

2020 级 第二学期 概率与统计

$$\begin{aligned}
 1. \text{ 解: } P(\bar{A} \cup B) &= P(\bar{A}) + P(B) - P(\bar{A}B) \\
 &= [1 - P(A)] + P(B) - [P(B) - P(AB)] \\
 &= 1 - P(A) + P(AB) \\
 &= 1 - 0.5 + 0.4 \\
 &= 0.9
 \end{aligned}$$

2. 解: 设 A_i 表示第一次取到新球的个数;

B 表示第二次取到新球.

$$\begin{aligned}
 (1). P(A_1) &= \frac{C_8^1 C_2^2}{C_{10}^3} = \frac{8}{120} = \frac{1}{15} \\
 P(A_2) &= \frac{C_8^2 C_2^1}{C_{10}^3} = \frac{28 \times 2}{120} = \frac{56}{120} = \frac{7}{15} \\
 P(A_3) &= \frac{C_8^3}{C_{10}^3} = \frac{56}{120} = \frac{7}{15}
 \end{aligned}$$

$$\begin{aligned}
 P(B) &= P(A_1) \cdot P(B|A_1) + P(A_2) P(B|A_2) + P(A_3) P(B|A_3) \\
 &= \frac{1}{15} \times \frac{7}{10} + \frac{7}{15} \times \frac{6}{10} + \frac{7}{15} \times \frac{5}{10} \\
 &= \frac{7+42+35}{150} \\
 &= \frac{84}{150} = \frac{42}{75} = \frac{14}{25}
 \end{aligned}$$

$$(2) P(A_2|B) = \frac{P(A_2B)}{P(B)} = \frac{P(A_2)P(B|A_2)}{P(B)}$$

$$= \frac{\frac{7}{15} \times \frac{6}{10}}{\frac{14}{25}} = \frac{1}{2}$$

故 (1) 第二次取到新球概率为 $\frac{14}{25}$

(2) 第一次取2个新球概率为 $\frac{1}{2}$

二、1. X可能取值有 0、1、2、3

$$P(X=0) = \frac{3}{6} = \frac{1}{2}$$

$$P(X=1) = \frac{3}{6} \cdot \frac{3}{5} = \frac{9}{30} = \frac{3}{10}$$

$$P(X=2) = \frac{3}{6} \cdot \frac{2}{5} \cdot \frac{3}{4} = \frac{3}{20}$$

$$P(X=3) = \frac{3}{6} \cdot \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{20}$$

X	0	1	2	3
P	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{3}{20}$	$\frac{1}{20}$

$$E(X) = 0 \times \frac{1}{2} + 1 \times \frac{3}{10} + 2 \times \frac{3}{20} + 3 \times \frac{1}{20} = \frac{3}{4}$$

$$2.(1) f_X(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{其他} \end{cases} \Rightarrow F_X(x) = \begin{cases} 1 - e^{-2x} & x > 0 \\ 0 & \text{其他} \end{cases}$$

$$(2) P(Y \leq y) = P(1 - e^{-2X} \leq y)$$

$$= P(e^{-2X} \geq 1-y) = \begin{cases} P(-2X \geq \ln(1-y)) & 0 < y < 1 \\ 0 & y \leq 0 \\ 1 & y \geq 1 \end{cases}$$

$$= \begin{cases} 0 & y \leq 0 \\ P(X \leq -\frac{1}{2} \ln(1-y)) & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}$$

$$\Rightarrow F_Y(y) = \begin{cases} 0 & y \leq 0 \\ 1 - e^{-2(-\frac{1}{2} \ln(1-y))} & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}$$

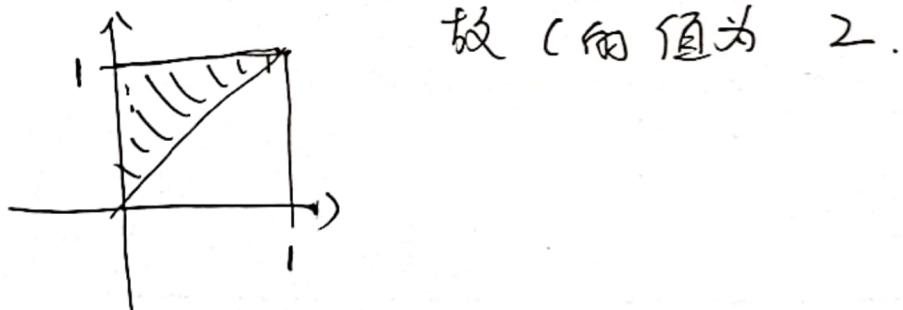
$$= \begin{cases} 0 & y \leq 0 \\ y & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}$$

$$\Rightarrow f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{其他} \end{cases}$$

三、1. A 成立. B 不成立.

附加条件: AB 相互独立.

2. (1) $C = \frac{1}{S_1} = \frac{1}{\frac{1}{2}} = 2$ (均匀分布).



故 C 的值为 2.

$$(2) f_X(x) = \int_{-\infty}^{+\infty} f(x,y) dy$$

$$= \begin{cases} \int_x^1 2 dy & 0 < x < 1 \\ 0 & \text{其他} \end{cases}$$

$$\Rightarrow f_X(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{其他.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y) dx$$

$$= \begin{cases} \int_0^y 2 dx & 0 < y < 1 \\ 0 & \text{其他} \end{cases}$$

$$\Rightarrow f_Y(y) = \begin{cases} 2y & 0 < y < 1 \\ 0 & \text{其他.} \end{cases}$$

$$13) f_X(x) \cdot f_Y(y) = \begin{cases} 2(1-x) \cdot 2y & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \quad 0 < y < 1.$$

$$\Rightarrow f(x,y) \neq f_X(x) f_Y(y)$$

故 X, Y 不相互獨立。

$$14) P(Z \leq z) = P(X+Y \leq z)$$

$$= \begin{cases} \frac{1}{2} \cdot z \cdot \frac{z}{2} \times 2 & 0 < z < 1 \\ 1 - (2-z)\left(\frac{z}{2}\right) \cdot \frac{1}{2} \times 2 & 1 < z < 2 \\ 1 & z \geq 2 \\ 0 & z < 0. \end{cases}$$

$$\Rightarrow F_Z(z) = \begin{cases} 0 & z < 0 \\ \frac{1}{2}z^2 & 0 < z < 1 \\ 1 - \frac{1}{2}(2-z)^2 & 1 < z < 2 \\ 1 & z \geq 2 \end{cases}$$

$$\Rightarrow f_Z(z) = \begin{cases} z & 0 < z < 1 \\ 2-z & 1 < z < 2 \\ 0 & \text{其他} \end{cases}$$

四、

1. 描述两个随机变量线性相关程度的系数。

$|P| \leq 1$ $|P|$ 越接近于1，表明两个随机变量线性相关性越强，反之越弱。

$P>0$ 表明二者正相关， $P<0$ 表明负相关。

2. A 成立。

B 不成立。

$$\text{Q} f(x,y) = \begin{cases} \frac{1}{\pi} & x^2+y^2 \leq 1 \\ 0 & \text{其他} \end{cases}$$

可以证明。① X 与 Y 不相关，但二者不独立。

$$3. E(X)=0, D(X)=1$$

$$E(Y)=2, D(Y)=4 \quad (Y \sim \chi^2(2))$$

$$(1) E(X-2Y) = E(X)-2E(Y) = -4$$

$$\text{独立} \Rightarrow D(X-2Y) = D(X) + 4D(Y) = 17$$

(2) 由 X, Y 相互独立

$$\Rightarrow \text{cov}(X, Y) = E(XY) - E(X)E(Y) = 0.$$

$$\Rightarrow E(XY) = 0$$

$$E(X^2) = D(X) + E^2(X) = 1 \quad E(Y^2) = D(Y) + E^2(Y) = 8$$

$$D(XY) = E(X^2Y^2) - E^2(XY)$$

$$= E(X^2)E(Y^2) - E^2(XY) = 8$$

$$(3) E(U) = E(X+Y) = E(X) + E(Y) = 2$$

$$D(U) = D(X+Y) = D(X) + D(Y) = 5$$

$$E(V) = E(X-Y) = E(X) - E(Y) = 2$$

$$D(V) = D(X-Y) = D(X) + D(Y) = 5.$$

$$\begin{aligned} E(UV) &= E((X+Y)(X-Y)) = E(X^2 - Y^2) \\ &= E(X^2) - E(Y^2) \\ &= 1 - 8 = -7 \end{aligned}$$

$$\begin{aligned} \text{cov}(U, V) &= E(UV) - E(U)E(V) \\ &= -7 + 4 = -3 \end{aligned}$$

$$\rho_{UV} = \frac{\text{cov}(U, V)}{\sqrt{D(U)} \sqrt{D(V)}} = \frac{-3}{\sqrt{5} \cdot \sqrt{5}} = -\frac{3}{5}$$

故 U, V 相关系数为 $-\frac{3}{5}$

(4) 由 $\rho_{UV} = -\frac{3}{5} \neq 0 \Rightarrow U, V$ 不独立。

五、由中心极限定理，这100件的总质量 X 服从正态分布 $N(100 \times 0.5, 100 \times 0.1)$

即 $X \sim N(50, 1)$.

$$\begin{aligned} P(X > 51) &= 1 - P(X \leq 51) \\ &= 1 - \phi\left(\frac{51-50}{1}\right) = 1 - \phi(1) = 0.1587 \end{aligned}$$

故总质量超过 51kg 的概率为 0.1587.

1. 由 $X \sim N(\mu, \sigma^2) \Rightarrow \bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

$$X_6 \sim N(\mu, \sigma^2)$$

$$\Rightarrow X_6 - \bar{X} \sim N(0, \frac{\sigma^2}{5})$$

$$\Rightarrow \frac{X_6 - \bar{X}}{\sqrt{\frac{\sigma^2}{5}}} \sim N(0, 1)$$

$$\alpha \frac{(n-1)s^2}{\sigma^2} = \frac{4s^2}{\sigma^2} \sim \chi^2(4)$$

$$\Rightarrow \frac{\frac{X_6 - \bar{X}}{\sqrt{\frac{\sigma^2}{5}}}}{\sqrt{\frac{4s^2}{\sigma^2}/4}} = \sqrt{\frac{5}{4}} \frac{X_6 - \bar{X}}{s} = \sqrt{\frac{5}{4}} T \sim t(4)$$

故 α 的值为 $\sqrt{\frac{5}{4}}$

2. 1. 由题意 $E(X) = \lambda$.

用样本矩估计 $E(X)$

$$\Rightarrow \lambda = \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i = \frac{1}{2} \times 2.1 = \frac{7}{4}$$

故 λ 的矩估计值为 $\frac{7}{4}$

2. 似然函数为 $L(\theta) = \theta^n (x_1 x_2 \dots x_n)^{\theta-1}$

$$\ln L(\theta) = n \ln \theta + (\theta-1) \left[\sum_{i=1}^n \ln x_i \right]$$

$$\frac{d \ln L(\theta)}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^n \ln x_i = 0$$

$$\Rightarrow \frac{1}{\theta} = -\frac{1}{n} \sum_{i=1}^n \ln x_i$$

$$\Rightarrow \theta = -\frac{n}{\sum_{i=1}^n \ln x_i}$$

故 θ 的最大似然估计量为 $\hat{\theta} = -\frac{n}{\sum_{i=1}^n \ln x_i}$

$\beta = e^{-\frac{1}{6}}$ 的最大似然估计量为 $e^{\frac{1}{n} \sum_{i=1}^n \ln x_i}$

$$= \sqrt[n]{\prod_{i=1}^n x_i}$$

$$3. \text{ 证明: } E(\bar{u}_1) = \frac{1}{3}[E(x_1) + E(x_2) + E(x_3)] = \frac{1}{3} \cdot 3u = u$$

$$\begin{aligned} E(\bar{u}_2) &= \frac{2}{3}E(x_1) - \frac{5}{9}E(x_2) + \frac{8}{9}E(x_3) \\ &= \frac{2}{3}u - \frac{5}{9}u + \frac{8}{9}u = u \end{aligned}$$

故 \bar{u}_1 和 \bar{u}_2 均为无偏估计.

$$(2) D(\bar{u}_1) = \frac{1}{9}(D(x_1) + D(x_2) + D(x_3)) = \frac{6^2}{3}$$

$$\begin{aligned} D(\bar{u}_2) &= \frac{4}{9}D(x_1) + \frac{25}{81}D(x_2) + \frac{64}{81}D(x_3) \\ &= \left(\frac{4}{9} + \frac{25}{81} + \frac{64}{81}\right)6^2 = \frac{125}{81}6^2 > \frac{6^2}{3} \end{aligned}$$

故 \bar{u}_1 更有效.

八、1. 第二类错误. 第一类错误.

$$2. \text{ 设 } H_0: u = u_0 = 500 \quad H_1: u \neq 500$$

检验统计量为 $\frac{\bar{x} - u_0}{6/\sqrt{n}}$. 拒绝域为 $\left| \frac{\bar{x} - u_0}{6/\sqrt{n}} \right| > z_{\alpha/2}$

查表 $z_{\alpha/2} = 1.96$, 计算得 $\left| \frac{\bar{x} - u_0}{6/\sqrt{n}} \right| = \frac{500 - 500}{6/\sqrt{50}} = 0.5 < 1.96$

故饮料符合要求.