Homework 3: Problem 5

i) The order of the δ-distribution is 0. To see this, let K be a compact subset of Ω and let $\phi \in \mathcal{D}(K)$. First, assume that $0 \in K$, then we have that

$$\langle \delta, \phi \rangle = \phi(0) \lesssim \sup_{x \in K} |\phi(x)|.$$

If $0 \notin K$, then we have that

$$\langle \delta, \phi \rangle = 0.$$

ii) The order of the regular distribution associated to $f \in L^1_{\mathrm{loc}}(\Omega)$ is 0. To see this let $\phi \in \mathcal{D}(K)$ and notice that

$$\langle f, \phi \rangle = \int_K f(x)\phi(x)\,\mathrm{d}x \lesssim \sup_{x \in K} |\phi(x)|\, \|f\|_{L^1(K)}.$$

iii) Call the distribution L. We claim that the order of L, which we denote m, is $|\alpha|$. For this, we let K be a compact subset of Ω and let $\phi \in \mathcal{D}(K)$. Notice that, in the same way as in part i), we can assume that $x_0 \in K$. Then, we have that

$$\langle L, \phi \rangle \leq |\partial^\alpha \phi(x_0)| \leq \sup_{x \in K} |\partial^\alpha \phi(x)|.$$

We now know that $m \leq |\alpha|$.

To see that $m = |\alpha|$, let's assume towards a contradiction that there exists $n < |\alpha|$ such that

$$\langle L, \phi \rangle = \partial^\alpha \phi(x_0) \leq \|\phi\|_{C^n(K)}, \tag{1}$$

for any $K \Subset \Omega$ and $\phi \in \mathcal{D}(K)$. Then for $\epsilon > 0$, we consider the test function $\phi_\epsilon(x) = \epsilon^{|\alpha|}\phi(x/\epsilon)$. Since $\partial^\beta \phi_\epsilon(x) = \epsilon^{|\alpha|-|\beta|}\partial^\beta \phi(x/\epsilon)$ for any multi-index β, if (1) were to hold then applying it to ϕ_ϵ would give:

$$\langle L, \phi_\epsilon \rangle = \partial^\alpha \phi(x_0/\epsilon) \leq \sum_{\beta\,:\,|\beta| \leq n} \sup_{x \in K} \epsilon^{|\alpha|-|\beta|}\partial^\beta \phi(x_0/\epsilon).$$

Notice that because $|\alpha| - |\beta| > 0$ since $|\beta| \leq n$, we can make $\epsilon^{|\alpha|-|\beta|}$ arbitrarily small by choosing ϵ small enough. This gives a contradiction.

iv) First notice that because $(x_j)_j$ does not have an accumulation point, for any compact set $K \Subset \Omega$ the sum in the definition of the distribution is actually a finite sum. (This is because on finitely many $x_j \in K$.) From part iii) it then follows that $m = \max_{j\,:\,x_j \in K} |\alpha_j|$.