21. Sei $v := (x, y, z)^T$.

$\frac{\partial f}{\partial x}(v) = y$, $\frac{\partial f}{\partial y}(v) = x + z \cdot \cos(yz)$, $\frac{\partial f}{\partial z}(v) = y \cdot \cos(yz)$;

$\frac{\partial^2 f}{\partial^2 x}(v) = 0$, $\frac{\partial^2 f}{\partial y \partial x}(v) = 1$, $\frac{\partial^2 f}{\partial z \partial x}(v) = 0$,

$\frac{\partial^2 f}{\partial^2 y} = -z^2 \cdot \sin(yz)$, $\frac{\partial^2 f}{\partial x \partial y}(v) = 1$, $\frac{\partial^2 f}{\partial z \partial y}(v) =$

$-z^2 \cdot \sin(yz) + \cos(yz)$,

$\frac{\partial^2 f}{\partial^2 z}(v) = -y \cdot \sin(yz)$, $\frac{\partial^2 f}{\partial x \partial z}(v) = 0$, $\frac{\partial^2 f}{\partial y \partial z}(v) =$

$-y^2 \cdot \sin(yz) + \cos(yz)$;

$$df(v) = \begin{pmatrix} y \\ x + z \cdot \cos(yz) \\ y \cdot \cos(yz) \end{pmatrix}^T.$$

$$[df(1,1,0)^T]\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = (1 \;\; 1 \;\; 1)\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 2.$$

23. $\frac{\partial f}{\partial x_1}(x) = 2x_1, \frac{\partial f}{\partial x_2}(x) = x_3, \frac{\partial f}{\partial x_3}(x) = x_2$

$\Rightarrow df(x) = (2x_1, x_3, x_2)$

$\Rightarrow df(1,1,1) = (2,1,1)$.

Seien $(2,1,1)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 1/2$ und $\sqrt{v_1^2 + v_2^2 + v_3^2} = 1$.

Also $2v_1 + v_2 + v_3 = 1/2$

$v_1^2 + v_2^2 + v_3^2 = 1$.

Für $v_1 := 1/4 \Rightarrow v_2 = -v_3$ und somit

$(1/4)^2 + (-v_3)^2 + v_3^2 = 1 \Leftrightarrow 2v_3^2 = 15/16$

$\Leftrightarrow v_3 = \sqrt{15/32}$.

Daher sei $v = (1/4, -\sqrt{15/32}, \sqrt{15/32})^T$.

24.
$$f(x,y) = \begin{cases} xy\,\frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2} \xrightarrow{y \to 0} 0 \xrightarrow{x \to 0} 0$$
$$\xrightarrow{x \to 0} -y \xrightarrow{y \to 0} 0$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2} \quad \dots \quad 0$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}.$$

$$\frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3} \xrightarrow{x \to 0} \frac{-y^6}{(y^2)^3} = -1$$
$$\xrightarrow{y \to 0} \frac{x^6}{(x^2)^3} = 1$$

$\Rightarrow$ Zweite Ableitungen im Punkt $(0,0)$ unstetig.

Satz von Schwarz gilt nicht.

„Widerspruch":

$$\left.\begin{array}{l} \frac{\partial f}{\partial x}(x,y) \xrightarrow{y \to 0} 0 \\ \qquad \xrightarrow{x \to 0} -y \xrightarrow{y \to 0} 0 \end{array}\right\} \text{lässt sich stetig auf 0 fortsetzen}$$

$\frac{\partial f}{\partial y}(x,y)$ analog

Abl. von $\frac{\partial f}{\partial x}(x,y)$ nach y an der Stelle $\begin{pmatrix}0\\0\end{pmatrix}$:

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{s \to 0} \frac{1}{s} \cdot \left(\frac{\partial f}{\partial x}\left(\begin{pmatrix}0\\0\end{pmatrix} + s\begin{pmatrix}0\\1\end{pmatrix}\right) - \frac{\partial f}{\partial x}\begin{pmatrix}0\\0\end{pmatrix}\right) = \frac{1}{s} \cdot \frac{-s^5}{s^4} = -1.$$

Analoges Argument für $\frac{\partial f}{\partial y} \mapsto \frac{\partial^2 f}{\partial x \partial y}$.

25. $F'(t) = (h(t, \beta(t))\beta'(t) -$

$h(t, \alpha(t))\,\alpha'(t) +$

$\int_{\alpha(t)}^{\beta(t)} \frac{\partial h}{\partial t}(t,s)\,ds$, wobei

$\alpha(t) = t$, $\alpha'(t) = 1$, $\beta(t) = 1+t^2$, $\beta'(t) = 2t$,

$h(s,t) = \sin(st)$.

$$F'(t) = \sin(t(1+t^2))\,2t - \sin(t^2)t + \int_t^{1+t^2} s\cdot\cos(ts)\,ds.$$

Und jetzt, jeweils die Stammfunktionen:

$$\int \sin(tx)dx \;\Big|\; \begin{matrix} u = tx \\ dx = 1/t\,du \end{matrix} \;= \frac{1}{t}\int \sin u\,du = -\frac{1}{t}\cdot\cos(tx).$$

$$\text{„}\int\text{“}(1+t^2) - \text{„}\int\text{“}(t) = \frac{1}{t}\left(\cos(t^2) - \cos(t+t^3)\right)$$

$$F'(t) = \frac{1}{t^2}\Big(\sin(t+t^3)(t+3t^3) + \cos(t+t^3) - \sin(t^2)\cdot 2t^2 - \cos(t^2)\Big).$$

Es kommt bei beiden Methoden dasselbe heraus.

26. Zz: $\forall f \in C^1(\mathbb{R}^2, \mathbb{R})\ \exists A: \mathbb{R}^2 \to \mathbb{R}$: T Tangentialebene.

Prop. 10.1.11. $f: \mathbb{R}^2 \to \mathbb{R}$ stetig partiell differenzierbar:

(ii) $v \in \mathbb{R}^2$, $v + u \in \mathbb{R}^2 \setminus \{v\}$

$\Rightarrow f(v+u) = f(v) + df(v)u + \|u\|_\infty \varepsilon(u)$

mit $\lim_{u \to 0} \varepsilon(u) = 0$.

Weil $f \in C^1(\mathbb{R}^2, \mathbb{R})$ differenzierbar ist, auch partiell.

Wähle $v := (x_0, y_0)$,

$u := (x, y) - (x_0, y_0) = (x - x_0, y - y_0)$,

$\Rightarrow v + u = (x, y)$,

$\Rightarrow (x_0, y_0, f(x_0, y_0)) = (x_0, y_0, z_0)$,

$A := df(v)$.

$$\lim_{u \to 0} \varepsilon(u) = \lim_{u \to 0} \frac{|f(v+u) - f(v) - df(v)u|}{\|u\|_\infty} = 0.$$

27. $\frac{\partial f}{\partial x}(x,y,z) = e^{x+yz}$, $\frac{\partial f}{\partial y}(x,y,z) = ze^{x+yz}$,

$\frac{\partial f}{\partial z}(x,y,z) = ye^{x+yz}$;

$df(x,y,z) = (e^{x+yz}, ze^{x+yz}, ye^{x+yz})$,

$df(1,0,1) = (e, e, 0)$.

$$d^2f(x,y,z) = \begin{bmatrix} \frac{\partial^2 f}{\partial x \partial x}(x,y,z) & \frac{\partial^2 f}{\partial x \partial y}(x,y,z) & \frac{\partial^2 f}{\partial x \partial z}(x,y,z) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y,z) & \frac{\partial^2 f}{\partial y \partial y}(x,y,z) & \frac{\partial^2 f}{\partial y \partial z}(x,y,z) \\ \frac{\partial^2 f}{\partial z \partial x}(x,y,z) & \frac{\partial^2 f}{\partial z \partial y}(x,y,z) & \frac{\partial^2 f}{\partial z \partial z}(x,y,z) \end{bmatrix}$$

$$= \begin{bmatrix} e^{x+yz} & ze^{x+yz} & ze^{x+yz} \\ ze^{x+yz} & z^2 e^{x+yz} & (yz+1)e^{x+yz} \\ ye^{x+yz} & (zy+1)e^{x+yz} & y^2 e^{x+yz} \end{bmatrix}.$$

$$\Rightarrow d^2f(1,0,1) = \begin{bmatrix} e & e & 0 \\ e & e & e \\ 0 & e & 0 \end{bmatrix}.$$

$$(1,1,0)\begin{bmatrix} e & e & 0 \\ e & e & e \\ 0 & e & 0 \end{bmatrix}\begin{pmatrix} e \\ 2e \\ e \end{pmatrix} = 3e.$$

28. $$f(x) = f(y) + \sum_{l=1}^{q} \frac{1}{l!} d^l f(y) \underbrace{(x-y, \dots, x-y)}_{l\text{-mal}} + R_q(x).$$

$$df(x,y,z) = \left(1, -\frac{1}{y^2 z(1 + 1/yz)}, -\frac{1}{yz^2(1+1/yz)}\right)$$

$$df(1,1,1) = (1, -1/2, -1/2)$$

$$d^2 f(x,y,z) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{2yz+1}{y^2(yz+1)^2} & \frac{1}{(yz^2+1)^2} \\ 0 & \frac{1}{(yz+1)^2} & \frac{2yz+1}{z^2(yz+1)^2} \end{bmatrix}$$

$$d^2 f(1,1,1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3/4 & 1/4 \\ 0 & 1/4 & 3/4 \end{bmatrix}$$

$$T_2(x,y,z) = (1 + \log 2) + (1, -1/2, -1/2)\begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} +$$

$$(x-1, y-1, z-1)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3/4 & 1/4 \\ 0 & 1/4 & 3/4 \end{bmatrix}\begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} \cdot 1/2 =$$

$$1 + \log 2 + x - \frac{y+z}{2} + \frac{(y-1)^2 + (z-1)^2}{2}.$$

29. $df(x,y) = (\cos x + 2x - y \cdot \sin(xy), -x \cdot \sin(xy))$

$df(0,1) = (1,0)$

$$d^2f(x,y) = \begin{bmatrix} -\sin x + 2 - y^2\cos(xy) & -\sin(xy) - xy\cdot\cos(xy) \\ -\sin(xy) - xy\cdot\cos(xy) & -x^2\cos(xy) \end{bmatrix}$$

$$d^2f(0,1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T_2(x,y) = f(0,1) + \frac{1}{1!}(1,0)\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) +$$

$$\frac{1}{2!}\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) =$$

$$1 + x + \frac{x^2}{2}.$$

30.

$$\frac{dh}{dx}(x) = \left(\cos x \cdot 2\sin x \cdot \cos x + \sin^2 x \cdot (-\sin x)\right) \frac{1}{\sin^2 x \cdot \cos x}$$

$$= \frac{2\cos^2 x - \sin^2 x}{\sin x \cdot \cos x}$$

Kettenregel: $d(f \circ g)(t) = df(g(t))\, dg(t) =$

$$df(\underbrace{\sin^2 t}_{=:a}, \underbrace{\cos t}_{=:b})\, dg(t) = d\ln(a \cdot b)\, dg(t) =$$

$$\left(\frac{1}{\cos t}, \frac{1}{\sin^2 t}\right) \begin{pmatrix} \cos t \cdot 2\sin t \\ -\sin t \end{pmatrix} = \ldots \checkmark$$