Stat. UE 10

(1.) Show that the one-parameter exponential family has a monotone likelihood ratio in a sufficient statistic $T(X)$ if $w(\vartheta)$ is a non-decreasing function in ϑ.

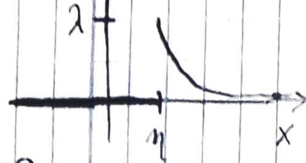
One-parameter exp. family: $f(x|\vartheta) = \overbrace{h(x)}^{\geq 0}\,\overbrace{c(\vartheta)}^{\geq 0}\, e^{w(\vartheta)t(x)}$. Let $\vartheta_1 < \vartheta_2$.

$$\Rightarrow L(\vartheta|x) = \left(\prod_{i=1}^n h(x_i)\right)(c(\vartheta))^n\, e^{w(\vartheta)\sum_{i=1}^n t(x_i)} \Rightarrow \frac{L(\vartheta_1|x)}{L(\vartheta_2|x)} = \underbrace{\left(\frac{c(\vartheta_1)}{c(\vartheta_2)}\right)^n}_{>0} e^{\underbrace{(w(\vartheta_1)-w(\vartheta_2))}_{>0}\sum_{i=1}^n t(x_i)}$$

We choose the statistic $T(X) := \sum_{i=1}^n t(X_i)$; obviously the LR is a non-decr. function of $T(x)$ if $\vartheta_1 < \vartheta_2$. It is also sufficient by the Fisher-Neyman factorization theorem.

(2.) $\alpha = 0{,}1$: $\left[\bar{X} - t_{\alpha/2}(n-1)\cdot\frac{s}{\sqrt{n}},\ \bar{X} + t_{\alpha/2}(n-1)\cdot\frac{s}{\sqrt{n}}\right] \overset{\text{(see R)}}{=} [148, 166]$. ($s$: empirical sd)

(3.) $X_1,\dots,X_n$ iid, $f(x|\lambda,\eta) = \begin{cases} \lambda e^{-\lambda(x-\eta)}, & x > \eta \\ 0, & \text{else} \end{cases}$, $\eta, \lambda > 0$, η known, λ unknown.



- MLE of λ: $\ell(\lambda|x) = \sum_{i=1}^n \log(f(x_i|\lambda)) = \sum_{i=1}^n [\log(\lambda) - \lambda(x_i - \eta)]$

$$= n(\log\lambda + \lambda\eta) - \lambda\sum_{i=1}^n x_i \Rightarrow \frac{\partial}{\partial\lambda}\ell(\lambda|x)\Big|_{\hat{\lambda}} = \frac{n}{\hat{\lambda}} + n\eta - \sum_{i=1}^n x_i \overset{!}{=} 0$$

$$\Longrightarrow n + \hat{\lambda}(n\eta - \sum_{i=1}^n x_i) \overset{!}{=} 0 \Rightarrow \hat{\lambda} = \frac{n}{\sum x_i - n\eta} = \frac{1}{\bar{x} - \eta}.$$

$$\frac{\partial^2}{\partial\lambda^2}\ell(\lambda|x) = -\frac{n}{\lambda^2} < 0 \Rightarrow \hat{\lambda} \text{ is MLE.}$$

- Construct a $(1-\alpha)100\%$ confidence intervall for λ when n is large.

For large n, we know that $\sqrt{n}(\hat{\lambda} - \lambda) \approx \mathcal{N}(0, \frac{1}{I(\lambda)})$.

By the calculation above, $I(\lambda) = -\mathbb{E}_\lambda(\ell''(\lambda|x)) = \frac{n}{\lambda^2}$. → continuous in λ

As λ is unknown, we can estimate $I(\lambda)$ by $I(\hat{\lambda})$.

As $I(\lambda)$ is continuous in λ ($\lambda > 0$!), we have $I(\hat{\vartheta}) \xrightarrow{P} I(\vartheta)$, therefore

$$\frac{n}{\hat{\lambda}}(\hat{\lambda} - \lambda) = \sqrt{\frac{I(\hat{\lambda})}{I(\lambda)}}\sqrt{n I(\lambda)}(\hat{\lambda} - \lambda) \xrightarrow{d} \mathcal{N}(0,1)$$

$$\Rightarrow \mathbb{P}_\lambda\left(-z_{\alpha/2} \leq \frac{n}{\hat{\lambda}}(\hat{\lambda} - \lambda) \leq z_{\alpha/2}\right) = \mathbb{P}_\lambda\left(\hat{\lambda} - z_{\alpha/2}\frac{\hat{\lambda}}{n} \leq \lambda \leq \hat{\lambda} + z_{\alpha/2}\frac{\hat{\lambda}}{n}\right) \longrightarrow 1 - \alpha$$

$\Rightarrow$ the asymptotic confidence interval is given by $\hat{\lambda} \pm z_{\alpha/2}\frac{\hat{\lambda}}{n}$.

(5.) (a), (b) ⇝ see R file

(c) Data: 2,2 3,5 7,6 9,0 13,7 23,7 34,2 44,

Def. $q_p \in \mathbb{R}$ empirical p-quantile $:\Leftrightarrow |(x_i \leq q_p)| \geq p \ \wedge\ |(x_i \geq q_p)| \geq 1-p$.

→ 0,5-quantile: $[9, \ 13{,}7]$

→ 0,25-quantile = 1st quartile: $8 \cdot 0{,}25 = 2 \Rightarrow [3{,}5, \ 7{,}6]$

→ $2/3$-quantile: $8 \cdot 2/3 = 5{,}\dot{3} \Rightarrow \{23{,}7\}$
$8 - 5{,}\dot{3} = 2{,}\dot{6}$

(4.) ⇝ See R file.
We know that the $(1-\alpha)$-CI is given by

$$\hat{\lambda} \pm z_{\alpha/2} \cdot \sqrt{\frac{\hat{\lambda}}{n}}, \quad \text{where } \hat{\lambda} = \overline{X}.$$