That is, the first unknown x_1 is the leading unknown in the first equation, the second unknown x_2 is the leading unknown in the second equation, and so on. Thus, in particular, the system is square and each leading unknown is *directly* to the right of the leading unknown in the preceding equation.

Such a triangular system always has a unique solution, which may be obtained by *back-substitution*. That is,

(1) First solve the last equation for the last unknown to get $x_4 = 4$.

(2) Then substitute this value $x_4 = 4$ in the next-to-last equation, and solve for the next-to-last unknown x_3 as follows:

$$7x_3 - 4 = 3 \quad \text{or} \quad 7x_3 = 7 \quad \text{or} \quad x_3 = 1$$

(3) Now substitute $x_3 = 1$ and $x_4 = 4$ in the second equation, and solve for the second unknown x_2 as follows:

$$5x_2 - 1 + 12 = 1 \quad \text{or} \quad 5x_2 + 11 = 1 \quad \text{or} \quad 5x_2 = -10 \quad \text{or} \quad x_2 = -2$$

(4) Finally, substitute $x_2 = -2$, $x_3 = 1$, $x_4 = 4$ in the first equation, and solve for the first unknown x_1 as follows:

$$2x_1 + 6 + 5 - 8 = 9 \quad \text{or} \quad 2x_1 + 3 = 9 \quad \text{or} \quad 2x_1 = 6 \quad \text{or} \quad x_1 = 3$$

Thus, $x_1 = 3$, $x_2 = -2$, $x_3 = 1$, $x_4 = 4$, or, equivalently, the vector $u = (3, -2, 1, 4)$ is the unique solution of the system.

Remark: There is an alternative form for back-substitution (which will be used when solving a system using the matrix format). Namely, after first finding the value of the last unknown, we substitute this value for the last unknown in all the preceding equations before solving for the next-to-last unknown. This yields a triangular system with one less equation and one less unknown. For example, in the above triangular system, we substitute $x_4 = 4$ in all the preceding equations to obtain the triangular system

$$\begin{aligned} 2x_1 - 3x_2 + 5x_3 &= 17 \\ 5x_2 - x_3 &= -1 \\ 7x_3 &= 7 \end{aligned}$$

We then repeat the process using the new last equation. And so on.

Echelon Form, Pivot and Free Variables

The following system of linear equations is said to be in *echelon form*:

$$\begin{aligned} 2x_1 + 6x_2 - x_3 + 4x_4 - 2x_5 &= 15 \\ x_3 + 2x_4 + 2x_5 &= 5 \\ 3x_4 - 9x_5 &= 6 \end{aligned}$$

That is, no equation is degenerate and the leading unknown in each equation other than the first is to the right of the leading unknown in the preceding equation. The leading unknowns in the system, x_1, x_3, x_4, are called *pivot* variables, and the other unknowns, x_2 and x_5, are called *free* variables.

Generally speaking, an *echelon system* or a *system in echelon form* has the following form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \cdots + a_{1n}x_n &= b_1 \\ a_{2j_2}x_{j_2} + a_{2,j_2+1}x_{j_2+1} + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{rj_r}x_{j_r} + \cdots + a_{rn}x_n &= b_r \end{aligned} \tag{3.5}$$

where $1 < j_2 < \cdots < j_r$ and $a_{11}, a_{2j_2}, \ldots, a_{rj_r}$ are not zero. The *pivot* variables are $x_1, x_{j_2}, \ldots, x_{j_r}$. Note that $r \leq n$.

The solution set of any echelon system is described in the following theorem (proved in Problem 3.10).

We now solve the new equation for y, obtaining $y = 2$. We substitute $y = 2$ into one of the original equations, say and solve for the other unknown x, obtaining

$$2x - 3(2) = -8 \quad \text{or} \quad 2x - 6 = 8 \quad \text{or} \quad 2x = -2 \quad \text{or} \quad x = -1$$

Thus, $x = -1, y = 2$, or the pair $u = (-1, 2)$ is the unique solution of the system. The unique solution is expected because $2/3 \neq -3/4$. [Geometrically, the lines corresponding to the equations intersect at the point $(-1, 2)$.]

EXAMPLE 3.6 (Nonunique Cases)

(a) Solve the system

$$\begin{aligned} L_1: \quad x - 3y &= 4 \\ L_2: \quad -2x + 6y &= 5 \end{aligned}$$

We eliminated x from the equations by multiplying L_1 by 2 and adding it to L_2—that is, by forming the new equation $L = 2L_1 + L_2$. This yields the degenerate equation

$$0x + 0y = 13$$

which has a nonzero constant $b = 13$. Thus, this equation and the system have no solution. This is expected because $1/(-2) = -3/6 \neq 4/5$. (Geometrically, the lines corresponding to the equations are parallel.)

(b) Solve the system

$$\begin{aligned} L_1: \quad x - 3y &= 4 \\ L_2: \quad -2x + 6y &= -8 \end{aligned}$$

We eliminated x from the equations by multiplying L_1 by 2 and adding it to L_2—that is, by forming the new equation $L = 2L_1 + L_2$. This yields the degenerate equation

$$0x + 0y = 0$$

where the constant term is also zero. Thus, the system has an infinite number of solutions, which correspond to the solutions of either equation. This is expected, because $1/(-2) = -3/6 = 4/(-8)$. (Geometrically, the lines corresponding to the equations coincide.)

To find the general solution, let $y = a$, and substitute into L_1 to obtain

$$x - 3a = 4 \quad \text{or} \quad x = 3a + 4$$

Thus, the general solution of the system is

$$x = 3a + 4, y = a \quad \text{or} \quad u = (3a + 4, \ a)$$

where a (called a *parameter*) is any scalar.

3.5 Systems in Triangular and Echelon Forms

The main method for solving systems of linear equations, Gaussian elimination, is treated in Section 3.6. Here we consider two simple types of systems of linear equations: systems in triangular form and the more general systems in echelon form.

Triangular Form

Consider the following system of linear equations, which is in *triangular form*:

$$\begin{aligned} 2x_1 - 3x_2 + 5x_3 - 2x_4 &= 9 \\ 5x_2 - x_3 + 3x_4 &= 1 \\ 7x_3 - x_4 &= 3 \\ 2x_4 &= 8 \end{aligned}$$

(2) *The system has no solution.*
Here the two lines are parallel [Fig. 3-2(b)]. This occurs when the lines have the same slopes but different y intercepts, or when

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} \neq \frac{C_1}{C_2}$$

For example, in Fig. 3-2(*b*), $1/2 = 3/6 \neq -3/8$.

(3) *The system has an infinite number of solutions.*
Here the two lines coincide [Fig. 3-2(c)]. This occurs when the lines have the same slopes and same y intercepts, or when the coefficients and constants are proportional,

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$$

For example, in Fig. 3-2(c), $1/2 = 2/4 = 4/8$.

Remark: The following expression and its value is called a *determinant of order two*:

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = A_1B_2 - A_2B_1$$

Determinants will be studied in Chapter 8. Thus, the system (3.4) has a unique solution if and only if the determinant of its coefficients is not zero. (We show later that this statement is true for any square system of linear equations.)

Elimination Algorithm

The solution to system (3.4) can be obtained by the process of elimination, whereby we reduce the system to a single equation in only one unknown. Assuming the system has a unique solution, this elimination algorithm has two parts.

ALGORITHM 3.1: The input consists of two nondegenerate linear equations L_1 and L_2 in two unknowns with a unique solution.

Part A. (Forward Elimination) Multiply each equation by a constant so that the resulting coefficients of one unknown are negatives of each other, and then add the two equations to obtain a new equation L that has only one unknown.

Part B. (Back-Substitution) Solve for the unknown in the new equation L (which contains only one unknown), substitute this value of the unknown into one of the original equations, and then solve to obtain the value of the other unknown.

Part A of Algorithm 3.1 can be applied to any system even if the system does not have a unique solution. In such a case, the new equation L will be degenerate and Part B will not apply.

EXAMPLE 3.5 (Unique Case). Solve the system

$$L_1: \; 2x - 3y = -8$$
$$L_2: \; 3x + 4y = \;\; 5$$

The unknown x is eliminated from the equations by forming the new equation $L = -3L_1 + 2L_2$. That is, we multiply L_1 by -3 and L_2 by 2 and add the resulting equations as follows:

$$\begin{aligned} -3L_1: &\quad -6x + 9y = 24 \\ 2L_2: &\quad \;\;\, 6x + 8y = 10 \\ \hline \text{Addition}: &\qquad\qquad 17y = 34 \end{aligned}$$

Linear Equation in One Unknown

The following simple basic result is proved in Problem 3.5.

THEOREM 3.5: Consider the linear equation $ax = b$.

(i) If $a \neq 0$, then $x = b/a$ is a unique solution of $ax = b$.

(ii) If $a = 0$, but $b \neq 0$, then $ax = b$ has no solution.

(iii) If $a = 0$ and $b = 0$, then every scalar k is a solution of $ax = b$.

EXAMPLE 3.4 Solve (a) $4x - 1 = x + 6$, (b) $2x - 5 - x = x + 3$, (c) $4 + x - 3 = 2x + 1 - x$.

(a) Rewrite the equation in standard form obtaining $3x = 7$. Then $x = \frac{7}{3}$ is the unique solution [Theorem 3.5(i)].

(b) Rewrite the equation in standard form, obtaining $0x = 8$. The equation has no solution [Theorem 3.5(ii)].

(c) Rewrite the equation in standard form, obtaining $0x = 0$. Then every scalar k is a solution [Theorem 3.5(iii)].

System of Two Linear Equations in Two Unknowns (2×2 System)

Consider a system of two nondegenerate linear equations in two unknowns x and y, which can be put in the standard form

$$A_1x + B_1y = C_1$$
$$A_2x + B_2y = C_2$$

Because the equations are nondegenerate, A_1 and B_1 are not both zero, and A_2 and B_2 are not both zero.

The general solution of the system (3.4) belongs to one of three types as indicated in Fig. 3-1. If $\mathbf{R}$ is the field of scalars, then the graph of each equation is a line in the plane $\mathbf{R}^2$ and the three types may be described geometrically as pictured in Fig. 3-2. Specifically,

(1) *The system has exactly one solution.*
Here the two lines intersect in one point [Fig. 3-2(a)]. This occurs when the lines have distinct slopes or, equivalently, when the coefficients of x and y are not proportional:

$$\frac{A_1}{A_2} \neq \frac{B_1}{B_2} \qquad \text{or, equivalently,} \qquad A_1B_2 - A_2B_1 \neq 0$$

For example, in Fig. 3-2(a), $1/3 \neq -1/2$.

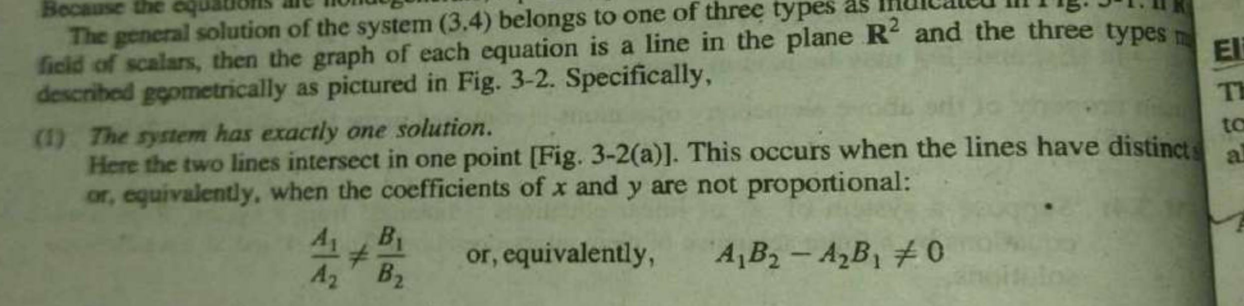

Figure 3-2

...e solution of such an equation depends only on the value of the constant b. Specifically,

(i) If $b \neq 0$, then the equation has no solution.
(ii) If $b = 0$, then every vector $u = (k_1, k_2, \ldots, k_n)$ in K^n is a solution.

The following theorem applies.

THEOREM 3.2: Let $\mathscr{L}$ be a system of linear equations that contains a degenerate equation L, say with constant b.

(i) If $b \neq 0$, then the system $\mathscr{L}$ has no solution.
(ii) If $b = 0$, then L may be deleted from the system without changing the solution set of the system.

Part (i) comes from the fact that the degenerate equation has no solution, so the system has no solution. Part (ii) comes from the fact that every element in K^n is a solution of the degenerate equation.

Leading Unknown in a Nondegenerate Linear Equation

Now let L be a nondegenerate linear equation. This means one or more of the coefficients of L are not zero. By the *leading unknown* of L, we mean the first unknown in L with a nonzero coefficient. For example, x_3 and y are the leading unknowns, respectively, in the equations

$$0x_1 + 0x_2 + 5x_3 + 6x_4 + 0x_5 + 8x_6 = 7 \quad \text{and} \quad 0x + 2y - 4z = 5$$

We frequently omit terms with zero coefficients, so the above equations would be written as

$$5x_3 + 6x_4 + 8x_6 = 7 \quad \text{and} \quad 2y - 4z = 5$$

In such a case, the leading unknown appears first.

3.3 Equivalent Systems, Elementary Operations

Consider the system (3.2) of m linear equations in n unknowns. Let L be the linear equation obtained by multiplying the m equations by constants $c_1, c_2, \ldots, c_m$, respectively, and then adding the resulting equations. Specifically, let L be the following linear equation:

$$(c_1a_{11} + \cdots + c_ma_{m1})x_1 + \cdots + (c_1a_{1n} + \cdots + c_ma_{mn})x_n = c_1b_1 + \cdots + c_mb_m$$

Then L is called a *linear combination* of the equations in the system. One can easily show (Problem 3.43) that any solution of the system (3.2) is also a solution of the linear combination L.

EXAMPLE 3.3 Let L_1, L_2, L_3 denote, respectively, the three equations in Example 3.2. Let L be the equation obtained by multiplying L_1, L_2, L_3 by $3, -2, 4$, respectively, and then adding. Namely,

$$\begin{array}{rl}
3L_1: & 3x_1 + 3x_2 + 12x_3 + 9x_4 = 15 \\
-2L_2: & -4x_1 - 6x_2 - 2x_3 + 4x_4 = -2 \\
4L_1: & 4x_1 + 8x_2 - 20x_3 + 16x_4 = 12 \\
\hline
(\text{Sum})\ L: & 3x_1 + 5x_2 - 10x_3 + 29x_4 = 25
\end{array}$$

solution of such an equation depends only on the value of the constant b. Specifically,

(i) If $b \neq 0$, then the equation has no solution.

(ii) If $b = 0$, then every vector $u = (k_1, k_2, \ldots, k_n)$ in K^n is a solution.

The following theorem applies.

THEOREM 3.2: Let $\mathscr{L}$ be a system of linear equations that contains a degenerate equation L, say with constant b.

(i) If $b \neq 0$, then the system $\mathscr{L}$ has no solution.

(ii) If $b = 0$, then L may be deleted from the system without changing the solution set of the system.

Part (i) comes from the fact that the degenerate equation has no solution, so the system has no solution. Part (ii) comes from the fact that every element in K^n is a solution of the degenerate equation.

Leading Unknown in a Nondegenerate Linear Equation

Now let L be a nondegenerate linear equation. This means one or more of the coefficients of L are not zero. By the *leading unknown* of L, we mean the first unknown in L with a nonzero coefficient. For example, x_3 and y are the leading unknowns, respectively, in the equations

$$0x_1 + 0x_2 + 5x_3 + 6x_4 + 0x_5 + 8x_6 = 7 \qquad \text{and} \qquad 0x + 2y - 4z = 5$$

We frequently omit terms with zero coefficients, so the above equations would be written as

$$5x_3 + 6x_4 + 8x_6 = 7 \qquad \text{and} \qquad 2y - 4z = 5$$

In such a case, the leading unknown appears first.

3.3 Equivalent Systems, Elementary Operations

Consider the system (3.2) of m linear equations in n unknowns. Let L be the linear equation obtained by multiplying the m equations by constants $c_1, c_2, \ldots, c_m$, respectively, and then adding the resulting equations. Specifically, let L be the following linear equation:

$$(c_1a_{11} + \cdots + c_ma_{m1})x_1 + \cdots + (c_1a_{1n} + \cdots + c_ma_{mn})x_n = c_1b_1 + \cdots + c_mb_m$$

Then L is called a *linear combination* of the equations in the system. One can easily show (Problem that any solution of the system (3.2) is also a solution of the linear combination L.

EXAMPLE 3.3 Let L_1, L_2, L_3 denote, respectively, the three equations in Exam equation obtained by multiplying L_1, L_2, L_3 by $3, -2, 4$, respectively, an

$$\begin{array}{ll}
3L_1: & 3x_1 + 3x_2 + 12x_3 + 9x_4 = 15 \\
-2L_2: & -4x_1 - 6x_2 - 2x_3 + 4x_4 = -2 \\
4L_3: & 4x_1 + 8x_2 - 20x_3 + 16x_4 = 12 \\
\hline
\text{(Sum) } L: & 3x_1 + 5x_2 - 10x_3 + 29x_4 = 25
\end{array}$$

The system (3.2) of linear equations is said to be *consistent* if it has one or more solutions, and it is said to be *inconsistent* if it has no solution. If the field K of scalars is infinite, such as when K is the real field $\mathbf{R}$ or the complex field $\mathbf{C}$, then we have the following important result.

THEOREM 3.1: Suppose the field K is infinite. Then any system $\mathscr{L}$ of linear equations has (i) a unique solution, (ii) no solution, or (iii) an infinite number of solutions.

This situation is pictured in Fig. 3-1. The three cases have a geometrical description when the system $\mathscr{L}$ consists of two equations in two unknowns (Section 3.4).

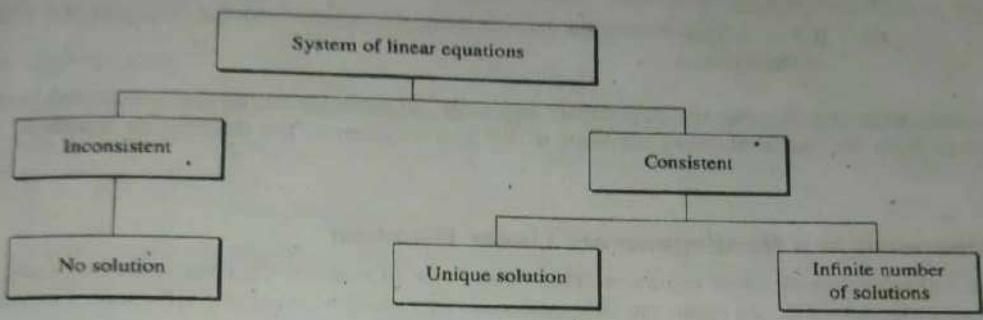

Figure 3-1

Augmented and Coefficient Matrices of a System

Consider again the general system (3.2) of m equations in n unknowns. Such a system has associated with it the following two matrices:

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The first matrix M is called the *augmented matrix* of the system, and the second matrix A is called the *coefficient matrix*.

The coefficient matrix A is simply the matrix of coefficients, which is the augmented matrix M without the last column of constants. Some texts write $M = [A, B]$ to emphasize the two parts of M, where B denotes the column vector of constants. The augmented matrix M and the coefficient matrix A of the system in Example 3.2 are as follows:

$$M = \begin{bmatrix} 1 & 1 & 4 & 3 & 5 \\ 2 & 3 & 1 & -2 & 1 \\ 1 & 2 & -5 & 4 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 4 & 3 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & -5 & 4 \end{bmatrix}$$

As expected, A consists of all the columns of M except the last, which is the column of constants.

Clearly, a system of linear equations is completely determined by its augmented matrix M, and vice versa. Specifically, each row of M corresponds to an equation of the system, and each column of M corresponds to the coefficients of an unknown, except for the last column, which corresponds to the constants of the system.

Degenerate Linear Equations

A linear equation is said to be *degenerate* if all the coefficients are zero—that is, if it has the form

$$0x_1 + 0x_2 + \cdots + 0x_n = b \tag{3.3}$$

HW **EXAMPLE 3.1** Consider the following linear equation in three unknowns x, y, z:

$$x + 2y - 3z = 6$$

We note that $x = 5, y = 2, z = 1$, or, equivalently, the vector $u = (5, 2, 1)$ is a solution of the equation. That is,

$$5 + 2(2) - 3(1) = 6 \quad \text{or} \quad 5 + 4 - 3 = 6 \quad \text{or} \quad 6 = 6$$

On the other hand, $w = (1, 2, 3)$ is not a solution, because on substitution, we do not get a true statement:

$$1 + 2(2) - 3(3) = 6 \quad \text{or} \quad 1 + 4 - 9 = 6 \quad \text{or} \quad -4 = 6$$

System of Linear Equations

A system of linear equations is a list of linear equations with the same unknowns. In particular, a system of m linear equations $L_1, L_2, \ldots, L_m$ in n unknowns $x_1, x_2, \ldots, x_n$ can be put in the *standard form*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{3.2}$$

where the a_{ij} and b_i are constants. The number a_{ij} is the *coefficient* of the unknown x_j in the equation L_i, and the number b_i is the *constant* of the equation L_i.

The system (3.2) is called an $m \times n$ (read: m by n) system. It is called a *square system* if $m = n$—that is, if the number m of equations is equal to the number n of unknowns.

The system (3.2) is said to be *homogeneous* if all the constant terms are zero—that is, if $b_1 = 0$, $b_2 = 0, \ldots, b_m = 0$. Otherwise the system is said to be *nonhomogeneous*.

A *solution* (or a *particular solution*) of the system (3.2) is a list of values for the unknowns or, equivalently, a vector u in K^n, which is a solution of each of the equations in the system. The set of all solutions of the system is called the *solution set* or the *general solution* of the system.

HW **EXAMPLE 3.2** Consider the following system of linear equations:

$$\begin{aligned} x_1 + x_2 + 4x_3 + 3x_4 &= 5 \\ 2x_1 + 3x_2 + x_3 - 2x_4 &= 1 \\ x_1 + 2x_2 - 5x_3 + 4x_4 &= 3 \end{aligned}$$

It is a 3×4 system because it has three equations in four unknowns. Determine whether (a) $u = (-8, 6, 1, 1)$ and (b) $v = (-10, 5, 1, 2)$ are solutions of the system.

(a) Substitute the values of u in each equation, obtaining

$$\begin{aligned} -8 + 6 + 4(1) + 3(1) = 5 \quad &\text{or} \quad -8 + 6 + 4 + 3 = 5 \quad \text{or} \quad 5 = 5 \\ 2(-8) + 3(6) + 1 - 2(1) = 1 \quad &\text{or} \quad -16 + 18 + 1 - 2 = 1 \quad \text{or} \quad 1 = 1 \\ -8 + 2(6) - 5(1) + 4(1) = 3 \quad &\text{or} \quad -8 + 12 - 5 + 4 = 3 \quad \text{or} \quad 3 = 3 \end{aligned}$$

Yes, u is a solution of the system because it is a solution of each equation.

(b) Substitute the values of v into each successive equation, obtaining

$$\begin{aligned} -10 + 5 + 4(1) + 3(2) = 5 \quad &\text{or} \quad -10 + 5 + 4 + 6 = 5 \quad \text{or} \quad 5 = 5 \\ 2(-10) + 3(5) + 1 - 2(2) = 1 \quad &\text{or} \quad -20 + 15 + 1 - 4 = 1 \quad \text{or} \quad -8 = 1 \end{aligned}$$

No, v is not a solution of the system, because it is not a solution of the second equation. (We do not need to substitute v into the third equation.)

CHAPTER 3

Systems of Linear Equations

3.1 Introduction

Systems of linear equations play an important and motivating role in the subject of linear algebra. In fact, many problems in linear algebra reduce to finding the solution of a system of linear equations. Thus, the techniques introduced in this chapter will be applicable to abstract ideas introduced later. On the other hand, some of the abstract results will give us new insights into the structure and properties of systems of linear equations.

All our systems of linear equations involve scalars as both coefficients and constants, and such scalars may come from any number field K. There is almost no loss in generality if the reader assumes that all our scalars are real numbers—that is, that they come from the real field $\mathbf{R}$.

3.2 Basic Definitions, Solutions

This section gives basic definitions connected with the solutions of systems of linear equations. The actual algorithms for finding such solutions will be treated later.

Linear Equation and Solutions

A *linear equation* in unknowns $x_1, x_2, \ldots, x_n$ is an equation that can be put in the *standard form*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{3.1}$$

where $a_1, a_2, \ldots, a_n$, and b are constants. The constant a_k is called the *coefficient* of x_k, and b is called the *constant term* of the equation.

A solution of the linear equation (3.1) is a list of values for the unknowns or, equivalently, a vector u in K^n, say

$$x_1 = k_1, \quad x_2 = k_2, \quad \ldots, \quad x_n = k_n \qquad \text{or} \qquad u = (k_1, k_2, \ldots, k_n)$$

such that the following statement (obtained by substituting k_i for x_i in the equation) is true:

$$a_1k_1 + a_2k_2 + \cdots + a_nk_n = b$$

In such a case we say that u *satisfies* the equation.

Remark: Equation (3.1) implicitly assumes there is an ordering of the unknowns. In order to avoid subscripts, we will usually use x, y for two unknowns; x, y, z for three unknowns; and x, y, z, t for four unknowns; they will be ordered as shown.