(2) **Sum of two independent distributions**

(a) Let $X \sim \mathcal{P}(\lambda_1)$ and $Y \sim \mathcal{P}(\lambda_2)$ be two independent Poisson random variables. Show that
$$X + Y \sim \mathcal{P}(\lambda_1 + \lambda_2).$$

(b) Let U and V be two independent random variables with exponential distribution $\exp(\lambda)$. Show that
$$U + V \sim Gamma(2, \lambda) \qquad \text{and}$$
$$\min\{U, V\} \sim \exp(2\lambda).$$

Hint: It is useful to use moment generating functions. Recall, the pdf of a random variable $X \sim Gamma(\alpha, \beta)$ is
$$f(x) = \begin{cases} \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\,\beta^{\alpha}} & x > 0 \\ 0, & x \leq 0 \end{cases},$$
and its mgf is of the form $\left(\frac{1}{1-\beta t}\right)^{\alpha}$ for $t < \frac{1}{\beta}$. Particularly, the pdf of a random variable $X \sim \exp(\lambda) = Gamma(1, \frac{1}{\lambda})$ is of the form
$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x \leq 0 \end{cases}.$$

a) $M_X(t) = \mathbb{E}(e^{tX}) = e^{-\lambda_1}\sum_{k=0}^{\infty} e^{tk}\frac{\lambda_1^k}{k!} = e^{-\lambda_1}\sum_{k=0}^{\infty}\frac{(\lambda_1 e^t)^k}{k!} = e^{-\lambda_1}e^{\lambda_1 e^t} = e^{\lambda_1(e^t-1)}$

$M_Y(t) = e^{\lambda_2(e^t-1)}$, $\quad M_X(t)\,M_Y(t) = e^{(\lambda_1+\lambda_2)(e^t-1)} = M_{X+Y}(t)$

Hence, $X+Y \sim \mathcal{P}(\lambda_1+\lambda_2)$

b) $M_U(t) = M_V(t) = \frac{1}{1-\lambda t}$, $\quad M_{U+V}(t) = M_U(t)M_V(t) = \frac{1}{(1-\lambda t)^2}$

Hence, $U+V \sim Gamma(2, \lambda)$, used $\exp(\lambda) = Gamma(1, \lambda)$!!!,

with pdf $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$, $0 < x < \infty$

$\mathbb{P}(\min(U,V) < z) = 1 - \mathbb{P}(U \geq z \wedge V \geq z) = 1 - \mathbb{P}(U \geq z)\,\mathbb{P}(V \geq z) = 1 - (1 - \mathbb{P}(U < z))^2$

$= 1 - (1 - (1 - e^{-\lambda z}))^2 = 1 - e^{-2\lambda z}$, where $F_U(x) = 1 - e^{-\lambda x}$, $0 < x < \infty$

is the cdf of an exponential distribution.