HW3

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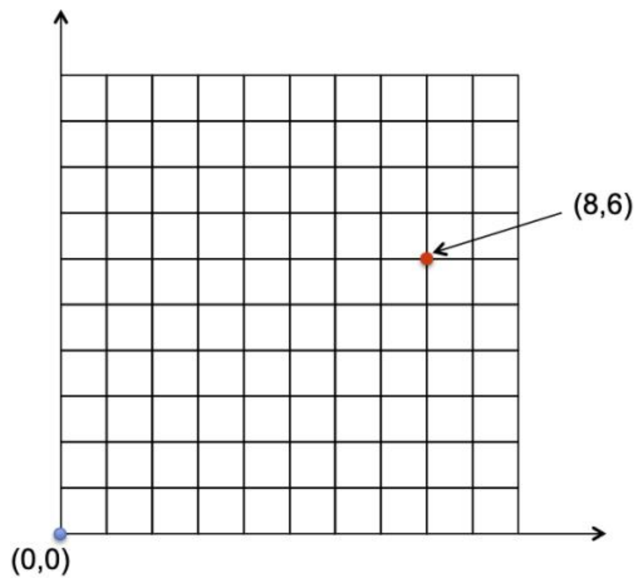
1. Random walk of a robot

A robot is placed at the origin (the point $(0,0)$) on a two-dimensional integer grid (see the figure below). Denote the position of the robot by (x, y). The robot can either move right to $(x+1, y)$ or move up to $(x, y+1)$.



(a) Suppose each time the robot randomly moves right or up with equal chance. What is the probability that the robot will ever reach the point $(8,6)$?
(b) Suppose another robot has a $\frac{2}{3}$ chance to move right and a $\frac{1}{3}$ chance to move up when $x+y$ is even, otherwise it has a $\frac{1}{4}$ chance to move right and a $\frac{3}{4}$ chance to move up. It stops whenever $|x-y| \geq 2$. Find the probability that $x-y=2$ when it stops.

Solution
(a) The robot can only ever get to the point $(8,6)$ if we get there after exactly 14 moves, 6 up and 8 to the right. After 14 moves we can be anywhere on the diagonal form $(0,14)$ to $(14,0)$. If we define the random variable

$$X = \text{Total number of times we moved right in 14 moves}$$

then $X \sim B(14, \frac{1}{2})$. With this get the following probabilities for any of the points on this diagonal:

$$P(\text{are at point (14-k,k)} \mid \text{have made exactly 14 moves}) = P(\text{are at point (k,14-k)} \mid \text{have made exactly 14 moves})$$
$$= P(X=k) = \binom{14}{k}\left(\frac{1}{2}\right)^{14} = \binom{14}{14-k}\left(\frac{1}{2}\right)^{14}$$

So we get the probablility

$$P(\text{ever reach point}(8,6)) = P(\text{are at}(8,6) \mid \text{have made exactly 14 moves}) = \binom{14}{8}\left(\frac{1}{2}\right)^{14}$$

(b) Since we can only make one move at a time we actually stop when $|x-y|=2$. We calculate

$$\begin{aligned}&P\left(x-y=2 \;\middle|\; |x-y|=2\right) = P(x-y=2 \mid x-y=2 \lor x-y=-2)\\ &= \frac{P(x-y=2 \land (x-y=2 \lor x-y=-2))}{P(x-y=2 \lor x-y=-2)} = \frac{P(x-y=2)}{P(x-y=2 \lor x-y=-2)} = \cdots\end{aligned}$$

It holds that $x-y=2$ exactly when we are at point $(x+2,x)$ for any $x \in \mathbb{N}$. Also $x-y=-2$ exactly when we are at point $(x,x+2)$ for any $x \in \mathbb{N}$. So we get

$$\cdots = \frac{\sum_{x\in\mathbb{N}} P(\text{are at point } (x+2,x))}{\sum_{x\in\mathbb{N}}\Big(P(\text{are at point } (x+2,x)) + P(\text{are at point } (x,x+2))\Big)} = \cdots$$

To calculate further we show two identities:

$$\begin{aligned}&P(\text{are at point } (x+2,x+2)) = P(\text{are at point } (x+2,x+1)) \cdot P(\text{move up} \mid x+y \text{ is odd})\\ &= P(\text{are at point } (x+2,x+1)) \cdot \frac{3}{4} = P(\text{are at point } (x+2,x)) \cdot P(\text{move up} \mid x+y \text{ is even}) \cdot \frac{3}{4}\\ &= P(\text{are at point } (x+2,x)) \cdot \frac{1}{4}\end{aligned}$$

Similarly we get

$$P(\text{are at point } (x+2,x+2)) = P(\text{are at point } (x,x+2)) \cdot \frac{1}{6}$$

Using these identities we get

$$\cdots = \frac{4\sum_{x\in\mathbb{N}} P(\text{are at point } (x+2,x+2))}{10\sum_{x\in\mathbb{N}} P(\text{are at point } (x+2,x+2))} = \frac{2}{5}$$

So the probability that $x-y=2$ when it stops is $\frac{2}{5}$.

2. Continous two-dimensional random variable

The joint pdf of two random variables X and Y is defined by

$$f(x,y) = \begin{cases} c(x+2y), & 0<y<1 \text{ and } 0<x<2\\ 0, & \text{otherwise}\end{cases}$$

(a) Find the value of c and the marginal distribution of Y.
(b) Find the joint cdf of X and Y.
(c) Find the marginal distribution of X and the pdf of $Z = \frac{9}{(X+1)^2}$.

Solution
(a) To find the value of c we stress, that $\int_{\mathbb{R}^2} f(x,y)dxdy = 1$ has to hold. So we just integrate over the values where f is not zero.

$$\int_0^1 \int_0^2 f(x,y)dxdy = c\int_0^1 (2+4y)dy = 4c$$

So we conclude $c = \frac{1}{4}$. To find the marginal of Y we have to integrate over x.

$$f_Y(y) = \int_0^2 \frac{x+2y}{4}dx = y + \frac{1}{2}, \quad \text{for } 0 < y < 1$$

(b) The joint cdf is given by

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(\xi,\tau)d\tau d\xi = \begin{cases} 0 & \text{for } x \le 0 \ \vee \ y \le 0 \\ \frac{yx^2+2xy^2}{8}, & \text{for } 0 < x < 2, 0 < y < 1 \\ 1, & \text{for } x \ge 2 \ \wedge \ y \ge 1 \end{cases}$$

(c) For the marginal of X we integrate over y.

$$f_X(x) = \int_0^1 \frac{x+2y}{4}dy = \frac{x+1}{4}, \quad \text{for } 0 < x < 2$$

To find the pdf of Z we use our transformation theorem. The transformation $g : (0,2) \to (1,9), x \mapsto \frac{9}{(x+1)^2}$ is invertible with differentiable inverse $h(z) = \frac{3}{\sqrt{z}} - 1$. Then the theorem states that the pdf of Z is given by:

$$f_Z(z) = f_X(h(z))|h'(z)| = \frac{3}{4z^{\frac{1}{2}}}| - \frac{3}{2z^{\frac{3}{2}}}| = \frac{9}{8z^2}, \quad \text{for} \ \ 1 < z < 9$$

3. Chi squared distribution

Let X and Y be independent and identically distributed (i.i.d.) $\mathcal{N}(0,1)$ random variables. Define $Z = \min\{X,Y\}$. Show that $Z^2 \sim \chi_1^2$, i.e. show that the pdf of Z^2 is given by

$$f_{Z^2}(z) = \frac{1}{\sqrt{2\pi}} \cdot z^{-\frac{1}{2}} \cdot e^{-\frac{z}{2}} \cdot \mathbf{1}_{\{z>0\}}.$$

Solution

We first aim to calculate the cdf of Z:

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(\min\{X,Y\} \le z) = 1 - P(\min\{X,Y\} > z) \\ &= 1 - P(X > z \ \wedge \ Y > z) = 1 - P(X > z)P(Y > z) \\ &= 1 - \big((1 - P(X \le z))(1 - P(Y \le z))\big) \\ &= P(X \le z) + P(Y \le z) - P(Y \le z)P(X \le z) \\ &= F_X(z) + F_Y(z) - F_X(z)F_Y(z) = 2\Phi(z) - \Phi(z)^2 \end{aligned}$$

With this we can easily get the cdf of Z^2:

$$\begin{aligned} F_{Z^2}(z) &= P(Z^2 \le z) = P(Z \le \sqrt{z}) - P(Z \le -\sqrt{z}) = 2\Phi(\sqrt{z}) - \Phi(\sqrt{z})^2 - 2\Phi(-\sqrt{z}) + \Phi(-\sqrt{z})^2 \\ &= 2\Phi(\sqrt{z}) - \Phi(\sqrt{z})^2 - 2(1 - \Phi(\sqrt{z})) + (1 - \Phi(\sqrt{z}))^2 = 2\Phi(\sqrt{z}) - 1 \end{aligned}$$

Now to get the pdf we differentiate

$$f_{Z^2}(z) = F'_{Z^2}(z) = \frac{\Phi'(\sqrt{z})}{\sqrt{z}} \cdot \mathbf{1}_{\{z>0\}} = \frac{f_X(\sqrt{z})}{\sqrt{z}} \cdot \mathbf{1}_{\{z>0\}} = \frac{1}{\sqrt{2\pi}} \cdot z^{-\frac{1}{2}} \cdot e^{-\frac{z}{2}} \cdot \mathbf{1}_{\{z>0\}}$$

4. Random variables on the unit disk

Let (X, Y) be uniformly distributed on the unit disk $f(x, y) : x^2 + y^2 < 1$. Let

$$R = \sqrt{X^2 + Y^2}$$

Find the cdf, pdf and the expectation of the random variable R.

Solution

Fist we compute $f(x, y)$, we know that it is constant in the unit disk and 0 outside of it. To fix the constant we note that the area of the unit disk is π so we get $f(x, y) = \frac{1}{\pi} \cdot \mathbf{1}_{B_1(0)}$. We know that $r = \sqrt{x^2 + y^2}$ is the radius, so we aim to use transformation to polar coordinates and take the marginal with respect to R. Using our transformation theorem and the differentiable inverse $h : [0, 1) \times [0, 2\pi) \to B_1(0), (r, \phi) \mapsto (r \cos\varphi, r \sin\varphi)$ with jacobian

$$\left| \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} \right| = r$$

we calculate:

$$f_R(r) = \int_0^{2\pi} f_{(R,\Phi)}(r, \varphi) d\varphi = \int_0^{2\pi} f_{(X,Y)}(h(r, \varphi)) \cdot r \; d\varphi = 2r, \quad r \in [0, 1)$$

We find our cdf by integrating our pdf:

$$F_R(r) = \int_{-\infty}^{r} f(\tau) d\tau = \begin{cases} 0, & r \leq 0 \\ r^2 & 0 < r < 1 \\ 1, & r \geq 1 \end{cases}$$

Last but not least we calculate the expected value of R.

$$E(R) = \int_{-\infty}^{\infty} f_R(r) \cdot r \; dr = 2 \int_0^1 r^2 dr = \frac{2}{3}$$

5. Transformations

Suppose X and Y are independent gamma distributed random variables with $X \sim Gamma(\alpha_1, \beta)$ and $Y \sim Gamma(\alpha_2, \beta)$. Consider the following two random variables

$$U = X + Y \quad \text{and} \quad V = \frac{X}{X + Y}$$

(a) Show that $U \sim Gamma(\alpha_1 + \alpha_2, \beta)$.
(b) Show that U and V are also independent random variables.

Solution
(a) To show this we use our transformation theorem. The transformation is undefined when $X+Y = 0$ but that

event occurs with probability zero and can therefore be ignored. The inverse is given by $X = VU,\ Y = U(1-V)$ with jacobian

$$\left| \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| = \left| \begin{pmatrix} v & u \\ 1-v & -u \end{pmatrix} \right| = -u$$

As reminder, the gamma function is defined via

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

and the beta function is related via

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

If we give the name h to our inverse function and use the independence of our random variables we get

$$\begin{aligned} f_U(u) &= -\int_{-\infty}^{\infty} f_{(X,Y)}(h(u,v)) \cdot u \ dv \\ &= -\int_{-\infty}^{\infty} \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)}(uv)^{\alpha_1-1} e^{-\beta uv} \cdot \mathbf{1}_{\{uv>0\}} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)}(u(1-v))^{\alpha_2-1} e^{-\beta(u-uv)} \cdot \mathbf{1}_{\{u(1-v)>0\}} \cdot u \ dv \\ &= \mathbf{1}_{\{u>0\}} \cdot \frac{\beta^{\alpha_1+\alpha_2} \cdot u^{\alpha_1+\alpha_2-1} e^{-\beta u}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 v^{\alpha_1-1}(v-1)^{\alpha_2-1} dv = \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} u^{\alpha_1+\alpha_2-1} e^{-\beta u} \cdot \mathbf{1}_{\{u>0\}} \end{aligned}$$

which is exactly what we wanted to show. \ (b) To show the independence of U and V we have to show that

$$f_{(U,V)}(u,v) = f_U(u) \cdot f_V(v)$$

So we calculate

$$\begin{aligned} f_V(v) &= -\int_{-\infty}^{\infty} f_{(X,Y)}(h(u,v)) \cdot u \ du \\ &= -\int_{-\infty}^{\infty} \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)}(uv)^{\alpha_1-1} e^{-\beta uv} \cdot \mathbf{1}_{\{uv>0\}} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)}(u(1-v))^{\alpha_2-1} e^{-\beta(u-uv)} \cdot \mathbf{1}_{\{u(1-v)>0\}} \cdot u \ du \\ &= \mathbf{1}_{\{0<v<1\}} \frac{v^{\alpha_1-1}(1-v)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty (\beta u)^{\alpha_1+\alpha_2-1} e^{-\beta u} du \\ &= \mathbf{1}_{\{0<v<1\}} \frac{v^{\alpha_1-1}(1-v)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \Gamma(\alpha_1+\alpha_2) \end{aligned}$$

All in all we see that

$$f_{(U,V)}(u,v) = -\frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{\alpha_1+\alpha_2-1} v^{\alpha_1-1}(1-v)^{\alpha_2-1} e^{-\beta u} \cdot \mathbf{1}_{\{u>0\}} \cdot \mathbf{1}_{\{0<v<1\}} = f_U(u) \cdot f_V(v)$$