Chapter 2

Sobolev Spaces and Poisson Problem

2.1 Sobolev Spaces on Domains

This section briefly recalls the definition of Sobolev spaces $H^m(\Omega)$, for integer order $m \in \mathbb{N}_0$, on domains $\Omega \subseteq \mathbb{R}^d$. While this section requires Ω only to be open and connected, the following sections will implicitly assume that Ω is a bounded Lipschitz domain.

Definition. A function $u \in L^1_{\ell oc}(\Omega) := \big\{ w : \Omega \to \mathbb{R} \text{ measurable} \,\big|\, \forall K \subset \Omega \text{ compact } w \in L^1(K) \big\}$ has a **weak partial derivative** $\partial_j u \in L^1_{\ell oc}(\Omega)$, if the pair $(u, \partial_j u)$ satisfies the integration by parts formula with smooth test functions that vanish on the boundary, i.e., it holds that

$$\int_\Omega u(\partial_j v)\,dx = -\int_\Omega (\partial_j u) v\,dx \quad \text{for all } v \in \mathcal{D}(\Omega) := C_c^\infty(\Omega). \tag{2.1}$$

Note that $\partial_j u$ is (so far) only a symbol, whereas $\partial_j v := \partial v / \partial x_j$ is the classical j-th derivative of $v \in \mathcal{D}(\Omega)$. We say that $u \in L^1_{\ell oc}(\Omega)$ is **weakly differentiable with weak gradient** $\nabla u \in L^1_{\ell oc}(\Omega)$, if all weak derivatives $\partial_j u$, for $j = 1, \ldots, d$, exist. □

The following main theorem of calculus, which will not be proven in this lecture, we infer that the weak derivative is unique, if it exists. Moreover, the weak derivative and the classical derivative coincide, if the classical derivative exists.

Theorem 2.1 (Fundamental Theorem of Calculus of Variations). *Let $f \in L^1_{\ell oc}(\Omega)$ satisfy $\int_\Omega fv\,dx = 0$ for all $v \in \mathcal{D}(\Omega)$. Then, it holds that $f = 0$ almost everywhere in Ω.* ■

Remark. Note that $C(\Omega) \subset L^1_{\ell oc}(\Omega)$. For $f \in C(\Omega)$, the fundamental theorem of calculus of variations can be proven by elementary calculus: Note that for any $x \in \mathbb{R}^d$ and any radius $\varepsilon > 0$, there is a function $\psi \in \mathcal{D}(\mathbb{R}^d)$ such that $\big\{ y \in \mathbb{R}^d \,\big|\, \psi(y) > 0 \big\} = U(x, \varepsilon) := \big\{ y \in \mathbb{R}^d \,\big|\, |x - y| < \varepsilon \big\}$; see the following Exercise 8. Provided $f \in C(\Omega)$ with $f(x) \neq 0$ for some $x \in \Omega$, we may assume $f(x) > 0$. By continuity, there is a small radius $\varepsilon > 0$ such that $U(x, \varepsilon) \subset \Omega$ and that $f(y) > 0$ for all $y \in U(x, \varepsilon)$. With the associated function $\psi \in \mathcal{D}(\Omega)$, we thus see that $\int_\Omega f\psi\,dx > 0$. Note that this argument provides the (logically equivalent) contraposition of the fundamental theorem of calculus of variations in the case of a continuous function f. □

Exercise 8. (i) Show that the following definition provides $\phi \in C^\infty(\mathbb{R})$ with $\operatorname{supp}(\phi) = [-1, 1]$:

$$\phi(t) := \begin{cases} \exp\big(-1/(1-t^2)\big), & \text{for } |t| < 1, \\ 0 & \text{else.} \end{cases}$$

(ii) For $\varepsilon > 0$ and $x \in \mathbb{R}^d$, define the function $\psi_{x,\varepsilon}(y) := \phi(|x-y|^2/\varepsilon)$. Show that $\psi_{x,\varepsilon} \in C^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(\psi_{x,\varepsilon}) = \big\{y \in \mathbb{R}^d \,\big|\, |x-y| \le \varepsilon\big\}$ and $\psi_{x,\varepsilon}(y) > 0$ for all $y \in \big\{y \in \mathbb{R}^d \,\big|\, |x-y| < \varepsilon\big\}$. □

Corollary 2.2. (i) *The weak derivative $\partial_j u$ is unique, if it exists: If $\partial_j u, \widetilde{\partial_j u} \in L^1_{\ell oc}(\Omega)$ satisfy* (2.1)*, it holds that $\partial_j u = \widetilde{\partial_j u}$ almost everywhere in Ω.*
(ii) *A function $u \in C^1(\Omega)$ is weakly differentiable, and the weak derivative coincides with the classical derivative.*

Proof. (i) It holds that $\int_\Omega (\partial_j u - \widetilde{\partial_j u}) v \, dx = 0$ for all $v \in \mathcal{D}(\Omega)$ and thus $\partial_j u - \widetilde{\partial_j u} = 0$ almost everywhere in Ω. (ii) follows from (i) and the integration by parts formula. ■

A deeper result is the following, which is somehow, nevertheless, quite natural and expected.

Theorem 2.3. *If $u \in L^1_{\ell oc}(\Omega)$ is weakly differentiable with $\nabla u = 0$, then the function u is constant, i.e., there is a constant $c \in \mathbb{R}$ such that $u = c$ almost everywhere in Ω.* ■

Definition. For $m = 0$, we define $H^0(\Omega) := L^2(\Omega)$ as the classical Lebesgue space of square integrable functions. For $m = 1$, the **Sobolev space** $H^1(\Omega)$ is defined by

$$H^1(\Omega) := \big\{u \in L^2(\Omega) \,\big|\, u \text{ weakly differentiable, } \nabla u \in L^2(\Omega)\big\} \tag{2.2}$$

and associated with the graph norm

$$\|u\|_{H^1(\Omega)} := \big(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\big)^{1/2}. \tag{2.3}$$

Higher-order Sobolev spaces of integer order $m \in \mathbb{N}$ may be defined inductively by

$$H^m(\Omega) := \big\{u \in L^2(\Omega) \,\big|\, u \text{ weakly differentiable, } \nabla u \in H^{m-1}(\Omega)\big\}, \tag{2.4}$$

with associated norm

$$\|u\|_{H^m(\Omega)} := \big(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{H^{m-1}(\Omega)}\big)^{1/2}. \tag{2.5}$$

Remark. Clearly, $C^1(\overline{\Omega}) \subseteq H^1(\Omega)$ and we note below that $C^1(\overline{\Omega})$ is even dense in $H^1(\overline{\Omega})$. □

Theorem 2.4. *For all $m \in \mathbb{N}_0$, the Sobolev space $H^m(\Omega)$ is a Hilbert space.*

Proof. The proof uses the (hopefully) well-known fact that $H^0(\Omega) = L^2(\Omega)$ is a Hilbert space. We shall proceed by induction on m. However, we explicitely consider the case $m = 1$ first: Obviously, the H^1-norm is induced by the scalar product

$$(u\,;\,v)_{H^1(\Omega)} := (u\,;\,v)_{L^2(\Omega)} + (\nabla u\,;\,\nabla v)_{L^2(\Omega)} \quad \text{for all } u, v \in H^1(\Omega),$$

i.e., $\|u\|^2_{H^1(\Omega)} = (u\,;\,u)_{H^1(\Omega)}$. Therefore, it only remains to prove the completeness of $H^1(\Omega)$. Let (u_n) be a Cauchy sequence in $H^1(\Omega)$. Note that, by definition of the H^1-norm, (u_n) as well as (∇u_n) are Cauchy sequences in $L^2(\Omega)$. Since $L^2(\Omega)$ is complete, there are unique $u \in L^2(\Omega)$ and $g \in L^2(\Omega)^d$ such that

$$\lim_{n\to\infty} \|u - u_n\|_{L^2(\Omega)} = 0 = \lim_{n\to\infty} \|g - \nabla u_n\|_{L^2(\Omega)}.$$

By definition of $H^1(\Omega)$, it thus only remains to prove that u is weakly differentiable with gradient $\nabla u = g$. Let $v \in \mathcal{D}(\Omega)$ be an arbitrary test function. From the weak differentiability of each u_n and L^2-convergence, we obtain that

$$(u\,;\,\partial_j v)_{L^2(\Omega)} = \lim_{n\to\infty} (u_n\,;\,\partial_j v)_{L^2(\Omega)} = -\lim_{n\to\infty} (\partial_j u_n\,;\,v)_{L^2(\Omega)} = -(g_j\,;\,v)_{L^2(\Omega)}.$$

Therefore, g_j is the j-th weak derivative of u and consequently $g = \nabla u$. This concludes the case $m = 1$. The induction step for $H^m(\Omega)$ is left to the reader, but obviously follows from the same arguments, where we replace $g \in L^2(\Omega)^d$ by $g \in H^{m-1}(\Omega)^d$. ■

2.2 Main Theorems on Sobolev Spaces

From now on, it will be important and thus assumed that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. By definition of the Sobolev spaces $H^m(\Omega)$, there holds $H^m(\Omega) \subset H^{m-1}(\Omega)$ with $\|u\|_{H^{m-1}(\Omega)} \le \|u\|_{H^m(\Omega)}$. In other words, the identity operator $id : H^m(\Omega) \to H^{m-1}(\Omega)$ is well-defined and continuous. The following Rellich theorem states that it is also compact. This is a pretty strong result. The impact of which will become clear in our proofs of the Poincaré inequality and the Friedrichs inequality.

Theorem 2.5 (Rellich Compactness Theorem). *For any integer order $m \in \mathbb{N}$, the embedding $H^m(\Omega) \subseteq H^{m-1}(\Omega)$ is compact.* ■

We recall that an operator $A \in L(X;Y)$ between normed spaces X and Y is compact, if and only if each bounded set $S \subseteq X$ is mapped to a pre-compact set $A(S) \subseteq Y$, i.e., $\overline{A(S)} \subseteq Y$ is compact.

Before the statement and the proof of the Poincaré inequality, we need a further technical lemma. The result is rather standard in the analysis of variational problems.

Lemma 2.6. *A continuous and convex functional $f : X \to \mathbb{R}$ on a normed space X is weakly lower semicontinuous, i.e., for each weakly convergent sequence (x_n) in X with $x_n \rightharpoonup x \in X$,* *it holds that*

$$f(x) \le \liminf_{n\in\mathbb{N}} f(x_n). \tag{2.6}$$

Proof. **1. step.** We prove that the epigraph $G := \{(x,\alpha) \in X \times \mathbb{R} \,|\, f(x) \le \alpha\}$ is convex: For $(x,\alpha), (y,\beta) \in G$ and $0 \le \theta \le 1$, the convexity of f proves that

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta)f(y) \le \theta\alpha + (1-\theta)\beta,$$

whence $\theta(x,\alpha) + (1-\theta)(y,\beta) \in G$, i.e., $G \subseteq X \times \mathbb{R}$ is convex.

2. step. We use the continuity of f to prove that G is also closed: Let (x_n,α_n) be a convergent sequence in G, i.e., it holds that $x_n \to x \in X$ and $\alpha_n \to \alpha \in \mathbb{R}$. We prove that $(x,\alpha) \in G$, which follows from

$$f(x) = \lim_{n\to\infty} f(x_n) \le \lim_{n\to\infty} \alpha_n = \alpha.$$

3. step. The following step in the proof is known as *Mazur's lemma*: We prove that the closed and convex set G is also weakly closed in $X \times \mathbb{R} =: Y$, i.e., closed with respect to the weak topology on Y. We argue by contradiction and assume that G is not weakly closed. Then, there is an element $y \in \overline{G}^\sigma \backslash G$, where $\overline{G}^\sigma$ denotes the weak closure of G. According to the Hahn-Banach separation theorem, there is a functional $\phi \in Y^*$ and a scalar $\lambda \in \mathbb{R}$ such that $\phi(y) < \lambda \le \inf \phi(G)$. Therefore $U := \phi^{-1}(-\infty,\lambda)$ is weakly open with $y \in U$ and $U \cap G = \emptyset$. This contradicts topologically that y is in the weak closure of G. Hence, $G = \overline{G}^\sigma$ is weakly closed, and we may proceed with the proof of (2.6).

4. step. We show the weak lower semicontinuity of f: Suppose that $x_n \rightharpoonup x \in X$. For $\alpha := \liminf_n f(x_n) = \infty$, (2.6) is trivial. We thus may assume $\alpha < \infty$. Let $\beta > \alpha$ and define $\alpha_n := \max\{\beta, f(x_n)\} \to \beta$. Clearly, $(x_n,\alpha_n) \in G$. Moreover, this sequence is weakly convergent $(x_n,\alpha_n) \rightharpoonup (x,\beta)$. We deduce $(x,\beta) \in G$. Thus, $f(x) \le \beta$ for all $\beta > \alpha$ and therefore finally $f(x) \le \alpha = \lim_{n\to\infty} f(x_n)$. ∎

A first consequence of the preceding abstract results is that one can easily construct equivalent norms on the Sobolev space $H^1(\Omega)$.

Proposition 2.7. *Let $|\cdot|_{H^1}$ be a continuous seminorm on $H^1(\Omega)$ which is definite on the constant functions, i.e., $|c|_{H^1} = 0$ implies $c = 0$ for all $c \in \mathbb{R}$. Then, there are constants $C_1, C_2 > 0$ such that*

$$|v|_{H^1} \le C_1 \|v\|_{H^1(\Omega)} \quad \textit{as well as} \quad C_2^{-1} \|v\|_{L^2(\Omega)} \le |||v||| := \|\nabla v\|_{L^2(\Omega)} + |v|_{H^1} \textit{ for all } v \in H^1(\Omega).$$

In particular, $|||\cdot|||$ defines an equivalent norm on $H^1(\Omega)$, i.e.,

$$(1 + C_1)^{-1} |||v||| \le \|v\|_{H^1(\Omega)} \le (1 + C_2)\, |||v||| \quad \textit{for all } v \in H^1(\Omega).$$

Proof. **1. step.** Existence of C_1: By definition of continuity, there exists an open neighborhood $O \subseteq H^1(\Omega)$ of 0 such that $|v|_{H^1} \le 1$ for all $v \in O$. Without loss of generality, we may choose a radius $r > 0$ sufficiently small such that $\overline{B_r(0)} \subset O \subset H^1(\Omega)$ for the closed ball with radius r and center zero. This implies

$$|v|_{H^1} = \frac{1}{r}\|v\|_{H^1(\Omega)}|r\frac{v}{\|v\|_{H^1(\Omega)}}|_{H^1} \le \frac{1}{r}\|v\|_{H^1(\Omega)}.$$

This proves existence of $C_1 := 1/r$.

2. step. Existence of C_2: We assume that there is no constant $C_2 > 0$ such that $\|v\|_{L^2(\Omega)} \le C_2 \,|||v|||$ for all $v \in H^1(\Omega)$. Therefore, there exists a sequence (v_n) in $H^1(\Omega)$ such that

$$\frac{1}{n}\,\|v_n\|_{L^2(\Omega)} > |||v_n||| = \|\nabla v_n\|_{L^2(\Omega)} + |v_n|_{H^1}$$

The definition of $w_n := v_n/\|v_n\|_{L^2(\Omega)}$ leads to to a sequence (w_n) in $H^1(\Omega)$ such that

$$\|w_n\|_{L^2(\Omega)} = 1, \quad \|\nabla w_n\|_{L^2(\Omega)} \le 1/n, \quad |w_n|_{H^1} \le 1/n.$$

Therefore, (w_n) is a bounded sequence in the Hilbert space $H^1(\Omega)$. A Hilbert space is reflexive. By virtue of the Banach-Alaoglou theorem, each bounded sequence thus has a weakly convergent subsequence. Therefore, we may assume that $w_n \rightharpoonup w \in H^1(\Omega)$. An application of Lemma 2.6 proves that

$$\|\nabla w\|_{L^2(\Omega)} \le \liminf_{n\to\infty} \|\nabla w_n\|_{L^2(\Omega)} = 0,$$

whence the weak limit w is constant. Another application of Lemma 2.6 proves that

$$|w|_{H^1} \le \liminf_{n\to\infty} |w_n|_{H^1} = 0$$

since a seminorm is always convex. Therefore, $w = 0$. On the other hand, the Rellich theorem states the strong convergence $w_n \to w \in L^2(\Omega)$ and thus $\|w\|_{L^2(\Omega)} = \lim_{n\to\infty} \|w_n\|_{L^2(\Omega)} = 1$. This contradiction concludes the existence of C_2. In particular, we hence observe $\|v\|_{H^1(\Omega)} \le \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)} \le (C_2 + 1)\,|||v|||$. ■

Corollary 2.8 (Poincaré Inequality). *It holds that*

$$\|v\|_{L^2(\Omega)} \le \widetilde{C}_P \left(\|\nabla v\|_{L^2(\Omega)} + \Big| \int_\Omega v\,dx \Big| \right) \quad \textit{for all } v \in H^1(\Omega), \tag{2.7}$$

where the constant $\widetilde{C}_P > 0$ depends only on Ω. Moreover, $|||v||| := \|\nabla v\|_{L^2(\Omega)} + \big|\int_\Omega v\,dx\big|$ defines even an equivalent norm on $H^1(\Omega)$.

Proof. According to Proposition 2.7, it only remains to show that

$$|v|_{H^1} := \Big| \int_\Omega v\,dx \Big| \quad \text{for } v \in H^1(\Omega)$$

defines a continuous seminorm on $H^1(\Omega)$ which is definite on the constant functions. The equality $|c|_{H^1} = |\Omega||c|$ for $c \in \mathbb{R}$ verifies the definiteness. Lipschitz continuity follows from

$$\big||v|_{H^1} - |w|_{H^1}\big| \le \Big| \int_\Omega v - w\,dx \Big| \le |\Omega|^{1/2}\|v - w\|_{L^2(\Omega)} \le |\Omega|^{1/2}\|v - w\|_{H^1(\Omega)}$$

and from the boundedness of Ω. ■

Corollary 2.9 (Poincaré Inequality). *There is a constant $C_P > 0$, which depends only on the shape of Ω but not on its diameter, such that*

$$\|v\|_{L^2(\Omega)} \leq C_P \operatorname{diam}(\Omega)\, \|\nabla v\|_{L^2(\Omega)} \quad \text{for all } v \in H^1_*(\Omega) := \big\{ w \in H^1(\Omega) \,\big|\, \textstyle\int_\Omega w\, dx = 0 \big\}, \tag{2.8}$$

where $\operatorname{diam}(\Omega) := \sup \big\{ |x-y| \,\big|\, x, y \in \Omega \big\}$ *denotes the diameter of* Ω.

Proof. The proof is a so-called **scaling argument**: We define $\lambda := \operatorname{diam}(\Omega)$ and $\widetilde{\Omega} := \lambda^{-1}\Omega$. Note that the scaled domain $\widetilde{\Omega}$ satisfies $\operatorname{diam}(\widetilde{\Omega}) = 1$ and depends only on the shape of Ω. We consider the affine bijection $\Phi : \Omega \to \widetilde{\Omega}$, $\Phi(x) := \lambda^{-1}x$. Recall the transformation theorem, which holds for arbitrary diffeomorphisms $\Phi : \Omega \to \widetilde{\Omega}$ and states that

$$\int_{\widetilde{\Omega}} \widetilde{f}\, dy = \int_\Omega \widetilde{f}(\Phi(x))\, |\det D\Phi(x)|\, dx \quad \text{for all } \widetilde{f} \in L^1(\widetilde{\Omega}).$$

Note that $\det D\Phi(x) = \lambda^{-d}$ since $D\Phi = \lambda^{-1}\boldsymbol{I}$ in our case. For $v \in H^1(\Omega)$, we define $\widetilde{v} := v \circ \Phi^{-1} \in H^1(\widetilde{\Omega})$. Then,

$$\|\widetilde{v}\|^2_{L^2(\widetilde{\Omega})} = \int_{\widetilde{\Omega}} |\widetilde{v}|^2\, dy = \lambda^{-d} \int_\Omega |v|^2\, dx = \lambda^{-d} \|v\|^2_{L^2(\Omega)}.$$

According to the chain rule, it holds that $\nabla\widetilde{v} = \lambda\, (\nabla v) \circ \Phi^{-1}$ and consequently that

$$\|\nabla\widetilde{v}\|^2_{L^2(\widetilde{\Omega})} = \lambda^{2-d} \|\nabla v\|^2_{L^2(\Omega)}.$$

With $\widetilde{C}_P > 0$ the Poincaré constant from (2.7) for $\widetilde{\Omega}$, we thus infer

$$\|v\|^2_{L^2(\Omega)} = \lambda^d\, \|\widetilde{v}\|^2_{L^2(\widetilde{\Omega})} \leq \lambda^d\, \widetilde{C}_P^2\, \|\nabla\widetilde{v}\|^2_{L^2(\widetilde{\Omega})} = \lambda^2\, \widetilde{C}_P^2\, \|\nabla v\|^2_{L^2(\Omega)}.$$

Note that $\widetilde{C}_P$ depends only on $\widetilde{\Omega}$ und thus only on the shape of Ω. This concludes the proof. ∎

Remark. We stress that $Iv := \int_\Omega v\, dx$ defines a linear and continuous functional on $H^1(\Omega)$. In particular, $H^1_*(\Omega) = \ker(I)$ is a closed subspace of $H^1(\Omega)$ and hence a Hilbert space. According to the Poincaré inequality, it holds that $\|\nabla v\|_{L^2(\Omega)} \leq \|v\|_{H^1(\Omega)} \leq (1 + \widetilde{C}_P^2)^{1/2} \|\nabla v\|_{L^2(\Omega)}$ for all $v \in H^1_*(\Omega)$. In particular, $\|\nabla v\|_{L^2(\Omega)}$ defines an equivalent Hilbert norm on $H^1_*(\Omega)$ with associated scalar product $(\nabla u\,;\, \nabla v)_{L^2(\Omega)}$. □

Theorem 2.10 (Meyers-Serrin). *For each integer order $m \in \mathbb{N}$, $C^\infty(\overline{\Omega})$ and, in particular, $C^\infty(\Omega) \cap H^m(\Omega)$ are dense subspaces of $H^m(\Omega)$.* ∎

Theorem 2.11 (Trace Operator). *There is a unique operator $\gamma \in L\big(H^1(\Omega); L^2(\Gamma)\big)$ such that $\gamma v = v|_\Gamma$ for all $v \in C^1(\overline{\Omega})$, i.e., γ extends the classical trace defined as restriction $v|_\Gamma$ on the boundary for smooth functions v.* ∎

As a first corollary to Theorem 2.11, we can prove that the integration by parts formula also holds for Sobolev functions $u, v \in H^1(\Omega)$.

Corollary 2.12 (Integration by Parts). *For all $u, v \in H^1(\Omega)$, it holds that*

$$\int_\Omega u \, \frac{\partial v}{\partial x_j} \, dx + \int_\Omega \frac{\partial u}{\partial x_j} \, v \, dx = \int_\Gamma \gamma u \, \gamma v \, n_j \, ds. \tag{2.9}$$

Proof. The formula (2.9) holds for $u, v \in C^1(\overline{\Omega})$. All three terms define continuous bilinear forms on $H^1(\Omega) \times H^1(\Omega)$. Therefore (2.9) follows, for arbitrary $u, v \in H^1(\Omega)$ from the density of $C^1(\overline{\Omega})$ in $H^1(\Omega)$: Given $u, v \in H^1(\Omega)$, there are sequences (u_n) and (v_n) in $C^1(\overline{\Omega})$ which converge to u resp. v in $H^1(\Omega)$. Therefore, if $a(\cdot, \cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is continuous, then it holds that $\lim\limits_{n\to\infty} a(u_n, v_n) = a(u, v)$. This concludes the proof. ■

The analytical treatment of the Dirichlet problem makes use of the so-called Friedrichs inequality, whereas the analytical treatment of the Neumann problem uses the previously proven Poincaré inequality.

Corollary 2.13 (Friedrichs Inequality). *Assume that the Dirichlet boundary $\Gamma_D \subseteq \Gamma$ has positive surface measure $|\Gamma_D| > 0$. Then, it holds that*

$$\|v\|_{L^2(\Omega)} \le \widetilde{C}_F \left(\|\nabla v\|_{L^2(\Omega)} + \|\gamma v\|_{L^2(\Gamma_D)}\right) \quad \textit{for all } v \in H^1(\Omega) \tag{2.10}$$

with a constant $\widetilde{C}_F > 0$, which depends only on Ω and Γ_D. Moreover, the right-hand side $|||v||| := \|\nabla v\|_{L^2(\Omega)} + \|\gamma v\|_{L^2(\Gamma_D)}$ even defines an equivalent norm on $H^1(\Omega)$.

Proof. We again apply Proposition 2.7. It only remains to show that

$$|v|_{H^1} := \|\gamma v\|_{L^2(\Gamma_D)} \quad \text{for } v \in H^1(\Omega)$$

defines a continuous seminorm on $H^1(\Omega)$ which is definite on the constant functions. The definiteness is again easily obtained from $|c|_{H^1} = |\Gamma_D|^{1/2}|c|$ for $c \in \mathbb{R}$. Lipschitz continuity follows from

$$\big||v|_{H^1} - |w|_{H^1}\big| \le \|\gamma v - \gamma w\|_{L^2(\Gamma_D)} = \|\gamma(v - w)\|_{L^2(\Gamma_D)} \le C \, \|v - w\|_{H^1(\Omega)}$$

according to the continuity of the trace operator $\gamma \in L(H^1(\Omega); L^2(\Gamma))$. ■

Definition. We define $H^1_0(\Omega) := \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{H^1}}$ and $H^1_D(\Omega) := \overline{C^1_D(\overline{\Omega})}^{\|\cdot\|_{H^1}}$, where the subscript D indicates the Dirichlet boundary Γ_D. By definition, $H^1_0(\Omega)$ as well as $H^1_D(\Omega)$ are closed subspaces of $H^1(\Omega)$ and thus Hilbert spaces. In particular, it holds that $H^1_0(\Omega) \subseteq H^1_D(\Omega)$. □

The same scaling argument as for the Poincaré inequality proves the following variant of the Friedrichs inequality, where we note that continuity of the trace operator γ proves that $\gamma v = 0$, for $v \in H^1_0(\Omega)$, as well as $(\gamma v)|_{\Gamma_D} = 0$, for $v \in H^1_D(\Omega)$.

Corollary 2.14 (Friedrichs Inequality). *It holds that*

$$\|v\|_{L^2(\Omega)} \le C_F \, \mathrm{diam}(\Omega) \, \|\nabla v\|_{L^2(\Omega)} \quad \textit{for all } v \in H^1_D(\Omega) \tag{2.11}$$

with a constant $C_F > 0$ that depends only on the shape of Ω and Γ_D. ■

We finally note the relation between $H^1_D(\Gamma)$ and the trace operator, cf. the Theorem of Meyers-Serrin.

Theorem 2.15. *There holds $H^1_0(\Omega) = \ker(\gamma)$ with $\gamma \in L(H^1(\Omega); L^2(\Gamma))$ the trace operator. Moreover, $H^1_D(\Omega) = \{v \in H^1(\Omega) \,|\, (\gamma v)|_{\Gamma_D} = 0\}$.* ■

Exercise 9. Usually, one defines the range of the trace operator as $H^{1/2}(\Gamma) := \mathrm{range}(\gamma) \subseteq L^2(\Gamma)$. This space is associated with the norm $\|v\|_{H^{1/2}(\Gamma)} := \inf\{\|\widehat{v}\|_{H^1(\Omega)} \,|\, \widehat{v} \in H^1(\Omega) \text{ with } \gamma\widehat{v} = v\}$. Prove that $H^{1/2}(\Gamma)$ associated with this norm is a Hilbert space with continuous inclusion $H^{1/2}(\Gamma) \subseteq L^2(\Gamma)$. **Hint:** Recall the definition and the standard results on quotient spaces and the associated quotient norm! □

For $X = H^1(\Omega)$ and $Y = L^2(\Omega)$, the following exercise shows that the L^2-scalar products $(f\,;\,\cdot)_{L^2(\Omega)}$ for $f \in L^2(\Omega)$ give (up to density) all linear and continuous functionals on $H^1(\Omega)$, i.e., the embedding $L^2(\Omega) \to H^1(\Omega)^*, f \mapsto (f\,;\,\cdot)_{L^2(\Omega)}$ is well-defined, linear, continuous, and injective with dense image.

Exercise 10. Let X and Y be Hilbert spaces with continuous embedding $X \subseteq Y$. Show that the mapping $I : Y^* \to X^*, Iy^* := y^*|_X$ is well-defined, linear, and continuous. Prove that $I(Y^*) \subseteq X^*$ is a dense subspace. Moreover, if $X \subseteq Y$ is dense with respect to $\|\cdot\|_Y$, then the embedding I is even injective. □

2.3 Weak Form of Laplace Problem

2.3.1 Dirichlet Problem

In this section, we generalize the variational form derived in the introductory section to our Hilbert space setting. We start with the homogeneous Dirichlet problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{2.12}$$

Recall that this formulation is called the **strong form** of the boundary value problem. The following proposition provides the — in some sense — equivalent and always uniquely solvable weak form of the boundary value problem.

Proposition 2.16. (i) *Provided that $u \in C^2(\overline{\Omega})$ solves (2.12) for a given source term $f \in C(\overline{\Omega})$, it holds that $u \in H^1_0(\Omega)$ as well as*

$$(\nabla u\,;\,\nabla v)_{L^2(\Omega)} = (f\,;\,v)_{L^2(\Omega)} \quad \textit{for all } v \in H^1_0(\Omega). \tag{2.13}$$

(ii) *Given $f \in L^2(\Omega)$, the* ***weak form*** (2.13) *has a unique solution $u \in H^1_0(\Omega)$. It holds that*

$$\|u\|_{H^1(\Omega)} \le C \sup_{v \in H^1_0(\Omega)\setminus\{0\}} \frac{(f\,;\,v)_{L^2(\Omega)}}{\|v\|_{H^1(\Omega)}} \le C\,\|f\|_{L^2(\Omega)}, \tag{2.14}$$

where the constant $C > 0$ depends only on Ω.

(iii) *Provided that $f \in C(\overline{\Omega})$ and that the weak solution $u \in H^1_0(\Omega)$ of* (2.13) *additionally satisfies $u \in C^2(\overline{\Omega})$, then u even solves the strong form* (2.12).

Proof. (i) We have already seen before that a strong solution $u \in C^2(\overline{\Omega})$ solves the variational form (2.13) for test functions $v \in C^1_0(\overline{\Omega}) := \{w \in C^1(\overline{\Omega}) \,|\, w|_\Gamma = 0\}$ replacing $H^1_0(\Omega)$; see Proposition 1.1. If we keep u fixed, the left-hand side as well as the right-hand side of (2.13) define continuous and linear functionals on $H^1(\Omega)$. Note that the closure of $C^1_0(\overline{\Omega})$ with respect to the H^1-norm leads to the Hilbert space $H^1_0(\Omega)$. Therefore, standard density arguments prove (2.13).

(ii) According to the Friedrichs inequality, it holds that

$$\|\nabla v\|^2_{L^2(\Omega)} \le \|v\|^2_{H^1(\Omega)} \le (1 + \widetilde{C}_F^2)\,\|\nabla v\|^2_{L^2(\Omega)} \quad \text{for all } v \in H^1_0(\Omega).$$

Therefore, the left-hand side of (2.13) defines an equivalent scalar product on $H^1_0(\Omega)$. The Riesz theorem thus provides a unique weak solution $u \in H^1_0(\Omega)$ of (2.13). Plugging-in $u = v \in H^1_0(\Omega)$, the weak form yields that

$$(1 + \widetilde{C}_F^2)^{-1}\|u\|^2_{H^1(\Omega)} \le \|\nabla u\|^2_{L^2(\Omega)} = (f\,;\,u)_{L^2(\Omega)} \le \sup_{v \in H^1_0(\Omega)\setminus\{0\}} \frac{(f\,;\,v)_{L^2(\Omega)}}{\|v\|_{H^1(\Omega)}}\,\|u\|_{H^1(\Omega)}$$

which results in the first estimate of (2.14). The second estimate follows from the Cauchy inequality

$$(f\,;\,v)_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}.$$

(iii) Since the weak solution u is smooth, we may use integration by parts to see that

$$(\nabla u\,;\,\nabla v)_{L^2(\Omega)} = (-\Delta u\,;\,v)_{L^2(\Omega)} \quad \text{for all } v \in H^1_0(\Omega).$$

The difference with the weak form (2.13) thus yields that

$$0 = (f + \Delta u\,;\,v)_{L^2(\Omega)} \quad \text{for all } v \in H^1_0(\Omega).$$

Note that $F := f + \Delta u \in C(\overline{\Omega})$. With $\mathcal{D}(\Omega) \subseteq H^1_0(\Omega)$, Theorem 2.1 proves $F = 0$; see also the remark right after Theorem 2.1. Consequently, it holds that $-\Delta u = f$ in Ω. The Dirichlet boundary conditions (in the strong form) follow from $0 = \gamma u = u|_\Gamma$. Altogether, u solves (2.12) ■

2.3.2 Mixed Boundary Value Problem

Second, we consider the mixed boundary value problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ \partial u/\partial n &= \phi \quad \text{on } \Gamma_N, \end{aligned} \tag{2.15}$$

with $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, $\Gamma_D \cap \Gamma_N = \emptyset$, and $|\Gamma_D| > 0$. The limit case $|\Gamma_D| = 0$ corresponds to the Neumann problem which is treated in Section 2.3.3. Recall the trace norm $\|\cdot\|_{H^{1/2}(\Gamma)}$ from Exercise 9. Then, the main proposition reads as follows:

Proposition 2.17. (i) *Suppose that Γ_N is smooth, i.e., the outer normal vector depends continuously on $x \in \Gamma_N$. Provided that $u \in C^2(\overline{\Omega})$ solves the* ***strong form*** (2.15) *for a given source term $f \in C(\overline{\Omega})$ and Neumann data $\phi \in C(\overline{\Gamma}_N)$, it holds that $u \in H^1_D(\Omega)$ as well as*

$$(\nabla u\,;\,\nabla v)_{L^2(\Omega)} = (f\,;\,v)_{L^2(\Omega)} + (\phi\,;\,\gamma v)_{L^2(\Gamma_N)} \quad \textit{for all } v \in H^1_D(\Omega). \tag{2.16}$$

(ii) *Given $f \in L^2(\Omega)$ and $\phi \in L^2(\Gamma_N)$, the* ***weak form*** (2.16) *has a unique solution $u \in H^1_D(\Omega)$. It holds that*

$$\begin{aligned}\|u\|_{H^1(\Omega)} &\le C_1 \Bigg(\sup_{v \in H^1_D(\Omega)\setminus\{0\}} \frac{(f\,;\,v)_{L^2(\Omega)}}{\|v\|_{H^1(\Omega)}} + \sup_{w \in H^{1/2}(\Gamma)\setminus\{0\}} \frac{(\phi\,;\,w)_{L^2(\Gamma_N)}}{\|w\|_{H^{1/2}(\Gamma)}} \Bigg) \\ &\le C_2 \left(\|f\|_{L^2(\Omega)} + \|\phi\|_{L^2(\Gamma_N)} \right)\end{aligned} \tag{2.17}$$

where the constants $C_1, C_2 > 0$ depend only on Ω and Γ_D.

(iii) *Provided that $f \in C(\overline{\Omega})$ and $\phi \in C(\overline{\Gamma}_N)$ and that the weak solution $u \in H^1_D(\Omega)$ of* (2.16) *additionally satisfies $u \in C^2(\overline{\Omega})$, then u even solves the strong form* (2.15).

Proof is done in the exercises. ■

2.3.3 Neumann Problem

Finally, we consider the Neumann problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \partial u/\partial n &= \phi \quad \text{on } \Gamma. \end{aligned} \tag{2.18}$$

Note that the solution u of (2.18) cannot be unique: If $u \in C^2(\overline{\Omega})$ solves the **strong form** (2.18), also $u + c$ solves (2.18), for all $c \in \mathbb{R}$. To fix the additive constant, we seek a solution which additionally satisfies, e.g., that

$$\int_\Omega u\,dx = 0. \tag{2.19}$$

Moreover, the Gauss divergence theorem shows

$$-\int_\Omega f\,dx = \int_\Omega \Delta u\,dx = \int_\Omega \operatorname{div}(\nabla u)\,dx = \int_\Gamma \frac{\partial u}{\partial n}\,ds = \int_\Gamma \phi\,ds.$$

Therefore, the data f and ϕ have to satisfy the compatibility condition

$$\int_\Omega f\,dx + \int_\Gamma \phi\,ds = 0 \tag{2.20}$$

to allow for the existence of (strong) solutions. Recall the trace norm $\|\cdot\|_{H^{1/2}(\Gamma)}$ from Exercise 9.

Proposition 2.18. (i) *Suppose that Γ is smooth, i.e., the outer normal vector depends continuously on $x \in \Gamma$. Provided that $u \in C^2(\overline{\Omega})$ solves (2.18) for a given source term $f \in C(\overline{\Omega})$ and Neumann data $\phi \in C(\Gamma)$, it holds that $u \in H^1(\Omega)$ and*

$$(\nabla u\,;\,\nabla v)_{L^2(\Omega)} = (f\,;\,v)_{L^2(\Omega)} + (\phi\,;\,\gamma v)_{L^2(\Gamma)} \quad \text{for all } v \in H^1(\Omega). \tag{2.21}$$

(ii) *Given $f \in L^2(\Omega)$ and $\phi \in L^2(\Gamma)$, the variational formulation*

$$(\nabla u\,;\,\nabla v)_{L^2(\Omega)} = (f\,;\,v)_{L^2(\Omega)} + (\phi\,;\,\gamma v)_{L^2(\Gamma)} \quad \text{for all } v \in H^1_*(\Omega) \tag{2.22}$$

has a unique solution $u \in H^1_*(\Omega) := \big\{ v \in H^1(\Omega) \,\big|\, \int_\Omega v\,dx = 0 \big\}$.

(iii) *Provided that the data $f \in L^2(\Omega)$ and $\phi \in L^2(\Gamma)$ satisfy (2.20), the unique solution $u \in H^1_*(\Omega)$ of (2.22) even solves the **weak form** (2.21). Moreover, it holds that*

$$\begin{aligned} \|u\|_{H^1(\Omega)} &\le C_1 \left(\sup_{v \in H^1(\Omega)\setminus\{0\}} \frac{(f\,;\,v)_{L^2(\Omega)}}{\|v\|_{H^1(\Omega)}} + \sup_{w \in H^{1/2}(\Gamma)\setminus\{0\}} \frac{(\phi\,;\,w)_{L^2(\Gamma)}}{\|w\|_{H^{1/2}(\Gamma)}} \right) \\ &\le C_2 \big(\|f\|_{L^2(\Omega)} + \|\phi\|_{L^2(\Gamma)} \big) \end{aligned} \tag{2.23}$$

where the constants $C_1, C_2 > 0$ depend only on Ω.

(iv) *Provided that $f \in C(\overline{\Omega})$ and $\phi \in C(\Gamma)$ satisfy (2.20) and that the weak solution $u \in H^1_*(\Omega)$ of (2.21) resp. (2.22) additionally satisfies $u \in C^2(\overline{\Omega})$, then u even solves the strong form (2.18).*

Proof. (i) The variational form (2.21) holds for test functions $v \in C^1(\overline{\Omega})$ according to integration by parts. For fixed u, the left-hand as well as the right-hand side define continuous linear functionals on $H^1(\Omega)$. Thus, (2.21) follows for $v \in H^1(\Omega)$ by density arguments. (ii) According to the Poincaré inequality, it holds that

$$\|\nabla v\|^2_{L^2(\Omega)} \le \|v\|^2_{H^1(\Omega)} \le (1 + \widetilde{C}_P^2)\|\nabla v\|^2_{L^2(\Omega)} \quad \text{for all } v \in H^1_*(\Omega).$$

Therefore, the left-hand side of (2.22) defines an equivalent scalar product on $H^1_*(\Omega)$. Note that $H^1_*(\Omega)$ is a closed subspace of $H^1(\Omega)$ and hence a Hilbert space. Therefore, (2.22) follows from the Riesz theorem. (iii) For a function $v \in H^1(\Omega)$, we define $\widetilde{v} := v - v_\Omega \in H^1_*(\Omega)$, where $v_\Omega \in \mathbb{R}$ denotes the integral mean $v_\Omega := (1/|\Omega|) \int_\Omega v\,dx \in \mathbb{R}$. Note that (2.20) implies that

$$(f\,;\,v_\Omega)_{L^2(\Omega)} + (\phi\,;\,v_\Omega)_{L^2(\Gamma)} = 0.$$

Thus, (2.22) proves that

$$(\nabla u\,;\,\nabla v)_{L^2(\Omega)} = (\nabla u\,;\,\nabla \widetilde{v})_{L^2(\Omega)} = (f\,;\,\widetilde{v})_{L^2(\Omega)} + (\phi\,;\,\gamma \widetilde{v})_{L^2(\Gamma)} = (f\,;\,v)_{L^2(\Omega)} + (\phi\,;\,\gamma v)_{L^2(\Gamma)},$$

i.e., u even solves (2.21). Plugging-in $u = v$, we see that

$$\|\nabla u\|^2_{L^2(\Omega)} \le \sup_{v \in H^1(\Omega)\setminus\{0\}} \frac{(f\,;\,v)_{L^2(\Omega)}}{\|v\|_{H^1(\Omega)}} \|u\|_{H^1(\Omega)} + \sup_{w \in H^{1/2}(\Gamma)\setminus\{0\}} \frac{(\phi\,;\,w)_{L^2(\Gamma)}}{\|w\|_{H^{1/2}(\Gamma)}} \|\gamma u\|_{H^{1/2}(\Gamma)},$$

where we have used that $H^{1/2}(\Gamma) = \mathrm{range}(\gamma)$. Note that the $H^{1/2}$-norm is defined in such a way that $\gamma \in L(H^1(\Omega); H^{1/2}(\Gamma))$ with $\|\gamma u\|_{H^{1/2}(\Gamma)} \le \|u\|_{H^1(\Omega)}$. Therefore,

$$\|\nabla u\|^2_{L^2(\Omega)} \le \|u\|_{H^1(\Omega)} \Big(\sup_{v \in H^1(\Omega)\setminus\{0\}} \frac{(f\,;\,v)_{L^2(\Omega)}}{\|v\|_{H^1(\Omega)}} + \sup_{w \in H^{1/2}(\Gamma)\setminus\{0\}} \frac{(\phi\,;\,w)_{L^2(\Gamma)}}{\|w\|_{H^{1/2}(\Gamma)}} \Big).$$

Together with $(1+\widetilde{C}_P^2)^{-1}\|u\|^2_{H^1(\Omega)} \le \|\nabla u\|^2_{L^2(\Omega)}$, this proves the first estimate in (2.23). As above, the first supremum may be estimated by $\|f\|_{L^2(\Omega)}$. With the continuous embedding $H^{1/2}(\Gamma) \subset L^2(\Gamma)$, the numerator of the second supremum can be dominated by

$$(\phi\,;\,w)_{L^2(\Gamma)} \le \|\phi\|_{L^2(\Gamma)}\|w\|_{L^2(\Gamma)} \le \widetilde{C}\,\|\phi\|_{L^2(\Gamma)}\|w\|_{H^{1/2}(\Gamma)}.$$

This provides the upper bound $\widetilde{C}\,\|\phi\|_{L^2(\Gamma)}$ for the second supremum. (iv) As above, we may use integration by parts to see that

$$(f + \Delta u\,;\,v)_{L^2(\Omega)} + (\phi - \partial u/\partial n\,;\,\gamma v)_{L^2(\Gamma)} = 0 \quad \text{for all } v \in H^1(\Omega).$$

From this, we first conclude $f = -\Delta u$ by use of Theorem 2.1 for test functions $v \in \mathcal{D}(\Omega) \subset H^1_0(\Omega) \subset H^1(\Omega)$. To prove $\phi = \partial u/\partial n$, one proceeds analogously to the remark right after Theorem 2.1. ■