(5) **Rayleigh distribution**

Let $X_1, \dots, X_n$ be a random sample with Rayleigh distribution

$$f(x|\theta) = \begin{cases} \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}}, & x \geq 0 \\ 0, & x < 0 \end{cases},$$

where $\theta > 0$ is unknown.

(a) Find the method of moments estimator of θ.

(b) Find the MLE of θ and its asymptotic variance.

Hint: Show that the first two moments are $\mathbb{E}X = \theta\sqrt{\frac{\pi}{2}}$ and $\mathbb{E}X^2 = 2\theta^2$.

$$\mathbb{E}(X_i) = \int_0^\infty \frac{x^2}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) dx = \theta \int_0^\infty u^2 \exp\left(-\frac{u^2}{2}\right) du = \frac{\theta}{2}\sqrt{2\pi} \int_{\mathbb{R}} u^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du$$

$$= \theta\sqrt{\frac{\pi}{2}}\, \mathbb{E}(Y^2) = \theta\sqrt{\frac{\pi}{2}}\left(\mathbb{V}\mathrm{ar}(Y) - (\mathbb{E}(Y))^2\right) = \theta\sqrt{\frac{\pi}{2}}, \text{ where } Y \sim \mathcal{N}(0,1)$$

$$\mathbb{E}(X_i^2) = \int_0^\infty \frac{x^3}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) dx = \int_0^\infty 2\theta^2 u \exp(-u)\, du = \theta^2 \int_0^\infty u e^{-u} du$$

$$\boxed{\begin{aligned} u &= \frac{x^2}{2\theta^2} \\ \frac{du}{dx} &= \frac{x}{\theta^2} \\ dx &= du \frac{\theta^2}{x} \end{aligned}}$$

$$= 2\theta^2 \left(\left[-u e^{-u}\right]_0^\infty + \int_0^\infty e^{-u} du \right) = 2\theta^2 \left[-e^{-u}\right]_0^\infty = 2\theta^2$$

a) $\overline{X} = \hat{\theta}\sqrt{\frac{\pi}{2}} \Leftrightarrow \hat{\theta} = \sqrt{\frac{2}{\pi}}\, \overline{X}$

b) $L(\theta|x) = \prod_{i=1}^n \frac{1}{\theta^2} \exp\left(-\frac{x_i^2}{2\theta^2}\right) \longrightarrow \ell(\theta|x) = -2n \log(\theta) + \sum_{i=1}^n \left(-\frac{x_i^2}{2\theta^2}\right)$

$$= -2n \log(\theta) - \frac{1}{2}\theta^{-2} \sum_{i=1}^n x_i^2$$

$$\ell'(\theta, x) = \frac{-2n}{\theta} + \theta^{-3} \sum_{i=1}^n x_i^2 \overset{!}{=} 0 \Leftrightarrow \theta^2 = \frac{1}{2n} \sum_{i=1}^n x_i^2 \Leftrightarrow \theta = \sqrt{\frac{1}{2n} \sum_{i=1}^n x_i^2}$$

$$\ell''(\theta, x) = 2n\theta^{-2} - 3\theta^{-4} \sum_{i=1}^n x_i^2 \quad \text{and}$$

$$\ell''\left((2n)^{-\frac{1}{2}}|x|\right) = \frac{4n^2}{|x|^2} - \frac{12n^2}{|x|^2} = \frac{-8n^2}{|x|^2} < 0$$

$$\boxed{\begin{aligned} &\text{computer} \\ &\mathbb{E}(X_i^4) = 8\theta^4 \\ &\Rightarrow \mathbb{V}\mathrm{ar}(X_i^2) = 8\theta^4 - 4\theta^4 = 4\theta^4 \end{aligned}}$$

By the CLT, $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n x_i^2 - 2\theta^2\right) \xrightarrow{d} \mathcal{N}(0, \mathbb{V}\mathrm{ar}(X_i^2))$

$g: \mathbb{R}_0^+ \to \mathbb{R}: x \mapsto \sqrt{\frac{x}{2}}$, $\quad g'(x) = \frac{1}{\sqrt{2}} \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{4}\sqrt{\frac{2}{x}}$

Applying the delta method, we obtain

$$\sqrt{n}\left(\sqrt{\frac{1}{2n}\sum_{i=1}^n x_i^2} - \theta\right) = \sqrt{n}\left(g\left(\frac{1}{n}\sum_{i=1}^n x_i^2\right) - g(2\theta^2)\right) \longrightarrow \frac{1}{4}\frac{1}{\theta}\mathcal{N}(0, \mathbb{V}\mathrm{ar}(X_i^2))$$

Thus, the asymptotic variance is $\frac{\mathbb{V}\mathrm{ar}(X_i^2)}{16\theta^2} = \frac{\theta^2}{4}$