Appendix A

Some Facts from Functional Analysis

I this appendix we collect some results from introductory functional analysis courses which are used throughout. We stick with the case of vector spaces over $\mathbb{R}$.

A.1 Main Theorems from Functional Analysis

Theorem A.1 (Hahn-Banach Extension Theorem). *Let $p : X \to \mathbb{R}$ be a sublinear functional on a linear space X, i.e. $p(x+y) \le p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for all $x, y \in X$ and $\lambda \ge 0$. If Y is a subspace of X and $f : Y \to \mathbb{R}$ is a linear functional with $f \le p$ on Y, there is a linear extension $F : X \to \mathbb{R}$ with $F|_Y = f$ and $F \le p$ on X.* ■

If X is a normed space and $f \in Y^*$, one may choose $p(x) = \|x\|_X \|f\|_{X^*}$ to prove the extension theorem for continuous linear functionals.

Corollary A.2. *If Y is the subspace of a normed space X and $f \in Y^*$, there is an extension $F \in X^*$ with $F|_Y = f$ and $\|F\|_{X^*} = \|f\|_{Y^*}$.* ■

One then considers the subspace $Y := \operatorname{span}\{x\}$ and $f(\lambda x) = \lambda \|x\|_X$ to derive the following corollary:

Corollary A.3. *If X is a normed space and $x \in X$, there is a linear functional $f \in X^*$ with $\|f\|_{X^*} = 1$ and $f(x) = \|x\|_X = \sup\limits_{\|f\|_{X^*}=1} |f(x)|$.* ■

Theorem A.4 (Hahn-Banach Separation Theorem). *Let X be a normed space, and let A and B be convex, nonempty subsets of X with $A \cap B = \emptyset$.*
(i) *If A is open, there is a linear functional $f \in X^*$ and a scalar $\lambda \in \mathbb{R}$ such that $f(x) < \lambda \le f(y)$ for all $x \in A$ and $y \in B$.*
(ii) *If A is compact and B is closed, there is a linear functional $f \in X^*$ and scalars $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $f(x) \le \lambda_1 < \lambda_2 \le f(y)$ for all $x \in A$ and $y \in B$.* ■

If Y is a subspace of X, one can use (ii) to characterize the closure $\overline{Y}$ of Y in X. The proof only needs that each bounded linear functional $f \in Y^*$ is trivial, i.e. $f|_Y = 0$.

Corollary A.5. *Let Y be a subspace of the normed space X. Then, $x \in X$ satisfies $x \in \overline{Y}$ if and only if $f(x) = 0$ for all $f \in X^*$ with $f|_Y = 0$.*

Proof. For $x \in \overline{Y}$ and $f \in X^*$ with $f|_Y = 0$, continuity yields $f(x) = 0$. The converse implication is proven by contradiction: We assume that $x \notin \overline{Y}$ and choose $f \in X^*$ such that $f(x) < \lambda \leq f(y)$ for all $y \in Y$ and some fixed $\lambda \in \mathbb{R}$. Using that Y is a vector space, we infer that $\lambda \leq f(\pm y) = -f(\mp y) \leq -\lambda$ and thus $f(y) \in [\lambda, -\lambda]$ for all $y \in Y$. As bounded linear functionals are trivial, we obtain $f|_Y = 0$. According to our assumptions, this implies $f(x) = 0$ and thus contradicts $f(x) < \lambda \leq f(0) = 0$. ∎

The following corollary is an immediate consequence of the last one.

Corollary A.6. *Let Y be a subspace of the normed space X. Then, Y is dense in X if and only if each functional $f \in X^*$ with $f|_Y = 0$ is trivial, i.e., $f = 0 \in X^*$.* ∎

For an operator $T \in L(X;Y)$, one defines $(T^*y^*)(x) := y^*(Tx)$ for all $y^* \in Y^*$ and $x \in X$. From the continuity of T, we see that $T^*y^* \in X^*$, and obviously $T^* : Y^* \to X^*$ is a linear operator. From the corollary of the Hahn-Banach extension theorem, we derive for the operator norm

$$
\begin{aligned}
\|T^*\| = \sup_{\|y^*\|_{Y^*}=1} \|T^*y^*\|_{X^*} &= \sup_{\|y^*\|_{Y^*}=1} \sup_{\|x\|_X=1} (T^*y^*)(x) \\
&= \sup_{\|x\|_X=1} \sup_{\|y^*\|_{Y^*}=1} (y^*)(Tx) = \sup_{\|x\|_X=1} \|Tx\|_Y = \|T\|,
\end{aligned}
$$

i.e. there holds $T^* \in L(Y^*; X^*)$ with operator norm $\|T^*\| = \|T\|$. The operator T^* is called the **adjoint operator** of T.

Theorem A.7 (Banach Closed Range Theorem). *For an operator $T \in L(X;Y)$ between Banach spaces X and Y and $T^* \in L(Y^*;X^*)$ its adjoint, the following is pairwise equivalent:*
(i) range(T) *is a closed subspace of Y.*
(ii) range$(T) = (\ker T^*)_\circ := \{y \in Y \mid \forall y^* \in \ker(T^*) \quad y^*(y) = 0\}$.
(iii) range(T^*) *is a closed subspace of X^*.*
(iv) range$(T^*) = (\ker T)^\circ := \{x^* \in X^* \mid \forall x \in \ker(T) \quad x^*(x) = 0\}$. ∎

A.2 Hilbert Spaces

A space X is called **Hilbert space** if it is a Banach space whose norm is induced by a scalar product.

Theorem A.8. *Let Y be the closed subspace of a Hilbert space X and $Y^\perp := \{x \in X \mid \forall y \in Y \quad (x\,;y)_X = 0\}$ the orthogonal complement. Then, there holds $X = Y \oplus Y^\perp$ in the sense of the linear algebra, i.e. every element $x \in X$ has a unique decomposition $x = y + y^\perp$ with some $y \in Y$ and $y^\perp \in Y^\perp$.* ∎

With the orthogonal decomposition $X = Y \oplus Y^\perp$, one can define a projection $\pi_Y : X \to Y$ by $x = y + y^\perp \mapsto y$.

Corollary A.9. *Let Y be the closed subspace of a Hilbert space X. Then, there is a unique linear operator $\Pi : X \to Y$ with $\Pi|_Y = id$ and $\ker(\Pi) = Y^\perp$, which is called* **orthogonal projection** *onto Y. This projection is continuous with operator norm $\|\Pi\| = 1$ and symmetric, i.e. $(x\,;\,y)_X = (\Pi x\,;\,y)_X$ for all $x \in X$ and $y \in Y$. Moreover, the orthogonal projection is the solution operator for the best approximation problem, $\|x - \Pi x\|_X = \min_{y \in Y} \|x - y\|_X$.* ■

The dual space X^* of a Hilbert space X has a straight-forward representation, and one can somehow identify X with X^*.

Theorem A.10 (Riesz). *For a Hilbert space X, the* **Riesz mapping** *$I_X : X \to X^*$, $I_X x := (x\,;\,\cdot)_X \in X^*$, is an isometric isomorphism.* ■