Stat. 7. UE

(1.) $X_1,\dots,X_n \sim \mathcal{U}(\vartheta, 1)$, $\vartheta<1$ unknown. We know that $f_\vartheta(x) = \mathbb{1}_{[\vartheta,1]}(x)\cdot\frac{1}{1-\vartheta}$.

(a) MLE of ϑ: $L(\vartheta|x) = \prod_{i=1}^n f(x_i|\vartheta) = \left(\frac{1}{1-\vartheta}\right)^n [\vec{x}\in[\vartheta,1]^n]$;

$$L(\hat\vartheta|x) \overset{!}{=} \max_\vartheta L(\vartheta|x) = \max_{\vartheta<\min_{i=1,\dots,n} x_i} \underbrace{\left(\frac{1}{1-\vartheta}\right)^n}_{>0} \Rightarrow \min_{i=1,\dots,n} x_i = \hat\vartheta.$$

strictly increasing in ϑ

(b) Is $\hat\vartheta$ asymptotically normal estimator? Yes → Find as. μ + V; No → Find (r_n), (a_n) s.t. $r_n(\hat\vartheta - a_n)\xrightarrow{d} X$, where X has a nondegenerate distr.

$$F_{\hat\vartheta}(x) = \mathbb{P}(\min x_i \le x) = 1 - \mathbb{P}(\min x_i > x) = 1-\prod_{i=1}^n \mathbb{P}(x_i>x) = 1-\left(\frac{1-x}{1-\vartheta}\right)^n \quad (\text{OBdA. } x\in[\vartheta,1]).$$

$\Rightarrow f_{\hat\vartheta}(x) = \frac{n}{(1-\vartheta)^n}(1-x)^{n-1}$. Defining $g(x) := r_n(x-a_n)$ we get $g^{-1}(x) = \frac{x}{r_n}+a_n$ and therefore

$$f_{r_n(\hat\vartheta-a_n)} = f_{\hat\vartheta}(g^{-1}(x))\,\frac{d}{dx}(g^{-1}(x)) = \frac{n}{(1-\vartheta)^n}\left(1-\frac{x}{r_n}-a_n\right)^{n-1}\frac{1}{r_n}.$$

With $a_n := \vartheta$, we get

$$f_{r_n(\hat\vartheta-\vartheta)} = \frac{n}{(1-\vartheta)^n}\frac{1}{r_n}\left(1-\frac{x}{r_n}-\vartheta\right)^{n-1} = \frac{n}{1-\vartheta}\frac{1}{r_n}\left(\frac{1-\vartheta-\frac{x}{r_n}}{1-\vartheta}\right)^{n-1} = \frac{n}{1-\vartheta}\frac{1}{r_n}\left(1-\frac{\frac{x}{r_n}}{1-\vartheta}\right)^{n-1}$$

and with $r_n := \frac{n-1}{1-\vartheta}$:

$$f_{\frac{n-1}{1-\vartheta}(\hat\vartheta-\vartheta)}(x) = \underbrace{\frac{n}{n-1}}_{\to 1}\underbrace{\left(1-\frac{x}{n-1}\right)^{n-1}}_{\to e^{-x}} \xrightarrow{n\to\infty} e^{-x},$$

so $r_n(\hat\vartheta - a_n)\sim\exp(1)$.

(4.) $X_1,\dots,X_n$ i.i.d. $\sim\mathcal{N}(\mu,1)$.

(a) Show: $\bar X^2-\frac1n$ is unbiased estimator of μ^2.

$$\mathbb{E}_\mu\left(\bar X^2 - \frac1n\right) = \mathbb{E}(\bar X^2)-\frac1n = \mathbb{V}(\bar X) + \mathbb{E}(\bar X)^2 - \frac1n = \frac{1}{n^2}n+\mu^2-\frac1n = \mu^2.$$

(b) Calculate its variance and show that it is > than the Cramér-Rao lower bound.

Steins Lemma: $X\sim \mathcal{N}(\mu,\sigma^2)$, g diff.able, $\mathbb{E}(g'(X))<\infty \Rightarrow \mathbb{E}(g(X)(X-\mu)) = \sigma^2\, \mathbb{E}(g'(X))$.

$$\mathbb{V}\left(\bar X^2-\frac1n\right) = \mathbb{V}(\bar X^2) = \mathbb{E}(\bar X^4) - \mathbb{E}(\bar X^2)^2 \overset{(a)}{=} \mathbb{E}(\bar X^4) - \left(\mu^2+\frac1n\right)^2 =$$

$$\mathbb{E}(\bar X^4-\mu^4) + \mu^4 - \left(\mu^2+\frac1n\right)^2 = \frac{4\mu^2}{n}+\frac{2}{n^2}.$$

$$\left[\begin{aligned}\mathbb{E}(\bar X^4-\mu^4) &= \mathbb{E}\Big((\bar X-\mu)\underbrace{(\bar X+\mu)(\bar X^2+\mu^2)}_{=:g(\bar X)}\Big) \overset{(\text{Stein})}{=} \frac1n\, \mathbb{E}(\mu^2+2\bar X\mu+3\bar X^2) = \\ &= \frac1n\left[\mu^2+2\mu^2+3\,\mathbb{V}(\bar X)+3\,\mathbb{E}(\bar X)^2\right] = \frac1n\left[3\mu^2+\frac3n+3\mu^2\right] = \frac{6\mu^2}{n}+\frac{3}{n^2}\end{aligned}\right]$$

$$\left[\begin{aligned}&\bar X\sim\mathcal{N}\left(\mu,\frac1n\right)\\ &g(x) := (x^2+\mu^2)(x+\mu) \Rightarrow g'(x) = \mu^2+2x\mu+3x^2\end{aligned}\right]$$

Cramér-Rao lower bound: $\hat{\vartheta}_n$ unbiased est. of $\vartheta \Rightarrow \mathbb{V}(\hat{\vartheta}_n) \geq \frac{1}{I_n(\vartheta)}$.

$I_n(\vartheta) = -\mathbb{E}_\vartheta(\ell_n''(\vartheta))$. $\quad [\vartheta = \mu^2.] \quad [f(x_i|\mu^2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2}]$

$\ell_n(\vartheta) = \sum_{i=1}^n \log f(x_i|\vartheta) = n \log(\frac{1}{\sqrt{2\pi}}) - \frac{1}{2}\sum_{i=1}^n (x_i - \sqrt{\vartheta})^2$

$\Rightarrow \ell_n'(\vartheta) = \frac{1}{2}\sum_{i=1}^n \frac{x_i - \sqrt{\vartheta}}{\sqrt{\vartheta}} \Rightarrow \ell_n''(\vartheta) = -\frac{1}{4}\sum_{i=1}^n \frac{x_i}{\vartheta^{3/2}} \Rightarrow \ell_n''(\mu^2) = -\frac{1}{4}\frac{1}{\mu^3}\bar{X}n)$

$\Rightarrow \mathbb{E}(\ell_n''(\mu^2)) = -\frac{1}{4\mu^2}n \Rightarrow I_n(\vartheta)^{-1} = \underline{\frac{4\mu^2}{n}}$.

Obviously, $\frac{4\mu^2}{n} + \frac{2}{n^2} > \frac{4\mu^2}{n}$. $\square$

(5.) Show: Family of $Poi(\lambda)$-distributions with unknown $\lambda > 0$ belongs to the exp. family.

Def. Family of pmfs called exp. fam. $:\Leftrightarrow \exists$ functions $h(x) \geq 0$, $t_1(x), \dots, t_k(x)$, $c(\vartheta) \geq 0$, $w_1(\vartheta), \dots, w_k$

PMF: $f(k|\lambda) = \frac{1}{k!}\lambda^k e^{-\lambda}$ $\quad$ s.t. $f(x|\vartheta) = h(x)\, c(\vartheta)\, e^{\sum_{i=1}^k w_i(\vartheta) t_i(x)}$.

$= \frac{1}{k!} e^{k\log(\lambda) - \lambda}$. $[c(\lambda) = 1;\ h(k) = \frac{1}{k!} \geq 0;\ w_1(\lambda) = \log\lambda;\ t_1(k) = k;\ w_2(\lambda) = -\lambda;\ t_2(k) = 1.]$

(3.) $W_1, \dots, W_k$ unbiased est. of ϑ with $\mathbb{V}(W_i) = \sigma_i^2$, $Cov(W_i, W_j) = 0$ for $i \neq j$.

- Show: $W^* := \frac{\sum W_i/\sigma_i^2}{\sum(1/\sigma_i^2)}$ has minimal $\mathbb{V}$ of all est. of form $\sum a_i W_i$ where a_i const. $\wedge$ $\mathbb{E}_\vartheta(\sum a_i W_i) = \vartheta$
- Show: $\mathbb{V}(W^*) = \frac{1}{\sum(1/\sigma_i^2)}$. $\quad W^* = \sum W_i \underbrace{\left(\frac{1}{\sigma_i^2 \sum(1/\sigma_j^2)}\right)}_{=: a_i^*}$

- $\mathbb{V}(W^*) \underset{\downarrow}{=} \left(\frac{1}{\sum(1/\sigma_i^2)}\right)^2 \sum \mathbb{V}\left(\frac{W_i}{\sigma_i^2}\right) = \left(\frac{1}{\sum(1/\sigma_i^2)}\right)^2 \sum \frac{1}{\sigma_i^4} \underbrace{\mathbb{V}(W_i)}_{\sigma_i^2} = \frac{1}{\sum(1/\sigma_i^2)}$.

$i \neq j \Rightarrow Cov(W_i, W_j) = 0$

- $\mathbb{V}(\sum a_i W_i) \underset{\downarrow}{=} \sum a_i^2 \mathbb{V}(W_i) = \sum a_i^2 \sigma_i^2$; $\quad \mathbb{E}(\sum a_i W_i) = \sum a_i \underbrace{\mathbb{E}(W_i)}_{=\vartheta} = \vartheta \sum a_i \overset{!}{=} \vartheta$

$i \neq j \Rightarrow Cov(a_i W_i, a_j W_j) = a_i a_j Cov(W_i, W_j) = 0$ $\quad \Rightarrow \sum a_i = 1 \ \forall \vartheta$

$\sum(a_i^* + \varepsilon_i)^2 \sigma_i^2 = \overbrace{\sum a_i^{*2}\sigma_i^2}^{>0} + \underbrace{2\sum a_i^* \varepsilon_i \sigma_i + \overbrace{\sum \varepsilon_i^2 \sigma_i^2}^{>0}}$ $\quad \underbrace{\sum a_i^*}_{=1} + \varepsilon_i = 1 \Rightarrow \underline{\sum \varepsilon_i = 0}$

$\sum a_i^* \varepsilon_i \sigma_i = \sum \frac{\varepsilon_i \cancel{\sigma_i}}{\sigma_i^{\cancel{2}} \sum(1/\sigma_j^2)} = \left(\sum \frac{\varepsilon_i}{\sigma_i}\right)\underbrace{\left(\frac{1}{\sum(1/\sigma_j^2)}\right)}_{>0}$

Assume that $\sum a_i^* \varepsilon_i \sigma_i < 0 \Rightarrow \sum \frac{\varepsilon_i}{\underbrace{\sigma_i}_{>0}} < 0 \Rightarrow \underline{\sum \varepsilon_i < 0}$ ↯. $\square$