(1) **Uniform distribution**
Let $X_1, \ldots, X_n$ be a random sample from uniform $(\theta, 1)$ distribution, where $\theta < 1$ is an unknown parameter.

(a) Find the MLE $\hat{\theta}$ of $\theta$.

(b) Is $\hat{\theta}$ asymptotically normal? If yes, find the asymptotic mean and variance. Otherwise, find a sequence $r_n$ and $a_n$ such that $r_n(\hat{\theta} - a_n)$ converges in distribution to a non-degenerate (not pointmass) distribution.

a)
$$L(\theta|x) = \prod_{i=1}^{n} f_\theta(x_i) = \begin{cases} 0 & , \text{if } \exists i \in \{1,\ldots,n\} : (x_i \leq \theta \vee x_i \geq 1\} \Leftrightarrow (\min\{x_i | 1 \leq i \leq n\} \geq \theta) \wedge (\max\{x_i|1\leq i\leq n\} \leq 1) \\ (1-\theta)^{-n} & , \text{else} \end{cases}$$

Let $\theta_1 < \theta_2 < 1$, then $(1-\theta_2)^{-n} > (1-\theta_1)^{-n} \Leftrightarrow \left(\frac{1-\theta_1}{1-\theta_2}\right)^n > 1 \Leftrightarrow \frac{1-\theta_1}{1-\theta_2} > 1 \Leftrightarrow 1-\theta_1 > 1-\theta_2 \Leftrightarrow \theta_1 < \theta_2$,

hence $L(\theta, x)$ is an increasing function for $\theta \in (-\infty, \min\{x_i | 1\leq i \leq n\})$ that clearly has it's maximum at $\hat{\theta}(x) := \min\{x_i | 1 \leq i \leq n\}$

b)
$$\mathbb{P}(\min\{X_i | 1 \leq i \leq n\} \leq x) = \prod_{i=1}^{n} \mathbb{P}(X_i \leq x) = \begin{cases} 0 & , \text{if } x < \theta \\ \left(\frac{x-\theta}{1-\theta}\right)^n & , \text{if } \theta \leq x < 1 \\ 1 & , \text{if } 1 \leq x \end{cases} \xrightarrow{n\to\infty} \begin{cases} 0 & , \text{if } x < 1 \\ 1 & , \text{if } x \geq 1 \end{cases}$$

hence, $\hat{\theta}(X)$ is not asymptotically normal distributed.

We choose $r_n := n$, $a_n := 1$ and obtain for all $x \in \mathbb{R}$

$$r_n(\theta - a_n) = n(\theta - 1) \xrightarrow{n\to\infty} -\infty, \qquad r_n(1-a_n) = n(1-1) = 0, \text{ and}$$

$$\left(\frac{x}{r_n(1-\theta)} + \frac{a_n - \theta}{1-\theta}\right)^n = \left(1 + \frac{x/(1-\theta)}{n}\right)^n \xrightarrow{n\to\infty} \exp\left(\frac{x}{1-\theta}\right).$$

Since $\frac{x}{r_n} + a_n < \theta \Leftrightarrow x < r_n(\theta - a_n)$ and $\frac{x}{r_n} + a_n < 1 \Leftrightarrow x < r_n(1-a_n)$, we have

$$\mathbb{P}(r_n(\min\{X_i | 1 \leq i \leq n\} - a_n) \leq x) = \begin{cases} 0 & , \text{if } \frac{x}{r_n} + a_n < \theta \\ \left(\frac{x}{r_n(1-\theta)} - \frac{a_n - \theta}{1-\theta}\right)^n & , \text{if } \theta \leq \frac{x}{r_n} + a_n < 1 \\ 1 & , \text{if } 1 \leq \frac{x}{r_n} + a_n \end{cases} \xrightarrow{n\to\infty} \begin{cases} \exp\left(\frac{x}{1-\theta}\right) & , \text{if } x < 0 \\ 1 & , \text{if } x \geq 0 \end{cases}$$

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(2) **Cramér-Rao lower bound**
Let $X_1, \dots, X_n$ be a random sample with the pdf $f(x|\theta) = \theta x^{\theta-1}$, where $0 < x < 1$ and $\theta > 0$ is unknown. Is there a function of $\theta$, say $g(\theta)$, for which there exists an unibiased estimator whose variance attains the Cramér-Rao lower bound? If there is, find it. If not, show why not.

$$\ell_n(\theta) = \sum_{i=1}^n \left(\log(\theta) + (\theta-1)\log(x_i)\right)$$

$$\ell_n'(\theta) = \sum_{i=1}^n \left(\frac{1}{\theta} + \log(x_i)\right) = \frac{n}{\theta} + \sum_{i=1}^n \log(x_i)$$

$$\ell_n''(\theta) = -\frac{n}{\theta^2} < 0$$

Since the MLE is the most efficient unbiased estimator, we want

$$0 = \ell_n'(\hat{\theta}) = \frac{n}{\theta} + \sum_{i=1}^n \log(x_i) \Leftrightarrow g(\theta) := \frac{1}{\theta} = -\frac{1}{n}\sum_{i=1}^n \log(x_i) =: h(x)$$

$$\mathbb{E}(h(X)) = -\frac{1}{n}\sum_{i=1}^n \mathbb{E}(\log(X_i)) = -\frac{1}{n}\sum_{i=1}^n \int_0^1 \log(x_i)\,\theta x_i^{\theta-1}\,dx_i$$

$$= -\frac{1}{n}\sum_{i=1}^n \left(\log(x_i)\,x_i^\theta\Big|_{x_i=0}^1 - \int_0^1 x_i^{-1} x_i^\theta\,dx_i\right) = \frac{1}{n}\sum_{i=1}^n \frac{1}{\theta} = \frac{1}{\theta} = g(\theta)$$

We have

$$\int_0^1\int_0^1 \log(x_i)\log(x_j)\,\theta^2 x_i^{\theta-1} x_j^{\theta-1}\,dx_i\,dx_j = \left(-\frac{1}{\theta}\right)^2 = \frac{1}{\theta^2}$$

$$\int_0^1 (\log(x_i))^2\,\theta\,x_i^{\theta-1}\,dx_i = (\log(x_i))^2 x_i^\theta\Big|_{x_i=0}^1 - 2\int_0^1 \log(x_i)\,x_i^{\theta-1}\,dx_i$$

$$= -\frac{2}{\theta}\int_0^1 \log(x_i)\,\theta x_i^{\theta-1}\,dx_i = \frac{2}{\theta^2} = \frac{1}{\theta^2} + \frac{1}{\theta^2}$$

Thus,

$$\mathbb{E}\left((h(X))^2\right) = \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \mathbb{E}\left(\log(x_i)\log(x_j)\right) = \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \frac{1}{\theta^2} + \frac{1}{n^2}\sum_{i=1}^n \frac{1}{\theta^2}$$

$$= \frac{1}{\theta^2} + \frac{1}{n\theta^2}$$

Hence, $\mathrm{Var}(h(X)) = \frac{1}{n\theta^2}$

$$\gamma := \frac{1}{\theta}, \quad \ell_n(\gamma) = \sum_{i=1}^n \left(-\log(\gamma) + (\gamma^{-1}-1)\log(x_i)\right)$$

$$\ell_n'(\gamma) = \sum_{i=1}^n \left(-\gamma^{-1} - \gamma^{-2}\log(x_i)\right)$$

$$\ell_n''(\gamma) = \sum_{i=1}^n \left(\gamma^{-2} + 2\gamma^{-3}\log(x_i)\right)$$

$$\mathbb{E}(\ell_n''(\gamma)) = \frac{n}{\gamma^2} + 2\gamma^{-3}\sum_{i=1}^n \mathbb{E}(\log(x_i)) = \frac{n}{\gamma^2} - 2\gamma^{-3} n\gamma = \frac{-n}{\gamma^2} = n\theta^2 = I_n(\theta)$$

$$\Rightarrow \mathrm{Var}(h(X)) = -\frac{1}{I_n(\theta)}$$

(3) **Minimum variance estimator**
Let $W_1, \dots, W_k$ be unbiased estimators of a parameter $\theta$ with $\mathbb{V}ar = \sigma_i^2$ and $\mathbb{C}ov(W_i, W_j) = 0$ if $i \neq j$. Show that, of all estimators of the form $\sum a_i W_i$ where $a_i$s are constant and $\mathbb{E}_\theta\left(\sum a_i W_i\right) = \theta$, the estimator

$$W^* = \frac{\sum W_i/\sigma_i^2}{\sum(1/\sigma_i^2)}$$

has minimum variance. Show that

$$\mathbb{V}ar\, W^* = \frac{1}{\sum(1/\sigma_i^2)}.$$

For $\theta \neq 0$, we have $\mathbb{E}\left(\sum_{i=1}^k a_i W_i\right) = \theta = \sum_{i=1}^k a_i \mathbb{E}(W_i) = \theta \sum_{i=1}^k a_i \Rightarrow \sum_{i=1}^k a_i = 1$,

hence we obtain for all $\theta \in \mathbb{R}$

$$\mathbb{E}\left(\left(\sum_{i=1}^k a_i W_i\right)^2\right) = \sum_{i=1}^k \sum_{j=1}^k a_i a_j \mathbb{E}(W_i W_j)$$

$$= \sum_{i=1}^k \sum_{j=1}^k a_i a_j \left(\underbrace{\mathbb{E}\left((W_i-\theta)(W_j-\theta)\right)}_{= Cov(W_i, W_j)} + \theta \underbrace{\mathbb{E}(W_i)}_{=\theta} + \theta \underbrace{\mathbb{E}(W_j)}_{=\theta} - \theta^2\right)$$

$$= \theta^2 \sum_{i=1}^k \sum_{j=1}^k a_i a_j + \sum_{i=1}^k a_i^2 \sigma_i^2 = \theta^2 + \sum_{i=1}^k a_i^2 \sigma_i^2$$

For $\theta \neq 0$, we define $b_i := a_i - \left(\sigma_i^2 \sum_{j=1}^k \frac{1}{\sigma_j^2}\right)^{-1}$, which fulfill $\sum_{i=1}^k b_i = 0$ and obtain

$$\mathbb{V}ar\left(\sum_{i=1}^k a_i W_i\right) = \mathbb{E}\left(\left(\sum_{i=1}^k a_i W_i\right)^2\right) - \left(\mathbb{E}\left(\sum_{i=1}^k a_i W_i\right)\right)^2 = \sum_{i=1}^k a_i^2 \sigma_i^2$$

$$= \sum_{i=1}^k \left(b_i + \left(\sigma_i^2 \sum_{j=1}^k \frac{1}{\sigma_j^2}\right)^{-1}\right)^2 \sigma_i^2$$

$$= \sum_{i=1}^k b_i^2 \sigma_i^2 + 2 \sum_{i=1}^k b_i \sigma_i^2 \left(\sigma_i^2 \sum_{j=1}^k \frac{1}{\sigma_j^2}\right)^{-1} + \sum_{i=1}^k \sigma_i^2 \left(\sigma_i^2 \sum_{j=1}^k \frac{1}{\sigma_j^2}\right)^{-2}$$

$$= \sum_{i=1}^k b_i^2 \sigma_i^2 + 2\left(\sum_{j=1}^k \frac{1}{\sigma_j^2}\right)^{-1} \underbrace{\sum_{i=1}^k b_i}_{=0} + \left(\sum_{i=1}^k \frac{1}{\sigma_i^2}\right)^{-1}$$

$$= \sum_{i=1}^k b_i^2 \sigma_i^2 + \left(\sum_{i=1}^k \frac{1}{\sigma_i^2}\right)^{-1} \geq \left(\sum_{i=1}^k \frac{1}{\sigma_i^2}\right)^{-1} = \sum_{i=1}^k \left(\sigma_i^2 \sum_{j=1}^k \frac{1}{\sigma_j^2}\right)^{-2} \sigma_i^2$$

$$= \mathbb{V}ar(W^*)$$

For $\theta = 0$, the values $a_i = 0$ are valid and $W^*$ does not have minimum Variance

(4) **Normal unbiased esimator of $\mu^2$**
Let $X_1 \dots X_n$ be i.i.d. $\mathcal{N}(\mu, 1)$.

(a) Show that $\bar{X}^2 - \frac{1}{n}$ is unbiased esimator of $\mu^2$.
(b) By using Stein's Lemma, calculate its variance and show that it is greater than the Cramér-Rao lower bound.

*Hint:* Recall, Stein's Lemma states that for $X \sim \mathcal{N}(\mu, \sigma^2)$ and a differentiable function $g$ satisfying $E|g'(X)| < \infty$ it holds $\mathbb{E}\left(g(X)(X-\mu)\right) = \sigma^2 \mathbb{E}g'(X)$.

a) For $i \neq j$ we have

$$\mathrm{Cov}(X_i, X_j) = \int_{\mathbb{R}}\int_{\mathbb{R}} xy \frac{1}{2\pi} e^{\frac{-x^2}{2}} e^{\frac{-y^2}{2}} dx\, dy = 0$$

Thus,

$$\mathbb{E}\left(\bar{X}^2 - \frac{1}{n}\right) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}(X_i X_j) - \frac{1}{n}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\mathbb{E}\left((X_i-\mu)(X_j-\mu)\right) + \mu^2\right) - \frac{1}{n}$$

$$= \mu^2 + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j) - \frac{1}{n} = \mu^2 + \frac{1}{n^2}\sum_{i=1}^{n} 1 - \frac{1}{n} = \mu^2$$

b) $$\mathbb{E}\left(\left(\bar{X}^2 - \frac{1}{n} - \mu^2\right)^2\right) = \mathbb{E}\left(\left((\bar{X}+\mu)(\bar{X}-\mu) - \frac{1}{n}\right)^2\right)$$

$$= \mathbb{E}\left((\bar{X}+\mu)^2(\bar{X}-\mu)^2\right) - \frac{2}{n}\mathbb{E}\left((\bar{X}+\mu)(\bar{X}-\mu)\right) + \frac{1}{n^2}$$

we define $g_1(x) := x+\mu$ and $g_2(x) := (x+\mu)^2(x-\mu)$ and have

$g_1'(x) = 1$ and $g_2'(x) = 2(x+\mu)(x-\mu) + (x+\mu)^2 = 2x^2 - 2\mu^2 + (x+\mu)^2$, hence

$\mathbb{E}(|g_1'(\bar{X})|) = \mathbb{E}(1) = 1$, and

$$\mathbb{E}(|g_2'(\bar{X})|) \leq 2\mathbb{E}(\bar{X}^2) + 2\mathbb{E}(\mu^2) + \mathbb{E}\left((\bar{X}+\mu)^2\right)$$

$$= 2\mathbb{E}(\bar{X}^2) + 2\mu^2 + \mathbb{E}(\bar{X}^2) + 2\mu\mathbb{E}(\bar{X}) + \mu^2 < \infty$$

We already know that $\bar{X} \sim \mathcal{N}\left(\frac{n\mu}{n}, \frac{n}{n^2}\right) = \mathcal{N}\left(\mu, \frac{1}{n}\right)$

By applying Stein's Lemma, we obtain

$$\mathrm{Var}\left(\bar{X}^2 - \frac{1}{n}\right) = \mathbb{E}\left(\left(\bar{X}^2 - \frac{1}{n} - \mu^2\right)^2\right) = \mathbb{E}\left(g_2(\bar{X})(\bar{X}-\mu)\right) - \frac{2}{n}\mathbb{E}\left(g_1(\bar{X})(\bar{X}-\mu)\right) + \frac{1}{n^2}$$

$$= \frac{1}{n}\mathbb{E}\left(g_2'(\bar{X})\right) - \frac{2}{n^2}\mathbb{E}\left(g_1'(\bar{X})\right) + \frac{1}{n^2}$$

$$= \frac{1}{n}\left(2\mathbb{E}(\bar{X}^2) - 2\mu^2 + \mathbb{E}(\bar{X}^2) + 2\mu\mathbb{E}(\bar{X}) + \mu^2\right) - \frac{1}{n^2}$$

$$= \frac{1}{n}\left(3\left(\frac{1}{n} + \mu^2\right) + \mu^2\right) - \frac{1}{n^2} = \frac{2}{n^2} + \frac{4\mu^2}{n}$$

We define $\theta := \mu^2$ and obtain the log-likelihood function

$$\ell_n(\theta) = \sum_{i=1}^n \log\left(f(x_i)\right) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^n (x_i - \sqrt{\theta})^2 \quad , \text{ hence}$$

$$\ell_n'(\theta) = -\frac{1}{2}\, 2 \sum_{i=1}^n (x_i - \sqrt{\theta})\left(-\frac{1}{2}\theta^{-\frac{1}{2}}\right) = \frac{1}{2}\sum_{i=1}^n \frac{x_i - \sqrt{\theta}}{\sqrt{\theta}} = \frac{1}{2}\sum_{i=1}^n \frac{x_i}{\sqrt{\theta}} - \frac{n}{2}$$

$$\ell_n''(\theta) = \frac{1}{2}\sum_{i=1}^n x_i \left(-\frac{1}{2}\right)\theta^{-\frac{3}{2}} = -\frac{1}{4}\theta^{-\frac{3}{2}}\sum_{i=1}^n x_i$$

We obtain

$$I_n(\theta) = \mathbb{E}\left(\ell_n''(\theta)\right) = -\frac{1}{4}\theta^{-\frac{3}{2}}\sum_{i=1}^n \mathbb{E}(X_i) = -\frac{1}{4}\mu^{-3} n\mu = -\frac{n}{4\mu^2} \, ,$$

and we conclude that

$$-\frac{1}{I_n(\theta)} = \frac{4\mu^2}{n} < \frac{2}{n^2} + \frac{4\mu^2}{n} = \mathbb{V}\mathrm{ar}\left(\bar{X}^2 - \frac{1}{n}\right)$$

(5) **Exponential family**

Show that a Poisson family of distributions $\mathcal{P}oi(\lambda)$, with unknown $\lambda > 0$ belongs to the exponential family.

$$h(x) := \begin{cases} \frac{1}{x!}, & \text{if } x \in \mathbb{N}_0 \\ 0, & \text{else} \end{cases}$$

$t_1(x) := x; \quad t_2(x) := -1$

$w_1(\lambda) := \log(\lambda); \quad w_2(\lambda) := \lambda$

$c(\lambda) := 1$

$\forall x \in \mathbb{N}_0:$

$$h(x)\, c(\lambda)\, e^{(w_1(\lambda) t_1(x) + w_2(\lambda) t_2(x))} = \frac{1}{x!} e^{(x \log(\lambda) - \lambda)} = \frac{\lambda^x}{x!} e^{-\lambda} = f(x|\lambda)$$