A M.1.2 Gg.: T VR über Polynomfunktionen $\mathbb{R} \to \mathbb{R}$,

für $f \in T$, $i \in \mathbb{N}$: $f^{(i)}$ i-te Ableitung v. f

(a) Zz.: $\iota : \begin{cases} T \times T \to \mathbb{R} \\ (f, g) \mapsto \sum\limits_{i \in \mathbb{N}} f^{(i)}(0) \cdot g^{(i)}(0) \end{cases}$ pos. def. SP auf T

d.h. ι ist ω-symmetrische ω-Sesquilinearform oder alternierende Bilinearform, sowie radikalfrei ($T^{\perp_\iota} = \{\mathcal{O}\}$)

(b) Geg.: $\left(f_i : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^i \end{cases} \right)_{i \in \mathbb{N}}$ Basis v. T,

$\forall n \in \mathbb{N}$: $B_n := (f_i)_{i=1}^{n}$, $U_n := [B_n]$, $\iota_n := \iota|_{U_n \times U_n}$

Ges.: $\iota_n(B_n, B_n)$

Ww.: $\iota_n(a, b) = \sum\limits_{k=1}^{n} a_k b_k k!^2 \stackrel{!}{=} a^T \iota_n(B_n, B_n) b$

$\Rightarrow \iota_n(B_n, B_n) = \mathrm{diag}(k!^2)_{k=1}^{n}$

A 11.1.3 Gg.: K Körper, (V, ι) 4-dim symplekt. VR/K

Zz.: $\forall U < V, \dim U = 2 : U \oplus U^\perp = V \;\dot\vee\; U = U^\perp$

Ww.: $\underbrace{\dim U}_{2} + \dim U^\perp = \underbrace{\dim V}_{4} + \underbrace{\cancel{\dim(U \cap V^\perp)}}_{V^\perp = \{0\}}$

$\Rightarrow \dim U^\perp = 2$

Sei $\{a, b\}$ Basis v. U, d.h. $U = [a, b]$.

$\Rightarrow [a,b] \cap [a,b]^\perp = ([a] + [b]) \cap ([a] + [b])^\perp =$
$([a] + [b]) \cap [a]^\top \cap [b]^\top = \cdots$

Beh.: $\cdots ([a] \cap [a]^\top \cap [b]^\top) + ([b] \cap [a]^\top \cap [b]^\top)$

Bew.: „$\subseteq$". Sei $x \in$ l.S., dann $\exists \tilde a \in [a], \tilde b \in [b]$:
$x = \tilde a + \tilde b \in ([a] + [b]), [a]^\top, [b]^\top$.

$\Rightarrow \underbrace{\iota(\tilde a + \tilde b, a)}_{\iota(\tilde a, a) + \iota(\tilde b, a)} = 0 \Rightarrow \iota(\tilde b, a) = 0 \Rightarrow \tilde b \in [a]^\top$

$\underbrace{\iota(\tilde a, a)}_{0} + \iota(\tilde b, a)$, und

$\Rightarrow \cdots \Rightarrow \tilde a \in [b]^\top$.

„$\supseteq$". ✓ $\square$

Fall 1. $[a] \cap [b]^\perp = \{0\}$

$\Leftrightarrow \iota(a, b) \neq 0 \Leftrightarrow \iota(b, a) \neq 0$
$\Leftrightarrow [b] \cap [a]^\perp = \{0\}$

$\Rightarrow U \cap U^\perp = \{0\} \Rightarrow \left(U \neq U^\perp,\ U \oplus U^\perp = V\right)$

Fall 2. $[a] \cap [b]^{\perp} \neq \{0\}$

$\Rightarrow [a] \subseteq [b]^{\perp}$

$\Rightarrow \iota(a, b) = 0 \Rightarrow \iota(b, a) = 0$

$\Rightarrow [b] \subseteq [a]^{\perp}$

$\Rightarrow U = U \cap U^{\perp} \Rightarrow \left(U = U^{\perp}, \neg (U \oplus U^{\perp} = V) \right)$

$$\underset{\substack{\leadsto \\ \text{SP auf } \mathbb{R}^{4\times 4}}}{\text{Sei } \iota(B,B)} = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \mathcal{O} \\ & \mathcal{O} & 0 & -1 \\ & & 1 & 0 \end{pmatrix}$$

- $U = [e_1, e_3]$

$$\left.\begin{aligned} \ker(\iota(e_1,\cdot)) &= [e_1, e_3, e_4] \\ \ker(\iota(e_3,\cdot)) &= [e_1, e_2, e_3] \end{aligned}\right\} \text{„}\cap\text{"} = [e_1, e_3] = U^\perp$$

$\Rightarrow \quad U = U^\perp$

- $U = [e_2, e_4]$

$$\left(\left.\begin{aligned} \ker(\iota(e_2,\cdot)) &= [e_2, e_3, e_4] \\ \ker(\iota(e_4,\cdot)) &= [e_1, e_2, e_4] \end{aligned}\right\} \text{„}\cap\text{"} = \overbrace{[e_2, e_4]}^{U} = U^\perp\right)$$

- $U = [e_1, e_2]$

$$\left.\begin{aligned} \ker(\iota(e_1,\cdot)) &= [e_1, e_3, e_4] \\ \ker(\iota(e_2,\cdot)) &= [e_2, e_3, e_4] \end{aligned}\right\} \text{„}\cap\text{"} = [e_3, e_4] = U^\perp$$

$\Rightarrow \quad U \oplus U^\perp = V$

A 11.5.1 Geg.: ι SP auf $\mathbb{R}^{3\times 1}$ mit ONB

$$b_1 = \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix},\quad b_2 = \begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix},\quad b_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

Ges.: $\iota(E, E)$

$$\langle E^*, B\rangle = \begin{array}{ccc} -5 & -6 & -2 \\ 3 & 3 & 1 \\ 1 & 2 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \mapsto \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 2 & 0 \\ -2 & -3 & -1 \\ 3 & 4 & 3 \end{array} = \langle E^*, B\rangle^{-1} = \langle B^*, E\rangle$$

$$\iota(E, E) = \langle B^*, E\rangle^T \underbrace{\cancel{\iota(B, B)}}_{E_3} \langle B^*, E\rangle$$

$$= \begin{pmatrix} 14 & 20 & 11 \\ 20 & 29 & 15 \\ 11 & 15 & 10 \end{pmatrix}.$$

A 11.5.2 Geg.: ($\mathbb{K}^{n \times 1}$, kanon. euklid. bzw. unit. SP),

$B = (b_1, \dots, b_n)$ Basis

Ges.: Orthogonalbasis, Orthonormalbasis aus B mit E. Schmidt

(β) Geg.: $n = 4$, $K = \mathbb{R}$,

$$b_1 = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 2 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 6 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad b_4 = \begin{pmatrix} 3 \\ -3 \\ -1 \\ 3 \end{pmatrix}$$

$$a_1 := b_1$$

$$a_2 := b_2 - \frac{a_1 \cdot b_2}{a_1 \cdot a_1} \cdot a_1$$

$$= \begin{pmatrix} 1 \\ 3 \\ 2 \\ 6 \end{pmatrix} - \cancel{\frac{2 \cdot 1 + 1 \cdot 3 + 4 \cdot 2 + 2 \cdot 6}{2^2 + 1^2 + 4^2 + 2^2}} \cdot \begin{pmatrix} 2 \\ 1 \\ 4 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 2 \\ -2 \\ 4 \end{pmatrix}$$

$\vdots$

$$a_{r+1} := b_{r+1} - \sum_{j=1}^{r} \frac{a_j \cdot b_{r+1}}{a_j \cdot a_j} a_j$$

$$\Rightarrow \quad a_3 = \begin{pmatrix} 4/5 \\ 2/5 \\ -2/5 \\ -1/5 \end{pmatrix}, \quad a_4 = \begin{pmatrix} 2 \\ -4 \\ -1 \\ 2 \end{pmatrix}$$

$$A = (a_1, \dots, a_4) \xmapsto{\frac{1}{\|a_i\|} a_i} \begin{pmatrix} 2 & -1 & 4 & 2 \\ 1 & 2 & 2 & -4 \\ 4 & -1 & -1 & -1 \\ 2 & 4 & 2 & 2 \end{pmatrix} \cdot 1/5 = \hat{A}$$

(γ) Geg.: $n = 3$, $K = \mathbb{C}$,

$$b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}$$

$$a_1 := b_1,$$

$$a_2 := b_2 - \frac{a_1 \cdot b_2}{a_1 \cdot a_1} \cdot a_1 = \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix} - \frac{\cancel{1 \cdot 0} + 1 \cdot i + \cancel{1 \cdot 0}}{1^2 + 1^2 + 1^2} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -i \\ 2i \\ -i \end{pmatrix} \cdot \frac{1}{3}$$

$$a_3 := b_3 - \left(\frac{a_1 \cdot b_3}{a_1 \cdot a_1} \cdot a_1 + \frac{a_2 \cdot b_3}{a_2 \cdot a_2} \cdot a_2 \right) = \begin{pmatrix} i \\ 0 \\ -i \end{pmatrix} \cdot \frac{1}{2}$$

$$A = (a_1, \dots, a_3) \xmapsto{\frac{a_i}{\|a_i\|}} \begin{pmatrix} 1/\sqrt{3} & 3\sqrt{6}/2 & -\sqrt{2}/4 \\ 1/\sqrt{3} & -2\sqrt{6}/2 & 0 \\ 1/\sqrt{3} & 3\sqrt{6}/2 & \sqrt{2}/4 \end{pmatrix} = \hat{A}$$

A 11.5.6 Geg.: $I = [-1,1]$, $(C^\circ(I), \iota)$, wobei

$\iota : (f,g) \mapsto \int_{-1}^{1} fg \, dx$

(a) Ges.: ONB von $U_3 := [f_i : x \mapsto x^i]_{i=0}^{3} < C^\circ(I)$

$$\iota(B_3, B_3) = \begin{pmatrix} 2/1 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 2/5 \\ 2/3 & 0 & 2/5 & 0 \\ 0 & 2/5 & 0 & 2/7 \end{pmatrix} \Rightarrow$$

$a_0 = f_0$

$a_1 = f_1$

$a_2 = f_2 - \frac{1}{3} f_0$

$a_3 = f_3 - \frac{3}{5} f_1$

$$A = (a_0, \dots, a_3) \xmapsto{\frac{a_i}{\|a_i\|}} \left(\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}, \sqrt{\frac{45}{8}}, \sqrt{\frac{175}{8}}\right) \odot A = \hat{A}$$

(b) Geg.: Orthogonalprojektion $p : C^\circ(I) \to U_2 = [f_i]_{i=0}^{2}$

Ges.: $p(\exp|_I) = \cdots$

Ww.: $p : f \mapsto \sum_{i \in I} \frac{a_i \cdot f}{a_i \cdot a_i} \cdot a_i$

$$\cdots = f_2 \cdot \left(\frac{45}{8} \cdot \frac{2e^2 - 14}{3e}\right) + x\left(\frac{3}{e}\right) + \left(\frac{1}{2}\left(e - \frac{1}{e}\right) - \frac{1}{3}\left(\frac{2e^2 - 14}{3e}\right)\right)$$

A 11.5.8 Gg.: $(V, \cdot)$ euklidisch

(a) Gg.: $(a_i)_{i\in I}$ ONS v. V, $J \in \mathcal{E}(I)$

Zz.: $\forall x \in V: \sum\limits_{i\in J} |a_i \cdot x|^2 \leq \|x\|^2$

$$0 \overset{\text{eukl.}}{\leq} \Big(x - \sum_{i\in J}(a_i\cdot x)a_i\Big)^2$$

$$= \|x\|^2 - 2\sum_{i\in J}\underbrace{(a_i\cdot x)a_i\cdot x}_{(a_i\cdot x)^2} + \underbrace{\Big(\sum_{i\in J}(a_i\cdot x)a_i\Big)^2}_{\dots}$$

$$\dots = \sum_{i,j\in J}\underbrace{(a_i\cdot x)a_i\cdot(a_j\cdot x)a_j}_{(a_i\cdot x)(a_j\cdot x)(a_i\cdot a_j)} \overset{\text{ONS}}{=} \sum_{i\in J}(a_i\cdot x)^2 .$$

(b) Gg.: $(b_i)_{i\in I}$ v. V

Zz.: $\forall x \in V: x\cdot y = \sum\limits_{i\in I}(x\cdot b_i)(b_i\cdot y)$

Beh.: $\forall i\in I: \langle b_i^*, x\rangle = b_i\cdot x$

Bew.: Ww.: $\exists!\,(x_j)_{j\in I}\in K^I: x = \sum\limits_{j\in I} x_j b_j$

$$\Rightarrow \langle b_i^*, x\rangle = \langle b_i^*, \sum_{j\in I} x_j b_j\rangle = \sum_{j\in I} x_j \underbrace{\langle b_i^*, b_j\rangle}_{\delta_{ij}} = x_i,$$

$$b_i\cdot x = b_i\cdot\sum_{j\in I}x_j b_j = \sum_{j\in I}x_j\underbrace{(b_i\cdot b_j)}_{\delta_{ij}} = x_i \qquad \square$$

$$\sum_{i\in I} (x\cdot b_i)(y\cdot b_i) = \sum_{i\in I} \langle b_i^*, x\rangle \langle b_i^*, y\rangle = \sum_{i\in I} x_i y_i ,$$

$$x\cdot y = \left(\sum_{i\in I} x_i b_i\right)\cdot\left(\sum_{j\in I} y_j b_j\right) = \sum_{i,j\in I} x_i y_j \underbrace{(b_i\cdot b_j)}_{\delta_{ij}} .$$