(5) **Sufficiency, bias, Rao-Blackwell theorem**
Let $X_1, \ldots, X_n$ be i.i.d. $Poi(\lambda)$, with unknown $\lambda > 0$.

(a) Show that $Y = \sum_{n=1}^{n} X_i$ is a sufficient statistic for λ.

(b) Find an unbiased estimator of $p_r = P(X = r)$, which depends only on X_1.
Find $P(X_1 = r|Y = k)$ both for $k \geq r$ and $k < r$.
Hence use the Rao-Blackwell theorem to improve your estimator of p_r.

a) $$L(x|\lambda) = \begin{cases} \lambda^{\sum_{i=1}^n x_i} \left(\prod_{i=1}^n x_i! \right)^{-1} \exp(-n\lambda) & , \text{if } x = 0,1,\ldots \\ 0 & , \text{otherwise} \end{cases}$$

From now on we only consider $x \in \{0,1,\ldots\}$. Let $g(z|\lambda) := \lambda^z \exp(-n\lambda)$, and $h(x) := \left(\prod_{i=1}^n x_i! \right)$. We have $L(x|\lambda) = g(Y,\lambda) h(X)$, hence Y is a sufficient statistic for λ.

b) $p_r = \mathbb{P}(X=r) = \frac{\lambda^r}{r!} e^{-\lambda}$, $\quad \hat{p}_r(x) := \begin{cases} 1, & \text{if } x_1 = r \\ 0, & \text{otherwise} \end{cases}$, then $\hat{p}_r(X)$ is an unbiased estimator of p_r, since $\mathbb{E}_\lambda(\hat{p}_r(X)) = \frac{\lambda^r}{r!} e^{-\lambda} = p_r$

Let $k < r$, if there was w with $k = Y(w) = \sum_{i=1}^n x_i(w)$ and $X_1(w) = r$, then $r = X_1(w) \leq \sum_{i=1}^n X_i(w) = k$ ↯, hence $\mathbb{P}(X_1 = r | Y = k) = 0$

Let $r \leq k$, then

$$\mathbb{P}(X_1 = r | Y = k) = \frac{\mathbb{P}(X_1 = r, \sum_{i=2}^n X_i = k - r)}{\mathbb{P}(Y=k)} = \frac{\mathbb{P}(X_1 = r)\,\mathbb{P}\left(\sum_{i=2}^n X_i = k-r\right)}{\mathbb{P}(Y=k)}$$

$$= \frac{\lambda^r}{r!} e^{-\lambda} \frac{((n-1)\lambda)^{k-r}}{(k-r)!} e^{-(n-1)\lambda} \left(\frac{(n\lambda)^k}{k!} e^{-n\lambda} \right)^{-1}$$

$$= \frac{k!}{r!(k-r)!} \frac{(n-1)^{k-r}}{n^k} = \binom{k}{r} \frac{(n-1)^{k-r}}{n^k}$$

We used that $\sum_{i=2}^n X_i \sim Poi((n-1)\lambda)$ and $Y \sim Poi(n\lambda)$

Hence $$\phi(k) = \mathbb{E}(X_1 = r | Y = k) = \mathbb{P}(X_1 = r | Y = k) = \begin{cases} \binom{k}{r} \frac{(n-1)^{k-r}}{n^k} & , \text{if } 0 \leq r \leq k \\ 0 & , \text{otherwise} \end{cases}$$

By the Rao-Blackwell Theorem, $\phi(Y)$ is an unbiased estimator of p_r with $\mathrm{Var}_{p_r}(\phi(Y)) \leq \mathrm{Var}(\hat{p}_r(X))$.