Electrodynamics - Problem Set-1.

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1: (a) Show that the proper-time ($d\tau$) is a scalar under Lorentz transformations!
Given $d\tau^2 = dt^2 - dx^2$ and $dx'^\alpha = \Lambda^\alpha{}_\beta dx^\beta$; Show that

$$\boxed{d\tau'^2 = d\tau^2}$$ - Eq: 2.1.5 of [SW]

(b) Show that $\Lambda^0{}_0 \geq 1$ & det $\Lambda = +1$ Eq: 2.1.9 of [SW]

(c) Show that $\Lambda^0{}_0 = \gamma$; $\Lambda^i{}_0 = \gamma v_i$ (Lorentz Transformation matrix element)
Eq: [2.1.17 + 2.1.18 + 2.1.19]

2. (a) The 4-momentum is given as ; $p^\alpha = m\frac{dx^\alpha}{d\tau}$ & $f^\alpha = \frac{dp^\alpha}{d\tau}$ [eq: 2.4.1 + 2.4.2]
starting with these and using (1a) above; Show that the non-relativistic limit.

$$\left.\begin{aligned} \vec{p} &= mv + \theta(v^3) \\ E &= m + \tfrac{1}{2}mv^2 + \theta(v^4) \end{aligned}\right\} \text{ in } c=1 \text{ - Natural units.}$$

(b) Prove that the massless particle (ν/γ - neutrino/photon) velocity $\equiv c$
(velocity of light $\equiv 1$ in natural unit).

(c) Show that for system of particle 'E' must be conserved if '$\vec{p}$' is conserved.
[SW → page-34 : 2nd paragraph].

3. (a) Prove that $\Lambda_\alpha{}^\gamma \Lambda^\alpha{}_\beta = \delta^\gamma{}_\beta$ (Eq: 2.5.7 - of SW) [

(b) Show $U'_\alpha V'^\alpha = U_\beta V^\beta$ [Lorentz Invariant ⇒ 4-vector dot (·) product]

(c) Show $\Box^2 = \nabla^2 - \frac{\partial^2}{\partial t^2}$ - Eq: (2.5.12)

(d) Show that $T^{\alpha\gamma} = T^\alpha{}_\beta{}^{\gamma\beta}$ (Contraction of Tensor - SW-page-37)

(e) Explain why the dT tells us immediately that '$\eta_{\alpha\beta}$' is a covariant tensor.

(f) Prove that $\boxed{\epsilon_{\alpha\beta\gamma\delta} = -\epsilon^{\alpha\beta\gamma\delta}}$ - eq: 2.5.15 of [SW]

4. (a) Given $f^\alpha = e\eta_{\beta\gamma}F^{\alpha\beta}\frac{dx^\gamma}{d\tau} = eF^\alpha{}_\gamma\frac{dx^\gamma}{d\tau} = eF^\alpha{}_\gamma u^\gamma$ (Eq: 2.7.9).
leads to $\frac{dp}{dt} = e[E + \vec{v}\times\vec{B}]$; where $\left\{\begin{array}{l} F^{\alpha\beta} = -F^{\beta\alpha} \quad F^{ij} = B_k \\ F^{0i} = E_i \\ \text{Eq: 2.7.5} \end{array}\right.$

5 (a) Show that $\partial_\mu A^\mu$ is a Lorentz Scalar! [for e.g., FM-page-128]

(b) Given $\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi$ and $\vec{B} = \vec{\nabla}\times\vec{A}$ [FM-pg-135]

Show that $E^i = -(\partial^0 A^i - \partial^i A^0)$ and.

$B_i = \varepsilon_{ijk}\partial^k A^j$

(c) Given $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$ with $F^{\alpha\beta} = \begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{pmatrix}$ - 5.115

Prove that $\boxed{\partial_\alpha F^{\alpha\beta} = \frac{4\pi J^\beta}{c}}$ $\xrightarrow{\text{leads to}}$ $\boxed{\begin{array}{l}\vec{\nabla}\cdot\vec{E} = 4\pi\rho \\ \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}J\end{array}}$

(d) $\boxed{\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0}$ $\xrightarrow{\text{leads to}}$ $\boxed{\begin{array}{l}\vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial B}{\partial t} = 0\end{array}}$

(e) Given $F'^{\alpha\beta} = \Lambda^\alpha{}_\gamma \Lambda^\beta{}_\mu F^{\gamma\mu}$ and for the Lorentz transformation in 1-D, say along Z-axis; write the (E', B')-component in term of (E, B) with boost factor!

[Get eq: 5.131 & 5.132 of FM].

6 Given $\Phi^\alpha = -\partial_\beta T^{\alpha\beta}_{em}$ & $4\pi T^{\alpha\beta}_{em} = F^\alpha{}_\mu F^{\mu\beta} + \frac{g^{\alpha\beta}F^{\lambda\mu}F_{\lambda\mu}}{4}$

and $\partial_\beta T^{\alpha\beta}_{em} = 0 \Rightarrow$

Show that (a) $T^{00}_{em} = \frac{1}{8\pi}(E^2+B^2)$

(b) $T^{0i}_{em} = \frac{1}{4\pi}(\vec{E}\times\vec{B})_i$

(c) $\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0$; $u = (E^2+B^2)/8\pi$; $S = \frac{c}{4\pi}(\vec{E}\times\vec{B})$.

(d) $\frac{\partial g^i}{\partial t} + \vec{\nabla}\cdot\vec{T}^i_{em} = 0$; $\vec{g} = \frac{1}{4\pi c}(\vec{E}\times\vec{B}) = \frac{\vec{S}}{c^2}$