1. Zeigen Sie, dass

$$u(x,y) = \ln\left(\ln \frac{1}{\sqrt{x^2+y^2}}\right) \in H^1(B_{1/2}(0))\,.$$

$$\partial_x u(x,y) = -\frac{x}{(x^2+y^2)\ln\left(\frac{1}{\sqrt{x^2+y^2}}\right)}$$ ← klassisch abgeleitet

$$\left(\left(\ln\left(\tfrac{1}{r}\right)\right)^{-1}\right)' = \left(\ln\left(\tfrac{1}{r}\right)\right)^{-2} r \frac{1}{r^2} = \frac{1}{r\left(\ln\left(\frac{1}{r}\right)\right)^2}$$

$$\int_\Omega |\nabla u|^2 \, d\lambda^2 = \int_\Omega \frac{x^2+y^2}{(x^2+y^2)^2\left(\ln\left((x^2+y^2)^{-\frac{1}{2}}\right)\right)^2} \, d\lambda^2(x,y) = \int_0^{2\pi}\int_0^{\frac{1}{2}} \frac{1}{r\left(\ln\left(\frac{1}{r}\right)\right)^2} \, dr\, d\varphi = 2\pi \int_0^{\frac{1}{2}} \frac{1}{r\left(\ln\left(\frac{1}{r}\right)\right)^2} \, dr = 2\pi\left(\frac{1}{\ln(2)} - \lim_{\varepsilon\to 0}\frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}\right) = \frac{2\pi}{\ln(2)} < \infty$$

Stimmt die klassische mit der distributionellen Ableitung überein ?

$$\partial_x u(r,\varphi) = -\frac{r\cos(\varphi)}{r^2\ln\left(\frac{1}{r}\right)} = -\frac{\cos(\varphi)}{r\ln\left(\frac{1}{r}\right)}$$

$$\lim_{r\to 0} r\ln\left(\tfrac{1}{r}\right) = \lim_{r\to 0}\frac{\ln\left(\frac{1}{r}\right)}{\frac{1}{r}} = \lim_{r\to 0}\frac{r\frac{1}{r^2}}{\frac{1}{r^2}} = \lim_{r\to 0} r = 0$$

$$\int_\Omega \partial_x u(x,y)\,\phi(x,y)\,d\lambda^2(x,y) = -\int_0^{2\pi}\int_0^{\frac{1}{2}} \frac{\cos(\varphi)}{r\ln\left(\frac{1}{r}\right)}\,\phi(r\cos(\varphi), r\sin(\varphi))\, r\, dr\, d\varphi = -\int_0^{\frac{1}{2}} \frac{1}{\ln\left(\frac{1}{r}\right)}\left(\left[\sin(\varphi)\,\phi(r\cos(\varphi), r\sin(\varphi))\right]_{\varphi=0}^{2\pi} - \int_0^{\frac{1}{2}}\right.$$

$$\left(\ln\left(\ln\left(\tfrac{1}{r}\right)\right)\right)' = -\frac{1}{\ln\left(\frac{1}{r}\right)}\frac{1}{r^2}\, r = \frac{1}{r\ln\left(\frac{1}{r}\right)}$$