(2) **Most powerful test 1**
Let $X_1, \ldots, X_n$ be iid Uniform$(0, \theta)$.

(a) Derive the most powerful (MP) test at level α for testing

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta = \theta_1, \theta_1 > \theta_0.$$

(b) Calculate the power of the MP test.

a) $L(\theta; x) = \begin{cases} \frac{n}{\theta}, & \text{if } \forall i \in \{1, \ldots, n\}: x_i \in (0, \theta) \\ 0, & \text{else} \end{cases}$, Therefore we obtain for $x \in (0, \theta_1)$

$$\lambda(X) = \frac{L(\theta_1, x)}{L(\theta_0, x)} = \begin{cases} \frac{\theta_0}{\theta_1}, & \text{if } \max\{x_i \mid 1 \le i \le n\} < \theta_0 \\ \text{„}\infty\text{“}, & \text{else} \end{cases}$$

We presume (*) that $T(x) = \max\{x_i \mid 1 \le i \le n\}$ is an appropriate Test-statistic for an MP

Assuming that $X_i \sim U(0, \theta_0)$, we have

$$\mathbb{P}(T(x) \ge c) = 1 - \mathbb{P}(T(x) < c) = 1 - \prod_{i=1}^{n} \mathbb{P}(X_i < c) = \begin{cases} 1 & , \text{if } c \le 0 \\ 0 & , \text{if } c \ge \theta_0 \\ 1 - \left(\frac{c}{\theta_0}\right)^n & , \text{if } 0 < c < \theta_0 \end{cases}$$

If $0 < \alpha < 1$, then $\mathbb{P}(T(X) \ge c) = \alpha \Leftrightarrow \alpha = 1 - \left(\frac{c}{\theta_0}\right)^n \Leftrightarrow \left(\frac{c}{\theta_0}\right)^n = 1 - \alpha \Leftrightarrow c = \theta_0 (1-\alpha)^{\frac{1}{n}}$

Hence, our test rejects H_0, if $T(x) \ge \theta_0 (1-\alpha)^{\frac{1}{n}}$

b) The power q of the test is

$$q = \mathbb{P}\left(T(X) \ge \theta_0 (1-\alpha)^{\frac{1}{n}} \mid X_i \sim U(0, \theta_1)\right) = 1 - \prod_{i=1}^{n} \mathbb{P}\left(X_i < \theta_0 (1-\alpha)^{\frac{1}{n}}\right) = 1 - \left(\frac{\theta_0 (1-\alpha)^{\frac{1}{n}}}{\theta_1}\right)^n$$

$$= 1 - (1-\alpha) \left(\frac{\theta_0}{\theta_1}\right)^n$$

(*) This presumption turns out to be true, since for any other test of level α' with rejection region R' we have

$$\mathbb{P}(X \in R' \mid X \sim U(0, \theta_1)) = \int_{R'} \frac{1}{\theta_1^n} \mathbb{1}_{[0,\theta_1]^n}(x)\, dx = \frac{\theta_0^n}{\theta_1^n} \int_{R'} \frac{1}{\theta_0^n} \mathbb{1}_{[0,\theta_0]^n}(x)\, dx + \int_{R'} \frac{1}{\theta_1^n} \mathbb{1}_{([0,\theta_0]^n)^c}(x)\, dx$$

$$\le \frac{\theta_0^n}{\theta_1^n} \alpha + \int_{([0,\theta_0]^n)^c} \frac{1}{\theta_1^n}\, dx = \frac{\theta_0^n}{\theta_1^n} \alpha + 1 - \frac{\theta_0^n}{\theta_1^n} = 1 - (1-\alpha)\left(\frac{\theta_0}{\theta_1}\right)^n = q$$