

2020级 第二学期 概率与统计.

一、

$$\begin{aligned} 1. \text{解: } P(\bar{A} \cup B) &= P(\bar{A}) + P(B) - P(\bar{A}B) \\ &= [1 - P(A)] + P(B) - [P(B) - P(AB)] \\ &= 1 - P(A) + P(AB) \\ &= 1 - 0.5 + 0.4 \\ &= 0.9 \end{aligned}$$

2. 解: 设 A_i 表示第一次取到新球的个数;
 B 表示第二次取到新球.

$$(1). P(A_1) = \frac{C_8^1 C_2^2}{C_{10}^3} = \frac{8}{120} = \frac{1}{15}$$

$$P(A_2) = \frac{C_8^2 C_2^1}{C_{10}^3} = \frac{28 \times 2}{120} = \frac{56}{120} = \frac{7}{15}$$

$$P(A_3) = \frac{C_8^3}{C_{10}^3} = \frac{56}{120} = \frac{7}{15}$$

$$\begin{aligned} P(B) &= P(A_1) \cdot P(B|A_1) + P(A_2) P(B|A_2) + P(A_3) P(B|A_3) \\ &= \frac{1}{15} \times \frac{7}{10} + \frac{7}{15} \times \frac{6}{10} + \frac{7}{15} \times \frac{5}{10} \\ &= \frac{7 + 42 + 35}{150} \\ &= \frac{84}{150} = \frac{42}{75} = \frac{14}{25} \end{aligned}$$

$$\begin{aligned}
 (2) \quad P(A_2|B) &= \frac{P(A_2 B)}{P(B)} = \frac{P(A_2) P(B|A_2)}{P(B)} \\
 &= \frac{\frac{7}{15} \times \frac{6}{10}}{\frac{14}{25}} = \frac{1}{2}
 \end{aligned}$$

故 (1) 第二次取到新球概率为 $\frac{14}{25}$

(2) 第一次取 2 个新球概率为 $\frac{1}{2}$

二、1、X 可能取值有 0、1、2、3

$$P(X=0) = \frac{3}{6} = \frac{1}{2}$$

$$P(X=1) = \frac{3}{6} \cdot \frac{3}{5} = \frac{9}{30} = \frac{3}{10}$$

$$P(X=2) = \frac{3}{6} \cdot \frac{2}{5} \cdot \frac{3}{4} = \frac{3}{20}$$

$$P(X=3) = \frac{3}{6} \cdot \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{20}$$

故 X 的分布律为

X	0	1	2	3
P	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{3}{20}$	$\frac{1}{20}$

$$E(X) = 0 \times \frac{1}{2} + 1 \times \frac{3}{10} + 2 \times \frac{3}{20} + 3 \times \frac{1}{20} = \frac{3}{4}$$

$$2. (1) f_X(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{其他} \end{cases} \Rightarrow F_X(x) = \begin{cases} 1 - e^{-2x} & x > 0 \\ 0 & \text{其他} \end{cases}$$

$$(2) P(Y \leq y) = P(1 - e^{-2X} \leq y) \\ = P(e^{-2X} \geq 1 - y) = \begin{cases} P(-2X \geq \ln(1-y)) & 0 < y < 1 \\ 0 & y \leq 0 \\ 1 & y \geq 1 \end{cases}$$

$$= \begin{cases} 0 & y \leq 0 \\ P(X \leq -\frac{1}{2} \ln(1-y)) & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}$$

$$\Rightarrow F_Y(y) = \begin{cases} 0 & y < 0 \\ 1 - e^{-2(-\frac{1}{2} \ln(1-y))} & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}$$

$$= \begin{cases} 0 & y \leq 0 \\ y & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}$$

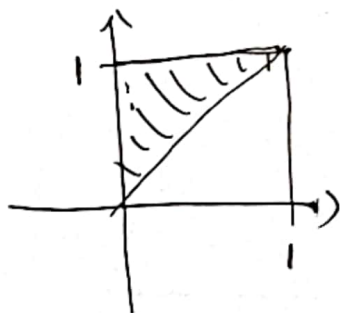
$$\Rightarrow f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{其他} \end{cases}$$

三、1. A 成立. B 不成立.

所加条件: A, B 相互独立.

2. (1) $C = \frac{1}{S_1} = \frac{1}{\frac{1}{2}} = 2$ (均匀分布).

故 C 的值为 2.



$$\begin{aligned} (2) \quad f_X(x) &= \int_{-\infty}^{+\infty} f(x, y) dy \\ &= \begin{cases} \int_x^1 2 dy & 0 < x < 1 \\ 0 & \text{其他} \end{cases} \end{aligned}$$

$$\Rightarrow f_X(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{其他} \end{cases}$$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{+\infty} f(x, y) dx \\ &= \begin{cases} \int_0^y 2 dx & 0 < y < 1 \\ 0 & \text{其他} \end{cases} \end{aligned}$$

$$\Rightarrow f_Y(y) = \begin{cases} 2y & 0 < y < 1 \\ 0 & \text{其他} \end{cases}$$

$$13) f_X(x) \cdot f_Y(y) = \begin{cases} 2(1-x) \cdot 2y & 0 < x < 1 \quad 0 < y < 1 \\ 0 & \text{其他} \end{cases}$$

$$\Rightarrow f(x, y) \neq f_X(x) f_Y(y)$$

故 X, Y 不相互独立.

$$14) P(Z \leq z) = P(X+Y \leq z)$$

$$= \begin{cases} \frac{1}{2} \cdot z \cdot \frac{z}{2} \times 2 & 0 < z < 1 \\ 1 - (2-z)\left(1-\frac{z}{2}\right) \cdot \frac{1}{2} \times 2 & 1 < z < 2 \\ 1 & z > 2 \\ 0 & z < 0 \end{cases}$$

$$\Rightarrow F_Z(z) = \begin{cases} 0 & z < 0 \\ \frac{1}{2} z^2 & 0 < z < 1 \\ 1 - \frac{1}{2} (2-z)^2 & 1 < z < 2 \\ 1 & z > 2 \end{cases}$$

$$\Rightarrow f_Z(z) = \begin{cases} z & 0 < z < 1 \\ 2-z & 1 < z < 2 \\ 0 & \text{其他} \end{cases}$$

四、

1. 描述两个随机变量线性相关程度的系数

$|\rho| \leq 1$ $|\rho|$ 越接近于 1, 表明两随机变量线性相关性越强, 反之越弱.

$\rho > 0$ 表明两二者正相关, $\rho < 0$ 表明负相关.

2. A 成立.

B 不成立.

$$\text{如 } f(x, y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \leq 1 \\ 0 & \text{其他.} \end{cases}$$

可以证明. $\because X$ 与 Y 不相关, 但二者不独立.

$$3. E(X) = 0, D(X) = 1$$

$$E(Y) = 2, D(Y) = 4 \quad (Y \sim \chi^2(2)).$$

$$(1) E(X - 2Y) = E(X) - 2E(Y) = -4$$

$$\text{独立} \Rightarrow D(X - 2Y) = D(X) + 4D(Y) = 17$$

(2) 由 X, Y 相互独立

$$\Rightarrow \text{COV}(X, Y) = E(XY) - E(X)E(Y) = 0.$$

$$\Rightarrow E(XY) = 0$$

$$E(X^2) = D(X) + E^2(X) = 1 \quad E(Y^2) = D(Y) + E^2(Y) = 8$$

$$D(XY) = E(X^2Y^2) - E^2(XY)$$

$$= E(X^2)E(Y^2) - E^2(XY) = 8$$

$$(3) \quad E(U) = E(X+Y) = E(X) + E(Y) = 2$$

$$D(U) = D(X+Y) = D(X) + D(Y) = 5$$

$$E(V) = E(X-Y) = E(X) - E(Y) = -2$$

$$D(V) = D(X-Y) = D(X) + D(Y) = 5.$$

$$\begin{aligned} E(UV) &= E((X+Y)(X-Y)) = E(X^2 - Y^2) \\ &= E(X^2) - E(Y^2) \\ &= 1 - 8 = -7 \end{aligned}$$

$$\begin{aligned} \text{cov}(U, V) &= E(U, V) - E(U)E(V) \\ &= -7 + 4 = -3 \end{aligned}$$

$$\rho_{UV} = \frac{\text{cov}(U, V)}{\sqrt{D(U)} \sqrt{D(V)}} = \frac{-3}{\sqrt{5} \cdot \sqrt{5}} = -\frac{3}{5}$$

故 U, V 相关系数为 $-\frac{3}{5}$

(4) 由 $\rho_{UV} = -\frac{3}{5} \neq 0 \Rightarrow U, V$ 不独立.

五、由中心极限定理, 这100件的总质量 X

服从正态分布 $N(100 \times 0.5, 100 \times 0.1^2)$

即 $X \sim N(50, 1)$.

$$P(X > 51) = 1 - P(X \leq 51)$$

$$= 1 - \Phi\left(\frac{51-50}{1}\right) = 1 - \Phi(1) = 0.1587$$

故总质量超过 51 的概率为 0.1587.

$$六、由 X \sim N(u, \sigma^2) \Rightarrow \bar{X} \sim N(u, \frac{\sigma^2}{5})$$

$$X_6 \sim N(u, \sigma^2)$$

$$\Rightarrow X_6 - \bar{X} \sim N(0, \frac{6}{5} \sigma^2)$$

$$\Rightarrow \frac{X_6 - \bar{X}}{\sqrt{\frac{6}{5} \sigma^2}} \sim N(0, 1)$$

$$又 \frac{(n-1)S^2}{\sigma^2} = \frac{4S^2}{\sigma^2} \sim \chi^2(4)$$

$$\Rightarrow \frac{\frac{X_6 - \bar{X}}{\sqrt{\frac{6}{5} \sigma^2}}}{\sqrt{\frac{4S^2}{\sigma^2} / 4}} = \sqrt{\frac{5}{6}} \frac{X_6 - \bar{X}}{S} = \sqrt{\frac{5}{6}} T \sim t(4)$$

故 a 的值为 $\sqrt{\frac{5}{6}}$

七、1. 由题意 $E(X) = \lambda$.

用样本矩估计 $E(X)$

$$\Rightarrow \lambda = \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i = \frac{1}{12} \times 21 = \frac{7}{4}$$

故 λ 的估计值为 $\frac{7}{4}$

2. 似然函数为 $L(\theta) = \theta^n (x_1 x_2 \cdots x_n)^{\theta-1}$

$$\ln L(\theta) = n \ln \theta + (\theta-1) \left[\sum_{i=1}^n \ln x_i \right]$$

$$\frac{d \ln L(\theta)}{d \theta} = \frac{n}{\theta} + \sum_{i=1}^n \ln x_i = 0$$

$$\Rightarrow \frac{1}{\theta} = -\frac{1}{n} \sum_{i=1}^n \ln x_i$$

$$\Rightarrow \theta = - \frac{n}{\sum_{i=1}^n \ln x_i}$$

故 θ 的最大似然估计量为 $\hat{\theta} = - \frac{n}{\sum_{i=1}^n \ln x_i}$

$\beta = e^{-\frac{1}{\theta}}$ 的最大似然估计量为 $e^{\frac{1}{n} \sum_{i=1}^n \ln x_i}$

$$= \frac{n}{\prod_{i=1}^n x_i}$$

3. 证明: $E(\hat{u}_1) = \frac{1}{3}[E(x_1) + E(x_2) + E(x_3)] = \frac{1}{3} \cdot 3u = u$

$$E(\hat{u}_2) = \frac{2}{3}E(x_1) - \frac{5}{9}E(x_2) + \frac{8}{9}E(x_3)$$

$$= \frac{2}{3}u - \frac{5}{9}u + \frac{8}{9}u = u$$

故 \hat{u}_1 和 \hat{u}_2 均为无偏估计.

(2) $D(\hat{u}_1) = \frac{1}{9}(D(x_1) + D(x_2) + D(x_3)) = \frac{6^2}{3}$

$$D(\hat{u}_2) = \frac{4}{9}D(x_1) + \frac{25}{81}D(x_2) + \frac{64}{81}D(x_3)$$

$$= \left(\frac{4}{9} + \frac{25}{81} + \frac{64}{81}\right)6^2 = \frac{125}{81}6^2 > \frac{6^2}{3}$$

故 \hat{u}_1 更有效.

1.1、第二类错误. 第一类错误.

2. 设 $H_0: u = u_0 = 500$ $H_1: u \neq 500$

检验统计量为 $\frac{\bar{x} - u_0}{s/\sqrt{n}}$ 拒绝域为 $\left| \frac{\bar{x} - u_0}{s/\sqrt{n}} \right| > z_{\frac{\alpha}{2}}$

查表 $z_{\frac{\alpha}{2}} = 1.96$, 计算得 $\left| \frac{\bar{x} - u_0}{s/\sqrt{n}} \right| = \frac{0.5}{0.4} = 1.25 < 1.96$

故饮料符合要求.