1, 2. $\mathcal{L} = -\frac{1}{4} F^{\lambda\nu} F_{\lambda\nu} - J^{\lambda} A_{\lambda}$

Recordemos: $T^{\mu}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} A_{\lambda})} (\partial_{\nu} A_{\lambda}) - \delta_{\nu}^{\mu} \mathcal{L}$

$$F^{\lambda\nu} = \partial^{\lambda} A^{\nu} - \partial^{\nu} A^{\lambda}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_{\mu} A_{\lambda})} = -F^{\mu\lambda} \therefore T^{\mu}_{\nu} = -F^{\mu\lambda}(\partial_{\nu} A_{\lambda}) + \frac{1}{4} F^{\lambda\nu} F_{\lambda\nu} + J^{\lambda} A_{\lambda}$$

$$T^{\mu}_{\nu} = -F^{\mu\lambda}(\partial_{\nu} A_{\lambda}) + \frac{1}{4} F^{\lambda 0} F_{\lambda 0} + \frac{1}{4} F^{\lambda i} F_{\lambda i} + J^{\lambda} A_{\lambda}$$

Ahora: $F_{\nu\lambda} = \partial_{\nu} A_{\lambda} - \partial_{\lambda} A_{\nu} \Rightarrow \partial_{\nu} A_{\lambda} = F_{\nu\lambda} + \partial_{\lambda} A_{\nu}$

$$T^{\mu}_{\nu} = -F^{\mu\lambda}(F_{\nu\lambda} + \partial_{\lambda} A_{\nu}) + \frac{1}{4} F^{\lambda 0} F_{\lambda 0} + \frac{1}{4} F^{\lambda i} F_{\lambda i} + J^{\lambda} A_{\lambda}$$

$$T^{\mu}_{\nu} = -F^{\mu\lambda} \partial_{\lambda} A_{\nu} - F^{\mu\lambda} F_{\nu\lambda} + \frac{1}{4} F^{\lambda 0} F_{\lambda 0} + \frac{1}{4} F^{\lambda i} F_{\lambda i} - J^{\lambda} A_{\lambda}$$

Para $\nu = 0$

$$T^{\mu}_{0} = -F^{\mu\lambda} \partial_{\lambda} A_{0} - F^{\lambda 0} F_{\lambda 0} + \frac{1}{4} F^{\lambda 0} F_{\lambda 0} + \frac{1}{4} F^{\lambda i} F_{\lambda i} - J^{\lambda} A_{\lambda}$$

$\therefore$

$$T^{\mu}_{0} = -F^{\mu\lambda} \partial_{\lambda} A_{0} - F^{\lambda 0} F_{\lambda 0} + \frac{1}{4} F^{\lambda 0} F_{\lambda 0} + \frac{1}{4} F^{\lambda i} F_{\lambda i} - J^{\lambda} A_{\lambda}$$

Note que Λ puede ser 0 o un j (i).:

$$T^{\mu}_{0} = -F^{0i}\partial_i A_0 - F^{i0}F_{i0} + \frac{1}{4}F^{i0}F_{i0} + \underbrace{\frac{1}{4}F^{i0}F_{i0} + \frac{1}{4}F^{ji}F_{ij}}_{\frac{1}{4}F^{\mu i}}$$
$$+ J^0 A_0 - \overline{J}\cdot\overline{A}$$

Entonces, note lo siguiente

$$-F^{0i}\partial_i A_0 + J^0 A_0 - \overline{J}\cdot\overline{A} = -\partial_i(A_0 F^{0i}) - A_0\partial_i F^{0i} + J^0 A_0 - \overline{J}\cdot\overline{A}$$

$\therefore$ Como $\partial_i F^{0i} = J^0$, con esto: tenemos:

$$-F^{0i}\partial_i A_0 + J^0 A^0 = -\partial_i(A_0 F^{0i}) - \overline{J}\cdot\overline{A}$$

Com esto, lemos:

$$T^{\mu}_{0} = -\partial_i(A_0 F^{0i}) - \overline{J}\cdot\overline{A} - \frac{1}{2}F^{i0}F_{i0} + \frac{1}{4}F^{ji}F_{ij}$$

Note que $F^{0i} = E^{i}$ -> lo mostramos en clase

y $F^{ml} = \epsilon_{lmk} B^{k}$ -> lembrar lo mostramos en clase.

$$T_0^{\,r} = -\partial_i (A_0 F^{0i}) + \frac{1}{2} F^{i0} F^{i0} + \frac{1}{4} F^{ji} F^{ji} - \overline{J} \cdot \overline{A}$$

$$T_0^{\,r} = \frac{1}{2} E^i E^i + \frac{1}{4} \epsilon_{ijk} B^k \epsilon_{ijl} B^l - \partial_i (A_0 E^i) - \overline{J} \cdot \overline{A}$$

$$T_0^{\,r} = \frac{1}{2} (\overline{E})^2 + \frac{1}{2} B^k B^l + \overline{\nabla} \cdot (A^0 \overline{E}) - \overline{J} \cdot \overline{A}$$

$$T_0^{\,r} = \frac{1}{2} (\overline{E})^2 + \frac{1}{2} (\overline{B})^2 + \overline{\nabla} \cdot (A^0 \overline{E}) - \overline{J} \cdot \overline{A}$$

$$\therefore \quad \mathcal{H} = \frac{1}{2} (\overline{E})^2 + \frac{1}{2} (\overline{B})^2 + \overline{\nabla} \cdot (A^0 \overline{E})$$ en ausencia de corrientes.

Recuerde que $T_\nu^{\,r}$ está asociado con $\delta_\nu^{\,r}$ $\therefore$ como elegimos $\nu = 0$ por ende $r = 0$

$$T_0^0 = \mathcal{H} = \frac{1}{2} (\overline{E})^2 + \frac{1}{2} (\overline{B})^2 + \overline{\nabla} (A^0 \overline{E})$$

• Para $T_i^0 = \dfrac{\partial \mathcal{L}}{\partial (\partial_0 A_\nu)} \partial_i A_\nu = -F^{0\nu} \partial_i A_\nu$.

Usando $F^{ij} = \partial_i A_j - \partial_j A_i$, tenemos:

$T_i^0 = -F^{0j} F_{ij} - F^{0j} \partial_j A_i$, usando la definición

$$\partial_j (F^{0j} A_i) = (\partial_j F^{0j}) A_i + F^{0j} \partial_j A_i$$

en conclusiones:

$$T_i^0 = E^j \epsilon_{jik} B^k + \partial_j (E^j A_i) + (J^0) A_i$$

$$= -(\bar{E} \times \bar{B})^i - \bar{\nabla} \cdot (A^i \bar{E}) - \rho A^i$$

En ausencia de corriente y cargas:

$$P^i = -\int_V d^3x \, T_i^0 = \int_V d^3x \, (\bar{E} \times \bar{B})^i + \int_V d^3x \, \bar{\nabla} \cdot (A^i \bar{E})$$

$\therefore$ Con Gauss $\int_V d^3x \, \bar{\nabla} \cdot (A^i \bar{E}) = \int ds \, A^i \bar{E} = 0$

$$P^i = \int_V d^3x \, (\bar{E} \times \bar{B})$$

.3. Temenos $\mathcal{L} = (D_\mu \phi)^* D^\mu \phi - m^2 \phi^* \phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$

Sabemos que el campo escalar transforma:

$$\phi \to \phi' = e^{-ig\theta}\phi \; ; \quad (\phi')^* = e^{ig\theta}\phi^*$$

$$\therefore \quad m^2 \phi^* \phi \to m^2 (e^{ig\theta}\phi^*)(e^{-ig\theta}\phi) = m^2 \phi^* \phi \checkmark$$

Ahora $D_\mu = \partial_\mu - igA_\mu$, Como $A_\mu \to A_\mu' = A_\mu - \partial_\mu \theta(x)$

entonces

$$D_\mu \phi \to \left[\partial_\mu - ig(A_\mu - \partial_\mu \theta(x))\right] e^{-ig\theta}\phi$$

$$D_\mu \phi \to \cancel{(-ig\partial_\mu \theta(x))e^{-ig\theta}\phi} + e^{-ig\theta}\partial_\mu \phi - igA_\mu e^{-ig\theta}\phi$$
$$+ \cancel{ig\partial_\mu \theta(x) e^{-ig\theta}\phi}$$
$$= e^{-ig\theta}\left[\partial_\mu - igA_\mu\right]\phi = e^{-ig\theta} D_\mu \phi$$

$$\therefore (D_\mu \phi)^* \to e^{ig\theta}(D_\mu \phi)^*$$

$$\therefore \quad (D_\mu \phi)^* (D^\mu \phi) \to \left[e^{ig\theta}(D_\mu \phi)^*\right]\left[e^{-ig\theta}(D^\mu \phi)\right]$$
$$\to (D_\mu \phi)^* (D^\mu \phi) \longleftarrow$$

Ahora, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$F^{\mu\nu} \to \partial^\mu \left(A^\nu - \partial^\nu \theta(x)\right) - \partial^\nu \left(A^\mu - \partial^\mu \theta(x)\right)$$

$$\to \partial^\mu A^\nu - \cancel{\partial^\mu \partial^\nu \theta(x)} - \partial^\nu A^\mu + \cancel{\partial^\nu \partial^\mu \theta(x)}$$

$$\to \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu} \leftarrow$$

De forma similar para $F_{\mu\nu}$.

Con esto, demostramos que la ecuación es invariante.

b) Necesitamos calcular: $\partial_\mu \left(\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \right) = \dfrac{\partial \mathcal{L}}{\partial \phi^*}$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \frac{\partial\left((D_\nu \phi^*) D^\nu \phi\right)}{\partial(\partial_\mu \phi^*)} = \delta_\nu^{\ \mu} D^\nu \phi = D^\mu \phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = -ig A_\mu D^\mu \phi - m^2 \phi$$

$$\therefore \quad \partial_\mu (D^\mu \phi) + ig A_\mu D^\mu \phi + m^2 \phi = 0$$

$$\underbrace{(\partial_\mu + ig A_\mu)}_{D_\mu} (D^\mu \phi) + m^2 \phi = 0$$

$$\therefore \quad (D_\mu D^\mu + m^2)\phi = \leftarrow$$

4. Recuerde que $\overline{J} = \overline{r} \times \overline{p} = \overline{r} \times (-i\overline{\nabla})$

En componentes: $J^K = [\overline{X} \times (-i\overline{\nabla})]^K = -i\sum_j \epsilon_{ijK} x^i \partial_j = i\epsilon_{ijK} x^i \partial^j$

Entonces: $[J^i, J^j]_- \psi = -[\epsilon_{ilm} x^l \partial_m, \epsilon_{jpq} x^p \partial_q] \psi$

$$= -\epsilon_{ilm}\epsilon_{jpq}[x^l \partial_m, x^p \partial_q]\psi$$

$$[J^i, J^j]_- = -\epsilon_{ilm}\epsilon_{jpq}\left(x^l\partial_m(x^p\partial_q\psi) - x^p\partial_q(x^l\partial_m\psi)\right)$$

$$= -\epsilon_{ilm}\epsilon_{jpq}\left(x^l\delta_{mp}\partial_q\psi + \cancel{x^l x^p \partial_m\partial_q}\psi - x^p\delta_{ql}\partial_m\psi - \cancel{x^p x^l \partial_q\partial_m}\psi\right)$$

$$[J^i, J^j]_- = -\epsilon_{ilm}\epsilon_{jpq}\left(x^l\delta_{mp}\partial_q\psi - x^p\delta_{ql}\partial_m\psi\right)$$

$$= -\epsilon_{ilm}\epsilon_{jmq}x^l\partial_q\psi + \epsilon_{ilm}\epsilon_{jpl}x^p\partial_m\psi.$$

podemos cambiar $\epsilon_{ilm} \to \epsilon_{iqm}$ usando δ_{ql}

y $\epsilon_{jpl} \to \epsilon_{jml} \to -\epsilon_{jlm}$, usando δ_{mp}

Ahora, note $\epsilon_{ilm}\epsilon_{jpl}x^p\partial_m \to -\epsilon_{iqm}\epsilon_{jlm}x^l\partial_q$

$$[J^i, J^j]_- = (\cancel{\delta_{ij}\delta_{lq}} - \delta_{iq}\delta_{lj} - \cancel{\delta_{ij}\delta_{ql}} + \delta_{il}\delta_{qj})x^l\partial_q\psi$$

$$= (\delta_{il}\delta_{qj} - \delta_{iq}\delta_{lj})x^l\partial_q\psi$$

$$[J^i, J^j]_- = \epsilon_{kij}\,\epsilon_{klq}\, x^l \partial_q \psi$$

$$= i\epsilon_{kij}\left(-i\epsilon_{klq}\, x^l \partial_q \psi\right)$$

$$[J^i, J^j] = i\epsilon_{kij} J^k \psi$$

5.

$$\mathrm{Tr}(\gamma^{n}) = 0 \quad \therefore \quad [\gamma^i, \gamma^j]_+ = -2\delta_{ij}$$

$$\gamma^i\gamma^i = -\mathbb{1} \; ; \quad \mathrm{Tr}(\gamma^{n}) = -\mathrm{Tr}(\gamma^{n}\gamma^i\gamma^i)$$

$$\mathrm{Tr}(\gamma^{n}) = \mathrm{Tr}(\gamma^i\gamma^{n}\gamma^i) = -\mathrm{Tr}(\gamma^{n}\gamma^i\gamma^i)$$

Como $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$

$\mathrm{Tr}(\gamma^{n}\gamma^i\gamma^i) = \mathrm{Tr}(\gamma^i\gamma^{n}\gamma^i)$; pero por la propiedad del anti-conmutador:

$$\mathrm{Tr}(\gamma^i\gamma^{n}\gamma^i) = -\mathrm{Tr}(\gamma^{n}\gamma^i\gamma^i) = \mathrm{Tr}(\gamma^{n})$$

$\therefore$ la unica solucion es que $\mathrm{Tr}(\gamma^{n}) = 0$

6 . Consideremos: $U_+ = N \begin{pmatrix} 1 \\ 0 \\ \frac{\overline{\sigma}\cdot\overline{p}}{E_p+m}\chi_\pm \end{pmatrix}$

$$\overline{\sigma}\cdot\overline{p} = \begin{pmatrix} 0 & p_x \\ p_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & -ip_y \\ ip_y & 0 \end{pmatrix} + \begin{pmatrix} p_z & 0 \\ 0 & -p_z \end{pmatrix}$$

$$\overline{\sigma}\cdot\overline{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \therefore \overline{\sigma}\cdot\overline{p}\,\chi_\pm = \begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix}$$

$$U_+ = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E_p+m} \\ \frac{(p_x+ip_y)}{E_p+m} \end{pmatrix} \therefore U_+^\dagger U_+ = N^2 \begin{pmatrix} 1 & 0 & \frac{p_z}{E_p+m} & \frac{p_x - ip_y}{E_p+m} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E_p+m} \\ \frac{p_x+ip_y}{E_p+m} \end{pmatrix}$$

$$\therefore U_+^\dagger U_+ = \frac{N^2}{(E_p+m)^2}\left(p_z^2 + p_x^2 + p_y^2\right) + N^2 = 2E_p$$

$$U_+^\dagger U_+ = \frac{N^2}{(E_p+m)^2}\left(p^2 + (E_p+m)^2\right) = 2E_p$$

Despejando $N = \sqrt{E_p + m}$