2nd year paper

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outline

1. Intro and Lit Review - Work on this section last. For the literature review I would focus on Walker (1981) and the Varian paper I gave you earlier this year.

2. Model - Define economic environment and assumptions on the utility functions/ production function/ etc. Use my draft as your main guide.

3. Generalized Lindahl Equilibrium - Define Generalized Lindahl Equilibria and provide an example (Rawlsian).

4. Generalized Walker Mechanism - define the mechanism and give your main result.

5. Discussion - I would talk about what you still need to do. For example, the other half of the mechanism result (all G. Lindahl Outcomes are achievable as NE of the game). Also, the set of NE is very large, we need to find a way to enforce a single outcome.

1 Introduction

2 Model

A fairly simple economic environment will provide an adequate setting. There will be $i = 1, ..., n \geq 2$ individuals in the economy, each represented by a convex consumption set $C_i = \mathbb{R}_i^2$ and an initial endowment of a private good $\omega_i > 0$. They consume bundles of a public good x a private good y_i, and in doing so gain utility according to utility functions of the following form.

$$u_i(x, y_i) = \int_0^x v_i(t)dt + y_i \tag{1}$$

This functional form simplifies later analysis, as $\frac{\partial u_i}{\partial x} = v_i(x) > 0$ and $\frac{\partial^2 u_i}{\partial x^2} = v_i'(x) < 0$ for $x \in \mathbb{R}_+$. v_i is the the marginal rate of substitution between the public good and the private good for individual i.

The public good is produced in a constant returns to scale production process that uses the private good as an input: for each unit of the public good produced, c of the private good must be sacrificed. If $\Omega = \sum_i \omega_i$, then the most public good that can be produced is $\frac{\Omega}{c}$. An allocation in this economy is then an $n+1$-tuple $(x, y_1, ..., y_n)$.

We shall assume that there is a unique Pareto optimal allocation in the economy, such that a unique level of provision of the public good is Pareto optimal. This is to say that there is a level of the public good that satisfies the Samuelson marginal condition: $\sum_i v_i(x^{PO}) = c$ for $x \in [0, \frac{\Omega}{c}]$.

2.1 Classical Lindahl Equilibrium

An unregulated market cannot efficiently provide a public good. A firm cannot earn revenue by selling the right to consume a public good because it cannot prevent the consumption of a public good by those who have not paid for it. Consumers face a strong incentive not pay anything for the provision of a public good because once the public good is provided the consumption of it by one individual does not prevent the consumption of it by another. These are the usual non-rival and non-excludable characteristics associated with a public good. Some sort of market regulation is required to elicit and finance the efficient provision of the public good. Individuals can be taxed and production can be centralized by a government. One theoretical equilibrium that allows a Pareto optimal amount of the public good to be provided is a classical Lindahl equilibrium.

A classical Lindahl equilibrium uses individualized taxes, known as Lindahl prices, to raise revenue for the provision of a public good. It finds the socially optimal level of the public good and then charges each customer their marginal valuation of the public good at the socially optimal level. With constant marginal costs of provision and downward sloping demand curves, a classical Lindahl equilibrium ensures that it is individually rational for individuals to pay the tax levied upon them and that the sum of such taxes exactly covers the provision of the public good.

For example, consider an instance of the economic setting outlined above with $n = 2$ and $c = 5$. The utility functions of the two individuals are

$$u_1(x, y_1) = \int_0^x (20 - 2t)\, dt + y_1$$

$$u_2(x, y_2) = \int_0^x (6 - 0.1t)\, dt + y_2$$
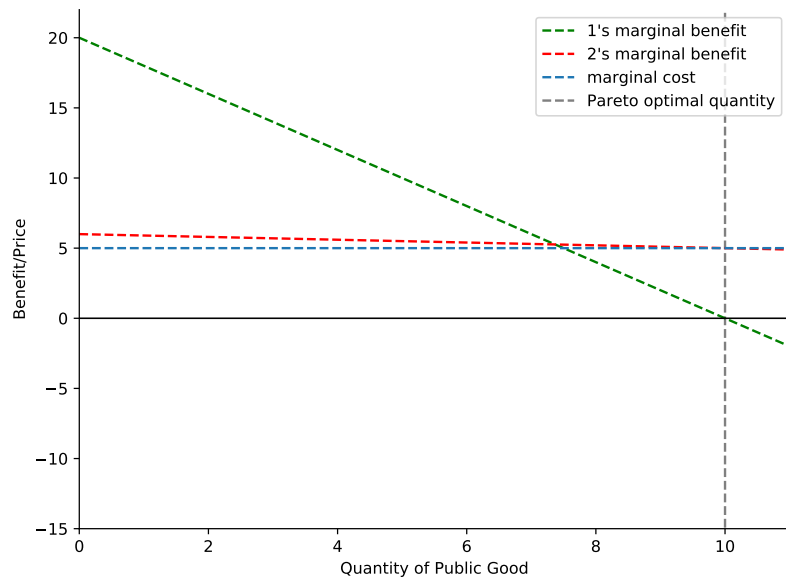
And so

$$v_1(x) = 20 - 2x$$
$$v_2(x) = 6 - 0.1x$$

Figure 1: An example of a classical Lindahl Equilbrium for $n = 2$ consumers. The vertical grey dotted line is located at the Pareto optimal level of the public good. Consumer 1 is charged nothing, while consumer 2 is charged 5 units of the private good.

This setting is shown graphically in figure 1. To construct a classical Lindahl equilibrium we would first find the Pareto optimal level of the public good, x^{PO}, via the Samuelson marginal condition, $\sum_i v_i(x^{PO}) = c$. Then, because the private good is a numeraire, the tax t_i levied upon each individual would be equal to their marginal rate of substitution evaluated at x^{PO}, $t_i = v_i(x^{PO})$. Here, the Pareto optimal amount of the public good is $x^{PO} = 10$. The marginal benefit of the public good at this level is zero for consumer 1 and 5 for consumer 2, and so $t_1 = 0$ and $t_2 = 5$.

We can see that the classical Lindahl equilibrium leaves both consumers better off than they would be if no public good was provided, so the outcome is indeed individually rational. It also raises tax revenue that covers the cost of providing the public good. But, it is also clear that consumer 1 is left far better off than consumer 2, even though consumer 2 incurs all of the costs.

This illustrates how a Lindahl equilibrium does not take into account the distribution of surplus among consumers when optimizing the provision of a public good. Even though the outcome is individually rational, some consumers may be left far better off than others. In order to address this flaw, we introduce the concept of a generalized Lindahl equilibrium.

3 Generalized Lindahl Equilibrium

A generalized Lindahl equilibrium results in each customer receiving a personalized price schedule. The outcome will still be individually rational and cover the cost of provision, as in the classical case, but now a central planner or government can manipulate the personalized price schedules to to distribute consumers surplus as they see fit. A personalized price schedule is defined as a function $p_i : \mathbb{R} \to \mathbb{R}$, such that the total tax revenue collected from individual i for a level of provision of the public good x is $T_i(x) = \int_0^x p_i(t)dt$.

The budget constraint of a customer in our economic setting is then $\int_0^x p_i(t)dt + y_i = \omega_i$. If we solve for the level of the private good and substitute this into their utility function we can express their utility in terms of their initial allocation and the level of public good provided.

$$u_i \left(x, \omega_i - \int_0^x p_i(t)dt \right) = \int_0^x \left[v_i(t) - p_i(t) \right] dt + y_i \tag{2}$$

Definition 1. *A profile of individualized price schedules induces a generalized Lindahl equilibrium if it is (1) individually rational for each individual to demand the Pareto optimal amount of the public good and (2) if the cost of providing the public good is exactly covered by taxes paid by each individual.*

For the tax revenues to cover the cost of provision, we must have

$$\sum_i \int_0^{x^{PO}} p_i(t)dt = cx^{PO}$$

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and for individual rationality we must have

$$u_i \left(x^{PO}, \omega_i - \int_0^{x^{PO}} p_i(t)dt \right) \geq u_i \left(0, \omega_i \right)$$

It is possible for there to be multiple generalized Lindahl equilibria capable of producing a certain level of the public good. In the next section a single example.

3.1 Generalized Lindahl Equilibrium under a CES Social Welfare Function

In this section I solve for a profile of individualized prices that obtains a generalized Lindahl equilibrium. The profile is a solution to a social planner's problem, in which a social planner with CES preferences attempts to maximize welfare in the economic setting described above. Specifically, the problem is

$$\max_{\{p_i(t)\}_{i=1}^n} \int_0^{x^{PO}} \left[\sum_{i=1}^n \beta_i (v_i(x) - p_i(x))^\alpha \right]^{\frac{1}{\alpha}} dx \tag{3}$$

$$s.t. \ v_i(x^{PO}) = p_i(x^{PO}) \tag{4}$$

$$\int_0^{x^{PO}} \sum_{i=1}^n p_i(x)dx = cx^{PO} \tag{5}$$

Where β_i is an arbitrary weight for individual i's surplus. Equation 4 ensures that the level of the public good provided is the level demanded by each individual, and equation 5 ensures that tax revenue will equal the cost of provision.

This is an optimal control problem. The necessary conditions to solve it are 4, 5, and n Euler equations, one for each individual:

$$\lambda = -\beta_j (v_j - p_j)^{\alpha-1} \left[\sum_{i=1}^n \beta_i (v_i - p_i)^\alpha \right]^{\frac{1-\alpha}{\alpha}} \quad \forall \ j = 1, ..., n \tag{6}$$

For the remainder of the derivation any dependency on x is suppressed. λ is the Lagrangian multiplier on the iso-perimetric constraint 5. If we set the Euler equations for individuals 1 and 2 equal to each other, we get

$$\left(\frac{\beta_1}{\beta_2} \right)^{\frac{1}{\alpha-1}} (v_1 - p_1) = v_2 - p_2$$

I will define $\Omega_{12} = (\beta_1/\beta_2)^{\frac{1}{\alpha-1}}$. Substituting this in and rearranging, we get

$$p_2 - \Omega_{12}p_1 = v_2 - \Omega_{12}v_1$$

If we perform a similar manipulation among all n Euler conditions, we are left with a system of $n-1$ equations

$$p_2 - \Omega_{12}p_1 = v_2 - \Omega_{12}v_1$$
$$p_3 - \Omega_{23}p_2 = v_3 - \Omega_{23}v_2$$
$$\vdots$$
$$p_n - \Omega_{n-1,n}p_{n-1} = v_n - \Omega_{n-1,n}v_{n-1}$$

One way to ensure that the iso-perimetric constraint holds is to assume that the prices levied at each marginal provision of the public good are always equal to the marginal cost of production.

$$p_1 + p_2 + ... + p_n = c \tag{7}$$

With this and the system of $n-1$ equations above, we now have a system of n equations that we can use to solve for closed-form solutions of individualized price schedules (the full solution is detailed in the appendix). Those solutions are

$$p_j = \left(\sum_{i=1}^{n} \left(\frac{\beta_j}{\beta_i} \right)^{\frac{1}{\alpha-1}} \right)^{-1} \left[c - \sum_{i \neq j} v_i + v_j \sum_{i \neq j} \left(\frac{\beta_j}{\beta_i} \right)^{\frac{1}{\alpha-1}} \right] \tag{8}$$

Figure 2 plots some examples of 8 in a two consumer setting for certain social welfare weights.

4 Generalized Walker Mechanism

In this section I introduce a mechanism that, when implemented, induces a generalized Lindahl equilibrium.

4.1 Mechanism

A mechanism is defined by the messages players are allowed to choose and the functions that convert these messages into outcomes. In our environment, an outcome is specified by a level of the public good x and a set of personalized tax schedules for each individual τ_i such that each individual pays in total $T_i(x) = \int_0^x \tau_i(t)dt$. A message sent by individual i is a continuous function $p_i : \mathbb{R} \to \mathbb{R}$. A message profile $\mathbf{p} = (p_1, ..., p_n)$ is the list of messages sent by all individuals. The mechanism processes a message profile with $n+1$ outcome functions $\mathcal{X} : \prod_i^n C(\mathbb{R}) \to \mathbb{R}$ and $\mathcal{T}_i : \prod_i^n C(\mathbb{R}) \to C(\mathbb{R})$ for $i = 1, ..., n$. $\mathcal{X}$ determines the level of the public good

$$\mathcal{X}(\mathbf{p}) = \arg\max_{t \geq 0} \int_o^t \sum_{i=1}^{n} p_i(m)dm \tag{9}$$

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Price Schedules, Marginal Benefits, and Marginal Cost in a 2 Consumer Setting for Various Values of α
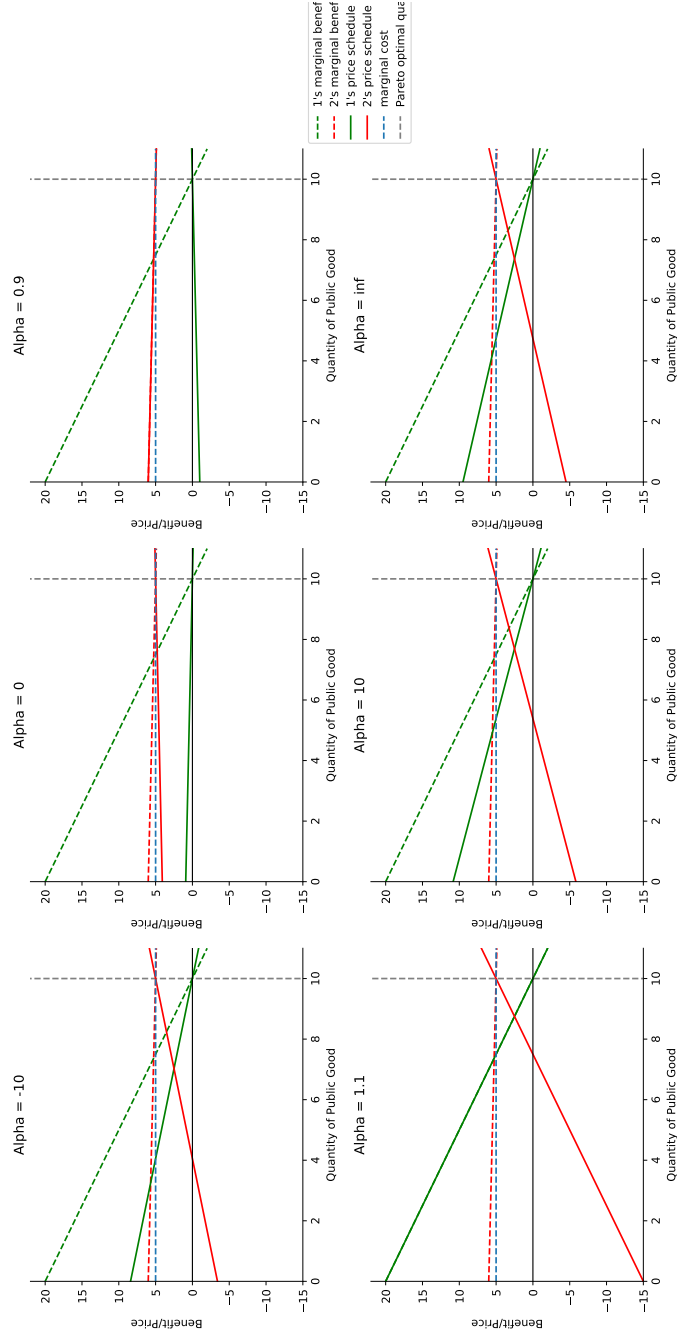

Figure 2: Above are several representations of price schedules leading to generalized Lindahl allocations for a setting with two individuals and various values of α. The marginal benefit functions are $v_1(x) = 20 - 2x$, $v_2(x) = 6 - x/10$. Social welfare weights are $\beta_1 = 10$ and $\beta_2 = 1$. Marginal cost of providing the public good is 5, and the Pareto optimal amount of it is 10. Note that α values of 0.9 and 1.1 are depicted because the weight ratios in 8 are raised to the power of $1/(\alpha - 1)$. In the limit, approaching $\alpha = 1$ from the left or right leads to vastly difference results.

And $\mathcal{T}_i$ determines individual i's personalized tax schedule.

$$\mathcal{T}_i(\mathbf{p}) = \frac{\beta}{n} + p_{i+2} - p_{i+1} \tag{10}$$

The subscript $i + j$ is understood to be a modulo operation, $n + j = j$. This mechanism defines a game form, adding preferences to it induces a game.

Proposition 1. *At any Nash equilibrium induced by our mechanism, any equilibrium outcome $(\vec{\tau}, x)$ coincides with a generalized Lindahl outcome and is Pareto optimal.*

Proof. To be a generalized Lindahl outcome, each individual must be given a personal price schedule, the cost of providing the public good must be covered, and the allocation $(\vec{\tau}, x)$ must be individually rational and Pareto optimal. To begin, each player faces the following utility maximization problem:

$$\max_{p_i(x)} \ u_i \left(\omega_i - \int_0^x \tau_i(t)dt, x \right)$$

$$s.t. \ x = \arg\max_{t \geq 0} \int_o^t \sum_{i=1}^n p_i(m)dm$$

or

$$\max_{p_i(x)} \ u_i \left(\omega_i - \int_0^x \left[\frac{\beta}{n} + p_{i+2}(t) - p_{i+1}(t) \right] dt, x \right) \tag{11}$$

$$s.t. \ x = \arg\max_{t \geq 0} \int_o^t \sum_{i=1}^n p_i(m)dm$$

The problem can be simplified further, as a choice over x. The objective function does not *directly* depend on an individual's choice of message p_i, but the amount of the public good provided does, via the constraint. Given any set of messages submitted by other individuals, a single individual's choice of p_i can demand any level of x, and only by this level of x does p_i affect their utility.

$$\max_x \ u_i \left(\omega_i - \int_0^x \left[\frac{\beta}{n} + p_{i+2}(t) - p_{i+1}(t) \right] dt, x \right) \tag{12}$$

The first order condition for utility maximization is:

$$\frac{u_{i,x}}{u_{i,y_i}} = \frac{\beta}{n} + p_{i+2}(x) - p_{i+1}(x) \tag{13}$$

A best responding individual submits a message p_i such that the level of the public good it demands is optimal given their personalized price at the margin.

The sum of all n first order conditions is

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$$\sum_{i=1}^{n} \frac{u_{i,x}}{u_{i,y_i}} = \beta \tag{14}$$

Because of the modulo dependence of the personalized prices, the sum of all personalized prices equals marginal cost at every marginal provision of the public good. The cost of providing x will always be covered. At any Nash equilibrium, any equilibrium outcome is a generalized Lindahl outcome: the outcome provides individualized price schedules, covers the cost of provision of the public good, and is Pareto optimal and individually rational.

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5   Discussion

6   Appendix

The solution will take place as follows: I will solve for p_1 in terms of p_2, v_1 and v_2, and then substitute p_1 into 7; I will iterate this process two more times, substituting equations for p_2 and p_3 into 7; at that point a pattern will emerge that will allow me to extrapolate a closed-form solution for to p_n, and then for p_{n-1}; finally, I will use the pattern observable in the solutions for p_n and p_{n-1} solution to deduce a closed form solution for p_i in general.

As stated, first solve for p_1

$$p_1 = \frac{-v_2}{\Omega_{12}} + v_1 + \frac{p_2}{\Omega_{12}}$$

Substitute into 7

$$\left(\frac{-v_2}{\Omega_{12}} + v_1 + \frac{p_2}{\Omega_{12}} \right) + p_2 + p_3 + ... + p_n = c$$

Combine terms, keeping prices and marginal benefits organized, and note that $\Omega_{ij} = \Omega_{ji}^{-1}$. We get our first instance of a pattern.

$$(1 + \Omega_{21}) p_2 + p_3 + ... + p_n + v_1 - \Omega_{21} v_2 = c$$

Iterating, solve for p_2 in terms of p_3, v_2 and v_3 using the equation relating p_2 to p_3 from our system of equalities. Substitute into above.

$$(1 + \Omega_{21}) \left(\frac{-v_3}{\Omega_{23}} + v_2 + \frac{p_3}{\Omega_{23}} \right) + p_3 + p_4 + ... + p_n + v_1 - \Omega_{21} v_2 = c$$

Combine terms again, and note that $\Omega_{Ki}/\Omega_{Kj} = \Omega_{ij}$. A nice cancellation occurs. We get a second instance of the pattern.

$$(1 + \Omega_{32} + \Omega_{31}) p_3 + p_4 + ... + p_n + v_1 + v_2 - (\Omega_{32} + \Omega_{31}) v_3 = c$$

Iterate one last time, substituting for p_3.

$$(1 + \Omega_{32} + \Omega_{31}) \left(\frac{-v_4}{\Omega_{34}} + v_3 + \frac{p_4}{\Omega_{34}} \right) + p_4 + p_5 + ... + p_n + v_1 + v_2 - (\Omega_{32} + \Omega_{31})v_3 = c$$

Combine terms, and we get a third instance of the pattern.

$$(1 + \Omega_{43} + \Omega_{42} + \Omega_{41}) p_4 + p_5 + ... + p_n + v_1 + v_2 + v_3 - (\Omega_{43} + \Omega_{42} + \Omega_{41})v_4 = c$$

Let's quickly look at all three instances of the pattern together.

$$(1 + \Omega_{21}) p_2 + p_3 + ... + p_n + v_1 - \Omega_{21}v_2 = c$$
$$(1 + \Omega_{32} + \Omega_{31}) p_3 + p_4 + ... + p_n + v_1 + v_2 - (\Omega_{32} + \Omega_{31})v_3 = c$$
$$(1 + \Omega_{43} + \Omega_{42} + \Omega_{41}) p_4 + p_5 + ... + p_n + v_1 + v_2 + v_3 - (\Omega_{43} + \Omega_{42} + \Omega_{41})v_4 = c$$

By extrapolation, and noting that $\Omega_{ii} = 1$, p_n must be implicitly defined by

$$p_n \sum_{i=1}^{n} \Omega_{ni} + \sum_{i=1}^{n-1} v_i - v_n \sum_{i=1}^{n-1} \Omega_{ni} = c$$

Solving for p_n, we get

$$p_n = \left(\sum_{i=1}^{n} \Omega_{ni} \right)^{-1} \left[c - \sum_{i=1}^{n-1} v_i + v_n \sum_{i=1}^{n-1} \Omega_{ni} \right] \tag{15}$$

To get a closed form solution for any p_i, I'll first use 6 to solve for p_{n-1} using the equation from our system of equations that relates p_{n-1} to p_n. Then with closed from solutions for both p_n and p_{n-1}, a pattern emerges that I use to solve for the general closed form solution for p_i.

First, solve for p_{n-1}. We know that

$$p_{n-1} = \frac{-v_n}{\Omega_{n-1,n}} + v_{n-1} + \frac{p_n}{\Omega_{n-1,n}}$$

Substitute in 6. Step by step simplification:

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$$p_{n-1} = \frac{-v_n}{\Omega_{n-1,n}} + v_{n-1} + \frac{1}{\Omega_{n-1,n}} \left(\sum_{i=1}^{n} \Omega_{ni} \right)^{-1} \left[c - \sum_{i=1}^{n-1} v_i + v_n \sum_{i=1}^{n-1} \Omega_{ni} \right]$$

$$= -\Omega_{n,n-1}v_n + v_{n-1} + \left(\sum_{i=1}^{n} \Omega_{n-1,n}\Omega_{ni} \right)^{-1} \left[c - \sum_{i=1}^{n-1} v_i + v_n \sum_{i=1}^{n-1} \Omega_{ni} \right]$$

$$= -\Omega_{n,n-1}v_n + v_{n-1} + \left(\sum_{i=1}^{n} \Omega_{n-1,i} \right)^{-1} \left[c - \sum_{i=1}^{n-1} v_i + v_n \sum_{i=1}^{n-1} \Omega_{ni} \right]$$

$$= \left(\sum_{i=1}^{n} \Omega_{n-1,i} \right)^{-1} \left[-\Omega_{n,n-1}v_n \sum_{i=1}^{n} \Omega_{n-1,i} + v_{n-1} \sum_{i=1}^{n} \Omega_{n-1,i} + c - \sum_{i=1}^{n-1} v_i + v_n \sum_{i=1}^{n-1} \Omega_{ni} \right]$$

$$= \left(\sum_{i=1}^{n} \Omega_{n-1,i} \right)^{-1} \left[-v_n \sum_{i=1}^{n} \Omega_{n,i} + v_{n-1} \sum_{i=1}^{n} \Omega_{n-1,i} + c - \sum_{i=1}^{n-1} v_i + v_n \sum_{i=1}^{n-1} \Omega_{ni} \right]$$

$$= \left(\sum_{i=1}^{n} \Omega_{n-1,i} \right)^{-1} \left[-v_n \sum_{i=1}^{n} \Omega_{ni} + v_{n-1} \sum_{i=1}^{n} \Omega_{n-1,i} + c - \sum_{i=1}^{n-1} v_i + v_n \sum_{i=1}^{n-1} \Omega_{ni} \right]$$

$$= \left(\sum_{i=1}^{n} \Omega_{n-1,i} \right)^{-1} \left[v_{n-1} \sum_{i=1}^{n} \Omega_{n-1,i} + c - \sum_{i=1}^{n-1} v_i + v_n \left(\sum_{i=1}^{n-1} \Omega_{ni} - \sum_{i=1}^{n} \Omega_{ni} \right) \right]$$

At this point, we need to break steps down term by term. Note that the term in the parenthesis within the brackets equals -1, and that

$$v_{n-1} \sum_{i=1}^{n} \Omega_{n-1,i} = v_{n-1} \left(1 + \sum_{i \neq n-1} \Omega_{n-1,i} \right) = v_{n-1} + v_{n-1} \sum_{i \neq n-1} \Omega_{n-1,i}$$

The lone v_{n-1} will cancel with the final term in the $\sum_{i=1}^{n-1} v_i$ summation. This leaves us with

$$p_{n-1} = \left(\sum_{i=1}^{n} \Omega_{n-1,i} \right)^{-1} \left[v_{n-1} \sum_{i \neq n-1} \Omega_{n-1,i} + c - \sum_{i=1}^{n-2} v_i - v_n \right]$$

Or

$$p_{n-1} = \left(\sum_{i=1}^{n} \Omega_{n-1,i} \right)^{-1} \left[c - \sum_{i \neq n-1} v_i + v_{n-1} \sum_{i \neq n-1} \Omega_{n-1,i} \right] \qquad (16)$$

Our closed form solution to p_{n-1}. Finally let's compare 7 to 6.

$$p_{n-1} = \left(\sum_{i=1}^{n} \Omega_{n-1,i} \right)^{-1} \left[c - \sum_{i \neq n-1} v_i + v_{n-1} \sum_{i \neq n-1} \Omega_{n-1,i} \right]$$

$$p_n = \left(\sum_{i=1}^{n} \Omega_{ni} \right)^{-1} \left[c - \sum_{i=1}^{n-1} v_i + v_n \sum_{i=1}^{n-1} \Omega_{ni} \right]$$

From the above pattern, it's clear that the closed form solution for any p_j is

$$p_j = \left(\sum_{i=1}^{n} \left(\frac{\beta_j}{\beta_i} \right)^{\frac{1}{\alpha-1}} \right)^{-1} \left[c - \sum_{i \neq j} v_i + v_j \sum_{i \neq j} \left(\frac{\beta_j}{\beta_i} \right)^{\frac{1}{\alpha-1}} \right] \qquad (17)$$