Physics 828: Homework Set No. 7

Due date: Friday, March 2, 2011, 1:00pm in PRB M2043 (Biao Huang's office)

Total point value of set: 80 points

Problem 1 (5 pts.): Exercise 14.3.8 (Shankar, p. 385)

Problem 2 (20 pts.): Exercise 14.4.3 (Shankar, p. 396)

Problem 3 (10 pts.): Exercise 14.5.1 (Shankar, p. 398)

Problem 4 (10 pts.): Exercise 14.5.2 (Shankar, p. 399)

Problem 5 (20 pts.): Exercise 15.1.2 (Shankar, p. 407)

Problem 6 (10 pts.): Exercise 15.2.2, only part (1) (Shankar, p. 413)

Problem 7 (5 pts.): Exercise 15.2.5 (Shankar, p. 415)

Note: a problem such as Exercise 14.5.3 or 14.5.4 might well appear on the next exam.

Exercise 14.3.8 (p.385)

(1) Proposition: if $[M, \sigma_i] = 0$, $i = 1, 2, 3$, then

$$M = \lambda \mathbb{1}$$

Proof: $M = \sum_\alpha m_\alpha \sigma_\alpha$ (14.3.42)

$\Rightarrow \sum_\alpha m_\alpha [\sigma_\alpha, \sigma_i] = 0$ since $[\sigma_0, \sigma_i] = 0$

$\Rightarrow \sum_k m_k [\sigma_k, \sigma_i] = 2i \sum_{jk} m_k \varepsilon_{kij} \sigma_j = 0$

$i=1$: $m_1 \underbrace{[\sigma_1, \sigma_1]}_{0} + m_2 [\sigma_2, \sigma_1] + m_3 [\sigma_3, \sigma_1] = 0$

$\Rightarrow -m_2 \sigma_3 + m_3 \sigma_2 = 0 \Rightarrow m_2 = m_3 = 0$
(since σ_3 and σ_2 are linearly independent)

$i=2$: (similarly) $\Rightarrow m_1 = m_3 = 0$

$i=3$: (similarly) $\Rightarrow m_2 = m_1 = 0$

$\Rightarrow m_1 = m_2 = m_3 = 0 \Rightarrow M = m_0 \sigma_0 = m_0 \mathbb{1}$ ✓

(2) Proposition: If $[M, \sigma_i]_+ = 0$ for $i = 1, 2, 3$ $\Rightarrow M = 0$

Proof: $M = m_\alpha \sigma_\alpha$

$$[M, \sigma_i] = m_\alpha [\sigma_\alpha, \sigma_i]_+ = m_0 [\sigma_0, \sigma_i]_+ + m_k [\sigma_k, \sigma_i]_+$$

$$= 2 m_0 \sigma_i + m_k 2 \delta_{ki} \sigma_0 = 2(m_0 \sigma_i + m_i \sigma_0) = 0$$

Since σ_0 and σ_i are linearly independent

$\Rightarrow m_0 = 0$ and $m_i = 0 \ \forall i$

$\Rightarrow \underline{M = 0}$ ✓

Exercise 14.4.3 (p. 396)

$|\psi(0)\rangle \leftrightarrow \begin{pmatrix}1\\0\end{pmatrix}$ $\qquad \vec{B} = B\left(\cos(\omega t)\vec{e_x} - \sin(\omega t)\vec{e_y}\right) + B_0\vec{e_z}$

$B \ll B_0$

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle = -\gamma\hat{\vec{S}}\cdot\vec{B}(t)|\psi(t)\rangle$$

Consider $|\psi_r(t)\rangle = e^{-i\omega t\hat{S}_z/\hbar}|\psi(t)\rangle$

$\Rightarrow |\psi(t)\rangle = e^{i\omega t\hat{S}_z/\hbar}|\psi_r(t)\rangle$

Plug into S. Eq:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = i\hbar\left(+\frac{i}{\hbar}\omega\hat{S}_z\right)e^{i\omega t\hat{S}_z/\hbar}|\psi_r(t)\rangle + e^{i\omega t\hat{S}_z/\hbar}i\hbar\frac{d}{dt}|\psi_r(t)\rangle$$

$$= -\gamma\left(\hat{S}_z B_0 + B\cos(\omega t)\hat{S}_x - B\sin(\omega t)\hat{S}_y\right)e^{i\omega t\hat{S}_z/\hbar}|\psi_r(t)\rangle$$

$$\Rightarrow i\hbar\frac{d}{dt}|\psi_r(t)\rangle = e^{-i\omega t\hat{S}_z/\hbar}\left[-(\omega-\gamma B_0)\hat{S}_z - \gamma B\cos(\omega t)\hat{S}_x + \gamma B\sin(\omega t)\hat{S}_y\right]\cdot e^{i\omega t\hat{S}_z/\hbar}|\psi_r(t)\rangle$$

$$= -(\omega-\gamma B_0)\hat{S}_z|\psi_r(t)\rangle - \gamma B\, e^{-i\omega t\hat{S}_z/\hbar}\left[\hat{S}_x\cos(\omega t) - \hat{S}_y\sin(\omega t)\right]\cdot e^{i\omega t\hat{S}_z/\hbar}|\psi_r(t)\rangle$$

Now from Eq. (14.3.44),

$$e^{-i\omega t\hat{S}_z/\hbar} = \cos\left(\frac{\omega t}{2}\right)\mathbb{1} - \frac{2i}{\hbar}\sin\left(\frac{\omega t}{2}\right)\hat{S}_z$$

$$\Rightarrow e^{-i\omega t S_z/\hbar}\hat{S}_x e^{+i\omega t S_z/\hbar} = \left(\cos\left(\frac{\omega t}{2}\right) - \frac{2i}{\hbar}\sin\left(\frac{\omega t}{2}\right)\hat{S}_z\right)\hat{S}_x\left(\cos\left(\frac{\omega t}{2}\right) + \frac{2i}{\hbar}\sin\left(\frac{\omega t}{2}\right)\hat{S}_z\right)$$

$$= \cos^2\left(\frac{\omega t}{2}\right)\hat{S}_x - \frac{2i}{\hbar}\sin\left(\frac{\omega t}{2}\right)\cdot\cos\left(\frac{\omega t}{2}\right)\underbrace{\left[\hat{S}_z,\hat{S}_x\right]}_{i\hbar\hat{S}_y} + \frac{4}{\hbar^2}\sin^2\left(\frac{\omega t}{2}\right)\hat{S}_z\hat{S}_x\hat{S}_z$$

Similarly

$$e^{-i\omega t\hat{S}_z/\hbar}\,\hat{S}_y\,e^{i\omega t\hat{S}_z/\hbar} = \cos^2\left(\frac{\omega t}{2}\right)\hat{S}_y - \frac{2i}{\hbar}\sin\left(\frac{\omega t}{2}\right)\overset{\cos\left(\frac{\omega t}{2}\right)}{}\underbrace{\left[\hat{S}_z, \hat{S}_y\right]}_{-i\hbar\hat{S}_x} + \frac{4}{\hbar^2}\sin^2\left(\frac{\omega t}{2}\right)\hat{S}_z\hat{S}_y\hat{S}_z$$

Now $\hat{S}_z\hat{S}_x\hat{S}_z = \hat{S}_z^2\hat{S}_x + \hat{S}_z\left[\hat{S}_x, \hat{S}_z\right] = \frac{\hbar^2}{4}\hat{S}_x - i\hbar\hat{S}_z\hat{S}_y$

$$= \frac{\hbar^2}{4}\hat{S}_x - \frac{1}{2}i\hbar\hat{S}_z\hat{S}_y + \frac{1}{2}i\hbar\hat{S}_y\hat{S}_z \qquad \text{since } \hat{S}_i\hat{S}_j = -\hat{S}_j\hat{S}_i \text{ for } i\neq j$$

$$= \frac{\hbar^2}{4}\hat{S}_x + \frac{i\hbar}{2}i\hbar\hat{S}_x$$

$$= -\frac{\hbar^2}{4}\hat{S}_x$$

and $\hat{S}_z\hat{S}_y\hat{S}_z = \frac{\hbar^2}{4}\hat{S}_y + \hat{S}_z\left[\hat{S}_y, \hat{S}_z\right] = \frac{\hbar^2}{4}\hat{S}_y + i\hbar\hat{S}_z\hat{S}_x = \frac{\hbar^2}{4}\hat{S}_y + \frac{1}{2}i\hbar\underbrace{\left[\hat{S}_z, \hat{S}_x\right]}_{+i\hbar\hat{S}_y}$

$$= -\frac{\hbar^2}{4}\hat{S}_y$$

So

$$e^{-i\omega t\hat{S}_z/\hbar}\left[\cos(\omega t)\hat{S}_x - \sin(\omega t)\hat{S}_y\right]e^{i\omega t\hat{S}_z/\hbar} =$$

$$= \cos(\omega t)\left[\cos^2\left(\frac{\omega t}{2}\right)\hat{S}_x + \sin(\omega t)\hat{S}_y - \sin^2\left(\frac{\omega t}{2}\right)\hat{S}_x\right]$$

$$-\sin(\omega t)\left[\cos^2\left(\frac{\omega t}{2}\right)\hat{S}_y - \sin(\omega t)\hat{S}_x - \sin^2\left(\frac{\omega t}{2}\right)\hat{S}_y\right]$$

$$= \cos^2(\omega t)\hat{S}_x + \cos(\omega t)\sin(\omega t)\hat{S}_y - \sin(\omega t)\cos(\omega t)\hat{S}_y + \sin^2(\omega t)\hat{S}_x$$

$$= \hat{S}_x \quad !$$

The Schrödinger equation for $|\psi_r(t)\rangle$ therefore reads

$$i\hbar\frac{d}{dt}|\psi_r(t)\rangle = \left(-(\omega-\gamma B_0)\hat{S}_z - \gamma B\hat{S}_x\right)|\psi_r(t)\rangle \equiv \hat{H}_r|\psi_r(t)\rangle$$

The effective Hamiltonian $\hat{H}_r$ controlling the time evolution of $|\psi_r(t)\rangle$ in the rotating frame is time independent! ✓

The time evolution operator in the rotating frame is

$$\hat{\mathcal{U}}_r(t) = e^{-i\hat{H}_r t/\hbar}$$

$$\Rightarrow |\psi_r(t)\rangle = e^{-i/\hbar \hat{H}_r t}|\psi_r(0)\rangle = e^{-i/\hbar \hat{H}_r t}|\psi(0)\rangle$$

and thus

$$|\psi(t)\rangle = e^{i\omega t \hat{S}_z/\hbar} e^{-i/\hbar \hat{H}_r t}|\psi(0)\rangle$$

We can write

$$\hat{H}_r = -\gamma \vec{B}_r \cdot \hat{\vec{S}} = \vec{\omega}_r \cdot \hat{\vec{S}} = \frac{1}{2}\hbar \vec{\omega}_r \cdot \hat{\vec{\sigma}}$$

where $\boxed{\vec{B}_r = (B_0 - \frac{\omega}{\gamma})\vec{e}_z + B\vec{e}_x}$ and $\vec{\omega}_r = -\gamma \vec{B}_r$

The magnitude of B_r is

$$B_r = \sqrt{B^2 + (B_0 - \tfrac{\omega}{\gamma})^2} \Rightarrow \omega_r = \gamma\sqrt{B^2 + (B_0 - \tfrac{\omega}{\gamma})^2}$$ (agrees with (14.4.30b))

$$\Rightarrow \boxed{|\psi(t)\rangle = e^{i\frac{\omega t}{2}\hat{\sigma}_z} e^{-i\frac{\omega_r t}{2}\tilde{n}\cdot\hat{\vec{\sigma}}}|\psi(0)\rangle} \quad \text{where } \tilde{n} \equiv \frac{\vec{\omega}_r}{\omega_r}$$

(14.3.44) $$\boxed{= e^{i\frac{\omega t}{2}\hat{\sigma}_z}\left(\cos\left(\frac{\omega_r t}{2}\right)\hat{\mathbb{1}} - i\sin\left(\frac{\omega_r t}{2}\right)\tilde{n}\cdot\hat{\sigma}\right)|\psi(0)\rangle}$$

Representing this in the $\hat{S}_z$-eigenbasis and using $|\psi(0)\rangle \longrightarrow \begin{pmatrix}1\\0\end{pmatrix}$ in that basis, we get

$$\psi(t) = \begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2}\end{pmatrix}\left[\cos\left(\frac{\omega_r t}{2}\right)\begin{pmatrix}1&0\\0&1\end{pmatrix} - i\sin\left(\frac{\omega_r t}{2}\right)\left(\tilde{n}_x\begin{pmatrix}0&1\\1&0\end{pmatrix} + \tilde{n}_z\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)\right]\begin{pmatrix}1\\0\end{pmatrix}$$

where $\tilde{n}_x = \frac{\omega_{r,x}}{\omega_r} = \frac{-B}{\sqrt{B^2+(B_0-\frac{\omega}{\gamma})^2}} = \frac{-\gamma B}{\omega_r}$; $\tilde{n}_z = \frac{\omega_{r,z}}{\omega_r} = -\frac{B_0-\frac{\omega}{\gamma}}{\sqrt{B^2+(B_0-\frac{\omega}{\gamma})^2}} = \frac{-\gamma B_0+\omega}{\omega_r}$

Work out the matrix multiplication:

$$\psi(t) = \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix} \begin{pmatrix} \cos(\frac{\omega_r t}{2}) - i\sin(\frac{\omega_r t}{2})\tilde{n}_z & -i\sin(\frac{\omega_r t}{2}) n_x \\ -i\sin(\frac{\omega_r t}{2})\tilde{n}_x & \cos(\frac{\omega_r t}{2}) + i\sin(\frac{\omega_r t}{2}) n_z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} [\cos(\frac{\omega_r t}{2}) - i\tilde{n}_z \sin(\frac{\omega_r t}{2})] e^{i\omega t/2} \\ -i\tilde{n}_x \sin(\frac{\omega_r t}{2}) e^{-i\omega t/2} \end{pmatrix}$$

Writing $\gamma B_0 = \omega_0$ and $\tilde{n}_x = -\gamma B/\omega_r$, $\tilde{n}_z = \frac{\omega_0 - \omega}{\omega_r}$ this gives

the desired result (14.4.36):

$$\boxed{\psi(t) = \begin{pmatrix} [\cos(\frac{\omega_r t}{2}) + i\,\frac{\omega_0-\omega}{\omega_r} \sin(\frac{\omega_r t}{2})] e^{i\omega t/2} \\ +i\,\frac{\gamma B}{\omega_r} \sin(\frac{\omega_r t}{2})\, e^{-i\omega t/2} \end{pmatrix}} \checkmark$$

For $\omega = \omega_0$ $(\gamma B_0 = \omega)$, $\omega_r = \gamma B$, and

$$\psi(t) = \begin{pmatrix} \cos(\frac{\omega_r t}{2})\, e^{i\omega_0 t/2} \\ i\sin(\frac{\omega_r t}{2})\, e^{-i\omega_0 t/2} \end{pmatrix} = \begin{pmatrix} \cos(\frac{\omega_r t}{2}) e^{i\omega_0 t/2} \\ \sin(\frac{\omega_r t}{2})\, e^{-i\omega_0 t/2} e^{i\pi/2} \end{pmatrix}$$

$$= e^{i\pi/4} \begin{pmatrix} \cos(\frac{\omega_r t}{2})\, e^{i(\frac{\omega_0 t}{2} - \frac{\pi}{4})} \\ \sin(\frac{\omega_r t}{2})\, e^{-i(\frac{\omega_0 t}{2} - \frac{\pi}{4})} \end{pmatrix} \longleftrightarrow |\vec{n},+\rangle \qquad (14.3.28a)$$

with $\vec{n} = (\sin\vartheta(t)\cos\varphi(t), \sin\vartheta(t)\sin\varphi(t), \cos\vartheta(t))$

where $\boxed{\vartheta(t) = \frac{\omega_r t}{2} \text{ and } \varphi(t) = -\frac{\omega_0 t}{2} + \frac{\pi}{4}}$

In this case, $\tilde{n}_z = 0$, so in the rotating frame $\langle\hat{\vec{S}}\rangle$ precesses in the y-z-plane.

Finally, we compute $\langle\hat{\mu}_z\rangle(t)$ in this state:

$$\langle\hat{\mu}_z\rangle(t) = \gamma \langle\hat{S}_z\rangle(t) = \frac{\gamma\hbar}{2}\langle\sigma_z\rangle(t)$$

$$= \frac{\gamma\hbar}{2}\langle\psi(t)|\,\hat{\sigma}_z\,|\psi(t)\rangle$$

$$= \underbrace{\frac{\gamma\hbar}{2}}_{\langle\hat{\mu}_z\rangle(0)}\left(\left[\cos\left(\frac{\omega_r t}{2}\right) - i\frac{\omega_0-\omega}{\omega_r}\sin\left(\frac{\omega_r t}{2}\right)\right]e^{-i\omega t/2},\ -i\frac{\gamma B}{\omega_r}\sin\left(\frac{\omega_r t}{2}\right)e^{i\omega t/2}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\cdot\begin{pmatrix}\left[\cos\left(\frac{\omega_r t}{2}\right) + i\frac{\omega_0-\omega}{\omega_r}\sin\left(\frac{\omega_r t}{2}\right)\right]e^{i\omega t/2}\\ i\frac{\gamma B}{\omega_r}\sin\left(\frac{\omega t}{2}\right)e^{-i\omega t/2}\end{pmatrix}$$

$$= \langle\hat{\mu}_z\rangle(0)\cdot\left\{\cos^2\left(\frac{\omega_r t}{2}\right) + \frac{(\omega_0-\omega)^2}{\omega_r^2}\sin^2\left(\frac{\omega_r t}{2}\right) - \frac{\gamma^2B^2}{\omega_r^2}\sin^2\left(\frac{\omega_r t}{2}\right)\right\}$$

$$= \frac{\langle\hat{\mu}_z\rangle(0)}{\omega_r^2}\left\{\left(\underbrace{\gamma^2B^2+(\omega_0-\omega)^2}_{\omega_r^2}\right)\cos^2\left(\frac{\omega_r t}{2}\right) + \left((\omega_0-\omega)^2-\gamma^2B^2\right)\sin^2\left(\frac{\omega_r t}{2}\right)\right\}$$

$$= \frac{\langle\hat{\mu}_z\rangle(0)}{\omega_r^2}\left\{(\omega_0-\omega)^2 + \gamma^2B^2\cos(\omega_r t)\right\}$$

$$= \langle\mu_z\rangle(0)\left\{\frac{(\omega_0-\omega)^2}{(\omega_0-\omega)^2+\gamma^2B^2} + \frac{\gamma^2B^2}{(\omega_0-\omega)^2+\gamma^2B^2}\cos(\omega_r t)\right\} \checkmark \qquad (14.4.31)$$

Exercise 14.5.1 (p. 398)

(1) Including both the electron's and proton's magnetic moments in the interaction with the external $\vec{B}$ field we would get

$$H_{int} = -\vec{\mu}_e^{\,orb.} \cdot \vec{B} - \vec{\mu}_e^{\,intr.} \cdot \vec{B} - \vec{\mu}_p^{\,orb} \cdot \vec{B} - \vec{\mu}_p^{\,intr.} \cdot \vec{B}$$

$$= \left(\frac{e}{2m_e c} L_z^e + \frac{e}{m_e c} S_z^e - \frac{e}{2Mc} L_z^p - \frac{g_p \cdot e}{2Mc} S_z^p \right) B$$

where $g_p = 5.6$. The terms involving L_z^p and S_z^p are multiplied by coefficients that are a factor $\frac{m_e}{M}$ smaller than the electron terms.

(2) Proton and electron orbit around their common center of mass. Due to its larger mass, the proton's distance from the center of mass is a factor $\frac{m_e}{M}$ smaller than that of the electron. There orbital velocities thus differ by the same factor (both have to complete an orbit around their ctr in the same time), which means that there orbital momenta $m_e v_e$ and $M v_p$ are the same. Hence their orbital angular momenta $r_e p_e$ and $r_p p_p$ differ by a factor $\frac{r_p}{r_e} = \frac{m}{M}$.

So $L_z^p \sim \frac{m_e}{M} L_z^e$ and $\boxed{\vec{\mu}_p^{\,orb} \sim \left(\frac{m_e}{M}\right)^2 \vec{\mu}_e^{\,orb}}$

Exercise 14.5.2 (p. 399)

(1) We showed in Eq. (14.5.6)

$$E_{n=1} = -R_y \pm \underbrace{\frac{e\hbar}{2mc}}_{\text{Bohr magneton}} B = -13.6\,eV \pm \underbrace{0.6\times10^{-8}\frac{eV}{G}}_{\text{Bohr magneton}} \cdot 10^6 G$$

for $B = 1000\,kG = 1\,MG$

$$= (-13.6 \pm 0.006)\,eV$$

$$\Rightarrow \frac{\Delta E}{E} \approx \frac{0.012}{13.6} \approx \frac{1.2\times10^{-2}}{1.36\times10} \simeq 9\times10^{-4}$$

(2) We kept the interaction term $-\frac{q}{2mc}(2\vec{A}\cdot\hat{\vec{P}}) = -\frac{q}{2mc}\hat{\vec{L}}\cdot\vec{B}$ but ignored the term $\frac{q^2}{c^2}\frac{\vec{A}^2}{2m} = \frac{q^2}{2mc^2}\frac{B^2}{4}(x^2+y^2)$.

The $n=1$ state has a spherically symmetric wavefunction, so $\langle\psi_{100}|\hat{X}^2+\hat{Y}^2|\psi_{100}\rangle = \frac{2}{3}\langle r^2\rangle_{n=1} \sim \frac{2}{3}a_0^2$.

Taking for $\langle\vec{\mu}\cdot\vec{B}\rangle_{n=1}$ the result from Eq. (14.5.6), $\langle\vec{\mu}\cdot\vec{B}\rangle \sim \frac{e\hbar}{2mc}B$ (this is entirely from the spin interacting with $\vec{B}$, not from the $\hat{\vec{L}}\cdot\vec{B}$-term above, since in the $n=1$ state $l=0$), we can evaluate as the figure of merit the ratio

$$R \equiv \frac{\frac{e^2B^2}{8mc^2}\frac{2}{3}a_0^2}{\frac{e\hbar}{2mc}B}.$$

The approximation of ignoring the B^2-term becomes bad when $R \gtrsim \frac{1}{2}$: $\frac{eBa_0^2}{6\hbar c} \gtrsim \frac{1}{2} \Rightarrow B \gtrsim \frac{3\hbar c}{e a_0^2} =$

$$\Rightarrow B \gtrsim \frac{3\hbar c\, m^2e^4}{e\hbar^4} = \underbrace{\frac{2mc}{e\hbar}}_{1/\mu_B \text{ (Bohr magneton)}} \cdot \underbrace{\frac{3me^4}{2\hbar^2}}_{3R_y} = \frac{3R_y}{0.6\times10^{-8}\,eV/G} = \frac{3\times13.6}{6\times10^{-9}}G = 6.8\times10^9 G$$

$$= 6.8\,GG \text{ (Giga-Gauss)}$$

Exercise 15.1.2 (p. 407)

(1) $\hat{H}_{hf} = A\,\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$ $\qquad \hat{\vec{S}}_1$ = electron spin

$\hat{\vec{S}}_2$ = proton spin

$$\hat{H} = \hat{H}_{Coulomb} + \hat{H}_{hf} = \hat{H}_{Coulomb} + \frac{A}{2}\left((\hat{\vec{S}}_1+\hat{\vec{S}}_2)^2 - \hat{S}_1^2 - \hat{S}_2^2\right)$$

$$= \hat{H}_{Coulomb} + \frac{A}{2}\left(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2\right)$$

This Hamiltonian is diagonalized by states of the form

$|nlm\rangle \otimes |sm; s_1 s_2\rangle$

i.e. by using the total-s basis in the spin sector.

Here $s_1 = s_2 = \frac{1}{2}$, and s can be either 1 or 0.

The eigenvalues of $\hat{H}_{hf} \sim \hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2$ do not depend on m, only on s, s_1, s_2.

We find

$$\underbrace{\hat{H}|100\rangle\otimes|1m;\tfrac{1}{2}\tfrac{1}{2}\rangle}_{} = \left(-R_y|100\rangle\right)\otimes|1m;\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$+|100\rangle\otimes\left(\frac{A}{2}\left(2\hbar^2 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2\right)\right)|1m;\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$= \left(-R_y + \frac{A\hbar^2}{4}\right)|100\rangle\otimes|1m;\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$= E_+\,|100\rangle\otimes|1m;\tfrac{1}{2}\tfrac{1}{2}\rangle$$

in the triplet (i.e. total-spin 1) states

and

$$\hat{H}|100\rangle\otimes|00;\tfrac{1}{2}\tfrac{1}{2}\rangle = \left(-R_y|100\rangle\right)\otimes|00;\tfrac{1}{2}\tfrac{1}{2}\rangle + |100\rangle\otimes\left(\frac{A}{2}\left(0-\frac{3}{4}\hbar^2-\frac{3}{4}\hbar^2\right)\right)|00;\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$= \left(-R_y - \frac{3A\hbar^2}{4}\right)|100\rangle\otimes|00;\tfrac{1}{2}\tfrac{1}{2}\rangle = E_-\,|100\rangle\otimes|00;\tfrac{1}{2}\tfrac{1}{2}\rangle$$

in the singlet (total-spin 0) state

(2) $\Delta E = E_{-} - E = A\hbar^2$

We must estimate A.

$$\text{Assume} \quad \mathcal{H}_{hf} \cong \left|\frac{\vec{\mu}_e \cdot \vec{\mu}_p}{a_0^3}\right| = \left|\left(\frac{-\overset{g_e}{2}e\hbar\vec{\sigma}_e}{4mc}\right)\left(\frac{+e\hbar}{4Mc} \times \overset{g_p}{5.6}\,\vec{\sigma}_p\right)\right|\frac{1}{a_0^3}$$

$$= \frac{2e}{2mc}\vec{S}_e \cdot \frac{5.6e}{2Mc}\vec{S}_p\frac{1}{a_0^3} \equiv A\,\vec{S}_1\cdot\vec{S}_2$$

$$\Rightarrow A\hbar^2 = \frac{1}{a_0^3}\,\frac{2e}{2mc}\,\frac{5.6e}{2Mc}\hbar^2 = \frac{11.2}{4}\,\frac{m}{M}\,\frac{e^2}{m^2c^2}\,\frac{m^3e^6}{\hbar^6}\hbar^2$$

$$= 5.6\,\frac{m}{M}\cdot\frac{me^4}{2\hbar^2}\,\underbrace{\frac{m^2e^4}{m^2c^2\hbar^2}}_{\alpha^2} = 5.6\,\frac{m}{M}\alpha^2 R_y$$

$$\Rightarrow \boxed{\frac{\Delta E}{E} \sim 5.6\,\frac{m}{M}\alpha^2} \quad \text{of order } \alpha^2\frac{m}{M} \text{ as stated} \checkmark$$

$\Delta E = \hbar\omega$

$$\lambda = \frac{2\pi c}{\omega} = \frac{2\pi\hbar c}{\hbar\omega} = \frac{2\pi\hbar c}{\Delta E} = \frac{2\pi\hbar c}{5.6\frac{m}{M}\alpha^2 R_y} = \frac{2\pi\cdot 200\times10^6\,\text{eV}\times10^{-15}\,\text{m}}{\underbrace{5.6\ \tfrac{1}{1836}\cdot\tfrac{1}{137^2}\cdot 13.6\,\text{eV}}_{\Delta E = 2.2\times10^{-6}\,\text{eV}}}$$

$= 0.56$ m

So the wavelength corresponding to this hyperfine transition is a few tens of centimeters. ✓

$$(3)\quad \frac{P(\text{triplet})}{P(\text{singlet})} = \frac{e^{-E_+/kT}}{e^{-E_-/kT}} = e^{-\Delta E/kT} = e^{-2.2\times10^{-6}\,\text{eV}/(1/40\,\text{eV})}$$

$$= 1 - 9\times10^{-5} \simeq 1 - 10^{-4}$$

The probability is very close to unity at room temperature.

Exercise 15.2.2 (p. 413)

(1) $1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}$

$\langle 1 m_1, \frac{1}{2} m_2 | \frac{3}{2} m \rangle = ?$ $\quad\quad \langle 1 m_1, \frac{1}{2} m_2 | \frac{1}{2} m \rangle = ?$

$\langle 11, \frac{1}{2}\frac{1}{2} | \frac{3}{2}\frac{3}{2} \rangle = 1$

$\left.\begin{array}{l} \langle 11, \frac{1}{2} -\frac{1}{2} | \frac{3}{2}\frac{1}{2} \rangle = \sqrt{\frac{1/2}{3/2}} = \sqrt{\frac{1}{3}} \\ \langle 10, \frac{1}{2}\frac{1}{2} | \frac{3}{2}\frac{1}{2} \rangle = \sqrt{\frac{1}{3/2}} = \sqrt{\frac{2}{3}} \end{array}\right\}$ (15.2.7), negative m: use (15.2.11)

$\left.\begin{array}{l} \langle 11, \frac{1}{2} -\frac{1}{2} | \frac{1}{2}\frac{1}{2} \rangle = \sqrt{\frac{1}{3/2}} = \sqrt{\frac{2}{3}} \\ \langle 10, \frac{1}{2}\frac{1}{2} | \frac{1}{2}\frac{1}{2} \rangle = -\sqrt{\frac{1/2}{3/2}} = -\sqrt{\frac{1}{3}} \end{array}\right\}$ (15.2.8) negative m: use (15.2.11)

(2) $1 \otimes 1 = 2 \oplus 1 \oplus 0$

$\langle 1 m_1, 1 m_2 | 2 m \rangle = ?$ $\quad \langle 1 m_1, 1 m_2 | 1 m \rangle = ?$ $\quad \langle 1 m_1, 1 m_2 | 00 \rangle$?

$\langle 11, 11 | 22 \rangle = 1$

$\langle 11, 10 | 21 \rangle = \sqrt{\frac{1}{2}} = \langle 10, 11 | 21 \rangle$ (15.2.7)

$|jm\rangle = |21\rangle = \frac{1}{\sqrt{2}}(|1 0\rangle + |0,1\rangle) \equiv \frac{1}{\sqrt{2}}(|m_1 = 1, m_2 = 0\rangle + |m_1 = 0, m_2 = 1\rangle)$

$|11\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$ (15.2.8)

$|10\rangle =$ (apply $\hat{J}_- = \hat{J}_{1-} + \hat{J}_{2-}$ on $|jm\rangle = |11\rangle$.)

⋮

(tedious)

Exercise 15.2.5 (p. 415)

(1) $\hat{\mathbb{P}}_1 \equiv \frac{3}{4}\hat{\mathbb{1}} + \frac{\hat{\vec{S}}_1\cdot\hat{\vec{S}}_2}{\hbar^2}$ $\qquad$ $\hat{\mathbb{P}}_0 \equiv \frac{1}{4}\hat{\mathbb{1}} - \frac{\hat{\vec{S}}_1\cdot\hat{\vec{S}}_2}{\hbar^2}$ $\quad$ definitions

$\hat{\vec{S}}_i = \frac{\hbar}{2}\hat{\vec{\sigma}}_i$

$$\hat{\mathbb{P}}_1^2 = \frac{1}{4}\left(3\hat{\mathbb{1}} + \hat{\vec{\sigma}}_1\cdot\hat{\vec{\sigma}}_2\right)\cdot\frac{1}{4}\left(3\cdot\hat{\mathbb{1}} + \hat{\vec{\sigma}}_1\cdot\hat{\vec{\sigma}}_2\right)$$
$$= \frac{1}{16}\left(9\cdot\hat{\mathbb{1}} + 6\,\hat{\vec{\sigma}}_1\cdot\hat{\vec{\sigma}}_2 + (\hat{\vec{\sigma}}_1\cdot\hat{\vec{\sigma}}_2)^2\right)$$

$$(\hat{\vec{\sigma}}_1\cdot\hat{\vec{\sigma}}_2)^2 = \hat{\sigma}_{1i}\,\hat{\sigma}_{2i}\,\hat{\sigma}_{1j}\,\hat{\sigma}_{2j} \qquad [\hat{\sigma}_{1i}, \hat{\sigma}_{2j}] = 0 \quad \forall i,j$$

(act on different Hilbert spaces)

$$= \hat{\sigma}_{1i}\,\hat{\sigma}_{1j}\,\hat{\sigma}_{2i}\,\hat{\sigma}_{2j}$$
$$= \sum_{i,j=1}^{3}(\vec{e}_i\cdot\hat{\vec{\sigma}}_1)(\vec{e}_j\cdot\hat{\vec{\sigma}}_1)\otimes(\vec{e}_i\cdot\hat{\vec{\sigma}}_2)(\vec{e}_j\cdot\hat{\vec{\sigma}}_2)$$
$$= \sum_{i,j=1}^{3}\left(\underbrace{(\vec{e}_i\cdot\vec{e}_j)}_{\delta_{ij}}\hat{\mathbb{1}}_1 + i\underbrace{(\vec{e}_i\times\vec{e}_j)\cdot\hat{\vec{\sigma}}_1}_{\varepsilon_{ijk}\sigma_{1k}}\right)\otimes\left(\delta_{ij}\hat{\mathbb{1}}_2 + i\,\varepsilon_{ijk}\hat{\sigma}_{2k}\right)$$
$$= 3\cdot\hat{\mathbb{1}}_{1\otimes 2} + 0 + 0 - \hat{\sigma}_{1k}\,\hat{\sigma}_{2k'}\underbrace{\varepsilon_{ijk}\varepsilon_{ijk'}}_{2\delta_{kk'}}$$
$$= 3\cdot\hat{\mathbb{1}} - 2\,\hat{\vec{\sigma}}_1\cdot\hat{\vec{\sigma}}_2$$

$$\Rightarrow \underline{\hat{\mathbb{P}}_1^2} = \left(\frac{12}{16}\hat{\mathbb{1}} + \frac{4}{16}\hat{\vec{\sigma}}_1\cdot\hat{\vec{\sigma}}_2\right) = \frac{3}{4}\hat{\mathbb{1}} + \frac{1}{4}\hat{\vec{\sigma}}_1\cdot\hat{\vec{\sigma}}_2 = \underline{\hat{\mathbb{P}}_1}\ \checkmark$$

$$\underline{\hat{\mathbb{P}}_2^2} = \frac{1}{16}\left(\hat{\mathbb{1}} - \hat{\vec{\sigma}}_1\cdot\hat{\vec{\sigma}}_2\right)^2 = \frac{1}{16}\left(\hat{\mathbb{1}} - 2\,\hat{\vec{\sigma}}_1\cdot\hat{\vec{\sigma}}_2 + 3\cdot\hat{\mathbb{1}} - 2\,\hat{\vec{\sigma}}_1\cdot\hat{\vec{\sigma}}_2\right)$$
$$= \left(\frac{4}{16}\hat{\mathbb{1}} - \frac{4}{16}\hat{\vec{\sigma}}_1\cdot\hat{\vec{\sigma}}_2\right) = \frac{1}{4}\hat{\mathbb{1}} - \frac{1}{4}\hat{\vec{\sigma}}_1\cdot\hat{\vec{\sigma}}_2 = \underline{\hat{\mathbb{P}}_2}\ \checkmark$$

$$\underline{\hat{\mathbb{P}}_1\cdot\hat{\mathbb{P}}_2} = \frac{1}{16}\left(3\cdot\hat{\mathbb{1}} + \hat{\vec{\sigma}}_1\cdot\hat{\vec{\sigma}}_2\right)\left(\hat{\mathbb{1}} - \hat{\vec{\sigma}}_1\cdot\hat{\vec{\sigma}}_2\right) = \frac{1}{16}\left(3\cdot\hat{\mathbb{1}} - 2\,\hat{\vec{\sigma}}_1\cdot\hat{\vec{\sigma}}_2 - (\hat{\vec{\sigma}}_1\cdot\hat{\vec{\sigma}}_2)^2\right)$$
$$\underline{= 0 = \hat{\mathbb{P}}_2\cdot\hat{\mathbb{P}}_1}$$

(2) The $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$ vector space is spanned by the 4 basis states

$$\frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) \in \mathbb{V}_0 \quad (s=0)$$

$$\left\{ |++\rangle, \tfrac{1}{2}(|+-\rangle + |-+\rangle), |--\rangle \right\} \in \mathbb{V}_1 \quad (s=1)$$

$$\hat{\mathbb{P}}_0\left(\frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)\right) = \left(\frac{1}{4}\hat{\mathbb{1}} - \frac{\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2}{\hbar^2}\right)\frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

$$= \left(\frac{1}{4}\hat{\mathbb{1}} - \frac{1}{2\hbar^2}\left(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2\right)\right)\frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{4} - \frac{1}{2\hbar^2}\left(0 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2\right)\right)\left(|+-\rangle - |-+\rangle\right) = \left(\frac{1}{4} + \frac{3}{4}\right)\frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

$$\hat{\mathbb{P}}_0 |++\rangle = \left(\frac{1}{4} - \frac{1}{2\hbar^2}\left(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2\right)\right)|++\rangle = \left(\frac{1}{4} - \frac{1}{2\hbar^2}\left(2\hbar^2 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2\right)\right)|++\rangle$$

$$= \left(\frac{1}{4} - \frac{1}{4}\right)|++\rangle = 0$$

$$\hat{\mathbb{P}}_0 |--\rangle = \left(\frac{1}{4} - \frac{1}{2\hbar^2}\left(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2\right)\right)|--\rangle = \left(\frac{1}{4} - \frac{1}{2\hbar^2}\left(2\hbar^2 - \frac{3}{4}\hbar^2 \cdot 2\right)\right)|--\rangle$$

$$= 0$$

$$\hat{\mathbb{P}}_0 \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right) = \cdots = 0$$

(all 3 states have same $\hat{S}^2 = 2\hbar^2$, $\hat{S}_1^2 = \hat{S}_2^2 = \frac{3}{4}\hbar^2$)

This proves that $\hat{\mathbb{P}}_0$ projects on $\mathbb{V}_0$. ✓

Now $\hat{\mathbb{P}}_1$:

$$\hat{\mathbb{P}}_1 \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) = \left(\frac{3}{4}\hat{\mathbb{1}} + \frac{1}{2\hbar^2}\left(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2\right)\right)\frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

$$= \left(\frac{3}{4} + \frac{1}{2\hbar^2}\left(0 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2\right)\right)\frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) = 0$$

The 3 $s=1$ states have $\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2 = \frac{1}{2}\hbar^2$, hence $\frac{3}{4}\hat{\mathbb{1}} + \frac{1}{2\hbar^2}\left(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2\right) = \hat{\mathbb{1}}$

$\Rightarrow$ $\hat{\mathbb{P}}_1$ acts like $\hat{\mathbb{1}}$ on the basis states of $\mathbb{V}_1$ ✓ $\Rightarrow \hat{\mathbb{P}}_1$ projects on $\mathbb{V}_1$ ✓