1. Ver solución en las notas de clase.

Problema 3: Corriente de Noether:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^+)}\delta\phi^+$$

$$\therefore \quad \delta\phi = -iq\theta \; ; \quad \delta\phi^+ = iq\theta$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi^+ \quad y \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^+)} = \partial_\mu \phi$$

$$\therefore \quad J^\mu = (\partial^\mu \phi^+)(-iq\theta\phi) + (\partial_\mu \phi)(iq\theta\phi^+)$$

Ahora $\phi^+ = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left(a^+(p)e^{ip\cdot x} + \hat{a}(p)e^{-ip\cdot x}\right)$

① $\partial^\mu \phi^+ = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left(ia^+(p)p^\mu e^{ip\cdot x} - i\hat{a}(p)p^\mu e^{-ip\cdot x}\right)$

② $\partial^\mu \phi = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left(-ia(p)p^\mu e^{-ip\cdot x} + i\hat{a}^+(p)p^\mu e^{ip\cdot x}\right)$

Note que si multiplicamos $\partial^\mu\phi^+$ por ϕ y $\partial^\mu\phi$ por ϕ^+

$$(\partial^\mu \phi)\phi^\dagger = \int \frac{d^3p\,d^3p'}{(2\pi)^3}\frac{1}{2\sqrt{E_p E_{p'}}}\, ip^\mu \Big(-a(p)a^\dagger(p')e^{-i(p-p')\cdot x} + \hat{a}^\dagger(p)\hat{a}(p')e^{i(p-p')\cdot x}$$
$$- a(p)\hat{a}(p')e^{-i(p+p')\cdot x} + \hat{a}^\dagger(p)a^\dagger(p')e^{i(p+p')\cdot x}\Big)$$

Similarmente:

$$(\partial^\mu \phi^\dagger)(\phi) = \int \frac{d^3p\,d^3p'}{(2\pi)^3}\cdot\frac{1}{2\sqrt{E_p\cdot E_{p'}}}\, ip^\mu \Big(a^\dagger(p)a(p')e^{i(p-p')\cdot x} - \hat{a}(p)\hat{a}^\dagger(p')e^{-i(p-p')\cdot x}$$
$$- \hat{a}(p)a(p')e^{-i(p+p')\cdot x} + a^\dagger(p)\hat{a}^\dagger(p')e^{i(p+p')\cdot x}\Big)$$

Ahora, multiplicando por $iq\theta$ o $-iq\theta$ y calculando la expresión para la corriente:

$$J^\mu(x) = iq\theta \int \frac{d^3p\,d^3p'}{(2\pi)^3}\frac{1}{2\sqrt{E_p E_{p'}}}\, ip^\mu \Big[\big(\hat{a}(p)a^\dagger(p') - a(p)a^\dagger(p')\big)e^{-i(p-p')\cdot x}$$
$$+ \big(a^\dagger(p)a(p') - a(p)a^\dagger(p')\big)e^{i(p-p')\cdot x} + \big(\hat{a}(p)a(p') - a^\dagger(p)a(p')\big)e^{-i(p+p')\cdot x}$$
$$+ \big(\hat{a}^\dagger(p)a^\dagger(p') - a^\dagger(p)a^\dagger(p')\big)e^{i(p+p')\cdot x}\Big]$$

Ahora, para hallar la carga conservada:

$$Q = \int d^3x\, J^\mu(x)$$

Note que van a aparecer terminos de la forma:

$$\int d^3x\, e^{\pm i(p\pm p')\cdot x} = \int d^3x\, e^{\pm i(E_p \pm E_{p'})\cdot t}\, e^{\mp i(\bar{p}\pm\bar{p}')\cdot\bar{x}}$$

$$= (2\pi)^3 e^{\pm i(E_p + E_{p'})}\, \delta^3(\bar{p}\pm\bar{p}')$$

Con esto, podemos simplificar la expresión, ya que resulta sólo una integral sobre d^3p.

$$\therefore \int d^3x\, J^\mu(x) = iq\theta \int d^3p\, \frac{1}{2E_p}\, i p^\mu \Big[\hat{a}\hat{a}^\dagger - aa^\dagger + a^\dagger a - a^\dagger a \Big] +$$

$$\Big[\big(a(p)a(-p) - a(p)\vec{a}(-p) \big) e^{-2iE_pt} +$$

$$\big(a^\dagger(p) a^\dagger(-p) - a^\dagger(p) a^\dagger(-p) \big) e^{2iE_pt} \Big]$$

Ahora, para $\mu = 0$; $p^\mu = E_p$.

$$Q = iq\theta \int d^3p\, \frac{i}{2} \Big(\hat{a}(p)\vec{a}^\dagger(p) - a(p)a^\dagger(p) + \hat{a}^\dagger(p)\vec{a}(p) - a^\dagger(p)a(p) \Big) +$$

$$\big(\hat{a}(p)a(-p) - \vec{a}(-p)a(p) \big) e^{-2iE_pt} +$$

$$\big(\hat{a}^\dagger(p) a^\dagger(-p) - \vec{a}^\dagger(-p) a^\dagger(p) \big) e^{2iE_pt} \Big)$$

• Note que las integrales de los términos exponenciales tienen la siguiente característica:

$$\int d^3p\, \hat{a}(-p)\, a(p)\, e^{-2iE_p t} \qquad \text{si} \quad p \to -p$$

entonces $$\int_{\mathbb{R}^3} d^3(-p)\, \hat{a}(p)\, a(-p)\, e^{-2iE_p t} = (-1)^3 \int_{-\mathbb{R}^3} d^3p\, \hat{a}(p)\, a(-p)\, e^{-2E_p t}$$

$$= \int_{\mathbb{R}^3} d^3p\, \hat{a}(p)\, a(-p)\, e^{-2E_p t}$$

Con esta propiedad, note que las integrales relacionadas con los exponenciales se anulan. Entonces:

$$Q = \frac{q\theta}{2} \int d^3p \left[a(p)a^{\dagger}(p) - \hat{a}(p)\hat{a}^{\dagger}(p) + a^{\dagger}(p)a(p) - \hat{a}^{\dagger}(p)\hat{a}(p) \right]$$

Como $$\left[a(\bar{p}), a^{\dagger}(\bar{p}') \right]_- = \delta^3(\bar{p} - \bar{p}')$$

$$\left[\hat{a}(p), \hat{a}^{\dagger}(p') \right] = \delta(\bar{p} - \bar{p}')$$

$$Q = \frac{q\theta}{2} \int d^3p \left[a^{\dagger}(p)a(p) + 1 - \hat{a}^{\dagger}(p)\hat{a}(p) - 1 + a^{\dagger}(p)a(p) - \hat{a}^{\dagger}(p)\hat{a}(p) \right]$$

$$Q = q\Theta \int d^3p \left[a^{\dagger}(p)\, a(p) - \hat{a}^{\dagger}(p)\, a + \right]$$

$$Q = qQ \int d^3p \left[N_a - N_{\hat{a}} \right]$$

Problema 2:

Sabemos que $T^{00} = \partial^0\phi\,\partial^0\phi - \delta^0_0\,\mathcal{L}$

$\therefore\ \partial^0\phi = \pi$, $\delta^0_0 = 1$, y $\mathcal{L} = \frac{1}{2}\left(\partial^\alpha\phi\,\partial_\alpha\phi - m^2\phi^2\right)$

$$T^{00} = \pi^2 - \frac{1}{2}\left(\partial^\alpha\phi\,\partial_\alpha\phi - m^2\phi^2\right)$$

Pero $\partial^\alpha\phi = \pi - \overline{\nabla}\phi$

$$T^{00} = \pi^2 - \frac{1}{2}\left[(\pi - \overline{\nabla}\phi)(\pi + \overline{\nabla}\phi) - m^2\phi^2\right]$$

$$T^{00} = \frac{1}{2}\left[\pi^2 + (\overline{\nabla}\phi)^2 + m^2\phi^2\right]$$

$$\therefore\ H = \frac{1}{2}\int d^3x\left[\pi^2 + (\overline{\nabla}\phi)^2 + m^2\phi^2\right]$$

Problema 4:

$$\mathcal{L} = \frac{i}{2}\bar{\psi}\gamma^\mu \partial_\mu \psi - \frac{i}{2}(\partial_\mu \bar{\psi})\gamma^\mu \psi - m\bar{\psi}\psi$$

$$\therefore \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}\right) = \frac{\partial \mathcal{L}}{\partial \bar{\psi}} ; \quad \text{Note que} \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = -\frac{i}{2}\gamma^\mu \psi$$

$$y \quad \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = \frac{i}{2}\gamma^\mu \partial_\mu \psi - m\psi$$

Con esto, tenemos: $\frac{i}{2}\gamma^\mu \partial_\mu \psi + \frac{i}{2}\gamma^\mu \partial_\mu \psi - m\psi = 0$

$$(\gamma^\mu \partial_\mu - m)\psi = 0$$

Ahora $J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\delta\psi + \delta\bar{\psi}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)}$

Consideramos una transformación $\psi \to \psi' = e^{-i\beta}\psi$

donde $\beta \to 0 \therefore \quad \delta\psi = -i\beta\psi \quad y$

$$\delta\bar{\psi} = i\beta\bar{\psi}$$

$$J^{\mu} = \left(\frac{i}{2}\overline{\psi}\gamma^{\mu}\right)\left(-i\beta\psi\right) + \left(i\beta\overline{\psi}\right)\left(\frac{-i}{2}\gamma^{\mu}\psi\right)$$

$$J^{\mu} = \beta\overline{\psi}\gamma^{\mu}\psi$$