MECANICA CUANTICA 2

LECTURA #7

ADISION DE DOS MOMENTUM ANGULARES ARBITRARIOS

Conciderемos $\vec{J}_1$ y $\vec{J}_2$ fijos $\Rightarrow \vec{J} = \vec{J}_1 + \vec{J}_2 \Rightarrow \vec{J}$ no es unico! los posibles valores de j son:

$|j_1 - j_2|, \ldots, j + j_2$

Demostremoslo:

Construyamos un posible estado con $j = j_1 + j_2$:

Recordemos que tenemos dos bases para el espacio de Hilbert $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$

$\{|j_1 m_1; j_2, m_2\rangle\}$ y $\{|j_1 j_2; JM\rangle\}$

Base desacoplada $\quad$ base acoplada.

$J_z |j_1 j_2; JM\rangle = M\hbar |j_1 j_2; JM\rangle$

$J_{1z} |j_1 m_1; j_2 m_2\rangle = m_1 \hbar |j_1 m_1; j_2 m_2\rangle$

$J_{2z} |j_1 m_1; j_2 m_2\rangle = m_2 \hbar |j_1 m_1; j_2 m_2\rangle$

$J_z = J_{1z} + J_{2z} \Rightarrow m = m_1 + m_2$

$J_z |j_1 m_1; j_2 m_2\rangle = (m_1 + m_2) |j_1 m_1; j_2 m_2\rangle$

Si $m_1 = m_{1max} = j_1$; $m_2 = m_{2max} = j_2 \Rightarrow m = (j_1 + j_2) \Rightarrow \boxed{M_{max} = (j_1 + j_2)}$

$\Rightarrow |j_1 j_2; J \underbrace{M_{max}}_{(j_1+j_2)}\rangle = |j_1 m_1 = j_1; j_2, m_2 = j_2\rangle$

De ahora en adelante vamos a omitir escribir los indices j_1 y j_2; ellos van a estar implicitos en la notación.

$|J, M\rangle = |j_1 j_2; JM\rangle$ (separado por una coma)

$|m_1; m_2\rangle = |j_1 m_1; j_2 m_2\rangle$ (separado por un punto y coma)

$M_{max} = j_1 + j_2 \Rightarrow$ este M_{max} debe corresponder a $\boxed{J_{max} = j_1 + j_2} \Longrightarrow$

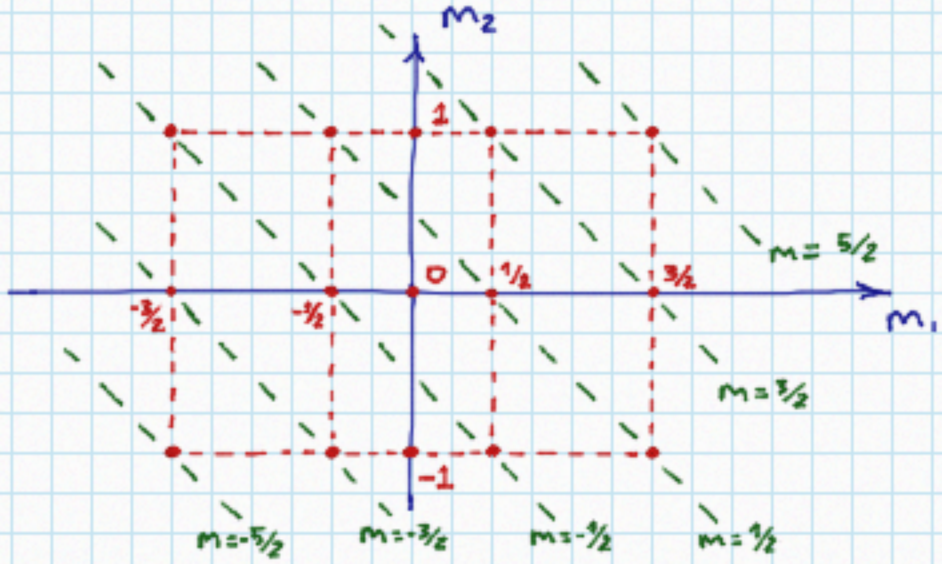
$\boxed{|J = j_1 + j_2; M = j_1 + j_2\rangle = |m_1 = j_1; m_2 = j_2\rangle}$

SUBESPACIOS DE J_z:

Tomemos la base desacoplada $\{|m_1; m_2\rangle\}$ y miremos el caso $j_1 = 3/2$; $j_2 = 1 \Rightarrow -3/2 \le m_1 \le 3/2$

$-1 \le m_2 \le 1$

Ahora vamos a graficar los posibles valores de m_1 y m_2 en un plano cartesiano.



$-5/2 \leq m \leq 5/2 \leftarrow$ 6 estados

Cada linea inclinada corresponde a $m = m_1 + m_2 = cte \rightarrow$ es decir, corresponde a un subespacio propio de J_z.

El punto en la esquina superior derecha corresponde a $|J = j_1+j_2, M = j_1+j_2\rangle = |m_1 = j_1; m_2 = j_2\rangle$

La dimensión total del espacio de Hilbert $\mathcal{E}$ asociado a valores fijos de j_1 y j_2 es:

$(2j_1+1)(2j_2+1) \longrightarrow (2 * 3/2 + 1)(2*1+1) = 4*3 = 12 \longleftarrow$ 12 puntos en el plano $\rightarrow$ 12 vectores en la base acoplada

$\{|J,M\rangle\}$ $J: J_{min}, \dots, J_{max}$; $M: -J, \dots, J$

$J_{max} = j_1 + j_2$ $\quad J_{min} = ?$

$$\left.\begin{array}{ll} J_{max} \longrightarrow & 2(j_1+j_2)+1 \\ J_{max}-1 \longrightarrow & 2(j_1+j_2-1)+1 \\ \vdots & \\ J_{min} \longrightarrow & 2(j_1+j_2-n)+1 \end{array}\right\} \text{n+1 terminos} \rightarrow n = ?$$

$J_{min} = J_{max} - n$
$= (j_1+j_2) - n$

$$\sum_{J=J_{min}}^{J_{max}} (2J+1) = (2j_1+1)(2j_2+1) \qquad J_{max} = j_1 + j_2 \implies J_{min} = ?$$

Supongamos $j_1 > j_2$ y $J_{min} = j_1 - j_2 \implies \displaystyle\sum_{J=j_1-j_2}^{j_1+j_2} (2J+1) = \sum_{i=0}^{2j_2} \left[2(j_1 - j_2 + i) + 1\right]$

$$= \left[2(j_1-j_2)+1\right](2j_2+1) + 2\,\frac{2j_2(2j_2+1)}{2}$$

$$= (2j_2+1)(2j_1+1) \checkmark$$

Si $j_2 > j_1 \implies$ Solo hay que intercambiar los indices 1 y 2 $\implies J_{min} = j_2 - j_1 \implies \boxed{J_{min} = |j_1 - j_2|}$

$\implies J: |j_1 - j_2|, \dots, (j_1+j_2) \qquad M: -J, \dots, J$

Ej: $j_1 = 3/2$, $j_2 = 5/2$ $\Rightarrow$ $j_{max} = 8/2 = 4$, $j_{min} = 2/2 = 1$ $\Rightarrow$ J : 1, 2, 3, 4

CALCULO DE LOS COEFICIENTES DE CLEBSH-GORDAN

$$\underset{j_1 m_1}{\vec{J}_1} , \underset{j_2 m_2}{\vec{J}_2} \longrightarrow \underset{J M}{\vec{J}} = \vec{J}_1 + \vec{J}_2$$

Fijamos j_1 y j_2 $\Rightarrow$ $\{|m_1 ; m_2\rangle\}$ $\leftarrow$ Base desacoplada

$\{|J, M\rangle\}$ $\leftarrow$ Base acoplada

$$|J, M\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} C^{JM}_{m_1 m_2} |m_1 ; m_2\rangle$$

j_1 y j_2 fijos e implícitos en esta notación simplificada

$C^{JM}_{m_1 m_2}$: Coeficientes de Clebsh-Gordan asociados a $j_1 \otimes j_2$

j_1 y j_2 definen un subespacio $\mathcal{E}(j_1, j_2)$. El proyector a este subespacio esta dada por:

$$\hat{P}_{j_1 j_2} = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |m_1 ; m_2\rangle\langle m_1 ; m_2| \qquad |J, M\rangle \in \mathcal{E}(j_1, j_2) \Rightarrow \hat{P}_{j_1 j_2} |JM\rangle = |J, M\rangle$$

$$|J, M\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |m_1 ; m_2\rangle\langle m_1 ; m_2 | J, M\rangle$$

$$\boxed{C^{JM}_{m_1 m_2} = \langle m_1 ; m_2 | J, M\rangle} \leftarrow \text{Coeficientes de Clebsh-Gordan asociados a } j_1 \otimes j_2$$

De ahora en adelante usaremos la notación $\langle m_1 ; m_2 | J, M\rangle$ para los coeficientes de Clebsh-Gordan

PROPIEDADES DE LOS COEFICIENTES DE CLEBSH-GORDAN

1) $\langle m_1 ; m_2 | J, M\rangle = 0$ si no se cumple que $M = m_1 + m_2$; $|j_1 - j_2| \leq J \leq (j_1 + j_2)$

2) $\langle m_1 ; m_2 | J, M\rangle = 0$ si no se cumple que $-J \leq M \leq J$; $-j_1 \leq m_1 \leq j_1$; $-j_2 \leq m_2 \leq j_2$

3) Ortogonalidad:
$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle m_1 ; m_2 | J, M\rangle^* \langle m_1 ; m_2 | J', M'\rangle = \delta_{JJ'} \delta_{MM'}$$

Veamos: $\langle J, M | J', M'\rangle = \delta_{JJ'} \delta_{MM'}$

$\langle J, M | \mathbb{1} | J', M'\rangle = \delta_{JJ'} \delta_{MM'}$

$$\hat{P}_{j_1 j_2} |J' M'\rangle = |J', M'\rangle = \mathbb{1} |J', M'\rangle \Rightarrow \langle J, M | \mathbb{1} | J', M'\rangle = \langle J, M | \hat{P}_{j_1 j_2} | J', M'\rangle = \delta_{JJ'} \delta_{MM'}$$

$$\langle J, M | \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |m_1 ; m_2\rangle\langle m_1 ; m_2 | J', M'\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle J, M | m_1 ; m_2\rangle \langle m_1 ; m_2 | J', M'\rangle$$

$$= \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle m_1 ; m_2 | J, M\rangle^* \langle m_1 ; m_2 | J', M'\rangle = \delta_{JJ'} \delta_{MM'}$$

RELACIONES DE RECURRENCIA

$$J_{\pm} = J_x + i J_y = J_{1\pm} + J_{2\pm}$$

$$J_{\pm} |J,M\rangle = \hbar \sqrt{J(J+1) - M(M\pm 1)}\, |J, M\pm 1\rangle$$

$$J_{1\pm} |m_1;m_2\rangle = \hbar \sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)}\, |m_1 \pm 1; m_2\rangle$$

$$J_{2\pm} |m_1;m_2\rangle = \hbar \sqrt{j_2(j_2+1) - m_2(m_2 \pm 1)}\, |m_1; m_2 \pm 1\rangle$$

$$|J,M\rangle = \sum_{m_1'=-j_1}^{j_1} \sum_{m_2'=-j_2}^{j_2} \langle m_1';m_2'|J,M\rangle\, |m_1';m_2'\rangle$$

$$J_-|J,M\rangle = (J_{1-} + J_{2-}) \sum_{m_1'=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle m_1';m_2'|J,M\rangle\, |m_1';m_2'|J,M\rangle |m_1';m_2'\rangle$$

$$\sqrt{J(J+1) - M(M-1)}\, |J, M-1\rangle = \sum_{m_1'=-j_1}^{j_1} \sum_{m_2'=-j_2}^{j_2} \sqrt{j_1(j_1+1) - m_1'(m_1'-1)}\, \langle m_1';m_2'|J,M\rangle |m_1'-1;m_2'\rangle \; +$$

$$+ \sum_{m_1'=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sqrt{j_2(j_2+1) - m_2'(m_2-1)}\, \langle m_1';m_2'|J,M\rangle |m_1';m_2'-1\rangle$$

Si actuamos con $\langle m_1;m_2|$ sobre esta expresión

- en la primera sumatoria solo sobrevive el término $m_1'-1 = m_1$; $m_2' = m_2$
  $m_1' = m_1 + 1$

- en la segunda sumatoria solo sobrevive el término $m_1' = m_1$; $m_2' - 1 = m_2$
  $m_2' = m_2 + 1$

$$\sqrt{J(J+1) - M(M-1)}\, \langle m_1;m_2|J,M\rangle = \sqrt{j_1(j_1+1) - m_1(m_1+1)}\, \langle m_1+1;m_2|J,M\rangle + \sqrt{j_2(j_2+1) - m_2(m_2+1)}\, \langle m_1;m_2+1|J,M\rangle$$

PRIMERA RELACION DE RECURRENCIA.

Si repetimos este procedimiento pero a partir de $J_+|J,M\rangle$ obtenemos la segunda relación de recurrencia

$$\sqrt{J(J+1) - M(M+1)}\, \langle m_1;m_2|J,M\rangle = \sqrt{j_1(j_1+1) - m_1(m_1-1)}\, \langle m_1-1;m_2|J,M\rangle + \sqrt{j_2(j_2+1) - m_2(m_2-1)}\, \langle m_1;m_2-1|J,M\rangle$$

SEGUNDA RELACION DE RECURRENCIA.

Las relaciones de recurrencia nos permiten calcular todos los coeficientes de Clebsh-Gordan

$\Rightarrow$ $J: |j_1 - j_2|, \dots, (j_1 + j_2)$ $\qquad M: -J, \dots, J$

Tomemos un $J \Rightarrow M_{max} = J$

$$|J, M=J\rangle = |J,J\rangle = \sum_{\substack{m_1=-j_1 \\ (m_1+m_2=M=J)}}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle m_1;m_2|J,J\rangle\, |m_1;m_2\rangle$$

Aplicando la segunda relación de recurrencia a este caso obtenemos

$$\sqrt{j_1(j_1+1)-m_1(m_1-1)}\,\langle m_1-1; m_2 | J,J\rangle = -\sqrt{j_2(j_2+1)-m_2(m_2-1)}\,\langle m_1; m_2-1 | J,J\rangle$$

RELACION ENTRE PAREJAS DE COEFICIENTES

Si a estas relaciones le agregamos

$$\langle J,J|J,J\rangle = 1 = \sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2} |\langle m_1; m_2 | J,J\rangle|^2$$

podemos calcular todos los coeficientes de Clebsh-Gordan asociados a un J dado.

Si hacemos eso para cada $J: |j_1-j_2|, \dots, (j_1+j_2)$, calculamos todos los coeficientes de Clebsh-Gordan asociados $j_1 \otimes j_2$.

Las relaciones entre los diferentes coeficientes de Clebsh-Gordan indican que todos poseen la misma fase compleja. Podemos escojer esa fase para que todos ellos sean reales $\Rightarrow$ es así como se definen y se usan

$\Rightarrow$ TODOS LOS COEFICIENTES DE CLEBSH-GORDAN SON REALES

Dado que el calculo de los coeficientes de Clebsh-Gordan es muy dispendioso, se usan tablas.

40. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			
.	.			

1/2×1/2

		1
		+1
+1/2	+1/2	1

		1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		1
		−1
−1/2	−1/2	1

1×1/2

		3/2
		+3/2
+1	+1/2	1

		3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2
		−3/2
−1	−1/2	1

2×1

		3
		+3
+2	+1	1

		3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

		3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

		3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

		3	2	1
		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

		3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

		3
		−3
−2	−1	1

1×1

		2
		+2
+1	+1	1

		2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

		2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

		2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

		2
		−2
−1	−1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

2×1/2

		5/2
		+5/2
+2	+1/2	1

		5/2	3/2
		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

		5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

		5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

		5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

		5/2
		−5/2
−2	−1/2	1

3/2×1/2

		2
		+2
+3/2	+1/2	1

		2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

		2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

		2
		−2
−3/2	−1/2	1

3/2×1

		5/2
		+5/2
+3/2	+1	1

		5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

		5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

		5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

		5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

		5/2
		−5/2
−3/2	−1	1

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

2×3/2

		7/2
		+7/2
+2	+3/2	1

		7/2	5/2
		+5/2	+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

		7/2	5/2	3/2
		+3/2	+3/2	+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

		7/2	5/2	3/2	1/2
		+1/2	+1/2	+1/2	+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

		7/2	5/2	3/2	1/2
		−1/2	−1/2	−1/2	−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

		7/2	5/2	3/2
		−3/2	−3/2	−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

		7/2	5/2
		−5/2	−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

		7/2
		−7/2
−2	−3/2	1

3/2×3/2

		3
		+3
+3/2	+3/2	1

		3	2
		+2	+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

		3	2	1
		+1	+1	+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

		3	2	1	0
		0	0	0	0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

		3	2	1
		−1	−1	−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

		3	2
		−2	−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

		3
		−3
−3/2	−3/2	1

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

2×2

		4
		+4
+2	+2	1

		4	3
		+3	+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

		4	3	2
		+2	+2	+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

		4	3	2	1
		+1	+1	+1	+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

		4	3	2	1	0
		0	0	0	0	0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

		4	3	2	1
		−1	−1	−1	−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

		4	3	2
		−2	−2	−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

		4	3
		−3	−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

		4
		−4
−2	−2	1

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 40.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).