(3) **Most powerful test 2**
Let $X_1, \dots, X_n$ be iid from a distribution with density

$$f_\theta(x) = \frac{x}{\theta} e^{-\frac{x^2}{2\theta}},\ x \geq 0,\ \theta > 0.$$

(a) Derive the MP test at level α for testing two simple hypoheses

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta = \theta_1,\ \theta_1 > \theta_0.$$

(b) Is there a uniformly most powerful (UMP) test at level α for testing the one-sided composite hypothesis

$$H_0 : \theta \leq \theta_0 \quad vs \quad H_1 : \theta > \theta_0$$

What is its power function?

Hint: Show $X_i^2 \sim \exp(1/2\theta)$, so that $\sum_i X_i^2 \sim \theta\chi^2(2n)$.

a)

$$L(\theta;x) = \begin{cases} \frac{1}{\theta^n} \prod_{i=1}^n x_i \exp\left(-\frac{1}{2\theta} \sum_{i=1}^n x_i^2\right), & \text{if } \min\{x_i \mid 1 \leq i \leq n\} \geq 0 \\ 0 & \text{, else} \end{cases}$$

For $x \in (\mathbb{R}^+)^n$ we have

$$\lambda(x) = \frac{L(\theta_1;x)}{L(\theta_0;x)} = \left(\frac{\theta_0}{\theta_1}\right)^n \exp\left(\left(\frac{1}{2\theta_0} - \frac{1}{2\theta_1}\right) \sum_{i=1}^n x_i^2\right)$$

Since $\theta_1 > \theta_0$, we obtain that the function $\lambda(x)$ is a monotone increasing function of $T(x) = \sum_{i=1}^n x_i^2$, which we choose as our test-statistic

We define $Y_i := X_i^2$, and have

$$f_{Y_i}(y) = f_{X_i}(\sqrt{y}) \frac{1}{2\sqrt{y}} + \overbrace{f_{X_i}(-\sqrt{y})}^{=0} \frac{1}{2\sqrt{y}} = \frac{\sqrt{y}}{\theta} e^{-\frac{y}{2\theta}} \frac{1}{2\sqrt{y}} = \frac{1}{2\theta} e^{-\frac{y}{2\theta}}$$

Thus $Y_i \sim \exp\left(\frac{1}{2\theta}\right)$, or equivalently $Y_i \sim \text{Gamma}(1, \frac{1}{2\theta})$, hence $T(X) \sim \text{Gamma}(n, \frac{1}{2\theta})$

Hence, $\sum_{i=1}^n Y_i \sim \text{Erlang}(n, \frac{1}{2\theta})$ and $\frac{1}{\theta} T(X) \sim \chi^2(2n)$, we write symbolically

$T(X) \sim \theta \chi^2(2n)$

We have $\alpha = \mathbb{P}(T(X) \geq c) \Leftrightarrow \alpha = 1 - \mathbb{P}(\frac{1}{\theta_0} T(X) < \frac{c}{\theta_0}) \Leftrightarrow \mathbb{P}(\frac{1}{\theta_0} T(X) < \frac{c}{\theta_0}) = 1 - \alpha$

$\Leftrightarrow F_{\chi^2(2n)}\left(\frac{c}{\theta_0}\right) = 1 - \alpha \Leftrightarrow \frac{c}{\theta_0} = F^{-1}_{\chi^2(2n)}(1-\alpha) \Leftrightarrow c = \theta_0 F^{-1}_{\chi^2(2n)}(1-\alpha)$

Our test rejects H_0, if $T(x) \geq c$.

b) The test from (a) is by the theorem at p. 33 from Lecture 10 an UMP

the power q is given by

$$q = \mathbb{P}\left(T(X) \geq c \mid \tfrac{1}{\theta} T(X) \sim \chi^2(2n)\right) = \mathbb{P}\left(\tfrac{1}{\theta} T(X) \geq \tfrac{c}{\theta}\right) = 1 - F_{\chi^2(2n)}\left(\tfrac{c}{\theta}\right)$$

$$= 1 - F_{\chi^2(2n)}\left(\frac{\theta_0}{\theta} F^{-1}_{\chi^2(2n)}(1-\alpha)\right)$$