Tarea 1 : Punto 3.

$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu$; $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

Euler - Lagrange: $\underbrace{\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\nu)}\right)}_{\mathrm{I}} - \underbrace{\frac{\partial \mathcal{L}}{\partial A^\nu}}_{\mathrm{II}} = 0$

$$F_{\mu\nu}F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$
$$= \partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu\partial^\nu A^\mu - \underbrace{\partial_\nu A_\mu \partial^\mu A^\nu}_{\nu\leftrightarrow\mu} + \underbrace{\partial_\nu A_\mu \partial^\nu A^\mu}_{\nu\leftrightarrow\mu}$$
$$= 2(\underbrace{\partial_\mu A_\nu \partial^\mu A^\nu}_{A} - \underbrace{\partial_\mu A_\nu \partial^\nu A^\mu}_{B})$$

A: $\frac{\partial}{\partial(\partial_\lambda A^\rho)}(\partial_\mu A_\nu \partial^\mu A^\nu) = \partial_\mu A_\nu \frac{\partial(\partial^\mu A^\nu)}{\partial(\partial_\lambda A^\rho)} + \frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\lambda A^\rho)}\partial^\mu A^\nu$

$$= \partial_\mu A_\nu\, g^{\mu\beta}\delta^\beta_\lambda \delta^\nu_\rho + \partial^\mu A^\nu g_{\nu\alpha}\delta^\alpha_\rho \delta^\mu_\lambda$$
$$= \partial^\beta A_\nu \delta^\beta_\lambda\delta^\nu_\rho + \partial^\mu A_\alpha \delta^\alpha_\rho \delta^\mu_\lambda = \partial^\lambda A_\rho + \partial^\lambda A_\rho = \boxed{2\partial^\lambda A_\rho}$$

B: $\frac{\partial}{\partial(\partial_\lambda A^\rho)}(\partial_\mu A_\nu \partial^\nu A^\mu) = \partial_\mu A_\nu \frac{\partial(\partial^\nu A^\mu)}{\partial(\partial_\lambda A^\rho)} + \frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\lambda A^\rho)}\partial^\nu A^\mu$

$$= \partial_\mu A_\nu\, g^{\nu\beta}\delta^\beta_\lambda\delta^\mu_\rho + \partial^\nu A^\mu g_{\nu\alpha}\delta^\alpha_\rho\delta^\mu_\lambda$$
$$= \partial_\mu A^\beta \delta^\beta_\lambda \delta^\mu_\rho + \partial_\alpha A^\mu \delta^\alpha_\rho \delta^\mu_\lambda = \partial_\rho A^\lambda + \partial_\rho A^\lambda = \boxed{2\partial_\rho A^\lambda}$$

I: $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\nu)}\right) = \partial_\mu\left(\frac{\partial[2(A-B)]}{\partial(\partial_\mu A^\nu)}\right)\cdot\left(\frac{-1}{4}\right) = -\frac{1}{2}\partial_\mu(2\partial^\mu A_\nu - 2\partial_\nu A^\mu)$

$$= \boxed{\partial_\mu(\partial_\nu A^\mu - \partial^\mu A_\nu)}$$

Para la segunda parte, solo se tiene en cuenta el segundo término de $\mathcal{L}$:

II: $\frac{\partial\mathcal{L}}{\partial A^\nu} = \frac{\partial}{\partial A^\nu}\left(\frac{m^2}{2}A_\mu A^\mu\right) = \frac{m^2}{2}\left(A_\mu\frac{\partial A^\mu}{\partial A^\nu} + A^\mu \frac{\partial A_\mu}{\partial A^\nu}\right)$

$$= \frac{m^2}{2}(A_\mu\delta^\mu_\nu + A^\mu g_{\mu\lambda}\delta^\lambda_\nu) = \frac{m^2}{2}(A_\nu + A_\nu) = \boxed{m^2 A_\nu}$$

Euler - Lagrange: $\partial_\mu(\partial_\nu A^\mu - \partial^\mu A_\nu) = m^2 A_\nu$

$$\partial_\mu\partial_\nu A^\mu = (\Box + m^2)A_\nu$$

$$\partial_\mu \partial_\nu A^\mu = (\Box + m^2) A_\nu$$

Aplico $(g^{\mu\nu})^3$ a ambos lados. Notar que $(g^{\mu\nu})^3 = g^{\mu\nu}$, dado que la inversa de $g^{\mu\nu}$ es $g^{\mu\nu}$. Además, notar $g^{\mu\nu} = g_{\mu\nu}$

$$\Rightarrow \quad g^{\mu\nu} \partial_\mu g^{\mu\nu} \partial_\nu g^{\mu\nu} A^\mu = (g^{\mu\nu})^3 (\Box + m^2) A_\nu$$

$$\partial^\nu \partial^\mu g_{\mu\nu} A^\mu = g^{\mu\nu} (\Box + m^2) A_\nu$$

$$\partial^\nu \partial^\mu A_\nu = g^{\mu\nu} (\Box + m^2) A_\nu$$

$$\Rightarrow \boxed{[g^{\mu\nu}(\Box + m^2) - \partial^\mu \partial^\nu] A_\nu = 0.}$$

Aplicamos ∂_μ a la ecuación anterior:

$$[g^{\mu\nu} \partial_\mu (\Box + m^2) - \partial_\mu \partial^\mu \partial^\nu] A_\nu = 0$$

$$[\partial^\nu (\Box + m^2) - \Box \partial^\nu] A_\nu = 0$$

$$[\Box \partial^\nu + m^2 \partial^\nu - \Box \partial^\nu] A_\nu = 0$$

$$m^2 \partial^\nu A_\nu = 0 \Rightarrow A^\nu A_\nu = \boxed{0 = A_\mu A^\mu.}$$