23

Scattering Theory

Figure 23.1 shows a typical scattering experiment. On the left is a source of particles such as an accelerator. The particles are directed at a target that scatters some of them; the scattered particles are detected by counters placed at a distance from the target, at an angle to the incident beam.

However, as you know, all theoretical analyses of two-body problems such as this are done in quantum mechanics using the center-of-mass reference frame.

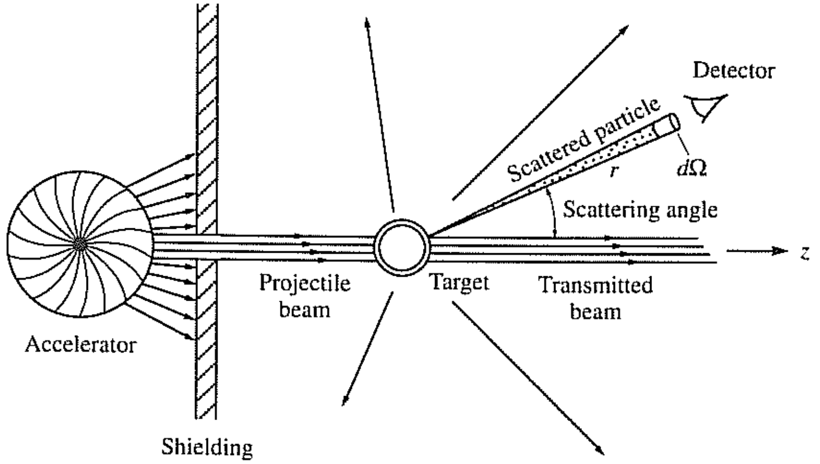


FIGURE 23.1
A scattering experiment (viewed classically). The scattered particles are detected at an angle θ to the incident beam within a solid angle $d\Omega$.

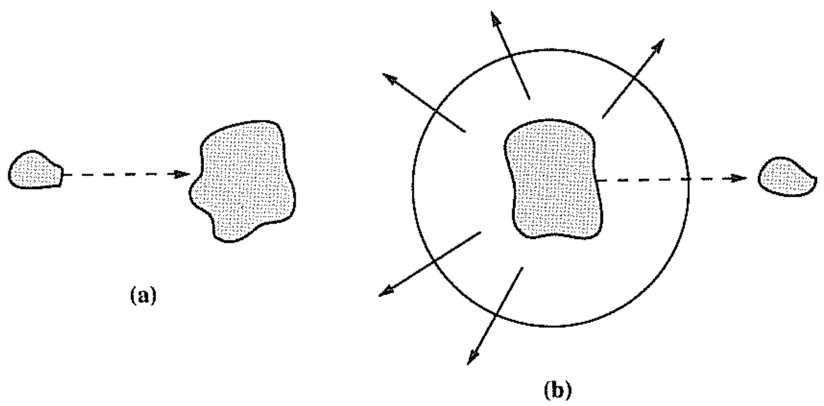
How does the above experiment look from the point of view of the center-of-mass system? In this frame, the two particle beams (projectile and target) come at one another but effectively we have a one-particle problem—a particle of reduced mass μ being scattered by a potential from an initial state to a final state. It is this one-particle scattering problem that we will analyze theoretically in most of this chapter; however, we will spend one section at the beginning establishing the connection between the center-of-mass quantities that we calculate and the laboratory quantities that we measure. (There are now some scattering experiments, using colliding beams, which are carried out in the center-of-mass frame, and the transformation from one set of coordinates to the other is rendered unnecessary.)

In order to develop theory in the simplest and most straightforward manner, we will make another assumption. The incident particle in the scattering experiment is most appropriately described by a wave packet, as is the final state consisting of the incident wave along with a scattered wave (fig. 23.2). However, the analysis of scattering in terms of wave packets is time-dependent; we have to take account of the spreading of the wave packets in time, and this becomes very complicated. So we resort to a time-independent approach.

Why do we do scattering experiments at all? One motivation is to learn about the interaction potential responsible for the scattering. However, our analysis is most straightforward with the added assumption that the potential is one of short range, as is the case in nuclear physics. We will not deal with the important Coulomb potential, because it needs special treatment.

We will begin with a discussion of what is measured and how the measurables are transformed to the center-of-mass system that our theory uses. Then we will develop a systematic solution of the Schrödinger equation for the scattering boundary conditions by the method of *partial waves*. And although most of the chapter is devoted to elastic scattering, some attention is given to inelastic scattering toward the end.

FIGURE 23.2
Wave packet description of scattering. (a) Wave packet of projectile incident on target; (b) transmitted wave packet along with spherical outgoing scattered wavefront.



23.1 SCATTERING AMPLITUDE, DIFFERENTIAL AND TOTAL CROSS SECTIONS, CENTER-OF-MASS, AND LABORATORY FRAMES

One constant challenge of quantum mechanics is to connect theory with experiment since we calculate quantities, wave functions, that are not directly measurable. So here we are in a scattering situation where the boundary conditions on the wave function are clear once we have made our basic assumptions. How do we relate these wave functions to the measurables, such as the differential cross section introduced in chapter 22?

Let's make the boundary conditions explicit. In the time-independent approach that we adopt, the incident particle is looked upon as a free particle and its wave function as a plane wave

$$\phi_k(\mathbf{r}) = \langle \mathbf{r} | \phi_k \rangle = \exp(ikz) \tag{23.1}$$

where $\hbar\mathbf{k}$ is the momentum of the incident particle, whose direction we have chosen as the z-axis, and where we have normalized the wave function so that there is just one particle per unit volume, which is the normalization we will choose for all our wave functions.

Now in the time-independent approach, we are looking at the scattering experiment of figure 23.1 in the way depicted in figure 23.3. If the scattering potential $V(\mathbf{r})$ is spherically symmetric (i.e., $V(\mathbf{r}) = V(|\mathbf{r}|)$), then asymptotically from the scattering source where we assume our detectors are, if we take the scattering source as the origin, we will have spherically outgoing scattered waves $\sim\exp(ikr)/r$ (see below for the justification of this nomenclature). The asymptotic form of the total wave function after the scattering by the potential is then represented by the sum of the incident and the scattered waves:

$$\begin{aligned}\psi_k(\mathbf{r}) &\underset{r\to\infty}{\longrightarrow} \phi_k(\mathbf{r}) + \psi_k^{sc}(\mathbf{r}) \\ &= \exp(ikz) + f(\theta,\phi)\exp(ikr)/r \end{aligned} \tag{23.2}$$

where the function $f(\theta,\phi)$ keeps account of the angular distribution of the scattered objects and is called the *scattering amplitude*. From the definition, we can tell that it has the dimension of length.

We will show below that the scattered wave function

$$\psi_k^{sc}(\mathbf{r}) \underset{r\to\infty}{\longrightarrow} f(\theta,\phi)\exp(ikr)/r \tag{23.3}$$

chosen in this fashion does satisfy the free-particle equation. In a sense this should be obvious to you except perhaps for the factor $1/r$, which is imperative so that the probability of particles being scattered into a solid angle $d\Omega$ does not depend on the radial distance.

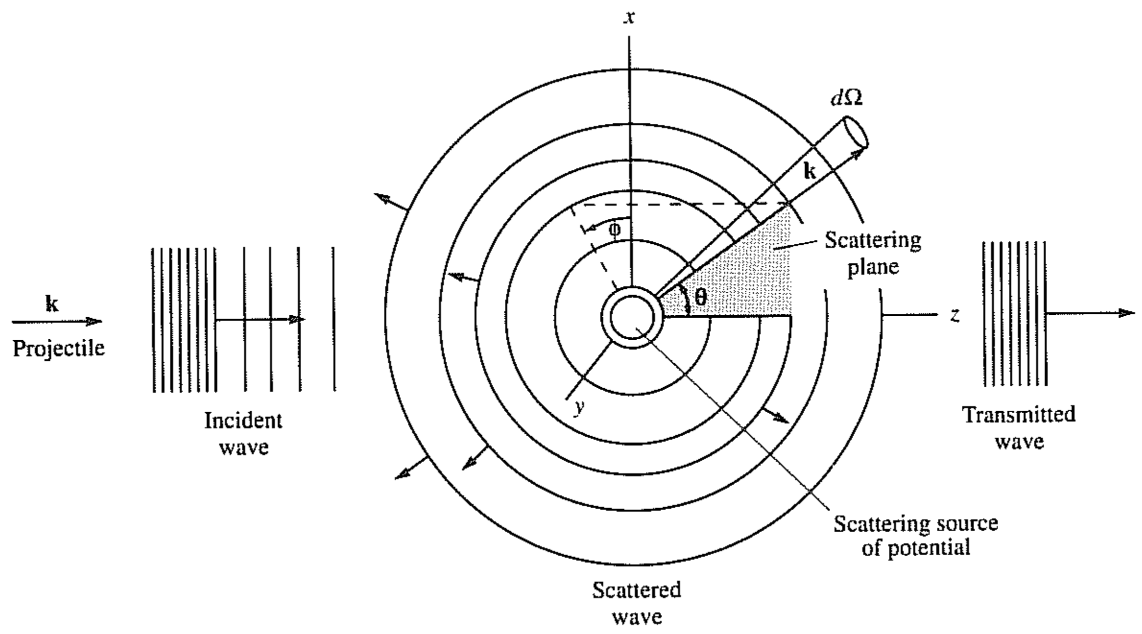


FIGURE 23.3
Idealized description of scattering: incident plane wave and spherical outgoing scattered waves. The scattering plane is formed by the incident and scattered wave vectors, k and k′; θ is the scattering angle and ϕ, the azimuth. For elastic scattering, |k′| = |k| = k.

Probability Currents for Elastic Scattering

Now let's calculate the probability currents associated with both the incident and the scattered waves above. The incident current is

$$\begin{aligned}\mathbf{j}_{\text{inc}} &= \frac{\hbar}{2i\mu}\left[\phi_k^*\nabla\phi_k - (\nabla\phi_k^*)\phi_k\right] \\ &= \frac{\hbar k}{\mu}\,\mathbf{e}_z \end{aligned} \tag{23.4}$$

where, as before, μ denotes the reduced mass of the effective one-body problem in the center-of-mass system.

The current scattered by the target is given as

$$\mathbf{j}_{\text{sc}} = \frac{\hbar}{2i\mu}\left[\psi_k^{sc*}\nabla\psi_k^{sc} - (\nabla\psi_k^{sc*})\psi_k^{sc}\right] \tag{23.5}$$

For large distances, we must use the asymptotic form, equation (23.3), for ψ^{sc}. It turns out that only the radial component of the scattered current is relevant. To see this, let's calculate an angular current:

$$\mathbf{e}_\theta \cdot \mathbf{j}_{sc} = \frac{\hbar}{2i\mu}\left[\psi_k^{sc*}\,\frac{1}{r}\,\frac{\partial}{\partial\theta}\,\psi^{sc} - \text{complex conjugate}\right]$$

This is of the order of $\sim 1/r^3$ if we use the asymptotic form of ψ_k^{sc}, equation (23.3). At large distances, such a current, even after multiplied by the area factor $dS = r^2\,d\Omega$ of the sphere of radius r that collects the particles, goes to zero $\sim 1/r$.

In contrast, the radial current is easily seen to give a term that is independent of r when multiplied by $r^2\,d\Omega$:

$$(\mathbf{j}_{sc})_r = \mathbf{j}_{sc}\cdot\mathbf{e}_r = \frac{\hbar}{2i\mu}\left[\psi^{sc*}(r)\,\frac{\partial}{\partial r}\,\psi^{sc} - \text{complex conjugate}\right]$$
$$\underset{r\to\infty}{\longrightarrow} \frac{\hbar k}{\mu}\,\frac{|f(\theta,\phi)|^2}{r^2} \tag{23.6}$$

where we ignore the term $\sim 1/r^3$. Clearly,

$$\mathbf{j}_{sc}\cdot d\mathbf{S} = (\mathbf{j}_{sc})_r r^2\,d\Omega = (\hbar k/\mu)|f(\theta,\phi)|^2\,d\Omega \tag{23.7}$$

is independent of r.

Differential Cross Section

The most convenient and universal measure of how much scattering is taking place in a given situation is the scattering cross section, which we must define in a way that is independent of detecting equipment and all that.

First let's define the differential cross section for elastic scattering. If N is the number of incident particles per unit area per unit time, and ΔN is the number scattered by the target (assumed infinitesimally thin) into the detector (assumed infinitesimally small) at the angle (θ,ϕ) into the solid angle $\Delta\Omega$ in unit time, then the differential cross section is

$$\sigma(\theta,\phi) = \frac{d\sigma}{d\Omega} = \lim_{\Delta\Omega\to 0}\frac{1}{N}\,\frac{\Delta N}{\Delta\Omega} \tag{23.8}$$

where the notation $\sigma(\theta,\phi)$ is sometimes used for the differential cross section purely for convenience. In terms of the currents, $\Delta N = \mathbf{j}_{sc}\cdot d\mathbf{S}$, $N = \mathbf{j}_{inc}\cdot\mathbf{e}_z$, and after substituting from equations (23.4) and (23.7), we obtain

$$\sigma(\theta,\phi) = |f(\theta,\phi)|^2 \tag{23.9}$$

The differential cross section has the unit of area (per steradian). For low-energy nuclear scattering processes, the barn (1 b = 10^{-24} cm^2) is a convenient unit of area (because it should be as easy to hit a nuclear target with low energy neutrons as, say, "hitting the broad side of a barn").

If the potential causing the scattering is central, as assumed here, the scattering must be axially symmetric, and therefore $\sigma(\theta,\phi)$ is independent of the azimuthal angle ϕ and varies only with θ and with the incident energy of the projectile.

Total Cross Section

If we integrate out all angular dependence, we get the integrated elastic cross section, or more commonly, the *total cross section* denoted as σ_T:

$$\sigma_T = \int d\Omega\,\frac{d\sigma}{d\Omega} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta\,\frac{d\sigma}{d\Omega} \tag{23.10}$$

The total cross section still depends on energy, of course. It is a useful quantity, since it represents how the strength and nature of scattering changes as a function of energy.

Then, there is also inelastic scattering in which some flux is lost (absorbed). When such processes occur, we enlarge the concept of total cross section to include both the integrated elastic and inelastic differential cross sections:

$$\sigma_T = \int d\Omega\left[\left(\frac{d\sigma}{d\Omega}\right)_{el} + \left(\frac{d\sigma}{d\Omega}\right)_{inel}\right]$$
$$= \sigma_{Tel} + \sigma_{Tinel} \tag{23.11}$$

Actually, the measurement of total cross section is easier than it seems from the above. In practice, all we have to measure is the depletion of the incident beam intensity upon making its journey through a target material which is of finite thickness (fig. 23.4). Suppose the target thickness is x and the density of

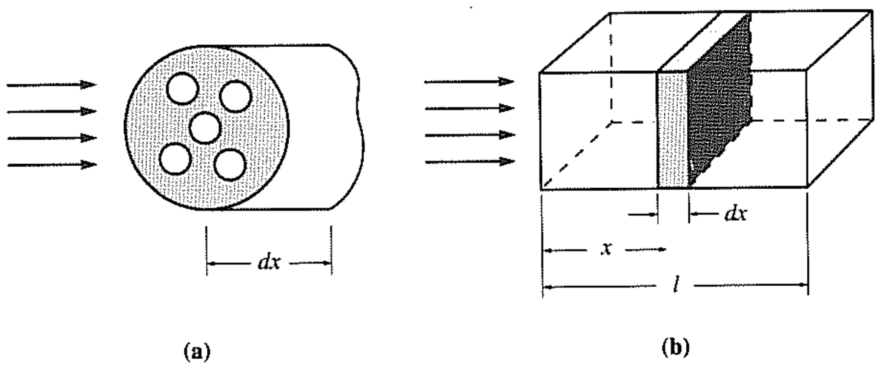


FIGURE 23.4
(a) The individual tiny scatterers in a target can be treated as if they each subtend an area equal to the total cross section.
(b) An elementary thickness dx at a distance x in a target of finite thickness.

the target material is ρ. Additionally, suppose that the incident beam intensity is I_0 and the intensity of the beam upon traveling a distance x of the target is $I(x)$. Then the depletion of the beam intensity corresponding to an infinitesimal thickness dx of the target is given as

$$dI = -\sigma_T \rho \, dx \, I \tag{23.12}$$

since ρ is the number of scattering centers per unit volume and σ_T is the effective area. As the beam traverses a finite thickness, the intensity decreases continuously. The final intensity $I(x)$ upon traversing a thickness x can consequently be obtained by integration of equation (23.12):

$$\int_{I(0)}^{I(x)} \frac{dI}{I} = -\int_0^x dx \, \sigma_T \rho$$

We obtain

$$\ln I(x) - \ln I(0) = -\sigma_T \rho x \tag{23.13}$$

Or

$$I(x) = I(0)\exp(-\sigma_T \rho x)$$

where we assume that ρ is uniform throughout the sample and that σ remains constant as well (that is, we are ignoring the change in the particle energies).

In using equation (23.13) for determining the total cross section, however, some practical considerations have to be given to things such as detector size, which makes it impossible not to count a few scattered particles while counting the transmitted ones, especially at small angles.

Transformation from the Center-of-Mass Frame to Laboratory Frame

As noted already, all the calculations in this chapter refer to the center-of-mass system; however, in this section we'll give the formulas necessary to convert center-of-mass quantities to laboratory quantities. Consider the scattering of two particles of mass m_1 and m_2 with m_2 being the stationary target and m_1 the projectile moving with velocity v_{1L} ($\ll c$) along the x-axis, where the subscript L denotes laboratory. The velocity of the center of mass (COM) of the system is along the x-axis and is equal to

$$v_{cm} = m_1 v_{1L}/(m_1 + m_2)$$

Let the collision take place in the xy-plane. Before collision, in the COM frame, the particles have velocities given by (the subscript c is used to distinguish the COM frame):

$$v_{1cx} = v_{1Lx} - v_{cm} = m_2 v_{1L}/(m_1 + m_2) \qquad v_{1cy} = 0$$

$$v_{2cx} = v_{2Lx} - v_{cm} = -m_1 v_{1L}/(m_1 + m_2) \qquad v_{2cy} = 0 \tag{23.14}$$

It is clear that in the COM system the two particles move with equal and opposite momenta before collision, and therefore, the same situation must persist, in this reference frame, after collision (fig. 23.5a). However, the situation in the laboratory is more complicated (fig. 23.5b).

If the center-of-mass scattering angle is θ_c and primes label velocities after collision, we have

$$v'_{1cx} = \frac{m_2}{m_1 + m_2} v_{1L} \cos\theta_c, \qquad v'_{1cy} = \frac{m_2}{m_1 + m_2} v_{1L} \sin\theta_c$$

$$v'_{2cx} = -\frac{m_1}{m_1 + m_2} v_{1L} \cos\theta_c, \qquad v'_{2cy} = -\frac{m_1}{m_1 + m_2} v_{1L} \sin\theta_c \tag{23.15}$$

Since $\mathbf{v}'_{1c} = \mathbf{v}'_{1L} - \mathbf{v}_{cm}$, equating x- and y-components of this equation and using equations (23.14) and (23.15), we get

$$v_{1c} \cos\theta_c = v'_{1L} \cos\theta_{1L} - v_{cm}$$

and

$$v_{1c} \sin\theta_c = v'_{1L} \sin\theta_{1L}$$

From these two equations, we get

$$\tan\theta_{1L} = \frac{\sin\theta_c}{\cos\theta_c + \gamma} \tag{23.16}$$

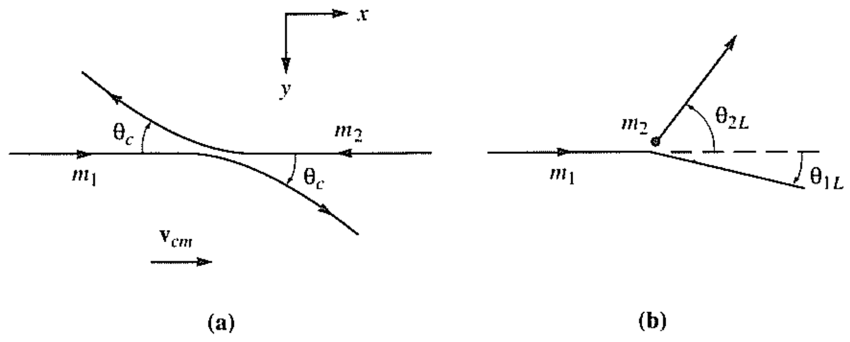


FIGURE 23.5
Scattering in (a) center-of-mass system and (b) in the laboratory system.

where

$$\gamma = v_{cm}/v_{1c} = m_1/m_2 \tag{23.17}$$

Similarly, it can easily be seen that

$$\tan\theta_{2L} = \cot\theta_c/2 \tag{23.18}$$

These equations can be generalized easily for inelastic collisions such as the reaction

$$m_1 + m_2 \to m_3 + m_4$$

If the amount of mass energy converted to kinetic energy of the final particles is Q (the Q-value of the reaction; $Q > 0$ for exothermic reactions, and $Q < 0$ for endothermic reactions), then γ obtains the value

$$\gamma = \left[\frac{m_1 m_3}{m_2 m_4}\frac{E}{E+Q}\right] \tag{23.19}$$

where E is the initial center-of-mass energy of m_1 and m_2, and equation (23.16) still holds.

The relation between the differential cross sections in the two frames is obtained by noting that the same number of particles that scatters into the solid angle $d\Omega_c$ at (θ_c, ϕ_c) goes into the solid angle $d\Omega_L$ at (θ_L, ϕ_L). Therefore,

$$\sigma_L(\theta_L, \phi_L)\sin\theta_L\, d\theta_L\, d\phi_L = \sigma_c(\theta_c, \phi_c)\sin\theta_c\, d\theta_c\, d\phi_c$$

In other words

$$\sigma_L(\theta_L, \phi_L) = \sigma_c(\theta_c, \phi_c)\,[d\cos\theta_c/d\cos\theta_L]\,d\phi_c/d\phi_L \tag{23.20}$$

Since we are assuming rotational symmetry, $\phi_c = \phi_L$. Using this and equations (23.16) and (23.18), we obtain the relation between the cross sections as (the details will be left as an exercise)

$$\sigma_L(\theta_{1L}, \phi_{1L}) = \frac{(1+\gamma^2+2\gamma\cos\theta_c)^{3/2}}{|1+\gamma\cos\theta_c|}\,\sigma_c(\theta_c, \phi_c) \tag{23.21}$$

$$\sigma_L(\theta_{2L}, \phi_{2L}) = 4\cos\theta_{2L}\,\sigma_c(\theta_c, \phi_c) \tag{23.22}$$

And as far as the total cross sections are concerned, they must be the same for both coordinate systems and for either outgoing particle—the total number of

scattering events has to be independent of the mode of description or reference frame used. You can also verify this using equations (23.21) and (23.22), which will be left as an exercise.

An important special case is $m_1 = m_2$ for which $\gamma = 1$, and the transformation equations between the reference frames simplify considerably. We get

$$\theta_{1L} = \frac{\theta_c}{2}, \qquad \theta_{2L} = \frac{\pi}{2} - \theta_{1L}$$

$$\sigma_L(\theta_{1L}, \phi_{1L}) = 4\cos\theta_{1L}\,\sigma_c(\theta_c, \phi_c), \qquad \sigma_L(\theta_{2L}, \phi_{2L}) = 4\cos\theta_{2L}\,\sigma_c(\theta_c, \phi_c) \tag{23.23}$$

23.2 CONTINUUM QUANTUM MECHANICS: PARTIAL WAVES

When we were calculating bound states back in chapter 12, we found it useful to incorporate rotational symmetry of the Hamiltonian into the solution and decompose the total wave function into eigenstates of orbital angular momentum. The same strategy of breaking up a wave into partial l waves will now be employed for the continuum problem starting with the case of a free particle.

Free-Particle Schrödinger Equation in Spherical Coordinates

The Schrödinger equation for a free particle

$$(\nabla^2 + k^2)\psi(r, \theta, \phi) = 0$$

upon substitution of the partial wave decomposition

$$\psi(r,\theta,\phi) = \sum_{lm} R_l(r) Y_{lm}(\theta\phi)$$

gives the following radial equation for R_l (see chapter 12, if you need a review; note that since we are dealing with continuum states, the quantum number n is dropped from the subscripts of R):

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}\right]R_l(r) + k^2 R_l(r) = 0 \tag{23.24}$$

Introducing a new variable $\rho = kr$, we rewrite equation (23.24) in the spherical Bessel form that you will recognize:

$$\left[\frac{d^2}{d\rho^2} + \frac{2}{\rho}\frac{d}{d\rho}\right]R_l(\rho) + \left[1 - \frac{l(l+1)}{\rho^2}\right]R_l(\rho) = 0 \tag{23.25}$$

The (unnormalized) solution of this equation, regular at the origin (which is the appropriate boundary condition here), is the spherical Bessel function

$$R_l(\rho) = R_l(kr) = j_l(kr) \tag{23.26}$$

You will recall from equation (12.48) that the asymptotic form of $j_l(kr)$ is given as

$$j_l(kr) \underset{r\to\infty}{\longrightarrow} \frac{1}{kr}\sin\left(kr - \frac{l\pi}{2}\right) \tag{23.27}$$

Therefore, the asymptotic form of R_l is

$$R_l \sim -\frac{1}{2ikr}\,[e^{-i(kr-l\pi/2)} - e^{i(kr-l\pi/2)}] \tag{23.28}$$

In this way, we see that not only is the outgoing spherical wave $\exp(ikr)/r$ an asymptotic solution of the Schrödinger equation, something that we have wanted to prove, but also the "incoming" spherical wave $\exp(-ikr)/r$ is a solution. That this last wave is incoming is easily established by calculating its current, which is equal to and the negative of the current of the outgoing wave. It follows that the net flux is zero, which it must be since there is no source of flux anywhere.

What happens when there is a potential? Since we assume that $V(r) \to 0$ asymptotically faster than $1/r^2$, the asymptotic solution is still a free-particle solution (i.e., a linear combination of incoming and outgoing waves), but the linear combination must be appropriately chosen so that the asymptotic solution continuously matches the solution that is regular at the origin, a solution that must be determined with the potential "on." However, such a linear combination must still conserve the flux; the incoming and outgoing flux cannot differ.

In general then, we can parameterize the $V(r) \neq 0$ asymptotic solution of the Schrödinger equation in the form

$$R_l(r) \sim -\frac{1}{2ikr}\,[e^{-i(kr-l\pi/2)} - S_l(k)e^{i(kr-l\pi/2)}] \tag{23.29}$$

with the constraint that

$$|S_l(k)|^2 = 1 \tag{23.30}$$

which is the constraint of flux conservation. The S-function satisfying the constraint, equation (23.30), can always be written as

$$S_l(k) = \exp[2i\delta_l(k)] \tag{23.31}$$

where the functions $\delta_l(k)$ are real. They are called *phase shifts* because the asymptotic radial function, equation (23.29), can be rewritten in the form

$$R_l(r) \sim e^{i\delta_l}\,\frac{\sin[kr - l\pi/2 + \delta_l(k)]}{kr} \tag{23.32}$$

Apart from the phase factor in front, this is the same as the asymptotic form of the free particle wave function, equation (23.28), except that it is shifted in phase by δ_l.

You can understand the phase shift qualitatively as follows: If the potential is attractive, it accelerates the particle as it scatters it, and consequently, the wavelength of the particle is shortened in the scattering region and the phase shift is positive (fig. 23.6). Conversely, if the potential is repulsive, the particle is decelerated, the wavelength is lengthened, the wave tends to be pushed out of the scattering region, and the phase shift is negative (fig. 23.7).

Expansion of a Plane Wave into Partial Waves

Now we will expand the incident plane wave of our scattering scenario in terms of the infinite hoard of partial waves by using the complete set $j_l(kr)Y_{lm}$. Since a plane wave $\exp(ikz) = \exp(ikr\cos\theta)$ does not depend on the azimuth ϕ, it is clear that the wave does not possess any angular momentum component along the z-axis, $m = 0$. Since $Y_{l0}(\theta\phi) \sim P_l(\cos\theta)$, we write our expansion as

$$\exp(ikz) = \sum_{l=0}^{\infty} a_l j_l(kr) P_l(\cos\theta) \tag{23.33}$$

where the coefficients of the expansion a_l are yet to be determined. To this end, we multiply equation (23.33) by $P_{l'}(\cos\theta)$ and integrate over the solid angle. This gives

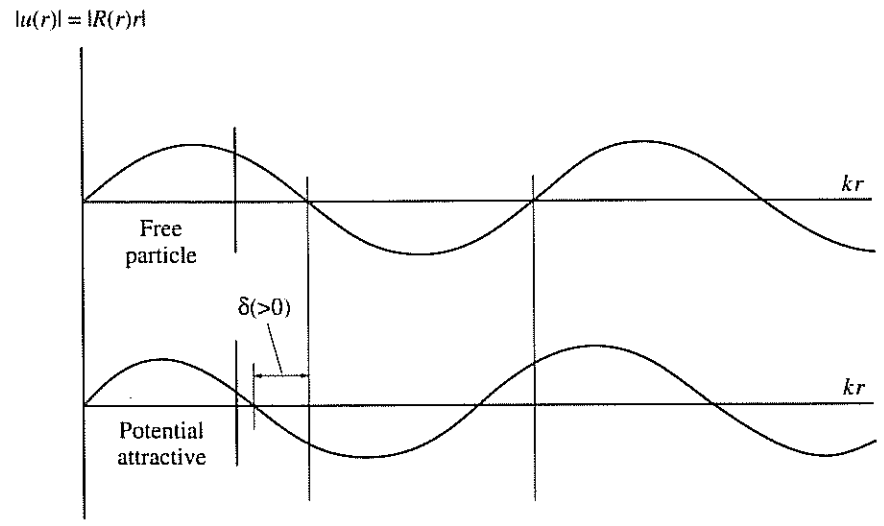


FIGURE 23.6
The phase shift due to an attractive potential.

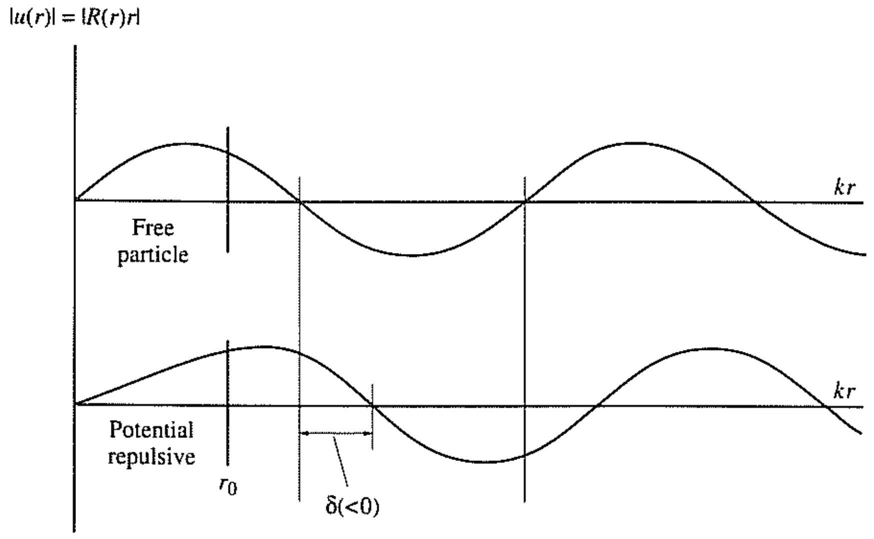


FIGURE 23.7
The phase shift due to a repulsive potential.

$$\int_0^\pi \sin\theta\, d\theta\, e^{ikr\cos\theta} P_{l'}(\cos\theta) = \sum_{l=0}^{\infty} a_l j_l(kr) \int_0^\pi \sin\theta\, d\theta\, P_l(\cos\theta) P_{l'}(\cos\theta)$$

But the last integral above is

$$(2/2l+1)\delta_{ll'}$$

by virtue of the orthogonality of the Legendre polynomials. It also turns out that the integral on the left-hand side is a well-known integral representation of the spherical Bessel function (look it up!):

$$\int_0^\pi \sin\theta\, d\theta\, e^{ikr\cos\theta} P_l(\cos\theta) = 2i^l j_l(kr)$$

Substituting, we obtain

$$a_l = (2l+1)i^l$$

Therefore, the partial wave expansion of a plane wave is given as

$$e^{ikz} = \sum_{l=0}^{\infty} i^l(2l+1) j_l(kr) P_l(\cos\theta) \tag{23.34}$$

For asymptotic distances, we have

$$e^{ikz} \underset{r\to\infty}{\to} \sum_{l=0}^{\infty} i^l[(2l+1)/kr]\sin(kr - l\pi/2) P_l(\cos\theta) \tag{23.35}$$

Partial Wave Expansion of the Scattering Amplitude

Strictly speaking, it makes sense to say only that a wave can be expanded in terms of partial waves; nonetheless, it is customary to talk about partial wave expansion of the scattering amplitude. How do we obtain such an expansion?

The trick is to invoke the change, the phase shift, that the wave function undergoes in the presence of the potential compared to the free particle. From equation (23.32) it is clear that the total wave function, after scattering, must have the asymptotic form

$$\psi_k(r) \underset{r\to\infty}{\to} \sum_{l=0}^{\infty} a_l i^l (2l+1) e^{i\delta_l(k)} \frac{\sin(kr - l\pi/2 + \delta_l)}{kr} P_l(\cos\theta) \tag{23.36}$$

where a_l is yet to be determined by comparing the expansion above with the scattering boundary condition we previously imposed upon ψ_k, namely,

$$\psi_k \underset{r\to\infty}{\to} \exp(ikz) + f(\theta,\phi)\exp(ikr)/r \tag{23.2}$$

Substituting for $\exp(ikz)$ from equation (23.35) and rearranging, we can write

$$\begin{aligned} f(\theta,\phi)\frac{e^{ikr}}{r} &= \sum_{l=0}^{\infty} a_l \frac{2l+1}{kr} i^l e^{i\delta_l} \sin\left(kr - \frac{l\pi}{2} + \delta_l\right) P_l(\cos\theta) \\ &\quad - \sum_{l=0}^{\infty} \frac{2l+1}{kr} i^l \sin\left(kr - \frac{l\pi}{2}\right) P_l(\cos\theta) \end{aligned}$$

The key point to note is that our boundary condition for scattering excludes the incoming wave (why? because nobody ever saw a scattering event where waves converge onto a center instead of diverging from it!) from the left-hand side. Consequently, the incoming wave must cancel from the right-hand side, too. By writing

$$\sin\left(kr - \frac{l\pi}{2} + \delta_l\right) = (1/2i)\left\{\exp\left[i\left(kr - \frac{l\pi}{2} + \delta_l\right)\right] - \exp\left[-i\left(kr - \frac{l\pi}{2} + \delta_l\right)\right]\right\}$$

we can easily see that the above condition can be true only if $a_l = 1$. This also gives the partial wave expansion of the scattering amplitude:

$$\begin{aligned} f(\theta,\phi) &= \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(e^{2i\delta_l} - 1)P_l(\cos\theta) \\ &= \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta) \\ &= \Sigma(2l+1)f_l P_l(\cos\theta) \end{aligned} \tag{23.37}$$

which defines f_l as the scattering amplitude for the lth partial wave.

Once we have the partial wave expansion for the scattering amplitude, the differential cross section follows from equation (23.9):

$$\sigma(\theta,\phi) = \frac{1}{k^2}\left|\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta)\right|^2 \tag{23.38}$$

What is the advantage of the partial wave expansion? The behavior of $j_l(kr)$ as $r \to 0$ is given as

$$j_l(kr) \sim (kr)^l \qquad (kr \to 0)$$

Therefore, each of the j_l's in the expansion of the plane wave, equation (23.34), is small until $kr \sim l$; this means that when a plane wave encounters a short-range scattering potential $V(r)$, some of its partial waves will not have any significant value inside the region of the potential and will thus be unaffected by it. Physically, it is the centrifugal potential that repels partial waves of large l from the region of the potential for low energy. At low energy (small k) then we need to include only a few partial waves in the expansion of the scattering amplitude, which is a great simplification.

The differential cross section, equation (23.38), determines the angular distribution of the scattered particles in the center-of-mass system. If only one partial wave l dominates (which is not unusual), the angular distribution is proportional to

$$|P_l(\cos\theta)|^2$$

At low energy, only the S-wave contributes, and the angular distribution is isotropic (fig. 23.8). As the energy of the incident particle increases, higher partial waves enter the picture, and the angular distribution reflects interference between the partial waves (fig. 23.9).

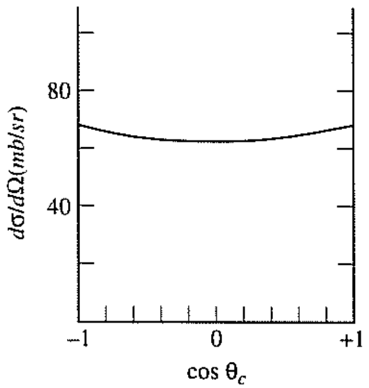
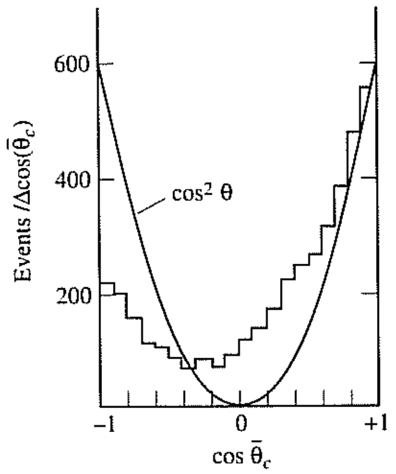


FIGURE 23.8
The COM differential cross section for neutron-proton scattering at a laboratory energy of 14.1 MeV (1 mb = 10^{-27} cm²).

FIGURE 23.9
$d\sigma/d\Omega$ for $\pi^+\pi^-$ scattering at 750 MeV (the jagged curve) compared with pure P-wave angular distribution (solid curve).



Total Cross Section and the Optical Theorem

The total cross section is obtained by integrating equation (23.38) over all solid angles:

$$\sigma_T = \int d\Omega\,\sigma(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\frac{2l+1}{k}\frac{2l'+1}{k}e^{i(\delta_l-\delta_{l'})}\sin\delta_l\sin\delta_{l'}$$
$$\times\int d\Omega\,P_l(\cos\theta)P_{l'}(\cos\theta)$$

Using the orthogonality of the Legendre polynomials once again, we obtain the following rather simple expression for σ_T:

$$\sigma_T = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l \tag{23.39}$$

Now compare this with the value of the forward scattering amplitude, $f(\theta,\phi)$ for $\theta = 0$, which can easily be calculated from equation (23.37):

$$f(0) = \sum_{l=0}^{\infty}\frac{2l+1}{k}e^{i\delta_l}\sin\delta_l P_l(1)$$

But $P_l(1) = 1$, so we get

$$\operatorname{Im} f(0) = \sum_{l=0}^{\infty}\frac{2l+1}{k}\sin^2\delta_l = \frac{k}{4\pi}\sigma_T$$

In other words,

$$\sigma_T = \frac{4\pi}{k}\,\mathrm{Im}\,f(0) \tag{23.40}$$

This relationship between the total cross section and the forward scattering amplitude is called the *optical theorem*. Although we have derived it for elastic scattering, it remains true even in the presence of inelastic processes, as we will see later.

If it puzzles you a little how $f(0)$ can occur linearly in a relationship involving the total cross section, consider this. The total cross section represents removal of flux from the incident beam. Such removal is the result of destructive interference between the incident current and the elastically scattered current in the forward direction, and the latter is proportional to the imaginary part of $f(0)$.

23.3 SCATTERING BY A SQUARE-WELL POTENTIAL AT LOW ENERGIES

Many of the important aspects of scattering can be illustrated by taking the scattering potential as a square well and by restricting ourselves to low energy when only the S-wave contributes. The attractive square-well potential, as you recall, is given by

$$\begin{aligned} V(r) &= -V_0 \qquad r < a \\ &= 0 \qquad\quad r > a \end{aligned}$$

Consequently, the S-wave radial equation for inside the well, $r < a$, is

$$\frac{d^2u}{dr^2} + k_{\text{in}}^2 u = 0, \qquad k_{\text{in}}^2 = \frac{2\mu}{\hbar^2}(E + V_0) \tag{23.41}$$

where, as in chapter 12, $u(r) = rR(r)$. The appropriate solution is the one that vanishes at $r = 0$; we have

$$u(r) = A \sin k_{\text{in}} r \tag{23.42}$$

For $r > a$, we have the free-particle radial equation

$$\frac{d^2u}{dr^2} + k^2 u = 0, \qquad k^2 = \frac{2\mu E}{\hbar^2} \tag{23.43}$$

The solution is phase shifted, however,

$$u(r) = B\sin(kr + \delta_0) \tag{23.44}$$

where the subscript 0 on δ denotes l, which is zero. (Note that the above is just another way of writing the most general solution of equation [23.43].)

Now we must ensure the continuity of the wave function and its derivative at the boundary $r = a$; the boundary conditions in their turn determine the phase shift δ_0. Also recall that the easiest way to incorporate the boundary conditions is to equate logarithmic derivatives of the outside and inside solutions at $r = a$:

$$k\cot(ka + \delta_0) = k_{\text{in}} \cot k_{\text{in}} a$$

or

$$(1/k)\tan(ka + \delta_0) = (1/k_{\text{in}})\tan k_{\text{in}} a \tag{23.45}$$

This is equivalent to

$$\frac{1}{k}\,\frac{\tan ka + \tan\delta_0}{1 - \tan ka \tan\delta_0} = \frac{1}{k_{\text{in}}}\tan k_{\text{in}} a$$

Rearranging,

$$\tan\delta_0(k_{\text{in}} + k\tan ka \tan k_{\text{in}} a) = k\tan k_{\text{in}} a - k_{\text{in}}\tan ka$$

or

$$\tan\delta_0 = \frac{(k/k_{\text{in}})\tan k_{\text{in}} a - \tan ka}{1 + (k/k_{\text{in}})\tan ka \tan k_{\text{in}} a} \tag{23.46}$$

To solve this for δ_0, define $\tan Ka$ such that

$$\tan Ka = (k/k_{\text{in}})\tan k_{\text{in}a}$$

We have

$$\tan\delta_0 = \frac{\tan Ka - \tan ka}{1 + \tan Ka \tan ka} = \tan(Ka - ka)$$

In this way we find δ_0,

$$\delta_0 = Ka - ka = \tan^{-1}\{(k/k_{\text{in}})\tan k_{\text{in}} a\} - ka \tag{23.47}$$

For low-energy scattering, $ka \ll 1$, $\tan ka \approx ka$; suppose additionally that the potential is shallow; then the denominator of equation (23.46) ≈ 1, and we get

$$\tan \delta_0 \approx \delta_0 \approx ka\left[\frac{\tan k_{\text{in}}a}{k_{\text{in}}a} - 1\right] \tag{23.48}$$

In this approximation, δ_0 is in the first quadrant. Now if we gradually increase the depth of the potential, at some point $k_{\text{in}}a$ will go through $\pi/2$. Recall from chapter 12 that this is the condition for the appearance of a bound state (at zero energy), that is, the potential is barely deep enough to bind the particle. So, what can we say now about the phase shift? From equation (23.46), since $\tan k_{\text{in}}a \to \infty$, we have

$$\tan \delta_0 = \cot ka \to \infty \tag{23.49}$$

This means that the phase shift δ_0 is going through $\pi/2$.

Suppose we increase the well depth again just a tad; now we are back to the same situation that led to equation (23.48); that is, $\tan \delta_0 \sim O(ka)$. The difference is that we must realize that δ_0 has to be in the third quadrant, where again the tangent is positive. We conclude that

$$\delta_0 \approx ka\left[\frac{\tan k_{\text{in}}a}{k_{\text{in}}a} - 1\right] \tag{23.48}$$

when there is no bound state, and

$$\delta_0 \approx \pi + ka\left[\frac{\tan k_{\text{in}}a}{k_{\text{in}}a} - 1\right] \tag{23.50}$$

when there is one bound state.

If we make the potential deeper still, a second bound state will appear when $k_{\text{in}}a$ goes through $3\pi/2$; then we will have to entertain a solution replacing equation (23.48) by

$$\delta_0 \approx 2\pi + ka\left[\frac{\tan k_{\text{in}}a}{k_{\text{in}}a} - 1\right]$$

and so forth. We are getting a glimpse of a general theorem of scattering theory known as *Levinson's theorem*:

$$\delta(E = 0) = (\text{number of bound states}) \times \pi \tag{23.51}$$

So what does all this mean? To investigate, let's look at the total cross section. When δ_0 is given by equation (23.48), the total cross section is given by

$$\begin{aligned}\sigma_T &= \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi}{k^2}\delta_0^2 \approx \frac{4\pi}{k^2}(ka)^2\left[\frac{\tan k_{\text{in}}a}{k_{\text{in}}a}\right]^2 \\ &= 4\pi a^2\left[\frac{\tan k_{\text{in}}a}{k_{\text{in}}a}\right]^2\end{aligned} \tag{23.52}$$

This is a constant. Of course, there is a $(ka)^2$ term here hiding behind all our approximations, so the total cross section is only approximately constant.

Now what happens when $\tan \delta_0 \to \infty$? Then

$$\sigma_T = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{4\pi}{k^2} \tag{23.53}$$

The total cross section approaches a maximum value (fig. 23.10). This is called a resonance (see below). If there is a bound state at zero energy and the phase shift goes through $\pi/2$, the cross section peaks, and we have a resonance.

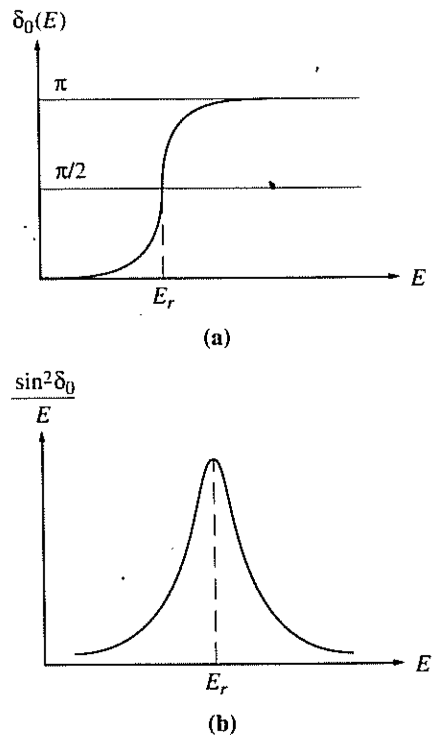


FIGURE 23.10
When the phase shift increases through $\pi/2$ (a), the corresponding partial wave cross section achieves a resonance (b).

Breit-Wigner Formula

Let's examine the behavior of a cross section near a resonance. At the resonant energy E_r, the phase shift increases through $\pi/2$, and we have

$$\cos\delta_0(E_r) = 0 \qquad \sin\delta_0(E_r) = 1$$

At $E \sim E_r$, we can expand $\sin\delta_0$ and $\cos\delta_0$ by means of Taylor series:

$$\sin\delta_0(E) = \sin\delta_0(E_r) + \cos\delta_0(E)\left.\frac{d\delta_0(E)}{dE}\right|_{E=E_r}(E-E_r) = 1 \tag{23.54}$$

$$\cos\delta_0(E) = \cos\delta_0(E_r) - \sin\delta_0(E)\left.\frac{d\delta_0(E)}{dE}\right|_{E=E_r}(E-E_r)$$
$$= -\left.\frac{d\delta_0(E)}{dE}\right|_{E=E_r}(E-E_r) = -\frac{2}{\Gamma}(E-E_r) \tag{23.55}$$

The last equation defines Γ, which we will interpret a little later.

The scattering amplitude is given by

$$f_0(\theta, E) = (1/k)\exp[i\delta_0(E)]\sin\delta_0(E) \tag{23.56}$$

The $1/k$-variation with energy is slow and uninteresting. The interesting, rapidly varying part comes from the phase shifts; let's call this part $f_0(\delta)$. We have

$$f_0(\delta) = e^{i\delta_0(E)}\sin\delta_0(E) = \frac{\sin\delta_0(E)}{\cos\delta_0(E) - i\sin\delta_0(E)}$$
$$\approx \frac{1}{-2(E-E_r)/\Gamma - i} = -\frac{\Gamma/2}{(E-E_r) + i\Gamma/2} \tag{23.57}$$

where we have used the Taylor-expanded values of $\cos\delta_0$ and $\sin\delta_0$. The total cross section is now given by

$$\sigma_T = \frac{4\pi}{k^2}|f_0(\delta)|^2 = \frac{4\pi}{k^2}\frac{\Gamma^2/4}{(E-E_r)^2 + \Gamma^2/4} \tag{23.58}$$

which is the Breit-Wigner resonance formula (the resonance curve is shown in fig. 23.10b). Clearly, Γ represents the width of the resonant curve.

Low-Energy Neutron-Proton Scattering: Scattering Length and Effective Range

Suppose we consider the scattering from a square-well potential with enough depth that there is a bound state with a small binding energy. This is a model for neutron-proton scattering; the neutron-proton system does indeed have a bound state near zero energy, namely deuteron. The square-well treatment of

deuteron was given in chapter 12, where we derived the following equation (changing the notation slightly):

$$k'_{\text{in}}\cot k'_{\text{in}}a = -k_B \tag{12.37}$$

with

$$k'^2_{\text{in}} = 2\mu(V_0 - |E|)/\hbar^2 \qquad k_B^2 = 2\mu|E|/\hbar^2$$

Since the binding energy $|E|$ is small, we can replace k'_{in} by $k_0 = [2\mu V_0/\hbar^2]^{1/2}$. Then we have

$$k_0\tan(\pi/2 - k_0a) = -k_B$$

Or, if k_0a is close to $\pi/2$, we have

$$\pi/2 - k_0a = -k_B/k_0$$

giving

$$k_0a = \frac{\pi}{2} + \frac{k_B}{k_0} \tag{23.59}$$

Coming back to low-energy scattering, the boundary conditions in the continuum case give us

$$(1/k)\tan(ka + \delta_0) = (1/k_{\text{in}})\tan k_{\text{in}}a \tag{23.45}$$

where now $k_{\text{in}}^2 = k^2 + k_0^2$. Since $ka \ll 1$, we have $k_{\text{in}}a \approx k_0a$ as well. Consequently, equation (23.45) can be written as

$$k\tan k_0a = k_0\tan(ka + \delta_0)$$

Substituting for k_0a from equation (23.59) and expanding the tangent on the right-hand side, we get

$$k\tan\left(\frac{\pi}{2} + \frac{k_B}{k_0}\right) = k_0\frac{\tan ka + \tan\delta_0}{1 - \tan ka\tan\delta_0}$$

Now

$$\tan(\pi/2 + k_B/k_0) = -\cot(k_B/k_0) = -[\tan(k_B/k_0)]^{-1} = -k_0/k_B$$

since $k_B/k_0 \ll 1$. Also $\tan ka \approx ka$. Substituting and rearranging, we get

$$-k(1 - ka\tan\delta_0) = k_Bka + k_B\tan\delta_0$$

Solving for $\tan\delta_0$, we obtain

$$\tan\delta_0 = \frac{k(k_B a + 1)}{k^2 a - k_B}$$

This last equation can be written as

$$k\cot\delta_0 = -\frac{k_B}{k_B a + 1} + \frac{a}{k_B a + 1}k^2 \tag{23.60}$$

And since $k_B a \ll 1$, we finally obtain

$$k\cot\delta_0 = -k_B + ak^2 \tag{23.61}$$

This is a special case of a very general result known as the *effective range expansion* for $k\cot\delta_0$ for low-energy potential scattering:

$$k\cot\delta_0 = -\frac{1}{\alpha} + \frac{1}{2}r_0 k^2 \tag{23.62}$$

The quantity α is called the *scattering length* and r_0, the *effective range*.

As $k \to 0$, we have from the effective range formula

$$\delta_0 \to -\alpha k \tag{23.63}$$

What happens to the scattering amplitude in this limit of zero energy? The near-zero energy ($l = 0$)-scattering amplitude is given as

$$f_0 = \frac{1}{k}e^{i\delta_0}\sin\delta_0 \underset{k\to 0}{\to} -\alpha \tag{23.64}$$

And the near-zero energy total cross section is given as

$$\sigma_T = (4\pi/k^2)\sin^2\delta_0 \underset{k\to 0}{\to} 4\pi\alpha^2 \tag{23.65}$$

The meaning of effective range is self-suggesting. Let's try to give a geometric interpretation of the scattering length α. We begin by noting that at very low incident energy, the Schrödinger radial equation outside the range of the potential

$$\frac{d^2u}{dr^2} + k^2 u = 0$$

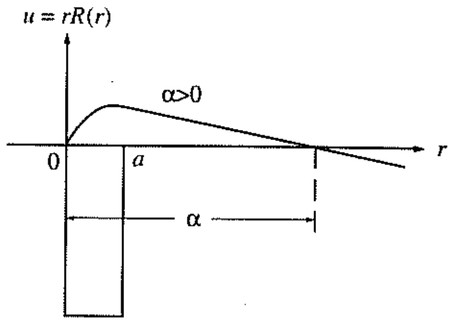


FIGURE 23.11
The scattering length α can be geometrically interpreted as the r-intercept of the asymptotic radial wave function $u = rR(r)$.

becomes

$$\frac{d^2u}{dr^2} \approx 0$$

Clearly, the asymptotic solution must be a linear function of r. To find the function, we look at the asymptotic form of the wave function in the limit of $k \to 0$:

$$r\psi = r\exp(i\mathbf{k}\cdot\mathbf{r}) + f_0\exp(ikr)$$
$$\to r - \alpha$$

as $k \to 0$. It is obvious that the scattering length is to be interpreted as the r-intercept of the asymptotic wave function $r\psi$ (fig. 23.11).

Scattering Lengths for Neutron-Proton Scattering

Coming back to the neutron-proton system, by comparing equations (23.60) and (23.62) we can see that

$$\alpha = \frac{k_B a + 1}{k_B} = a + \frac{1}{k_B}$$

For the deuteron binding energy of $|E| = 2.23$ MeV, noting that the reduced mass $\mu = m/2$, where m = nucleon mass = 940 MeV, we have

$$k_B^{-1} = [\hbar^2/m|E|]^{1/2} = (\hbar/mc)[mc^2/|E|]^{1/2}$$
$$\approx 4.3 \times 10^{-13} \text{ cm}$$

where we have substituted for the proton Compton wavelength $\hbar/mc$. Taking the range of the potential $a = 1.2 \times 10^{-13}$ cm, we get

$$\alpha = 5.5 \times 10^{-13} \text{ cm} = 5.5 \text{ F}$$

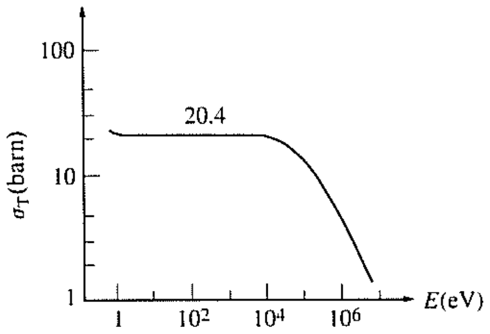


FIGURE 23.12
Experimental neutron-proton total scattering cross section at very low energies of 1 eV to 10 keV is constant; however, it is much greater in magnitude than the theoretical prediction. The answer is spin dependence of scattering.

The near-zero energy cross section is then given from equation (23.65) as

$$\sigma_T = 4\pi\alpha^2 \approx 4 \times 10^{-24}\ \text{cm}^2 = 4 \text{ barns}$$

Scattering cross section measurement with thermal neutrons gives a value of (fig. 23.12)

$$\sigma_T^{\text{exp}} \approx 20 \text{ barns}$$

This is a huge discrepancy!

The explanation is that the potential is different for the two different possible spin-states of the neutron-proton system, the singlet $S = 0$ and the triplet $S = 1$. When the spins are aligned, the neutrons scatter with a different cross section from the protons than when they are not. In a sense, this should be obvious since only the triplet potential is sufficiently deep to have a bound state, which is the deuteron ground state. The resultant cross section is, therefore, equal to the weighted sum of the triplet and singlet cross sections:

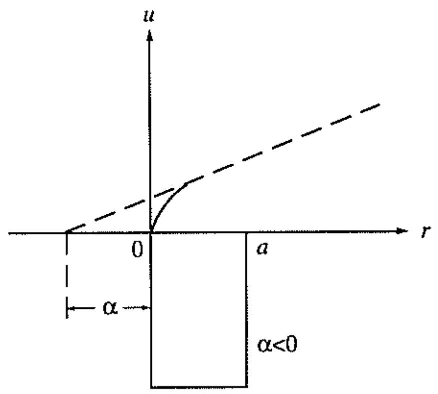
$$\sigma_T = 3\sigma_t/4 + \sigma_s/4 \tag{23.66}$$

where the subscripts t and s denote triplet and singlet, respectively. The 4 barns we estimated above are really the triplet cross section (since we used deuteron ground state for the estimate). So to obtain agreement with experiment, we must have

$$\sigma_s = 4\sigma_T - 3\sigma_t = 68 \text{ barns}$$

Such a large cross section suggests a resonance phenomenon. The wisdom from this is that the singlet potential barely misses producing a bound state at zero energy, hence the resonant enhancement of the near-zero energy cross section. The singlet wave function just misses turning over to have a node, and thus its

FIGURE 23.13
The neutron-proton scattering length for the singlet state.



r-intercept—the scattering length—is large and negative (fig. 23.13). The negative scattering length of the singlet n-p state has been verified by other means.

23.4 INELASTIC SCATTERING

Inelastic scattering occurs when the incident beam is robbed of flux, when there is a net loss of flux. Under this circumstance, the S-function introduced in equation (23.31) cannot be a pure phase, $\exp(2i\delta_l)$ with δ_l real. So how do we modify our formalism to account for the absorbed flux? The answer is to redefine the S-function:

$$S_l(k) = \eta_l(k)\exp[2i\delta_l(k)] \tag{23.67}$$

with $0 \le \eta_l(k) \le 1$. With this modification, the partial wave amplitude $f_l(k)$ is given by

$$\begin{aligned} f_l(k) &= \frac{S_l(k) - 1}{2ik} = \frac{\eta_l(k)\exp[2i\delta_l(k)] - 1}{2ik} \\ &= \frac{\eta_l \sin 2\delta_l}{2k} + i\,\frac{1 - \eta_l \cos 2\delta_l}{2k} \end{aligned} \tag{23.68}$$

The total elastic cross section is now easily obtained

$$\begin{aligned} \sigma_{Tel} &= 4\pi \sum_l (2l + 1)|f_l(k)|^2 \\ &= \pi \sum_l (2l + 1)\,\frac{1 + \eta_l^2 - 2\eta_l \cos 2\delta_l}{k^2} \end{aligned} \tag{23.69}$$

But there is also the job of calculating the cross section for the inelastic processes; fortunately, the total effect of all inelastic processes can be calculated without specifying precise mechanisms for inelasticity. We will calculate this total inelastic cross section by calculating the difference of the incoming flux and the outgoing flux and dividing the difference by the incident flux.

First note that the asymptotic total wave function can be written in terms of the $S_l(k)$'s as follows:

$$\psi_k(\mathbf{r}) \to \frac{i}{2k} \sum_l (2l+1) i^l \left(\frac{e^{-i(kr - l\pi/2)}}{r} - S_l(k) \frac{e^{i(kr - l\pi/2)}}{r} \right) P_l(\cos\theta) \quad (23.70)$$

where $S_l(k)$ is now given by equation (23.67). The radial flux $\int j_r r^2 \, d\Omega$ of the incoming lth partial wave

$$(2l+1)(i/2k)[\exp(-ikr)/r] P_l(\cos\theta)$$

is given as

$$-(1/4k^2)(\hbar k/\mu) 4\pi (2l+1)$$

where we have used the results of the angular integration:

$$\int P_l^2(\cos\theta) \, d\Omega = \frac{4\pi}{2l+1} \int Y_{l0}^2 \, d\Omega = \frac{4\pi}{2l+1}$$

Likewise, the radial flux of the outgoing part of the wave function, equation (23.70),

$$(2l+1)(iS_l/2k)[\exp(ikr)/r] P_l(\cos\theta)$$

is given as

$$[|S_l|^2/4k^2](\hbar k/\mu) 4\pi (2l+1)$$

But $|S_l|^2 = \eta_l^2$. In this way we see that the net flux of the lth partial wave is given by

$$(2l+1) \frac{\pi}{k^2} \frac{\hbar k}{\mu} (\eta_l^2 - 1)$$

The negative of this quantity is the flux removed from the incident beam. Dividing that with the incident flux $\hbar k/\mu$, we get the contribution of the lth partial wave to the inelastic total cross section. Summing over l, we get σ_{Tinel}:

$$\sigma_{Tinel} = \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1)(1 - \eta_l^2(k)) \quad (23.71)$$

The grand total of elastic and inelastic total cross sections is

$$\sigma_T = \sigma_{Tel} + \sigma_{Tinel} = (2\pi/k^2) \sum_l (2l+1)(1 - \eta_l \cos 2\delta_l) \quad (23.72)$$

Now let's verify a comment made earlier that the optical theorem remains valid even when inelastic processes are included. Calculate $\operatorname{Im} f(0)$, the forward scattering amplitude, with the present form of f_l, equation (23.68),

$$\begin{aligned} \operatorname{Im} f(0) &= \sum_l (2l+1) \operatorname{Im} f_l(k) \\ &= (1/2k) \sum_l (2l+1)(1 - \eta_l \cos 2\delta_l) \\ &= (k/4\pi)\sigma_T \end{aligned}$$

It just works out this way.

Example: Scattering from a Black Disc

A black disc is a perfectly absorbing disc with all

$$\eta_l(k) = 0$$

Additionally, we will assume that only l-values up to some l_{max} need to be considered (this is equivalent to assuming that the disc has a sharp edge). We also consider the situation of large k, thus

$$l_{max} = ka$$

With these caveats, the total inelastic cross section is given by the sum

$$\sigma_{Tinel} = \frac{\pi}{k^2} \sum_{l=0}^{ka} (2l+1) = \frac{\pi}{k^2} (ka+1)^2 \approx \pi a^2 \quad (23.73)$$

Perhaps it's a little surprising since we are considering total absorption, but equation (23.69) makes it clear that there is still elastic scattering going on; we have for the total elastic cross section

$$\sigma_{Tel} = \frac{\pi}{k^2} \sum_{l=0}^{ka} (2l+1) \approx \pi a^2 \quad (23.74)$$

Consequently, the total cross section

$$\sigma_T = \sigma_{Tel} + \sigma_{Tinel} = 2\pi a^2 \qquad (23.75)$$

The total cross section is twice the geometrical cross section of πa^2. This result, unexpected from a classical physics point of view, can be explained with the idea of *shadow scattering*. The point is that the shadow cast by the absorbing disc extends only up to a finite distance; far away we can't see a shadow, so the shadow must get filled in. But how so? Only if there is scattering of some of the wave at the edge of the disc. And this scattered flux must have the same magnitude as the flux that is taken out of the incident beam in order to do the job of filling up the shadow. You can see that the scattering and inelastic scattering cross sections would have to be the same, both must be πa^2, and their sum therefore exceeds the classically expected value by a factor of two.

23.5 OUTLOOK

We are going to end our discussion of scattering theory here. The main omission is the effect of spin degrees of freedom, best incorporated using a matrix formulation for the scattering amplitude, which is beyond the scope of this book.

With an exposition of the partial wave analysis of scattering, we have also formally treated most of the quantum phenomena of importance and most that is new. There are, of course, more surprises in store for the reader, for example, the relativistic equation of Dirac. And there are many more phenomena to calculate, some requiring more sophisticated mathematics than the level assumed in this book. But nobody expects any major revision of the basic principles of quantum mechanics introduced here.

Anyway, this is a good place to end the formal presentations of this book. There is, however, a final chapter, an unfinished one at that, which is an informal presentation of some of the ideas that may interest you. What is the meaning of the radical quantum principles? How do we interpret quantum mechanics? Is the ontological question (this is the one that turned on Einstein) of the philosophy of quantum mechanics answerable or worth answering?

PROBLEMS

1. Consider the scattering of particles of mass m from a target of objects of mass $M = 2m$. Suppose the beam has a laboratory kinetic energy of mc^2 and that we are observing scattering at an angle of 30°. (a) What is the relation of the scattering angles in the laboratory and the center-of-mass frames? (b) What is the relation between the differential cross sections?

2. Neutrons are scattered by protons at such energy that only S and P waves need be considered. Assume that the scattering potential is spherically symmetric.
   (a) Show that the differential cross section can be written in the form
   $$\sigma(\theta,\phi) = a + b\cos\theta + c\cos^2\theta$$
   (b) What are the values of a, b, c in terms of the phase shifts?
   (c) What is the value of the total cross section in terms of a, b, and c? What is the value of the forward scattering amplitude?
3. Derive equations (23.21) and (23.22). Verify by direct integration of equations (23.21) and (23.22) that the total cross sections are the same for both laboratory and center-of-mass coordinate systems.
4. Consider S-wave neutron-proton scattering in the square-well potential model (depth V_0 and range a). (a) Draw graphs showing the variation of phase shift with energy for (i) $2\mu V_0 a^2/\hbar^2 < \pi^2/4$, (ii) $\pi^2/4 < 2\mu V_0/\hbar^2 < 9\pi^2/4$, and (iii) $2\mu V_0 a^2/\hbar^2 = \pi^2/4$. (b) Show that if there is a bound state at zero energy, the scattering length α becomes ∞ and the effective range $r_0 = a$.
5. Calculate the S-wave phase shift for a repulsive square-well potential. Discuss the limit of large and small depth of potential and the case of very low incident energy.
6. Calculate the S-wave phase shift for the *hard core* potential:
   $$V(r) \to \infty \qquad r < c$$
   $$V(r) = 0 \qquad r > c$$
   What is the value of the S-wave scattering length?
7. If the neutron-neutron potential is the same as the neutron-proton potential (for which the only bound state exists for l even state), give an argument against a dineutron bound state.
8. What is the S-wave phase shift for a repulsive hard-core (of radius c) plus attractive square-well potential (fig. 23.14)?
9. Consider the S-wave neutron scattering by the delta-function potential
   $$2\mu V/\hbar^2 = -(s/c)\delta(r-c)$$
   Show that
   $$\tan\delta_0(k) = \frac{s\sin^2 kc/kc}{1 - s\sin kc\cos kc/kc}$$

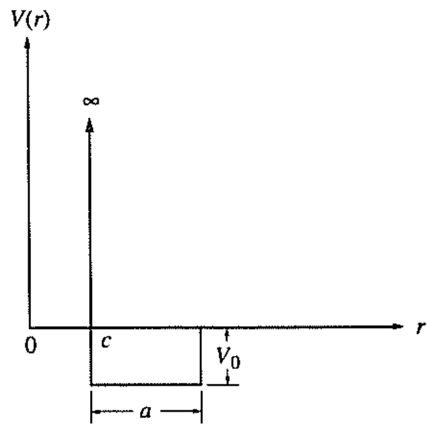


FIGURE 23.14

10. Consider the scattering of 6 MeV neutrons from a heavy black nucleus of diameter 5 F.
(a) What is the approximate number of partial waves that are affected?
(b) Calculate and make a simple plot of the elastic differential cross section $\sigma(\theta,\phi)$ as a function of θ.
(c) Calculate the total elastic, total inelastic, and total cross sections.

ADDITIONAL PROBLEMS

A1. Show that for a central potential $V(r)$ the scattering amplitude in the Born approximation can be written as

$$f_B(\theta) = -\frac{2\mu}{\hbar^2}\int_0^\infty r^2\,dr\,\frac{\sin qr}{qr}\,V(r)$$

where q is the momentum transfer as defined in figure 22.7.

A2. Using the formula

$$\frac{\sin qr}{qr} = \sum_l (2l+1)[j_l(kr)]^2 P_l(\cos\theta)$$

in the Born expression above for the scattering amplitude, show that for small phase shifts δ_l (for which $\exp[2i\delta_l] - 1 \approx 2i\delta_l$), the Born approximation for the phase shifts is given as

$$\delta_l^B \approx \frac{2\mu}{\hbar^2}\,k\int_0^\infty V(r)j_l^2(kr)r^2\,dr$$

A3. Explain why for two identical spin-$\frac{1}{2}$ fermions, the scattering amplitude should be written in the form

$$f = [f_s(\theta) + f_s(\pi-\theta)] + [f_t(\theta) - f_t(\pi-\theta)]$$

where the subscripts s and t refer to singlet and triplet spin states, respectively.

REFERENCES

A. Das and A. C. Mellissinos. *Quantum Mechanics: A Modern Introduction.*
S. Gasiorowicz. *Quantum Physics.*