4. Betrachten Sie die eindimensionale Wärmeleitungsgleichung mit gemischten Randbedingungen

$$\begin{cases} u_t - u_{xx} = 0 & \text{für } x \in (0,\pi)\,,\ t > 0\,,\\ u(0,t) = 0 & \text{für } t > 0\,,\\ u_x(\pi,t) = 0 & \text{für } t > 0\,,\\ u(x,0) = u_0(x) & \text{für } x \in (0,\pi)\,, \end{cases}$$

wobei $u_0 \in L^2(0,\pi)$.

(i) Bestimmen Sie ein vollständiges Orthonormalsystem $(\phi_n)_{n\in\mathbb{N}} \subset L^2(0,\pi)$ mit $\phi_n'' = \lambda_n \phi_n$ in $(0,\pi)$ mit Randbedingungen $\phi_n(0) = \phi_n'(\pi) = 0$.

(ii) Konstruieren Sie aus $(\phi_n)_{n\in\mathbb{N}}$ eine Lösungsformel für das obige parabolische Problem.

(iii) Welche Abklingrate (für $t \to \infty$) hat die Wärmeenergie $E(t) := \int_0^\pi u(x,t)dx$ für eine Lösung u?

(i) Allg. Lsg. der Dgl. $\phi_n(x) = c_1 e^{\sqrt{\lambda_n}x} + c_2 e^{-\sqrt{\lambda_n}x}$

Fall 1: $\lambda_n > 0$; $\phi_n(x) = c_1 \cosh(\sqrt{\lambda_n}x) + c_2 \sinh(\sqrt{\lambda_n}x)$

$\phi_n(0) = c_1 \overset{!}{=} 0 \Rightarrow \phi_n'(\pi) = c_2\sqrt{\lambda_n}\cosh(\sqrt{\lambda_n}\pi) \overset{!}{=} 0$ ↯

Fall 2: $\lambda_n < 0$: $\phi_n(x) = c_1 \sin(\sqrt{|\lambda_n|}x) + c_2 \cos(\sqrt{|\lambda_n|}x)$

$\phi_n(0) = c_2 \overset{!}{=} 0 \Rightarrow \phi_n'(\pi) = c_1\sqrt{|\lambda_n|}\cos(\sqrt{|\lambda_n|}x) \overset{!}{=} 0 \Leftrightarrow \sqrt{|\lambda_n|} = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

$\phi_n(x) = c \sin\left(\frac{2n-1}{2}x\right)$, $\phi_n(0) = 0$,

$\phi_n'(x) = c\left(\frac{2n-1}{2}\right)\cos\left(\frac{2n-1}{2}x\right)$, $\phi_n'(\pi) = 0$

$\phi_n''(x) = c\underbrace{-\left(\frac{2n-1}{2}\right)^2}_{\lambda_n}\sin\left(\frac{2n-1}{2}x\right)$

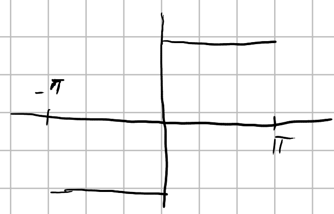
Fall 3: $\lambda_n = 0$: $\phi_n'' = 0 \Rightarrow \phi_n(x) = c_1 x + c_2$, $\left.\begin{array}{l}\phi_n(0) = c_2 \overset{!}{=} 0\\ \phi_n'(\pi) = c_1 \overset{!}{=} 0\end{array}\right\} \Rightarrow \phi_n \equiv 0$

Also: $\forall n \in \mathbb{N}$: $\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin\left(\frac{2n-1}{2}x\right)$; $\|\phi_n\|_{L^2} = \sqrt{\frac{\pi}{2}}$

(ii) $u(x,t) = \sum_{k=1}^{\infty} e^{\lambda_k t}(u_0, \phi_k)_{L^2}\sqrt{\frac{2}{\pi}}\sin(\sqrt{-\lambda_k}x)$ konvergiert unbedingt

$u_t - u_{xx} = 0$, $u(0,t) = 0$, $u_x(\pi,t) = 0$ (weil $\phi_n'(\pi) = 0$)

$u(x,0) = \sqrt{\frac{2}{\pi}}\sum_{k=1}^{\infty}(u_0,\phi_k)_{L^2}\sin(\sqrt{-\lambda_n}x) \overset{?}{=} u_0(x)$



$\{\sin(nx) \mid n \in \mathbb{N}\}$ ist vollst. in $L^2(0,\pi)$

Sei $f \in L^2(0,\pi)$ geg.

Def. $f:]-\pi,\pi[\to \mathbb{R}$: $\begin{cases} f(x), & x > 0\\ -f(-x), & x < 0\end{cases}$

also, f symmetrisch, d.h. $(f, \cos(n\cdot))_{L^2} = 0$ für alle $n \in \mathbb{N}$

$$\sin\left(\frac{2n-1}{2}x\right) = \sin\left(nx - \frac{1}{2}x\right) = \sin(nx)\cos\left(\tfrac{1}{2}x\right) - \cos(nx)\sin\left(\tfrac{1}{2}x\right)$$

$$\sin\left(\frac{2n+1}{2}x\right) = \sin(nx)\cos\left(\tfrac{1}{2}x\right) + \cos(nx)\sin\left(\tfrac{1}{2}x\right)$$

$$\Rightarrow \sin\left(\frac{2n-1}{2}x\right) + \sin\left(\frac{2n+1}{2}x\right) = 2\sin(nx)\cos\left(\tfrac{1}{2}x\right)$$

$$\forall f \in L^2(0,\pi): \quad \|f\|_{L^2}^2 = \sum_{k=1}^{\infty} |(f, \sin(k\,\cdot))_{L^2}|^2 = \sum_{k=1}^{\infty} \left|\left(f, \sin\left(\frac{2k-1}{2}\,\cdot\right)\right)_{L^2}\right|^2$$

(iii) $$E(t) = \int_0^\pi u(x,t)\,dx = \int_0^\pi \sum_{k=1}^\infty e^{\lambda_k t}(u_0, \phi_k)_{L^2} \sqrt{\frac{2}{\pi}} \sin\left(\sqrt{-\lambda_k}\,x\right) dx =$$

$$= \sqrt{\frac{2}{\pi}} \sum_{k=1}^\infty e^{\lambda_k t}(u_0, \phi_k)_{L^2} \underbrace{\int_0^\pi \sin\left(\sqrt{-\lambda_k}\,x\right) dx}$$

$$\left.\begin{array}{l} u = \sqrt{-\lambda_k}\,x \\ \frac{du}{dx} = \sqrt{-\lambda_k} \end{array}\right| = \frac{1}{\sqrt{-\lambda_k}} \int_0^{\sqrt{-\lambda_k}\pi} \sin(u)\,du = \frac{1}{\sqrt{-\lambda_k}}\left(\cos(0) - \cos\left(\sqrt{-\lambda_k}\,\pi\right)\right) = \frac{1}{\sqrt{-\lambda_k}}$$

$$\|u(\cdot,t)\|_{L^2}^2 = \int_0^\pi (u(x,t))^2\,dx = \int_0^\pi \sum_{k=1}^\infty e^{2\lambda_k t}(u_0,\phi_k)_{L^2}^2 \frac{2}{\pi}\left(\sin\left(\sqrt{-\lambda_k}\,x\right)\right)^2 dx \overset{\|\phi_k\|_{L^2}^2 = \frac{\pi}{2}}{\leq} e^{2\lambda_1 t}\|u_0\|_{L^2}^2$$

$$|E(t)| \leq \int_0^\pi |u(x,t)|\,dx \leq \sqrt{\pi}\,\|u(\cdot,t)\|_{L^2} \leq \sqrt{\pi}\,e^{\lambda_1 t}\|u_0\|_{L^2}^2$$