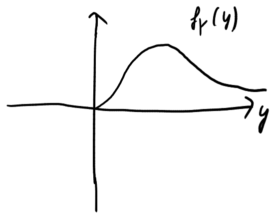
(3) **Uniform-exponential relationship**

(a) Let Y be an exponential random variable $Y \sim \exp(\lambda)$, i.e. its pdf is given by

$$f_Y(y) = \begin{cases} \lambda\, e^{-\lambda y}, & y \geq 0 \\ 0, & \text{else} \end{cases}$$

and its mean equals $\frac{1}{\lambda}$. Compute $P(Y > y)$.

(b) Let X be a random variable, uniformlly distributed on $(0, 1)$. Find the cumulative distribution function of X. What is the distribution of $Z = -\log X$?



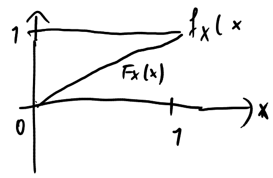
$z := \max\{y, 0\}$

a) $\mathbb{P}(Y > y) = \int_z^\infty \lambda e^{-\lambda \eta}\, d\eta = \lambda \int_z^\infty e^{-\lambda \eta}\, d\eta = \lambda \left[-\frac{1}{\lambda} e^{-\lambda \eta}\right]_{\eta = z}^{\infty}$

$= \lambda \frac{1}{\lambda} e^{-\lambda z} = \begin{cases} 1, & \text{if } y \leq 0 \\ e^{-\lambda y}, & \text{else} \end{cases}$

$\left(-\frac{1}{\lambda} e^{-\lambda \eta}\right)' = e^{-\lambda \eta}$

b)



$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x > 1 \\ x, & \text{else} \end{cases}$

$\mathbb{P}(Z \leq z) = \mathbb{P}(-\log(X) \leq z) = \mathbb{P}(X \geq e^{-z}) = 1 - \mathbb{P}(X < e^{-z}) = 1 - e^{-z} = F_Z(z)$

$f_Z(z) = F_Z'(z) = e^{-z}$ Hence, $Z \sim \exp(1)$