HW7

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1. Uniform distribution

Let $X_1, \dots, X_n$ be a random sample from uniform $(\theta, 1)$ distribution, where $\theta < 1$ is an unknown parameter.

(a) Find the MLE $\hat{\theta}$ of θ.

(b) Is $\hat{\theta}$ asymptotically normal? If yes, find the asymptotic mean and variance. Otherwise find a sequence r_n and a_n such that $r_n(\hat{\theta} - a_n)$ converges in distribution to a non-degenerate (not pointmass) distribution.

Solution:

(a) The likelihood function is given by

$$L(\theta \mid x) = \prod_{i=1}^{n} f(x_i \mid \theta) = \prod_{i=1}^{n} \frac{1}{1-\theta} \mathbf{1}_{[\theta,1]}(x_i) = \frac{1}{(1-\theta)^n} \mathbf{1}_{(-\infty,1]}\Big(\max_{i=1,\dots,n} x_i\Big) \mathbf{1}_{[\theta,\infty)}\Big(\min_{i=1,\dots,n} x_i\Big)$$

We see that the indicator of the maximum does not depend on θ, the fraction is a growing function in θ and the indicator of the minimum is a falling function in θ. So the maximum of the likelihood function is reached when we take

$$\hat{\theta} = \min_{i=1,\dots,n} X_i$$

(b) We first calculate the cdf of $X_{(1)} = \min_{i=1,\dots,n} X_i$ where we use the fact that the random variables are independent

$$F_{X_{(1)}}(x) = \mathbb{P}(X_{(1)} \le x) = 1 - \mathbb{P}(X_{(1)} > x) = 1 - \mathbb{P}(X_1 > x)^n = 1 - (1 - \mathbb{P}(X_1 \le x))^n = \begin{cases} 0, & x < \theta \\ 1 - (1 - \frac{x-\theta}{1-\theta})^n, & \theta \le x \le 1 \\ 1, & x > 1 \end{cases}$$

This looks very similar to the first example of HW 5. If we choose $r_n = \frac{n}{1-\theta}$ and $a_n = \theta$ we get

$$\begin{aligned}
\mathbb{P}(r_n(X_{(1)} - a_n) \le x) = \mathbb{P}(X_{(1)} \le \frac{1-\theta}{n} x + \theta) &= \begin{cases} 0, & \frac{1-\theta}{n} x + \theta < \theta \\ 1 - (1 - \frac{x}{n})^n, & \theta \le \frac{1-\theta}{n} x + \theta \le 1 \\ 1, & \frac{1-\theta}{n} x + \theta > 1 \end{cases} \\
&= \begin{cases} 0, & x < 0 \\ 1 - (1 - \frac{x}{n})^n, & 0 \le x \le n \\ 1, & x > n \end{cases} \\
&\xrightarrow{n \to \infty} \begin{cases} 1 - e^{-x}, & x \ge 0 \\ 0, & x < 0 \end{cases}
\end{aligned}$$

So it converges to $\exp(1)$ in distribution.

2. Cramér-Rao lower bound

Let $X_1, \ldots, X_n$ be a random sample with the pdf $f(x \mid \theta) = \theta x^{\theta-1}$, where $0 < x < 1$ and $\theta > 0$ is unknown. Is there a function of θ, say $g(\theta)$, for which there exists an unbiased estimator whose variance attains the Cramér-Rao lower bound? If there is, find it. If not, show why not.

Solution:

3. Minimum variance estimator

Let $W_1, \ldots, W_k$ be unbiased estimators of a parameter θ with $\mathbb{V}ar = \sigma_i^2$ and $\mathbb{C}ov(W_i, W_j) = 0$ if $i \neq j$. Show that, of all estimators of the form $\sum a_i W_i$ where a_is are constant and $\mathbb{E}_\theta\Big(\sum a_i W_i\Big) = \theta$, the estimator

$$W^* = \frac{\sum W_i/\sigma_i^2}{\sum(1/\sigma_i^2)}$$

has minimum variance. Show that

$$\mathbb{V}ar W^* = \frac{1}{\sum(1/\sigma_i^2)}$$

Solution:

Since the random variables have covariance 0 we can calculate the variance rather easily

$$\mathbb{V}ar(\sum a_i W_i) = \sum a_i^2 \mathbb{V}ar(W_i) = \sum a_i^2 \sigma_i^2$$

We can see that $\sum a_i = 1$ has to hold:

$$\mathbb{E}_\theta(\sum a_i W_i) = \sum a_i \mathbb{E}_\theta(W_i) = \theta \sum a_i \stackrel{!}{=} \theta$$

So we are dealing with the optimization problem of minimizing $f(a) = \sum a_i^2 \sigma_i^2$ where $a = (a_1, \ldots, a_k) \in \mathbb{R}^n$ under the constraint $g(a) := \sum a_i - 1 = 0$. We can immediately observe that $D^2 f(a)$ is positive definite for any a, so we are dealing with a convex problem. This means we can use the method of Lagrange multipliers to not only get a necessary but also a sufficient condition for the minimum. Since $\nabla g(a) = (1, \ldots, 1) \neq 0$ for any a the constraint qualification holds. We now define

$$\Lambda(a, \lambda) = f(a) - \lambda g(a)$$

and note that for the optimum a has to fulfill

$$\begin{aligned}
\partial_{a_i} \Lambda(a, \lambda) &= 2 a_i \sigma_i^2 - \lambda = 0 \\
\partial_\lambda \Lambda(a, \lambda) &= \sum a_i - 1 = 0
\end{aligned}$$

From the first group of equations we get that for any $i = 1, \ldots, k$

$$2 a_i \sigma_i^2 = \lambda \iff a_i = \frac{\lambda}{2\sigma_i^2}$$

has to hold. From the last equation we then get

$$\sum \frac{\lambda}{2\sigma_i^2} = 1 \iff \lambda = \frac{1}{\sum 1/2\sigma_i^2}$$

So in the end we get

$$a_i = \frac{\lambda}{2\sigma_i^2} = \frac{\frac{1}{\sum 1/2\sigma_i^2}}{2\sigma_i^2} = \frac{1}{\sigma_i^2 \sum 1/\sigma_i^2}$$

This choice of a now gives not only a local but also global minimum. With this choice we can also see that

$$\sum W_i a_i = W^*$$

so this estimator indeed has minimum variance. To now show the variance of this estimator we use the formula from above

$$\mathbb{V}ar(W^*) = \sum a_i^2 \sigma_i^2 = \sum \frac{1}{\sigma_i^2 \big(\sum 1/\sigma_i^2\big)^2} = \frac{1}{\big(\sum 1/\sigma_i^2\big)^2} \cdot \sum \frac{1}{\sigma_i^2} = \frac{1}{\sum 1/\sigma_i^2}$$

4. Normal unbiased estimator of μ^2

Let $X_1, \ldots, X_n$ be i.i.d. $\mathcal{N}(\mu, 1)$.

(a) Show that $\bar{X}^2 - \frac{1}{n}$ is an unbiased estimator of μ^2.

(b) By using Stein's Lemma, calculate its variance and show that it is greater that the Cramèr-Rao lower bound.

Hint: Recall, Stein's Lemma states that for $X \sim \mathcal{N}(\mu, \sigma^2)$ and a differentiable function g satisfying $E|g'(X)| < \infty$ it holds $\mathbb{E}\Big(g(X)(X-\mu)\Big) = \sigma^2 \mathbb{E} g'(X)$.

Solution:

(a) Since the random variables are independent it holds that $\mathbb{V}ar(\bar{X}) = \frac{1}{n^2}\sum_{i=1}^n \mathbb{V}ar(X_i)$ with that and $\mathbb{E}(\bar{X}) = \mu$ we calculate

$$\mathbb{E}(\bar{X}^2 - \frac{1}{n}) = \mathbb{E}(\bar{X}^2) - \frac{1}{n} = \mathbb{V}ar(\bar{X}) + \mathbb{E}(\bar{X})^2 - \frac{1}{n} = \frac{1}{n} + \mu^2 - \frac{1}{n} = \mu^2$$

So this estimator is indeed unbiased. 8 (b) From (a) we can infer that $\mathbb{E}\big(\bar{X}^2\big) = \mu^2 + \frac{1}{n}$. We will use Stein's Lemma (with $g(x) = x^2$) to calculate

$$\mathbb{E}(\bar{X}^3) = \mathbb{E}(\bar{X}^2(\bar{X}-\mu)) + \mu\mathbb{E}(\bar{X}^2) = \frac{2}{n}\mathbb{E}(\bar{X}) + \mu\big(\mu^2 + \frac{1}{n}\big) = \mu\big(\mu^2 + \frac{3}{n}\big)$$

With the help of this and again using Stein's Lemma (this time with $g(x) = x^3$) we now calculate

$$\mathbb{E}(\bar{X}^4) = \mathbb{E}(\bar{X}^3(\bar{X}-\mu)) + \mu\mathbb{E}(\bar{X}^3) = \frac{3}{n}\mathbb{E}(\bar{X}^2) + \mu^2(\mu^2 + \frac{3}{n}) = \frac{3}{n}(\mu^2 + \frac{1}{n}) + \mu^2(\mu^2 + \frac{3}{n}) = \mu^4 + \frac{6}{n}\mu^2 + \frac{3}{n^2}$$

Finally we can calculate the variance

$$\mathbb{V}ar(\bar{X}^2 - \frac{1}{n}) = \mathbb{V}ar(\bar{X}^2) = \mathbb{E}\big(\bar{X}^4\big) - \mathbb{E}\big(\bar{X}^2\big)^2 = \mu^4 + \frac{4}{n}\mu^2 + \frac{3}{n^2} - \big(\mu^2 + \frac{1}{n}\big)^2 = \mu^2\frac{4}{n} + \frac{2}{n^2}$$

To now show that this is greater that the Cramèr-Rao lower bound we obviously have to calculate that bound. Our estimator is unbiased w.r. to μ^2, so we actually have to use the Cramér-Rao inequality with $g(x) = x^2$

$$\mathbb{V}ar_\mu(T) \geq \frac{g'(\mu)^2}{nI(\mu)} = \frac{4\mu^2}{nI(\mu)}$$

Where $I(\mu)$ is the Fisher information. To get this function we note that the log likelihood function for our random variables is (see page 62, Lecture 6)

$$\ell(\mu \mid x) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^2$$

We take the second derivative with respect to μ:

$$\partial_\mu \ell(\mu \mid x) = \sum_{i=1}^{n}(x_i - \mu)$$
$$\partial_\mu^2 \ell(\mu \mid x) = -n$$

The Fisher information is now

$$I(\mu) = -\mathbb{E}(-n) = n$$

So the lower bound is given by

$$\frac{4\mu^2}{n^2}$$

and with this we get

$$\mathbb{V}ar(\bar{X}^2 - \frac{1}{n}) = \mu^2\frac{4}{n} + \frac{2}{n^2} \overset{!}{>} \frac{4\mu^2}{n^2} \iff 4\mu^2 n + 2 > 4\mu^2 \iff 4\mu^2(n-1) + 2 > 0$$

and see that the variance is indeed greater than this bound.

5. Exponential family

Show that a Poisson family of distributions $\mathcal{P}oi(\lambda)$, with unknown $\lambda > 0$ belongs to the exponential family.

Solution:

We say a family of distributions belong to the exponential family if it can be represented in the form

$$f(x \mid \theta) = h(x)c(\theta)e^{\sum w_i(\theta)t_i(x)}$$

where $h(x) > 0, t_1(x), \ldots, t_k(x)$ are real valued functions that do not depend on θ and $c(\theta) > 0, w_1(\theta), \ldots, w_k(\theta)$ are real valued functions that do not depend on x. We recall that the pmf of a Poisson random variable is given by

$$f(k \mid \lambda) = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{e^{k\ln(\lambda) - \lambda}}{k!}$$

So it belongs to the exponential family with

$$
\begin{aligned}
h(k) &= \frac{1}{k!} \\
c(\lambda) &= 1 \\
w_1(\lambda) &= \ln(\lambda) \\
t_1(k) &= k \\
w_2(\lambda) &= -\lambda \\
t_2(k) &= 1
\end{aligned}
$$