Lecture 12 Scribe

MCS 2-Linear Algebra
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1 Linear Transformation

1.1 Linear Functional

If V is a vector space over the field F, a linear transformation f from V into the scalar field F is also called a linear functional on V.

Example-

$$f : F^n -> F$$

$$f(x_1, \cdots, x_n) = \sum_{n=1}^{n} x_i a_i$$

where $a_1 \cdots a_n$, $x_1 \cdots x_n \in F$

1.2 Hyper Space

In a vector of dimension **n**, a subspace of dimension **(n-1)** is called a hyper space.

Null Space(f) is a hyper space.

1.3 The Annihilator of a Set $S \subseteq V$.

Annihilator of a set $S \subseteq V$ is the set of all linear functional on V such that

$$f(\alpha) = 0 \quad \forall \quad \alpha \in S \tag{1}$$

$$S^0 = Annihilator \quad of \quad S$$

$\mathbf{S} = \{0\}$

$\mathbf{S}^0$ = Collection of all linear functional = V*

If $\mathbf{S} = V$
$\mathbf{S}^0 = \{0\}$ is a zero subspace of $\mathbf{V}^*$

Theorem 1- If V be a vector space over F and W be a subspace of V ,then.

$$\dim(W) + \dim(W^0) = \dim(V) \tag{2}$$

Proof. Let $\{\alpha_1, \cdots \alpha^k\}$ be a basis of W

Now choose $\{\alpha_k + 1, \cdots \alpha^n\}$ such that

$\{\alpha_1, \cdots \alpha^n\}$ is a basis of W.

Then define B^* =$\{f_1, \cdots f_n\}$ be the basis of $V^* =$

such that $f_i(\alpha_j) = \delta_i j$

Claim. $\{f_k + 1, \cdots f_n\}$ is a basis for W^0

Proof. 1.

$$\begin{gathered} f_i \in W^0 \\ \forall \alpha \in W, \quad f_i(\alpha) = 0 \\ \alpha = \sum_{j=1}^{k} x_j \alpha_j \\ f_i(\alpha) = \sum_{j=1}^{k} x_j f_i(\alpha_j) = 0 \end{gathered}$$

2. $\{f_k + 1, \cdots f_n\}$ are linearly independent.

By definition as $\{f_1, \cdots f_n\}$ is a basis function

3. $\{f_k + 1, \cdots, f_n\}$ spans W^0. For any functional f in

$$V^* : f = \sum_{j=1}^{k} x_j f_i(\alpha_j)$$

If $f \in W^0$ then

$$f = \sum_{j=1}^{k} f(\alpha_i) f_i + \sum_{j=k+1}^{n} f(\alpha_i) f_i$$

$$\alpha_i \quad (i = 1 \cdots k) \in W$$

$$f = \sum_{j=k+1}^{n} f(\alpha_i) \cdot f_i$$

dim(W^0)=n-K
dim(W^0)=dim V - dim W

dim(W)+dim(W^0)=dim(V)

1.4 The Double Dual

$$V^{**} = \textbf{Dual of } V^*$$
$$\dim(V) = \dim(V^*) = \dim(V^{**})$$

If α is a vector in **V**, then α induces a linear functional $\mathbf{L}_\alpha$ on V^*

$L_\alpha(f) = f(\alpha) \forall f \in V^*$

$L_\alpha : V^* \to F$

Note: If $\alpha \neq 0$, then $\exists$ a linear functional f such that L_α

Proof. $\beta = \{\alpha_{k+1}, \cdots, \alpha^n\}$ is a basis for V,
where $\alpha_i \neq 0$ for all $i = 1, \cdots, n$.

Defining $\alpha = \alpha_1$.

f is a linear functional which arranges each vector in V by its first coordinate in β.

$f(\alpha) = 1 \neq 0$

Theorem. Let V be a vector space over F. For each vector $\alpha \in V$, define

$L_\alpha(f) = f(\alpha)$ for all $f \in V^*$.
Then the mapping $\alpha \to L_\alpha$ is an isomorphism from V onto V^{**}.

Proof. Linearity

$L_{c\alpha+\beta}(f) = cL_\alpha(f) + L_\beta(f)$

$L_{c\alpha+\beta}(f) = f(c\alpha + \beta) = cf(\alpha) + f(\beta)$

$L_{c\alpha+\beta}(f) = f(c\alpha + \beta) = cL_\alpha(f) + L_\beta(f)$

or

$\alpha \to L_\alpha$ is a linear transformation from V into V^{**}

Note $L_\alpha = 0$ if and only if $\alpha = 0$
$\Rightarrow \alpha \mapsto L(\alpha)$ is a non-singular transformation.

1.5 Transposition of Linear Transformation

V, W are finite dimensional vector spaces over F

and $T : V \to W$

Goal: We want to define the transpose of T.

Suppose g is a linear functional on W, i.e., $g \in W^*$.

Define $f(\alpha) = g(T(\alpha))$ for all $\alpha \in V$, where $f(\alpha) \in \text{range}(T)$.
$f \in V^*$, $g \in W^*$

$V^* =$ Dual of V
$W^* =$ Dual of W

Then we can define $f = T^t \cdot g$ for each $g \in W^*$. Thus $T^t : W^* \to V^*$
such that

$f = T^t \cdot g.$
$f(\alpha) = (T^t \cdot g) \cdot (\alpha)$
$f(\alpha) = (g(T(\alpha))$

Note

$T^t(cg_1 + g_2) = cT^t g_1 + T^t g_2 \forall g_1, g_2 \in W^*$

Proof.

$$T^t(cg_1 + g_2)(\alpha) = (cg_1 + g_2)T(\alpha)$$

$$T^t(cg_1 + g_2)(\alpha) = c(g_1T(\alpha)) + (g_2T(\alpha))$$

$$T^t(cg_1 + g_2)(\alpha) = (cT^tg_1)(\alpha) + (T^tg_2)(\alpha)$$

$T^t \cdot g(\alpha) = g(T(\alpha))$ is a linear transformation and
is called the transpose of T.

Theorem. V, W are vector spaces, $T : V \to W$. Then

1) The null space of T^t is the annihilator of $\beta(T) = \text{Range of } T$.
2) $\text{rank}(T^t) = \text{rank}(T)$.
3) The range of T^* is the annihilator of the null space of T.

Proof. If g is in W^*, then by definition

$$(T^tg)(\alpha) = g(T\alpha) \tag{3}$$

for each α in V. The statement that g is in the null space of T^t means that $g(T\alpha) = 0$ for every α in V. Thus, the null space of T^t is precisely the annihilator of the range of T.

Suppose that V and W are finite-dimensional, say $\dim V = n$ and $\dim W = m$.
For (1): Let r be the rank of T, i.e., the dimension of the range of T. By theorem, the annihilator of the range of T then has dimension $(m - r)$. By the first statement of this theorem, the nullity of T^t must be $(m - r)$. But then since T^t is a linear transformation on an m-dimensional space, the rank of T^t is $m - (m - r)$, and so T and T^t have the same rank.

For (2): Let N be the null space of T. Every functional in the range of T^t is in the annihilator of N; for, suppose $f = T^tg$ for some g in W^*; then, if α is in N,

$$f(\alpha) = (T^tg)(\alpha) = g(T\alpha) = g(0) = 0 \tag{4}$$

Now the range of T^t is a subspace of the space N^0, and

$$\dim N^0 = n - \dim N = \text{rank}(T) = \text{rank}(T^t) \tag{5}$$

so that the range of T^t must be exactly N^0.

Example- Let V be the vector space of all polynomial functions over the field of real numbers. Let a and b be fixed real numbers and let f be the linear functional on V defined by

$$f(p) = \int_a^b p(x)dx$$

If D is the differentiation operator on V, what is

$$D^t f?$$

Solution- Let $p(x) = c_0 + c_1 x + \ldots + c_n x^n$.

$D^t f(p) = f(D(p))$

$D^t f(p) = f(c_1 + 2c_2 x + 3c_3 x^2 + \ldots + nc_n x^{n-1})$

$D^t f(p) = c_1 + c_2 x^2 + \ldots + c_n x^n \mid b_0$

$D^t f(p) = p(b) - p(a)$