(2) **Cramér-Rao lower bound**

Let $X_1, \dots, X_n$ be a random sample with the pdf $f(x|\theta) = \theta x^{\theta-1}$, where $0 < x < 1$ and $\theta > 0$ is unknown. Is there a function of θ, say $g(\theta)$, for which there exists an unibiased estimator whose variance attains the Cramér-Rao lower bound? If there is, find it. If not, show why not.

$$\ell_n(\theta) = \sum_{i=1}^{n} \left(\log(\theta) + (\theta-1)\log(x_i) \right)$$

$$\ell_n'(\theta) = \sum_{i=1}^{n} \left(\frac{1}{\theta} + \log(x_i) \right) = \frac{n}{\theta} + \sum_{i=1}^{n} \log(x_i)$$

$$\ell_n''(\theta) = -\frac{n}{\theta^2} < 0$$

Since the MLE is the most efficient unbiased estimator, we want

$$0 = \ell_n'(\hat{\theta}) = \frac{n}{\theta} + \sum_{i=1}^{n} \log(x_i) \Leftrightarrow g(\theta) := \frac{1}{\theta} = -\frac{1}{n}\sum_{i=1}^{n} \log(x_i) =: h(X)$$

$$\mathbb{E}(h(X)) = -\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(\log(X_i)) = -\frac{1}{n}\sum_{i=1}^{n} \int_0^1 \log(x_i)\, \theta x_i^{\theta-1}\, dx_i$$

$$= -\frac{1}{n}\sum_{i=1}^{n} \left(\log(x_i)\, x_i^{\theta} \Big|_{x_i=0}^{1} - \int_0^1 x_i^{-1} x_i^{\theta}\, dx_i \right) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\theta} = \frac{1}{\theta} = g(\theta)$$

We have

$$\int_0^1 \int_0^1 \log(x_i)\log(x_j)\, \theta^2 x_i^{\theta-1} x_j^{\theta-1}\, dx_i\, dx_j = \left(-\frac{1}{\theta}\right)^2 = \frac{1}{\theta^2}$$

$$\int_0^1 \left(\log(x_i)\right)^2 \theta x_i^{\theta-1}\, dx_i = \left(\log(x_i)\right)^2 x_i^{\theta} \Big|_{x_i=0}^{1} - 2\int_0^1 \log(x_i)\, x_i^{\theta-1}\, dx_i$$

$$= -\frac{2}{\theta}\int_0^1 \log(x_i)\, \theta x_i^{\theta-1}\, dx_i = \frac{2}{\theta^2} = \frac{1}{\theta^2} + \frac{1}{\theta^2}$$

Thus,

$$\mathbb{E}\left((h(X))^2\right) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\left(\log(x_i)\log(x_j)\right) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\theta^2} + \frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{\theta^2}$$

$$= \frac{1}{\theta^2} + \frac{1}{n\theta^2}$$

Hence, $\mathbb{V}\mathrm{ar}(h(X)) = \frac{1}{n\theta^2}$

$$\gamma := \frac{1}{\theta}, \quad \ell_n(\gamma) = \sum_{i=1}^{n} \left(-\log(\gamma) + (\gamma^{-1}-1)\log(x_i) \right)$$

$$\ell_n'(\gamma) = \sum_{i=1}^{n} \left(-\gamma^{-1} - \gamma^{-2}\log(x_i) \right)$$

$$\ell_n''(\gamma) = \sum_{i=1}^{n} \left(\gamma^{-2} + 2\gamma^{-3}\log(x_i) \right)$$

$$\mathbb{E}(\ell_n''(\gamma)) = \frac{n}{\gamma^2} + 2\gamma^{-3}\sum_{i=1}^{n}\mathbb{E}(\log(x_i)) = \frac{n}{\gamma^2} - 2\gamma^{-3} n\gamma = \frac{-n}{\gamma^2} = n\theta^2 = I_n(\theta)$$

$$\Rightarrow \mathbb{V}\mathrm{ar}(h(X)) = -\frac{1}{I_n(\theta)}$$