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Improving Phase Estimation with Leakage Minimization

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Abstract – *The paper presents possibilities of an error reduction of the phase estimation with an interpolated discrete Fourier transform (DFT). Properties of interpolations are studied for the rectangular and the Hanning windows with respect to their ability for correction systematic effects. The correction is improved with considering the leakage effect of the component spectrum. Uncertainties of the phase estimations have been studied. The simulation and experimental results are presented showing the effectiveness in estimating the phase of the signal component.*

Keywords – *non-coherent sampling, phase estimation, DFT and leakage effect, uncertainty.*

I. INTRODUCTION

Analysis of periodic signals is often performed in the frequency domain. To estimate parameters of the time depended signals, containing any periodicity, it is most suitable to use a transformation of the signal in the frequency domain. For sampled signals discrete Fourier transformation (DFT) is generally used. The basic parameters of periodicity are (1): frequency of the energy kernel f_m, amplitude of the frequency main lobe A_m and phase φ_m, i.e. the time position of the signal structure [1]. A sampled periodic band limited analog multi-frequency signal $g(t)$ can be written as follows:

$$g(k\Delta t)_N = \sum_{m=0}^{M} A_m \sin(2\pi f_m k\Delta t + \varphi_m) \qquad (1)$$

Evaluations of the time-discrete spectra, however, are hampered by leakage, which occurs if a non-integer number of periods is presented in the sampled data set. The DFT of signal on N sampled points (1) at the spectral line i is given by:

$$G(i) = -\frac{\mathrm{j}}{2}\sum_{m=0}^{M} A_m \left(W(i-\theta_m)\mathrm{e}^{\mathrm{j}\varphi_m} - W(i+\theta_m)\mathrm{e}^{-\mathrm{j}\varphi_m}\right) \qquad (2)$$

where θ_m is the component frequency related to base frequency resolution depending on the window span $\Delta f = 1/N\Delta t$ and can be written in two parts:

$$\theta_m = \frac{f_m}{\Delta f} = i_m + \delta_m \qquad -0.5 < \delta_m \le 0.5\,, \qquad (3)$$

where i_m being an integer value. The non-coherent sampling causes the displacement term δ_m.

Tones of the sampled signal at θ_m do not always coincide with the basic set of periodic components of the DFT at i. Minimization of spectral leakage (the second part in (2)) is an important prerequisite for spectrum analysis. The position of the measurement component δ_m between DFT coefficients $G(i_m)$ and $G(i_m+1)$ surrounding the component can be estimated by means of the interpolation [2-5]. In this paper we try to show interpolations of the DFT also for improving the phase estimation. In estimations the rectangular window and the Hanning window have been used.

II. ANALYSIS OF THE DFT COEFFICIENTS

The DFT coefficients surrounding one signal component are due to the short-range leakage contribution of the window spectrum weighted by the amplitude of the component (from the first term in (2)) and the long-range leakage contributions. For one component only, the expression (2) can be reduced:

$$G(i) = -\mathrm{j}\frac{A_m}{2}\left(W(i-\theta_m)\mathrm{e}^{\mathrm{j}\varphi_m} - W(i+\theta_m)\mathrm{e}^{-\mathrm{j}\varphi_m}\right) \qquad (4)$$

If the function $W(\theta)$ of window used is analytically known, the parameters of the signal can be estimated.

For the rectangle window, the following equation is valid, where the Dirichlet kernel is used [6]:

$$W_{\mathrm{rect.}}(\theta) = \frac{\sin(\pi\theta)}{N\sin(\pi\theta/N)}\cdot\mathrm{e}^{-\mathrm{j}\pi\left(\frac{N-1}{N}\right)\theta} \qquad (5)$$

The largest DFT coefficient, which is mostly composed of the short-range leakage contribution of the investigated component m, can be deduced from (4) and (5) using $a = \pi(N-1)/N$ and $-\mathrm{j} = \mathrm{e}^{-\mathrm{j}\pi/2}$:

$$G(i_m) = \frac{A_m}{2}\left(\frac{\sin(\pi(i_m-\theta_m))}{N\sin(\pi(i_m-\theta_m)/N)}\mathrm{e}^{\mathrm{j}\left(a(\theta_m-i_m)+\varphi_m-\frac{\pi}{2}\right)} - \frac{\sin(\pi(i_m+\theta_m))}{N\sin(\pi(i_m+\theta_m)/N)}\mathrm{e}^{-\mathrm{j}\left(a(\theta_m+i_m)+\varphi_m+\frac{\pi}{2}\right)}\right) \qquad (6)$$

The component phase φ_m is referred to the start point of the window (not to the middle point [7]). Since N is usually large $N \gg 1$ and considering (3), equation (6) can be rewritten:

$$G(i_m) = \frac{A_m}{2}\left(\frac{\sin(\pi\delta_m)}{\pi\delta_m}\mathrm{e}^{\mathrm{j}\left(a(\delta_m)+\varphi_m-\frac{\pi}{2}\right)} - \frac{\sin(\pi\delta_m)}{\pi(2i_m+\delta_m)}\mathrm{e}^{-\mathrm{j}\left(a(2i_m+\delta_m)+\varphi_m+\frac{\pi}{2}\right)}\right) \qquad (7)$$

Both, amplitude and phase have additional disturbing components from the second part in (7):

$$G(i_m)=|G(i_m)|\mathrm{e}^{\mathrm{j}\left(a(\delta_m)+\varphi_m-\frac{\pi}{2}\right)}\pm\Delta(i_m)$$

$$\arg(G(i_m))=\varphi_m+a\delta_m-\frac{\pi}{2}\pm\Delta\varphi(i_m) \qquad (8)$$

If the displacement term is positive $0.5>\delta_m\geq 0$, than the second largest DFT coefficient is $G(i_m+1)$ and if the displacement term is negative $0>\delta_m\geq -0.5$, than the second largest DFT coefficient is $G(i_m-1)$. The sign of displacement $s=\mathrm{sign}(\delta_m)$ can be estimated with the difference of the phase DFT coefficients $s=\mathrm{sign}(\arg[G(i_m)]-\arg[G(i_m+1)]-\pi/2)$. The largest side coefficient can be expressed in common as:

$$G(i_m+s)=\frac{A_m}{2}\left(\frac{\sin(\pi(s-\delta_m))}{\pi(s-\delta_m)}\mathrm{e}^{\mathrm{j}\left(a(\delta_m-s)+\varphi_m-\frac{\pi}{2}\right)}-\frac{\sin(\pi(2i_m+s+\delta_m))}{\pi(2i_m+s+\delta_m)}\mathrm{e}^{-\mathrm{j}\left(a(\delta_m-s)+\varphi_m-\frac{\pi}{2}\right)}\right) \qquad (9a)$$

$$G(i_m+s)=|G(i_m+s)|\mathrm{e}^{\mathrm{j}\left(a(\delta_m-s)+\varphi_m-\frac{\pi}{2}\right)}\mp\Delta(i_m+s)$$

$$\arg(G(i_m+s))=\varphi_m+a(\delta_m-s)-\frac{\pi}{2}\mp\Delta\varphi(i_m+s) \qquad (9b)$$

The Hanning window spectrum can be obtained by shifting and weighted summations of the rectangular window spectrum [2,6].

$$W_{\mathrm{Hann.}}(\theta)=\frac{1}{2}W_{\mathrm{rect.}}(\theta)-\frac{1}{4}(W_{\mathrm{rect.}}(\theta+1)+W_{\mathrm{rect.}}(\theta-1)) \quad (10)$$

Using (5) in (10), it can be written:

$$W_{\mathrm{H}}(\theta)=\frac{\sin(\pi\theta)}{2}\cdot\mathrm{e}^{-\mathrm{j}a\theta}\cdot\left(\frac{1}{N\sin(\pi\theta/N)}-\frac{1}{2}\left(\frac{-1}{N\sin(\pi(\theta+1)/N)}\cdot\mathrm{e}^{-\mathrm{j}a}+\frac{-1}{N\sin(\pi(\theta-1)/N)}\cdot\mathrm{e}^{\mathrm{j}a}\right)\right)$$

If we have a lot of points in the measurement set $N>>1$, the sine function can be approximated by $\sin(\pi\theta/N)\approx\pi\theta/N$. Considering also $\mathrm{e}^{\mathrm{j}a}\approx -1$ and $\mathrm{e}^{-\mathrm{j}a}\approx -1$, the expression in brackets can be simplify $\left(\frac{1}{\pi\theta}-\frac{1}{2}\left(\frac{1}{\pi(\theta+1)}+\frac{1}{\pi(\theta-1)}\right)\right)=\frac{1}{\pi\theta(1-\theta^2)}$ and we finally get:

$$W_{\mathrm{H}}(\theta)=\frac{1}{2}\frac{\sin(\pi\theta)}{\pi\theta(1-\theta^2)}\cdot\mathrm{e}^{-\mathrm{j}a\theta} \qquad (11)$$

The largest DFT coefficient can be deduced from (4) and (11) considering $i_m-\theta_m=-\delta_m$ and $i_m+\theta_m=2i_m+\delta_m$:

$$G_{\mathrm{H}}(i_m)=\frac{A_m}{4}\left(\frac{\sin(\pi\delta_m)}{\pi\delta_m(1-\delta_m^2)}\mathrm{e}^{\mathrm{j}\left(a(\delta_m)+\varphi_m-\frac{\pi}{2}\right)}-\frac{\sin(\pi\delta_m)}{\pi(2i_m+\delta_m)(1-(2i_m+\delta_m)^2)}\mathrm{e}^{-\mathrm{j}\left(a(2i_m+\delta_m)+\varphi_m+\frac{\pi}{2}\right)}\right) \qquad (12)$$

The second part in (12) causes additional disturbing components:

$$G_{\mathrm{H}}(i_m)=|G_{\mathrm{H}}(i_m)|\mathrm{e}^{\mathrm{j}\left(a(\delta_m)+\varphi_m-\frac{\pi}{2}\right)}\pm\Delta(i_m); \qquad (13a)$$

$$|G_{\mathrm{H}}(i_m)|=\frac{A_m}{4}\frac{\sin(\pi\delta_m)}{\pi\delta_m(1-\delta_m^2)};$$

$$\arg(G_{\mathrm{H}}(i_m))=\varphi_{i_m}=\varphi_m+a\delta_m-\frac{\pi}{2}\pm\Delta\varphi(i_m) \qquad (13b)$$

The largest side coefficient can be expressed in short form as:

$$G_{\mathrm{H}}(i_m+s)=|G_H(i_m+s)|\mathrm{e}^{\mathrm{j}\left(a(\delta_m-s)+\varphi_m-\frac{\pi}{2}\right)}\mp\Delta(i_m+s)$$

$$|G_{\mathrm{H}}(i_m+s)|=\frac{A_m}{4}\frac{\sin(\pi(s-\delta_m))}{\pi(s-\delta_m)(1-(s-\delta_m)^2)}$$

$$\arg(G_{\mathrm{H}}(i_m+s))=\varphi_{i_m+s}=\varphi_m+a(\delta_m-s)-\frac{\pi}{2}\mp\Delta\varphi(i_m+s) \quad (14)$$

III. PHASE ESTIMATION

A comparison of the algorithms of the phase estimation shows that estimations via DFT are among the best compared under identical disturbing conditions [8]. In the first approximation using rectangular window, the second term in (7) and (9a) can be neglected and the phase of component can be estimated by:

$$\varphi^{\mathrm{I}}_{m,\mathrm{R}}=\arg[G(i_m)]+a\delta_m+\frac{\pi}{2} \qquad (15)$$

$$\varphi^{\mathrm{II}}_{m,\mathrm{R}}=\arg[G(i_m+s)]+a(s-\delta_m)+\frac{\pi}{2} \qquad (16)$$

Another possibility is to estimate the component phase only by the phase DFT coefficient itself where φ_m is referred to the middle point of the measurement window [7]. But this method has the weak point since it doesn't consider the long-range contributions of the window (Fig. 4e).

We can improve the estimation by considering the long-range contributions. Because the disturbing angle components in (8) and (9b) are small, they can be exchanged by sine functions and approximated by quotients:

$$\frac{\Delta\varphi(i_m)}{\Delta\varphi(i_m+s)}\cong\frac{\sin[\Delta\varphi(i_m)]}{\sin[\Delta\varphi(i_m+s)]}=\frac{|\Delta(i_m)|}{|G(i_m)|}\frac{|G(i_m+s)|}{|\Delta(i_m+s)|} \qquad (17)$$

$$\frac{|\Delta(i_m)|}{|\Delta(i_m+s)|}\cdot\frac{|G(i_m+s)|}{|G(i_m)|}=\frac{2i_m+s+\delta_m}{2i_m+\delta_m}\cdot\frac{|G(i_m+s)|}{|G(i_m)|} \qquad (18)$$

If $i_m>>1$ is large enough, we can equalize $|\Delta(i_m)|\approx|\Delta(i_m+s)|$ and (17) can be rewritten as:

$$\frac{\Delta\varphi(i_m)}{\Delta\varphi(i_m+s)}\cong\frac{|G(i_m+s)|}{|G(i_m)|}=\frac{\delta_m}{s-\delta_m};\quad (s-\delta_m)\Delta\varphi(i_m)\cong\delta_m\Delta\varphi(i_m+s) \qquad (19)$$

The multiplication of (8) and (9b) by the correction (19) and summation give us estimation of the phase as an averaging of the two arguments $\varphi_{i_m}=\arg[G(i_m)]$ and $\varphi_{i_m+s}=\arg[G(i_m+s)]$ surrounding the component (Fig. 2b):

$$\varphi^{\mathrm{III}}_{m,\mathrm{R}} = \left(1-|\delta_m|\right)\varphi_{i_m} + |\delta_m|\varphi_{i_m+s} + \frac{\pi}{2} \tag{20}$$

Better estimation can be attained by considering also the long-range contributions (Fig. 2c):

$$\frac{\Delta\varphi(i_m)}{\Delta\varphi(i_m+s)} \cong \frac{2i_m+s+\delta_m}{2i_m+\delta_m}\cdot\frac{|G(i_m+s)|}{|G(i_m)|} = b\cdot\frac{|G(i_m+s)|}{|G(i_m)|} \tag{21}$$

$$\varphi^{\mathrm{IV}}_{m,\mathrm{R}} = \frac{|G(i_m)|\varphi_{i_m} + b|G(i_m+s)|\varphi_{i_m+s}}{|G(i_m)| + b|G(i_m+s)|} + \\ + sa\left(\frac{b|G(i_m+s)|}{|G(i_m)|+b|G(i_m+s)|} - |\delta_m|\right) + \frac{\pi}{2} \tag{22}$$

The systematic errors of the phase estimations $E = \varphi_m - \varphi_0$ (φ_0 - is the true value of the phase) are phase dependent (Fig. 1: The error curves are very close to sine like function). In simulations, the absolute maximum values of the errors at a given relative frequency have been searched when phase has been changed in interval $-\pi/2 \le \varphi \le \pi/2$ (Fig. 2). The estimation errors drop with the increasing relative frequency.

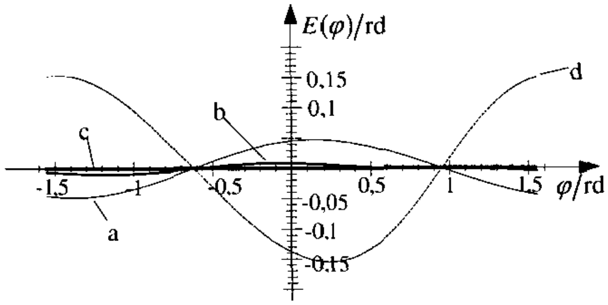

Fig. 1. Phase dependency errors at $\theta = 2.2$, $-\pi/2 \le \varphi \le \pi/2$; Estimations: a – by (15), b – by (20), c – by (22), d – by (16)

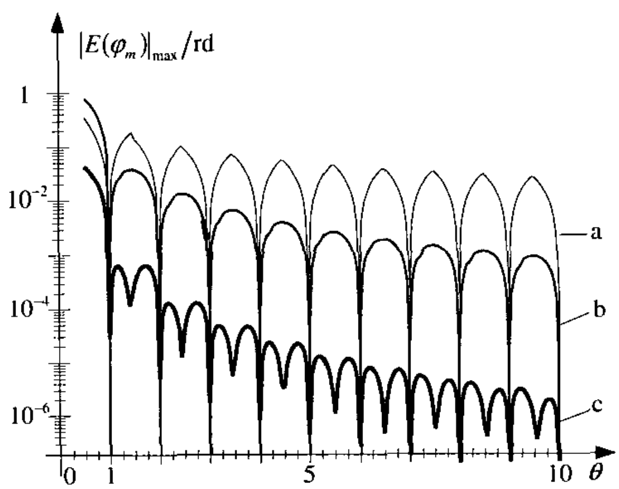

Fig. 2. Maximal systematic errors of the phase estimations with rectangular window: a – by (15), b – by (20), c – by (22); θ is known

Fig. 3 shows the importance of the frequency estimation accuracy. If the frequency is estimated by a known two-point estimation (23) [2] the overall errors increase (Fig. 3: $E_{c^*}/E_c \approx 200$).

$$\theta = i + \delta_m = i + s\frac{|G(i_m+s)|}{|G(i_m)| + |G(i_m+s)|} \tag{23}$$

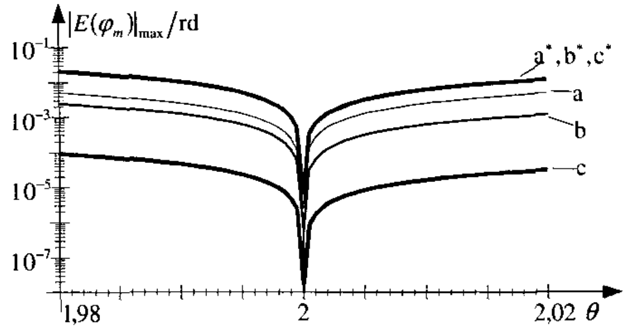

Fig. 3. Maximal systematic errors of the phase estimations with rect. window in the interval $1.98 \le \theta \le 2.02$: a – by (15), b –by (20), c –by (22); a*, b*, c* - θ is estimated by (23)

Using Hanning window the expressions for phase have the same forms as for the rectangular window when the second term in (12) and (14) is neglected:

$$\varphi^{\mathrm{I}}_{m,\mathrm{H}} = \varphi_{i_m} - a\delta_m + \frac{\pi}{2} \tag{24}$$

$$\varphi^{\mathrm{II}}_{m,\mathrm{H}} = \varphi_{i_m+s} + a(s-\delta_m) + \frac{\pi}{2} \tag{25}$$

We can again improve the estimation by considering the long-range contributions, which have following properties:

$$\frac{\Delta\varphi(i_m)}{\Delta\varphi(i_m+s)} \cong \frac{\sin(\Delta\varphi(i_m))}{\sin(\Delta\varphi(i_m+s))} = \frac{|\Delta(i_m)|}{|G_{\mathrm{H}}(i_m)|}\frac{|G_{\mathrm{H}}(i_m+s)|}{|\Delta(i_m+s)|} = \\ = \frac{(2i_m+s+\delta_m)\left(1-(2i_m+s+\delta_m)^2\right)}{(2i_m+\delta_m)\left(1-(2i_m+\delta_m)^2\right)}\cdot\frac{|G_{\mathrm{H}}(i_m+s)|}{|G_{\mathrm{H}}(i_m)|} \tag{26}$$

We can equalize $|\Delta(i_m)| \approx |\Delta(i_m+s)|$ and (26) can be rewritten as:

$$\frac{\Delta\varphi(i_m)}{\Delta\varphi(i_m+s)} \cong \frac{|G_{\mathrm{H}}(i_m+s)|}{|G_{\mathrm{H}}(i_m)|} = \frac{\delta_m(1-\delta_m^2)}{(s-\delta_m)\left(1-(s-\delta_m)^2\right)} = \frac{1+s\delta_m}{2-s\delta_m} \tag{27}$$

If we equalize $(2-s\delta_m)\Delta\varphi(i_m) \cong (1+s\delta_m)\Delta\varphi(i_m+s)$, the equations (13b) and (14) can be multiplied by corrections and added:

$$\varphi^{\mathrm{I}}_{m,\mathrm{H}} = \varphi_{i_m} - a\delta_m + \frac{\pi}{2} \pm \Delta\varphi(i_m) \quad |\cdot(2-s\delta_m)$$

$$\varphi^{\mathrm{II}}_{m,\mathrm{H}} = \varphi_{i_m+s} + a(s-\delta_m) + \frac{\pi}{2} \mp \Delta\varphi(i_m+s) \quad |\cdot(1+s\delta_m)$$

$$\varphi^{\mathrm{III}}_{m,\mathrm{H}} = \frac{(2-s\delta_m)\varphi_{i_m} + (1+s\delta_m)\varphi_{i_m+s}}{3} + \frac{a}{3}(s-2\delta_m) + \frac{\pi}{2} \tag{28}$$

When the sign of the displacement is positive $s=1$, the phase estimation (28) is made with arguments φ_{i_m} and φ_{i_m+1}:

$$\varphi^{\mathrm{III}}_{m,\mathrm{H}}\left(\varphi_{i_m},\varphi_{i_m+1}\right)=\frac{(2-\delta_m)\varphi_{i_m}+(1+\delta_m)\varphi_{i_m+1}}{3}+\frac{a}{3}(1-2\delta_m)+\frac{\pi}{2},\quad(29)$$

and when the sign is negative $s=-1$, the estimation is done with φ_{i_m} and φ_{i_m-1}:

$$\varphi^{\mathrm{III}}_{m,\mathrm{H}}\left(\varphi_{i_m},\varphi_{i_m-1}\right)=\frac{(2+\delta_m)\varphi_{i_m}+(1-\delta_m)\varphi_{i_m-1}}{3}-\frac{a}{3}(1+2\delta_m)+\frac{\pi}{2}\quad(30)$$

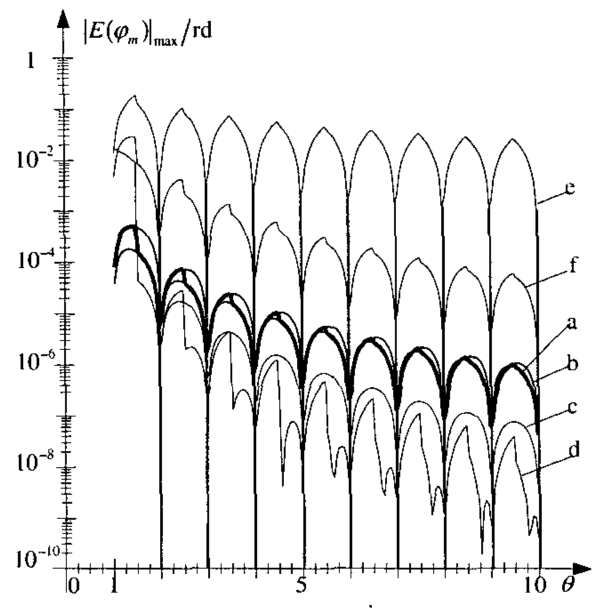

Fig. 4. Maximal systematic errors of the phase estimations with Hanning window: a – by (24), b – by (25), c – by (28), d – by (32), θ is known; and estimations as in [7]: e – the rectangular window, f – the Hanning window.

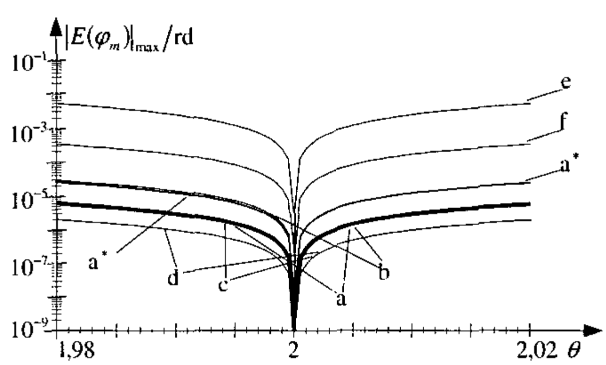

Fig. 5. Maximal systematic errors of the phase estimations with Hanning window in the interval $1{,}98 \le \theta \le 2{,}02$: a – by (24), b – by (25), c – by (28), d – by (32), θ is known; a^* - θ is estimated by (33); and estimations only by DFT coefficients [7]: e – the rectangular window, f – the Hanning window.

The phase estimation can be improved even better with averaging of the estimations by (29) and (30). With this averaging we get the three-point estimation (Fig. 4d):

$$\varphi^{\mathrm{IV}}_{m,\mathrm{H}}\left(\varphi_{i_m},\varphi_{i_m+1},\varphi_{i_m-1}\right)=\frac{\varphi^{\mathrm{III}}_{m,\mathrm{H}}\left(\varphi_{i_m},\varphi_{i_m+1}\right)+\varphi^{\mathrm{III}}_{m,\mathrm{H}}\left(\varphi_{i_m},\varphi_{i_m-1}\right)}{2}\quad(31)$$

$$\varphi^{\mathrm{IV}}_{m,\mathrm{H}}=\frac{(1-\delta_m)\varphi_{i_m-1}+4\varphi_{i_m}+(1+\delta_m)\varphi_{i_m+1}}{6}-\frac{2a\delta_m}{3}+\frac{\pi}{2}\quad(32)$$

In Fig. $5a^*$ the frequency is estimated by the three-point interpolation [9]:

$$\theta=i+\delta_m=i+2\frac{|G_{\mathrm{H}}(i_m+s)|-|G_{\mathrm{H}}(i_m-s)|}{|G_{\mathrm{H}}(i_m+s)|+2\cdot|G(i_m)|+|G(i_m-s)|}\quad(33)$$

IV. UNCERTAINTY PROPAGATION

The uncertainty propagation through the DFT procedure is well known $\sigma_R=\sigma_I=\sigma_{|G(i)|}=\sigma_{\mathrm{DFT}}=\sigma_t\big/\left(N\sqrt{2}\right)\sqrt{\sum_{k=0}^{N-1}w^2(k)}$ [10], where we use $R(i)=\mathrm{Re}[G(i)]$ and $I(i)=\mathrm{Im}[G(i)]$ for the real and imaginary parts of the DFT and $|G(i)|=\sqrt{R^2(i)+I^2(i)}$ for the amplitude and $\varphi(i)=\arg[G(i)]=\tan^{-1}(I(i)/R(i))$ for the phase, respectively. The phase uncertainty is equal to the uncertainty of the DFT procedure scaled by the amplitude coefficient $\sigma_{\varphi(i)}=\sigma_{\mathrm{DFT}}/|G(i)|$:

$$c_R(i)=\frac{\partial\varphi(i)}{\partial R(i)}=-\frac{I(i)}{|G(i)|^2}\ ;\ c_I(i)=\frac{\partial\varphi(i)}{\partial I(i)}=\frac{R(i)}{|G(i)|^2}\quad(34)$$

$$\sigma^2_{\varphi(i)}=(c_R\sigma_R)^2+(c_I\sigma_I)^2=\sigma^2_{\mathrm{DFT}}\frac{I^2+R^2}{|G(i)|^4}=\frac{\sigma^2_{\mathrm{DFT}}}{|G(i)|^2}\quad(35)$$

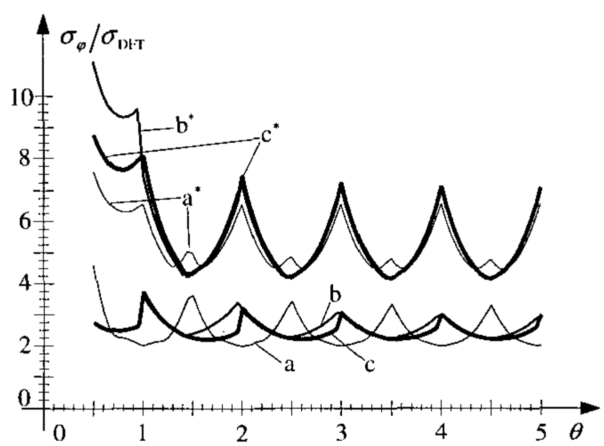

Fig. 6. Ratios of the uncertainties of the phase estimations related to the standard uncertainty of the amplitude DFT coefficient using the rectangular window (dimensions in rd/V). Estimations: a – by (15), b – by (20), c – by (22), θ is known; a^*, b^*, c^* - θ is estimated by (23)

It is evident that the standard uncertainty of the phase depends on the amplitude of the component. Moreover, in the non-coherent sampling it changes with displacement δ as can be seen from Fig. 6. Ratios of the uncertainties of the phase estimations related to the standard uncertainty of the amplitude DFT coefficient using the rectangular window are between 2 and 3,5 symmetrically depending on the term δ at higher values of the relative frequency θ.

The uncertainties of the estimations increase if one needs to estimate also the displacement term $\delta(|G(i)|,|G(i+s)|)$. In estimations algorithms (15), (17), and (22) one needs partial sensitivity coefficients also for the displacement term $\partial\delta/\partial|G(i)|$ and $\partial\delta/\partial|G(i+s)|$. Fig 6 shows that uncertainties of the estimations increase for a factor $2 \div 2{,}3$ if frequency is estimated by (23).

V. EXPERIMENTAL RESULTS

The methods were also tested by a real measurement system. In experiment we use sampling DVM (HP3458: $f_S = 25\,\text{kHz}$, $U_{\text{range}} = 10\,\text{V}$, $N = 1024$, time base accuracy: 0.01%, jitter $< 100\,\text{ps}$) and synthesizer/function generator (HP3325A: $f = 100\,\text{Hz}$, $u_{pp} = 20\,\text{V}$, frequency accuracy: $5 \cdot 10^{-6}$). Interpolation algorithms have been validated with the sine shape signal (Figs. 7 and 8).

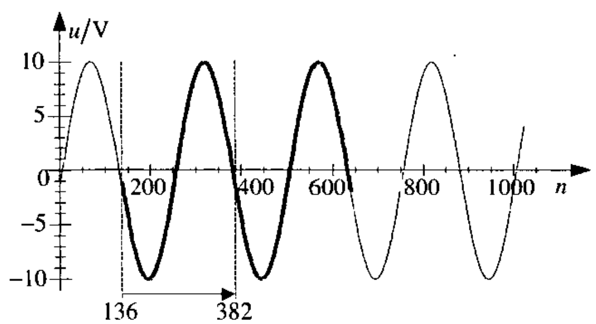

Fig. 7. Sampled sine function $N = 1024$ and truncated part $N_{\text{window}} = 505$ ($\theta = 2.02$)

The truncated window (the bold line in Fig. 7) has been moved for approximately one period (from point 136 to point 382). The estimations with interpolations algorithms (15), (20), (22), and (20) with the estimated displacement by (23) have been compared to the estimation by the coherent sampling window $N = 500 = 2 * f_S/f$ ($\theta = 2$) with the same first sampling point. The maximal errors of phase estimations $E_\varphi = \varphi(\theta = 2.02) - \varphi(\theta = 2)$ (Fig. 8) are close to the expected values in Fig. 3. The best results give us the estimation by (22) (Fig. 8c: $|E_\varphi|_{\max} \approx 4 \cdot 10^{-5}\,\text{rd} \approx 2\,\text{m}^\circ$).

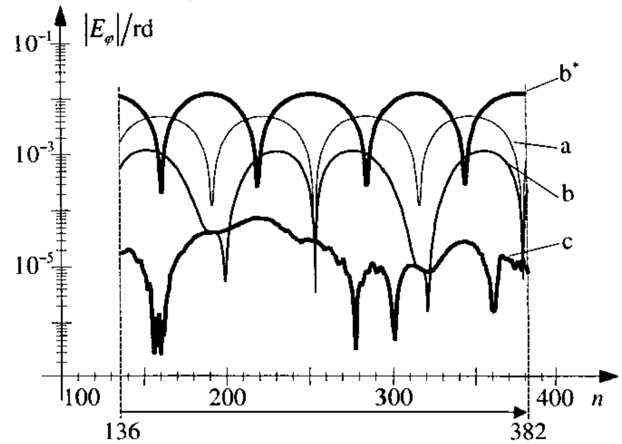

Fig. 8. The absolute values of errors of the phase estimations using the rectangular window: a – by (15), b – by (20), c – by (22), b* – by (20) and θ is estimated by (23)

VI. CONCLUSIONS

In the paper, we have pointed out the advantages of the DFT interpolations for the phase estimation. Interpolations where the long-range leakage is considered decrease systematic effects. One possibility is an averaging of the two arguments surrounding the component ((20) using the rectangular window and (28) using the Hanning window). This estimation is independent of the number of the sampling points.

Better estimation can be attained by considering also the long-range contributions (22) using the rectangular window. The error bound of the phase estimation is lower than $|E|_{\max} < 1\,\text{m}^\circ$, if we have enough periods of the signal in the measurement interval $\theta > 5$. When the measurement window is shortened to around two cycles of the signal errors increase to $|E|_{\max} < 2\,\text{m}^\circ$.

In the case of the three-point estimation using Hanning window (32) the error bound drops down to $|E|_{\max} < 0.2\,\text{m}^\circ$ if the measurement window is shortened to around two cycles of the signal.

The simulation and experimental results show that, in the non-coherent sampling, the systematic errors and uncertainties are changing almost symmetrically to the integer values of the relative frequency with the displacement term δ.

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