(2) **Continuous two-dimensional random variable**
The joint pdf of two random variables X and Y is defined by

$$f(x,y) = \begin{cases} c(x+2y), & 0<y<1 \text{ and } 0<x<2 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find the value of c and the marginal distribution of Y.
(b) Find the joint cdf of X and Y.
(c) Find the marginal distribution of X and the pdf of $Z = \frac{9}{(X+1)^2}$.

a) $$1 \stackrel{!}{=} \int_{\mathbb{R}^2} f \, d\lambda^2 = \int_0^1 \int_0^2 c(x+2y)\,dx\,dy = c\int_0^1 (2+4y)\,dy = c\,(2+2) = 4c \Leftrightarrow c = \frac{1}{4}$$

$$\forall y =]0,1[: \int_0^2 f(x,y)\,dx = \frac{1}{2} + y = \frac{1+2y}{2}$$

$$f_Y(y) = \begin{cases} y+\frac{1}{2} & , \text{if } 0<y<1 \\ 0 & , \text{otherwise} \end{cases}$$

b) $$\int_0^y \int_0^x f(\xi,\eta)\,d\xi\,d\eta = \int_0^y \int_0^x \left(\frac{\xi}{4}+\frac{\eta}{2}\right)d\xi\,d\eta = \int_0^y \left(\frac{x^2}{8}+\frac{\eta x}{2}\right)d\eta = \frac{x^2 y}{8} + \frac{y^2 x}{4}$$

$$F(x,y) = \begin{cases} 0 & , \text{if } x<0 \text{ or } y<0 \\ \frac{x^2y}{8}+\frac{y^2x}{4} & , \text{if } 0\le x\le 2 \text{ and } 0\le y\le 1 \\ \frac{x^2}{8}+\frac{x}{4} & , \text{if } 0\le x\le 2 \text{ and } y>1 \\ \frac{y}{2}+\frac{y^2}{2} & , \text{if } x>2 \text{ and } 0\le y\le 1 \\ 1 & , \text{if } x>2 \text{ and } y>2 \end{cases}$$

c) $$\forall x \in]0,2[: \int_0^1 f(x,y)\,dy = \int_0^1 \left(\frac{x}{4}+\frac{y}{2}\right)dy = \frac{x}{4}+\frac{1}{4}$$

$$f_X(x) = \begin{cases} \frac{1}{4}(x+1) & , \text{if } 0<x<2 \\ 0 & , \text{otherwise} \end{cases}$$

$$g :]0,2[\to]1,9[: x \mapsto \frac{9}{(1+x)^2}$$

$$\forall x \in]0,2[\ \forall z \in]1,9[: z = \frac{9}{(1+x)^2} \Leftrightarrow z(1+x)^2 = 9 \Leftrightarrow 1+x = \sqrt{\frac{9}{z}} \Leftrightarrow x = \sqrt{\frac{9}{z}} - 1$$

$$h :]1,9[\to]0,2[: z \mapsto \sqrt{\frac{9}{z}} - 1 = 3z^{-\frac{1}{2}} - 1 \Rightarrow h'(z) = -\frac{3}{2} z^{-\frac{3}{2}}$$

$$f_X(h(z))\,|h'(z)| = \frac{1}{4}\left(-\frac{3}{2}z^{-\frac{3}{2}} + 1\right)\frac{3}{2}z^{-\frac{3}{2}} = \frac{3}{8}z^{-\frac{3}{2}} - \frac{9}{16}z^{-3} = \frac{3}{8}\left(z^{-\frac{3}{2}} - \frac{3}{2}z^{-3}\right)$$

$$f_Z(z) = \begin{cases} \frac{3}{8}\left(z^{-\frac{3}{2}} - \frac{3}{2}z^{-3}\right) & , \text{if } 1<z<9 \\ 0 & , \text{otherwise} \end{cases}$$