(2) **Box of candles**
There are blue and red candles in a box. Probability that a randomly chosen candle is blue is $\frac{1}{1+2a}$, for $a > 0$. Based on a sample of sample size n, find the maximum likelihood estimator (MLE) $\hat{a}$ of the parameter a.

Für $x \in \{0,1\}^n$: $\quad S_n := \sum_{i=1}^{n} x_i$

$$L_n(a) = \prod_{i=1}^{n} \left(\frac{1}{1+2a}\right)^{x_i} \left(1 - \frac{1}{1+2a}\right)^{1-x_i} = \left(\frac{1}{1+2a}\right)^{\sum_{i=1}^n x_i} \left(\frac{2a}{1+2a}\right)^{n - \sum_{i=1}^n x_i}$$

$$l_n(a) = -S_n \log(1+2a) + (n-S_n)\left(\log(2a) - \log(1+2a)\right) = \log(2a)(n-S_n) - n\log(1+2a)$$

$$l_n'(a) = \frac{n-S_n}{a} - \frac{2n}{1+2a} \overset{!}{=} 0 \Leftrightarrow (n-S_n)(1+2a) = 2na \Leftrightarrow n + 2na - S_n - 2aS_n = 2na$$
$$\Leftrightarrow n - S_n = 2aS_n \Leftrightarrow a = \frac{n-S_n}{2S_n}$$

$0 < S_n < n$

$$l_n''(a) = -\frac{n-S_n}{a^2} + \frac{4n}{(1+2a)^2}$$

$$l_n''\left(\frac{n-S_n}{2S_n}\right) = -\frac{4S_n^2}{n-S_n} + 4n\left(\frac{S_n + n - S_n}{S_n}\right)^{-2} = \frac{4S_n^2}{n} - \frac{4S_n^2}{n-S_n} = 4S_n^2\left(\frac{1}{n} - \frac{1}{n-S_n}\right) < 0$$

If $S_n = 0$, then $L_n(a) = \left(\frac{2a}{1+2a}\right)^n$ is infinitely increasing, hence "$\hat{a} = \infty$"

If $S_n = n$, then $L_n(a) = \left(\frac{1}{1+2a}\right)^n$ is decreasing, hence $\hat{a} = 0$

In general: $\hat{a} = \frac{n-S_n}{2S_n}$