The problems are to be presented on **May 11, 2021**. They should be ticked and solution paths uploaded by **23:59 on May 10, 2021**.

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(1) **Uniform distribution**
Let $X_1, \dots, X_n$ be a random sample from uniform $(\theta, 1)$ distribution, where $\theta < 1$ is an unknown parameter.

(a) Find the MLE $\hat{\theta}$ of θ.

(b) Is $\hat{\theta}$ asymptotically normal? If yes, find the asymptotic mean and variance. Otherwise, find a sequence r_n and a_n such that $r_n(\hat{\theta} - a_n)$ converges in distribution to a non-degenerate (not pointmass) distribution.

(2) **Cramér-Rao lower bound**
Let $X_1, \dots, X_n$ be a random sample with the pdf $f(x|\theta) = \theta x^{\theta - 1}$, where $0 < x < 1$ and $\theta > 0$ is unknown. Is there a function of θ, say $g(\theta)$, for which there exists an unibiased estimator whose variance attains the Cramér-Rao lower bound? If there is, find it. If not, show why not.

(3) **Minimum variance estimator**
Let $W_1, \dots, W_k$ be unbiased estimators of a parameter θ with $\mathbb{V}ar = \sigma_i^2$ and $\mathbb{C}ov(W_i, W_j) = 0$ if $i \neq j$. Show that, of all estimators of the form $\sum a_i W_i$ where a_is are constant and $\mathbb{E}_\theta\Big(\sum a_i W_i\Big) = \theta$, the estimator

$$W^* = \frac{\sum W_i/\sigma_i^2}{\sum (1/\sigma_i^2)}$$

has minimum variance. Show that

$$\mathbb{V}ar\, W^* = \frac{1}{\sum (1/\sigma_i^2)}.$$

(4) **Normal unbiased esimator of μ^2**
Let $X_1 \dots X_n$ be i.i.d. $\mathcal{N}(\mu, 1)$.

(a) Show that $\bar{X}^2 - \frac{1}{n}$ is unbiased esimator of μ^2.

(b) By using Stein's Lemma, calculate its variance and show that it is greater than the Cramér-Rao lower bound.

Hint: Recall, Stein's Lemma states that for $X \sim \mathcal{N}(\mu, \sigma^2)$ and a differentiable function g satisfying $E|g'(X)| < \infty$ it holds $\mathbb{E}\Big(g(X)(X - \mu)\Big) = \sigma^2\, \mathbb{E} g'(X)$.

(5) **Exponential family**
Show that a Poisson family of distributions $\mathcal{P}oi(\lambda)$, with unknown $\lambda > 0$ belongs to the exponential family.