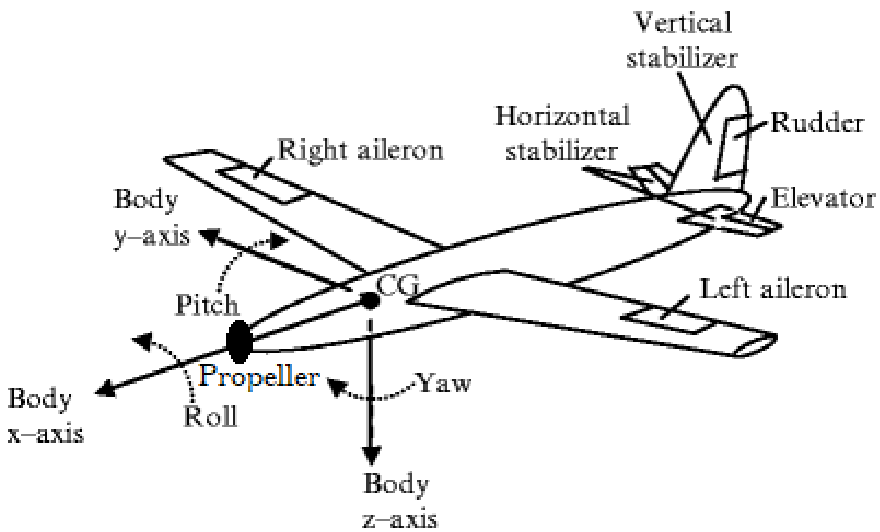
AEM 591 Project 3

Optimal Airplane Control

This project is intended to introduce the use of the robust servomechanism linear-quadratic regulator (RSLQR) for the control of an airplane.

To that end, consider a traditional airplane configuration



where the inputs to the airplane are the control surface deflections from the ailerons, δ_a, the elevator, δ_e, and the rudder, δ_r, as well as the thrust from the propeller, δ_t.

One can show that the 6-degree-of-freedom (6-DOF) rigid body airplane equations of motion as defined in the body-fixed frame

$$
\begin{aligned}
\begin{bmatrix} mX - g\sin\theta \\ mY + mg\sin\phi\cos\theta \\ mZ - mg\cos\phi\cos\theta \end{bmatrix} &= m\begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \end{bmatrix} \\
\begin{bmatrix} I_{xx}L \\ I_{yy}M \\ I_{zz}N \end{bmatrix} &= \begin{bmatrix} I_{xx}\dot{p} + (I_{zz} - I_{yy})qr - I_{xz}(\dot{r} + pq) \\ I_{yy}\dot{q} + (I_{xx} - I_{zz})pr + I_{xz}(p^2 - r^2) \\ I_{zz}\dot{r} + (I_{yy} - I_{xx})pq - I_{xz}(\dot{p} - qr) \end{bmatrix}
\end{aligned} \tag{1}
$$

where the states of the airplane are the linear velocity of the airplane body-fixed frame, $[u\ v\ w]^T$, the angular velocity of the airplane body-fixed frame, $[p\ q\ r]^T$, and the $3-2-1$ Euler angles, $(\phi,\ \theta,\ \psi)$, representing the three sequential rotations to relate the body-fixed frame and the North-East-Down (NED) "flat Earth" frame. The Euler angle rates depend explicitly on the angular velocity by the relationship

$$\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} = \begin{bmatrix}1 & \sin\phi\tan\theta & \cos\phi\tan\theta\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta\end{bmatrix}\begin{bmatrix}p\\ q\\ r\end{bmatrix} \tag{2}$$

From inspection, one can see that the gravitational force, mg, acts on the vehicle in the down direction of the airplane. The normalized (by mass, m) aerodynamic and propulsive forces, X, Y, Z, as well as the normalized (by moments of inertia I_{xx}, I_{yy}, and I_{zz}) aerodynamic and propulsive moments, L, M, N, are typically represented using linear functions of the different airplane states, whose coefficients are known as the stability and control derivatives and are typically represented by state subscripts to the forces and moments.

It should be noted that the body-fixed frame forces can be related to the thrust and aerodynamic lift, drag, and side forces, L, D, S, on the airplane through a rotation matrix based on the angles of attack, α, and sideslip angle, β, which define the relative direction of the airflow to the airplane. This can be shown to be

$$m\begin{bmatrix}X\\ Y\\ Z\end{bmatrix} = \begin{bmatrix}\delta_t\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}-D\cos\alpha\cos\beta - S\cos\alpha\sin\beta + L\sin\alpha\\ S\cos\beta - D\sin\beta\\ -D\sin\alpha\cos\beta - S\sin\alpha\sin\beta - L\sin\alpha\end{bmatrix} \tag{3}$$

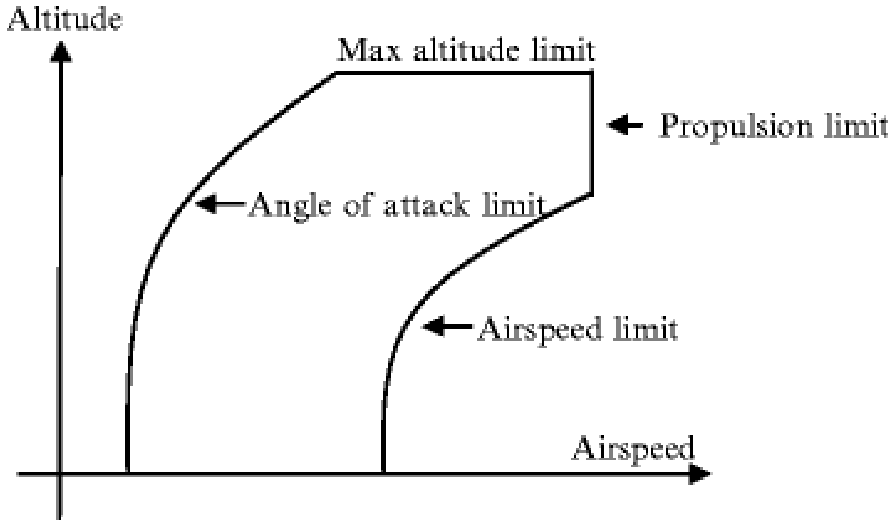
Lastly, one can also include the position over the North-East-Down (NED) flat Earth frame by integrating the inertial velocity components through a rotation matrix using the $3-2-1$ Euler angles as

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ -\dot{h}\end{bmatrix} = \begin{bmatrix}\cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi\\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi\\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix} \tag{4}$$

where x is the north position, y is east position, and h is the altitude of the airplane.

Linearized Flight Dynamics for Control

As derived above, the general airplane flight dynamics are highly nonlinear and represent much complexity with relation to the model parameters. Thus, for control purposes one can use operate near equilibrium flight conditions, known as trimmed steady flight, which allows one to linearize the dynamics of the airplane for the purposes of control design. This act of trimming the airplane occurs when one balances the forces, moments, and inputs which are typically represented using overbars. Given the typical operating flight envelope of airplanes, i.e.



based on the aerodynamic, structural, and engine limits, one can compute a dense set of trim points over the entire envelope for which linearized dynamics and fixed-gain controllers can be designed. Then, one can interpolate between these trim points based on the flight conditions for suitable linear controllers. This approach is known as gain scheduling.

When this trimmed steady flight condition is straight, coordinated flight, one can decouple the dynamics into two sets of linearized dynamics, the longitudinal linearized dynamics and the lateral-directional linearized dynamics. The longitudinal linearized dynamics of an airplane are

$$\begin{aligned}\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{\alpha}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix} &= \begin{bmatrix} X_u & X_\alpha & 0 & -g\cos\bar{\theta} \\ \frac{Z_u}{\bar{u}} & \frac{Z_\alpha}{\bar{u}} & 1 & -\frac{g}{\bar{u}}\sin\bar{\theta} \\ M_u + M_{\dot{\alpha}}\frac{Z_u}{\bar{u}} & M_\alpha + M_{\dot{\alpha}}\frac{Z_\alpha}{\bar{u}} & M_q + M_{\dot{\alpha}} & -M_{\dot{\alpha}}\frac{g}{\bar{u}}\sin\bar{\theta} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix}\Delta u\\ \Delta\alpha\\ \Delta q\\ \Delta\theta\end{bmatrix} \\ &+ \begin{bmatrix} 0 & X_{\delta_t} \\ \frac{Z_{\delta_e}}{\bar{u}} & \frac{Z_{\delta_t}}{\bar{u}} \\ M_{\delta_e} + M_{\dot{\alpha}}\frac{Z_{\delta_e}}{\bar{u}} & M_{\delta_t} + M_{\dot{\alpha}}\frac{Z_{\delta_t}}{\bar{u}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix}\Delta\delta_e\\ \Delta\delta_t\end{bmatrix}\end{aligned} \tag{5}$$

and the lateral-directional linearized dynamics of the airplane are

$$\begin{aligned}\begin{bmatrix}\Delta\dot{\beta}\\ \Delta\dot{p}\\ \Delta\dot{r}\\ \Delta\dot{\phi}\end{bmatrix} &= \begin{bmatrix} \frac{Y_\beta}{\bar{u}} & \frac{Y_p}{\bar{u}} & \frac{Y_r}{\bar{u}} - 1 & \frac{g}{\bar{u}}\cos\bar{\theta} \\ L^*_\beta + \frac{I_{xz}}{I_{xx}}N^*_\beta & L^*_p + \frac{I_{xz}}{I_{xx}}N^*_p & L^*_r + \frac{I_{xz}}{I_{xx}}N^*_r & 0 \\ N^*_\beta + \frac{I_{xz}}{I_{zz}}L^*_\beta & N^*_p + \frac{I_{xz}}{I_{zz}}L^*_p & N^*_r + \frac{I_{xz}}{I_{zz}}L^*_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix}\Delta\beta\\ \Delta p\\ \Delta r\\ \Delta\phi\end{bmatrix} \\ &+ \begin{bmatrix} 0 & \frac{Y_{\delta_r}}{\bar{u}} \\ L^*_{\delta_a} + \frac{I_{xz}}{I_{xx}}N^*_{\delta_a} & L^*_{\delta_r} + \frac{I_{xz}}{I_{xx}}N^*_{\delta_r} \\ N^*_{\delta_a} + \frac{I_{xz}}{I_{zz}}L^*_{\delta_a} & N^*_{\delta_r} + \frac{I_{xz}}{I_{zz}}L^*_{\delta_r} \\ 0 & 0 \end{bmatrix} \begin{bmatrix}\Delta\delta_a\\ \Delta\delta_r\end{bmatrix}\end{aligned} \tag{6}$$

where it should be noted that in this derivation, $\Delta\alpha$ and $\Delta\beta$ have been used to eliminate Δv and Δw from the dynamics and $u = \bar{u} + \Delta u$ is the overall airspeed of the airplane.

Project Assignment and Deliverables

For an airplane flying at mean sea level, i.e. $h = 0$, and an airspeed of 250 ft/s, the continuous-time LTI state-space model has been derived for the longitudinal linearized dynamics as

$$\begin{aligned}\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{\alpha}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix} &= \begin{bmatrix}-0.038 & 18.984 & 0 & -32.174\\ -0.001 & -0.632 & 1.0 & 0\\ 0 & -0.759 & -0.518 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta\alpha\\ \Delta q\\ \Delta\theta\end{bmatrix}\\ &\quad + \begin{bmatrix}0 & 10.1\\ -0.0086 & 0\\ -0.011 & 0.025\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Delta\delta_e\\ \Delta\delta_t\end{bmatrix}\end{aligned} \tag{7}$$

and the lateral-directional linearized dynamics as

$$\begin{aligned}\begin{bmatrix}\Delta\dot{\beta}\\ \Delta\dot{p}\\ \Delta\dot{r}\\ \Delta\dot{\phi}\end{bmatrix} &= \begin{bmatrix}-0.0829 & 0 & -1 & 0.0487\\ -4.546 & -1.699 & 0.1717 & 0\\ 3.382 & -0.0654 & -0.0893 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}\Delta\beta\\ \Delta p\\ \Delta r\\ \Delta\phi\end{bmatrix}\\ &\quad + \begin{bmatrix}0 & 0.0116\\ 27.276 & 0.5758\\ 0.3952 & -1.362\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Delta\delta_a\\ \Delta\delta_r\end{bmatrix}\end{aligned} \tag{8}$$

where the airspeed is in ft/s, the Euler angles and control surface deflections are in radians, the thrust is in lbs, and the angular velocities are in radians/sec. Note that $\bar{\theta} = 0$ in this case.

For this project, perform the following tasks in Python:

1. Design an RSLQR to track a constant airspeed command and a constant angle-of-attack command;
2. Design an RSLQR to track a constant roll angle command;
3. Simulate the linear airplane dynamics with the decoupled control systems with a step response on the commands.

In Python, the function

```
scipy.linalg.solve_continuous_are(a, b, q, r, e=None, s=None, balanced=True)
```

can be used to solve the CARE for the unconstrained infinite horizon continuous-time LQR. This function uses a solver that forms the extended Hamiltonian matrix pencil, and then uses a

QZ decomposition method. It should be noted that the `e` in this function is not used for LQR control.

An example script of this is

```
import numpy as np

from scipy import linalg as la

A = np.array(...)

B = np.array(...)

Q = np.array(...)

R = np.array(...)

la.solve_continuous_are(A, B, Q, R, S=np.array(...))

K = la.inv(R)@(B.T@P + S.T)
```