Tarea 3: Física de Partículas
SOLUCIÓN

(1) • $[a(p), a^{\dagger}(p')]_{-} = \delta^3(p-p')$

Para saber el conmutador, se deben extraer los términos $a(p)$ y $a^{\dagger}(p)$ de la definición de ϕ y Π. Por simplicidad, denotaré $a(p)$ como a_p y $a(p')$ como a_q.

Recordar:

$$\Rightarrow [\phi(x), \Pi(y)]_{-} = i\delta^3(x-y) = [\phi^*(x), \Pi^*(y)]_{-} \quad \text{(los otros son cero)}$$

$$\Rightarrow \phi(x) = \frac{1}{\sqrt{(2\pi)^3}} \int \frac{d^3p}{\sqrt{2E_p}} \left(a_p e^{-ip\cdot x} + a_p^{\dagger} e^{ip\cdot x}\right) = \phi^*(x)$$

$$\Rightarrow \Pi(x) = \dot{\phi}(x) = \frac{i}{\sqrt{(2\pi)^3}} \int d^3p \sqrt{\frac{E_p}{2}} \left(a_p^{\dagger} e^{ip\cdot x} - a_p e^{-ip\cdot x}\right) = \Pi^*(x)$$

Inversa de Fourier:

$$\phi(x) \overset{\mathcal{F}}{\rightsquigarrow} \frac{1}{\sqrt{(2\pi)^3}} \int d^3x\, e^{iq\cdot x} \phi(x) = \frac{1}{(2\pi)^3} \int d^3x\, e^{iq\cdot x} \int \frac{d^3p}{\sqrt{2E_p}} \left(a_p e^{-ip\cdot x} + a_p^{\dagger} e^{ip\cdot x}\right)$$

$$= \frac{1}{(2\pi)^3} \int d^3x \int \frac{d^3p}{\sqrt{2E_p}} \left(a_p e^{i(q-p)\cdot x} + a_p^{\dagger} e^{i(p+q)\cdot x}\right) = \frac{1}{\sqrt{(2\pi)^3}} \int \frac{d^3p}{\sqrt{2E_p}} \left(\delta^3(q-p) a_p + \delta^3(q+p) a_p^{\dagger}\right)$$

$$= \boxed{\frac{1}{\sqrt{2E_q}} \left(a_q + a_{-q}^{\dagger}\right)} \quad \textbf{(a)} \rightarrow \tilde{\phi}(q)$$

Para el campo $\Pi(x)$ se tiene:

$$\Pi(x) \overset{\mathcal{F}}{\rightsquigarrow} \frac{i}{(2\pi)^3} \int d^3x\, e^{iq\cdot x} \int d^3p \sqrt{\frac{E_p}{2}} \left(a_p^{\dagger} e^{ip\cdot x} - a_p e^{-ip\cdot x}\right) = \frac{i}{(2\pi)^3} \int d^3x \int d^3p \sqrt{\frac{E_p}{2}} \left(a_p^{\dagger} e^{i(p+q)\cdot x} - a_p e^{i(q-p)\cdot x}\right)$$

$$= \frac{i}{\sqrt{(2\pi)^3}} \int d^3p \sqrt{\frac{E_p}{2}} \left(\delta^3(p+q) a_p^{\dagger} - \delta^3(q-p) a_p\right) = \boxed{i\sqrt{\frac{E_q}{2}} \left(a_{-q}^{\dagger} - a_q\right)} \quad \textbf{(b)} \rightarrow \tilde{\Pi}(q)$$

$*$ $\sqrt{2E_q}\,(a) + i\sqrt{\frac{2}{E_q}}\,(b) = 2a_q$

$*$ $\sqrt{2E_q}\,(a) - i\sqrt{\frac{2}{E_q}}\,(b) = 2a_{-q}^{\dagger}$

Con lo anterior,

$a_q = \frac{\sqrt{2E_q}}{2} \tilde{\phi}(q) + \frac{i}{2}\sqrt{\frac{2}{E_q}}\, \tilde{\Pi}(q) \longrightarrow$ conjugado, tenemos $a_q^{\dagger}$.

Con la definición de las transformadas:

$$a_q = \frac{\sqrt{2E_q}}{2}\,\frac{1}{\sqrt{(2\pi)^3}}\int d^3x\, e^{iq\cdot x}\phi(x) + \frac{i}{2}\sqrt{\frac{2}{E_q}}\int d^3x'\, e^{iq\cdot x'}\pi(x')\left(\frac{1}{\sqrt{(2\pi)^3}}\right)$$

$$\boxed{a_q = \frac{1}{\sqrt{(2\pi)^3}}\int d^3x\, e^{iq\cdot x}\left[\sqrt{\frac{E_q}{2}}\,\phi(x) + i\sqrt{\frac{1}{2E_q}}\,\pi(x)\right]}$$

$\downarrow$ conjugar

$$\boxed{a_q^\dagger = \frac{1}{\sqrt{(2\pi)^3}}\int d^3x\, e^{-iq\cdot x}\left[\sqrt{\frac{E_q}{2}}\,\phi(x) - i\sqrt{\frac{1}{2E_q}}\,\pi(x)\right]}$$

Ahora sí, saquemos el primer conmutador:

$$[a_p, a_q^\dagger] = a_p a_q^\dagger - a_q^\dagger a_p = \frac{1}{(2\pi)^3}\Big\{\int d^3x\, e^{ip\cdot x}\left[\sqrt{\frac{E_p}{2}}\,\phi(x) + i\sqrt{\frac{1}{2E_p}}\,\pi(x)\right]\int d^3y\, e^{-iq\cdot y}\left[\sqrt{\frac{E_q}{2}}\,\phi(y) - i\sqrt{\frac{1}{2E_q}}\,\pi(y)\right]$$

$$-\int d^3y\, e^{-iq\cdot y}\left[\sqrt{\frac{E_q}{2}}\,\phi(y) - i\sqrt{\frac{1}{2E_q}}\,\pi(y)\right]\int d^3x\, e^{ip\cdot x}\left[\sqrt{\frac{E_p}{2}}\,\phi(x) + i\sqrt{\frac{1}{2E_p}}\,\pi(x)\right]\Big\}$$

$$= \frac{1}{(2\pi)^3}\int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)}\Big\{\Big[\frac{\sqrt{E_pE_q}}{2}\,\phi(x)\phi(y) - \frac{i}{2}\sqrt{\frac{E_p}{E_q}}\,\phi(x)\pi(y) + \frac{i}{2}\sqrt{\frac{E_q}{E_p}}\,\pi(x)\phi(y) + \frac{1}{2}\sqrt{\frac{1}{E_pE_q}}\,\pi(x)\pi(y)\Big]$$

$$-\Big[\frac{1}{2}\sqrt{E_qE_p}\,\phi(y)\phi(x) - \frac{i}{2}\sqrt{\frac{E_q}{E_p}}\,\phi(y)\pi(x) + \frac{i}{2}\sqrt{\frac{E_p}{E_q}}\,\pi(y)\phi(x) + \frac{1}{2}\sqrt{\frac{1}{E_qE_p}}\,\pi(y)\pi(x)\Big]\Big\}$$

$$= \frac{1}{(2\pi)^3}\int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)}\Big\{\frac{\sqrt{E_pE_q}}{2}\overset{0}{\cancel{[\phi(x),\phi(y)]}} + \frac{i}{2}\sqrt{\frac{E_p}{E_q}}\,[\pi(y),\phi(x)] + \frac{i}{2}\sqrt{\frac{E_q}{E_p}}\,[\pi(x),\phi(y)]$$

$$+ \frac{1}{2}\sqrt{\frac{1}{E_pE_q}}\,\overset{0}{\cancel{[\pi(x),\pi(y)]}}\Big\}$$

$$= \frac{1}{2(2\pi)^3}\int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)}\Big\{\sqrt{\frac{E_p}{E_q}}\,\delta^3(x-y) + \sqrt{\frac{E_q}{E_p}}\,\delta^3(y-x)\Big\}$$

$$= \frac{1}{2(2\pi)^3}\int d^3x\, e^{i(p-q)\cdot x}\Big\{\sqrt{\frac{E_p}{E_q}} + \sqrt{\frac{E_q}{E_p}}\Big\} = \frac{1}{2}\,\delta^3(p-q)\Big\{2\sqrt{\frac{E_p}{E_p}}\Big\} = \delta^3(p-q)$$

• $[a(p), a(p')]_- = 0$

$$a_pa_q - a_qa_p = \frac{1}{(2\pi)^3}\int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)}\Big\{\left[\sqrt{\frac{E_p}{2}}\,\phi(x) + i\sqrt{\frac{1}{2E_p}}\,\pi(x)\right]\left[\sqrt{\frac{E_q}{2}}\,\phi(y) + i\sqrt{\frac{1}{2E_q}}\,\pi(y)\right]$$

$$-\left[\sqrt{\frac{E_q}{2}}\,\phi(y) + i\sqrt{\frac{1}{2E_q}}\,\pi(y)\right]\left[\sqrt{\frac{E_p}{2}}\,\phi(x) + i\sqrt{\frac{1}{2E_p}}\,\pi(x)\right]\Big\}$$

$$= \frac{1}{(2\pi)^3}\int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)}\Big\{\frac{1}{2}\sqrt{E_pE_q}\,[\cancel{\phi(x),\phi(y)}]^{0} + \frac{i}{2}\sqrt{\frac{E_p}{E_q}}[\phi(x),\pi(y)] + \frac{i}{2}\sqrt{\frac{E_q}{E_p}}\underbrace{[\pi(x),\phi(y)]}_{=-[\pi(x),\phi(y)]}$$

$$+ \frac{i}{2}\sqrt{\frac{1}{E_pE_q}}[\cancel{\pi(y),\pi(x)}]^{0}\Big\}$$

$$= \frac{1}{2(2\pi)^3}\int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)}\Big\{-\sqrt{\frac{E_p}{E_q}}\,\delta^3(x-y) + \sqrt{\frac{E_q}{E_p}}\,\delta^3(x-y)\Big\} = \frac{1}{2(2\pi)^3}\int d^3x\, e^{i(p-q)\cdot x}\left(\sqrt{\frac{E_q}{E_p}} - \sqrt{\frac{E_p}{E_q}}\right)$$

$$= \frac{1}{2}\,\underset{\substack{\downarrow \\ \text{cero si } p\neq q}}{\delta(p-q)}\underset{\substack{\downarrow \\ \text{cero si } p=q}}{\left(\sqrt{\frac{E_q}{E_p}} - \sqrt{\frac{E_p}{E_q}}\right)} = 0$$

- $[a_p^\dagger, a_q^\dagger] = 0$

Si $[a_p, a_q] = 0 \rightarrow a_p a_q - a_q a_p = 0 \rightarrow$ conjugar $\rightarrow a_q^\dagger a_p^\dagger - a_p^\dagger a_q^\dagger = 0$

$$\downarrow$$

$$a_p^\dagger a_q^\dagger - a_q^\dagger a_p^\dagger = 0$$

$$\downarrow$$

$$[a_p^\dagger, a_q^\dagger] = 0$$

② • $H = \frac{1}{2}\int d^3p\, E_p\,[a_p^\dagger a_p + a_p a_p^\dagger]$

Si la densidad hamiltoniana es $\mathcal{H} = \frac{1}{2}(\pi^2 + (\nabla\phi)^2 + m^2\phi^2)$, H es su integral en el espacio. Así pues, hay que integrar cada término usando las definiciones de los campos cuantizados.

a) $$\int d^3x\, \frac{\pi(x)^2}{2} = \int \frac{d^3x}{2}\left(\frac{-1}{(2\pi)^3}\right)\int d^3p\sqrt{\frac{E_p}{2}}\left(a_p^\dagger e^{ip\cdot x} - a_p e^{-ip\cdot x}\right)\int d^3q\sqrt{\frac{E_q}{2}}\left(a_q^\dagger e^{iq\cdot x} - a_q e^{-iq\cdot x}\right)$$

$$= \int \frac{d^3x}{4}\left(\frac{-1}{(2\pi)^3}\right)\int d^3p\, d^3q\sqrt{E_pE_q}\left(a_p^\dagger a_q^\dagger e^{i(p+q)\cdot x} - a_p^\dagger a_q e^{i(p-q)\cdot x} - a_p a_q^\dagger e^{i(q-p)\cdot x} + a_p a_q e^{-i(p+q)\cdot x}\right)$$

$$= -\frac{1}{4}\int d^3p\int d^3q\,\sqrt{E_pE_q}\left(a_p^\dagger a_q^\dagger e^{i(E_p+E_q)t}\delta^3(p+q) - a_p^\dagger a_q e^{i(E_p-E_q)t}\delta^3(p-q) - a_p a_q^\dagger e^{i(E_q-E_p)t}\delta^3(q-p) + a_p a_q e^{i(-E_p-E_q)t}\delta^3(p+q)\right)$$

$$= -\frac{1}{4}\int d^3p\,(E_p)\left(a_p^\dagger a_p^\dagger e^{2E_pit} - a_p^\dagger a_p - a_p a_p^\dagger + a_p a_p e^{-2iE_pt}\right)$$

$$= \boxed{\frac{1}{4}\int d^3p\, E_p\left(a_p^\dagger a_p + a_p a_p^\dagger - a_p^\dagger a_p^\dagger e^{2E_pit} - a_p a_p e^{-2E_pit}\right)}$$

b) $\int d^3x \frac{(\nabla\phi)^2}{2} = \frac{1}{4(2\pi)^3}\int\frac{d^3p\,d^3q}{\sqrt{E_pE_q}}\int d^3x\,\left(a_p(-i\vec{p})e^{-ip\cdot x}+a_p^\dagger(i\vec{p})e^{ip\cdot x}\right)\left(a_q(-i\vec{q})e^{-iq\cdot x}+a_q^\dagger(i\vec{q})e^{iq\cdot x}\right)$

$= \frac{1}{4}\int\frac{d^3p\,d^3q}{\sqrt{E_pE_q}}\int\frac{d^3x}{(2\pi)^3}\left(-a_pa_q\,\vec{p}\cdot\vec{q}\,e^{i(-p-q)\cdot x}+a_pa_q^\dagger\,\vec{p}\cdot\vec{q}\,e^{i(q-p)\cdot x}+a_p^\dagger a_q\,\vec{p}\cdot\vec{q}\,e^{i(p-q)\cdot x}-a_p^\dagger a_q^\dagger(\vec{p}\cdot\vec{q})e^{i(p+q)\cdot x}\right)$

$= \frac{1}{4}\int\frac{d^3p\,d^3q}{\sqrt{E_pE_q}}\vec{p}\cdot\vec{q}\left(-a_pa_q\,e^{i(-E_p-E_q)t}\underbrace{\delta^3(p+q)}_{p=-q}+a_pa_q^\dagger e^{i(E_p-E_q)t}\delta^3(q-p)+a_p^\dagger a_q\,e^{i(E_p-E_q)t}\delta^3(p-q)-a_p^\dagger a_q^\dagger e^{i(E_p+E_q)t}\underbrace{\delta^3(p+q)}_{p=-q}\right)$

$= \boxed{\frac{1}{4}\int\frac{d^3p}{E_p}\vec{p}^2\left(a_pa_p\,e^{-2iE_pt}+a_pa_p^\dagger+a_p^\dagger a_p+a_p^\dagger a_p^\dagger e^{2iE_pt}\right)}$

c) $\frac{m^2}{2}\int d^3x\,\phi^2(x)$ queda igual que el anterior pero sin el $(\vec{p})^2$, cambiando algunos signos y con m^2:

$= \boxed{\frac{m^2}{4}\int\frac{d^3p}{E_p}\left(a_pa_p\,e^{-2iE_pt}+a_pa_p^\dagger+a_p^\dagger a_p+a_p^\dagger a_p^\dagger e^{2iE_pt}\right)}$

$\Rightarrow\ H = a)+b)+c) = \frac{1}{4}\int d^3p\Big\{E_p\left(\underline{a_p^\dagger a_p}+\underline{a_pa_p^\dagger}-\underline{a_pa_p\,e^{-2iE_pt}}-\underline{a_p^\dagger a_{-p}^\dagger e^{2iE_pt}}\right)$

$+\frac{\vec{p}^2}{E_p}\left(\underline{a_pa_{-p}e^{-2iE_pt}}+\underline{a_pa_p^\dagger}+\underline{a_p^\dagger a_p}+\underline{a_p^\dagger a_p^\dagger e^{2iE_pt}}\right)$

$+\frac{m^2}{E_p}\left(\underline{a_pa_pe^{-2iE_pt}}+\underline{a_pa_p^\dagger}+\underline{a_p^\dagger a_p}+\underline{a_p^\dagger a_p^\dagger e^{2iE_pt}}\right)$

$= \frac{1}{4}\int\frac{d^3p}{E_p}\left[(E_p^2+\vec{p}^2+m^2)a_p^\dagger a_p+(E_p^2+\vec{p}^2+m^2)a_pa_p^\dagger+\overset{0}{\cancel{(-E_p^2+\vec{p}^2+m^2)}}e^{-2iE_pt}a_pa_p+\overset{0}{\cancel{(-E_p^2+\vec{p}^2+m^2)}}e^{2iE_pt}a_p^\dagger a_p^\dagger\right]$

Note que por definición, $E_p^2=\vec{p}^2+m^2$, por lo cual se cancelan varios términos y queda:

$$H = \frac{1}{4}\int\frac{d^3p}{E_p}\left(2E_p^2a_p^\dagger a_p+2E_p^2a_pa_p^\dagger\right) = \frac{1}{2}\int d^3p\,E_p\left(a_p^\dagger a_p+a_pa_p^\dagger\right)$$

3 • $[H, a_{p'}]_- = -E_p a_{p'}$

$[H, a_q] = \frac{1}{2}\int d^3p\,E_p\{[a_p^\dagger a_p+a_pa_p^\dagger, a_q]$

$= \frac{1}{2}\int d^3p\,E_p\{[a_p^\dagger a_p, a_q]+[a_pa_p^\dagger, a_q]\} = \frac{1}{2}\int d^3p\,E_p\left(\cancel{a_p^\dagger[a_p,a_q]}^{\,0}+[a_p^\dagger,a_q]a_p+a_p[a_p^\dagger,a_q]+\cancel{[a_p,a_q]}^{\,0}a_p^\dagger\right)$

$= \frac{1}{2}\int d^3p\,E_p\left\{\delta^3(p-q)(-a_p)-a_p\,\delta^3(p-q)\right\}$

$= \frac{1}{\cancel{2}}E_p(-\cancel{2}a_q) = \boxed{-E_p a_q}$

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• $[H, a_q^\dagger] = E_p a_q^\dagger$

$$[H, a_q^\dagger] = \frac{1}{2}\int d^3p\, E_p\left([a_p^\dagger a_p, a_q^\dagger] + [a_p a_p^\dagger, a_q^\dagger]\right) = \frac{1}{2}\int d^3p\, E_p\left(a_p^\dagger[a_p, a_q^\dagger] + [a_p, a_q^\dagger]a_p^\dagger\right)$$

$$= \frac{1}{2}\int d^3p\, E_p\left(a_p^\dagger \delta^3(p-q) + \delta^3(p-q)a_p^\dagger\right) = \boxed{E_q a_q^\dagger}$$

④ • $\gamma_\lambda \gamma^\lambda = 4$

la identidad básica para las matrices γ es que $[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}\mathbb{1}_4$.

Así pues:

$$\gamma_\mu \gamma^\mu = g_{\mu\nu}\gamma^\nu\gamma^\mu = \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu})\gamma^\nu\gamma^\mu$$ → porque $g_{\mu\nu} = g_{\nu\mu}$ podemos reescribirlo así.

$$= \frac{1}{2}(g_{\mu\nu}\gamma^\nu\gamma^\mu + \underbrace{g_{\nu\mu}\gamma^\nu\gamma^\mu}_{\mu\leftrightarrow\nu}) = \frac{1}{2}(g_{\mu\nu}\gamma^\nu\gamma^\mu + g_{\mu\nu}\gamma^\mu\gamma^\nu) = \frac{g_{\mu\nu}}{2}\underbrace{(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu)}_{=[\gamma^\mu, \gamma^\nu]_+}$$

→ matriz 4 $\mathbb{1}_4$

$$= \frac{g_{\mu\nu}}{2}2g^{\mu\nu}\mathbb{1}_4 = \underbrace{g_{\mu\nu}g^{\mu\nu}}_{g_{\mu\nu} = g^{\mu\nu}}\mathbb{1}_4 = \boxed{4}$$

→ suma implícita: $\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}(g_{\mu\nu})^2 = g_{00}^2 + g_{11}^2 + g_{22}^2 + g_{33}^2 = 4.$

↓ nulo excepto si $\mu = \nu$

• $\gamma_\lambda \gamma_\mu \gamma^\lambda = -2\gamma_\mu$

$$\gamma_\lambda\gamma_\mu\gamma^\lambda = (2g_{\lambda\mu}\mathbb{1}_4 - \gamma_\mu\gamma_\lambda)\gamma^\lambda = 2g_{\lambda\mu}\gamma^\lambda - \gamma_\mu\underbrace{\gamma_\lambda\gamma^\lambda}_{=4} = 2\gamma_\mu - \gamma_\mu 4 = \boxed{-2\gamma_\mu}$$

• $\gamma_\lambda\gamma_\mu\gamma_\nu\gamma^\lambda = 4g_{\mu\nu}$

$$\gamma_\lambda\gamma_\mu\gamma_\nu\gamma^\lambda = ([\gamma_\lambda, \gamma_\mu]_+ - \gamma_\mu\gamma_\lambda)\gamma_\nu\gamma^\lambda = 2g_{\lambda\mu}\mathbb{1}_4\gamma_\nu\gamma^\lambda - \gamma_\mu\gamma_\lambda\gamma_\nu\gamma^\lambda$$

$$= 2\gamma_\nu\gamma_\mu - \gamma_\mu([\gamma_\lambda, \gamma_\nu]_+ - \gamma_\nu\gamma_\lambda)\gamma^\lambda = 2\gamma_\nu\gamma_\mu - \gamma_\mu 2g_{\lambda\nu}\mathbb{1}_4\gamma^\lambda + \gamma_\mu\gamma_\nu\gamma_\lambda\gamma^\lambda$$

$$= 2\gamma_\nu\gamma_\mu - 2\gamma_\mu\gamma_\nu + \gamma_\mu\gamma_\nu 4$$

$$= 2\gamma_\nu\gamma_\mu + 2\gamma_\mu\gamma_\nu = 2(\gamma_\nu\gamma_\mu + \gamma_\mu\gamma_\nu) = 2(2g_{\nu\mu}) = \underset{\nu\leftrightarrow\mu}{4g_{\nu\mu}} = \boxed{4g_{\mu\nu}}$$

5) $\Pi_{\pm} \to \frac{1}{2}(1 \pm \gamma_5)$ si $m \to 0$.

$\Pi_{\pm}(\vec{p}) \to$ Operadores de proyección de helicidad.

$\Pi_{\pm} = \frac{1}{2}(1 \pm \Sigma_p)$ donde $\Sigma_p = \frac{\vec{\Sigma}\cdot\vec{P}}{p}$ con $\vec{\Sigma} = (\sigma^{23}, \sigma^{31}, \sigma^{12})$

$\sigma^{ij} = \frac{i}{2}[\gamma^i, \gamma^j]_+ = i\gamma^i\gamma^j$ si $i \neq j$

De Dirac: $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \to m=0$

$i\gamma^\mu\partial_\mu\psi(x) = 0$

Sol: $\psi = u(\vec{p})e^{-ip\cdot x}$

En este caso, $u(\vec{p})$ satisface que $\not{p}\, u(\vec{p}) = 0 = \gamma^\mu p_\mu u(\vec{p}) = (\gamma^0 p^0 - \vec{\gamma}\cdot\vec{p})u(\vec{p}) = 0$

$(\gamma^0 p^0)u(\vec{p}) = (\vec{\gamma}\cdot\vec{p})u(\vec{p})$

$\Rightarrow \gamma^5 \underbrace{\gamma^0\gamma^0}_{=\mathbb{1}} p^0 u(\vec{p}) = \gamma^5\gamma^0(\vec{\gamma}\cdot\vec{p})u(\vec{p})$

$\gamma^5 p^0 u(\vec{p}) = \gamma^5\gamma^0\gamma^i p^i u(\vec{p}) = \underbrace{i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^i}_{I} p^i u(\vec{p})$

Usaremos $[\gamma^\mu, \gamma^\nu]_+ = 2g_{\mu\nu}\mathbb{1}_4 \to$ Si $\mu \neq \nu$, anticonmutan: $\gamma^\mu = -\gamma^\nu$.

Si $i=1$:

$I = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1 = i(-1)^3\gamma^1\gamma^2\gamma^3\underbrace{(\gamma^0)^2}_{1}\gamma^1 = -i\gamma^1\gamma^2\gamma^3\gamma^1 = -i\underbrace{(\gamma^1)^2}_{-1}\gamma^2\gamma^3 = i\gamma^2\gamma^3 = \sigma^{23}$

Si $i=2$:

$I = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^2 = i\gamma^0\gamma^1\underbrace{(\gamma^2)^2}_{-1}\gamma^3\gamma^0 = -i\gamma^0\gamma^1\gamma^3\gamma^0 = -i\gamma^1\gamma^3 = i\gamma^3\gamma^1 = \sigma^{31}$

Si $i=3$:

$I = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^3 = -i\gamma^0\gamma^1\gamma^2\underbrace{(\gamma^3)^2}_{-1}\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^0 = i\gamma^1\gamma^2 = \sigma^{12}$

Entonces: $\gamma^5 p^0 u(\vec{p}) = \underbrace{(\sigma^{23}p^1 + \sigma^{31}p^2 + \sigma^{12}p^3)}_{=\vec{\Sigma}\cdot\vec{p}}u(\vec{p})$

$\Rightarrow \gamma^5 p^0 = \vec{\Sigma}\cdot\vec{p} \to$ si $m=0$, $p^0 = E = p \to \boxed{\gamma^5 = \frac{\vec{\Sigma}\cdot\vec{p}}{p}}$

Por lo cual $\Pi_{\pm} := \frac{1}{2}\left(1 \pm \frac{\vec{\Sigma}\cdot\vec{p}}{p}\right) = \boxed{\frac{1}{2}(1 \pm \gamma_5)}$ si $m=0$.