(4) **Minimal sufficient statistic 2**

Let $X_1, \dots, X_n$ be a random sample from a population with pdf

$$f(x|\theta) = \begin{cases} \frac{2x}{\theta}, & 0 < x < \theta \\ 0, & \text{otherwise} \end{cases},$$

with unknown parameter $\theta > 0$. Find a minimal sufficient statistic for θ.

$$L(x|\theta) = \begin{cases} \left(\frac{2}{\theta}\right)^n \prod_{i=1}^{n} x_i &, \text{if } 0 < \min\{x_i \mid i=1,\dots,n\} \leq \max\{x_i \mid i=1,\dots,n\} < \theta \\ 0 &, \text{otherwise} \end{cases}$$

If we constrain x to $(\mathbb{R}^+)^n$ and define $T(x) := \max\{x_i \mid i=1,\dots,n\}$, $h(x) = \prod_{i=1}^{n} x_i$, and

$$g(z|\theta) = \begin{cases} \left(\frac{2}{\theta}\right)^n z_1 &, \text{if } z_2 < \theta \\ 0 &, \text{else} \end{cases}$$

then $L(x|\theta) = g(T(x)|\theta)\, h(x)$, hence $T(X)$ is sufficient.

If $T(x) = T(y)$, then $\frac{L(x|\theta)}{L(y|\theta)} = \frac{h(x)}{h(y)}$ is constant as a function of $\theta \in (T(x), \infty)$.

If $T(y) < T(x)$, then we choose $\theta_1, \theta_2 \in \mathbb{R}^+$ such that $T(x) < \theta_2$ and $T(y) < \theta_1 < T(x)$, hence

$\frac{L(x,\theta_1)}{L(y,\theta_2)} = 0$ and $\frac{L(x,\theta_2)}{L(x,\theta_2)} = \frac{h(x)}{h(y)} > 0$, hence $\frac{L(x|\theta)}{L(y|\theta)}$ is not constant as a function of θ. We conclude that $T(X)$ is a minimal sufficient statistic.