1u(x+h)$. Construct an approximation of the form

$$u''(x) = \frac{1}{h^2}\,[c_{-2} \quad c_{-1} \quad c_0 \quad c_1 \quad c_2]\, u(x) + \mathcal{O}(h^4)$$

with suitable constants $c_{-2}, \ldots, c_2 \in \mathbb{R}$ and the 5-point stencil

$$[c_{-2} \quad c_{-1} \quad c_0 \quad c_1 \quad c_2]\, u(x) := \sum_{i=-2}^{2} c_i u(x+ih).$$

Exercise 60:

The aim of this exercise is the construction of a finite difference method for curved boundaries. To this end, let $E = (x_i, y_j) \in \mathbb{R}^2 \in \Omega$ be a mesh node in the interior of Ω in such a way that the nodes left $(x_i - h, y_j)$ and below $(x_i, y_j - h)$ of E lie in Ω, but the nodes right $(x_i + h, y_j)$ and above $(x_i, y_j + h)$ of E do not lie in Ω. Analogously to exercise 59, construct a 5-point stencil for $\Delta u(E)$, which uses E, the neighboring nodes $(x_i - h, y_j)$ and $(x_i, y_j - h)$ that lie in Ω, as well as the boundary nodes $(x_i + \delta_x h, y_j), (x_i, y_j + \delta_y h) \in \partial\Omega$ with $\delta_x, \delta_y \in (0, 1)$ (cf. Fig. 2). Which order can be achieved?

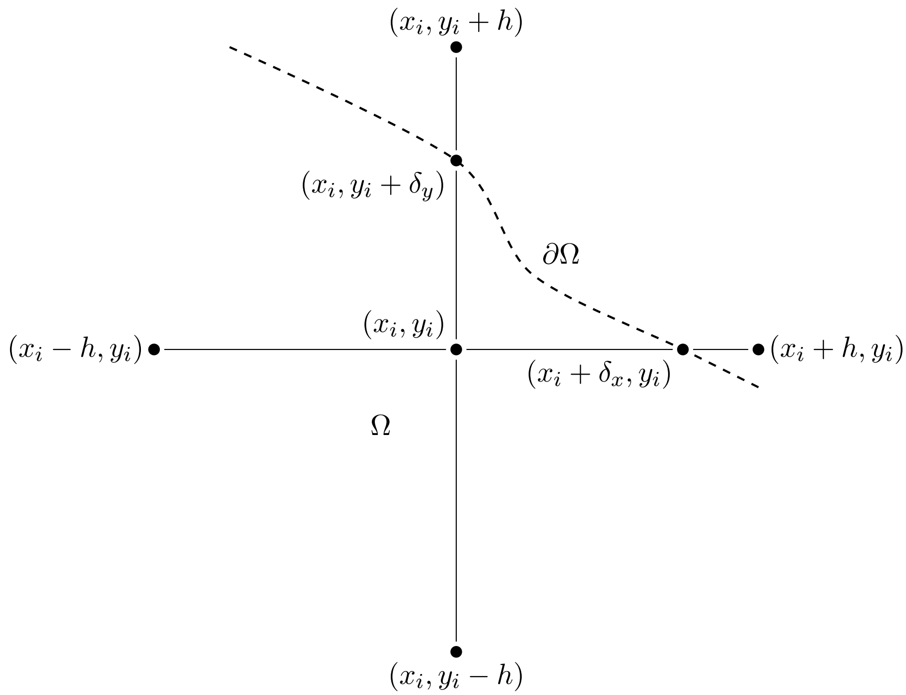

Figure 2: Stencil for curved boundaries.