(1) **Method of moment estimator**
Let $X_1, \dots, X_n$ be a random sample from a population with pdf

$$f(x) = \begin{cases} \frac{\theta x^{\theta-1}}{3^\theta}, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

where $\theta \in \mathbb{R}^+$ is unknown parameter.

(a) Show that the method of moments estimator for θ is $T_n = \frac{\bar{X}}{3-\bar{X}}$.

(b) Find the limiting distribution of $\frac{T_n - \theta}{\frac{1}{\sqrt{n}}}$ as $n \to \infty$.

a) $\mu(\theta) = \int_0^3 \frac{\theta x^{\theta-1}}{3^\theta} x \, dx = \theta 3^{-\theta} \left[\frac{x^{\theta+1}}{\theta+1} \right]_{x=0}^{3} = \frac{\theta}{\theta+1} 3 \overset{!}{=} \bar{X} \Leftrightarrow 3\theta = \bar{X}\theta + \bar{X} \Leftrightarrow \theta(3-\bar{X}) = \bar{X}$

Thus, $T_n = \frac{\bar{X}}{3-\bar{X}}$.

b) $g:]0,3[\longrightarrow \mathbb{R}: y \mapsto \frac{y}{3-y}$; $\quad g'(y) = (3-y)^{-1} + y(3-y)^{-2} = \frac{3-y+y}{(3-y)^2} = \frac{3}{(3-y)^2}$

$(\sigma(\theta))^2 - (\mu(\theta))^2 = \int_0^3 \frac{\theta x^{\theta-1}}{3^\theta} x^2 \, dx = \theta 3^{-\theta} \left[\frac{x^{\theta+2}}{\theta+2} \right]_0^3 = \frac{9\theta}{\theta+2} \Rightarrow (\sigma(\theta))^2 = \frac{9\theta}{\theta+2} - \frac{9\theta^2}{(\theta+1)^2}$

$\Rightarrow (\sigma(\theta))^2 = \left((\theta+2)(\theta+1)^2\right)^{-1} \left(9\theta(\theta+1)^2 - 9\theta^2(\theta+2) \right)$

$= \left((\theta+2)(\theta+1)^2\right)^{-1} \left(9\theta^3 + 18\theta^2 + 9\theta - 9\theta^3 - 18\theta^2 \right) = 9\theta\left((\theta+2)(\theta+1)^2\right)$

By CLT, we have $\sqrt{n}\left(\bar{X} - \frac{3\theta}{\theta+1}\right) \xrightarrow{d} Y \sim \mathcal{N}(0, (\sigma(\theta))^2) = \mathcal{N}\left(0, \frac{9\theta}{(\theta+2)(\theta+1)^2}\right)$.

$g\left(\frac{3\theta}{\theta+1}\right) = \frac{\cancel{3\theta}/\theta+1}{3 - \frac{3\theta}{\theta+1}} = \frac{3\theta}{3(\theta+1) - 3\theta} = \frac{3\theta}{3\theta + 3 - 3\theta} = \theta$

We apply the delta method and obtain

$\sqrt{n}(T_n - \theta) = \sqrt{n}\left(g(\bar{X}) - g\left(\frac{3\theta}{\theta+1}\right)\right) \longrightarrow \mathcal{N}\left(0, (\sigma(\theta))^2 \left(g'\left(\frac{3\theta}{\theta+1}\right)\right)^2\right)$

$g'\left(\frac{3\theta}{\theta+1}\right) = \frac{3}{\left(3 - \frac{3\theta}{\theta+1}\right)^2} = \frac{3(\theta+1)^2}{(3(\theta+1) - 3\theta)^2} = \frac{(\theta+1)^2}{3}$

$\Rightarrow (\sigma(\theta))^2 \left(g'\left(\frac{3\theta}{\theta+1}\right)\right)^2 = \frac{9\theta}{(\theta+2)(\theta+1)^2} \frac{(\theta+1)^4}{9} = \frac{\theta(\theta+1)^2}{\theta+2}$

Thus, $\sqrt{n}(T_n - \theta) \longrightarrow \mathcal{N}\left(0, \frac{\theta(\theta+1)}{\theta+2}\right)$