(4) **Most powerful test for the normal variance - $\mu$ is known**
Let $X_1, \dots, X_n$ be iid $\mathcal{N}(\mu, \sigma^2)$, where $\mu$ is known.

(a) Find an MP test at level $\alpha$ for testing two simple hypoheses

$$H_0 : \sigma^2 = \sigma_0^2 \quad vs \quad H_1 : \sigma^2 = \sigma_1^2,\ \sigma_1 > \sigma_0 .$$

(b) Show that the MP test is a UMP test for testing

$$H_0 : \sigma^2 \leq \sigma_0^2 \quad vs \quad H_1 : \sigma^2 > \sigma_0^2.$$

*Hint:* $\sum_i (X_i - \mu)^2 \sim \sigma^2 \chi^2(n)$.

a) $L(\mu, \sigma; x) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2\right)$

$\lambda(x) = \frac{L(\mu, \sigma_1; x)}{L(\mu, \sigma_0; x)} = \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{n/2} \exp\left(\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) \frac{1}{2} \sum_{i=1}^n (x_i - \mu)^2\right)$ is an MP-test

$T(x) := \sum_{i=1}^n (x_i - \mu)^2$, we reject $H_0$ if $T(x) \geq C$, where $\mathbb{P}(T(X) \geq C) = \alpha$.

$\frac{X_i - \mu}{\sigma} \sim \mathcal{N}(0,1)$, hence $\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2 = \frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \mu)^2 \sim \chi^2(n)$

We write symbolically $T(X) \sim \sigma^2 \chi^2(n)$

Therefore, $\alpha = \mathbb{P}(T(X) \geq C) = 1 - \mathbb{P}\left(\frac{1}{\sigma_0^2} T(X) < \frac{C}{\sigma_0^2}\right) = 1 - F_{\chi^2(n)}\left(\frac{C}{\sigma_0^2}\right)$

$\Leftrightarrow F_{\chi^2(n)}\left(\frac{C}{\sigma_0^2}\right) = 1 - \alpha \Leftrightarrow C = \sigma_0^2 F_{\chi^2(n)}^{-1}(1-\alpha)$

Our test rejects $H_0$, if $T(x) \geq C$.

b) The test from (a) is by the theorem at p. 33 from Lecture 10 also an UMP

For all $\sigma \in (0, \sigma_0]$ we have

$\mathbb{P}_\sigma(T(X) \geq C) = \mathbb{P}_\sigma\left(\frac{1}{\sigma^2} T(X) \geq \frac{C}{\sigma^2}\right) = 1 - F_{\chi^2(n)}\left(\frac{C}{\sigma^2}\right) = 1 - F_{\chi^2(n)}\left(\frac{\sigma_0^2}{\sigma^2} F_{\chi^2(n)}^{-1}\left(\frac{C}{\sigma^2}\right)\right)$

$\leq 1 - \frac{C}{\sigma_0^2} \Leftrightarrow \frac{C}{\sigma_0^2} \leq F_{\chi^2(n)}\left(\frac{\sigma_0^2}{\sigma^2} F_{\chi^2(n)}^{-1}\left(\frac{C}{\sigma^2}\right)\right)$

and since $\sigma_0^2 > \sigma^2$ we have $F_{\chi^2(n)}\left(\frac{\sigma_0^2}{\sigma^2} F_{\chi^2(n)}^{-1}\left(\frac{C}{\sigma^2}\right)\right) \geq F_{\chi^2(n)}\left(F_{\chi^2(n)}^{-1}\left(\frac{C}{\sigma^2}\right)\right) = \frac{C}{\sigma^2}$

Hence $\sup\{\mathbb{P}_\sigma(T(X) \geq C) \mid \sigma \in (0, \sigma_0]\} = \mathbb{P}_{\sigma_0}(T(X) \geq C) = \alpha$

To show the remaining property we consider $\sigma_1 > \sigma_0$ and any test at level $\alpha' \leq \alpha$ with rejection region $R'$. Since our test is an MP for (a), we have

$\mathbb{P}_{\sigma_1}(T(X) \geq C) \geq \mathbb{P}(X \in R)$