PROBLEMA 1

$V(r) = \alpha \delta(r-a)$

$f(\theta) = \frac{1}{k} \sum_{\ell} (2\ell+1) e^{i\delta_\ell} \sin \delta_\ell \; P_\ell(\cos\theta)$

Como solo sobrevive $\ell=0$ ya que $ka \ll 1 \Rightarrow f(\theta) \cong \frac{1}{k} e^{i\delta_0} \sin \delta_0$

Resolvemos la parte radial de la ecuación de schrödinger por $\ell=0$:

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \alpha \delta(r-a) u = E u$$

Para determinar condiciones de frontera integramos esta ecuación alrededor de a:

$$-\frac{\hbar^2}{2m} \int_{a-\epsilon}^{a+\epsilon} \left(\frac{d^2 u}{dr^2}\right) r^2 dr + \alpha \int_{a-\epsilon}^{a+\epsilon} r^2 \delta(r-a) u \, dr = E \int_{a-\epsilon}^{a+\epsilon} r^2 u \, dr \qquad \epsilon \to 0$$

Por partes

$$-\frac{\hbar^2}{2m}\left(r^2 \frac{du}{dr} \Big|_{a-\epsilon}^{a+\epsilon} - 2 \int_{a-\epsilon}^{a+\epsilon} r \frac{du}{dr} dr \right) + \alpha a^2 u(a) = 0$$

$$-\frac{\hbar^2}{2m}\Big(a^2 (u'(a^+) - u'(a^-)) \underbrace{- 2ur \Big|_{a-\epsilon}^{a+\epsilon} + 2 \int_{a-\epsilon}^{a+\epsilon} u dr}_{0 \qquad 0} \Big) + \alpha a^2 u(a) = 0$$

Por continuidad de $u(r)$

$$\Rightarrow -\frac{\hbar^2 a^2}{2m} \left(\frac{du}{dr}\Big|_{r=a^+} - \frac{du}{dr}\Big|_{r=a^-} \right) = -\alpha a^2 u(a)$$

$$\Rightarrow \frac{du}{dr}\Big|_{r=a^+} - \frac{du}{dr}\Big|_{r=a^-} = \frac{2m\alpha}{\hbar^2} u(a)$$

Sea $\phi = \frac{2ma\alpha}{\hbar^2} \Rightarrow \boxed{\frac{du}{dr}\Big|_{r=a^+} - \frac{du}{dr}\Big|_{r=a} = \phi \frac{u(a)}{a}}$ (*)

Para $r \neq a$ $\qquad \dfrac{d^2 U}{dr^2} + K^2 U = 0 \qquad$ como $K^2 = \dfrac{2mE}{\hbar^2}$

$$\Rightarrow \begin{cases} U(r) = A_1 \sin(Kr) + A_2 \cos(Kr) & r < a \\ U(r) = B_1 \sin(Kr) + B_2 \cos(Kr) & r > a \end{cases}$$

$R(r) = \dfrac{U(r)}{r}$

$R(0)$ debe ser finita $\Rightarrow$ requiere $\underline{A_2 = 0}$

Continuidad de $U(r)$ en $r = a$:

① $A_1 \sin(Ka) = B_1 \sin(Ka) + B_2 \cos(Ka)$

Discontinuidad de la derivada ec. (*)

② $-K A_1 \cos(Ka) + K B_1 \cos(Ka) - K B_2 \sin(Ka) = \phi A_1 \dfrac{\sin(Ka)}{a}$

Hallamos B_1 y B_2 en terminos de A_1:

De ① : $B_1 = A_1 - B_2 \cot(Ka)$

en② $\therefore -KA_1 \cancel{\cos}(Ka) + KA_1 \cancel{\cos}(Ka) - KB_2 \cos(Ka)\cot(Ka) - KB_2 \sin(Ka)$

$$= \phi A_1 \frac{\sin(Ka)}{a}$$

$$\Rightarrow -KB_2 \left[\sin(Ka) + \frac{\cos^2(Ka)}{\sin(Ka)}\right] = A_1 \left[\frac{\phi}{a} \sin(Ka)\right]$$

$$\Rightarrow -KB_2 = A_1 \frac{\phi}{a} \sin^2(Ka) \Rightarrow \boxed{B_2 = -A_1 \frac{\phi}{Ka} \sin^2(Ka)}$$

$$\Rightarrow B_1 = A_1 \left[1 + \frac{\phi}{Ka} \sin(Ka)\cos(Ka)\right]$$

$\Rightarrow$ Para $r > a$:

$$U(r) = A_1 \left[1 + \frac{\phi}{Ka} \sin(Ka)\cos(Ka)\right] \sin(Kr) - \frac{A_1 \phi}{Ka} \sin^2(Ka) \cos(Kr)$$

y $R(r) = U(r)/r$ se puede escribir como:

$$R(r) = \frac{C}{r} \sin(Kr + \delta_0)$$

$$C = A_1 \left\{\left(1 + \frac{\phi}{Ka} \sin(Ka)\cos(Ka)\right)^2 + \left(\frac{\phi}{Ka} \sin^2(Ka)\right)^2\right\}^{1/2}$$

$$-\delta_0 = \tan^{-1}\left\{\frac{\frac{\phi}{Ka} \sin^2(Ka)}{1 + \frac{\phi}{Ka}\sin(Ka)\cos(Ka)}\right\} = \tan^{-1}\left\{\frac{1}{\frac{Ka}{\phi \sin^2(Ka)} + \cot(Ka)}\right\}$$

$$Ka << 1 \Rightarrow \sin^2(Ka) \approx K^2a^2$$

$$\cot an(Ka) \approx \frac{1}{Ka}$$

$$\Rightarrow -\delta_0 = \tan^{-1}\left\{ \frac{1}{\frac{Ka}{\phi K^2 a^2} + \frac{1}{Ka}} \right\} \cong \tan^{-1}\left\{ \frac{\phi Ka}{1+\phi} \right\}$$

$$\underline{\sin(\tan^{-1} u)} \Rightarrow \tan u = \frac{\sin u}{\cos u} = \frac{\sin u}{\sqrt{1-\sin^2 u}} \Rightarrow \sin^2 u = \tan^2 u\,(1-\sin^2 u)$$

$$\Rightarrow \sin^2 u = \frac{\tan^2 v}{1+\tan^2 u}$$

$$\sin^2(\tan^{-1} u) = \frac{u^2}{1+u^2} \quad \cos(\tan^{-1} u) = \sqrt{1-\frac{u^2}{1+u^2}} = \frac{1}{\sqrt{1+u^2}}$$

$$f(\theta) \approx \frac{1}{K}\left[\cos\delta_0 \sin\delta_0 + i\sin^2\delta_0\right]$$

$$\approx \frac{1}{K}\left[\frac{i\phi^2K^2a^2/(1+\phi)^2}{1+\frac{\phi^2K^2a^2}{(1+\phi)^2}} - \frac{\phi Ka/(1+\phi)}{1+\frac{\phi^2K^2a^2}{(1+\phi)^2}} \right]$$

$$\boxed{f(\theta) \simeq \frac{1}{K}\frac{\phi Ka}{(1+\phi)^2+\phi^2K^2a^2}\left[-(1+\phi)+i\phi Ka\right]}$$

$$\sigma_T = \frac{4\pi}{K^2}\sin^2\delta_0 = \frac{4\pi}{K^2}\left\{ \frac{\phi^2K^2a^2/(1+\phi)^2}{1+\frac{\phi^2K^2a^2}{(1+\phi)^2}} \right\}$$

$$\sigma_T = 4\pi a^2 \left\{ \frac{\phi^2}{(1+\phi)^2+\phi^2K^2a^2} \right\} \quad \text{para } \phi^2K^2a^2 << \phi^2$$

$$\Rightarrow \boxed{\sigma_T \approx \frac{4\pi a^2 \phi^2}{(1+\phi^2)}}$$

PROBLEMA 2

2 $$f_0 = -\frac{\alpha K[1 - i\alpha K]}{K[1+\alpha^2 K^2]}$$

para $f(\theta) = \frac{1}{K}\sum_{\ell}(2\ell+1)\, e^{i\delta_\ell}\sin\delta_\ell\; P_\ell(\cos\theta)$

$$\Rightarrow f_0 = \frac{1}{K}\, e^{i\delta_0}\sin\delta_0 = -\frac{\alpha[1 - i\alpha K]}{1+\alpha^2 K^2}$$

igualando partes Re e Im :

$$\left\{\begin{array}{ll} ① & \frac{1}{K}\cos\delta_0\sin\delta_0 = -\frac{\alpha}{1+\alpha^2K^2} \\ ② & \frac{1}{K}\sin^2\delta_0 = \frac{\alpha^2 K}{1+\alpha^2K^2} \end{array}\right\} \begin{array}{l} ②/① : \\ \Rightarrow \tan\delta_0 = -\alpha K \\ \Rightarrow \boxed{\delta_0 = \tan^{-1}[-\alpha K]} \end{array}$$

PROBLEMA 3

$\boxed{3}$ $\delta_0 = k(a + bk^2)$

$\delta_1 = ck^3$

$\delta_i = 0 \quad i > 1$

$\boxed{a}$ Dado que $f(\theta) = \sum_l (2l+1) \dfrac{e^{i\delta_l} \operatorname{sen}\delta_l}{k} P_l(\cos\theta)$

Para verificar que $\delta_l = 0$ para $l > 1$ medimos $\dfrac{d\sigma}{d\Omega}$ en función de θ y hacemos un ajuste a $\dfrac{d\sigma}{d\Omega}$ con un polinomio en $\cos\theta$ hasta orden 2, esto es porque si solo hay contribución hasta $l=1$:

$$f(\theta) = \left(e^{i\delta_0} \frac{\operatorname{sen}\delta_0}{k} + e^{i\delta_1} \frac{\operatorname{sen}\delta_1}{k} \cos\theta \right)$$

$\dfrac{d\sigma}{d\Omega} = |f(\theta)|^2$ solo puede tener términos: constante1 + Constante2 $\cos\theta$ + Const3 $\cos^2\theta$

estas mediciones se hacen a un valor específico de k.

Para medir las constantes se hace un fit a $\dfrac{d\sigma}{d\Omega}$ vs k, manteniendo θ constante.

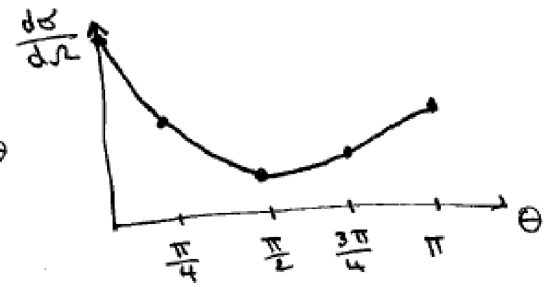
$\boxed{b}$ $a = b = c = \pi/4$, $k = 1$

$$\Rightarrow \frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{1}{k^2} \left| (\cos\delta_0 \operatorname{sen}\delta_0 + 3\cos\delta_1 \operatorname{sen}\delta_1 \cos\theta)^2 + (\operatorname{sen}^2\delta_0 + 3\operatorname{sen}^2\delta_1 \cos\theta)^2 \right|$$

$$= \frac{1}{k^2} \left(\operatorname{sen}^2\delta_0 + 9\cos^2\theta \operatorname{sen}^2\delta_1 + 6\cos\theta \operatorname{sen}\delta_0 \operatorname{sen}\delta_1 \cos(\delta_1 - \delta_0) \right)$$

$\delta_0 = \pi/2$, $\delta_1 = \pi/4$

$$\frac{d\sigma}{d\Omega} = 1 + 3\cos\theta + \frac{9}{2}\cos^2\theta$$



PROBLEMA 4

$\boxed{4}$ $$V(r) = \begin{cases} -V_0 \,, & r \leq R \\ 0 \,, & r > R \end{cases}$$

De acuerdo a la primera aproximación de Born:

$$f(\theta) = -\frac{2m}{\hbar^2 q}\int_0^{\infty} r' V(r') \operatorname{Sen}(q r')\, dr'$$

$$= \frac{2mV_0}{\hbar^2 q}\int_0^{R} r' \operatorname{Sen}(q r')\, dr'$$

$$= \frac{2mV_0}{\hbar^2 q^3}\int_0^{qR} u \operatorname{Sen}(u)\, du$$

$$= \frac{2mV_0}{\hbar^2 q^3}\Big[\operatorname{Sen}(u) - u\cos(u)\Big]\Bigg|_0^{qR}$$

$$f(\theta) = \frac{2mV_0}{\hbar^2 q^3}\Big[\operatorname{Sen}(qR) - qR\cos(qR)\Big]$$

PROBLEMA 5

Usando la primera aproximación de Born.

ⓐ $V(r) = V_0 \exp(-\alpha r)$

$$f(\theta) = \frac{-2m}{\hbar^2 K}\int_0^\infty r\, V(r)\sin(Kr)\,dr$$ porque el potencial es esfericamente simétrico

$$\Rightarrow f(\theta) = -\frac{mV_0}{\hbar^2 K\sin\left(\frac{\theta}{2}\right)}\int_0^\infty r e^{-\alpha r}\sin(kr)\,dr$$

$$= -\frac{mV_0}{\hbar^2 K\sin\left(\frac{\theta}{2}\right)}\frac{1}{2i}\left\{\int_0^\infty r e^{-(\alpha - ik)r}dr - \int_0^\infty r e^{-(\alpha+ik)r}dr\right.$$

$$u_1 = (\alpha - ik)r \Rightarrow du_1 = (\alpha = ik)\,dr$$

$$= -\frac{mV_0}{\hbar^2 K\sin\left(\frac{\theta}{2}\right)}\frac{1}{2i}\left\{\frac{1}{(\alpha-ik)^2}\underbrace{\int_0^\infty u e^{-u}du}_{\Gamma(2)} - \frac{1}{(\alpha+ik)}\underbrace{\int_0^\infty u e^{-u}du}_{\Gamma(2)}\right\}$$

$$\Rightarrow f(\theta) = -\frac{mV_0}{2i\hbar^2 K\sin\left(\frac{\theta}{2}\right)}\left\{\frac{1}{(\alpha-ik)^2} - \frac{1}{(\alpha+ik)^2}\right\} \qquad \Gamma(2) = 1$$

$$= \frac{mV_0}{2i\hbar^2 K\sin\left(\frac{\theta}{2}\right)}\left\{\frac{\alpha^2 - k^2 + 2i\alpha k - \alpha^2 + k^2 + 2i\alpha k}{(\alpha^2+k^2)^2}\right\}$$

$$\Rightarrow f(\theta) = -\frac{4mV_0\alpha}{\hbar^2}\;\frac{1}{(\alpha^2+k^2)^2}$$

$$\Rightarrow \boxed{f(\theta) = -\frac{4mV_0\alpha}{\hbar^2}\;\frac{1}{(\alpha^2+4k^2\sin^2(\theta/2))^2}}$$

$$\frac{d\sigma}{d\Omega} = \frac{16\,m^2V_0^2\alpha^2}{\hbar^4}\;\frac{1}{[\alpha^2+4K^2\sin^2\left(\frac{\theta}{2}\right)]^4}$$

$$\sigma_T = \int_0^{2\pi}\int_0^{\pi}\left(\frac{d\sigma}{d\Omega}\right)\sin\theta\,d\theta\,d\phi$$

(b) Potencial Gaussiano

$$V(r) = V_0 \, e^{-\alpha^2 r^2}$$

$$\Rightarrow f(\theta) = -\frac{2m}{\hbar^2 k}\int_0^\infty r e^{-\alpha^2 r^2} \sin(kr)\, dr$$

$$= -\frac{m}{i\hbar^2 k}\int_0^\infty r \left[e^{-\alpha^2 r^2 + ikr} - e^{-\alpha^2 r^2 - ikr}\right] dr$$

Completamos cuadrados en los argumentos de las exponenciales:

$$e^{-\alpha^2 r^2 + ikr} = e^{-(\alpha r - \frac{ik}{2\alpha})^2} e^{-k^2/4\alpha^2} \quad ; \quad e^{-\alpha^2 r^2 - ikr} = e^{-(\alpha r + \frac{ik}{2\alpha})^2 - k^2/4\alpha^2}$$

$$\Rightarrow f(\theta) = \frac{-m}{i\hbar^2 k} e^{-k^2/4\alpha^2}\left\{\int_0^\infty r e^{-(\alpha r - \frac{ik}{2\alpha})^2} dr - \int_0^\infty r e^{-(\alpha r + \frac{ik}{2\alpha})^2} dr\right\}$$

sea $u = \alpha r - ik/2\alpha \Rightarrow dr = \frac{1}{\alpha} du$

$$f(\theta) = -\frac{m}{i\hbar^2 k \alpha^2} e^{-k^2/4\alpha^2}\left\{\cancel{\int_0^\infty u e^{-u^2} du} + \frac{ik}{2\alpha}\int_0^\infty e^{-u^2} du - \cancel{\int_0^\infty u e^{-u^2} du} + \frac{ik}{2k}\int_0^\infty e^{-u^2} du\right\}$$

$$\boxed{f(\theta) = -\frac{m\sqrt{\pi}\, e^{-k^2/4\alpha^2}}{2\hbar^2 \alpha^3}}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{m^2 \pi}{4\hbar^4 \alpha^6} \exp\left(-\frac{2\sin^2(\theta/2)}{\alpha^2}\right)}$$

(c) $V(r) = \dfrac{V_0}{(r^2 + d^2)^2}$

$$f(\theta) = \frac{-2mV_0}{\hbar^2 k}\int_0^\infty r \frac{\sin(kr)}{(r^2 + d^2)^2} dr = -\frac{2mV_0}{\hbar^2 k}\frac{1}{4d} e^{-kd} \pi k$$

$$\Rightarrow -\frac{\pi m V_0}{2\hbar^2 d} e^{-kd} \qquad k = 2K\sin\left(\frac{\theta}{2}\right)$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{\pi^2 m^2 V_0^2}{4\hbar^4 d^2} e^{-2kd}$$

$$\sigma_T = \frac{\pi^3 m^2 V_0^2}{2\hbar^4 d^2}\int_0^\pi \sin\theta \, e^{-4Kd\sin(\frac{\theta}{2})} d\theta$$

$$\Rightarrow \frac{\pi^3 m^2 V_0^2}{2\hbar^4 d^2}\frac{1}{4d^2 K^2}\left(1 - (1 + 4dK)e^{-4dK}\right)$$

$$\boxed{\sigma_T = \frac{\pi^3 m^2 V_0^2}{8\hbar^4 d^4 K^2}\left[1 - (1 + 4dK)e^{-4dK}\right)}$$